PME 43

CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

07-12 July 2019 **Pretoria South Africa**

Proceedings

43rd Annual Meeting of the International Group for the Psychology of Mathematics Education

VOLUME 1

Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminar, Colloquium, National Presentation, Regional Conference Report

Editors:
Mellony Graven, Hamsa Venkat, Anthony A Essien and Pamela Vale

Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education

Pretoria, South Africa

7 – 12 July 2019

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Mellony Graven

Hamsa Venkat

Anthony A Essien

Pamela Vale

Volume 1

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PREFACE

We are delighted to welcome you to the 43rd Annual Conference of the International Group for the Psychology of Mathematics Education in South Africa. PME is among the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. This year we have 340 presentations from 56 different countries. This number includes a substantial number of papers from across Africa, and we are particularly proud of this broadening base of participation in mathematics education research. We expect to host in the region of 430 delegates at PME 43.

PME 43 is hosted by the University of Pretoria. Pretoria (now often known as Tshwane), is the administrative capital of South Africa. The conference is on during the South African winter, but winter in Pretoria is sunnier than summer in many countries! This is the second time PME is organized in South Africa (PME 22 was held at Stellenbosch University in the Western Cape in 1998), and we look forward to providing participants with a warm South African welcome.

The theme of PME 43 is 'Improving access to the power of mathematics'. This theme was selected to reflect the priorities of many of those working in mathematics education in our region. While significant progress has been made in relation to providing physical access to schooling, access to high quality learning experiences that reveal the power of mathematics remain elusive. Many of the papers submitted to PME 43 indicate that this theme remains relevant in many parts of the world. We look forward to conversations across borders that can help us to move forward in relation to shared goals of improving access.

We are delighted to have leading scholars as plenary speakers and panelists from a wide range of countries with vastly different education contexts. This diversity in these plenary sessions we believe will strengthen our deliberations throughout the conference on improving access to the power of mathematics globally.

PME 43 continues with sessions across a range of formats: Research reports, Oral Communications and Poster Presentations at the individual level, and, Research Forums, Working Groups, Seminars and Colloquia at the group level. We include a Regional Presentation this year, pulling together inputs from the African and Southern African region, and providing insights into growing participation in mathematics and mathematics education research in Africa. In addition we include a regional conference report on the first PME conference held in South America.

The papers in the four volumes of these proceedings are organized according to type of presentation. Volume 1 contains the Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminars, Colloquia, the Regional Presentation and a PME Regional Conference Report. Volumes 2 & 3 contain the Research Reports, while Volume 4 consists of the Oral Communications and Poster Presentations.

The organization of PME 43 is a collaborative effort involving colleagues from the University of Pretoria, the University of the Witwatersrand, Rhodes University, and the University of Johannesburg. The conference is organized with the support of three committees: the International Program Committee for PME 43, the International Committee of PME together with the PME Administrative Manager, and the Local Organizing Committee. We acknowledge the support and effort of all involved in making the conference possible and offer our heartfelt thanks to all the people who have given so generously of their time and expertise.

We also thank each PME participant for making the journey to PME 43 in Pretoria and for your contributions to this conference. As Co-Chairs of the Programme Committee, we have already read your contributions as part of our work compiling the Programme and the Proceedings and we are excited for what is to come as the conference unfolds. We look forward to interacting further with you at the Conference and learning more about your work in mathematics education.

We hope that you enjoy PME 43 intellectually and socially. On the scientific side, this will mean going away with new ideas, insights and research directions. On the social side, this will mean going away with new friendships and opportunities for ongoing collaborations. We fully expect that PME 43 will deliver on both of these goals. Thanks once again for your participation.

Mellony Graven and Hamsa Venkat

PME 43 International Program Committee Chairs

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THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976. Its former presidents have been:

Efraim Fischbein, Israel
Stephen Lerman, UK
Richard R. Skemp, UK
Gilah Leder, Australia
Gerard Vergnaud, France
Rina Hershkowitz, Israel
Kevin F. Collis, Australia
Chris Breen, South Africa
Pearla Nesher, Israel
Fou-Lai Lin, Taiwan
Nicolas Balacheff, France
João Filipe Matos, Portugal
Kathleen Hart, UK
Barbara Jaworski, UK
Carolyn Kieran, Canada

The current president is **Peter Liljedahl** (Canada). The president-elect is **Markku Hannula** (Finland).

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting on August 17, 1980 and changed by the Annual General Meetings on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005 and on July 21, 2012. The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

All information concerning PME and its constitution can be found at the PME website: www.igpme.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the PME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution, and past conferences can be found at the PME website: www.igpme.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)

Hans Freudenthal (Deceased)

Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager:

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Paderborn University
Postal address:
Institut für Mathematik
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INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference. The IC work is led by the PME president who is elected by PME members for three years.

President

Peter Liljedahl (Canada)

President-Elect

Markku Hannula (Finland)

Vice-President

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Secretary

Einat Heyd-Metzuyanim (Israel)

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Richard Barwell	Canada
Yiming Cao	China
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Anika Dreher	Germany
Anthony A Essien	South Africa
Mellony Graven	South Africa
Maitree Inprasitha	Thailand
Berinderjeet Kaur	Singapore
Maria Mellone	Italy
Miguel Ribeiro	Brazil
Kai-Lin Yang	Taiwan

PROCEEDINGS OF PREVIOUS PME CONFERENCES

The table includes the ERIC numbers, links to download, ISBN/ISSN of the proceedings, and/or the website address of annual PME.

No.	Year	Location	ERIC number, ISBN/ISSN and/or website address
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	1983	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130 (vol.1) ED411131 (vol. 2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532, ISSN 0771-100X
12	1988	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	1989	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	1990	Oaxtepex, Mexico	ED411137 (vol. 1)

			ED411138 (vol. 2) ED411139 (vol. 3)
15	1991	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)
20	1996	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	1998	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN 0771-100X
23	1999	Haifa, Israel	ED436403, ISSN 0771-100X
24	2000	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4)

			ISSN 0771-100X
25	2001	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	2002	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	2003	Honolulu, Hawai'i, USA	ED500857 (vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN 0771-100X http://www.hawaii.edu/pme27
28	2004	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN 0771-100X www.emis.de/proceedings/PME28
29	2005	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN 0771-100X
30	2006	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN 0771-100X http://class.pedf.cuni.cz/pme30
31	2007	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2)

			ED499416 (vol. 3) ED499418 (vol. 4) ISSN 0771-100X
32	2008	Morelia, Mexico	ISBN 978-968-9020-06-6 ISSN 0771-100X http://www.pme32-na30.org.mx
33	2009	Thessaloniki, Greece	ISBN 978-960-243-652-3 ISSN 0771-100X
34	2010	Belo Horizonte, Brazil	ISSN 0771-100X http://pme34.lcc.ufmg.br
35	2011	Ankara, Turkey	ISBN 978-975-429-262-6 ISSN 0771-100X http://www.arber.com.tr/pme35.org
36	2012	Taipei, Taiwan	ISSN 0771-100X http://tame.tw/pme36
37	2013	Kiel, Germany	ISBN 978-3-89088-287-1 ISSN 0771-100X http:// http://www.pme2013.de/
38	2014	Vancouver, Canada	ISBN 978-0-86491-360-9 ISSN 0771-100X http://www.pme38.com/
39	2015	Hobart, Australia	ISBN 978-1-86295-829-6 ISSN 0771-100X http://www.pme39.com
40	2016	Szeged, Hungary	ISSN 0771-100X http://pme40.hu
41	2017	Singapore	ISBN 978-981-11-3742-6 http://math.nie.edu.sg/pme41
42	2018	Umeå, Sweden	ISSN 0771-100X ISBN (Vol 1) 978-91-7601-902-3 ISBN (Vol 2) 978-91-7601-903-0

			ISBN (Vol 3) 978-91-7601-904-7 ISBN (Vol 4) 978-91-7601-905-4 ISBN (Vol 5) 978-91-7601-906-1 http://www.pme42.se/

THE PME 43 CONFERENCE

Two committees are responsible for the organization of the PME 43 Conference: the International Program Committee (IPC) and the Local Organizing Committee (LOC).

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

Mellony Graven	Rhodes University (South Africa) Co-chair, PME representative
Hamsa Venkat	University of the Witwatersrand (South Africa) Co-chair, LOC representative
Anthony A Essien	University of the Witwatersrand (South Africa) LOC representative
Pamela Vale	Rhodes University (South Africa) Co-opted LOC representative
Johann Engelbrecht	University of Pretoria (South Africa) LOC representative and Conference Chair
Peter Liljedahl	Simon Fraser University (Canada) PME President
Ugorji Ogbonnaya	University of Pretoria (South Africa) LOC representative
Maithree Inprasitha	Khon Kaen University (Thailand) PME representative
Miguel Ribeiro	University State of Campinas – UNICAMP (Brazil) PME representative
Markku Hannula	University of Helsinki (Finland) Co-opted as incoming president

THE LOCAL ORGANISING COMMITTEE (LOC)

Johann Engelbrecht (Conference chair, University of Pretoria)

Sonja van Putten (Vice chair, University of Pretoria)

Mellony Graven (Rhodes University)

Hamsa Venkat (WITS University)

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Pamela Vale (Rhodes University)

Adri van der Nest (University of South Africa)

Harry Wiggins (University of Pretoria)

ACKNOWLEDGMENTS

We thank all of our reviewers and the IPC for the detailed review work that has led to the presentations in these proceedings. In particular, we also thank Caroline Long and Arindam Bose for their editorial support on the OCs and PPs.

REVIEW PROCESS OF PME 43

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The number of submitted RR proposals was 213, and 128 of them were accepted. Of those not accepted as RR proposals, 70 were invited to be re-submitted as Oral Communication (OC) and 18 as Poster Presentation (PP).

As in previous years, every RR submission underwent a fully independent double-blind peer review by three international experts in the field in order to decide acceptance for the conference.

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted OC proposals was 116, and 81 of them were accepted. Of those not accepted as OC proposals, 36 were invited to be re-submitted as Poster Presentation (PP). In the end, considering re-submissions of Research Reports as Oral Communications, 124 OCs were accepted for presentation at PME 43.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted PP proposals was 72, and 60 of them were accepted. In the end, considering re-submissions of Research Reports and Oral Communications as Poster Presentations, 82 PPs were accepted for presentation at PME 43.

COLLOQUIUM (CQ)

The goal of a Colloquium is to provide the opportunity to present a set of three papers that are interrelated in a particular way (e.g., they are connected through related or contrasting theoretical stances, use identical instruments or methods, or focus on closely related research questions), and to initiate a discussion with the audience on the interrelated set.

The number of submitted CQ proposals was 1, and it was accepted.

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering PME members more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field.

The number of submitted RF proposals was 3, and all of them were accepted.

WORKING GROUPS (WG)

The aim of Working Group is that PME participants are offered the opportunity to engage in exchange or to collaborate in respect to a common research topic (e.g., start a joint research activity, share research experiences, continue or engage in academic discourse). A Working Group may deal with emerging topics (in the sense of newly developing) as well as topics that are not new but possibly subject to changes. It must provide opportunities for contributions of the participants that are aligned with a clear goal (e.g. share materials, work collaboratively on texts, and discuss well-specified questions). A Working Group is not supposed to be a collection of individual research presentations (see Colloquium format), but instead is meant to build a coherent opportunity to work on a common research topic. In contrast to the Research Forum format that is meant to present the state of the art of established research topics, Working Groups are considered to involve fields where research topics are evolving.

The number of submitted WG proposals was 14 and 10 of them were accepted.

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing, or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively.

The number of submitted SE proposals was 1, and this was accepted.

LIST OF PME 43 REVIEWERS

The International Program Committee of PME 43 thanks the following people for their help in the review process.

A

Abdulhamid,Lawan (South Africa)

Adler, Jill (South Africa)

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Ofri, Ofra (Israel)

Ogbonnaya, Ugorji Iheanachor (South Africa)

Okazaki, Masakazu (Japan)

Okumuş, Samet (Turkey)

Olteanu, Lucian (Sweden)

Orellana, Claudia Andrea (Australia)

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Patsiomitou, Stavroula (Greece)

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Surith, Dhanya (United Arab Emirates)

Suurtamm, Chris (Canada)

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Sztajn, Paola (United States)

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Tso, Tai-Yih (Taiwan)

Tzekaki, (Marianna Greece)

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Van Zoest, Laura (United States)

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Viirman, Olov (Sweden)

Vogel, Rose F. (Germany)

Voica, Cristian (Romania)

Vollstedt, Maike (Germany)

Vos, Pauline (Norway)

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Warner, Lisa (United States)

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Webb, Lyn (South Africa)

Weigand, Hans-Georg (Germany)

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Wilson, Kirsty (United Kingdom)

Wilson, Sue (Australia)

Wu, Der-Bang (Taiwan)

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PLENARY LECTURES

INSTITUTIONAL NORMS: THE ASSUMED, THE ACTUAL, AND THE POSSIBLE

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Much of what happens in classrooms is predicated on an assumption that the foundational normative structures that define what a classroom looks like are non-negotiable. These normative structures transcend the classroom, the school, and even international boundaries, and have led to classrooms from around the world to look more alike than they look different. In this paper I look at some of the implications of these normative structures on student behaviour and present some of what is possible when we are willing to tear down these norms.

INTRODUCTION

In 2003 Jane had already been teaching mathematics for over 10 years. At the time, the province in which Jane worked was getting ready to bring in a new mathematics curriculum which, it was being reported, was going to be placing a greater emphasis on reasoning, numeracy, and problem solving. Jane, who had always managed to avoid these aspects of curricula in her teaching decided that maybe it was time to get ahead of the change and begin to implement problem solving in her grade 8 classroom. Not knowing where or how to begin with this, she reached out to me.

At the time, I was a doctoral student working on a thesis tangentially related to problem solving. As a former high school mathematics teacher, I was missing being in the classroom. So, I was overjoyed by the prospect of working with Jane to co-plan and co-teach some lessons using problem solving. Jane, as it turned out, was not interested in either co-planning or co-teaching a lesson – and she made this very clear during our first meeting. All that she wanted from me was a collection of *good* problems that she could begin to experiment with in her classroom. After some tense negotiations we came to the agreement that I would provide Jane with problems and she would let me observe her use them. With this concession in hand, the first problem I gave Jane was one that I had used in my own teaching with grade 8 students with much success.

> If 6 cats can kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes? (Lewis Carroll, cited in Wakeling, 1995, p. 34)

June accepted this problem in good faith and, with me observing, used it the next day. It did not go well. Immediately after posting the question on the board a forest of arms shot up and Jane began to frantically move around the room to answer questions. As the period went on, students became discouraged and began to give up. In response, Jane, while continuing to answer the questions of those who were stuck, began to motivate those who had quit. This continued for the whole 40-minute period. In the

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 1-16). Pretoria, South Africa: PME.

end, one pair of students, with much help and encouragement form Jane, had managed to solve the problem. The rest of the students were, for the most part, not even close to solving the problem, and most had given up completely. But Jane was tenacious and asked for a new problem. The results on the second day were as abysmal as they had been on the first. The same was true on the third day.

Over the course of three 40-minute classes we had seen no improvement in either the students' efforts to solve the problem or their abilities to do so. So, Jane decided it was time to give up. Her efforts to bring problem solving to her students had been met with resistance and challenge and resulted in few, if any, rewards. This was as disappointing to me as it was to Jane. But, I was not ready to give up yet. I wanted to understand why the results had been so poor and why the students were so quick to give up. So, I asked Jane if I could remain in the classroom to observe her and her students in their normal classroom routines. Jane agreed. So, I sat in the back of Jane's classroom and watched her teach a variety of courses to a variety of students, including the grade 8 students who had performed so poorly in the problem solving activities.

After three full days of observation I began to discern a number of patterns. The first pattern that emerged was that Jane was clearly a dedicated and committed teacher - she cared about her students, her teaching, and the curriculum. The second pattern to emerge was that, despite Jane's commitment, the students seemed to be lacking in effort. At first, this manifested itself in my observations of widespread apathy and lethargy, but as the days went on I realized that what I was seeing was really a lack of thinking. As this realization dawned, I began to reflect back on my observations and field notes and realized that over the course of three days and 21 lessons, I had not once seen Jane's students be asked to think. This is not to say that there was not activity. There was lots of activity – notes, worksheets, and exercises – none of which required the students to think. This realization did not take away from my initial observations that Jane was a dedicated teacher. She cared deeply about delivering the curriculum to her students, that her students learn this curriculum, and that they perform well on assessment.

Jane, as it turned out, was in a difficult position. It was nearing the end of the year, she still had content to cover, and her students were not thinking. As a result, Jane was planning her teaching on the assumption that her students either couldn't or wouldn't think. This was the third pattern I observed.

Once these realizations had set in I decided to see if they were unique to Jane and her classroom. So, I visited another teacher's classroom in the same school. I saw the same things – students not thinking and the teacher planning on the assumption that students either couldn't or wouldn't think. I visited another two teachers – same thing. The fact that I saw the same thing – something I had never noticed before – in four classrooms in the same school was surprising to me. Each of these teachers had their own style of teaching and relating to the students. But, these differences aside, what I observed through all the activity was that, at a fundamental level, students were not thinking, and the teachers were planning on the assumption that students either couldn't or wouldn't think. Instead, they were structuring activities, sometimes very carefully planned activities, that would allow them to move through curriculum without requiring students to have to think.

I wanted to know if this was a phenomenon that was unique to the school, the grade band, and/or the school demographic. So, over the course of many months I began to visit different classrooms in different schools. The way I found these classrooms was through the technique of snowballing (Mishler, 1986; Noy, 2008; Patton,1990). At the outset, I asked friends and former colleagues to recommend some teachers that they had heard were good. I contacted these teachers and asked if I could visit their classroom for a day or part of a day. During these visits I would then ask the host teacher to recommend a teacher in a different school that they had heard was good. Through this process I visited classrooms of every grade from kindergarten to grade 12. I visited regular classes, advanced classes, and remedial classes. I visited classrooms in both low socioeconomic and high socioeconomic neighbourhoods. And I visited classrooms taught in English and in French. In all, I visited 40 different classrooms in 40 different schools. And everywhere I went, regardless or school, grade, or demographics, I saw the same thing – students not thinking and teachers planning on the assumption that the students either couldn't or wouldn't think. This phenomenon could no longer be overlooked. I wanted to understand it – to explain it (Niss, 2018).

Immediately this phenomenon fractured into two separate research questions for me. The first question had to do with student behaviours. If the students are not thinking, as I had observed, then what exactly are they doing. This question led me down a branch of inquiry that I call *studenting* and I will discuss this in the next section. The second question that emerged for me was centred on the desire to find a way to change what I was seeing – to find a way to transform these non-thinking classrooms into thinking classrooms. This led me down a path of research I came to call *Building Thinking Classrooms* and will be discussed later in the article.

STUDENTING

In 1986, Gary Fenstermacher introduced the term *studenting* to serve as the analogue to *teaching* and to describe the classroom activities of students that include, but are not exclusive to learning.

> The concept of studenting or pupiling is far and away the more parallel concept to that of teaching. Without students, we would not have the concept of teacher; without teachers, we would not have the concept of student. Here is a balanced ontologically dependent pair, coherently parallel to looking and finding, racing and winning. There are a range of activities connected with studenting that complement the activities of teaching. For example, teachers explain, describe, define, refer, correct, and encourage. Students recite, practice, seek assistance, review, check, locate sources, and access material. (p. 39)

In essence, studenting is what students do in a learning setting.

> … there is much more to studenting than learning how to learn. In the school setting, studenting includes getting along with one's teachers, coping with one's peers, dealing with one's parents about being a student, and handling the non-academic aspects of school life. (p. 39)

In 1994, Fenstermacher expanded this definition to include some of the darker side of studenting behaviours.

> [T]hings that students do such as 'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments, threading the right line between curricular and extra-curricular activities, and determining what is likely to be on the test and what is not. (p. 1)

Taken together, the understanding of studenting as what students do while in a learning situation expands our ability to talk about student behaviour in classroom settings. More specifically, it gives us a name for the autonomous actions of students that may or may not be in alignment with the goals of the teacher. As such, studenting extends constructs such as the didactic contract (Brousseau, 1997) and classroom norms (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996) to encompass a broader spectrum of classroom behaviours – behaviours that are not necessarily predicated on an assumption of intended learning. In collaboration with Darien Allan, we used studenting as a lens to explore what students are actually doing in a classroom when they are not, as I had observed, thinking (Allan, 2017; Liljedahl & Allan, 2013a, 2013b, in press).

A lesson is comprised of a wide array of teaching and student activity including, but not limited to, direct instruction, notes, individual or group work on assignments, homework, worksheets, small group or whole class discussion, student presentations, review, and assessment. Darien and I used these discrete instances of activity to disaggregate the complexity of a lesson into its discrete parts – or activity settings – for the purpose of documenting studenting behaviour within the specific instances of a lesson.

The data we collected for this research consisted of classroom videos, field notes, and post observation interviews with students. Using a grounded theory (Charmaz, 2006) approach these data were continually analysed between observations. From this analysis, over time, a number of interesting studenting behaviours began to emerge within each activity settings. As these behaviours emerge and clarity was gained, coding for these now known studenting behaviours in subsequent observations became easier. Over time a form of saturation was reached as new observations of these activity settings no longer revealed new studenting behaviours. When this occurred, we said that a taxonomy of studenting behaviour in that activity setting had been reached.

Once such a taxonomy had been achieved, for subsequent observations no video was used. Instead, we simply used our pre-established codes to annotate observed student behaviour on a supplied seating chart of the classroom during the targeted activity setting. Immediately after these observations, while students began to work on their assigned homework, as well as for a few minutes after class, we collected very brief interview data from a number of students selected based on the different behaviours we saw exhibited during our observations. The interviews were short (1-4 minutes) and were audio recorded using a portable digital recorder. For the most part these interviews consisted of a brief declaration of what we had observed them doing and one or two questions regarding their reasons for their behaviour. Added to this were lengthier interviews with the teacher before and after the lesson in order to understand his/her goals for the lesson and to gain greater perspective on some of the activity setting behaviours we observed.

In what follows I present the catalogues of behaviours Darien and I documented in two mathematics classroom activity settings – *now you try one* tasks and notes.

Now You Try One

Now you try one tasks (as we came to call them) are the tasks assigned, usually one at a time, by classroom teachers immediately after they have done some direct instruction and presented some worked examples. This method of teaching is the most prevalent method we encountered at the grade 10-12 levels. The data for what I present here came from a single lesson on completing the square as a way to graph quadratic functions being taught in a grade 11 classroom (n = 32). Our earlier work had previously established a taxonomy of five studenting behaviours within this activity setting – *slacking, stalling, faking, mimicking,* and *thinking*.

Slacking is the behaviour assigned to students who display a general lack of engagement and interest towards the lesson. Visibly these students pay little attention, took no notes, and when they are asked to try to solve an example on their own, they make no attempt to either do so or seek help. *Stalling* behaviours, on the other hand, are actions that can be seen as legitimate and not out of place in a lesson – going to the bathroom, going to get a drink of water, sharpening a pencil, looking in their backpack for a calculator. Although outwardly legitimate, interviews reveal that the purpose behind these behaviours is to avoid the task at hand. Like stalling, *faking* behaviour has an outward appearance of legitimacy. From the front of the class it appears as though the students are working on the assigned task – their heads are down, and their pencils are moving. But from the vantage point of the back of the classroom, however, it is clear that this is nothing more than a façade and no work is being done.

Behaviour	n	%
Slacking	3	9
Stalling	4	13
Faking	2	6

Mimicking	17	53
Thinking	6	19

Table 1: Distribution of student behaviours

Unlike the previous three behaviours, *mimicking* results from a legitimate effort to solve the *now you try one* tasks. This behaviour is punctuated by frequent references to notebooks, textbooks, or the notes on the board in an effort to map characteristics of a worked example onto the current task. Although these students are trying to solve the *now you try one* tasks, not one of the teachers whose classes the research was done in valued this behaviour. They wanted their student to *think* their way through the task – relying on reasoning, understanding, and connections to solve it.

The prevalence and distribution of each of these five behaviours within classes in general and a particular grade 11 class (n=32, see table 1)[1] gives an accounting of student behaviour in the *now you try one* activity setting and allowed us to name and nuance the non-thinking behaviour I had observed in the aforementioned 40 lessons.

Notetaking

Along with *now you try one* tasks, the practice of having students write notes was also prevalent in the classrooms in which the studenting research was being conducted, with some teachers devoting as much as 45 minutes in a single lesson to this activity. Most often this involved the students writing down what the teacher is writing on the board. Given both the frequency and duration of notetaking within the lessons we observed, saturation of studenting behaviours within this activity setting was quickly reached and consisted of three basic behaviours – *not taking notes, keeping up,* and *not keeping up*.

Like with slacking in the previous section, *not taking notes* is often a behaviour which overtly defies the wishes of the teacher and is often exhibited by the same students who slack during *now you try one* tasks. On rare occasions, however, it is also a behaviour exhibited by students who would rather listen to the teacher than write notes. Regardless, in classrooms where note taking is expected, almost all students take notes.

Of those who do take notes, studenting behaviour bifurcates into two sub-behaviours – *keeping up* and *not keeping up*. Although both of these behaviours result in the same notes being recorded, the way in which students who are keeping up engage with the process is very different from those who do not. To explain this, I need to differentiate between, what I came to call *live notes* and *dead notes*. Live notes are the real time generation of notes where the teacher is working through an example, demonstrating the sequence of how something is to be done, and providing a narrative as the work is unfolding. Whereas live notes are a chronologically linear process, they are often spatially non-linear.

[1] For a more detailed analysis see Liljedahl and Allan (2013a).

For example, consider a grade 10 teacher demonstrating how to make a graph of a function (see figure 1). Chronologically, the teacher would first write the function or relationship. Then they would make a table of values, perhaps generating a list of values for the x variable. This would be followed by calculating and recording the y-value for each x-value. The teacher would then draw an appropriate set of x and y coordinates with consideration of the domain and range of the values in their table of values. This would be followed by a labelling and numbering of the axis, plotting of each ordered pair from the table of value, and finally the drawing of the curve. All the while the teacher would be narrating what is happening and why certain choices and decisions are being made and how and why certain actions are being performed.

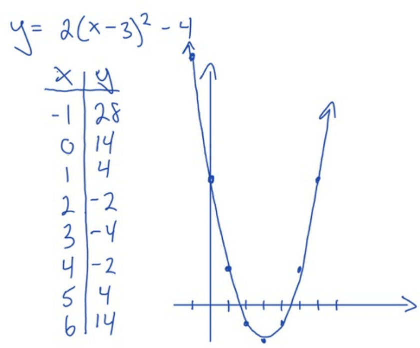

Figure 1: Graphing a function

A student who is keeping up the notes is able to engage in both the chronological and spatial unfolding of the process being demonstrated. A student who is not keeping up with the notes, however, is left to copy down the resultant static images – the dead notes – in which neither the chronological nor spatial sequencing is preserved. Copying an image without the benefit of chronological and spatial sequencing requires them to look repeatedly between the board and their page to discern all the details of, for example, the table of values, the axis, the plotted ordered pairs, and the graph into the correct place. Just plotting the points, without the benefit of the teacher's narrative of which ordered pair each point is associated with, takes a lot of precision – especially if they are just copying rather than thinking about how they are connected to the table of values. This takes more time, and so they fall further behind. And this is repeated until the point where they just stop listening to the teacher, stop thinking about what they are writing, and just copy what is on the board.

Behaviour	n	%
Don't take notes	3	10
Don't keep up	16	53
Keep up	11	37

Table 2: Distribution of student behaviours during note taking

In the grade 10 classroom where the data being presented here was gathered, only 11 students were keeping up with the notes (see table 2) while 16 were not keeping up. Interviews and deeper analysis showed that these 16 students were completely disengaged and mindlessly copying the notes. As with the research on the *now you try one* tasks, and all of the different activity settings, these results name and nuance the non-thinking behaviours I had observed in my visits to the aforementioned 40 classrooms.

Motives Behind Studenting Behaviour

These behaviours, regardless of how incongruous they are with the intentionality of the activity settings in which they occur, all stem from a common motive.

> Activity does not exist without a motive; 'non-motivated' activity is not activity without a motive but activity with a subjectively and objectively hidden motive. (Leontiev, 1978, p. 99).

We wanted to understand these motives, to explain the diversity of the studenting behaviours we had observed across the variety of activity settings. Activity theory (Leontiev, 1978) gave us the means to do this.

For the studenting research presented above we had placed the activity setting at the centre and studied the variety of behaviours surrounding it. To get at motives, however, we needed to place the individual student at the centre and, using an "*actions first*" strategy (Kaptelinin & Nardi, 2012), looked closely at their specific studenting behaviours across a wide range of activity settings. Coupling this with extended interviews we were able to discern the goals of an individual student and extrapolate upwards from there to higher goals and ultimately the motives behind all their studenting behaviours. In this regard activity theory served as both a theoretical lens and an analytical tool.

From this analysis emerged five primary motives, the two most prevalent of which were *learning* and *getting good grades*. Once in hand, these motives allowed us to nuance studenting behaviours that, on the surface, appeared to be the same. One of the main outcomes of this was the emergence of a distinction between continuous and discrete behaviours. What we noticed was that students with a motive of learning viewed every activity setting as an opportunity to learn leading to a continuous and contiguous pursuit of this motive. On the other hand, students with a motive of getting good grades only engaged in a specific activity setting if their participation could, in some way, and at that time, contribute to their mark. For example, although students of either motives would engage in *now you try one* tasks, students with the motive to learn did so authentically in every lesson. On the other hand, students with a motive to get good marks would engage with such tasks only during review sessions immediately prior to a test, while exhibiting non-thinking behaviours in other, less proximal, lessons[2].

The results of our research into studenting behaviours, and the motives behind them, confirmed my earlier observations that students were, by and large, not thinking. This is a problem. As mentioned earlier, this puts teachers in a difficult situation. Tasked with the responsibility of delivering a set curriculum these teachers were having to plan their teaching on the assumption that students either wouldn't, or couldn't, think. This is also a problem.

These problems eclipse many of our field's collective efforts to improve mathematics education writ large. No amount of nuancing of curricular content, for example, is going to have an impact on student learning if the students aren't thinking. I wanted to find a solution to this primary issue.

[2] See Allan (2017) and Liljedahl and Allan (in press) for a more detailed analysis.

INSTITUTIONAL NORMS

Back in 2003, when I was visiting the 40 classrooms, I noticed another, not yet mentioned, pattern – something that, until that point, I had not noticed before. Everywhere I went, irrespective of grade or demographic, classrooms looked more alike than they looked different. And what happened in those classrooms looked more alike than they looked different. There were differences, to be sure, but the majority of what I was seeing was the same. There were desks or tables, usually oriented towards a discernible front of the classroom. Towards this front was a teacher desk, some sort of vertical writing space, and some sort of a vertical projection space. Students sat, while the teacher stood. Students wrote horizontally while the teacher wrote vertically. And the lessons mostly followed the same rhythm – beginning with some sort of teacher led activity like a lecture or note taking, perhaps shifting to some sort of small or big group discussion, and then culminating in some sort of individual work. Even in the few more progressive classrooms I observed the physical space looked the same and the rhythm of the lesson was the same. What was different was the duration and nature of the activity in the middle of the lesson.

These normative structures that permeate classrooms in North America, and around the world, are so robust, so entrenched, that they transcend the idea of classroom norms (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996) and can only be described as institutional norms (Liu & Liljedahl, 2012) – norms that transcend the classroom, even individual schools, and have become ensconced in the very institution of school. When public education was created 100 to 200 years ago (depending on the country) it was essentially modelled on some combination of three institutions that were, at the time, seen as successful – the church, the factory, and the prison (Egan, 2002). Since its inception, not much has changed. Yes, desks look different now, and we have gone from blackboards to greenboards to whiteboards to smartboards, but students are still sitting, and teachers are still standing. There have been a lot of innovations in assessment, technology, and pedagogy, but much of the foundational structures that constitute what we think of as school has not changed.

From the studenting research we learned that, also irrespective of the grade or demographic, student behaviours were remarkably similar across the different activity settings. These studenting behaviours seemed to be impervious to the particular, yet minor, differences I saw from class to class. So, I began to wonder if these behaviours were somehow rooted in the very fabric of the institution of school – if they were implicated in, and by, the institutional norms I was seeing everywhere I went. If this was true, then the only way to change the non-thinking studenting behaviours, I reasoned, was to change the institutional norms in which they were occurring – to radically alter their experience of the classroom. Hence, the *Building Thinking Classrooms* project was born, a project that I have been pursuing for the last 16 years, involved the participation of hundreds of teachers, and has resulted in the foundational transformation of thousands of classrooms.

BUILDING THINKING CLASSROOMS

The Building Thinking Classroom project was centred around the idea of creating a radically different classroom experience within which, and because of which, students could allow themselves to begin to think. My initial efforts in this regard were unstructured and chaotic. I was creating change. And that change was having an effect on student thinking. But I was losing control of the ability to associate changes in student behaviour to particular changes in teacher practice. So, I decide to atomize teaching practice – to break it down into its components – and to experiment with each of these components independently.

Using a grounded theory approach (Charmaz, 2006) I reengaged with the data from my initial visits to 40 classrooms. I was looking for, and documenting, evidence of instances within a lesson that could be categorized as discrete, what I came to call, routines – for example, giving notes. Some of these *routines* overlapped with the activity settings discussed in the studenting research – notes, homework, assessment, etc. – whereas others were more subtle aspects of what teachers do and how they do it. This was followed by visits to another 20+ classrooms where these routines were used to code teaching while, at the same time, allowing for the emergence of new codes. Once saturation was reached, I had a list of 14 routines that could be used to encode all of what happened in the lessons I had observed.

- choice of task
- giving the task
- answering questions
- forming groups
- student workspace
- room organization
- student autonomy
- notes
- homework
- hints and extensions
- consolidation
- formative assessment
- summative assessment
- reporting out

I viewed each of these 14 routines as an opportunity to get students to think and my goal was to find a single practice, or set of practices, that would optimize this opportunity. So, using a combination of design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003) and action research (Greenwood & Levin, 2007), I began to work with teams of teachers to systematically look for practices within each routine that increased student thinking.

Over the course of the next 13 years I worked with over 400 teachers in two week cycles, testing, evaluating, and refining practices that could get students to think. Depending on the routine we were focused on we used a variety of observable proxies for thinking – enjoyment, engagement, participation, etc. – to gauge the effectiveness of what we were testing. For some routines we ran upwards of 20 iterations before we felt we were zeroing in on something that could be deemed to be an optimal practice for thinking. When these optimal practices emerged, I would often do a more in-depth comparative or case study of the practice in use in classrooms to try to document the effectiveness and to look for ancillary benefits – beyond thinking – of these new practices[3]. In the end, what emerged were 14 practices (one for each routine) that teachers could use to build a thinking classroom. Although a few of these were well-known, although not necessarily implemented, practices the majority were radical departures from institutionally normative practices entrenched in the classrooms I had been observing early on in my work.

1. *The type of tasks used*: Lessons should begin with good problem solving tasks. In the beginning of the school year these tasks need to be highly engaging, non-curricular tasks. After a period of time (1-2 weeks) these are gradually replaced with curricular problem solving tasks that then permeate the entirety of the lesson.

2. *When, where, and how tasks are given*: The first task of the day should be given verbally in the first 3-5 minutes of class with the students standing around the teacher in an open area of the room. If there are data, diagrams, or long expressions in the task then these can be written or projected on a wall or given as a handout, but the instructions pertaining to the activity of the task should still be given verbally.

3. *How groups are formed*: At the beginning of every class a visibly random method should be used to create groups of three students who will work together for the duration of the class.

4. *Student workspace*: Groups should stand and work on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This makes the work visible to the teacher and other groups.

5. *Room organization*: The classroom should be de-fronted with desks placed in a random configuration around the room (but away from the walls) and the teacher addressing the class from a variety of locations within the room.

6. *How questions are answered*: It turns out that students only ask three types of questions: (1) *proximity questions* – asked when the teacher is close; (2) *stop thinking questions* – most often of the form “is this right?” or "will this be on the test?"; and (3) *keep thinking questions* – questions that students ask so they can get back to work. The teacher should only answer keep thinking question.

7. *Hints and Extensions*: The teacher should maintain student engagement through a judicious and timely use of hints and extensions to maintain *flow* (Csíkszentmihályi, 1998, 1996, 1990) – a perfect balance between the challenge of the current task and the abilities of the students working on it.

8. *Autonomy*: Students should be encouraged to interact with other groups extensively, both for the purposes of extending their work and getting help. As much as possible, the teacher should occasion this interaction by directing students towards other groups when they are stuck or need an extension.

9. *Consolidation*: When every group has passed a minimum threshold the teacher should pull the students together to debrief what they have been doing. This debrief should begin at a level that every student in the room can participate in.

[3] See Liljedahl (2019, 2018, 2014) for examples of these.

10. *Meaningful notes*: Notes should consist of thoughtful notes written by students to their future selves. The students should have autonomy of what goes in these notes and how they are formatted and should be based on the work that is already existing on the boards from their own work, another group's work, or a combination of work from many groups.

11. *Check your understanding questions*: Rather than assigning homework or practice questions, students should be assigned 4-6 questions for them to check their understanding. The students should have the freedom to work on these questions in self-selected groups or on their own, and on the vertical non-permanent surfaces or on their desks. These questions should not be marked or checked for completeness – they are for the students' self-evaluation.

12. *Formative assessment*: Formative assessment should be focused primarily on informing students about where they are and where they are going in their learning. This will require, by necessity, a number of different activities from observation to *check your understanding questions* to unmarked quizzes where the teacher helps students to decode their demonstrated understandings.

13. *Summative assessment*: Summative assessment should honour the activities of a thinking classroom (evaluate what you value) through a focus on the processes of learning more so than the products of learning and should include the evaluation of both group work and individual work. Summative assessment should not in any way have a focus on ranking students.

14. *Reporting out*: Reporting out of students' performance should be based on the analysis of the data, rather than the counting of points, that is collected for each student within a reporting cycle. These data need to be analysed on a differentiated basis and be focused on discerning the learning that a student has demonstrated.

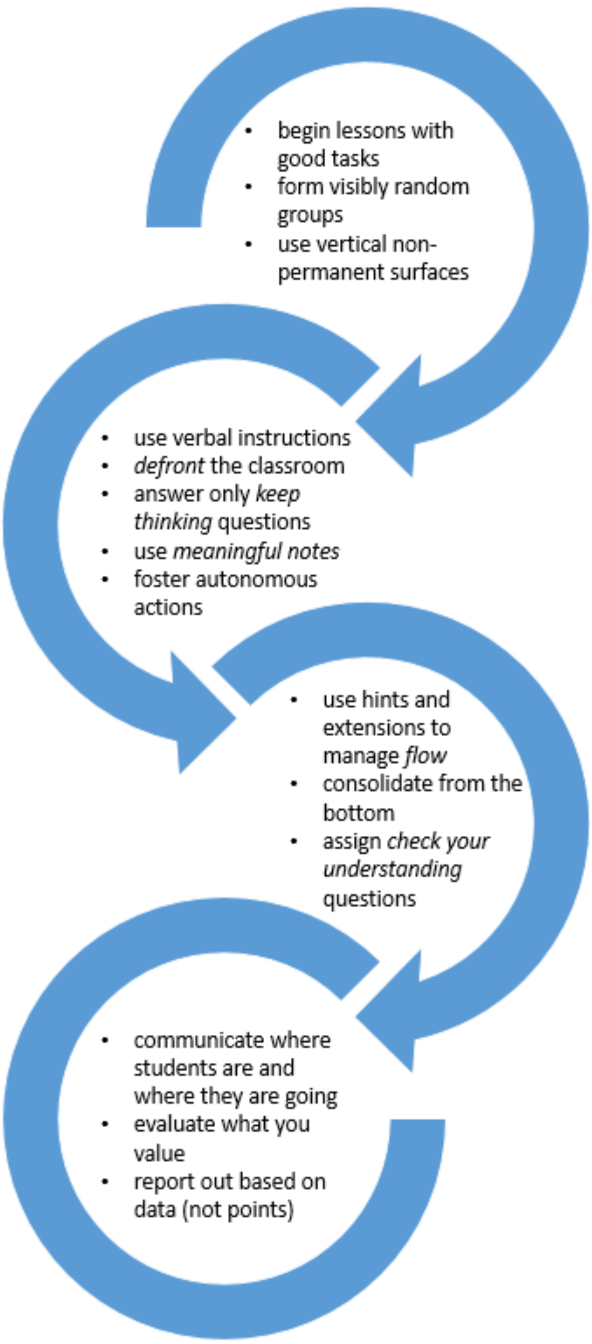

Figure 2: The Building Thinking Classrooms framework

What is presented above does not begin to capture the nuances and complexities of these practices. Nor does it capture the nuance and complexities of the research that went into emerging these practices. For example, in the third practice (how groups are formed), the research clearly showed that groups of three created an environment conducive to thinking much more so than either groups of two or four. For student workspace, we learned that having one marker or piece of chalk produced more thinking than if every student had their own. We learned that if students are allowed to present their own solutions during consolidation, the engagement of the rest of the class is greatly diminished. We learned so much about what practices get students to engage, to participate, and to think. And in every instance, we learned these things long before we had an explanation for why this was the case.

The Building Thinking Classrooms Framework

But now I had a new problem. After 13 years of research I had a toolkit comprised of 14 practices accompanied by 100's of micro-practices. How to disseminate all this knowledge to teachers? More than this, however, I was concerned with how students would react when teachers implemented these, sometimes radical, practices in their classrooms. Students are the biggest stakeholders in the classroom and their initial reactions to change have a tremendous impact on a teacher's ability to sustain a new practice with confidence and fidelity.

This created a new branch of my research in which I wanted to learn if there was a sequence for implementing the aforementioned 14 practices that would maximize student acceptance and teacher uptake. To find this out, 20 teams of 20 to 30 teachers were given professional development sessions on the 14 practices with each team receiving a different sequence of the practices to implement. What emerged from this research was a pseudo-hierarchy where the 14 practices organized themselves into four tool kits (see figure 2). The order within each toolkit, for the most part, does not matter. What matters is that the practices in any one toolkit are all implemented before the teacher moves onto the next toolkit.

Like with the research that emerged the 14 practices, this research produced results that preceded explanations. And like the previous research, these results are too nuanced and complex to be fully explained here. What can be stated with clarity, however, is that when a teacher begins their lesson with a good task, forms visibly random groups, and has groups work on vertical non-permanent surfaces, two things happen. The first is that there is a radical transformation of student engagement and thinking. The second is that there is a radical transformation in the willingness and desire of the teacher to continue with this practice. Research along this line showed that, of the 124 inservice teachers exposed to this combination of practices in a professional development setting, 100% left the session intended to implement the practices, 94% implemented them within the first week, and 97% were still implementing them after six weeks with all of them intending to continue with these practices (Liljedahl, 2019).

With regards to the entire framework, baseline assessment of student engagement and thinking showed that in a given class, on average, about 20% of the students spend approximately 10% of the class thinking – with the rest of the students spending close to zero time thinking. After full implementation of the Building Thinking Classroom framework, these same classes have, on average, 85% of the students thinking for 85% of the lesson. This is a massive change in student behaviour from both the baseline data and the studenting research.

By my estimate, there are currently thousands of classroom teachers who are implementing the Building Thinking Classroom to some extent. These teachers are primarily in Canada and the US, but there are also teachers in Chile, Iceland, Italy, Israel, Sweden, Finland, Norway, Denmark, and Germany who are implementing the framework to some extent within their classroom practice.

CONCLUSIONS

PME in particular, and the field of mathematics education in general, has been engaged in the pursuit of improved mathematics teaching and learning for over 40 years. In that time, we have made some amazing progress in our ability to foster and research change. But much of this research has been built around an assumption that the institutional norms that define what constitutes classroom and a lesson are, to a great extent, non-negotiable.

The Building Thinking Classrooms research and resultant framework has shown what is possible when we tear down these institutional norms. But this is only one possibility – one possible alternate reality. When we stop assuming that what is always has to be, a world of possibility emerges. My research placed student thinking at the centre and then tore down and built up practice with increased thinking as the only desirable outcome. What would happen if we put student equity, student identity, or student self-efficacy at the centre? What new practices and what new possibilities would emerge then?

PME, as an organization, could lead the way in this research. But first, we would need to re-examine the institutional norms that define and bind us. From what we consider research, to what we consider a research paper, to how that paper is reviewed, to the very name of our organization, we have been straining against our norms for some time. Perhaps it is time to break them?

References

Allan, D. (2017). *Student actions as a window into goals and motives in the secondary mathematics classroom*. Unpublished doctoral thesis, Simon Fraser University, Canada.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Charmaz, K. (2006). *Constructing grounded theory*. London; Thousand Oaks, CA: Sage Publications.

Cobb, P., Confrey, J. diSessa, A., Lehrer, R. and Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1) 9-13.

Cobb, P., Wood, T., & Yackel, E. (1991). Analogies from the philosophy and sociology of science for understanding classroom life. *Science Education, 75*(1), 23-44.

Csíkszentmihályi, M. (1998). *Finding Flow: The Psychology of Engagement with Everyday Life*. New York, NY: Basic Books

Csíkszentmihályi, M. (1996). *Creativity: Flow and the Psychology of Discovery and Invention*. New York, NY: Harper Perennial.

Csíkszentmihályi, M. (1990). *Flow: The Psychology of Optimal Experience*. New York, NY: Harper and Row.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5-8, 35-37.

Egan, K. (2002). *Getting it wrong from the beginning: Our progressivist inheritance from Herbert Spencer, John Dewey, and Jean Piaget*. New Haven, CT: Yale University Press.

Fenstermacher, G. (1986). Philosophy of research on teaching: Three aspects. In M.C. Whittrock (Ed.), *Handbook of Research on Teaching* (3rd edition, pp. 37-49). New York, NY: Macmillan.

Fenstermacher, G. (1994, revised 1997). *On the distinction between being a student and being a learner*. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Greenwood, D. J. & Levin, M. (2007). *Introduction to action research* (2nd edition). Thousand Oaks, CA: Sage Publications.

Kaptelinin, V., & Nardi, B. (2012). Activity theory in HCI: Basic concepts and applications. In J. Carroll (Ed.), *Synthesis lecture series*. Penn State University, PA: Morgan & Claypool.

Leontiev, A. (1978). *Activity, consciousness, and personality*. Englewood Cliffs, NJ: Prentice Hall.

Liljedahl, P. (2019). Conditions for Supporting Problem Solving: Vertical Non-Permanent Surfaces. In P. Liljedahl & M. Santos-Trigo (Eds.), *Mathematical problem solving: Current themes, trends, and research* (pp. 289-310). New York, NY: Springer.

Liljedahl, P. (2018). On the edges of flow: Student problem solving behavior. In S. Carreira, N. Amado, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect* (pp. 505-524). New York: Springer.

Liljedahl, P. (2014). The affordances of using visibly random groups in a mathematics classroom. In Y. Li, E. Silver, & S. Li (Eds.), *Transforming Mathematics Instruction: Multiple Approaches and Practice* (pp. 127-144). New York, NY: Springer.

Liljedahl, P. & Allan, D. (in press). Studenting: the behavior, the psychology, and the possibility. In B. Pieronkiewicz (Ed.), *Different perspectives on transgressions in mathematics and its education*. Towarzystwo Autorów i Wydawców Prac Naukowych Universitas.

Liljedahl, P., & Allan, D. (2013a). Studenting: The case of "now you try one". In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 257-264). Kiel, Germany: PME.

Liljedahl, P., & Allan, D. (2013b). Studenting: The case of homework. In M. V. Martinez & A. Castro Superfine (Eds.), *Proceedings of 35th Conference of North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 489-492). Chicago, USA: PMENA.

Liu, M. & Liljedahl, P. (2012). 'Not normal' classroom norms. In T.Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 300). Taipei, Taiwan.

Mishler, E. G. (1986). *Research interviewing: Context and narrative*. Cambridge, MA: Harvard University Press.

Niss, M. (2018). The very multi-faceted nature of mathematics education research. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 35-50). Umeå, Sweden: PME.

Noy, C. (2008). Sampling knowledge: The hermeneutics of snowball sampling in qualitative research. *International Journal of Social Research Methodology, 11*(4), 327–344.

Patton, M. Q. (1990). *Qualitative evaluation and research methods*. Beverly Hills, CA: Sage.

Wakeling, E. (1995) *Rediscovered Lewis Carroll Puzzles*. New York, NY: Dover Publications, Inc.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in mathematics Education, 27*(4), 458-477.

TRANSITION ZONES IN MATHEMATICS EDUCATION RESEARCH FOR THE DEVELOPMENT OF LANGUAGE AS RESOURCE

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Research has been able to detect systematic inequities of access to mathematics education in many parts of the world. In my work, I address inequity between socio-economic language groups and the dramatic extent to which it enters the mathematics classroom in the form of unequal opportunities to learn. In this text, I will focus on the concept of language as resource, specifically on the growing body of work regarding the pedagogic, epistemic and political value of the languages of learners and teachers in mathematics teaching and learning. The guiding questions that will help me in this endeavour are: 1) When and why did the concept of language as resource originate? 2) What does the use of the concept imply today in mathematics education research? 3) How is it particularly applicable to improve access to mathematics learning?

INTRODUCTION

As a mathematics teacher in a multilingual secondary school of Barcelona in the 1990s, I became involved in the teaching of mathematics in a language that was not a home language for a majority of learners who in addition came from socio-economically deprived areas of the city. I was lucky to undertake teaching duties that forced me to rethink much of what I believed at that time about a sort of spontaneous pedagogic power of *the* mathematical language. Since then, I have gained insight into the significance, prospects and implications of the languages of learners and teachers in school mathematics. Conversations with learners, teachers and teacher educators, classroom research and my own experience as a child learning in a language that was not my home language have been decisive. A major concern of my research program today in the domain of mathematics education and language is the understanding of the features of language in multilingual mathematics lessons and of uses of language for the generation of more democratic spaces where all learners have a voice (Civil & Planas, 2004). For this, the topic of language as resource and the contentious when not elusive use of this expression in mathematics education research (Chronaki & Planas, 2018; Planas, 2018; Planas & Schütte, 2018) is especially important. In line with my program and the theme of the conference, I attempt to answer the following questions:

- When and why did the concept of language as resource originate?
- What does the use of the concept imply today in mathematics education research?
- How is it particularly applicable to improve access to mathematics learning?

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-31). Pretoria, South Africa: PME.

The study of the inception, field-specific development and gradual maturity of the topic of language as resource illuminates ongoing processes of theorization of language in mathematics education research. The evidence that these processes are unfinished and open to revision (Barwell & Pimm, 2016; Planas, Morgan & Schütte, 2018; Radford & Barwell, 2016) is probably one of the reasons why the intricate conceptualization of language as resource is also unfinished. The location of language in the field continues to present many ontological, epistemological and analytical challenges. Given these challenges and the multifaceted approaches to language as resource, my central argument is that the search for conceptual clarity requires building work at a complicate metatheoretical level. Such level takes place in the transition zones of ontological assumptions about the nature of language, epistemological stances about the production of knowledge, and analytical frameworks about ways of studying language. Much of this building work is concurrently happening in the transition zones of a variety of cross-disciplines like discourse analysis, social semiotics and philosophy of language. All these transition zones together are unique and complement each other in their capacity to support mathematics education research on language.

WHEN AND WHY DID LANGUAGE AS RESOURCE ORIGINATE?

Almost forty years ago, scholars, policy-makers and practitioners began to read about distinct orientations to language planning called “language-as-resource”, “language-as-right” and “language-as-problem” in the work of Ruíz (1984). Drawing upon the idea that languages are often framed as resources, rights or problems, Ruíz’s classical orientations inform about three broad intersecting groupings of attitudes, preferences and responses of the language policy and practice in education (e.g., the language of instruction), in the workplace, in the mass media, in the governmental services… Drawing on the earlier work of Fishman (1974), Ruíz presented language-as-resource at the double-edged level of integration and identification for diaspora and indigenous communities in the linguistically diverse United States of that time. He conceived the use of English as a resource for practical integration into the sociopolitical mainstream, and the use of minority languages as a right for legal preservation of the identities, experiences and foregrounds of all groups in society either belonging or not to the privileged classes. Through the favourable position on languages other than English, Fishman and Ruíz took position on the protection of non-state languages, and on the design of policies that held integration and cohesion but also pluralism and diversity. These authors and the discipline of language planning itself, however, have become the object of criticism for being more technicist than ethical-based, as well as for only considering the rights of the named and standardised languages in the culture (Ricento, 2005). While this critique deserves serious attention, Ruíz’s orientations remain of historical importance since they helped to move societal discourses beyond the static divisive representation of some languages as problems and some others as resources.

Education is likely the most influential terrain of language planning, and the terrain in which controversies, conflicts and debates surrounding Ruíz’s orientations have been notably visible in policy and practice. Over the last half century, we have witnessed bilingual education programs that use and recognize more than one language in teaching and learning, as well as multilingual initiatives of cultivation of the home languages of learners. In the United States but also in Europe and other parts of the world like Southern Africa, we find bilingual programs differently termed within multilingual education policies. Paradoxically, the provision of bilingual programs – shadowed by the resource and right orientations – often coexists with remedial language programs – shadowed by the problem orientation – for some learners to learn the state/official language so that the use of their home languages at school becomes gradually reduced to minimal expression. At the level of research, the tracking of discourses of remediation of language diversity, and hence of some languages as problems per se, leads us back to the beginnings of the domain. By the 1960s and into the 1970s, research on bilingualism and mathematics learning evolved in connection to assumptions of cognitive confusion and decrease of the ability to think (see the survey review of Austin & Howson, 1979). Some of these assumptions have endured for more than fifty years though subtly formulated (see the discussion of Chronaki & Planas, 2018). In different ways, the problem orientation prevails or at least survives even in approaches to language diversity and mathematics learning that are more complex and socially grounded. It is unclear whether such orientation simply behaves as cause or effect of remedial directions in policy, practice and research but it might be involved in the difficulty of thinking pedagogies out of the limits of monolingualism.

While the problem orientation selectively applied to languages and cultures of some groups of learners and families still is an influential context of representation of language in mathematics education research, from the 2000s onwards the domain brings up counterbalanced frames of interpreting language. The resource orientation towards the languages of learners gains ground (see, e.g., Adler, 2001) in parallel to the advance of critical studies of power in education (see, e.g., Popkewitz, 2000). In the representation of the home languages of all learners as pedagogic resources for mathematics learning, we can situate the start of the groundwork for the field-specific development of the concept of language as resource (hereafter, the LAR concept). In In the configuration of the LAR concept, the resource orientation *towards* language goes together with resource orientations *about* language through the questions of what language is and how languages (of mathematics included) relate to each other. Under the rationale that we cannot think of forms, effects and possibilities of language use if we do not know what language is and do not examine relationships between languages, today orientations *towards* and *about* language sustain the interpretation and realization of language as resource. For two decades now, we can talk of research in which the languages of a) mathematics, b) learners and c) teachers constitute a prototypical resource for mathematics education, and in which the problem of study is the interpretation of features of these languages in mathematics teaching and learning. In the next section, I will present some of the contemporary forms of this research.

WHAT DOES THE LAR CONCEPT IMPLY IN CONTEMPORARY MATHEMATICS EDUCATION RESEARCH?

The LAR concept may not be an organizing major focus of contemporary mathematics education research on language yet, but its relevance, expansion and status are growing in the scientific community, specifically supported by the widespread thinking of language as social (see the survey work in Planas, Morgan & Schütte, 2018). Even though the domain may not be ready to situate the LAR concept at its core, the topic of language as resource is certainly contemporary because it reflects current issues, ongoing challenges and historical concerns of mathematics education, some of which were identified time ago but remain unresolved. It particularly allows us to further consider the superdiversity of mathematics classrooms (Barwell et al., 2016), the needs of dislocated groups of learners (Chronaki & Planas, 2018), and the wider demands of humanization of school mathematics (Subramaniam, 2015) in ways that combined approaches with language as right and language as problem of the past cannot.

The current representation of the topic inside the field implies much more than attitudes *towards* the place and role of languages in mathematics teaching and learning. A number of socially embedded conceptualizations of language, in which language is integral to its use in social activity, underpin LAR research. In this respect, language is not distinct from language use. Neither makes sense without the other; rather, the reciprocal relationship between language and language use lies in both (Planas, 2018). Language does not exist solely in the mind or in the cognitive, logical, psychological and linguistic senses, therefore, nor is it only a phenomenon of communication and exchange of information. It expresses and arises in activity involving multiple individuals, structures and systems in the culture. Mathematical notations, for example, are static inscriptions of a representational linguistic system, which is instrumental in developing and capturing thinking processes. In the social phenomenon called language, the power of notations is not automatic and understandable separately from the situated use of spoken, signed, written, digital languages, their grammars, their lexical manifestations and connections to action. Consequently, knowledge of standardised notations alone does not guarantee access to and successful performance of school mathematics. Learners still need to know when, how, with whom and for which contents of the interaction a mathematical notation is due in the culture.

Given the width of socially embedded orientations *about* language, nowadays a range of theoretical-thematic emphases informs LAR research. Different authors take different emphases when adopting the stance that the languages of learners, of teachers and of mathematics, both its facilitation and impedance, are potential resources of pedagogic, epistemic and political value. While the idea of the value of language itself suggests a sense of common direction to what LAR researchers do and believe in, the specific qualities of value are distinct and complementary in the inquiry into the effects produced and the capacity to influence in language use. Overall, the pedagogic, the epistemic and the political value respectively refer to the following emphases:

i) The orchestration and fostering of teaching and learning (pedagogic value)
ii) The creation and exchange of knowledge (epistemic value)
iii) The representation and distribution of ways of knowing (political value)

As I will argue in the next subsections, it is not realistic to track or operationalize these emphases on their own in the field-specific development and international expansion of the topic of language as resource. Any selection and compilation of publications, as a whole, points to some oscillation between emphases, and the vast majority of individual publications oscillate between two or three emphases with diverse amplitudes of oscillation (see an integration of emphases in Robertson & Graven, 2018). There are indeed numerous overlaps and links underlying how authors think and write about languages that are resources in mathematics education, and how collaborations among authors take shape. This being the case and drawing on the three emphases above, a map of LAR research studies must include lines of work on:

- Meaning making that facilitates content-specific learning
- Participation in multilingual discourses of explanation and argumentation
- Navigation between scales of mathematical languages and knowing

Criteria and decisions about the studies to include in a picture of a map of research are personal and representative of work done in collaboration. For the configuration of a map of LAR research studies, my decisions reflect work with colleagues from different parts of the world with whom I share language-based theories of mathematics teaching and learning. Beyond unintended omissions, possible flaws and inevitable gaps, the map suggested (made of theoretical-thematic emphases and lines of work) reflects the heterogeneous building of the LAR concept through diverse interpretations of common terms. Several concurrent responses to what constitutes language coexist in the transition zones of discourse analysis, social semiotics and philosophy of language in the most thorough applications of the LAR concept that I know. The term ‘tools’ in LAR research, for example, may refer to ways of speaking to get different sorts of jobs and identities done in work that concurs with discourse analysis (especially Gee, 1996). The same word may refer to systems of meaning potential for communication and interpretation of the world in work that concurs with social semiotics (especially Halliday, 1978), or still to dialectic relationships mediated by words in social activity when philosophy of language (especially Bakhtin, 1981) is considered. A similar reasoning applies to the diverse interpretations of the term ‘discourse’. While in traditional linguistics, discourse means language in use as opposed to the surface of language in the linguistic system, the pervasive notion of discourse in LAR research primarily diverges in the analytical approaches to the linguistic and supra-linguistic levels of text. Another interesting example is the Hallidayan notion of mathematical register – the meanings that belong to *the* language of mathematics – and how it is serving to discuss what *the* language of mathematics itself means, and what the implications of representing it in the singular as a unique specialized discourse are.

Meaning making that facilitates content-specific learning

Language-responsive experimental research with the preparation, implementation and validation of mathematically content-oriented teaching and learning sequences and lesson materials, together with the eventual motivation of newer recursive experiments, is rapidly growing in the domain (Prediger, 2019). In the context of the pedagogic demands of the language in use by teachers and learners, although not necessarily intertwined with design research frameworks, we find LAR research studies relative to 'micro-situations' of teaching and learning mathematical contents such as fractions, functions, equations or probability. Here language is a resource of pedagogic value because it contributes to educational processes of mathematical meaning making, which in turn contribute to mathematically content-specific teaching and learning.

Under the theoretical influence of social semiotics and applied linguistics, especially Halliday (1978) but not only, this line of work examines how language functions to make and exchange meaning through a variety of potential choices available in the linguistic system(s) of the language(s) in place. Choices of lexicon and grammar relevant in the culture and in the immediate situation are thus key to the use of the system(s) of language in ways that support content-specific learning. Accordingly, mathematical meaning is the result of choice to the extent that change of choice leads to change of meaning. In Planas (2014, p. 59), for example, "It is as if they were two areas" and "They are two areas" are two different instances of oral language with respect to linguistic choice as well as to the meanings produced in the classroom discourse (see the detailed discussion and context of the instances in the paper mentioned). Regarding the choice of language, the clause "as if" – interpreted in the linguistic system of the original data – functions to add the meaning that what follows may not be feasible. It does so by combining an element of similarity, "as", with an element of hypotheticality, "if", to signal some kind of comparison. The introduction of "two areas" through a dependent subordinate phrase further suggests unfeasibility. The reference to "two areas" in the option with the verb modality "they are" expresses factual evaluation and has the contrastive effect to indicate that the clause is true. The analysis of instances of language in terms of choices undertaken is also present in Kazima (2007), Alshwaikh and Morgan (2018), and Prediger and Wessel (2013).

In this line of work, mathematics teachers and learners become agents over the course of several processes of meaning making through the selection, deliberate or not, of single unique texts from the uncountable possible texts in the language systems available. Awareness of the lexico-grammatical choices may thus help mathematics teachers to develop learning environments through answers to the question of how language must be instantiated given the needs of the learning content and the constraints embedded in the language itself and in the culture (educational policies, curricular guidelines, language of instruction, programs for "language learners", etc.). In her paper, Kazima (2007) addresses word meanings and the distance between wanting to communicate a mathematical idea and having the linguistic means to express it. Her semantic analysis of meanings made by learners in Chichewa and

English for probability words shows the role of lexical choices in the learning of probability. In the examination of written and spoken mathematical texts offered to learners, Alshwaikh and Morgan (2018) analyse actual effects of linguistic choices (e.g., thematising procedures and calculations, naming relational processes) as well as alternative texts and meanings that might have been produced instead. Prediger and Wessel (2013) examine the possibilities and realizations of meaning production for fractions in lessons with learners from low-socioeconomic status and a variety of home languages. These authors treat the levels of verbal organization and of mathematical performance in an integrated way in the different phases of designing, implementing and analysing their didactical proposal. Whereas whether verbal organizations are 'grammatical' or not in a certain system can be decided regardless of context, equally important in all these studies is the fact that choice, meaning and learning are interpreted in the specific mutual dependencies of language and culture.

Differently to the earlier research on mathematics education and language focused on linguistic proficiency and cognition (see Austin & Howson, 1979), LAR research concerned with grammar and lexicon embraces an ecological view of mathematics teaching and learning that is context-sensitive and, as such, addresses the production of ways of being, thinking, communicating and performing in context. The range for this notion of context includes both interactional and sociological lenses. The integration of the immediate communication as it happens and the larger institutional societal structures as they enter the mathematics classroom are particularly addressed in the lines of work with major emphasis on the creation and representation of knowledge and knowing. The next two subsections illustrate some examples.

Participation in multilingual discourses of explanation and argumentation

LAR research comprise studies that primarily considers the epistemic value of all languages and speakers in the mathematics classroom. Following Gee (1996), Moschkovich (2007), Planas and Setati (2009), and Setati, Molefe and Langa (2008) have investigated the production of academic languages, classroom cultures and lesson practices for inquiry and knowledge discussion in the discourse of the multilingual mathematics classroom. Here references to discourse generally function to expand views of language as material text. While the focus is on the quality of participation in the discourse in mathematics, the uses of language that allow learners to enter their arguments, reasons and cultures into this discourse is of outmost importance. Deliberate language choice and flexible language switching are uses researched with accumulated evidence of impact on the quality and continuity of learners' participation in discourse practices of explanation, argumentation, justification, generalization… (Setati & Adler, 2000). Hence, language is an epistemic resource because it has the potential to promote participation in discourse practices that contribute to the creation and exchange of knowledge. Participation in such discourse practices operates as a process of initiating learners in the culture and language expertise of specific mathematical communities, not without taking into account the home culture and language expertise for which the learners are themselves representatives.

There is a line of work in which the epistemic value of language is mostly equated with the relationship between language switching and participation in conceptual discourse practices (i.e., practices of explanation, argumentation, justification, generalization… oriented to understand the concepts and principles of the mathematical content of teaching and learning). Setati et al. (2008) show how the planned, proactive and strategic switching between the learners' home languages and the language of the teacher in instruction leads to reciprocal appreciation and mutual sharing of experiences and conceptual knowledge. Planas and Setati (2009) show how flexible language switching provides continuity to group work where bilingual learners draw on their languages and knowledge to communicate mathematically by means of selecting one of the languages and then shifting to the other depending on the conceptual complexity of the task. While bilingual learners may have difficulties with one of the languages, they may use the other for enhancing participation in joint activity. Importantly, these studies link the linguistic presence of the languages of learners to social participation in rich discourse practices of explanation and argumentation, with time for experience-based peer learning, group negotiation of language expertise and joint building of activity. This is consistent with the formulation of school mathematics learning as the accomplishment of the language of performative actions (e.g., the language of long division) and of conceptual knowledge for reflective understanding of practices (e.g., the language of argumentation of division methods).

In the interpretation of language switching, switching registers is also a key issue in the studies most influenced by Vygotskian sociocultural perspectives, with precise interest in the representation of everyday and specialized discourses in mathematics (e.g., Moschkovich, 2007). In this way, it is emphasized the relevance of what the learners and their communities of reference bring to any learning situation as active meaning makers. Such position goes further in the broader uptake of the mathematical register to imply discourse practices in which everyday registers and academic registers alternate across different language systems. In the examination of discourse practices of explaining, arguing, reporting and describing, Planas and Setati (2009) find that participation in these practices develop operational and technical academic registers in the language of instruction, whereas conceptual discourses develop with the help of everyday registers and home languages in interaction. Switching across, within and between language systems and registers, thus, impact on participation in quality discourse practices. This evidence does not go, however, with the absence of tensions and interruptions in participation. Overall, the ideas of who has expertise in certain registers and languages of mathematics and what the discourse practices of significance for mathematics teaching and learning are, are political. The last subsection addresses LAR research committed to the identification and institutionalization of mathematical languages and ways of knowing in the routines and rationalities of school mathematics.

Navigation between scales of mathematical languages and knowing

A third major emphasis on the map of LAR research studies deals with the political value of language. Now the capacity of language of enabling us to mean and to know comes along with the capacity of stratifying meaning, ways of knowing and knowers. The attention to the potential of language to (re)define the epistemic status of language systems, communities and learners is very much in consonance with how Gee (1996) and Halliday (1978) see knowledge (of mathematics) as a human construction that circulates in discourse and culture. An important line of work under this emphasis adds Bakhtin (1981) to interpret the tensions and forces intertwined in the realization of the political. While the historically accepted forms of mathematical languages and ways of knowing in school mathematics remain at the centre (Radford & Barwell, 2016), the unifying uses of language pushing toward this centre are seen tightened to permanent tensions and subjected to eventual changes. The possibilities of deploying language differently act as a centrifugal force and sustains the understanding of the political in terms of the plural realization of language. More particularly, if the situated conditions of realization of language vary, the situated ordering of meaning, ways of knowing and knowers becomes unstable and may vary as well. Beyond the description of difficulties and constraints in language, therefore, this line of work brings up the discussion of how voices of the past and official discourses of school mathematics and ways of knowing may be either maintained or resisted and changed (Barwell & Pimm, 2016).

Barwell (2018) posits that examining the stratified nature of language and its capacity of stratifying is necessary for understanding and transforming inequality of access to school mathematics learning. Speaking and being seen as a speaker of specific social languages (i.e., ways of using language to enact socially situated identities and carry out socially situated activities) are themselves stratifying processes. In the study of these and other stratifying processes operated in the language of the multilingual mathematics classroom, Barwell traces three main sources of meaning: literal texts, supra-textual discourses and historical voices manifested in discourses and texts. More generally, such approach arises a number of analytical-methodological challenges regarding how to deconstruct language since a substantial part of what it does is not audible, not readable, not printed or written… Whereas most LAR studies under the pedagogic and the epistemic emphases work with verbal texts as data, those under the political emphasis distinctively add discourses and voices as data. Most often varieties of sociolinguistics and critical discourse analysis frameworks serve to meet and deal with analytical-methodological challenges. Indexical signs, iconic traces and nonverbal cues, which take place inside the linguistic system and inform of the outside, may lead to uncover the static gesture of pointing to someone as something else than a reference to the body or location indicated. Pointing may be an index of someone who plays a role of knower in the mathematics classroom, and hence a source of meaning in mathematics teaching and learning about the learner to be heard, the mathematical language to be learned and the ways of knowing to be recognized and practised.

The study of language and of scale-based discourses of comparison and stratification, which locate a language and those who speak it within the same strata, is sustained to describe traditional mathematical languages and fixed ways of knowing but importantly to reveal unexpected mathematical languages and visions of the world.

LAR research-generated insights in this respect are present in the chapters of Hunter, Civil, Herbel-Eisenmann, Planas and Wagner (2017), with reports and proposals of educational innovation. Throughout the chapters, a variety of analytical frameworks lead to identify alternative languages and ways of knowing, which in turn set the scene for creative experiences of mathematics education grounded on heterogeneous realizations of textuality outside the limitations of standardized, rigid views of mathematics teaching and learning. Multilingual interaction, culturally responsive curricula and discourse practices of argumentation between forms of knowledge and knowing support voiceless groups of learners in the use of their languages and cultures in mathematics learning. To this end, teachers and teacher educators reframe the opening of norms and the conditions of participation by overtly referring to what counts as mathematical and why and fostering explorative and dialogic types of talk.

In the last section that follows, I introduce and discuss data from an experience of small variations in the language of a mathematics teacher in her mathematics classroom. The significance of small variations strongly relies on the capacity of language to create meaning and navigate between scales of languages and knowing. I will finish by making the point that small variations in the language of the mathematics teacher can result into appreciable pedagogic, epistemic and political benefits for learners.

HOW IS THE LAR CONCEPT PARTICULARLY APPLICABLE TO IMPROVE ACCESS TO MATHEMATICS LEARNING?

During all these years, I have learned from visits to schools, observations of lessons and discussions with groups of mathematics teachers. These collaborations often start with the joint identification of content-specific challenges of conceptual understanding in the classroom and continue with the thinking and implementation of language responsive proposals of mathematics teaching (see a reflection in Planas, 2017). When applying methods of transformation to solve equations, the fact that most learners do not realize that the transformed equation is equivalent to the original one is an example of the challenges addressed so far. Learners often learn to perform transformational operations from one equation to another but do not regard them as entities of the same mathematical object. Guided by the broader challenge of teaching and learning the meaning of equation as equivalence class, I visited two Catalan-Spanish bilingual lessons in Barcelona devoted to practices of solving quadratic equations in the attempt to respond to: Why does not the solving of equations promote the understanding of the concept of equation? I collected data for discussion with the teachers of the classrooms about the languages of mathematics and discourse practices in their teaching.

Since the construction of the meaning of equation as equivalence class was in focus and any equivalence class involves a relationship (i.e., entities related and criteria of equivalence), the first attempt was to study the semantic specification with lexical complements into the content-specific mathematical register of verbs such as compare, change, modify, map or connect. Preliminary findings actually revealed similarities regarding the weak lexicalization of relational verbs in the languages of both teachers.

In the teaching of the procedural details of the chosen method of solving quadratic equations, mathematically important relational verbs were under-lexicalized over the course of the lessons. The creation of lexicon adequate for the semantic domain of algebraic equations where some words are given specialized meaning was thus undermined. In discussion with the teachers, it was suggested that some conceptual challenges related to algebraic equations might derive from the weak lexicalization of the idea of relationship. These are examples of pieces of the languages of the teachers commented with them, transcribed with instances of relational verbs in bold:

> **Anar fent canvis** pas a pas és bàsic. **Canvieu** cada equació per la següent i obteniu una seqüència. Cada ecuación la **cambiáis** un poco. Tenéis que utilizar las reglas de transposición. ¿Sí? Si añades un número a un lado, entonces añades el mismo número en el otro lado. ¿Sí?
>
> ***Making changes*** *step by step is key. You* ***change*** *every equation into the next and get a sequence. Every equation, you* ***change*** *it a bit. You have to use the transposition rules. Okay? If you add a number on one side, then you add the same on the other side. Okay?*
>
> Una de les tasques per resoldre una equació quadràtica serà **modificar** l'equació escrita inicial i, llavors, **aneu associant** una forma amb una altra fins l'expressió general. **Váis asociando** una forma escrita con otra hasta llegar a la expresión general de la pizarra.
>
> *One of the tasks for solving a quadratic equation will be* ***modifying*** *the initial written equation, and then you* ***go mapping*** *one form to another up to the general expression. You* ***go mapping*** *one written form to another up to the general expression on the board.*

Relational verbs normally appear in the absence of adjacent complements in colloquial registers expressed in the linguistic systems of either Catalan or Spanish. The talk of the teachers is thus 'grammatical' in the systems in which it is developed. The fact that the complements are not grammatically obligatory and do not have to be literal in the texts is indeed a manifestation of the economy strategies common to all linguistic systems. Nonetheless, the creation of specific meanings adequate for the conceptual understanding of equations may require the revision of specific economy strategies and choices. Still, the literal absence of complements may work well when texts of the past communicate the meanings implied in the particular semantic domain. For some of the relational verbs in use the complements did remain implicit and inferable from prior instantiations in that same lesson. This was not the case, however, with all the verbs that could have functioned to co-develop the meaning of equivalence class. The verbs 'to change', 'to modify' and 'to map', for example, were always modulated and instantiated in the absence of the criteria of equivalence in the change, modification or mapping. Not only were these verbs uncompleted in the mathematical register related to the semantic domain of algebraic equations. One of them ("to change") was often nominalized ("changes") and meanings attached to the verbal forms partly covered (e.g., "making changes" contains less information than "you change every equation").

Relational verbs need to be meant in the particular mathematical register at some organizational point of the texts used in teaching and learning equations. The adjacent complements that semantically complete these verbs, therefore, need to be explicit.

Otherwise, the exploitation of the potential in language to create the meaning of equation as equivalence class is hindered. Pronouns and verbs-turned-into-nouns can certainly compensate unnecessary repetition, although in the multilingual classroom and spite of the evidence of regular language switching in the teaching, this option deserves close attention. As documented in Planas (2014, 2018), it is not unproblematic whether and how mathematically important clauses move across different language systems or literally remain within the language of instruction only. The tension between what is literally said in each language is consistent with the political view of language switching in Planas (2018). On the other hand, the more general demands of lexical elaboration and explicitness are consistent with how Halliday (1978) refers to lexicalization as a prominent feature of registers. In a similar sense, Prediger and Wessel (2013) introduce linguistic means as a cognitive tool for relating registers in the ordering of fractions, and Alshwaikh and Morgan (2018) identify the organizational relevance of naming relational processes in mathematical texts offered to learners.

In collaborative work with teachers, instead of listing a large number of issues to implement in the language for lessons with the same mathematical content, we decided to plan and pursuit two small variations, which were both substantial to the instantiation of the meaning of equation as equivalence class. The first variation was the deliberate lexicalization of verbs that express a relationship in the performative language of solving quadratic equations. The second variation consisted of the deliberate introduction of linguistic clauses for the activation of conceptual discourses of explanation and argumentation. “Transformations that keep the equivalence because”, “not equal but equivalent”, “neither equal, nor equivalent” and “two equations are equivalent to each other if” were some of the phrases anticipated. While the first variation would strengthen the pedagogic value of the language of the teacher, the second variation would strengthen the epistemic and political value of the languages of the learners through participation in practices of explanation. Most bilingual teachers in the group had learners of Latin American families who were in the process of learning the language of instruction, and hence special attention went to avoid loss of mathematical meaning when switching between languages.

The systematic discussion with the group of teachers influenced the adoption of the variations during the teaching of quadratic equations, and such variations presumably influenced the language of the learners, as the following transcript seems to indicate:

Teacher: El que ha de passar és que comparem les equacions escrites diferent i que siguin equivalents perquè les arrels són les mateixes. Equivalents, eh? No iguals. Digue-me equacions diferents que siguin equivalents i per què?

What has to happen is that you compare the equations written differently, and they are equivalent because the roots are the same. Equivalents, eh? Not equal. Tell me different equations that are equivalent and why?

Learner 1: Diferents però equivalents? Diferents perquè s’escriuen diferent… Podem agafar qualsevol equació i afegir el mateix número a cada banda.

Different but equivalent? Different because we write them differently. We can take any equation and add the same number on each side.

Learner 2: Però això són regles per resoldre l'equació… Qui diu que siguin equivalents?

But these are rules for solving the equation… Who says they are equivalent?

Teacher: Sí, són les regles per resoldre una equació anar canviant l'equació original per una equació més simple amb les mateixes solucions. Perquè cada pas del mètode és un canvi en una equació equivalent. En parelles digue-me equacions diferents que siguin equivalents i vosaltres em direu perquè.

Yes, they are rules for solving an equation by changing the original equation into a more simple equation with the same solutions. Because each step of the method is a change into an equivalent equation. Tell me in pairs different equations that are equivalent and you will say why.

I see this piece of classroom data as a powerful choice for the end of the present text. Not only because the language of the teacher reflects the application of the type of small variations suggested (e.g., "Because each step of the method is a change into an equivalent equation"), but also because some of the learning opportunities and benefits can be uncovered in the languages of the learners. First, it is a piece that illustrates the pedagogic value of language in how the transposition rules become expanded and represented at the service of both solving an equation (performative actions) and generating equivalent equations (conceptual knowledge). Second, it illustrates the epistemic value of language in how the meanings of equation as equivalence class and of equivalence class as relation become lexicalized as the starting point of discourse practices of exemplifying and explaining. Third, it illustrates the political value of language in how it is established who is able to do the constructing of the explanation. Learners in pairs are proposed the constructing work in the role of producers and distributors. The seemingly intended negation in "Who says they are equivalent?", and the task assigned in "Tell me in pairs different equations that are equivalent and you will say why" are instances of lexicalization of who says what and who can do what.

Tensions between lexicalization and implicitness in this piece of language could be discussed as well. In the process of putting some meanings to the front, some other meanings are always omitted or delayed. In my own academic text, I may have omitted distinctive meanings related to the semantic domain of contemporary LAR research. This is not important if, by means of the inestimable use of language, I have succeeded in showing the vitality of the topic of language as resource in mathematics education research and the fruitful theoretical debates and investigations going on at many levels.

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References

Austin, J. L., & Howson, A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics, 10*, 161-197.

Adler, J. (2001). *Teaching mathematics in multilingual classrooms*. Dordrecht, Netherlands: Kluwer Academic Publishers (Springer).

Alshwaikh, J., & Morgan, C. (2018). A framework for the study of written and spoken discourse: School mathematics in Palestine. *ZDM, 50*, 1041-1051.

Austin, J. L., & Howson, A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics, 10*, 161-197.

Bakhtin, M. M. (1981). *The dialogic imagination*. Austin, TX: University of Texas Press.

Barwell, R. (2018). From language as a resource to sources of meaning in multilingual mathematics classrooms. *The Journal of Mathematical Behavior, 50*, 155-168.

Barwell, R., Clarkson, P. C., Kazima, M., Halai, A., Moschkovich, J., Phakeng, M. S., Planas, N., Valero, P., & Villavicencio, M. (2016). Introduction: An ICMI Study on language diversity in mathematics education. In R. Barwell et al. (Eds), *Mathematics education and language diversity. The 21st ICMI Study* (pp. 1-22). New York: Springer.

Barwell, R., & Pimm, D. (2016). Bakhtin and some dilemmas of mathematics-language. In M. Phakeng & S. Lerman (Eds.), *Mathematics education in a context of inequity, poverty and language diversity* (pp. 7-18). Cham, Switzerland: Springer.

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: Does every student have a voice? *For the Learning of Mathematics*, *24*(1), 7-13.

Chronaki, A., & Planas, N. (2018). Language diversity in mathematics education research: A move from language as representation to politics of representation. *ZDM, 50*, 1101-1111.

Fishman, J. A. (1974). Language planning and language planning research: The state of the art. In J. A. Fishman (Ed.), *Advances in language planning* (pp. 15-33). The Hague, Netherlands: Mouton.

Gee, J. P. (1996). *Sociolinguistics and literacies: Ideology in discourses* (2nd ed.). London, England: Taylor & Francis.

Halliday, M. A. K. (1978). *Language as social semiotics*. London, England: Edward Arnold.

Hunter, R., Civil, M., Herbel-Eisenmann, B., Planas, N., & Wagner D. (2017). *Mathematical discourses that breaks barriers and create space for marginalized learners.* Rotterdam, Netherlands: Sense Publishers (Brill).

Kazima, M. (2007). Malawian students' meanings for probability vocabulary. *Educational Studies in Mathematics, 64*, 169-189.

Moschkovich, J. N. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*, 121-144.

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics, 87*, 51-66.

Planas, N. (2017). The adventure of moving mathematics education forward. A commentary. In J. Adler & A. Sfard (Eds.), *Research for education change. Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 195-207). London, England: Routledge.

Planas, N. (2018). Language as resource: A key notion for understanding the complexity of mathematics learning. *Educational Studies in Mathematics, 98*, 215-229.

Planas, N., Morgan, C., & Schütte, M. (2018). Mathematics education and language. Lessons from two decades of research. In T. Dreyfus et al. (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 196-210). London, England: Routledge.

Planas, N., & Schütte, M. (2018). Research frameworks for the study of language in mathematics education. *ZDM, 50*, 965-974.

Planas, N., & Setati, M. (2009). Bilingual students using their languages in the learning of mathematics. *Mathematics Education Research Journal, 21*(3), 36-59.

Popkewitz, T. S. (Ed.) (2000). *Educational knowledge. Changing relationships between the state, civil society and the educational community*. New York: SUNY Press.

Prediger, S. (2019). Theorizing in design research. Methodological reflections on developing and connecting theory elements for language-responsive mathematics classrooms. *Avances de Investigación en Educación Matemática, 15*, 5-27.

Prediger, S., & Wessel, L. (2013). Fostering German-language learners' constructions of meanings for fractions. Design and effects of a language- and mathematics- integrated intervention. *Mathematics Education Research Journal, 25*(3), 435-456.

Radford, L., & Barwell, R. (2016). Language in mathematics education research. In Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 275-313). Rotterdam, Netherlands: Sense Publ.

Ricento, T. (2005). Problems with the "language-as-resource" discourse in the promotion of heritage languages in the USA. *Journal of Sociolinguistics, 9*(3), 348-368.

Robertson, S.-A., & Graven, M. (2018). Using a transdisciplinary framework to examine mathematics classroom talk taking place in and through a second language. *ZDM, 50*, 1013-1027.

Ruíz, R. (1984). Orientations in language planning. *NABE Journal, 8*(2), 15-34.

Setati, M., & Adler, J. (2000). Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa. *Educational Studies in Mathematics, 43*, 243-269.

Setati, M., Molefe, T., & Langa, M. (2008). Using language as a transparent resource in the teaching and learning of mathematics in a Grade 11 multilingual classroom. *Pythagoras, 1*, 14-25.

Subramaniam, J. (2015). One mathematics for all: Can it be realized in a multicultural, multilingual country? *Intercultural Education, 26*(4), 266-277.

REPRESENTATIONAL COHERENCE IN INSTRUCTION AS A MEANS OF ENHANCING STUDENTS' ACCESS TO MATHEMATICS

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Curricular reforms have promoted the use of a variety of representations to support learners' access to mathematics. This gives rise to concerns about the coherence among these representations. I explore theoretical accounts that aim to enhance the coherence among the key representations of symbol, context and model, which support learners in extending their understanding of whole number arithmetic to signed numbers and fractions. Both these extensions are difficult for learners to navigate. I draw on a recent theory of whole number arithmetic, and on our prior work with teachers and learners to explore how coherence among representations may be achieved through interconnections of meaning. Models are representations that mediate between contexts and symbols. A careful choice of models is important for various representations to cohere. I identify the double number line model as one that is important for the coherence of meaning for the arithmetic of rational numbers and multiplicative thinking in general.

INTRODUCTION

The theme of the PME 43 conference, "Improving access to the power of mathematics", articulates what I take to be one of the central goals of mathematics education. There are many factors that are relevant to achieving this goal. I will address only one of them here – the instructional scaffolds that learners receive to make sense of mathematics and the coherence among these scaffolds. Curriculum designers and teachers recognise that mathematics is abstract and difficult and try to scaffold learners' access to mathematics using a variety of representations or what have been called "mediational means" (Adler & Ronda, 2015). These include tasks, problems, contexts, concrete manipulatives, diagrams, displays, gestures, teacher talk and so on. My focus here is on those mediational means that have an explanatory or justificatory role, that is, they function as a source of warrant as learners reason, implement procedures, solve problems, etc.

In some of our earlier work, we aimed at improving learners' access to algebra in the middle school (Banerjee & Subramaniam, 2012). Algebra opens up the power of mathematics by providing tools for generalising, proving, expressing relationship between quantities and using these to make predictions and inferences. Algebra is both a generalised form of thinking, and a powerful notation to express and support such thinking. We argued, as have others, that access to algebra can be improved by using number sense gained from arithmetic as a foundation for a structural understanding of

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 33-52). Pretoria, South Africa: PME.

algebraic symbolism, by having learners develop a sense for the "operational composition" of a number as encoded in a numerical or algebraic expression (Subramaniam & Banerjee, 2011). Here I focus on two extensions of whole number arithmetic that are critical for access to algebra – the extension to negative numbers and the extension to fractions. Both these extensions are difficult for learners to navigate.

Traditional approaches to mathematics instruction, which emphasise learning to manipulate symbolic representations on the basis of memorised rules, are widely thought to lead to a view of mathematics as devoid of meaning. The broad trend in curricular revisions around the world is to emphasise sense making in learning mathematics. Researchers have studied the role representations play in sense making from a variety of perspectives. Research oriented towards understanding student thinking has revealed the spontaneous representations used by learners as they attempt to make sense of mathematics. Fresh theoretical perspectives have brought to the fore hitherto neglected representational modes such as gesture and body movement. Design oriented research has sought to identify the most effective representations that support mathematical learning and to describe their affordances and limits.

As the repertoire of representations expands and diversifies, it is natural for concerns about coherent and efficient ways of using representations in mediating learning to emerge. Venkat and Askew (2018) point to the role of coherence in effective instruction, regardless of whether such instruction is teacher led or learner centred. Mathews, Venkat and Askew (2018, p. 263) express the "need for careful signifier choices, assemblies and sequences that support students in making sense of situations in ways that allow access to increasingly formal, flexible and efficient ways of working." A related concern expressed by these researchers is that learners may become bound to specific concrete representations, which while being closer to learners' intuitions, may hinder the development of more powerful, abstract strategies. A similar concern is expressed by Ng (2015) in relation to the "Model Method", widely used in primary mathematics in Singapore. Many Singapore students, who are exposed to the Model Method and enter secondary school "continue to use the concrete and visual model method to solve related algebra word problems" instead of adopting more general letter-symbolic techniques (p. 1011).

There is a need to develop accounts of the structure of the representations used in mathematics instruction and to the inter-connections between representations, that aid in enhancing their coherence and unity. Representations must allow for generalization beyond context and must be capable of hybridizing with symbolic representations and segueing into them. I will take some steps towards such an account for negative numbers and fractions. In preparation for this, I will briefly discuss the model for multiplication in whole number arithmetic, which will be relevant to the extension to fractions. The representations that I will focus on here could broadly be described as those that connect intuitive understanding of quantity to more formal representations of quantity. The latter serve as a bridge to algebra, whose intuitive basis is a sense for numbers and operations, which is not necessarily connected to quantities in realistic contexts (Subramaniam & Banerjee, 2011).

REPRESENTATIONS IN MATHEMATICS INSTRUCTION

The category of representations used in mathematics instruction is broad. The quote below reflects their range and diversity:

> Mathematical representations are visible or tangible productions – such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator – that encode, stand for, or embody mathematical ideas or relationships. (Goldin, 2014, p. 409)

This list of representations is still a subset of what are considered mediational means (for e.g., Adler & Ronda, 2015) – a broader category including tasks, examples, teacher talk, etc. However, I intend to focus on an even smaller subset of representations in order to restrict the scope of the discussion, without prejudice to claims on the importance of other mediational means. In particular, language and teacher talk are crucial for how representations support learning. Representations are embedded in a discursive network from which they derive their meaning and capacity to support reasoning. Meanings associated with representations find their roots in the culture and in learners' experiences outside school. Hence although attention to language and culture is important, it brings in a complexity that we will avoid in the present discussion. Occasionally, I will make some remarks relating representations to culture and history for specific purposes, especially to indicate the extended time over which some mathematical representations have evolved and point to significant changes in cultural supports for fractions over the last century.

Further, my discussion will focus only on "external" representations and will not be concerned with "internal" or mental/embodied representations. Nor will spontaneous or idiosyncratic representations invented or used by learners and teachers receive attention. I will focus instead on representations designed for pedagogical purposes and for appropriation by the community of learners and teachers. Such representations are not always a part of standard mathematical conventions, i.e., shared by the mathematical community. Some may belong specifically to the pedagogy of mathematics, hence theories and discussions concerning such representations may belong exclusively to the domain of mathematical education.

I will discuss three kinds of key representations below – symbol, context and model. Mathematical symbols are conventional, compact signs, designed for economy and efficiency, whose use is governed by rules. For our purposes, the symbols concerned are numerals (including signed numbers and letter symbols), operation signs, the fraction symbol, brackets, "equal to" sign, concatenations of these in expressions, etc. Contexts refer to realistic situations that learners may be familiar with or be able to imagine. Models are more abstract and formal; they are figural, i.e., depend on form and spatial arrangement to convey meaning and afford certain actions and transformations. Models may be instantiated using concrete tokens or through visual displays and diagrams.

Since our concern is with the coherence of representations, we will look for grounding contexts and models that undergird relatively extended parts of the learning progression. All representations will eventually reach a point of limitation and must yield place to more general representations. Symbolic representation has the advantage of going much further from this perspective since it has evolved over longer historical time scales. However, for this very reason, it is less accessible. Our focus hence will be on grounding contexts and models, including diagrammatic representations, the connections between them and the connections to symbols. One of my main arguments will be that while contexts, as representations used in instruction, are variable, models play a unifying role by extracting generalised mathematical meanings that are common across contexts. They serve as bridges to symbolic representations and a careful choice of models is crucial to achieving coherence among these representations. By representational coherence, I mean the interconnectedness of representations via meanings, which not only support learners, but also teachers in making effective use of representations. This will be elaborated further in the context of the discussion on representations for signed numbers.

Examples of grounding models are the bead string and the Empty Number Line (Klein, Beishuizen & Treffers, 1998), the horizontal bars used to denote quantity in the "Model Method" used in Singapore (Ng & Lee, 2009), the number line in its various avatars, etc. Figural representations such as these can be powerful in supporting reasoning. (Recall the figural arrangement that demonstrates that each square is the sum of consecutive odd numbers starting with 1, or the figures frequently used in generalisation tasks in school algebra.) Models are powerful also as communicative devices – for learners to communicate their thinking. Moreover, they can facilitate connections with contexts better than symbols alone, as in the case of the "Model Method". Further, they can aid computation as in the case of the Empty Number Line. Figural arrangements are frequently static or frozen, but dynamism ensues with actions done on them. This may include physical manipulation or just eye scanning procedures (Sfard, 2009). Dynamism is also introduced by the use of notational devices such as arrows which indicate gesture, movement, transformation, etc. We also note that figural arrangements are often hybridized with symbols, or with text. In the sections that follow, I discuss in turn representations relevant to the topic domains of whole number arithmetic (WNA), signed numbers and fractions. For WNA, I focus only on the model for multiplication, which will be relevant to the extension to fractions.

WHOLE NUMBER ARITHMETIC

Whole number arithmetic is the foundation for the mathematics of quantity and has been the focus of education research and reform efforts for long. Curricular reforms initiated in India and many other countries include increased emphasis on sense making, connection to contexts, exploration, developing mathematical thinking, etc.

The increased use of concrete manipulatives and figural representations in WNA are also a part of the changes. Researchers working from cognitive perspectives have attempted to synthesise long range learning progressions for WNA (Sarama & Clements, 2009). Complementary to these efforts are recent historically informed attempts to describe an underlying theory that provides a unity to WNA instruction. Chambris (2018) has analysed the changes over several decades in the theory underlying the teaching of decimal numeral notation in the French elementary curriculum. Ma and Kessel (2018) have developed an underlying theory for WNA that has roots in theories formulated in the 19th Century in the West and incorporates developments over the last Century in China and other countries. It is noteworthy that the notion of "unit", which occupies a central place in both these analyses is under-emphasised in modern approaches to the pedagogy of WNA.

The theory of WNA presented by Ma and Kessel is a quasi-deductive theory with definitions, basic rules and laws. It is expressed through "words and diagrams" (Ma & Kessel, 2018, p. 18), and seeks to provide a carefully designed vocabulary and narrative that can support instruction. In contrast to the underlying theory proposed by Ma and Kessel, my attempt here seeks to secure coherence in instruction by way of an account of representations and meanings. Thus it has a semiotic flavour, which I believe complements the quasi-deductive approach of these authors. Ma and Kessel do support their theory with models of addition and multiplication, which are important from a semiotic point of view. Addition and subtraction operations are represented via the part-part-whole model, which also underlies the Singapore approach described by Ng and Lee (2009). The model for multiplication is important since it foregrounds the correspondence relation basic to the multiplication operation and to multiplicative thinking in general, as emphasised by Nunes and Bryant (2015). Both the models for addition and for multiplication are important for the extensions to signed numbers and fractions that I will discuss below. As a preparation for the discussion, I present an extension of the model for multiplication from discrete to continuous measures.

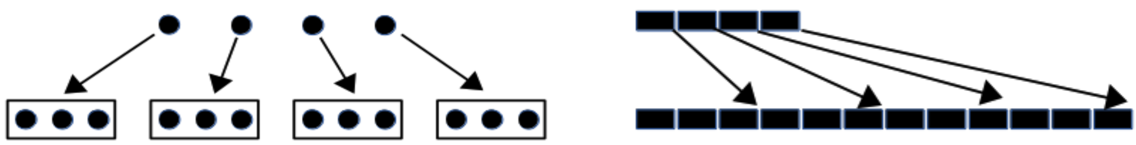

Figure 1: Models for multiplication of 3×4 (a 3-unit "taken 4 times")

The model on the left in Figure 1 represents the multiplication operation "3×4" for discrete units, and is similar to the model presented in Ma and Kessel (2018), where the product of two numbers is defined as "a third number which contains as many units as one number [multiplicand] taken as many times as the units in the other [multiplier]" (p. 452). Here the product contains four copies of a 3-unit, expressed as 12 basic units. This formulation, which emphasises the correspondence between units taken in two measure spaces has distinct advantages over the definition of multiplication as addition repeated a certain number of times. For one, the definition as well as the model can be extended easily to continuous units and hence can support the multiplication of fractions. The model on the right in Figure 1 shows a "continuous" unit version for "3×4". This is essentially a double number line representing the correspondence between measures, which I will return to later. This connection is important, since the number line is a unifying representation, which provides a home for whole numbers and all the extensions of whole numbers that learners will encounter in school (Bass, 2018).

The model on the right can also support division in the following two meanings:

1. If 12 units of the product correspond to 4 units of the multiplier, how many units of the product correspond to 1 unit of the multiplier? (Partitive meaning)
2. Haw many 3-units are contained in 12 units (or alternatively, "in a 12-unit")? (Quotitive meaning)

I make some further observations. In partitive division, the correspondence is usually between two different kinds of measures. For example, "if 12 bars of chocolate are shared equally among 4 children (i.e., correspond to 4 units), how much will each get (i.e., how much is one unit)?" In quotitive division, the relation sought is a within measure space relation. How many packets containing 3 bars of chocolate each will make up 12 chocolates? The fact that both within and across measure space multiplicative relations can be represented suggests that the model can also represent proportionality relations.

EXTENSION TO NEGATIVE NUMBERS

In the school curriculum, the arithmetic of signed numbers is essential for symbolising and operating with expressions, both numerical and algebraic, and hence forms a foundation for algebra. The extension from whole numbers to negative integers is based on the reification of the subtraction operation or bringing the "operator" meaning of subtraction to the fore. In the curricular sequence, the extension from whole numbers to fractions precedes the extension to negative numbers. Historically too, fractions occur in several cultures and precede negative numbers. I have reversed the discussion sequence here since the addition operation is simpler than the multiplication operation. There is no implication, however, that negative integers must be taught before fractions.

Negative numbers arose in history first in the context of developing general methods of solving linear and quadratic equations in China and India. In both cultures, signed numbers were frequently interpreted in terms of money, as credits and debits or assets and debts. In astronomical calculation by Indian mathematicians, angles measured from the vertical in two opposite directions (to the North and South) were shown with positive and negative signs (Mumford, 2010). In the 12th Century CE, the Indian mathematician Bhaskara II, while solving for the position of the foot of a perpendicular in a triangle, interpreted a negative value for the position as indicating that the foot of the perpendicular lay in the contrary direction, i.e., outside the triangle. Beginning with the early centuries CE, Indian mathematicians routinely used negative numbers and operations with them in equation solving in algebra and in astronomical computation involving trigonometric functions. The rules for operating with signed numbers were first stated fully by the Indian mathematician Brahmagupta in the 7th Century CE (Plofker, 2009). In contrast to the routine use of negative numbers by Indian mathematicians, there was resistance to the use of negative numbers by Islamic and European mathematicians, surprisingly till as recently as the mid-nineteenth Century (Mumford, 2010).

It is not surprising then that learners have difficulty mastering the operations with signed numbers in middle and secondary school. Teachers and textbooks aim to teach the rules for integer operations and offer representations that draw on learners' prior knowledge or intuitive understanding as scaffolds. The representations range from contexts where negative numbers acquire meaning, to models such as the number line. In a study conducted with in-service teachers (Kumar, Subramaniam & Naik, 2017), we found that the teachers' use of models to support integer operations reflected varying degrees of coherence. Despite this, the teachers had a developed sense of representational adequacy and were often critical of the lack of coherence in their own and their colleagues' use of representations. (I am indebted to the notion of representational adequacy developed by Kumar, 2018, on which my own reflections on representational coherence are based.) This led us to explore a more coherent framework of meaning that could provide a foundation for the use of representations to teach integer operations. I will describe elements of this framework below, referring occasionally to teachers' responses as well as learners' classroom responses as reported by the teachers. More details may be found in Kumar et al. (2017).

Since we are looking for grounding representations, it is useful to keep in sight the outcomes for the topic domain of signed numbers as a whole. Representations of signed numbers and operations should support the learners in handling a range of tasks. The basic capabilities include interpreting signed numbers, and their addition and subtraction in contexts, ordering signed numbers, symbolic addition/subtraction of integers and understanding that subtraction can be rewritten as addition of the additive inverse. Developing a number sense in the context of integers would involve being able to flexibly evaluate expressions encoding two or more addition/subtraction operations of integers, recognising that reordering additive units leaves the value unchanged, and using symbolic expressions to express and justify reasoning based on number sense (flexibly use of partitioning, compensation, etc.). (See Banerjee & Subramaniam, 2012.) Further capabilities include interpreting multiplication and division of integers in contexts, symbolic multiplication/division of integers, using the distributive property and integrating this with the understanding of fractions to understand rational numbers. The representations designed for instruction should support learners in their reasoning as they progress through the range of tasks related to these outcomes. By support I mean that learners should be able to draw warrant for their reasoning from features encoded in the representations. Representational coherence requires that there must be inter-connectedness of meanings between the representations that are used.

I will discuss below in turn, the meanings at the level of symbol, context and model. The discussion will highlight the connections between the meanings. The articulation of general features or meanings, as indicated in our study on integer meanings (Kumar et al., 2017), enables teachers to construct contexts and suitably configure models on their own to support learners' understanding.

Symbol

By the phrase "meaning of symbols", one often indicates the referents that symbols have in contexts or models. However, here I wish to focus on significations within the symbolic order. We note first that the key symbol for signed numbers, the minus sign, acquires new meanings as learners move beyond the arithmetic of whole numbers. While the learners are familiar with the minus sign signifying the subtraction operation, it now also indicates a negative number. Correspondingly, the plus sign now indicates either the addition operation or a positive number. Learners encounter unfamiliar juxtapositions of signs such as "4 + (–5)" which they need to parse by choosing the appropriate signification. In the context of letter symbols, a third meaning of the minus sign comes to the fore: the unary operation of taking the inverse, as for example, when x takes a negative value, say "–3" in the expression "$–x$ + 5" (Vlassis, 2004). The distinction between these significations is essential in the initial stages, and some approaches use distinct signs for the subtraction operation and negative integer. In general, learners must recognise the distinct significations from the arrangement of the expressions, sometimes aided by the use of brackets. It is also important that learners appreciate why a common sign is used for the subtraction operation and a negative integer, and the connection between the two via the reification of subtraction (Kumar et al., 2017).

Context

The use of negative numbers in real life contexts is uncommon. Hence there is little direct experience of negative numbers that learners can draw on for support. However, when signed numbers are interpreted as denoting increase or decrease, i.e., as transformations, they can be related to several contexts, which can then be used to support instruction.

The usual interpretation of a negative number is that it denotes a state. As state, a signed number may, for example, refer to the ambient temperature in degrees Celsius. In tropical countries however, learners may have no acquaintance with negative temperatures. But one may focus on a ''derived'' quantity, namely change in temperature, which may be positive or negative. For example, the change in ambient temperature from hour to hour may be represented by a signed number. This seemingly small shift, from viewing the signed number as denoting a state, to viewing it as denoting a transformation, remarkably led the teachers in our study to discover a number of contexts that could potentially be used for instruction. Some of these were, pebbles added and removed from a bowl containing an unknown number of pebbles, positive and negative scores in a quiz game, changes in a baby's weight in the first few weeks after birth, etc. Further it was possible to meaningfully speak of addition and subtraction of signed numbers by interpreting them as transformations with learners able to reason about the outcome of addition. Subtraction could be interpreted in multiple ways – in some contexts as a cancellation of a transformation, and in others as the transformation needed to go from an initial to a final state.

We had shared Vergnaud's (1982) framework of integer meanings with the teachers, which includes the meanings of state and transformation as discussed above, and also "static relation" (for e.g., tracking by an airplane in the air of the relative altitudes of other planes in the air nearby). The teachers noted that the three meanings were inter-connected. When signed numbers denote a state, a reference point is fixed by convention and the number specifies a relation or a (realized) transformation from the reference state. In the case of transformations, as when change in temperature by the hour is recorded using signed numbers, the reference point is continuously changing. The importance of the role of a reference point was part of the teachers' own construction of the meaning of integers in contexts (Kumar et al. 2017).

Model

The models used for signed numbers are broadly classified into "neutralisation" and "number line" models (Stephan & Akyuz, 2012). The neutralisation model is often represented in the form of concrete tokens of two colours denoting positive and negative quantities. A pair of different coloured tokens cancel each other resulting in a net value of zero. Contexts to which the neutralisation model applies are positive and negative electric charges, or assets and debts. The number line model makes the order aspect more salient. Contexts such as height above and below sea level, floors in a building, are examples where the number line model applies. One of the characteristics of the number line model is that it does not readily make sense to add two states, that is, two points on the number line (such as two floor numbers in a building), while subtraction can be readily interpreted as, for e.g., the directed distance between two floors.

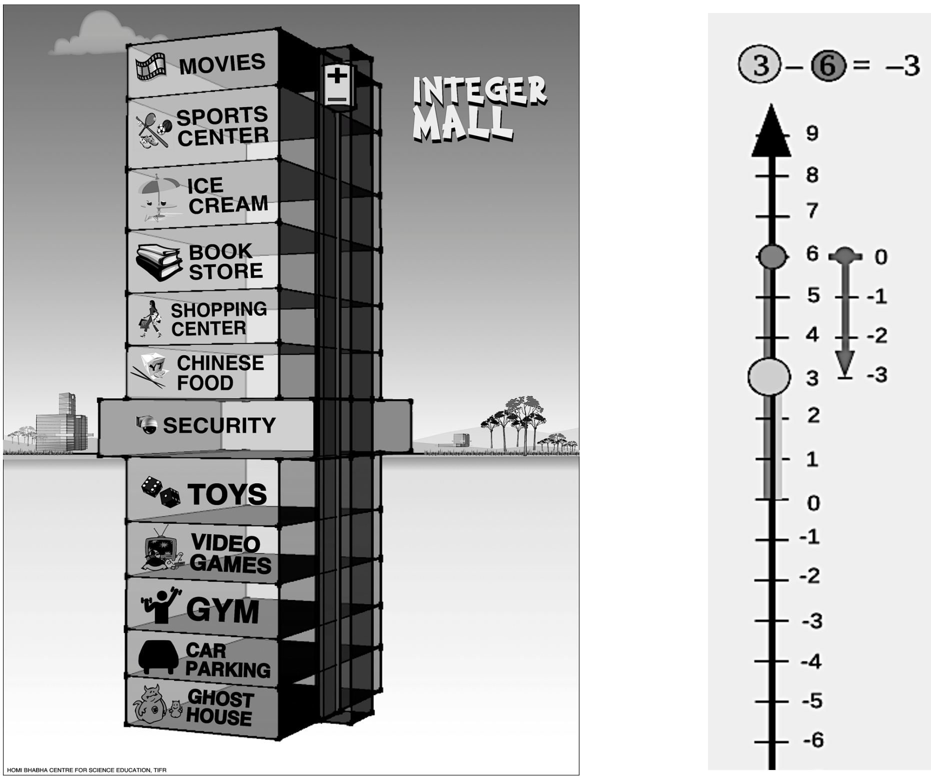

Figure 2: Context and Model for integers (image on left from Kumar et al., 2017)

Figure 2 shows the context of an "integer mall" that was developed together with the teachers in the study. Here the interpretation of integers as state is salient, i.e., as the position or "name" of the floor in the building, which suggests an underlying number line model. The context has an additional feature – an elevator with only two buttons marked "+" and "–" respectively. The number of presses of each button determines the total upward or downward movement of the elevator and can be expressed as an integer. This corresponds to the change meaning of an integer. We note that changes can be combined (added) or cancelled (subtracted). Subtraction may also be interpreted as the change needed to reach a target floor from a starting floor. This is indicated in the vertical number line on the right in Figure 2, which shows the result of "3 – 6" as a vector that takes 6 to 3, which is adjoined to the number line.

In general, although a particular context may appear to be described by one of the two models, a deeper examination may reveal the relevance of both models. For example, debts and assets seem to be best described by the neutralisation model since a debt and an asset of equal value cancel one another. However, combining assets and debts makes sense only in relation to the notion of "net worth," which is the sum of the assets and debts taken with their proper sign. "Net worth" is a state variable and fits more closely with a number line model.

In an array of two coloured tokens, the net value of the array (after cancelling pairs of opposite colours) corresponds to net worth and is a state, while individual tokens may be taken to represent unit increases or decreases, which are transformations. The visual presentation of a number line can represent both meanings more transparently, where points correspond to states and free vectors placed alongside the number line correspond to transformations as in Figure 2. Addition of signed numbers corresponds to movement on the number line, with the first addend representing a state and the second an "increase" or "decrease". Addition may also be interpreted in terms of combining changes to obtain a net change, corresponding to a neutralisation interpretation. Subtraction is usually interpreted in the neutralisation model as take away. However, this poses a difficulty for expressions such as 5 – (–3), where there are no negative tokens to begin with, and hence none available to take away. This is resolved by adding three zero pairs to the array without changing the net value. Taking away –3 now yields the result, +8. In our study, teachers were uncomfortable with this resolution since they thought that the original problem of 5 – (–3) had been changed. Subtraction may be interpreted more readily in terms of comparison or a missing addend. On the number line, this results in a vector that will take the subtrahend to the minuend as in Figure 2. Subtraction may also be interpreted in terms of cancellation of a vector, however this encounters similar problems as subtraction in the neutralisation model.

The models can also be adapted to support reasoning in regard to other tasks. An extended version of the integer mall that allows the use of bigger numbers is the context of a mine that extends both below and above ground (into a rocky hill). Signed numbers here denote the height above or below the ground in metres. Here too, elevators are appropriate. In fact, fast moving elevators may be needed and a new feature emerges as salient: the velocity of the elevator, which can be represented as a signed number indicating both speed and direction.

The research literature on the teaching and learning of multiplication of signed numbers is surprisingly sparse in comparison to the literature on addition and subtraction. An approach that has been taken is to interpret multiplication by a positive integer n as starting with zero and adding n copies of a multi-unit, which may be positive or negative. Multiplication by a negative integer $-n$ is then interpreted as starting with zero and *taking away n* copies of a multi-unit, which may be positive or negative. Of course, in order to take away anything from zero, one must first put down a sufficient number of zero pairs. It is difficult to give a plausible context corresponding to this interpretation but has been attempted by Menon (2015). A limitation of this interpretation is that it does not allow extension to multiplication by a negative fraction, or at least, such an extension is yet to be developed.

The context of a mine where elevators move with a certain velocity affords a better interpretation of multiplication of negative numbers. As discussed in the previous section, multiplication is more coherently interpreted as a correspondence between two measure spaces. The measure spaces in this context are velocity and time, both of which can be denoted by signed numbers and their correspondence can be modelled as a pair of number lines. Here the multiplication of two negatives corresponds to an intuitive interpretation in the relation *velocity* × *time* = *distance*, which also yields the product with the correct sign. If the elevator is moving with a velocity of –3 m/s (i.e., in the downward direction), at what distance was the elevator 5 seconds ago (i.e., at –5s)? This gives "(–3) × (–5) = +15". We also note that since speed and time are continuous measures, it is possible to extend this interpretation to rational numbers.

To summarise, the meaning of an integer as a transformation unlocks the possibility of using diverse contexts to support instruction on signed numbers. Models provide a representation for the underlying meanings that unify the diverse contexts. The number line is a powerful model that can support further extensions. On the number line, points are represented by signed numbers and correspond to states. Transformations and relations can be shown as free vectors adjoining the number line. Extension to the multiplication operation is made possible by the correspondence model, represented by a pair of number lines.

In the next section, I will use the discussion of representations and meanings at the levels of symbol, context and model for signed numbers as a guide to evolve a possible framework for the extension to fractions.

EXTENSION TO FRACTIONS

Fractions extend the scope of quantification to continuous magnitudes and are hence fundamental to measurement. Unit fractions (i.e., fractions of the form *1/n*) are obtained by sub-divisions of the unit, that is, they are obtained through the division operation. Fractional measures are composed of unit fractions and hence are multiplicatively related to the unit fractions. This is encoded in the vocabulary used for fractions: the denominator "names" the unit fraction, while the numerator enumerates the number of copies of the unit fraction contained in the measure. Allowing for fractional measures also allows one to move beyond division with remainder in WNA to complete division without remainder. Indeed, the fraction notation can also be read as a compact notation for the result of the division operation as captured vividly in this quote from the mathematician William Thurston.

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to *134* divided by *29* is *134/29* (and so forth). What a tremendous labor-saving device! To me, "*134* divided by *29*" meant a certain tedious chore, while *134/29* was an object with no implicit work. I went excitedly to my father to explain my discovery. He told me that of course this is so, *a/b* and *a* divided by *b* are just synonyms. To him, it was just a small variation in notation. (Thurston quoted in Sfard, 2008, p. 163)

The fraction notation hence reflects the process-product duality or reification (Sfard, 2008) characteristic of algebraic notation, i.e., *a/b* is both a notation for the division operation of *a* divided by *b*, as well as the result of that operation. It is this capacity to represent division in a compact manner that makes the fraction notation useful in dealing with proportionality. Proportionality signifies an invariant multiplicative relation between two varying quantities. The "between measure" invariant relation, emerges through pairs of corresponding values, one taken from each varying quantity. The invariant relation obtains for each pair and is usually interpreted as a rate (e.g., Rs

20 per kg of potatoes). When only two pairs of corresponding values are considered, a second "local" invariant relation also obtains, the within measure or scalar ratio. (For e.g., Rs 40 for 2 kg, Rs 80 for 4 kg, hence 4/2 = 80/40.) The fraction notation may be used to denote both the between and the within measure invariant relation in the form of a ratio. This connects the invariant relation to the division operation, although this connection is often not made explicit in instruction.

Historically, fractions arose and were in use much earlier than negative numbers because of the ubiquity of measurement in commerce. One must note however that the complete set of fractions of the form m/n, where m and n are any positive integers with $m < n$, is not needed for measurement. A sub-set of the fractions is sufficient to quantify continuous magnitudes to the desired accuracy. In modern measurement, the decimal fractions, i.e., fractions obtained by dividing the unit recursively into ten equal parts, are sufficient to represent measures to the required accuracy. In the British colonial period in India, the binary fractions (fractions obtained by repeated halving) were used to designate lengths smaller than an inch, which still continues in some contexts. The denominations of Indian currency in the pre-Independence period used base four fractions (recursive sub-division by 4). Hence in these contexts, the generalized arithmetic of fractions is not needed. The appropriate rules for converting between different fractional units, or between different positional values is sufficient. We also note that complete division, i.e., division without remainder, can be carried out to any desired accuracy with a subset of the fractions. The division algorithm for whole numbers can be easily extended to yield decimal fractions.

Thus the principle reason for including the complete arithmetic of fractions in the curriculum is that this is needed for algebraic manipulation (Subramaniam, 2013). This is also the reason why it appears full blown, that is, with the complete rules for operations with fractions, in the work of Brahmagupta, mentioned earlier in the context of the rules for operations with negative numbers (Plofker, 2009).

A secondary reason, also important, is that fractions provide conceptual support to understanding proportionality and multiplicative relations between quantities. Several studies have shown that adults unschooled in arithmetic can reason about quantities in proportionality contexts using complex build-up strategies (Nunes & Bryant, 2015). Children can also do so independent of formal school learning in limited contexts, when simple whole number ratios obtain between quantities (Subramaniam, 2013). Hence grasping invariant multiplicative relations via correspondence of values has an intuitive basis. The fraction notation, as a shorthand for division, can denote the multiplicative relation between any two whole numbers. This notational support and an assurance that multiplication and division is possible for any pair of fractions can secure and generalise the intuitive understanding of multiplicative relations.

Further, fraction instruction itself provides ample opportunity to repeatedly draw on and extend the intuitive grasp of multiplicative relations and of proportional reasoning.

Many examples of tasks connected with the learning of fractions that highlight the multiplication relation between units and quantities may be found in Lamon (2005).

Although fractions play a central role in measurement and proportional relations, the route to mastering the arithmetic of fractions is complex and difficult for young learners. There are several sources of difficulty. A part of the difficulty stems from the fraction notation, which is presented as a pair of positive integers separated by a horizontal bar. Learners tend to interpret this as a pair of whole numbers rather than as a single number that denotes the relation between the pair of whole numbers. Traditional fraction instruction, which emphasises the part-whole interpretation of fraction may actually reinforce whole number thinking (Streefland, 1993).

Moreover, unlike whole numbers, fractions typically have multiple equivalent notations that occur frequently in working with fractions. The whole numbers follow a fixed ordinal sequence, which is not the case for fractions, which are densely ordered. The fact that between any two fractions, one can find another fraction is a property that is completely unfamiliar. Further, cultural supports that existed even a century earlier in the form of binary and other fractions that were used in commerce have all but disappeared and decimal fractions follow a notation different from fractions in general.

Many researchers have also pointed out that fractions have multiple interpretations in contexts of application. This was articulated by the influential sub-construct theory proposed by Thomas Kieren in the 1970s. Kieren suggested that the encounter between fractions and real world contexts is mediated by different sub-constructs or “meanings” of fractions (Kieren, 1988). The five meanings that most researchers accept are part-whole, measure, quotient, ratio and operator. This research led to an exploration of approaches to teaching fractions that emphasized sub-constructs other than part-whole as a starting point. In particular, approaches using equal sharing situations, which correspond to the quotient sub-construct, have proved to be effective and powerful in creating a robust initial understanding of fractions, fraction ordering, and equivalence (Streefland 1993; Subramanian, Umar & Verma, 2015). Some researchers also explored instruction that focused on integrating different sub-constructs, for example, reasoning about the connection between the quotient and measure interpretations (Naik & Subramaniam, 2008).

However, exploring all the sub-constructs within the limited space offered by a curriculum appears impossible, let alone working towards ways of integrating the various interpretations. Although approaches that prioritize the quotient sub-construct as a grounding interpretation of a fraction are effective in initial fraction learning, they do not offer support for fraction arithmetic as a whole. One needs a theory that works with inter-connected meanings to begin with. I will present the outline of such a theory by using the one sketched in the previous section dealing with signed numbers as a guide.

Symbol

As suggested earlier, the fraction notation has two inter-connected significations. First, it denotes a number, specifying a unit and enumerating the number of such units. The encoding is different from the decimal place value encoding for whole numbers. In the fraction *m/n*, *1/n* names the unit (unit fraction) and *m* enumerates the number of copies of this unit. Symbolically this may be represented as

$$\frac{m}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{\text{taken m times}} = \frac{1}{n} \times m$$

The decomposition of a fraction via the unit fractions can provide a warrant for general relations such as $k/n + m/n = (k+m)/n$ or $k/n - m/n = (k-m)/n$. It can also be seen that a fraction of the form *m/m* is equal to the unit, and that when $m>n$ in *m/n*, it can be decomposed into the sum of whole units and fractional units, that is, can be re-notated as a mixed number. All of these relations can receive additional support and warrant from contexts and models.

The second signification of a fraction is via the reification of the division operation – the fraction *m/n* denotes the result obtained when *m* is divided by *n*. This may be expressed as $m \div n = m/n$. This signification also provides a ready warrant for the relation $km/m = k$.

A striking aspect of the fraction symbolism is that multiple fraction notations may be equivalent. Both the measure and the division interpretations provide some support for learners here. Familiarity with division may lead learners to see that 16/4 and 8/2 are equivalent.

Context

As remarked earlier, contexts can be powerful in creating an initial understanding of fractions. However a framework that unifies the diverse meanings that fractions take in contexts is needed to support effective use of representations for fraction instruction over the entire range of tasks that constitute fraction arithmetic. In a context involving continuous magnitudes, a fraction (i.e., a positive rational number) denotes a measure or quantity. In this interpretation, the fraction *m/n* is composed of *m* copies of the unit fraction *1/n*. The measure interpretation provides the foundation for all the four basic operations with fractions – addition, subtraction, multiplication and division. Fraction addition and subtraction can be sensibly interpreted in terms of the measure meaning. Once the parsing of a fraction notation in terms of the unit fraction is secure, addition and subtraction are similar to the corresponding operations in WNA. The “like units” principle (Ma & Kessel, 2018) applies here to the unit fractions – fractions composed of the same unit fractions can be added or subtracted.

The measure interpretation of a fraction is the foundation also for the multiplication and division operations. This is best understood in terms of the correspondence relation and we postpone further discussion to the subsection on models below. Division of a fraction and by a fraction has the usual quotitive and partitive interpretations as in

WNA, although the quotitive interpretation is more natural in contexts involving division by a fraction.

The other two interpretations of a fraction stem from the two interpretations of the division operation, where the fraction *m/n* is interpreted as the result of the division $m \div n$. One must distinguish between *fraction division* and the interpretation of the *fraction as division*; the latter is a (complete) division of whole numbers. In the quotitive interpretation of fraction as division, the quotient *m/n* specifies how many units of size *n* are contained in *m*. In the partitive or unit share interpretation of fraction as division, *m/n* specifies the measure (or size) of one unit if *m* is the measure of *n* units. Note that now, in both interpretations *n* may be larger than *m*. The unit share interpretation presented in the context of equal sharing is especially powerful in identifying equivalent fractions and in ordering fractions as mentioned earlier. The unit share or partitive interpretation can be connected to the measure interpretation of a fraction by interpreting *1/n* as dividing one unit measure into *n* equal shares or parts and *m/n* as *m* copies of the equal share thus obtained (Naik & Subramaniam, 2008). The unit share interpretation can be applied to proportion problems via the "unitary method", i.e., by finding the unit rate or the measure corresponding to one unit. The quotitive interpretation also applies in proportion problems in the form of a scalar ratio.

Model

A common model for a fraction is the part-whole area model presented through plane figures such as the circle or the rectangle. A more general model is the number line, where the fraction corresponds to a position on the number line and denotes the length from zero up to that point. This corresponds to the measure interpretation of a fraction. The model for addition and subtraction on the number line is not different from that discussed in section on signed numbers.

The model for multiplication and division is the correspondence between a pair of number lines, similar to the presentation shown in Figure 1. However, we are now in a position to clarify the meaning of correspondence. An initial correspondence is set up between two number lines by pairing two values, one from each number line. (A second pair of corresponding values always are the zeroes on each line). Now, each partition and replication when mirrored on the two lines creates new pairs of corresponding values. This notion of correspondence provides an initial scheme to locate the unit fraction on the number line. The fraction 1/8 may be located by interpreting it as $1 \div 8$, obtaining 8 equal parts of the unit. Further, we obtain an interpretation for the multiplication of *p* by a unit fraction *1/n* analogous to multiplication by a positive integer *k*. The simple case is multiplying by half. Since half is obtained by a "2-partitioning" of 1 unit, when this partition is mirrored in the target measure space, we obtain the value of $\frac{1}{2} \times p$ by a 2-partitioning of p. To obtain the result of multiplication by a fraction in general, one goes via the unit fraction. Thus to find the product of a quantity *q* by *m/n*, one first finds the quantity corresponding to *1/n* (i.e., the product $1/n \times q$) and then takes *m* copies of this quantity.

We are also now in a position to understand why multiplication needs to be represented as a correspondence between two lines. Signed numbers obtained through the reification of addition and subtraction could be shown as free vectors adjoining the number line because of the additive construction of the number line. Thus a vector of a fixed length is an invariant operator wherever it is moved to on the number line. A vector representing +4 can be freely moved along the number line and will still indicate the result of adding +4 to any number accurately.

However the reification of multiplication (i.e., a multiplication operator) cannot be represented as a free vector of a fixed length, unless the number line is constructed using a logarithmic scale. This is the reason why multiplication is better modelled as the correspondence between two number lines. Geometric properties allow for a diversity of ways of representing multiplicative correspondence. These are shown in Figure 3 for the multiplication operator “×2”.

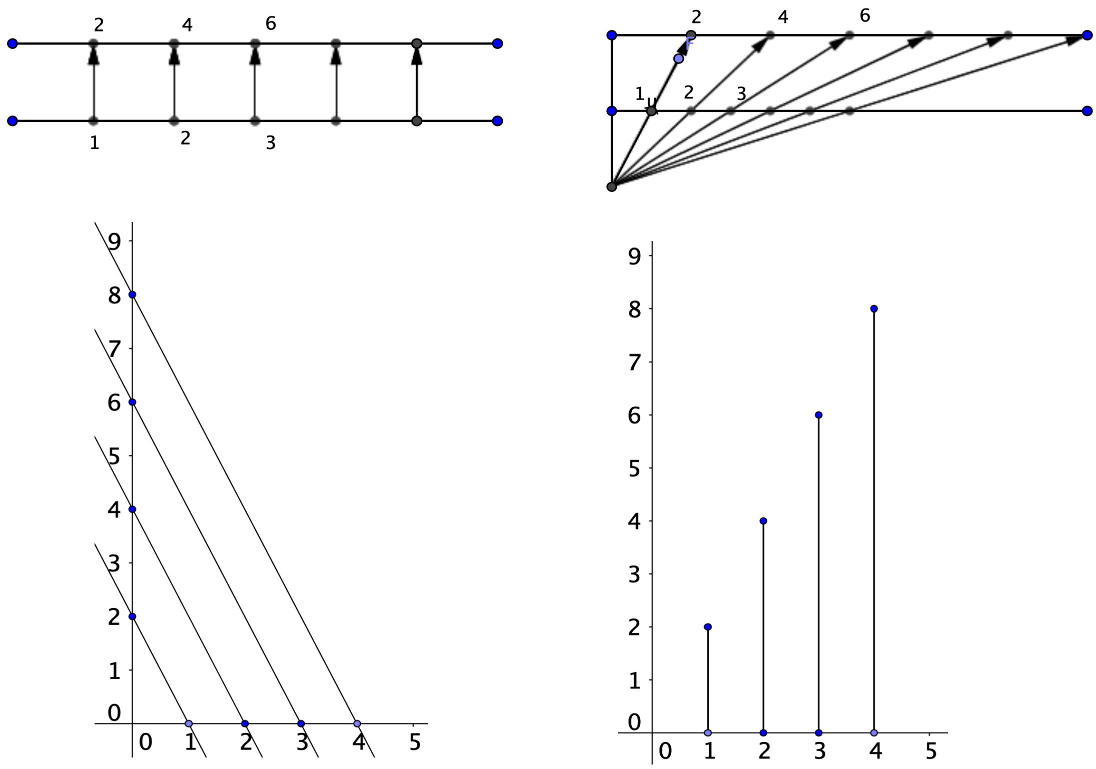

Figure 3: Correspondence between two number lines for the multiplication operator “×2”. (Compare also Freudenthal, 1983, p. 199)

In both diagrams on top in Figure 3, the number lines are parallel to each other. In the top left diagram, the scales in the two number lines are different and adjusted so that the lines of correspondence are all vertical and parallel. In the other three diagrams, the two number lines have the same scale. In the diagrams at the bottom, the two number lines aligned in orthogonal directions: the one on the left is (parallel sun rays) is taken from Freudenthal (1983). The one on the bottom right is the representation invented by Oresme to show motion in time (Mumford, 2010), and is a version of the familiar Cartesian graph. The various diagrams showing correspondence clearly have different affordances but have the same underlying meaning.

CONCLUDING REMARKS

The exploration that we have undertaken shows that it may be possible to construct powerful and coherent representations that support learners in extending their understanding of WNA to signed numbers and fractions. The representations achieve coherence by inter-connections between the meanings associated with symbols, contexts and models. This may be easier for the extension to signed numbers since the addition operation is relatively simpler to represent on the number line, which is constructed additively. For the extension to fractions, a powerful representation for the multiplication operation is needed. This does not seem possible with a single number line but needs a representation as a pair of number lines with a correspondence relation. How well this representation can support learners, in understanding fractions and rational numbers and multiplicative thinking in general, needs to be explored. The account given above is founded on a more tightly integrated framework of meanings for fractions and operations with fractions compared to other theories such as the sub-construct theory, and hence prima facie is more coherent. It recognises the measure interpretation of fraction, corresponding to position on the number line, as fundamental to understanding all the operations and hence provides a unifying meaning for fraction arithmetic. It indicates how the measure interpretation may be connected with the interpretation of the fraction notation as a reification of the division operation. However, the multiplicative conceptual field, as Vergnaud pointed out, is complex. Whether these representations succeed in helping learners navigate this field will become clearer with further design and intervention studies.

References

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education*, *19*(3), 237-254.

Banerjee, R., & Subramaniam, K. (2012). Evolution of a teaching approach for beginning algebra. *Educational Studies in Mathematics*, *80*(3), 351-367.

Bass, H. (2018). Quantities, numbers, number names and the real number line. In M.G. Bartolini Bussi and X.H. Sun (Eds.), *Building the Foundation: Whole Numbers in the Primary Grades* (pp. 465-475). Cham: Springer.

Chambris, C. (2018). The influence of theoretical mathematical foundations on teaching and learning: a case study of whole numbers in elementary school. *Educational Studies in Mathematics*, *97*(2), 185-207.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel.

Goldin, G. A. (2014). Mathematical representations. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 409-413). Dordrecht: Springer.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, Virginia: Lawrence Erlbaum.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education*, 443-464.

Kumar, R. S., Subramaniam, K. & Naik, S. (2017). Teachers' construction of meanings of signed quantities and integer operation. *Journal of Mathematics Teacher Education*, *20*(6), 557-590.

Kumar, R. S. (2018). *Supporting in-service professional development of mathematics teachers: the role of beliefs and knowledge*. Unpublished doctoral dissertation submitted to the Tata Institute of Fundamental Research.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding* (2nd Edition). Mahwah, New Jersey: Lawrence Erlbaum.

Ma, L., & Kessel, C. (2018). The theory of school arithmetic: Whole numbers. In M. G. Bartolini Bussi and X. H. Sun (Eds.), *Building the foundation: Whole numbers in the primary grades* (pp. 439-463). Cham: Springer.

Mathews, C., Venkat, H., & Askew, M. (2018). Primary Teachers' Semiotic Praxis: Windows into the Handling of Division Tasks. In N. Presmeg, L. Radford, W. Roth & G. Kadunz (Eds.), *Signs of Signification* (pp. 257-274). Cham: Springer.

Menon, U. (2015). Multiplying integers: Mathematizing to make sense. In S. Chandrasekharan, S. Murthy, G. Banerjee & A. Muralidhar (Eds.), Proceedings of the *epiSTEME 6 International Conference to Review Research on Science, Technology and Mathematics Education* (pp. 93-101). India: Cinnamonteal.

Mumford, D. (2010). What's so baffling about negative numbers? – A cross-cultural comparison. In C.S. Seshadri (Ed.), S*tudies in the History of Indian Mathematics.* (pp. 113-143). New Delhi: Hindustan Book Agency.

Naik, S. and Subramaniam, K. (2008). Integrating the measure and quotient interpretation of fractions. In O. Figueras et al. (Eds.), *Proc. 32nd Conf. of the Int. Group for the Psychology of Mathematics Education and PME-NA XXX*, (Vol 4, pp. 17-24). Morelia, Mexico: PME.

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, *40*, 282-313.

Ng, S. F. (2015). A theoretical framework for understanding the different attention resource demands of letter-symbolic versus model method. In B. Sriraman, J. Cai, K-H. Lee, L. Fan, Y. Shimizu, C. S. Lim, & K. Subramaniam (Eds.), *First sourcebook on Asian research in mathematics Education* (Vol. 2, pp. 1011-1043). Charlotte, NC: Information Age Publishers.

Nunes, T., & Bryant, P. (2015). The development of mathematical reasoning. In R. M. Lerner (Ed.), *Handbook of child psychology and developmental science* (Vol 2, pp. 715-762). New York: Wiley.

Plofker, K. (2009). *Mathematics in India*. Princeton, NJ: Princeton University Press.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Abingdon-on-Thames: Routledge.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education*, *43*(4), 428–464.

Streefland, L. (1993). Fractions: A Realistic Approach. In T. Carpenter, E. Fennema, and T.A. Romberg (Eds.) *Rational Numbers: An integration of research* (pp. 289-325), Hillsdale, NJ: Lawrence Erlbaum.

Subramanian, J., Umar, & Verma, S. (2015). Some ethical concerns in designing the upper primary mathematics curriculum. In B. Sriraman, J. Cai, K-H. Lee, L. Fan, Y. Shimizu, C. S. Lim, & K. Subramaniam (Eds.), *First sourcebook on Asian research in mathematics Education* (Vol. 2, pp. 1583-1606). Charlotte, NC: Information Age Publishers.

Subramaniam, K. (2013). Research on the learning of fractions and multiplicative reasoning: A review. In S. Chunawala (Ed.), *The epiSTEME Reviews: Research Trends in Science, Technology and Mathematics Education*. Vol. 4. New Delhi: Macmillan.

Subramaniam, K. & Banerjee, R. (2011). The arithmetic-algebra connection: A historical-pedagogical perspective. In J. Cai, & E. Knuth (Eds.), *Early Algebraization: A Global Dialogue from Multiple Perspectives* (pp. 87-107). Cham: Springer.

Subramaniam, K. & Naik, S. (2010) Attending to Language, Culture and Children's thinking as They Learn Fractions. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 333-337). Belo Horizonte, Brazil: PME.

Venkat, H., & Askew, M. (2018). Mediating primary mathematics: theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics*, *97*(1), 71-92.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39–59). London: Lawrence Erlbaum.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction*, *14*(5), 469–484.

PLENARY PANEL

WHAT IS PROVEN TO WORK (ACCORDING TO INTERNATIONAL COMPARATIVE STUDIES) IN SUCCESSFUL COUNTRIES SHOULD BE IMPLEMENTED IN OTHER COUNTRIES

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The PME 43 (2019) plenary panel session takes the form of a debate. The two members of the affirmative team will argue in favour of the claim "What is proven to work (according to international comparative studies) in successful countries should be implemented in other countries." The negative team will argue against that proposition. I will chair the debate.

INTRODUCTION AND BACKGROUND TO THE PLENARY PANEL

The question posed for the panel team is one that we have each probably faced at some point, either in our classrooms, in the news, or even at social gatherings. In the United States, certainly, each release of the international scores in mathematics is followed by questions such as "Why don't we use the same curriculum they use in country x?" "Why aren't we teaching math the way they do in country y?" Sometimes local news people ask me to comment on what I think about the curriculum used in country x, which has just been adopted in a local school district. In my undergraduate course, as soon as students read about math instruction in other countries, they also wonder "Why don't we just teach math the way they do?" This is certainly an issue that is relevant to mathematics education and the panel welcomes the opportunity to discuss it directly during the panel debate.

The first task for the panel members was to accept the proposition as written. After some discussion about attempting to revise the claim, we accepted it as stated. The second task was to choose a team, for or against. This was not easy, since it seemed simplest to argue against the proposition. However, we made the decision to have one member from the "for" report on interviews with education officials' responses to the claim, we moved forward to prepare for and against papers. In the words of the PME Panel chair from 2016 "As is traditional in debates, it is quite conceivable that the members of a team may not actually believe the proposition for which they are arguing, but argue they must, with whatever reasoning and evidence they have at their disposal (Chick, 2016)."

At first glance, I thought I could only see one side of the argument: How can I claim that because something works in one setting it should be implemented in another, knowing the nuances and details of how setting, context, structural issues, and local practices can impact and change any educational intervention? Upon further reflection,

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 55-60). Pretoria, South Africa: PME.

I could also see the sense in not re-inventing wheels and came to see the other side of the argument. When I am attempting to make some improvement in teaching/learning, the first place I look to is what have others done who were successful, how did they do it, and what will it take to bring that improvement to my local setting.

SOME QUESTIONS

In my experience, debates can become mired in talking *at* each other rather than *with* each other. In order to support a debate that involves talking *with* the other side, I will propose a few questions, hoping these will help all of us better hear both sides of the argument.

Question 1: How are we defining when an intervention has been proven to work?

There are multiple ways to think that an intervention has been proven to work. One would be that it increases student achievement according to agreed-upon results such as an international assessment. Another way to think that an intervention has been proven to work is if the results are more equitable in some way (i.e. access to more courses, higher completion rates, etc.) than the results that preceded the intervention.

Question 2: How are we defining the elements of an intervention?

Any of the multiple essential ideas in any educational intervention can be, and will be, re-interpreted, by both researchers and practitioners. My favourite example is the concept of scaffolding. Although many researchers and practitioners have written extensively on what precisely are the distinguishing characteristics of scaffolding, the term is continually used to refer to any and all types of support or guidance without considering the details of what is scaffolded or how the scaffolding happens. Other examples include multiple interpretations and meanings for phrases such as “learner centered,” “knowledge centered,” or “community centered” as described by Bransford, Brown, & Cocking (2000).

Question 3: What theoretical assumptions did the first implementers of an intervention make?

The people who implement any educational intervention make assumptions which reflect their theoretical stances. Implementers at another location may or may not share those assumptions. For example, in considering an intervention that might be relevant to language and mathematics learning, it is important to distinguish between psycholinguistics and sociolinguistics stances because these two perspectives differ in how they conceptualize language. While sociolinguistics stresses the social nature of language and its use in varying contexts, psycholinguistic studies have been limited to an individual view of performance in experimental settings. From a sociolinguistic perspective, psycholinguistic experiments provide only limited knowledge about how people actually use language in a given social setting and depending on a group’s collective linguistic norms (Hakuta & McLaughlin, 1996).

Question 4: How are we expecting an intervention to travel?

There are multiple assumptions we can make about how an intervention can move across settings. One would be we can replicate the results in one setting when the intervention is implemented in another setting. Another is that, although replication is impossible, we may be able to have some aspect travel to another setting. This second stance would ask questions such as: What might be the relevant differences among settings, students, and communities?

For example, an important resource for innovations in learning/teaching mathematics is research carried out in geographic settings with student populations other than the target population for a particular study. Researchers have studied language, bilingualism, and mathematics learning in South African multilingual classrooms (e.g., Adler, 1998, 2006; Setati, 1998). This work can be an important resource for research with other student populations, as long as researchers note differences among settings that might be relevant to issues of language and learning mathematics for the student population for a particular research study. To think about the relevance of work from other settings for Latinx mathematics learners in the U.S., anyone implementing an intervention would need to consider the historical, political, and linguistic differences between the U.S. and other countries. Before applying an intervention from other countries to U.S. settings and student populations, researchers should carefully consider relevant differences among settings, students, languages, and communities. One difference might be that the U.S. Latinx population of school age children can be largely described as bilingual in Spanish or as monolingual English speakers, although there is also a small yet growing percentage of Latino children and adults in the U.S. who also speak an indigenous language as their first language, Spanish as a second language, and English as a third language. In contrast, the majority of students (as well as teachers) in South African classrooms speak multiple languages at home. Another contrasting example is Pakistan, where the language of schooling is usually not spoken at home but reserved for activities related to school or government related activities.

Question 5: What systems are the classrooms embedded in?

Educational interventions happen in classrooms that are embedded in larger systems and these systems vary across settings. For example, the United States has a decentralized education system with multiple locations for teacher preparation. In contrast, Singapore's system is completely centralized and there is only one School of Education which prepares all of the nation's teachers. Educational interventions will be affected by those systems and those systems need to be included when considering how an intervention might travel to another setting.

Question 6: How are we expecting teachers to support the travel of an intervention?

Educational interventions do not happen by themselves, they require the commitment and hard work of the teachers. Any task, lesson, curriculum, or intervention is enacted by the teachers who implement them. Before asking an intervention to travel, these seem like crucial questions to ask: What specialized knowledge is required of teachers for travel to be possible? What teacher preparation is required for travel to be possible?

CLOSING THOUGHTS

Cross-cultural research and implementations require that we make our assumptions about the nature of cultural practices explicit. Definitions of culture are contested and vary across academic disciplines. A definition of culture or an account of debates around its definition is beyond the scope of this introduction. However, educational anthropology and cultural psychology (i.e. Cole, 1996) provide assumptions to ground transnational or cross-cultural studies and interventions. One of my favourite definitions is by Rogoff (1990) as the "organized and common practices of particular communities in which people live," which may be different than the practices of the people's nation, their geographic location, or ancestry.

We cannot assume "cultural uniformity or a set of harmonious and homogeneous set of shared practices (González, 1995, p. 237)" about any cultural group. To avoid essentializing cultural practices or describing culture as individual traits Gutiérrez & Rogoff (2003) propose that we focus not on individual traits but on what they call "repertoires of practice" using the assumptions that individuals develop, communities change, and learners have access to multiple practices. Lee (2003) argues that we should "neither attribute static qualities to cultural communities nor assume that each individual within such communities shares in similar ways those practices that have evolved over generations."

Researchers have also described pitfalls to avoid in cross-cultural research. One major pitfall to avoid is using deficit views. Learners from non-dominant groups have been characterized by a deficit model in which their failures in schools are related to their home environment. In particular, research on Latinx students in the U.S. has often focused on "…language genres, behaviour patterns, motivations, attitudes, and expectations that are either unacknowledged by the schools or seen as developmental deficits that must be 'remediated' or proscribed before learning can begin (Garcia & Gonzalez, 1995, p. 422). In designing transnational research studies and interventions we need to move from deficit models of students' homes and communities to frameworks that value the resources that students bring to the mathematics classroom from their previous experiences. Only then can interventions be designed that honour their experiences.

PLAN FOR THE DEBATE

The plan proposed for the PME 43 Plenary Panel session debate will comprise a Chairperson (Judit Moschkovich) and four presenters/debaters; two will be on the affirmative team arguing in favour of the topic (Mercy Kazima and Robyn Jorgensen), and the other two will be on the negative team arguing against the topic (Yeping Li and Kim Heejeong). As Chair, I will open proceedings, outline the rationale for the topic, and describe the procedure and rules of the debate. Each speaker will have a maximum of 10 minutes to speak. There will be a warning bell at 9.5 minutes; the speaker will be asked to sit down at 10 minutes. The summation talks will have a 5-minute time limit, with a warning bell at 4.5 minutes. As has been the case with previous Plenary Panel debates, we expect an informative and scholarly dialogue which may also include some light-hearted exchange or humor.

The opening speaker is from the affirmative team. Next is the first speaker for the negative; this speaker begins with a brief rebuttal of the affirmative speaker’s argument, including a comment on any definitions and whether the negative team accepts the definitions or has alternative interpretations of the terms. This person then proceeds to open the negative team’s argument. Then the second speaker for the affirmative begins with a brief rebuttal of the negative team’ s opening argument and then continues with the remaining arguments of the affirmative team. Next, the second negative speaker repeats this process i.e. brief rebuttal followed by continuing the negative team's argument. To close the debate the first affirmative speaker has 5 minutes to rebut the negative team's argument and summarize the affirmative team’s case; no new arguments are allowed. Finally, the first speaker from the negative team does the same; 5 minutes to rebut followed by summary of the negative team’ s case and no new material allowed. I will then invite questions and comments from the floor for about 15 to 20 minutes. We will then take a vote from the floor to determine the winning team.

References

Adler, J. (1998). A language of teaching dilemmas: Unlocking the complex multilingual secondary mathematics classroom. *For the learning of mathematics*, *18*(1), 24-33.

Adler, J. B. (2006). *Teaching mathematics in multilingual classrooms* (Vol. 26). Cham: Springer.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How people learn* (Vol. 11). Washington, DC: National Academy Press.

Cole, M. (1996b). *Cultural psychology: A once and future discipline*. Cambridge, Mass: Harvard University Press.

Garcia, E., & Gonzalez, R. (1995). Issues in systemic reform for culturally and linguistically diverse students. *Teachers College Record, 96*(3), 418-431.

González, N. (1995). Processual approaches to multicultural education. *Journal of Applied Behavioral Science*, *31* (2), 234-244.

González, N., Andrade, R., Civil, M., & Moll, L.C. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk*, *6*, 115-132.

Gutiérrez, K., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice? *Educational Researcher, 32*(5), 19-25.

Hakuta, K., & McLaughlin, B. (1996). Bilingualism and second language learning: Seven tensions that define research. In D. Berliner & R. C. Calfe (Eds.), *Handbook of Educational Psychology* (pp. 603-621). New York: Macmillan.

Lee, C. (2003). Why we need to rethink race and ethnicity in educational research. *Educational Researcher*, *32*(5), 3–5.

Rogoff, B. (1990). *Apprenticeship in thinking*. NY: Oxford University Press.

Setati, M. (1998). Code-switching in a senior primary class of second-language mathematics learners. *For the Learning of Mathematics*, *18*(1), 34-40.

DIRECT INSTRUCTION: A CASE OF WHAT IS PROVEN TO WORK IN CONTEXTS IN OTHER COUNTRIES

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As part of the affirmative team of the PME 43 (2019) plenary panel "What is proven to work (according to international comparative studies) in successful countries should be implemented in other countries," this paper draws on the example of an extensive program, Direct Instruction, which has a long history and was initially developed for learners from impoverished backgrounds in the USA. The program now has considerable traction in remote Indigenous communities in Australia and from the success of its implementation, is being adopted by schools in low SES and underperforming schools.

In preparing for this panel, I have opted to take the affirmative stance for implementation of a curriculum that has been developed in one context and applied in another. This is counter to my ideological stance and what I have tried to achieve in researching for this panel is to develop a better understanding of how a program developed in the USA, which was developed nearly 60 years ago and adopts principles that I personally see as counter to good mathematics teaching, is now seen to be highly successful in some of the most disadvantaged contexts in the Australian educational landscape. There is strong media and political interest in the perceived success of Direct Instruction in remote Indigenous contexts, so much so, that other state governments have commenced rolling the program out in other communities including low SES urban communities. My focus in this paper is to better understand why this is occurring and to challenge my own personal assumptions of why this should not happen. I adopt the questions posed by the Chair to frame my response.

INTRODUCTION AND BACKGROUND TO DIRECT INSTRUCTION AND THE REMOTE INDIGENOUS CONTEXT

Direct Instruction (DI) is a particular program that was developed in the 1960s. Since that time, the materials and approach of DI has been expanded and modified based on research outcomes that have reshaped the program and include elements of systemic and explicit instruction such that the current approach has deviated from the original program. These modified programs may have elements of the original DI but due to the deviation are referred to as direct instruction (lower case di). DI was original conceived to assist students from very impoverished backgrounds in the USA (Bereiter & Engelmann, 1966).

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Foundational Principles of DI

The premises behind DI focus on instruction and that well-designed instruction is the key to learning. In contrast to other theories of learning that are based on stages of learning (developmental or stage theories) or the individual construction of meaning (constructivism); DI is premised on the assumption that all students can learn provided they mastered prerequisite knowledge and skills and that instruction is unambiguous (Stockard, Wood, Coughlin, & Khoury, 2018). Originators of the program (Engelmann & Carnine, 1991) have identified a number of key principles. Integral to instruction is the careful selection and design of examples – unambiguous, sequenced and minimal steps in processing. Mastery of learning is central to DI since a solid foundation of concepts allows students to learn new knowledge and skills. Positive reinforcement that is continuous followed by celebration of success/es provides a rewarding learning context of learners.

Applying these principles, curriculum materials have been designed for teachers. Scripted lessons that are fast paced and include consistent reinforcement for students are part of the daily routine and include regular checking of students' learning along with regular mastery testing (Engelmann, 2014). In their comprehensive review of DI, Stockard and colleagues (2018) reported that there were consistently positive results and statistically significant outcomes when using traditional psychological measures from studies using DI.

The Context of Remote Indigenous Education in Australia

Within the educational context of Australia, Indigenous students living in remote areas are the most educationally at-risk students in the educational landscape. The context is complex and beyond the scope of this paper. Suffice to say at this point, variables include, but are not limited to language; culture; isolation; teacher recruitment and retention; and curriculum relevance. All these variables impact the learning environments and opportunities for success for Indigenous learners. DI was first taken up by the Cape York Aboriginal Australian Academy (CYAAA) which is a multibase college across a number of schools in the remote regions of the Cape York Peninsula. The College currently operates from two communities – Coen and Hopevale. The mantra of the College is to provide an education that is quality in the best of both worlds. The College is now part of the Federal Government's Good to Great Schools initiative. Under the guidance of a leading Aboriginal activist – Noel Pearson who strongly advocated for the uptake of DI in this region - the government initially funded the implementation of DI across three campuses. The College was funded for various amounts of money at different periods of time. DI was first implemented at the College in 2010 when the College commenced operation. In 2014 Killan (2014) reported the Federal government allocated a further \$22million to expand the use of DI in Cape York schools. While enrolments can vary, to give a sense of the profile of the College, there were 174 students listed as enrolled in 2017 with the equivalent of 25 full time teachers and 17 non-teaching staff (ACARA, 2019). Considerable funding has been invested in the use of Direct Instruction at the College. Exact figures could not be found since there is often cross-sectorial funding – such as the initial establishment of the College or the subsequent funding for the *Good to Great Schools* initiative. Separating exact funding for DI is not possible.

Before progressing further, I take Moschkovich's (2019) note for a need to consider what is meant by 'success' when considering the implementation of reforms such as DI.

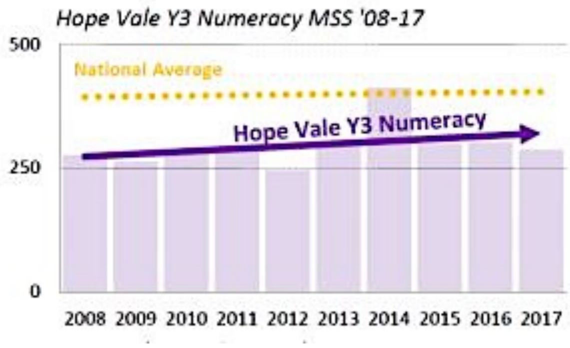

Figure 1: CYAAA numeracy success (source: 2017 Annual Report)

From the Annual report of the Cape York academy annual report (Cape York Aboriginal Australian Academy, 2018) they report growth in numeracy outcomes over a ten-year period. This can be seen in Fig 1. By contrast, the national testing scheme (NAPLAN) reports school outcomes. In numeracy, the My School website reports the numeracy averages for the Cape York Academy. These can be seen in Table 1 below.

	2013	2014	2015	2016	2017
Year 3	271	317	314	317	285
	290/397	286/402	309/398	286/402	294/409
Year 5	359	363	361	356	390
	287/486	372/488	405/493	378/493	392/494

Table 1: Numeracy averages

The protocol used by the website is to compare schools against the national average or similar schools. The data in table 1 is the comparison against similar schools since comparison with all schools would have resulted in all sections being dark red (substantially below). The colour coding refers to dark red (the 2015 cell with '361') = substantially below; pink (the 2013 cells with '271' and '359' and the 2016 '351' cell) = below; white = close to; light green (the 2014 and 2016 '317' cells) = above; dark green = substantially above. The numbers in the cells below the school refers to the average similar schools score and the national average school score respectively. This gives a numerical comparison to the mean scores. While NAPLAN data are based on mean scores, the evaluation of the CYAAA (ACARA, 2013) indicated that there were increasing numbers of students who were reaching or exceeding benchmarks since the introduction of DI. Proponents of DI argue that NAPLAN cannot be used as a measure of success of DI since it does not cater to the factors that impact learning for remote Indigenous students. The continuous data feedback model of DI that uses school and student data to inform instruction is seen to offer a better way of measuring success.

The numeracy scores are only one measure used by the school. Other benchmarks are undertaken including measures of student attendance, confidence, behaviour, and transition to secondary school. On these measures the CYAAA reported positive outcomes as well.

Teacher Professional Learning

A further measure of success is in the teacher learning at the College. The CYAAA has adopted a model of school practice that aligns with the National School Improvement Tool and supports teacher learning around DI. The model is an 8-cycle model:

- School Professional development – roles and models to support instructional curriculum and pedagogy.
- Teacher coaching and feedback – teacher leaders work with teaching teams through collaborative processes.
- School Data Review – weekly reviews of school data.
- Classroom feedback data – teacher provided and receive feedback in relation to mastery and progress of learners.
- Professional conversations – teachers provided with opportunities to engage in conversations with regard to best practices and the results of school and classroom data.
- Peer collaboration – collaboration with other teachers to improve their own classroom practice – working as a team to solve classroom problems.
- Community-School Improvement partnership – formal systems in place to ensure mutual accountability for school improvement with the community.
- Classroom family engagement – positive relationships with students and families outside the classroom.

Data provided in the CYAAA Annual Report (2017) indicate that the teachers reported very strong and positive reactions which suggest that the teachers felt that the support for professional learning was strong, and the program met their (and their students') needs.

The successes of DI that have been reported from the CYAAA have been taken seriously to the point where other jurisdictions have taken DI and implemented it in other Indigenous contexts. In 2017, 39 remote schools across the three states were participating in the expansion of the CYAAA DI model. The government funded further a $23.5million trial of the expansion of DI through the *Good to Great Schools* initiative. While there are very mixed reactions to the possibilities of DI in these contexts, there are some figures cited by the *Good to Great* organisation that purport that in 2015 at the commencement of the DI trial, only 8% of the prep, primary and high school students placed at Year 2 or above. That figure had risen to 30% by 2017, suggesting that DI was improving outcomes (Walker, 2017). DI has been successful in many schools but there is also a lot of criticism of the approach. Sarra, a leading Aboriginal educator, has been highly critical of the program through its 'drill to kill' approach.

SOME THINKING OF WHY DI WORKS IN THESE CONTEXTS

Pearson has been a strong advocate for DI in Indigenous schools. He is motivated by the highly structured program with its scripted text. Where most of the teachers in remote areas are in the early stages of their career (often first year out); where there is high teacher turnover; significant issues around student attendance; and where leadership at the school can be challenging, Pearson sees the structure of DI being of high value in the remote Indigenous teaching context. Pearson advocates that Indigenous learners should receive the best instruction rather than itinerate teaching by neophyte teachers who have little to no experience. Pearson further argues that when the leadership team buys into the DI program and gives the program full support, there is the strongest potential for success.

DI is well supported financially by the Federal government. Tens of millions of dollars have been invested in the CYAAA initiative and then the expanded *Good to Great Schools* Initiative, both of which are supported by Pearson. Aside from financial support, teachers are also supported through a plethora of resources readily available – including teacher resource books (Kinder, Rolf, & Carnine, 2017), teacher resources to support DI, tests, and so forth. There has been considerable investment in the schools to support the roll out of the program in terms of support personnel.

CLOSING THOUGHTS

A principal at one of the successful NT schools was reported to have said- "This is the first time I have had … a lot of kids learning …because of the rigour. The children feel safe with it. The content stays the same even if the teacher comes and goes" (cited in (Killan, 2014)). The rollout of DI, or any other program, may impact positively in some schools or classrooms for a wide range of reasons, some directly related to the initiative, others may be more indirect. The significant investment of money, time and goodwill help to ensure a new program of success. The successful uptake of DI in remote schools has been attributed to the knowledge of incoming teachers that the school is a DI school and that to be part of the school, there is an expectation that new teachers are part of that program. Some teachers actively seek to be employed by DI schools because they are implementing the DI program (Australian Council for Educational Research, 2013).

References

Australian Council for Educational Research. (2013). *Evaluation of the Cape York Aboriginal Academy Initiative*. Melbourne: ACER.

Bereiter, C., & Engelmann, S. (1966). *Teaching disadvantaged children in the preschool*. Englewood Cliffs, NJ: Prentice Hall.

Cape York Aboriginal Australian Academy. (2018). *Annual Report 2017: Queensland State School Reporting*. Cairns: Queensland Government.

Engelmann, S. (2014). Research from the inside: The development and testing of DI programs. In J. Stockard (Ed.), *The science and success of Engelmann's Direct Instruction* (pp. 3–24). Eugene, OR: NIFDI Press.

Engelmann, S., & Carnine, D. (1991). *Theory of instruction: Principles and applications*. Eugene, OR: ADI Press.

Killan, S. (2014). Schools get $22 million to implement direct instruction. *The Australian Society for Evidence Based Teaching - Turning research into practical advice, on line* (accessed 23//4/2019).

Kinder, D., Rolf, K., & Carnine, D. (2017). *Direct instruction mathematics*. New York: Pearson.

Stockard, J., Wood, T. W., Coughlin, C., & Khoury, C. R. (2018). The effectiveness of Direct Instruction curricula: A meta-analysis of a half century of research. *Review of Educational Research, 88*(4), 479-507. doi:https://doi.org/10.3102/0034654317751919

SHOULD WHAT WORKS IN SUCCESSFUL COUNTRIES BE IMPLEMENTED IN OTHER COUNTRIES?

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This paper sets the stage to oppose the proposition that "what is proven to work (according to international comparative studies) in successful countries should be implemented in other countries." To further discussion and research development in international comparative studies, I highlight and discuss three aspects related to this proposition, supported with existing studies, to illustrate why the proposition should be refuted and what advantages and limitations international comparative studies may have.

INTRODUCTION

It would be nice if what is proven to work (according to international comparative studies) in successful countries could simply be implemented in other countries to 'wishfully' replicate the success. The challenge of improving students' learning in different countries would then likely be simplified in two steps: (a) conducting international comparative studies to identify which countries are successful, if not yet known; and (b) identifying what works in those successful countries and then implementing them in other not so successful countries. However, the history of educational research and practice over the past several decades tells us that this is not that simple. To further discussion, I specify possible questions about the proposition: (1) the proposition contains some words, such as "work" and "successful," that need definition and clarification. It is unclear if the term "successful" is defined by the term "work", or by other criteria but informed by "what works" as the only (key) factor. (2) the proposition is a very strong statement using the phrase "… should be implemented …", which should be refuted. In the following sections, I highlight and discuss three aspects related to the proposition, using existing studies to illustrate these disputable questions in the proposition.

Not all successful countries are the same

If "what is proven to work" is used to define "success", every country can be in a position to identify something important that works for their students in certain aspect including, academic performance, self-confidence, interest, attitude, team collaboration, or skills on specific tasks. However, this would use a broad meaning of "success", and this is not what international comparative studies have intended to do.

Many large-scale international comparative studies have focused on students' academic performance, and other associated factors, as assessed by specifically designed tests and instruments. The results of students' performance in these studies

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 67-72). Pretoria, South Africa: PME.

have typically been used to rank participating countries and those highly-ranked countries have often been termed as high-performing or successful countries in IEA's curriculum-based studies (e.g., TIMSS) or OECD's skill-based studies (e.g., PISA). Such high-performing countries or systems often include those located in East Asia such as Singapore, South Korea, Hong Kong, and Japan; those in Europe, including Finland and the Netherlands. However, these high-performing countries or systems differ in many ways, including culture and education system. One may point out that these high-performing systems all have high GDPs that likely contributed to high outcomes in education. It should be noted that such a quick over-generalization carries great risk. Specifically, research does find that educational outcome is often connected closely with resources, the more resources, the better students' cognitive and non-cognitive development (e.g., Garcia & Weiss, 2017). Across nations, resources can be measured in terms of GDP, and one would assume that the higher GDP, the better educational quality and outcome. However, recent PISA results revealed that Vietnam students' performance in math is better than their counterparts in the U.S. and Vietnam also has far smaller percentage of low-performing students than the U.S. At the same time, average income in Vietnam, adjusted for purchasing power, stands at just one-tenth of the U.S. average (e.g., Tucker, 2016). Thus, it is important to look beyond students' performance ranking itself and any single contributing aspect or factor. International comparative studies can help reveal what is possible to achieve in an education system, but they are not a recipe for success.

What works in different successful countries are not the same

Learning about contributing factors to students' high performance is certainly in researchers' mind, when they carry out large-scale international comparative studies. The reason is simply that researchers want to know what one might learn from the practices of those high-performing and rapidly improving countries (e.g., Duncan, 2011). For example, a special report was prepared by OECD, at the request of U.S. education secretary (Arne Duncan at that time), to find out what the U.S. could learn from high-performing countries based on PISA studies (OECD, 2011). The report provides a qualitative analysis of selected high-performing and rapidly-improving education systems to uncover contextual influences on educational performance. Specifically, five high-performing education systems were included in the report: Canada (Ontario), China (Hong Kong and Shanghai), Finland, Japan, and Singapore; plus, two rapidly improving systems: Brazil and Germany. The analyses revealed diverse pictures about what had likely contributed to the success of these different systems. For example, Finland is characterized as having "slow and steady reform for consistently high results", Japan as "a story of sustained excellence", Singapore as having "rapid improvement followed by strong performance", whereas Germany as a case of "once weak international standing prompts strong nationwide reforms for rapid improvement" (OECD, 2011). The results should not be surprising as schooling is a social-cultural activity, which is influenced and shaped by many different aspects that are often unique to an educational system's history and context.

Revealing a diverse picture of various high-achieving systems was only part of these efforts. The OECD's report also presented results, drawn and summarized from examining the successful policies and practices of these selected high-performing systems, as some broader lessons for the United States (OECD, 2011). A list of 12 broad lessons was provided including, "Developing a commitment to education and a conviction that all students can achieve at high levels", "Establishing ambitious, focused and coherent education standards that are shared across the system and aligned with high-stakes gateways and instructional systems", and "Investing resources where they can make the most difference". These lessons are so broad that they don't provide any specific suggestions or steps for implementation. Just as OECD has indicated in the report, "the intent of this volume is not to specify a formula for success. This volume does not contain policy prescriptions. Rather the objective is to describe the experience of countries whose education systems have proven exceptionally successful to help identify policy options for consideration. It is intended as a resource for decision making." (p. 22)

One may argue that such large-scale suggestions prevent possible implementation of successful practices and policies at the system level. After identifying those small-scale successful practices and policies, are they likely to achieve success in other countries if implemented? Again, one difficulty is that what works in different successful countries are not the same. For example, Li and Shimizu (2009) put together a special issue on the topic of exemplary mathematics instruction that is valued in each of six selected education systems in East Asia (i.e., Chinese Mainland, Hong Kong, Japan, Singapore, South Korea, and Taiwan). Although these are all high-performing education systems located in East Asia, papers published in this special issue revealed many differences across these education systems. In fact, there has not been a clear agreement about what can be counted as exemplary mathematics instruction internationally or across these six education systems in particular. Thus, eight papers in this special issue were structured into three clusters based on the ways used by authors to identify high-quality mathematics instruction in their own education systems: (1) high-quality classroom instruction was identified through public evaluation and/or teachers' joint development efforts, (2) high-quality classroom instruction was identified as taught by mathematics teachers for their locally-defined "teaching competence", (3) high-quality classroom instruction was identified as in line with recommended instructional practices. Given these different definitions, it is not surprising that a variety of practices were reported by researchers as valued exemplary mathematics instruction and a variety of approaches were used for teaching improvement in these six different education systems.

Likewise, Leung and Li (2010) published a book that presents and shares changes and issues in mathematics curriculum and teacher education in each of these same six high-achieving education systems in East Asia. For example, these systems shared a general trend in upgrading mathematics teacher education programs, requiring the completion of a 4-year B.A. or B.Sc. program (and even an all-graduate professional degree) for elementary and secondary teachers. However, an examination of their curricula revealed a diverse picture of what pre-service teachers were required to learn within (e.g., China, see Li, Zhao, et al., 2008) or across education systems (Li, Ma, & Pang, 2008). In the concluding chapter, Li & Leung (2010) indicated that "we now know that different practices in curriculum or teacher education can all possibly work well to contribute to students' high achievement in different system contexts. Effective educational practices are not necessarily universal and can also change from time to time even within the same education system." (p. 242).

Learning about what works in one country differs from implementing it for similar success in another country

Identifying and learning what works in a successful country is one important contribution of international comparative studies. However, this should not be confused with the decision to simply adopt and use what works in one country in another education system. Here I question such a direct adoption from two aspects: (1) what works in one country can be viewed differently when examined from different perspectives, (2) what works in one country is often constrained by that country's specific context, which can be very different from the context in another country, thus making the feasibility of a direct adoption questionable.

For example, Japan is commonly viewed as a successful country with students' performance in mathematics often ranked at the top in international comparative studies (e.g., OECD, 2011). In examining what works in Japan, researchers have discovered different factors including, teaching practices in classrooms and lesson study for teachers' professional development (e.g., Stigler & Hiebert, 1999). At the same time, some claimed that exam preparation schools played a big role in pushing students in Japan to high achievement (e.g., Manzo, 2002). The existence of such diverse factors makes it very difficult to single out those practices that can work without other additional factors. If focusing on classroom teaching practice alone, educators and researchers in different countries commonly agree that improving teaching practice is very important for students' performance improvement. However, students in Japan, Taiwan and the U.S. perceived differently what counts as the most important influence on their mathematics achievement (see Stigler, Thompson, & Ji, 2013). The majority of the U.S. students believed that having a good teacher is the most important factor, in contrast to students in Japan and Taiwan who overwhelmingly responded that "studying hard" is the most important factor. Such dramatic cross-cultural differences in students' beliefs has great implications for what teachers can and shall do in their teaching across these systems and how changes in teaching practices might impact student achievement. Similar argument can be made for the case of acquiring and improving teachers' knowledge, where some approaches and practices can work well but they are unique to a specific system and cultural context such as China (Li & Huang, 2018).

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It should be pointed out that refuting the proposition is, by no means, to devalue international comparative studies. In fact, I highly value international comparative studies, and believe they can allow us to learn something that are otherwise not possible to know. At the same time, simply assuming that we can identify what works in successful countries, implement those practices in another country, and then expect success in another country is naïve and is not an appropriate use of what international comparative studies can and shall do. In light of what I have discussed above, we may consider an alternative proposition: "What practices are proven to be effective in improving students' mathematics performance in one country can be taken as options for consideration in other countries."

References

Duncan, A. (2011). *Lesson from high-achieving countries*. Remarks at National Center on Education and the Economy National Symposium. Retrieved 22 April 2019 from *https://www.ed.gov/news/speeches/lessons-high-performing-countries*

Garcia, E. & Weiss, E. (2017). *Education inequalities at the school starting gate*. Economic Policy Institute report, Washington, DC.

Leung, F. K. S. & Li, Y. (Eds.). (2010). *Reforms and issues in school mathematics in East Asia – Sharing and understanding mathematics education policies and practices*. Rotterdam, the Netherlands: Sense Publishers.

Li, Y., & Huang, R. (Eds.) (2018). *How Chinese acquire and improve mathematics knowledge for teaching*. Leiden, The Netherlands: Brill, Sense Publishers.

Li, Y. & Leung, F. K. S. (2010). Practices and changes in mathematics curriculum and teacher education in selected education systems in East Asia: What can we learn? In F. K. S. Leung, & Y. Li, (Eds.), *Reforms and issues in school mathematics in East Asia – Sharing and understanding mathematics education policies and practices* (pp. 233-244). Rotterdam, the Netherlands: Sense Publishers.

Li, Y., Ma, Y., & Pang, J. (2008). Mathematical preparation of prospective elementary teachers. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education: Knowledge and beliefs in mathematics teaching and teaching development* (pp. 37-62). Rotterdam, The Netherlands: Sense.

Li, Y., & Shimizu, Y. (2009). Exemplary mathematics instruction and its development in selected education systems in East Asia. *ZDM-The International Journal on Mathematics Education, 41*, 257-262.

Li, Y., Zhao, D., Huang, R., & Ma, Y. (2008). Mathematical preparation of elementary teachers in China: Changes and issues. *Journal of Mathematics Teacher Education, 11*, 417–430.

Manzo, K. K. (2002). *Japanese schoolchildren 'cram' to boost achievement*. Retrieved 22 April, 2019 from https://www.edweek.org/ew/articles/2002/08/07/43japan.h21.html

OECD. (2011). *Lessons from PISA for the United States*, Strong Performers and Successful Reformers in Education, OECD Publishing.

Stigler, J., & Hiebert, J. (1999). *The teaching gap - Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Stigler, J., Thompson, B., & Ji, X. (2013). This book speaks to us. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 223-231). New York: Routledge.

Tucker, M. (2016). *Asian countries take the U.S. to school*. Retrieved 22 April 2019 from https://www.theatlantic.com/education/archive/2016/02/us-asia-education-differences/471564/

WHAT IS PROVEN TO WORK IN SUCCESSFUL COUNTRIES SHOULD BE IMPLEMENTED IN OTHER COUNTRIES: THE CASE OF MALAWI AND ZAMBIA

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The PME 43 (2019) plenary panel debate focuses on whether "what is proven to work (according to international comparative studies) in successful countries should be implemented in other countries." I am part of the affirmative team and argue in favour of the proposition. I use two examples of Malawi and Zambia, the two countries that fall at the bottom of SACMEQ assessments. Both countries have implemented interventions proven to work in Japan and Kenya. The interventions started as projects in some schools and then later rolled out to other schools. Evaluations of the interventions suggest that they are being successful.

INTRODUCTION AND BACKGROUND TO MALAWI AND ZAMBIA

Malawi and Zambia are two neighbouring land locked countries with many similarities including languages and cultures as well as concerns about low achievements in mathematics at all school levels. There are also differences in particular in terms of economic development; Zambia is considered a low-middle income country, while Malawi is low income, and Malawi is far more densely populated than Zambia (the World Bank, 2019a, 2019b).

Both Malawi and Zambia are members of the Southern and Eastern Africa Consortium for Monitoring Education Quality (SACMEQ) and have participated in assessments in mathematics.. Below is a summary of the performance of participating countries and the rankings from top to bottom in three SACMEQ assessments.

SACMEQ II (2000)			**SACMEQ III (2007)**			**SACMEQ IV (2013)**	
Country	**Score**		**Country**	**Score**		**Country**	**Score**
Mauritius	585		Mauritius	623		Mauritius	694
Kenya	**563**		**Kenya**	**557**		**Kenya**	**651**
Seychelles	554		Tanzania	553		Seychelles	630
Mozambique	530		Seychelles	551		Swaziland	601
Tanzania	522		Swaziland	541		Botswana	598
Swaziland	517		Botswana	521		South Africa	587
Botswana	513		Zimbabwe	520		Uganda	580

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 73-78). Pretoria, South Africa: PME.

Uganda	506		South Africa	495		Zimbabwe	566
South Africa	486		Zanzibar	486		Lesotho	559
Zanzibar	478		Mozambique	484		Mozambique	558
Lesotho	447		Uganda	482		Namibia	558
Zambia	**435**		Lesotho	477		**Malawi**	**522**
Malawi	**433**		Namibia	471		**Zambia**	**522**
Namibia	431		**Malawi**	**447**			
			Zambia	**435**			
SACMEQ average	500		SACMEQ average	507		SACMEQ average	584

Table 1: Mathematics achievement of grade 6 learners in SACMEQ countries (SACMEQ (2006, 2010, 2014)

As can be seen from the table, Kenya was the second best performing in all the three assessments and thus a successful country according to the definition this panel debate is taking. Malawi and Zambia were among the three lowest performing countries in 2000, and the two lowest performing countries in 2007 and 2013. The low achievement is also observed in post primary schools as evidenced by national examination results of the two countries. This has been a cause for concern and therefore the two governments have embraced interventions that aim at improving the teaching and learning of mathematics in schools. Malawi has adopted from Kenya the Strengthening Mathematics and Science in Secondary Education (SMASSE) in secondary schools while Zambia has adopted Lesson Study from Japan in primary and secondary schools. I discuss each case below.

THE CASE OF SMASSE IN MALAWI

SMASSE started in Kenya in 1998 as a project of the Kenya Ministry of Education in collaboration with, and with funding from, the Japanese International Corporation Agency (JICA). This was an attempt to improve performance in mathematics and science in schools. The project started with a baseline study which explored the causes of poor student performance in mathematics and science. Findings from the baseline study documented that teaching methods were teacher centred and not effective for student learning, many teachers lacked mastery of content, and both students and teachers held negative attitudes towards mathematics and science (Kenya Science Teachers College, 2002). To address these issues, the SMASSE project developed an in-service curriculum to build the capacity and competence of teachers. The curriculum addressed teachers' attitudes, teaching methods, mastery of content, and development of teaching and learning materials (Kenya Science Teachers College, 2002). In this paper, I will focus on teaching methodology because this was a problem solving approach adapted from Japan. The project recommended what they called ASEI principles for teaching: Activity (to incorporate activities that actively engage learners in their learning), Student-centred (shifting from teacher centred to learner centred teaching), Experiment (shifting from demonstrations to investigations by students) and Improvisation (improvising resources for teaching). To achieve the ASEI principles, the project suggested a reflective approach to teaching which they called the Plan, Do, See, Improve (PDSI) approach. The teaching approach was referred to as ASEI/PDSI approach (Nui & Wahome, 2008).

The project was piloted in 9 districts in Kenya in 1999. The pilot study reported success in terms of teachers' shift from teacher centred to student-centred approaches and students' achievement at secondary school level. In 2000, the project was expanded to 15 districts and continued to register successes, then finally in 2004 it was rolled out to the rest of Kenya. In 2008, the programme was extended to primary schools. Consequently, the name of programme was changed to Strengthening of Mathematics and Science Education (SMASE) to accommodate the primary school activities (CEMASTEA, 2019).

SMASSE Malawi

Following the successes reported by SMASSE Kenya, other countries in sub-Sahara Africa adapted SMASSE with the support of JICA. Countries included Ghana, Malawi, Rwanda, Swaziland, Tanzania, Uganda, Zambia and Zimbabwe. These were referred to as SMASSE Western, Eastern, Central and Southern Africa (WECSA) and most of the countries registered successes in the pilot phase (Kenya Science Teachers College, 2010). I will focus on Malawi only.

SMASSE Malawi started in 2004 as a pilot in one of six education divisions in the country. According to the Ministry of Education, Malawi adopted SMASSE because the problems facing Malawi were similar to those in Kenya before SMASSE, hence the solutions suggested by SMASSE Kenya were promising (Domasi College of Education, 2003). SMASSE Malawi was therefore a transfer of the ASEI/PDSI approach to teaching mathematics and science from Kenya to Malawi. Similar to Kenya, mathematics teachers were introduced to the approach through in-service trainings which were offered to all mathematics and science teachers in the pilot education division. Malawi reported success in the pilot phase and therefore decided to roll out the in-service trainings to all mathematics and science teachers throughout the country (Department of Teacher Education, 2009). The trainings were rolled out from 2009 and to date all mathematics and science teachers in secondary schools in Malawi are required to attend the SMASSE trainings at least once a year. Teachers also get supervised periodically by the trainers to see and evaluate how they are using the ASEI/PDSI approach. Successes reported so far include a shift from teacher centred to learner centred teaching, increase in student motivation, and improved student achievement (Department of Teacher Education, 2016). This is an instance of an intervention that worked in a more successful country being transferred to a less successful country, where it seems to be working.

THE CASE OF LESSON STUDY IN ZAMBIA

Lesson study is a relatively well known model of teachers' professional development where teachers work together to study their own teaching and their students' learning. In Japan, where lesson study originated, teachers have been practicing it for more than a hundred years and it is part of their teaching profession (Fujii, 2014). The success of lesson study in Japan has motivated implementation in many other countries outside Japan. Lesson study is therefore a good example of what works in successful countries being implemented in other countries. Many implementers of lesson study outside Japan have reported success in terms of benefits (da Ponte, 2017).

Zambia is one of the countries that has been implementing lesson study on a large scale. Lesson study was introduced to Zambia in 2005 by the Ministry of General Education with support from JICA. This was in response to the many challenges that the teachers were facing in schools and the low achievement by students (Ministry of General Education & JICA, 2016). Lesson study was therefore adopted to support the teachers by building their capacity and hence improve their practice. Zambia lesson study process has eight steps; (1) defining problems, (2) planning a lesson, (3) conducting the lesson, (4) reviewing the lesson, (5) planning a lesson again, (6) conducting revised lesson, (7) reviewing the lesson again, and (8) compiling learning (Ministry of General Education & JICA, 2016). Introduction of lesson study started with 213 schools and about two thousand teachers in one province, then was extended to three provinces in 2008, and later in 2015 extended to 3121 schools in 10 provinces where they reached more than forty thousand teachers. The target is to reach to all schools and all teachers by 2023 (Ministry of General Education & JICA, 2016).

The use of lesson study in Zambia is reported to be successful in terms of improving teachers' knowledge and skills, shifting from teacher centred to learner centred methods, student achievement, student motivation and positive attitudes. Below is what some implementers said:

> "Because of the practice of lesson study, teachers have come to value working together as an ever-learning community of practitioners, exchanging ideas, classroom experiences, challenges and best practices through interacting and learning from each other continuously."
>
> Head of Department
>
> "Through lesson study, my analytical skills in planning, implementation, observation and monitoring lessons have improved greatly . . . My teacher's planning and lesson delivery skills learnt in lesson study have started bearing fruits. Through learner-centred methodologies, learners are acquiring lifelong knowledge, skills, value and positive attitude towards science." Deputy Headteacher
>
> "Evidence exists to showcase the gains from Lesson Study. Teachers are collaborating more through collaborative lesson planning and are using learner centred pedagogies. The

Lesson Study has continued to build teachers' analytical skills with a definite shift from a focus on superficial aspects of lessons to productive teaching and learning."

Assistant Director - Teacher Education, Ministry of General Education

(Ministry of General Education & JICA, 2016, page 8).

CLOSING THOUGHTS

The two cases summarised above demonstrate that the transfer of an intervention from successful countries to other less successful countries works. There is a case here in favour of the proposition. It is not necessary to 'reinvent the wheel' when there is an intervention that is already proven to work that can fit the conditions and needs of the other countries needing the intervention. In addition, designing quality and effective interventions requires time, expertise and resources, all of which might not be available in the less successful countries, especially where the situation of student learning is of great concern and there is urgency to intervene.

References

da Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6* (2), 169–181.

Department of Teacher Education. (2009). *A Report on the Fourth SEED SMASSE INSET.* Lilongwe: Ministry of Education Science and Technology.

Department of Teacher Education. (2016). *Final monitoring and evaluation report for 2016.* Lilongwe: Ministry of Education Science and Technology.

Domasi College of Education. (2003). *The establishment of SMASSE INSET Malawi Pilot Programme-needs assessment survey & the 1st and 2nd stakeholders' meetings*. Lilongwe: Ministry of Education Science and Technology.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development*, *16*(1), 2–18.

Kenya Science Teachers College (2008). *SMASSE WECSA Kenya Report*. Kenya: Ministry of Education.

Kenya Science Teachers College. (2010). *SMASSE-WECSA 10th Anniversary*. Nairobi: Ministry of Education.

Ministry of General Education & JICA. (2016). *Lesson Study in Zambia: for effective teacher professional growth &improvement of student learning*. Lusaka: Republic of Zambia.

Nui, W. N., & Wahome, N. A. (2008). *SMASSE Baseline Studies and Intervention Strategies*. Retrieved from http://www.criced.tsukuba.ac.jp/math/sympo_2006/nui.pdf

SACMEQ. (2006). *SACMEQ II project results: Pupil achievement levels in reading and mathematics.* www.sacmeq.org (http://www.sacmeq.org/). Retrieved on 29 March 2019.

SACMEQ (2010). *SACMEQ III project results: Pupil achievement levels in reading and mathematics*. www.sacmeq.org (http://www.sacmeq.org/). Retrieved on 29 March 2019.

SACMEQ (2016). *SACMEQ IV project results: Pupil achievement levels in reading and mathematics*. www.sacmeq.org (http://www.sacmeq.org/). Retrieved on 29 March 2019.

The World Bank. (2019a). *The World Bank in Malawi*. Retrieved 2 May 2019. https://www.worldbank.org/en/country/malawi/overview

The World Bank. (2019b). *The World Bank in Zambia*. Retrieved 2 May 2019. https://www.worldbank.org/en/country/zambia/overview

ASK AGAIN, "WHY SHOULD WE IMPLEMENT WHAT WORKS IN SUCCESSFUL COUNTRIES?"

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This paper argues against the claim "What is proven to work (according to international comparative studies) in successful countries should be implemented in other countries." In support of my refutation of this claim, I highlight two key aspects to be considered before and when implementing any successful educational interventions in other countries, grounded in cultural theories and international research.

Introduction

International comparison studies such as the Programme for International Student Assessment (PISA; OECD, 2011) or the Trends in International Mathematics and Science Study (TIMSS, 1999) report that student achievement in mathematics is higher in some countries than in other countries. Cross-national studies such as TIMSS report that the countries with higher mathematics achievements also have a higher quality of teaching (Stigler & Hiebert, 1999; Stigler & Stevenson, 1991). These studies also report that the countries with higher achievement in mathematics have a nationally-coherent curriculum, often one that is nationally-controlled (e.g., Schmidt, McKnight, Houang, Wang, Wiley, Cogan & Wolfe, 2001). Researchers and policymakers have studied the methods and mechanisms that help students achieve highly in those countries. If the proposition "What is proven to work (according to international comparative studies) in successful countries should be implemented in other countries" is to be accepted, the countries which show low achievement in mathematics performance can simply implement the intervention(s) that work in successful countries. However, as we all know, magic does not happen that simply. In this paper, I highlight two key aspects to be considered before implementing interventions proven to work in "successful" countries in secondary countries.

Ask the Core Question, "Why?"

When implementing "proven" new educational interventions, the question that we must ask first is "*Why* do we want to implement a successful intervention in this country?" The proposition "What is proven to work in successful countries should be implemented in other countries" does not provide any context about the values and vision behind the original intervention, which is the most critical aspect to consider before arguing "*Should* we?" Why do we want to—or *should* we even want to—implement an intervention? What do we know about the mechanism and cultural context that made the intervention successful, and is it something that could be replicated in this country? Because these kinds of questions should be considered first,

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but the proposition leaves no gray area to contextualize or contemplate them, I argue that the proposition should be refuted.

If a successful educational intervention in one country is valuable enough to implement in another country, then I also argue that we should consider whether the values and visions of the first country are shared with most people in the second country. Many countries attempt to learn from and implement other countries' "successes," especially well-known "successes" in educational achievement or in educational reform reported in international comparison studies. However, such transfers are not always successful. According to Fullan (2007, 2016), one of the critical aspects of educational reform, or (as in the case of this paper) implementing new educational interventions from other countries, is to lead a change in "culture." As my colleague Judit Moschkovich cited in her introduction, culture is the "organized and common practices of particular communities in which people live (Rogoff, 1990)." That is, culture is an organized system where members of the particular communities share common values and beliefs as well as common practices. I argue that we should ask the question, "Do most members of the particular community share the values and visions of the educational intervention?" when implementing new interventions in a community or a country.

Implementing new interventions in a community requires change, and change in culture is neither as simple nor easy as just saying, "We should implement what works in successful countries." Leading cultural change requires understanding shared values and visions among members of a community as a priority (Fullan, 1993; Senge, 1992). If the members of a community do not share any of the values and visions that made the intervention successful originally, then the intervention might not be successful or meaningful in the second community.

Dynamics and Complexities in a Culture

International comparative studies on academic performance such as TIMSS (1999) reveal that the average mathematics performance of eighth graders is high in some countries in East Asia (e.g., Singapore, South Korea, Hong Kong, and Japan), in Europe (e.g., Belgium, Finland and Netherlands), and Australia. Based on the results of these studies, many researchers have also investigated the factors that may influence these performances using cross-cultural studies. Curriculum (e.g., Schmidt et al., 2001), teacher preparation and professional development (e.g., Akiba, LeTendre & Scribner, 2007; Darling-Hammond & Cobb, 1995; Wang, Coleman, Coley & Phelps, 2003), instructional practices (e.g., Stigler & Hiebert, 1999; Stigler & Stevenson, 1991) are some examples of influential factors. Although similar patterns can be found across countries, these studies show that different countries have different systems and complexities in their cultures. For example, East Asian countries have national curricula that are more coherent and contain focused sets of mathematical contents (Schmidt et al., 2001), but Australia is different, with greater decentralization in its curricula than in East Asian countries. In teacher hiring practices, local schools are responsible for hiring teachers in Australia, Hong Kong, and the Netherlands, while

Japan, Singapore, and Korea have more centralized teacher hiring systems (Wang et al., 2003). This goes to show that what works in different countries can vary, and that each country has its own complex and dynamic system for effective practices.

To date, there has not been a model that clearly explains what would be universal success in any country. Instead, various approaches tailored to different regions have been studied. For example, the Learners' Perspective Study (LPS; Clarke, Keitel & Shimizu, 2006) describes differences in teaching practices in China, Japan, and South Korea. They attribute these differences to the diverse cultures and traditions of those countries. In another example, Son, Watanabe, and Lo (2017) provide insights into what makes students' performances better in several East Asian countries such as China, Hong Kong, Singapore, South Korea, and Japan, as well as in Australia. The authors point out the different approaches to curriculum, teacher education and professional development, and politics and economic systems across East Asian countries. Conducting international comparative studies in different countries that have dynamic and complex systems is challenging, and the authors cite Torsten Husén's 1983 notion, "comparing the incomparable" (p. 455).

Closing Thoughts

Considering shared values and cultural practices are key aspects to implementing new educational interventions in another setting. It is not quite as simple as saying that "what is proven to work in successful countries should be implemented in other countries." International studies have also pointed out the complexity and cultural dynamics at play. Prior to adopting the proposition, I would consider necessary questions to be "Why should we consider to implementing what works in successful countries?" and "What specific things do we want to learn from the international studies?"

References

Akiba, M., LeTendre, G. K., & Scribner, J. P. (2007). Teacher quality, opportunity gap, and national achievement in 46 countries. *Educational Researcher, 36*(7), 369-387.

Clarke, D. J., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics classrooms in twelve countries: The insider's perspective.* Rotterdam: Sense Publishers.

Darling-Hammond, L., & Cobb, V. L. (1995*) Teacher preparation and professional development in APEC Members: A comparative study.* Washington, DC: U.S. Department of Education.

Fullan, M. (1993). *Change forces: Probing the depth of educational reform*. London: Routledge.

Husen, T. (1983). Are standards in US schools really lagging behind those in other countries? *The Phi Delta Kappan*, *64*(7), 455-461.

OECD (2011). *Lessons from PISA for the United States*, Strong Performers and Successful Reformers in Education, OECD Publishing.

Rogoff, B. (1990). *Apprenticeship in thinking*. NY: Oxford University Press.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., & Wolfe, R. G. (2001). *Why Schools Matter: A Cross-National Comparison of Curriculum and Learning. The Jossey-Bass Education Series*. San Francisco, CA: Jossey-Bass.

Senge, P. (1992) *The Fifth Discipline*. New York, Doubleday.

Son, J. W., Watanabe, T., & Lo, J. J. (2017). *What Matters? Research Trends in International Comparative Studies in Mathematics Education*. Springer, Cham. Switzerland.

Stigler, J., & Hiebert, J. (1999). *The teaching gap - Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Stigler, J. W., & Stevenson, H. W. (1991). How Asian teachers polish each lesson to perfection. *American Educator*, (Spring), 15(1), 43-47.

Wang, A. H., Coleman, A. B., Coley, R. J., & Phelps, R. P. (2003). *Preparing Teachers around the World*. Policy Information Report.

RESEARCH FORUMS

RESEARCHING MATHEMATICS EDUCATION AND LANGUAGE DIVERSITY: THE THEORIES WE USE AND DEVELOP, AND WHAT THEY ENABLE US TO SEE, SAY OR DO

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INTRODUCTION TO THE RESEARCH FORUM: THEORIES AND THEORETICAL CHALLENGES IN MATHEMATICS EDUCATION RESEARCH ON LANGUAGE DIVERSITY

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This Research Forum focuses on and discusses a range of theories and approaches to theory used in research on mathematics education and language diversity. This topic is particularly appropriate for a PME conference held in South Africa, a country with 11 official languages, and a leader on research in this field for several decades. A Research Forum on this topic, ‘Researching mathematics education in multilingual contexts: Theory, methodology and the teaching of mathematics’, was last held in 2004 during PME28 in Bergen, Norway, under the coordination of Richard Barwell and Philip Clarkson, in which Judit Moschkovich and Mamokgethi Phakeng (Setati) also participated in the roles of contributor and discussant. The coordinators and whole group of contributors at that time were concerned with the significance of theory in research into multilingualism within mathematics education and its interdisciplinary nature. In the introduction and discussion of theories of language mostly close to applied linguistics but also linguistic anthropology and other fields, Barwell and Clarkson (2004, pp. 227-228) specifically asked, “What theories are relevant to work in mathematics education? How might these theories be applied in mathematics education? What are the challenges which arise from working with theories from other disciplines?” One of the challenges exemplified was the visibility of mathematics in linguistic analyses of mathematics classroom interaction, and the related more general challenge of working with both language and mathematics while keeping them both in view in mathematics education and mathematics education research.

Investigation on this topic has received much attention and grown since then, with manuscripts, edited volumes and new terrains and initiatives of study oriented to interrogating and addressing the mesh of affordances, limitations and objectives of

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different theories to move research and practice forward. In the debates on which language(s) to use and why, how and when to use them in, for example, multilingual mathematics classrooms (Bose & Phakeng, 2017), research has interpreted and adopted broadly sociocultural and discursive perspectives. Nonetheless, recent reviews of research on mathematics education, language and language diversity (Barwell et al., 2016; Planas, Morgan & Schütte, 2018; Planas & Schütte, 2018; Radford & Barwell, 2016; and others) show the complexity of perspectives and theoretical grounds contributing to the domain. We thus continue to struggle with the major endeavour and need of articulating and developing theory. In this endeavour, a careful discussion of the tapestry of theories and epistemologies can help to uncover traditions and frameworks, both outside and inside mathematics education, with a possible goal to more clearly delimit the understanding, scope and range of the area of study and of some of its substantial transformations and distinctive changes.

In this Research Forum, we pose to ourselves, to the community of our common area of study and to the broader PME audience and readers the following guiding questions:

- What theories have been used and developed in research on mathematics education and language diversity and why?
- What have the theories made visible/invisible and what have they enabled us to say and/or to do as researchers and/or practitioners?

Four decades since the inception of the PME conferences, the world is becoming more multilingual and increasing awareness of the importance of the languages of all groups and communities. It is timely to examine what theories have been used or whose development has emerged to explore the complexities of language diversity in mathematics classrooms. PME43 is an opportune time for us as researchers to reflect on and interrogate the theories that inform our mathematics education research in settings of language diversity. Systemic functional linguistics, anthropological linguistics and sociolinguistics, sociocultural and discourse theories, social and cultural semiotics, structural and applied linguistics and more increasingly often combinations of them, to name a few, show that a single theory cannot fully explain the progress of this area of study. Considering this, it is crucial to create opportunities for conversation among researchers using existing theories, and for the generation of theoretical revision, refinement and articulation. While the contributors to this Forum have developed well-recognized expertise using specific theories, their reflections serve to frame and open a discussion of the ways in which theories may influence one another in order to deepen and extend insights. As the named theories largely stem from linguistics, the specific need emerges to develop theories further for grasping the relation between mathematics learning and language. In spite of the maturity of some networking work in the broader field (Bikner-Ahsbahs & Prediger, 2014), our knowledge of the research community of mathematics education and language diversity suggests the need for further theoretical elaboration and discussion of relationships between theories in order to proceed from imported theories to also include mathematics-specific and topic-specific contributions to theorizing.

Theory is a lens through which one views and understands events, behaviour and/or situations in a systematic way. Theory guides how we think about a situation and influences what we see and what we do not see when we analyse data or take instructional design decisions. As researchers in the area of mathematics education and language diversity, we know this very well and do not expect that further theoretical elaboration guides us to some uniquely correct theory. The theories we use determine the limits or boundaries of the distinct ways we talk about language and language issues, the basic questions we ask, the data we collect and the conclusions we draw from that data. These theories are sometimes difficult to compare because they use different constructs and methods to address different research problems. Importantly, theoretical frameworks guide practical judgement and steer new developments. They allow us as researchers to make links between the abstract and the concrete; the theoretical, the analytical and the empirical; thought and observational statements, etc. Being explicit about the theories we choose to use (and describing why we choose them over others) can assist those who might use our research to maintain a critical awareness of how the research is used, of the illusion of reaching a neutral, definitive observation language, and of the inexistence of a theoretical vacuum for the claims we make. These are central reasons for the significance of being explicit about our theories in mathematics education research on language diversity and being able to explain our choices and positions.

After this introduction, we compile the four concrete contributions and finish with some synthesising concluding remarks. As our colleagues explain and emphasize, the study, interpretation and building of theoretical lenses is a pervasive challenge in their empirical research in mathematics education and language diversity. Personal reviews of our own research over the years in the area are an excellent source of empirical evidence of how and why we choose some specific theories and conceptualisations, and what they allow us to see, say and do. This Research Forum is an opportunity to revisit these questions in terms of how they underpin our research but importantly in terms of how they collectively underpin research in the area. Thus, the process of addressing the two questions mentioned above begins with providing responses on the basis of personal research agendas and from here moves to the wider discussion of the theoretical challenges and insights that we share as part of the community of mathematics education researchers of issues of language and language diversity.

For this, we adopt and summarise four perspectives:

- Heteroglossia, stratification and mathematics (*Richard Barwell)*
- An ecological approach to academic literacy in mathematics in linguistically diverse settings (*Judit Moschkovich)*
- Ethnography and critical theory lens to look into language diversity as funds of knowledge in mathematics classrooms (*Arindam Bose*)

- Theoretical frameworks in topic-specific design research on supporting language learners (*Susanne Prediger)*

HETEROGLOSSIA, STRATIFICATION AND MATHEMATICS

Richard Barwell, University of Ottawa

Tensions I

Throughout my career, I have adopted a broadly sociocultural, discursive perspective on the role of language in the learning and teaching of mathematics. I have generally focused on the way language is used in mathematics classrooms, particularly by students. This broad perspective includes some widely used assumptions, such as that language plays an important mediating role in the learning of mathematics, that language is a social phenomenon and therefore reflects collective ways of talking and thinking, and that language also reflects social and institutional structures, and so privileges or marginalises some participants. Learning mathematics is about learning to use the language of mathematics appropriately, where "appropriately" is situated, collective and in constant evolution.

In my initial work on language diversity in mathematics classrooms, including my collaborations with various colleagues (e.g. Barwell, 2009), it became clear to me that language diversity is itself diverse. I noticed that there was an increasing amount of research on language diversity in mathematics classrooms, that this work was conducted in a variety of sociolinguistic contexts, and that these contexts were in many ways very different. For example, the use of more than one language occurred in some situations but not in others. Despite this diversity, I began to see some common patterns. In many contexts, there was some kind of tension between the presence of multiple languages and the possibility of their use in mathematics classrooms. Several studies have reported situations in which only one language was supposed to be used, when in fact students or teachers or both made use of more than one language. Meanwhile in my own research in the UK, while there was increasing recognition that students' proficiency in their home languages could be important for their schooling in English, I observed little opportunity for students to use languages other than English in mathematics. Nevertheless, I did observe occasions in which students' home languages appeared to influence their use of English in mathematics (Barwell, 2005). I thus identified several tensions including: between home and school languages, between language policy and mathematics classroom practice, and between language for learning mathematics and language for getting on the world (Barwell, 2009).

Tensions II

In subsequent work, my thinking has been strongly informed by Bakhtin's (1981, 1986) theoretical ideas about the dialogic nature of language. Why? Because Bakhtin's ideas explain *why* the tensions I had started to notice arise and relate these tensions to a range of other important phenomena that need to be understood, including the stratified and stratifying nature of language, and the role of the other in learning mathematics. From this perspective, the phenomenon of language use is organised around the kinds of tensions I had noticed. There is a deeper underlying tension between standardising or unifying on the one hand, and diversifying or heteroglossia on the other (Bakhtin, 1981). Languages depend on the idea that there are standard forms of grammar, vocabulary, pronunciation, genre, discourse and so on. This idea is common in mathematics: we speak of 'formal mathematical discourse', as though there is a clearly defined way of using language in mathematics. But while there are certainly forms of language use that are recognisably mathematical (e.g., "by treating rational numbers as ordered pairs, we can show that they are isomorphic to the natural numbers"), we must also recognise that mathematical language shows considerable variation (between sub-domains, in different contexts, for different audiences, and so on). Indeed, language only happens through a constant process of using language in new and original ways (including in adapting previous forms of language to develop new mathematics). This heteroglossia is apparent in any mathematics classroom, in the wide variety of ways in which students and teachers in fact use language(s) to do mathematics. The presence of standardisation and heteroglossia and the constant tension between them is an inherent feature of language. Moreover, this tension shapes every utterance (Bakhtin, 1981). Heteroglossia can be understood in terms of three overlapping dimensions: languages, discourses and voices (Busch, 2014). In my research, for example, I have described instances in which a second language learner discusses polygons with his teacher showing how the discussion is shaped by the student's Spanish-inflected French (languages), non-standard mathematical expressions (discourses), and appropriation of the teacher's words (voices) (all illustrative of heteroglossia), as well as by the teacher's use of a more standard form of French, more formal mathematical discourse and her re-voicing of the student's words (Barwell, 2016).

These ideas have made possible a more complete picture of language use and language diversity in mathematics classrooms. This perspective involves four principles: language is agentive; meaning is relational; language is diverse; and language is stratified and stratifying (Barwell, 2018). From this perspective, language itself makes a difference: participants respond to words, rather than to ideas or thoughts. Mathematical meaning arises from the relations between forms of language and is shaped by the inherent tensions in language. There is, for example, always a tension around multiple languages (there is scarcely anywhere in the world now in which multiple languages are not present). It is not sufficient only to pay attention to the use of multiple languages when it occurs, nor to focus research in contexts in which multiple languages are routinely used. In many contexts, multiple languages are present, a potential source of meaning in the mathematics classroom, but the prevailing combination of institutional and political forces means that they are not used. In a similar way, the interaction between languages, discourses and voices is highlighted as an important feature of mathematical meaning making. This perspective, applied in comparative ethnographic research, has highlighted a distinction between mathematics classrooms in which language and its different forms is explicitly discussed and recognised ('language positive mathematics classrooms'), and classrooms in which language is largely treated in implicit ways ('language neutral mathematics classrooms'). The former seem to offer greater scope for supporting students to appropriate the contentions of mathematical discourse in the contexts in which they find themselves, while still recognising the diversity of their own languages, discourses and voices.

Superdiversity

In the context of the increasing diversity of diversity, referred to as *superdiversity* (Vertovic, 2007), in mathematics classrooms, most recently I have begun to work with theoretical ideas from the contemporary sociolinguistics of multilingualism (e.g. Blommaert, 2010). These ideas extend the preceding perspective of heteroglossia to theorise the stratified and stratifying nature of language. All instances of language use rely on what linguists call indexicality: for example, the use of certain words or ways of talking indexes mathematical activity. The ordered nature of indexicality is part of how we interpret beyond the meanings of individual words to understand, for instance, which of several adults in a classroom is the teacher. This indexicality is, however, also stratified, so that some forms or patterns of language use are perceived as more or less valuable, and are thus linked to issues of authority, control and marginalisation (Blommaert, 2010). These ideas explain why, for example, English is preferred as a language of instruction in many post-colonial contexts, even though this choice may disadvantage many students (e.g. Setati, 2008): English indexes social advancement, power and success. They also explain why some forms of mathematical discourse are privileged over others, and why some voices predominate in mathematics classrooms. Language diversity in mathematics classrooms is socially, institutionally and politically stratified. Contemporary sociolinguistic theory can help to untangle how this stratification shapes students' learning in mathematics classrooms.

Superdiversity is a way of recognising the great complexity of multilingual mathematics classrooms, wherever they are and whatever their sociolinguistic profile. In such classrooms, students are learning language and learning mathematics. These processes interact and are shaped by the tension between uniformity and heteroglossia, a tension that leads to linguistic and social stratification. Learning mathematics and opportunities to learn mathematics are influenced by these linguistic processes. A key insight in Bakhtin's work is that these processes are a fundamental feature of language. The question, therefore, is not how to solve the problem of multilingualism (why does no-one ever raise the problem of monolingualism), but how to work with the tensions in productive ways. My recent research suggests that the fostering of language positive classrooms, in which teachers and learners pay explicit attention to different features of language and its use may be a productive way forward.

AN ECOLOGICAL APPROACH TO ACADEMIC LITERACY IN MATHEMATICS IN LINGUISTICALLY DIVERSE SETTINGS

Judit Moschkovich, University of California Santa Cruz

Ecological approach

As a researcher in mathematics education, I specifically bring the lenses of the learning sciences and the field of mathematics education to my research. My particular focus is on mathematical thinking, learning and communicating in monolingual and bilingual settings, so I have had to read across several sets of research literature such as the work of Gee. While I remained grounded in my own field, I had to learn how to use theoretical perspectives from fields in which I had little training, such as linguistics, bilingualism and second language acquisition. My experiences of learning a second language in elementary school and later becoming an immigrant as an adolescent (and learning to live in both bilingual and monolingual modes) sparked my curiosity about bilingualism and second language acquisition. My commitment to improving the education of learners from non-dominant groups provided my motivation and sustains my dedication to tackling issues of language diversity in mathematics education. This is the more general context of my current work within an ecological approach.

Researchers in education have recently called for ecological approaches that integrate a dynamic view of cultural practices into the study of learning and development and document the resources in everyday thinking (Gutiérrez & Rogoff, 2003; Lee, 2008). These ecological approaches are based on the ecological framework proposed by Bronfenbrenner (1989), on seminal studies that examined cross-cultural learning and development and documented the complexity of reasoning in everyday settings (e.g. Lave, 1988; Saxe, 1991), and on studies of learning and development among youth from non-dominant communities (e.g. Lee, 2008; Nasir, 2000; Gutierrez et al, 1999; Gutiérrez & Rogoff, 2003). The study of students' mathematical reasoning practices and language diversity requires such ecological approaches not only because this work is cross-cultural, but also because it involves several interacting levels of analysis. These approaches provide theoretical notions and methods that simultaneously address the cognitive, domain specific, cross-cultural, and linguistic nature of mathematical reasoning in multilingual and multicultural settings.

My research has focused on examining bilingual students' mathematical reasoning practices in middle school classrooms in the United States. Such studies needed to be carefully framed by theoretical notions that simultaneously address the cognitive, domain specific, cross-cultural, and linguistic nature of this work. I use an ecological approach to frame the complex endeavour of examining mathematical reasoning practices in settings with language diversity. This approach has three components. 1) An ethno-mathematical perspective grounds the analysis of multiple and hybrid ways to reason in the domain of mathematics. 2) Educational anthropology and cultural psychology ground the cross-cultural aspects of my work. 3) Research in sociolinguistics, especially approaches to bilingualism and multilingualism, ground linguistic aspects of my work.

I summarize the three components of an ecological approach (Moschkovich, 2011) as using an ethno-mathematical stance, a naturalistic paradigm, and a situated view of language, that serve to simultaneously address the cognitive, domain specific, cross-cultural, and linguistic nature of this work. This ecological approach provides the central assumptions and overall framing to studies with learner populations who use one, two, or more languages in mathematics classrooms. The fundamental assumptions of an ecological approach include that context matters, that routine practices count, and that the cognitive, social, and linguistic dimensions of both individuals and contexts interact in important ways). These three components provide strategies for avoiding deficit models of learners such as providing a full picture of learners' competencies and considering and valuing hybrid practices.

The three components of the ecological approach I use for my research are: a naturalistic paradigm (Moschkovich and Brenner, 2000), a situated view of language and Discourse (Gee, 1996 and 1999; Moschkovich, 2002), and an ethno-mathematical perspective on mathematical activity (D'Ambrosio, 1991). These components contribute several theoretical notions-- including culture, context, and mathematical activity-- and assumptions. These three conceptual frameworks provide an integrated ecological approach for analysing mathematical reasoning practices in the following ways: 1) A naturalistic paradigm considers the *ecological validity of problems, tasks, and questions* used to explore mathematical reasoning. 2) Situated views of language and discourse provide *an ecological approach to language*, in particular to the meaning of utterances, texts, and inscriptions used during mathematical reasoning. 3) Ethno-mathematics provides an *ecological view of mathematical practices* because it assumes that mathematical reasoning practices are multiple, heterogeneous, and connected to other cultural practices.

Academic literacy in mathematics for bilingual and multilingual learners

The second conceptual framework that I use in my work provides an integrated view of academic literacy in mathematics for bilingual/multilingual learners. Although there are many labels used to refer to students who are learning an additional language, I will use the term bilingual/multilingual learners.

The proposed definition of academic literacy in mathematics includes three integrated components: mathematical proficiency, mathematical practices, and mathematical discourse. The framework includes the following assumptions: the three components of academic literacy in mathematics are intertwined, academic literacy in mathematics is situated, and participants engaged in academic literacy in mathematics use hybrid resources. A sociocultural perspective of academic literacy in mathematics provides a complex view of mathematical proficiency as participation in discipline-based practices that involve conceptual understanding and mathematical discourse. The sociocultural perspective of academic literacy in mathematics described here builds on previous work that appeared in several publications where I described a sociocultural view of mathematics learners who are bilingual and/or learning English (Moschkovich, 2002 & 2007a), of mathematical discourse (Moschkovich, 2007a), and of mathematical practices (Moschkovich, 2013). In Moschkovich (2008, 2009) I described how mathematical discourse is situated, involves coordinated utterances and focus of attention, and combines every-day and academic registers. The definition of academic literacy in mathematics used here (Moschkovich, 2015) brings together and builds on different aspects of those analyses. This sociocultural theoretical framework draws on situated perspectives of learning mathematics (Brown, Collins & Duguid, 1989; Greeno, 1998) as a discursive activity (Forman, 1996) that involves participating in a community of practice (Forman, 1996; Lave & Wenger, 1991), developing classroom socio-mathematical norms (Cobb et al., 1993, 2001), and using multiple material, linguistic, and social resources (Greeno, 1998). Mathematical activity thus involves not only mathematical knowledge, but also mathematical practices and discourse.

The view of academic literacy in mathematics presented here is different than previous approaches to academic language in several ways. First, the definition includes cognitive aspects of mathematical activity—i.e. mathematical reasoning, thinking, concepts and metacognition—but also sociocultural aspects—participation in mathematical practices—and discursive aspects—participation in mathematical discourse. Importantly for learners who are bilingual/multilingual, this integrated view, rather than separating academic language from mathematical proficiency or practices, takes the three components as working together. Separating language from mathematical thinking and practices can have direct consequences for this student population. Such a separation can make these students seem more 'deficient' than they might actually be, since they may not express their mathematical ideas through the language of instruction in the classroom, but may still be engaging in correct mathematical thinking and participating in mathematical practices that are less language intensive---for example using objects or drawings to show a result, finding regularity in data, or using gestures to illustrate a mathematical concept.

This sociocultural perspective expands academic literacy in mathematics beyond simplified views of language as words. Simplified views of academic language focus on words, assume that meanings are static and given by definitions, separate language from mathematical knowledge and practices, and limit mathematical discourse to formal language. In contrast, the view of academic literacy in mathematics described here sees meanings for academic mathematical language as socio-culturally situated in mathematical practices and the classroom setting. A complex view of mathematical discourse also means that mathematical discourse draws on hybrid resources and involves not only oral and written text, but also multiple modes, representations (gestures, objects, drawings, tables, graphs, symbols, etc.), and registers (school mathematical language, home languages and the everyday register).

Beyond the assumption that mathematical activity is simultaneously cognitive, social, and cultural, a sociocultural perspective brings two other assumptions to a definition of academic literacy in mathematics. First, the focus is on the potential for progress in what learners say and do (Vygotsky, 1978), not on learner deficiencies or misconceptions. Second, participants bring multiple perspectives to a situation, representations and utterances have multiple meanings for participants, meanings for words are situated and constructed while participating in practices, and multiple meanings are negotiated through interaction.

Shifting from a simplified view of academic language as words to a view of academic literacy in mathematics that integrates mathematical proficiency and practices is crucial for the mathematics education of students from non-dominant communities and groups. Instruction for this student population needs to move beyond defining academic literacy in mathematics as low-level language skills (i.e. vocabulary) or mathematical skills (i.e. arithmetic computation) and use an expanded definition of academic literacy in mathematics to describe and prescribe instruction that supports academic literacy in mathematics. Such instruction (a) includes the full spectrum of mathematical proficiency, balancing computational fluency with high-cognitive-demand tasks that require conceptual understanding and reasoning; (b) provides opportunities for students to participate in mathematical practices; (c) allows students to use multiple modes of communication, symbol systems, registers, and languages as resources for mathematical reasoning, and (d) supports students in negotiating situated meanings for mathematical language that is grounded in mathematical activity, instead of giving students definitions divorced from mathematical activity.

ETHNOGRAPHY AND CRITICAL THEORY LENS TO LOOK INTO LANGUAGE DIVERSITY AS FUNDS OF KNOWLEDGE IN MATHEMATICS CLASSROOMS

Arindam Bose, Tata Institute of Social Sciences

I try to explore the role of language in fostering meaning making in mathematics classrooms and in learners' work-contexts. I started by analysing multilingual mathematics classroom discourse to understand how languages are negotiated in student-teacher conversations under the assumption that language-use is a socially embedded process. The attempt was to comprehend in what different ways languages of learning and teaching (LOLT), home language and language of practice are mixed and switched to arrive at better clarity and understanding of the mathematical contexts and discursive practices (Bose & Choudhury, 2010).

The theoretical notion of *funds of knowledge* (Gonzalez, Andrade, Civil & Moll, 2001) developed by neo-Vygotskian theories largely frames my research. It helps me in understanding the potential language resource available in the community in the form of embedded mathematical practices in the work contexts that students are exposed to, and in illuminating the nature and extent of language negotiation at the interface of knowledge drawn from cultural embeddings and different sites of mathematical learning. Funds of knowledge framework helps in understanding shared pattern of language and belief system to enquire into the connections between work practices and opportunities available for gathering everyday mathematical knowledge.

Complementarily, the lens of critical theory makes visible that funds of knowledge and language resources of people from underprivileged groups are often not leveraged in classroom pedagogy and that, hierarchical social structure (e.g. linguistic, caste, class division in Indian society) has bearings on academic achievements including mathematics learning (Bose & Kantha, 2014). The language of mathematics textbooks and the language of mathematics classroom practice often do not connect with the language resources and language repertoires of the learners. This disconnect becomes more apparent and causes debilitating academic effect particularly for learners from the non-dominant and underprivileged backgrounds. With the classroom space becoming more and more multilingual due to constant migration of population (often belonging to disadvantaged conditions) from one province to the other in search of better livelihood, classroom pedagogy arising from such disconnects becomes even more alarming given the bearings of the hierarchical social structure as described above. Therefore, it is necessary and important to understand the complexities and challenges arising in such classroom space in multilingual societies of the third world (like India and other similar countries) to find ways forward that can support mathematics teaching and learning.

India although has 22 official languages used in its different states/provinces, each state uses only one or two official languages and a few more languages are spoken there. English remains the subsidiary working language (associate official language) in most part of the country. Hindi is spoken as a mother tongue by a little more than 40% of India's population but it is the official language of only 10 out of 29 states. Trilingual characteristic is a unique feature of most states (barring a few) which also prevails in most classrooms in different states. Although "three-language formula" is promoted by the national curriculum (NCERT, 2006, p. 12-14), often learners' language repertoire and "spaces for participation" (Phakeng, Planas, Bose & Njurai, 2018, p. 292) remain unnoticed and unrecognised in the policy and in classroom practice. It is therefore of importance to explore and understand the interplay and complexities of languages of different forms (mother tongue or home language and languages of practice) and in different contexts. Ethnography as a lens offers a mediating tool to unpack not just some of the language practices in the multilingual contexts in the third world but also to look into the issues of rights, access and social justice concerns.

Following ethnography and critical theory lenses helps in pointing to the disconnection between students' identities formed in out-of-school contexts as well as those formed during formal classroom learning. Formal mathematics learning in school facilitates shaping of students' identity as learners. However, their exposure and experience in the out-of-school work-contexts help build their identities as knowers and learners as well as doers. Students draw out-of-school mathematical knowledge from their own work practices or by observing others work, which helps them look at themselves as “knowers” of some body of knowledge that is valued in the community. The identity of a “doer” is shaped when children reflect on themselves as doing certain tasks in the out-of-school contexts, even if those tasks are fragmented, piecemeal or routine household chores. It can be argued that when classroom teaching practices acknowledge students’ language and knowledge resource and allow merging of their identities then such practices facilitate building powerful connections between out-of-school and school knowledge (of mathematics) and strengthen understanding. Merger or negotiation between students’ identities help in transfer of learning and therefore in enhancing learning opportunities. These processes happen through language negotiation and are part of the socio-cultural role of language in the social settings (Bose & Clarkson, 2016). Such bridging of the socio-cultural role of language and students’ identities goes beyond the contemporary research on connecting students’ identities with classroom norms and argues for bringing together students’ identities and funds of knowledge. Ethnographic exploration through extensive fieldwork in terms of a prolonged period of observation helped in understanding how the groups functioned and unpacked community’s language resource and settings.

Another aspect of my work has been to understand language policy formulations and ways in which the policies play out. Understanding language-in-education policy (LiEP) in cross-cultural contexts helped in analysing language complexities in multilingual mathematics classrooms in developing countries with similar socio-cultural-economic milieu by looking at similarities and differences of language practices in mathematics classrooms. My work has looked into India and South Africa’s contexts and explored how such practices shape learners' mathematical communication (Bose & Feza, 2018; Bose & Phakeng, 2017).

It is not just the use of learners' home language, LOLT and the language of mathematics or a mix of them that is critical for facilitating effective mathematical communication – necessary for developing sound conceptual understanding. Different languages function differently at the interplay with mathematical language depending upon the language’s intonation, syntax and diction. For example, in the case of South Africa and India, uniform policy formulation may not be effective in their multilingual mathematics contexts (Bose & Phakeng, 2017). As emerging economies, these two countries have to deal with learners' identities in the classrooms that emerge from the language settings. The socio-economic statuses of these two countries are vastly different from the developed countries or other developing nations. The languages that the learners use and how they use them during mathematics lessons often serve as an indicator of the social class they belong to or the social context they grew up in. Often learners' familiarity with everyday mathematical registers came from their exposure to the micro-enterprises around them. Their justification and reasoning revolved around their identities drawn from the work practices. In South African context, learners' identities emerge from their racial identities. However, there is a dearth of research that explores links between identities and language use and how they influence learners' communication of mathematics.

There is an emergent possibility arising from the present studies to inquire into the connection between language, culture and mathematical cognition. Below is a list of possible areas in language and communication that can be explored in similar studies:

- How are communicative activity and mathematical thinking linked and in what ways does language negotiation support (or not) such activities in work-contexts and in mathematics classrooms?
- Different representations of mathematical concepts in different languages and their connection in building mathematical understanding drawing from out-of-school mathematics learning in multicultural and multilingual settings.
- Language negotiation at the interface of knowledge drawn from cultural embeddings and formal, academic knowledge – mutual impact on different sites of mathematical learning.
- A look at the curriculum and policy planning taking on board (or not) the connection between out-of-school mathematics learning and language diversity and ways in which such integration can be achieved.

THEORETICAL FRAMEWORKS IN TOPIC-SPECIFIC DESIGN RESEARCH ON SUPPORTING LANGUAGE LEARNERS

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Theoretical backgrounds and need for theory building

The MuM-research group in Dortmund (currently consisting of four postdocs, one professor, five teachers and nine PhD students) works on designing and investigating topic-specific learning opportunities for monolingual and multilingual language learners in different research designs and within different theoretical perspectives. Four basic assumptions about relevant ingredients influence the project-specific choice of theoretical perspectives:

- the epistemic function of language and a functional perspective on language and mathematics learning
- the emphasis on the discursive level of language and their social embeddedness
- the need for design research to combine different theoretical frameworks
- the topic-specificity of language demands

The epistemic function of language and the functional perspective

Language is crucial for mathematics learning due to its function as a "tool for thinking". The tight connection between language and thinking has been theorized by different theoretical approaches. It goes along with functional perspectives on language like in systemic functional linguistics (Halliday, 1985) or in functional pragmatics (Redder, 2008). The different functional perspectives on language share their focus not on the forms of language alone, but on the interplay of forms and function (function in communication and thinking). The functional perspective is hence crucial for avoiding the treatment of language forms as an end in itself.

The emphasis on the discursive level

Language can be considered on the word level (comprising lexical features such as words and chunks), the sentence level (comprising syntactical features to structure the relation between the words), or on the discursive level on which to locate the more complex language features. The socially embedded more complex features on the discourse level comprise discourse practices like arguing, explaining, reporting, but also the sociomathematical norms which regulate which practice counts as valuable in a specific classroom culture. The relevance of the discursive level has been emphasized by many mathematics education researchers (e.g. Moschkovich, 2015; Barwell, 2012), even if different conceptualizations of discourse are applied in mathematics education research. For our work, the Interactional Discourse Analysis (IDA by Quasthoff, Heller & Morek, 2017) has proven insightful as it provides a powerful framework to link the social, interactional phenomena of discourse practices with the individual discourse competence of each students to participate in these joint practices (Erath et al., 2018).

The need for design research to combine different theoretical frameworks

The mentioned general (mostly linguistic) theoretical approaches provide very insightful topic-independent frameworks for grasping general language demands and for investigating the role of language for mathematics learning within a certain focus. However, each perspective is also blind for other aspects, and that is why different theoretical perspectives must be combined in order to grasp the complexity of language in mathematics teaching and learning. Especially for designing and investigating learning opportunities for language learners in mathematics classrooms, the linguistic frameworks must be combined with frameworks grasping the mathematical topic to be learned and typical learning pathways towards these topics. These topic-specific frameworks change for each project, e.g. functional relationships (Prediger & Zindel, 2017), percentages (Pöhler & Prediger, 2015), fractions (Prediger et al., submitted), or logical deductions (Prediger & Hein, 2017).

The topic-specificity of language demands requires more specific theory building

When supporting language learners in a language- and content-integrated way, there is not only a need to juxtapose theoretical frameworks for language with topic-specific theory on learning the specific topic. Instead, for each of the projects, there was a need to *generate* theory for disentangling the topic-specific language demands in more details. For this purpose, the combination of theoretical frameworks must be complemented by careful empirical reconstructions in order to identify the language demands and a theoretical framework in which they are captured best. So far, we have only found local solutions, but there is still a huge need for theory building which goes beyond importing linguistic theories to mathematics education. *So the question is not only:*

Which theories can be imported from other disciplines? But also: Which mathematics-specific and even topic- specific theories must we develop in order to find a sound theoretical base?

Design research for building topic-specific theories on relating registers and languages

Design research is a suitable research methodology in which we can design learning opportunities for language learners and then investigate the interplay of language demands and conceptual demands when studying the initiated learning processes for the aim of contributing to more integrated theory building (Prediger, 2019). For example, the project on *functional relationships* (Prediger & Zindel, 2017) developed a theory for capturing students' processes of unpacking conceptual demands within the highly condensed concept of function. The project on *percentages* (Pöhler & Prediger, 2015) could successfully rely on a well-established learning trajectory towards percentages and combine it with matching language learning opportunities which focus on constructing meanings; the empirical trace-analysis of students' language uses shows the complexities of moving from the informal resources to the meaning-related academic language and the technical language. The project on *logical deductions* (Prediger & Hein, 2017) drew upon mathematics education research on logical structures and identified (in Halliday's 1985 framework) the interplay to the syntactical language demands for expressing these logical structures. The project on multilingual students' learning of *fractions* (Prediger et al., submitted) showed how students' learning pathways towards the part-whole concept can be shaped by translanguaging and connecting different language-related nuances. In all four projects, the instructional approach of macro-scaffolding (Gibbons, 2002) was elaborated in topic-specific ways. In all four projects, first theoretical contributions about the role of languages and registers in topic-specific learning pathways were constructed which require further elaboration and transfer to other topics (Prediger, 2019).

The common core is that all four projects provide concretizations for the often-repeated claim for building upon the students' everyday experiences and informal language resources as well as for actively supporting their systematic relation to more formal language demands. However, the results also show the definite need to proceed in theorizing these processes in a much finer way on the micro-level.

REFLECTIONS, FUTURE ISSUES AND DIRECTIONS

In the Research Forum coordinated by Richard Barwell and Philip Clarkson fifteen years ago, some concluding remarks were importantly related to "where to look further" as well as to all that "remains to be done to take account of multilingualism at substantive, methodological or theoretical levels of our research" (Barwell & Clarkson, 2004, p. 252). The contributions to the current Research Forum reveal the vitality of the theoretical debate in the mathematics education research on language diversity, together with the emergence of newer strong foci such as content-specific approaches to multilingual mathematics teaching and learning, cognitive and sociocultural integrated ways of addressing academic literacy in mathematics in linguistically diverse settings, ethnography-based models of curricular design, or the significance of the forces and tensions involved in language use over the course of mathematics teaching and learning. A look at the changing picture of the area of study throughout these years shows that some concepts are still used –e.g. academic literacy in mathematics– but in theoretically modified ways. This is consistent with the discussion in Planas and Schütte (2018) about the same terms taking on more or less different meanings and being used to indicate belonging to more or less different combinations of theories. The meaning of academic literacy in mathematics, for example, depends on the theories used to interpret it and actually varies in the understandings of language that articulate cognition and participation. This same phenomenon can be possibly made in relation to other areas of study as well as the need for being more explicit in determining the role of theory in research.

In the different contributions put together on the occasion of this Research Forum, some important theoretical challenges deal with ontological positions regarding the nature of language, of language negotiation, of mathematical reasoning, and the relationships between them. Not only is the relevance of such positions acknowledged, but the necessity of dealing with them in detail is also considered.

The four theoretical perspectives showcased in this Research Forum display several similarities and points of connection. In particular, they all adopt a view of language that positions variation, multiplicity and difference as valuable (rather than as problematic), whether in terms of sources of meaning, resources, or funds of knowledge. In this common orientation, language is understood to be a tool, a mediational means or a set of discursive practices or literacy practices through which or with which learners and teachers construct or create mathematical meaning. In all four approaches, language and mathematics are understood to be socially shared and reproduced. This position, which can be traced back to research in the 1980s (see Barwell, Moschkovich & Setati Phakeng, 2017, for a discussion) in turn leads to an assumption that pedagogical innovation in the face of language diversity must take this social dimension of language and mathematics into account.

While there is a shared ontological position in how the world is represented as socially produced, a number of differences (at least in emphasis) appear in how this position impacts the analytical proposals and methods for investigating language and mathematics in settings of language diversity and mathematics teaching and learning. Barwell's theoretical proposal emphasises the dialogic nature of language, for example, whereas Moschkovich emphasises academic literacy. The two perspectives are similar and can be related, but the former leads to more attention on the (dialogic) relationships between forms of language (different languages, informal expression, genres), while the latter suggests more attention to relationships between modes (e.g. gestures, diagrams, spoken language). Similarly, Bose's approach arguably has a more material perspective on mathematical knowledge (in the form of funds) and tends to highlight the different nature of knowledge in school and out of school. Prediger, meanwhile, emphasises didactics, as developed through pragmatic cycles of task design, where a key relation is that between actual and desired mathematical practices (including language and discursive practices).

Didactical design research, critical ethnography, sociolinguistic indexicality and culturally situated cognition are some examples of the very different analytical proposals linked to the adoption of very similar positions regarding the nature of language, of mathematical reasoning and of their relationship. In this respect, we can also identify the fundamental question of the extent to which our analytical choices refer to the theories in use and under development. Given the strong theoretical dimension of any methodological proposal, it is important to acknowledge the diversity of analytical strands within very close views of language.

Not less significant and indeed very relevant to all this discussion is what can be planned next or, as posed by Barwell and Clarkson (2004, p. 252), “where to look further”. Among possible future issues and paths to consider we include those that are open and relevant to theory use and theory development, particularly to theory that offers a lens for making sense of, impacting and giving direction to research design and practice in mathematics education settings with language diversity. At least six unresolved questions are important here:

- Which aspects of learning and teaching mathematics in multilingual classrooms need most urgent attention?
- Which theoretical and empirical findings from outside mathematics education seem most relevant to research and practice in mathematics education?
- How can researchers decide which theoretical frameworks to use for their research?
- How can researchers develop expertise in theories that are outside of mathematics education?
- How can curriculum designers include attention to language when designing materials?
- Which kind of topic-specific theory elements are crucial for the foundation of curriculum design?
- How can teachers learn to include attention to language in their mathematics lessons?

It is not possible to think of all these questions and make them intelligible without theory, nor is it possible to investigate tentative responses without some clarity about how theory informs the process of inquiry. We hope that this Research Forum has given some insight into the relationship between different theoretical perspectives used in this area of study and the kinds of questions that researchers are working on.

Collected references

Bakhtin M. M. (1986). *Speech genres and other late essays.* Austin, TX: University of Texas Press.

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays.* (Ed., M. Holquist; Trans., C. Emerson & M. Holquist). Austin, TX: University of Texas Press.

Barwell, R. (2005). Integrating language and content: Issues from the mathematics classroom. *Linguistics and Education, 16*(2), 205-218.

Barwell, R. (Ed.) (2009). *Multilingualism in mathematics classrooms: Global perspectives.* Bristol, UK: Multilingual Matters.

Barwell, R. (2012). Discursive demands and equity in second language mathematics classrooms. In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education* (pp. 147-163). Dordrecht, Netherlands: Springer.

Barwell, R. (2016). Formal and informal mathematical discourses: Bakhtin and Vygotsky, dialogue and dialectic. *Educational Studies in Mathematics, 92*(3), 331- 345.

Barwell, R. (2018) From language as a resource to sources of meaning in multilingual mathematics classrooms. *Journal of Mathematical Behavior, 50*, 155-168.

Barwell, R., & Clarkson, P. (2004). Introduction. Researching mathematics education in multilingual contexts: Theory, methodology and the teaching of mathematics. In M. Johnsen-Høines, & A. B. Flugestad (Eds.), *Proceedings 28th PME* (vol. 1, pp. 227-228). Bergen, Norway: PME.

Barwell, R., Moschkovich, J. N., & Setati Phakeng, M. (2017) Language diversity and mathematics: Second language, bilingual, and multilingual learners. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 583-606). Reston, VA: National Council of Teachers of Mathematics.

Barwell, R., & Pimm, D. (2016). Bakhtin and some dilemmas of mathematics–language. In M. Phakeng, & S. Lerman (Eds.), *Mathematics education in a context of inequity, poverty and language diversity* (pp. 7-18). New York: Springer.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J. N., Phakeng, M., Planas, N., Valero, P., & Villavicencio, M. (Eds.) (2016). *Mathematics education and language diversity. The 21st ICMI Study*. New York: Springer.

Blommaert, J. (2010). *The sociolinguistics of globalization*. Cambridge, UK: Cambridge University Press.

Bose, A., & Kantha, V. K. (2014). Influence of socio-economic background and cultural practices on mathematics education in India: A contemporary overview in historical perspective. *ZDM – Mathematics Education, 46*(7), 1073-1084.

Bose, A., & Clarkson, P. (2016). Students' use of their languages and registers: An example of the socio-cultural role of language in multilingual classrooms. In A. Halai & P. Clarkson (Eds.), *Teaching and learning mathematics in multilingual classrooms: Issues for policy and practice*. Rotterdam, Netherlands: Sense Publishers (Brill).

Bose, A., & Feza, N. N. (2018). Additive model of language policy and hybridity: Glimpses from numeracy learning in early grades in a South African province. In S. Ladage, & S. Narvekar (Eds.), *Proceedings of epiSTEME7 (*pp. 97-106). India: Cinnamon Teal.

Bose, A., & Phakeng, M. (2017). Language practices in multilingual mathematics classrooms: Lessons from India and South Africa. In B. Kaur, W.K. Ho, T.L. Toh, & B.H. Choy (Eds.), *Proceedings 41st PME* (vol. 2, pp. 177-184). Singapore: PME.

Bronfenbrenner, U. (1989). Ecological systems theory. In R. Vasta (Ed.), *Annals of Child Development* (pp. 187-248). Greenwich, CT: JAI Press Inc.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Busch, B. (2014). Building on heteroglossia and heterogeneity: The experience of a multilingual classroom. In A. Blackledge, & A. Creese (Eds.), (2014). *Heteroglossia as practice and pedagogy* (pp. 21-40). Dordrecht, Netherlands: Springer.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, *10*(1&2), 113-164.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & C. A. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development* (pp. 91-119). New York: Oxford University Press.

D'Ambrosio, U. (1991). Ethnomathematics and its place in the history and pedagogy of mathematics. In M. Harris (Ed.), *Schools, mathematics and work* (pp. 15-25). Bristol, PA: Falmer Press.

Erath, K., Prediger, S., Quasthoff, U., & Heller, V. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: The case of explaining to learn and learning to explain. *Educational Studies in Mathematics, 99*(2), 161-179.

Forman, E. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 115-30). Mahwah, NJ: Lawrence Erlbaum Associates.

Gee, J. (1996). *Social linguistics and literacies: Ideology in Discourses* (3rd ed.). London, UK: The Falmer Press.

Gee, J. (1999). *An introduction to Discourse Analysis: Theory and method*. New York: Routledge.

Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*(1), 5.

Gutiérrez, K., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice? *Educational Researcher, 32*(5), 19-25.

Gutiérrez, K., Baquedano-Lopez, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind, Culture, and Activity, 6*(4), 286-303.

Gibbons, P. (2002). *Scaffolding language, scaffolding learning. Teaching second language learners in the mainstream classroom*. Portsmouth, UK: Heinemann.

Halliday, M. A. K. (1985). *An introduction to Functional Grammar*. London, UK: Edward Arnold.

Lave, J. (1988). *Cognition in practice.* New York: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Lee, C. (2008). The centrality of culture to the scientific study of learning and development: How an ecological framework in education research facilitates civic responsibility (2008 Wallace Foundation Distinguished Lecture). *Educational Researcher, 37*(5), 267-279.

Moschkovich, J. N. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, Special issue on Diversity, Equity, and Mathematical Learning, N. Nasir & P. Cobb (Eds.), *4*(2&3), 189-212.

Moschkovich, J. N. (2007a). Bilingual mathematics learners: How views of language, bilingual learners, and mathematical communication impact instruction. In N. Nasir & P. Cobb (Eds.), *Diversity, equity, and access to mathematical ideas* (pp. 89-104). New York: Teachers College Press.

Moschkovich, J. N. (2007b). Examining mathematical Discourse practices. *For the Learning of Mathematics*, *27*(1), 24-30.

Moschkovich, J. N. (2008). "I went by twos, he went by one:" Multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences*, *17*(4), 551-587.

Moschkovich, J. N. (2011). Ecological approaches to transnational research on mathematical reasoning. In R. Kitchen & M. Civil (Eds.), *Transnational and borderland studies in mathematics education* (pp. 1-22). New York: Routledge, Taylor & Francis Group, 1-22.

Moschkovich, J. N. (2013). Issues regarding the concept of mathematical practices. In Y. Li, & J. N. Moschkovich (Eds.), *Proficiency and beliefs in learning and teaching mathematics: Learning from Alan Schoenfeld and Günter Toerner* (pp. 257-275). Rotterdam, Netherlands: Sense Publishers (Brill).

Moschkovich, J. N. (2015). Academic literacy in mathematics for English Learners. *Journal of Mathematical Behavior, 40*, 43-62.

Moschkovich, J. N., & Brenner, M. (2000). Integrating a naturalistic paradigm into research on mathematics and science cognition and learning. In R. Lesh, & A. Kelly (Eds.), *Handbook of Research Design in Mathematics & Science Education* (pp. 457-486). Mahwah, NJ: Lawrence Erlbaum Associates.

Nasir, N. (2000). Points ain't everything: Emergent goals and average and percent understandings in the play of basketball among African-American students. *Anthropology and Education*, *31*(3), 283-305.

National Council of Educational Research and Training (NCERT). (2006). *Position Paper of National Focus Group on Teaching of Indian Languages*. New Delhi, India: NCERT.

Phakeng, M., Planas, N., Bose, A., & Njurai, E. (2018). Teaching and learning mathematics in trilingual classrooms: Learning from three different continents. In R. Hunter, M. Civil, B. Herbel-Eisenmann, D. Wagner & N. Planas (Eds.), *Mathematical discourse that breaks* *barriers and creates space for marginalized learners* (pp. 277-304). Rotterdam, Netherlands: Sense Publishers (Brill).

Planas, N., Morgan, C., & Schütte, M. (2018). Mathematics education and language. Lessons from two decades of research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe*. Oxon, UK: Routledge.

Planas, N., & Schütte, M. (2018). Research frameworks for the study of language in mathematics education. *ZDM – Mathematics Education*, *50*(6), 965-974.

Pöhler, B., & Prediger, S. (2015). Intertwining lexical and conceptual learning trajectories. *Eurasia Journal of Mathematics, Science & Technology Education, 11*(6), 1697-1722.

Prediger, S., Kuzu, T., Schüler-Meyer, A. & Wagner, J. (2019, in press). One mind, two languages: Separate conceptualisations? A case study on students' bilingual modes for dealing with language-related conceptualisations of fractions. *Research in Mathematics Education*. doi:10.1080/14794802.2019.1602561.

Prediger, S., & Hein, K. (2017). Learning to meet language demands in multi-step mathematical argumentations. *European Journal of Applied Linguistics, 5*(2), 309-335.

Prediger, S., & Zindel, C. (2017). School academic language demands for understanding functional relationships. *Eurasia Journal of Mathematics, Science & Technology Education, 13*(7b), 4157-4188.

Prediger, S. (2019). Theorizing in Design Research: Methodological reflections on developing and connecting theory elements for language-responsive mathematics classrooms. *Avances de Investigación en Educación Matemática* (Special Issue on Task Design), *15*, 5-25.

Quasthoff, U., Heller, V., & Morek, M. (2017). On the sequential organization and genre-orientation of discourse units in interaction. *Discourse Studies, 19*(1), 84-110.

Radford, L., & Barwell, R. (2016). Language in mathematics education research. In Á. Gutiérrez, G. H. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education. The journey continues* (pp. 275-313). Rotterdam, Netherlands: Sense Publishers (Brill).

Redder, A. (2008). Functional Pragmatics. In G. Antos, & E. Ventola (Eds.), *Handbook of interpersonal communication* (pp. 133-178). Berlin: Mouton de Gruyter.

Saxe, G. B. (1991). *Culture and cognitive development: Studies in mathematical understanding.* Hillsdale, NJ: Lawrence Erlbaum.

Setati, M. (2008). Access to mathematics versus access to the language of power: The struggle in multilingual mathematics classrooms. *South African Journal of Education, 28*, 103-116.

Vertovec, S. (2007). Super-diversity and its implications. *Ethnic and Racial Studies, 30*(6), 1024-1054.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

THE USE OF VIDEO FOR THE LEARNING OF TEACHERS OF MATHEMATICS

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INTRODUCTION AND RATIONALE

The use of video, relating to teachers' learning, is on the increase across the globe (Gaudin and Chalies, 2015). Video can allow the same example of teaching to be observed by many people, perhaps multiple times, and specific aspects of teaching and learning episodes to be focussed upon and discussed. It can allow access to a broader range of classroom contexts, mathematical content, and teaching approaches than otherwise might be available (Star & Strickland, 2008). Video can, more effectively than live observation, help teachers move from focussing on aspects of teaching and learning episodes that confirm their existing beliefs about 'good' teaching, towards attention to student learning and evidence thereof (Philipp et al., 2007). The role of the facilitator is crucial in realising the potential of video (Beswick & Muir, 2013).

The use of video in mathematics education has been the subject of significant research, making this Research Forum timely. We will not be considering methodological utilizations of video for broad research purposes. We are, rather, interested in the use of video with mathematics teachers for the purpose of enhancing teacher learning (within which we would include research using video, conducted by teachers). We note a trend in several parts of the world towards teachers supporting other teachers via using video. This trend can be seen, for instance, in Israel and in the USA, where teachers might be trained as mentors or facilitators of other teachers in the scaling up of research programmes (e.g. Beisiegel, 2016; Borko et al., 2017; Karsenty, 2016). It is therefore more important than ever to elaborate our understandings of how video can be used effectively in relation to teacher learning.

We have attempted a *mapping of the terrain* of work on video in relation to mathematics teacher learning and draw on specific projects (our own and others') in order to exemplify trends and possibilities. This mapping takes the form of five dimensions of variation of video use with teachers of mathematics; elaborating these dimensions forms the bulk of this report. Before getting to these dimensions, we first look back over some past reviews of work on video in mathematics education and beyond in order to offer a sense of the *state of the art* as well as unresolved issues or questions.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 107-135). Pretoria, South Africa: PME.

VIDEO AS A TOOL FOR TEACHER LEARNING: A BRIEF ACCOUNT OF PAST REVIEWS

Although video has been used in mathematics pre-service teacher education (TE) and in-service professional development (PD) for more than 50 years, it is only since the turn of the millennium that substantial reviews on this issue began to accumulate. In 2004, Elsevier published the volume *Using Video in Teacher Education* (Brophy, 2004), opening with a review chapter by Sherin (2004), which became a highly cited source in research on video work with teachers. Sherin described trends in using video, from microteaching in the 1960s and interaction analysis in the 1970s, through modelling expert teaching and the use of video-based cases in the 1980s and the 1990s, to hypermedia programs that emerged as a result of the digital revolution we have experienced since the 2000s.

Since the turn of the century, the body of research on video use in TE and PD programs for mathematics teachers has been growing rapidly, with hundreds of papers published. Hall and Wright (2007) provided a literature review on video as a resource for supporting the learning of both pre-service and practicing mathematics teachers. Hall and Wright's review focused on the affordances and limitations of using video. They noted the (then) apparent gap in the literature regarding detailed accounts of video use, particularly for in-service development. Marsh and Mitchell's (2014) review of the literature similarly focused on the possible gains of video use (with no specific reference to mathematics teaching), adding in the component of asynchronous versus synchronous viewing. Marsh and Mitchell noted the need for more research on how learning environments for teachers may be constructed to optimize the power of video.

Tripp and Rich (2012) conducted a review of 63 studies on pre-service and practicing teachers reflecting on their own videotaped lessons (only few of these studies related to mathematics teaching). They suggested 6 dimensions for analyzing studies: (a) the type of reflection tasks (e.g., completing a pre-constructed checklist, participating in an interview, writing an open reflection, etc.); (b) how reflection is facilitated (i.e., what is the focus set for observing and analyzing the video); (c) is the reflection individual or collaborative; (d) the length of the video used; (e) the number of reflection events; and (f) how the effectiveness of reflection is measured (e.g., changes in teaching practices, self-assessments, scores on pre- and post-tests of teaching skills, etc.). Tripp and Rich posited that although existing evidence point to the advantages of video use, further research is needed to examine the ways in which video-aided teacher reflection impacts teacher practice.

Gaudin and Chalies (2015) conducted a comprehensive review of the literature published from 2003 onwards, on video viewing in TE and PD programs. About 25% of the 255 reviewed articles were related to mathematics teaching. Gaudin and Chalies conceptualized their review around four categories: (a) teachers' activity as they view a classroom video; (b) the objectives of video viewing; (c) the types of videos viewed; and (d) the effects of video viewing on teacher education and professional development. Each category was further broken down into several sub-categories, thus a broad view was obtained on issues at the core of video viewing, for example interpreting and reflecting on classroom events (sub-category of (a)), or the effects of video viewing on teacher motivation (sub-category of (d)).

One of the interesting points noticed when reading Gaudin and Chalies' (2015) review, concerns the multitude of theoretical frameworks offered in various studies, to either direct or interpret teachers' video viewing. At least twenty-five different theories and frameworks are mentioned, some more general (e.g., situated learning (Lave & Wenger, 1992); or, the constructivist perspective), others more specific (e.g., the Learning and Teaching Geometry model (Seago, Driscoll & Jacobs, 2010)). This range reflects the centrality ascribed to frameworks in video-based teacher learning, an issue to be further elaborated as one of the dimensions we offer here. Gaudin and Chalies (2015) raised several questions as possible directions for future research, among them the important question of creating a "teaching continuum": how can we connect TE and PD programs in regards to video viewing, so that video becomes a routine and familiar professional tool throughout an entire teaching career?

Major and Watson (2018) provide the most recent review of studies focusing on video use with teachers, although their review includes research on PD only and excludes TE programs. The review covers 82 empirical studies published between 2005 and 2015, with some overlap with Gaudin and Chalies' (2015) review, and, similarly to the latter, about 30% of the reviewed papers relate to mathematics teaching. Major and Watson present interesting statistical mapping of the studies, thus it can be seen that more than half of them are from the USA, and that the most common mode of video use, reported in about two-thirds of the studies, is teachers watching videotaped lessons from their own classrooms and/or those of peers. In terms of methodology, Major and Watson found that about two-thirds of the studies they reviewed employ qualitative research methods, and that in nearly half of them $n<10$, where n is the number of participating teachers reported (and in the next 30%, $10 \leq n < 20$). They conclude that while small-scale qualitative studies are important and contribute to developing theory, "the field is reaching saturation point and we are approaching a limit on what might be learned from such research" (Major & Watson, 2018, p. 65). They further claim that "fresh thinking is now needed to advance understanding of how professional learning is supported through the use of video. […] For the field to develop further, it is necessary to look in particular at how video-based teacher PD impacts on students' learning" (ibid).

We note here that apart from Hall and Wright's (2007) review, all the other literature reviews mentioned in this section do not pertain to mathematics teaching specifically. However, the interest of the international community of mathematics education in video use, in both PD and TE programs, is apparent. This can be seen from (a) the abundance of articles published (a recent search on Google Scholar showed more than 200 relevant results since the beginning of 2018 only); (b) the two recent special issues dedicated to video use: *Video as a catalyst for mathematics teachers' professional* *growth*, a special issue in Journal of Mathematics Teaching Education (co-edited by Karsenty and Sherin, 2017), and *Designing, facilitating, and scaling-up video-based professional development: Supporting complex forms of teaching in science and mathematics*, a special issue in International Journal of STEM Education (co-edited by Tekkumru-Kisa and Stein, 2017); (c) PME Working Groups on video; and (d) conferences (e.g., the symposium *Video Resources for Mathematics Teacher Development,* www.weizmann.ac.il/conferences/video-lm2014).

DIMENSIONS OF VIDEO USE

There is no single theoretical framework within which work on video is situated and conceptualized and, as authors who have studied uses of video, we come from different backgrounds and perspectives. We focus therefore on how video is used in practice. Of course, it is not possible to separate the enactment of video use from a theoretical orientation (Coles et al., 2018), but we have chosen to organise our thinking by *use* first and, given the range of practices we identify, we point to different theoretical backgrounds. In order to undertake a mapping of current trends in research we have identified five dimensions of variation, across uses of video. The dimensions we identify are, in some sense, from the point of view of a teacher and draw on our experience of the kinds of things that are different across various uses of video with teachers of mathematics. We elaborate on the five dimensions, below, which are: (1) what is the purpose of using video?; (2) who watches the video?; (3) what is being watched?; (4) what framework is being used for watching?; and, (5) who leads and guides the use of video?

In discussing each dimension, we have paid attention to common or consistent findings and have tried to point to a range of opportunities and choices, with their affordances and constraints. We also identify further possibilities for development and research in the field.

1. WHAT IS THE PURPOSE OF USING VIDEO?

The reasons for videoing teacher practice are myriad and not all are associated with practicing teachers' professional learning as is the primary focus here. There is for example quite a long-standing use of videos in pre-service teacher training (Kazemi, Franke, & Lampert, 2009; Ho, Leong, & Ho, 2015), which Oates and Evans (2017) note has been a predominant use of video in the past. The use of videos has been much less frequently extended to observe the practice of more experienced teachers outside of their role as exemplar-teachers (e.g. in Ho et al., 2015), and is especially limited at the university level, where perhaps mathematics lecturers have commonly been seen as experts, at least from a mathematical perspective (Oates & Evans, 2017).

Even in situations where videos were designed for developmental purposes, Oates and Evans (2017) note that they are increasingly being used for evaluative purposes, such as to measure teacher effectiveness for organizational prerogatives (e.g., promotion, annual performance reviews, etc.). In the latter respect, Hill, Blazar, Humez et al. (2013) describe the drive by policymakers and school leaders to measure teachers' performance in the USA, and the role of videos to achieve such ratings. Barton et al. (2015) note that such evaluative practices can be counter-productive to the goals of teacher learning. At no stage was it envisaged that any element of the observations made by the participants in their study would be used to make judgments about teaching practice. Indeed, all members of the study had to feel comfortable that their practice was being observed in a caring and empathetic manner, with the aim of informing, as opposed to judging their practice

Thus we focus here on the uses of video that are essentially developmental in nature, although research published in recent years shows that, even within the overarching goal of teacher learning, many different objectives are defined. Video might be used for: (a) disseminating reform materials or modelling certain teaching approaches (e.g., Borko, Koellner, Jacobs & Seago, 2011); (b) developing teachers' ability to notice and understand students' mathematical thinking (e.g., Choy, 2013; Sherin, Jacobs & Philipp, 2011); (c) encouraging reflection on practice (e.g., Geiger, Muir & Lamb, 2016; Karsenty & Arcavi, 2017); (d) providing feedback on the quality of teaching and offering ways to improve it (e.g., Hollingsworth & Clarke, 2017); and other emerging objectives such as following development of teachers in their classrooms over time (Boston, Bostic Lesseig and Sherman, 2015), assessing the success of a PD initiative (Oates & Evans, 2017), and supporting the upscaling of PD programs (for example the longitudinal *DATUM* study using videos of lectures at Auckland University, Barton et al., 2015). There is a general distinction here between using video to develop particular teaching strategies and techniques (e.g., linked to reasoning) and using video to develop particular skills related to teaching (e.g., reflection). Consistent findings across this dimension point to the potential power of video for teacher learning, but also highlight the subtlety of making effective use of video (Coles, 2014; Jaworski, 1990). In the Auckland study, there was evidence of both types of development. With respect to particular teaching strategies, Barton et al. (2015) note that discussions often tended to focus on mathematical and epistemological aspects of the lectures and provided informative insights into lecturer behaviour in mathematics, and the theoretical lens through which this was being framed. Hannah, Stewart and Thomas (2013) used videos to examine the role of language and visualisation in the teaching of linear algebra.

One example of the use of video for critical reflection is the study by Geiger, Muir and Lamb (2016) who explored the use of video-stimulated recall as a catalyst for teacher professional learning. They compare and contrast the outcomes of two studies that used different formats of video-supported professional learning, as measured against a framework that details three levels of reflection (*technical, deliberate* and *critical*) against the object of the reflective response (*self, practice* and *students*). They conclude that "video-stimulated recall can be an effective medium for promoting teacher professional learning, providing quality reflection and questioning are included as crucial elements of the processes" (Geiger et al., 2016, p. 457). A further example of the power of videos to provoke critical reflection is evident for one of the participants in the University of Auckland *DATUM* study (Oates & Evans, 2017; Paterson & Evans, 2013). In Paterson and Evans (2013), a new lecturer provides a highly personal account of the dramatic changes she instigated in her teaching, partly in response to student evaluations which identified some issue, but more directly provoked by watching videos of her own and colleagues' practice as part of the study. While she notes that some of the changes may be due to increasing confidence and experience, and are also undoubtedly a consequence of her personal interest in improving student learning, she still largely attributes the motivation, context and vehicle for instigating the change to the discussions and observations she experienced within the *DATUM* forum. Paterson and Evans (2013) conclude that the (*DATUM*) model of intra-departmental professional development in which she had the opportunity both to observe others teach and to examine and discuss her own practice provided a supportive opportunity for productive professional development.

2. WHO WATCHES THE VIDEO?

The literature on video-based programmes around the world shows that there is considerable variation in target teacher populations (e.g., pre-service school teachers; practicing teachers (including university teachers); teacher leaders; facilitators who work with teachers). In certain cases, video is watched by mixed audiences, for instance pre-service and practicing teachers together (e.g., Koc, Peker & Osmanoglu, 2009), or teachers together with mathematicians and teacher educators (McGraw et al., 2007). The size of the groups watching a particular video at a specific time also vary from a single individual to a large group. All of these variations in audience occur also in online contexts where video is used in both synchronous and asynchronous modes. In this section we explore the range of audiences that have featured in video research in order of the degree or closeness of facilitation, where one teacher with one facilitator (e.g., Muir, Beswick, & Williamson, 2010) represents the closest facilitation, and freely available online video without facilitation represents the other end of the spectrum.

Watching video of one's own teaching has been shown to be effective in promoting teachers' reflection (Muir et al., 2010), allowing them to view their practice with a degree of objectivity (Shepherd & Hannafin, 2009). While most studies report teachers reacting positively to opportunities to view their own teaching (Downey, 2008; Muir et al., 2010), many other teachers feel threatened by the prospect, especially when the video is to be viewed with a group of peers (Borko, Jacobs, Eiteljorg, & Pittman, 2008). Individually facilitated video viewing constitutes a substantial investment of time that does not necessarily lead to change in participating teachers' beliefs or practices (e.g., Muir et al., 2010). Nevertheless, Muir et al. (2010) argued that the model they described allows for complete personalization of professional development, and the reflection that it stimulates may lead to change in time. Despite presenting as a barrier for some teachers, viewing videos of one's teaching alongside peers can serve to ameliorate the tendency for teachers to be overly critical of themselves in videos, as well as affording the opportunity for additional and new perspectives on the observed teaching to be offered (van Es, 2012).

The extent to which these activities are successful depends upon the degree of trust that exists among the teachers and between the teachers and the facilitator. Van Es (2012) described video clubs whose practice was framed by collegiality and collaboration, appropriate norms for interaction, and a commitment to maintaining the focus on the teaching and students' learning. More recently Hundley et al. (2018) described video clubs, based on earlier work of van Es and Sherin (2008), as one of four 'signature pedagogies' that distinguished their program for pre-service teachers from the usual university and school-based activity that constitute initial teacher education. Hundley et al. (2018) described preparatory work that they undertook, beginning with viewing videos of other teachers and including careful attention to conversational norms. Their experience confirms that using video of the teaching of audience members requires more preparation in terms of building trust and establishing norms than does viewing video of unknown teachers (Kleinknecht & Schneider, 2013).

Relatively few studies have included audiences comprising a mixture of participant groups. In these examples the diversity of backgrounds and perspectives seems to have enriched and broadened the range of foci of discussions, with mathematicians, for example, more inclined to focus on the task whereas other participants attended more readily to issues of implementation (McGraw et al., 2007). Koc et al. (2009) found that using video cases with pre-service and practicing teachers together assisted teachers to connect theory and practice. Barton et al. (2015) described a group of teachers of undergraduate university-level mathematics that included both mathematicians and mathematics educators, engaged in a practice similar to video club as described by van Es (2012). They emphasized the co-learner status of those involved in the community of inquiry (Jaworski, 2003) that facilitated acceptably critical responses to each other's videos.

The prevalence of recorded lectures, video-conferencing and online learning is blurring the line between face-to-face and online learning for pre-service teachers (Australian Institute for Teaching and School leadership, 2018) and increasingly for practicing teachers as well. Recent Australian government funding initiatives in Science, Technology, Engineering, and Mathematics (STEM) education (e.g., Australian Government, Department of Education and Training, 2019), for example, have mandated the production of professional development resources, including video, that can be used in face-to-face professional development activities but also be available freely online after the project end. Massive Online Open Courses (MOOCs) and open online video sharing sites represent further avenues whereby videos of teaching can be made very widely available. The ultimate audience for the online resources will include individual teachers, alone or with colleagues, teacher educators, and PD facilitators including commercial consultants. Each of these potential audiences may engage with the resources with or without a facilitator although facilitator notes may be available.

Ethical and quality issues arise when videos are used in settings that are beyond the influence of either the teachers portrayed or those who produced the videos. Editing and re-framing of videos could present the teaching in quite different lights and without the teacher or others close to source able either to know about various ways in which the teaching depicted is being interpreted or to respond to erroneous interpretations. The scope that freely available online video offers for individual viewing also brings into relief the role of facilitation (addressed in Dimension 5) and of group interactions in shaping what teachers learn from viewing videos.

3. WHAT IS BEING WATCHED?

Research in the use of video for mathematics teacher learning has outlined at least three models of *videos being watched* based on the level of acquaintance between those who are watching the video and the filmed teacher. These are: (1) watching teachers with similar backgrounds; (2) watching one's own teaching and (3) watching remote and unfamiliar experiences of unknown teachers. These models will now be described, in turn, and compared and contrasted.

In their study, Sherin and van Es (2009) facilitated a video-based professional development, video clubs, where they met with teachers from the same schools monthly across one school year. During these meetings, they engaged in watching video clips from the participating teachers' classrooms to ensure involvement and motivation, and focus attention, in particular, to student mathematical thinking. Researchers as facilitators took responsibility for choosing the video clips that would be shown at the meetings. Findings revealed that engaging in such video clubs, watching teachers in similar backgrounds (from the same school) supported the teachers' development of professional vision, i.e., their ability to notice and interpret significant features of classroom interactions. Regarding the kinds of video clips that may be useful to watch in a professional development context, Sherin, Linsenmeier and van Es (2009) identified three characteristics of video clips that may promote productive discussion of student mathematical thinking: (a) the extent to which the video clip provides windows into student thinking; (b) the depth of thinking shown; and (c) the clarity of student thinking shown in the video. Interestingly, they found that video clips that were high in depth did not always lead to productive discussions.

Second, is watching video of one's own teaching, and engaging in self-reflection (Hollingsworth & Clarke, 2017; Geiger, et al., 2016), mostly in a form of video-stimulated recall to analyze and reflect on practice. The teacher and a researcher watch video clips of the teacher's own classroom with the aim of allowing the teacher to recall what had transpired in the lesson, and to stimulate discussion. Two questions are normally asked by the researcher as initial framing of the discussion: "What did you see?" and "What are your thoughts about what you saw?" (Hollingsworth & Clarke, 2017, p.467). Previous research findings reveal that, as a tool for teacher professional learning, watching teacher's own video clips and engaging with the teacher in a discussion about her own practice provide an effective mechanism for identifying and examining teachers' thoughts and decisions, and the reasons for their actions in the classroom (Muir, 2010; Muir & Beswick, 2007). Consequently, reflection-oriented questions that probe teachers' thought process, create opportunities for teacher learning. There is also evidence that this approach has been effective in enhancing mathematics teaching (Geiger et al., 2016), as well as providing a powerful context for reflection on and in practice (Geiger et al., 2016). Teachers often regarded the video clip of their own teaching as a catalyst for their learning as it both provided insight into their teaching practice and demanded a response from them.

Third, (and different from the first two) teachers watch video of remote and unfamiliar experiences of unknown teachers (Karsenty & Schwarts, 2016; Arcavi & Schoenfeld, 2008), sometimes even from another culture, with the hope that enough detachment is created to allow for free speaking and commenting. Karsenty and Arcavi (2017) argue that watching unknown teachers allow for the emergence of "vicarious experiences", which are the indirect explorations of one's own perceptions, ideas and credos through the observation of a third person. Vicarious experiences are a powerful mechanism for achieving reflection; comparing and contrasting one's own practices with the practices observed on the screen, are almost unavoidable (Karsenty, 2018). Referring to cross-cultural video cases, Hollingsworth and Clarke (2017) note that watching lessons from one's own culture might seem to teachers as 'too familiar' and hence reduce the power of video to catalyze teacher reflection, whereas while watching lessons from another culture, teachers' assumptions about what is accepted and expected no longer apply, and the unfamiliarity of what is viewed challenges teachers' perspectives about what is a competent teaching practice. Hollingsworth and Clarke argue that in this kind of video-based PD "teachers are more inclined to interrogate the videotape and, by implication, their own practice" (Hollingsworth and Clarke, 2017, p. 460).

Depending on how they are used, video recordings of teachers' own classrooms and recordings of other classrooms can provide teacher learning opportunities to see and consider teaching practices in different ways. Question of agency naturally arises in the model of watching teacher's own video clips. Who should decide what material will be filmed and watched? To address this question, Hollingsworth and Clarke (2017) designed an observation framework that provides teachers with opportunity and agency for considering different elements of their practices, and use it to select foci for their own professional learning. The observation framework allows negotiated observation foci, which balances between the researcher dictating what will be watched, and simply allowing the teacher to take control.

We note here but have not explored the use of various platforms for creating and viewing animations of classroom events (either fictional or re-constructed), for example LessonSketch (Herbst, Aaron & Chieu, 2013) and GoAnimate (Estapa et al., 2018). There is also use being made of student-filmed video (e.g., from head mounted cameras). We imagine these will be growing areas of interest and research in the future.

4. WHAT FRAMEWORK IS BEING USED FOR WATCHING?

There are a range of methodological approaches to using video, including as stimulated recall (e.g., Geiger et al., 2016), as alluded to in our account of past reviews of research. It is not our intention here to conduct a systematic review of frameworks. It is also not our intention to attempt to generalise across video use, say, within a country or within the use of a particular framework. Instead we will set out four frameworks and particular uses made of them. These frameworks were chosen because they were ones known to us and which we see as common or influential in the field. We firstly set out these frameworks (Schoenfeld's ROG; Mason's noticing; the VIDEO-LM framework; and the TDS) and then we will consider similarities and differences.

Resources, Orientations and Goals

Barton et al. (2015) describe the development of a professional learning model developed for university-level mathematics teachers, grounded in Schoenfeld's (2008; 2010) theoretical framework for goal-oriented decision-making in school teaching. Schoenfeld's (2010) framework employs the resources, orientations, and goals (ROG's) that teachers call upon to investigate how these are linked to in-the-moment decisions made in the classroom.

Resources, according to Schoenfeld (2010), primarily comprise teacher procedural and conceptual knowledge, along with heuristics, or problem-solving strategies. This may include knowledge of the subject material; knowledge of the levels of the students; knowledge of how the content fits in the overall course structure, consistent with other models (e.g. Ball, Thames & Phelps, 2008), but also available physical entities, such as pens and whiteboards, textbooks, models, digital technology and attributes such as time and energy.

Orientations is "an inclusive term encompassing a group of related terms such as dispositions, beliefs, values, tastes and preferences" towards mathematics teaching and learning (Schoenfeld, 2010, p.29). These might for example include the use of technology, student-centered or inquiry-based learning approaches, or the relative importance of skills and conceptual understanding. Orientations are crucial, since "What people perceive, how they interpret it, and how they prioritize the ways they might respond to what they see are all shaped in fundamental ways by their orientations." (Schoenfeld, 2010, p.44).

The goals may be immediate or long term, conscious or unconscious. They may for example include 'to keep the students engaged throughout the lesson', 'to appreciate the interconnectedness of mathematics', or 'to develop conceptual understanding of content or techniques'.

The relationship between resources, orientations and goals, views the orientations as shaping the goals, setting "the prioritization of the goals … and the prioritization of the knowledge that is used in the service of those goals" (Schoenfeld, 2010, p. 29). Once the teacher has oriented themselves and set goals for the current situation, they decide on the direction to take and call on their repertoire of resources to achieve the goals (Barton et al., 2015). Thus, decision-making is viewed as complex interactions of an individual's ROGs for a given situation, a fine balance between mathematical and pedagogical prerogatives, with the quality of the decision-making affecting how successful a teacher is in attaining their goals.

Schoenfeld (2008, 2010) conducted a fine-grained analysis of entire lesson videos to develop his framework. However, as described earlier (Dimension 1), the study at the University of Auckland (Barton et al., 2015) examined small, selected snippets of lessons to encourage development of the skills of reflective practice for university mathematics lecturers. The ROGs have been used effectively in this study as a basis to analyse for example why a teacher chose a particular approach to the lesson, or as the comparative value of an unplanned decision to deviate from the lecture plan to improve understanding, versus the mathematical value in pursuing this (Paterson, Thomas & Taylor, 2011). Hannah, Stewart and Thomas (2011) have used the ROG framework to analyse lecturer practice and describe the role of orientations and goals in shaping this. As described earlier, the ROG framework has also proved valuable in provoking critical reflection and changes in practice for new mathematics lecturers (Paterson & Evans, 2013).

The Art of Noticing

The practice of having teachers watch videos of their own or other teachers' lessons can have mixed benefits. Hill, Beisiegel, and Jacob (2013) for example describe two studies where it seemed that while teachers watching their own videos were engaged and reported high levels of satisfaction, on the other hand they were sometimes non-critical of their own practice, noticed few consequences for student learning and showed limited improvement on some other objective measures (e.g., skill acquisition). Beswick and Muir (2013) found similar results in their trial using video excerpts of mathematics teaching with pre-service primary teachers. While the pre-service teachers were positive about the use of video excerpts in their course, they often struggled to see beyond readily evident aspects of teaching, such as the use of concrete materials. Most participants reported that the videos showed teaching that was similar to teaching they had observed and that confirmed their existing beliefs (Beswick & Muir, 2013). An awareness of such mixed benefits can be seen in the development of a framework for "productive noticing" by Choy (2013; 2014). The act of 'noticing' suggested here is a commonly used notion in observations of teacher practice. Mason (2002; 2010) conducted the foundational work in this area and views noticing as central to all mathematical teaching practices. Schoenfeld (2011) likewise sees noticing as essential for improving teaching. For Mason, noticing is seen as a set of practices that enhance teachers' awareness of classroom activity, and prompt them to act and respond differently during teaching situations. Choy (2013) however notes that there are different notions of mathematical noticing; some researchers for example focus solely on what teachers attend to (Sherin, Russ & Colestock, 2011). One commonly used operationalisation of the idea of noticing, views it as consisting of two main processes: "attending to particular events and making sense of events in an educational setting" (Sherin, Jacobs, & Philipp, 2011, p.5).

Several frameworks or protocols have been devised to encourage and examine effective noticing, for example in the study with pre-service teachers by Beswick and Muir (2013) described earlier. Ho et al. (2015) have also used lesson videos with pre-service teachers. They asked the pre-service teachers to critically reflect on their observations in respect of their own experiences and beliefs about learning, directed through a series of questions formulated around the lessons demonstrated in the videos. They then used the framework of five reflective-noticing levels developed by Manouchehri (2002) to analyse the responses. These levels are: (i) *describing*, (ii) *explaining*, (iii) *theorizing*, (iv) *confronting* and, (v) *restructuring*. Ho et al. (2015) conclude that in general, pre-service teachers mostly engaged in high-level reflections, which they attribute to a number of factors, including the "authenticity of the classroom practices captured in the videos", the innovative nature of the instruction, the guided opportunities for reflection within the design of the program, and its accessibility online (Ho et al., 2015, pp 19-20).

Choy (2014) proposes a "three-point" protocol to provoke what he coins "teachers' productive mathematical noticing". He suggests that productive noticing occurs when teachers are able to:

> attend to specific details related to the key point, difficult point or critical point that could potentially lead to new responses;
> relate these details to prior knowledge and experiences to gain new understanding for instruction (key point and difficult point);
> combine this new understanding to decide how to respond (critical point) to instructional events.

This characterization of productive mathematical noticing uses the *three points* not only to direct teachers' attention to specific details of what they notice, but also to highlight the need to connect the critical point of a lesson to the key point and difficult point (Choy, 2014, p. 299). Choy (2014) concludes that teachers' noticing is most productive when it goes beyond the specificity of what teachers notice to include justification based on what they have noticed about students' thinking. His study demonstrates the usefulness of this construct in analysing what teachers notice when planning (p. 297).

The framework for using video that is perhaps most closely based on Mason's (2002) insights about noticing was developed by Mason and Jaworski at the Open University in the UK (Jaworski, 1990). This method has more recently been described and developed by Coles (2013). Coles (2019) suggests that the distinction between making an observation about a video recording, compared to making a judgment or evaluation about a video recording, is one that could be usefully applied and used across frameworks of using video. The distinction is based on Mason's (2002) notions of offering accounts *of* phenomena (which aim to be ones that can be agreed by different observers, e.g., what was said by the teacher at a particular time on the video) and accounts *for* phenomena (which involve values and interpretation, e.g., why someone said what they did on a video). The method works by having a short continuous clip of a video recording to observe (maximum 4 minutes) and initially discussing only a reconstruction of the events that took place (i.e., avoiding evaluation). Only after a prolonged period of time in this initial phase, will discussion move to an interpretation of events. Coles (2019) provides evidence that the distinction between observation and judgment, in itself, is one that is potentially generative for teachers.

The VIDEO-LM framework for video-based peer discussions

The VIDEO-LM project (Viewing, Investigating and Discussing Environments of Learning Mathematics), developed at the Weizmann Institute of Science in Israel, is aimed at enhancing secondary mathematics teachers' reflection skills and their mathematical knowledge for teaching (MKT, as defined by Ball et al., 2008). The project uses a collection of videotaped lessons (about 70 lessons filmed in Israel between 2012-2016, and additionally several lessons from other countries, with Hebrew subtitles, such as the famous TIMSS Japanese lesson *Changing shapes without changing area*). These videos serve as learning objects and sources for discussions with groups of teachers. Since teachers participating in VIDEO-LM PD courses watch videos of teachers unknown to them, the videos are taken as "vicarious experiences", in the sense described earlier under the dimension of "what is being watched", i.e., an indirect exploration of one's own perceptions on teaching, through the observation of "remote" teaching events (Karsenty & Arcavi, 2017). The PD sessions are usually conducted in groups of 10-15 teachers, in a supportive atmosphere which does not focus on evaluative feedbacks (Karsenty, 2018). To date, more than 75 VIDEO-LM PD courses (30 hours each) were held across Israel, including the Arab, Druze and Ultra-orthodox sectors, and more than 1000 teachers participated in these courses.

At the heart of the project is a unique analytic framework (SLF; see Table 1) used as a directive tool for teachers' observation of the videos and for the peer-discussions following it. This framework is inspired by Schoenfeld's (1998; 2010) ROG model, as described earlier in the section *Resources, Orientations and Goals*. The VIDEO-LM framework also draws on a practical implementation model: Lesson Study (e.g., Fernandez & Yoshida, 2004). The VIDEO-LM designers came to understand that their video-based PD should meaningfully involve explicit teachers' awareness to resources, orientations and goals in and around practice, and bring to the fore what (consciously or unconsciously) underlies the conceptualization of practice and its implementation. Initial experimentation with these ideas, conducted by Arcavi and Schoenfeld (2008), yielded analytical tools with which mathematics teachers can reflect upon their own practice while watching videotaped lessons of colleagues unknown to them. In VIDEO-LM, these tools were refined and extended into a framework consisting of six 'viewing lenses': (1) mathematical and meta-mathematical ideas around the lesson's topic; (2) explicit and implicit goals that may be ascribed to the teacher; (3) the tasks selected by the teacher and their enactment in class; (4) the nature of the teacher-student interactions; (5) teacher dilemmas and decision-making processes; and (6) beliefs about mathematics, its learning and its teaching as inferable from the teacher's actions and reactions. Table 1 details the focus of activities around each of these lenses.

VIDEO-LM viewing lenses	The focus of activities around each lens
Mathematical and meta-mathematical ideas	Scanning the space of relevant ideas, concepts and procedures, as well as meta-mathematical ideas (e.g., one counter example is sufficient to refute a conjecture) that may be associated with the lesson's topic.
Explicit and implicit goals	Attributing goals that may underlie the teacher's actions or decisions, on the basis of what was observed in the video. Rather than "scientifically verifying true goals", the aim is to sharpen awareness to different possible goals and negotiate the pros and cons of preferring certain goals over others.
Tasks and activities	Conducting an "a posteriori task analysis": discussing features of the task and how it was enacted by the filmed teacher and students. Noticing if and when it develops differently than expected.
Interactions with students	Observing and analyzing if and how the filmed teacher: poses further questions to those of the task; listens to (or ignores) comments or difficulties raised by students; manages discussions; delegates responsibilities in the process of knowledge generation.
Dilemmas and decision-making	Uncovering situations of dilemma (i.e., when there is no evident optimal course of action) that the filmed teacher seemed to have faced during the lesson. Discussing the decisions taken in order to resolve these dilemmas, and their consequent tradeoffs.
Beliefs about mathematics teaching	Eliciting orientations, beliefs and values that may be attributed to the filmed teacher on the basis of the video. Unpacking implicit messages that may be conveyed to students through the teacher's communications and actions.

Table 1: The six-lens framework (SLF) used in VIDEO-LM

This six-lens framework (henceforth: SLF) and its utilization in VIDEO-LM PD sessions are characterized by the following features (Arcavi & Karsenty, 2018):

- *The teacher is at the centre*. This does not mean abandonment of the consensually adopted principle of student-centred pedagogies. Rather, in the pursuit of the goal to promote teacher reflection, the observations and discussions are centred on the filmed teacher's actions, utterances, choices, etc., and within this focus, enactment of student-centred approaches is certainly also included.
- *Acknowledging different "best" practices*. The underlying belief here is that although there may be some agreed features of good teaching, for different teachers there may exist different best practices, contingent upon personal traits, contextual factors and cultural settings. Thus, the choice of lessons for observation is based on whether they can serve as springboards for meaningful discussions on different aspects of practice, rather than on alignment with criteria of how teaching should look.
- *Shunning evaluation*. In line with the works of Jaworski (1990) and Coles (2013), the use of SLF attempts to establish non-judgmental norms of discussion, through the redirection of highly evaluative comments into "issues to think about".

- *Whole lessons*. Discussions are conducted around whole lessons, or large pieces of a lesson, rather than short episodes. A whole lesson enables teachers to follow the development of a topic from opening to closure, depicting a 'story' rather than an isolated event.
- *Focus on the mathematics*. Although mathematics teaching shares many general features of the teaching of any subject matter, it has unique specific characteristics. SLF centres on what lies at the heart of mathematics teaching, that is, mathematical concepts, processes, and meta-mathematical issues, and de-emphasizes generic issues of teaching not closely tied to the mathematics (e.g., general classroom management).

Research shows that repeated and guided use of SLF in PD sessions supports the development of a reflective language, with which teachers can engage in deep discussions about core issues embedded in the mathematics teaching profession (Karsenty & Arcavi, 2017; Karsenty, 2018). There is also evidence of teachers' MKT growth (Karsenty, Arcavi, & Nurick, 2015). Several possible mechanisms were suggested to explain these outcomes (Karsenty, 2017). The group's research shows also that these effects are not limited to teachers watching unknown teachers from their own culture (i.e., Israeli teachers watching Israeli lessons), but that they also take place, and are sometimes even intensified, when teachers watch lessons from a culture very different than their own (Karsenty & Schwarts, 2016).

Another finding relates to the gradual exposure to VIDEO-LM norms. These include accepting a basic working assumption that the filmed teacher is acting in the best interest of his/her students; practicing the exercise of "stepping into the shoes" of the filmed teacher in an attempt to understand his/her goals, decisions and beliefs; maintaining a non-evaluative and respectful conversation; justifying suggested alternatives not as better or worse courses of action, but rather as a way to enrich the span of possible options, while considering the trade-offs involved. There is evidence that in time, teachers internalize these norms and judgemental comments gradually decrease (Karsenty, Peretz, & Heyd-Metzuyanim, 2019).

The Theory of Didactical Situations

We review, more briefly, research in France on video that has been reported on, in connection to both pre-service and in-service teachers of mathematics (Coles et al., 2018). A range of variation can be observed within an overall orientation towards the Theory of Didactical Situations (TDS) (Brousseau, 1997) and the Double Approach (Robert & Rogalski, 2002). At the most research-oriented, some work with video is conducted with the aim of introducing pre-service teachers to elements of the TDS. In other words, the TDS is used as a framework to support teachers in analysing the events on the video. The video chosen needs to have quite specific characteristics that make what takes place amenable to analysis via the TDS, requiring a detailed and specific delineation of intended mathematical subject knowledge to be gained by students. A less research-oriented use of video is one in which the actions of the teacher educator are informed by the TDS and the processes of the TDS are used, but where the focus is on effective teaching strategies rather than elements of the TDS itself.

However, an important idea within the TDS that would be present across all uses of video is that of the *a priori* analysis of mathematical tasks. An a priori analysis involves detailed work on the mathematics that students might be offered in the classroom. There is a need to identify likely existing knowledge and how this knowledge will be mobilised in a novel situation in a way that will lead to the development of new knowledge. The a priori analysis involves a mapping from the known (to students) to the unknown and an identification of likely barriers or difficulties that students might face. This a priori analysis serves a purpose of sensitising viewers to what is on any video recording. The need for such an analysis places a constraint on the kinds of video that are suitable for video watching, i.e., the classroom needs to be one where there is a sense of the teacher aiming to mobilise students' existing knowledge in order to meet a challenge that entails the development of a specific new item of knowledge.

Similarities and differences across frameworks

We initially consider the origins of the frameworks reviewed above. ROG and VIDEO-LM share a sense of having come out of a need to support observations of classrooms. In other words, they are frameworks for analysing classrooms which is, of course, quite appropriate given the intention to support the observation of video recordings of classrooms. A distinction here is that ROG is potentially applicable across school subjects, whereas VIDEO-LM has mathematics as a core and explicit focus.

The TDS is a theory about the learning of mathematics in classrooms and when it is employed in the context of video with teachers, there are aims around making elements of this theory explicit and operational for teachers. Using video guided by the TDS therefore shares some similarity with use of ROG and VIDEO-LM in that it provides a framework for analysing a mathematics classroom, which can be used by teachers to discuss video recordings of lessons and, in time, support practice in classrooms.

As Brown (2018) has commented, the discipline of noticing was conceived by Mason (2002) as a method of introspection; it is often however taken up by researchers (e.g., Sherin, Jacobs & Philipp, 2011) as a method for analysing the noticing of others. This latter use of noticing is then quite similar in purpose to those described for the frameworks above, i.e., the discipline of noticing becomes a framework for analysing the work of mathematics teachers on video. Where noticing is used in a sense that is closer to Mason's articulation (e.g., Coles, 2013) then the framework that is operationalised is a distinction between observation and interpretation (accounts of/accounts for). This is not a framework that has come out of the particularities of mathematics teaching and so, like ROG, can potentially be used across school subjects.

Across articulations of working with video from within the different frameworks, we identify some commonalities around the importance of: (a) working on the mathematics that is being done by students observed in the video; (b) having some kind of focus on the detail of events; (c) the need for establishing shared norms for discussion; (d) the need for a shared language for talking about the video.

We do not in any sense want to propose that some frameworks are "better" than others nor that there would be any benefit in somehow working towards consistent frameworks. The multiplicity of ways of working with video seems to us a strength of the field. What we have been concerned to do in this section is point to some of the similarities and differences and their affordances and constraints, in order to support decision making around the use of video.

5. WHO LEADS AND GUIDES THE USE OF VIDEO?

It seems self-evident that simply viewing video is insufficient to improve practice. Rather, "accompanying high-quality support is a prerequisite if video is to realise its transformative potential in supporting in-service teachers and in improving classroom practice" (Major & Watson, 2018, p. 65). Because of this, video observation and analysis carried out by teachers is most commonly guided by a facilitator, who can be a researcher, a teacher educator, or a lead teacher. However, we know relatively little about how these actors gain the skills to lead or guide video-based PD sessions although studies have gradually accumulated in recent years (e.g., Borko, Koellner & Jacobs, 2014; Lesseig et al., 2017; van Es, Tunney, Goldsmith & Seago, 2014). Issues around the facilitation of video-viewing connect with broader questions about the ways in which teacher educators learn and develop. How do the skills required to learn from video relate to those needed to learn from observing live classrooms? To what extent are these skills transferable?

A significant contribution to thinking in this area is an articulation of a framework of facilitator 'moves' (van Es et al., 2014) defined as shown in Table 2.

Central facilitation practice	**Facilitation move**	**Definition**
Orienting the group to the video analysis task	Launching	Pose general prompts to elicit participant ideas
	Contextualizing	Provide additional information about the classroom context and mathematics lesson
Sustaining an inquiry stance	Highlighting	Direct attention to noteworthy student ideas in the videos
	Lifting up	Identify and important idea that a participant raised in the discussion for further discussion
	Pressing	Prompt participants to explain their reasoning and/or elaborate on their ideas
	Offering an explanation	Provide and interpretation of an event, interaction, or a mathematical idea, from a stance of inquiry

	Countering	Offer an alternative point of view
	Clarifying	Restate and revoice to ensure common understanding of an idea
Maintaining a focus on the video and the mathematics	Redirecting	Shift the discussion to maintain focus on the video analysis
	Pointing to evidence	Contribute substantively to the conversation, using evidence to reason about teaching and learning with the video
	Connecting ideas	Make connections between ideas raised in the discussion
Supporting group collaboration	Standing back	Allow the group members time to discuss and issue
	Distributing participation	Invite participants to share different ideas based on who is (and is not) participating
	Validating participant ideas	Confirm and support participant contributions

Table 2: Framework for facilitation of video-based discussion (From Table 1: Van Es et al., 2014, p. 347)

It could be argued that the moves that van Es et al. (2014) described apply to any facilitated discussion whether or not prompted by video viewing. That is, these are arguably things that teacher educators do in all of their work with pre-service or practicing teachers. Questions about how facilitators of teachers' video viewing acquire the necessary knowledge and skills, therefore, mirror those asked more generally about the ways in which mathematics teacher educators (MTEs) learn and develop. The increasing body of literature conceptualizing the knowledge needed by MTEs can shed light upon the knowledge that facilitators need in order to be able to make the moves that van Es et al. list in ways and at moments that will enhance the discussion. Many of these conceptualizations of MTE knowledge are extensions or meta-versions of models of the knowledge required by school mathematics teachers in which school mathematics is replaced as the relevant content by the knowledge that teachers need to teach school mathematics (Beswick & Goos, 2018). The importance of the knowledge that underpins facilitators' capacity to effectively make the moves that van Es et al. (2014) described also answers possible questions about how these moves may or may not be specific to facilitating video-viewing of mathematics teaching with teachers of mathematics. Although mathematics is mentioned in only one central practice and in none of the moves, the framework was derived from observations of video viewing facilitation as part mathematics teaching focused professional development (van Es et al., 2014) and so the extent to which it might apply to other subject areas would need to be investigated. Regardless of whether or not it is generic in the sense of applying to any video-viewing activity, the knowledge that underpins the moves, that allows the facilitator to know which contextual information is most salient, which student ideas are worth highlighting, which participant contributions warrant foregrounding, what constitutes evidence of student thinking, and so on, are deeply mathematical questions in the context of viewing a video of mathematics teaching and require the knowledge of an MTE.

The ways in which MTEs acquire the knowledge they need is typically through reflection on their practice (Beswick & Goos, 2018), sometimes alone (e.g., Krainer, 2008) and sometimes supported by more experienced colleagues (e.g., Chen, Ling, & Yang, 2018; Masingila, Olanoff, & Kimani, 2018; Zazkis & Mamolo, 2018). Several researchers in this field have also pointed to opportunity for MTEs to learn about their own practice from their research with teachers (e.g., Chapman, 2008; Chen et al., 2018; Jaworski, 2003). There are also a small number of studies that have reported on the impacts of formalized programs designed to enhance MTEs' capacities for their roles (e.g., Childs, Hillier, Thornton, & Watson, 2014).

Beswick and Muir (2013) reported on the use of a video viewing protocol for pre-service teachers that was designed to allow a tutor to facilitate pre-service teachers viewing of video clips, and also to be used online without a facilitator. The protocol comprised time points at which to pause the video along with questions to stimulate discussion and to assist the pre-service teachers to focus on the students' thinking. They reported only on the face-to-face facilitation and acknowledged that the tutor's facilitation task was a demanding one. They observed that further refinement of the protocol, as well as providing the pre-service teachers with more practice at identifying evidence of student understanding would be helpful. Yung, Yip, Lai, and Lo (2010) similarly concluded that the challenges of facilitating video-based teacher learning, including in sophisticated online environments that offer supports for discussion including the ability to annotate video, appear to have been under-estimated.

In studies where the facilitators have not been closely involved in designing the research around the video-based professional development (e.g., Beswick & Muir, 2013; Santagata, 2009), considerable time and effort is needed to ensure that the facilitator's role is adequately supported. Supports include meetings with the researchers to identify the learning outcomes that it is intended the teachers will achieve (Santagata, 2009), and ensuring the facilitator has a detailed and carefully structured plan for their facilitation (Beswick & Muir, 2013; Santagata, 2009).

As alluded to in relation to Dimension 2 the increasing availability of online video of teaching that is either accompanied only with facilitator notes, that may or may not be used as intended or at all, or for which there is no facilitation support available raises important questions about what teachers viewing these videos might learn. These situations are, however, not unlike more familiar situations such as when pre-service teachers observe live teaching in the practicum components of their programs, perhaps without adequate preparation to make sense of what they are seeing.

Borko et al. (2014) suggested that in contexts like video clubs where the objectives of video viewing are driven by the participants, facilitators need to draw upon their knowledge of the context and participants. In some of these situations a formal facilitator may not be needed. When, for example, all participants in a video-viewing group are sufficiently skilled at providing appropriate critique (e.g., Barton et al., 2015) facilitation of the viewing can be a shared task led by the individual whose teaching is represented in the video. In these contexts, the subject of the video typically selects the excerpt to be viewed and may suggest a focus for the critical discussion that occurs in response to the viewing.

The facilitation of video is one of the growing areas of research in the context of video use with mathematics teachers and we have pointed to some of the open questions, above. The suggestion from the research in this area so far is that the skills needed for successful facilitation are independent of who is doing that facilitation. We are conscious, however, that the bulk of this research to date (at least, that we are aware of) has been conducted in the relatively wealthy countries, often with a shared language of English and therefore is not sensitive to the role of influence of different cultural contexts and traditions of teacher leadership or development.

CONCLUSIONS AND QUESTIONS FOR FUTURE RESEARCH

In concluding this Research Forum document, we will attempt to summarise and tabulate the five dimensions of video use we have explored above and the suggestions for future areas of research (see Table 3).

Dimension	**Variation**	**Future areas**
What is the purpose of using video?	disseminating reform materials or modelling certain teaching approaches; developing teachers' ability to notice and understand students' mathematical thinking; encouraging reflection on practice; providing feedback on the quality of teaching and offering ways to improve it	following the development of teachers in their classrooms over time; assessing the success of a PD initiative; supporting the upscaling of PD programs
Who watches the video?	pre-service teachers; practicing teachers; teacher leaders; facilitators who work with teachers; mathematics education researchers; mathematicians; mixed audiences	online learning and MOOCs, which blur the line between face-to-face and online learning for pre-service and in-service teachers and others
What is being watched?	teachers with similar backgrounds; teachers' own teaching; remote and unfamiliar experiences of unknown teachers	teachers creating or engaging with animations of teaching scenarios; student-filmed perspectives

What framework is being used for watching?	Resources Orientations Goals; Noticing; VIDEO-LM; Theory of Didactical Situations	theories that cross levels of the system
Who leads and guides the use of video?	a researcher; a teacher educator; a lead teacher; the teacher whose lesson is being observed	how do facilitators gain skills?; how do the skills required to learn from video relate to those needed to learn from observing live classrooms?; to what extent are these skills transferable?

Table 3: Summary of dimensions of video use

We want to highlight that many of the questions being flagged as relevant now, within the field of video use, point us towards wider issues of mathematics teacher learning or the preparation of pre-service mathematics teachers. For instance, in the use of MOOCs and animations of lesson scenarios, we see a blurring of the boundaries of what it means to work with video compared to a PD task or course not linked directly to a classroom. Video recordings and animations can now be mobilised with relative ease in many parts of the world and used in learning environments or PD sessions. Or, to take another example, our understanding of the roles and moves of facilitators working with mathematics teachers on video begins to feel like it has little separation from the kinds of skills, roles and moves needed to work with mathematics teachers in any scenario. We see the developments described here as positive ones and it is perhaps the case that interest in the use of video with mathematics teachers is drawing focus and attention towards under-researched questions about ways of working productively with mathematics teachers more generally.

We are mindful of the quotation cited earlier, from the most recent review of work on using video, namely, “the field is reaching saturation point and we are approaching a limit on what might be learned from [small scale qualitative] research” (Major & Watson, 2018, p. 65). We hope that the Research Forum itself will be an opportunity for some of the new thinking needed and, as Major and Watson suggest, linking professional development through to changes in classrooms and student learning. There is a hope here that professional development, using video, will lead to improved skills or knowledge on the part of the teacher, that will lead to teaching that is more effective and that in turn will result in greater student attainment. The complexity of interactions is immense, within this aspirational story of cascading change.

Given that the aims of video use centre around change for teachers, leading to change in the classroom and change for learners, some theory of change or learning must be present, either implicitly or explicitly. When thinking about research that might map work with facilitators of video, into work with teachers of mathematics, into the classroom and the learning of students, therefore, a legitimate question is whether the same theory of change or learning is being used across all those different sites.

Coles et al. (2018) raise the question of whether the theories of learning espoused by facilitators are the same as those enacted in training sessions and whether they are the same as, or different from, any explicit theoretical ideas about change which it is the intention that teachers adopt or use, during or as a result of work with video. These questions become important when we start to think about the use of video and its link into the classroom, simply in terms of the coherence of what is being proposed or researched. Coles et al. (2018) propose a meta-framework of: espoused, enacted and intended theories, in order to frame questions about the theory or theories of change being mobilised during work with teachers on video and into the classroom. It is an open question as to whether there might be any common results from programmes where theories of change are either more or less explicit, or any common results from programmes where the theory of change is either more or less consistent with the framework for analysis of video.

An implicit principle that we observe across work within the ROG, VIDEO-LM, noticing, and TDS frameworks is that the distinctions or frameworks operationalised in working with video will be productive ones for teachers when planning or teaching. The sense from Major and Watson (2018) of questioning the impact of these distinctions on teachers' classroom practice seems therefore to be an important one. Is there anything we can learn from the ways in which teachers do, or do not, take up the distinctions offered in the course of work with video? Are some distinctions, when set up in the context of work with video, more generative than others, in terms of use in the classroom? Are some distinctions easier for teachers to adopt, than others? We also would like to raise the question of whether more can usefully be articulated about the theory of change within video use. Are there differences in how it is envisaged that work that takes place in the context of viewing a video recording will translate into teachers' actions in their own classrooms?

In the Research Forum meetings during the Psychology of Mathematics Education (PME) annual conference, we will be working with participants on selections of video clips, drawn from each of our respective lines of research. We will focus on the following questions: 'What are the dimensions associated with the use of video in contexts of mathematics teachers' learning and development?', drawing on the ideas presented above, leading to consideration of: 'What are the implications for video use?'. And, in the second session: 'What do we know about effective facilitation of discussion using video with teachers of mathematics?', leading to consideration of: 'What are the implications for researching video use?'. We hope that the Research Forum will provide new insights into these questions and impetus to work towards a further joint publication, including any participants who are interested.

References

Arcavi, A., & Karsenty, R. (2018). Enhancing mathematics teachers' reflection and knowledge through peer-discussions of videotaped lessons: A pioneer program in Israel. In N. Movshovitz-Hadar (Ed.), K-12 *Mathematics Education in Israel - Issues and*

Challenges (Chapter 33, pp. 303-310). Series on Mathematics Education (Volume 13). Singapore: World Scientific.

Arcavi, A., & Schoenfeld, A. H. (2008). Using the unfamiliar to problematize the familiar: The case of mathematics teacher in-service education. *Canadian Journal of Science, Mathematics, and Technology Education, 8*(3), 280-295.

Australian Government, Department of Education and Training. (2019). *Support for Science, Technology, Engineering and Mathematics (STEM).* Retrieved from https://www.education.gov.au/support-science-technology-engineering-and-mathematics

Australian Institute for Teaching and School Leadership. (2018). *The rise of online initial teacher education: What do we know?* Retrieved from https://www.aitsl.edu.au/docs/default-source/research-evidence/spotlight/spotlight_ite_online__.pdf?sfvrsn=22a8f73c_2

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Barton, B., Oates, G., Paterson, P., & Thomas, M. O. J. (2015). A marriage of continuance: Professional development for mathematics lecturers. *Mathematics Education Research Journal, 27*(2), 147-164.

Beisiegel, M. (2016). Aligning mathematics GTA training with research findings. Presented at the *Joint Mathematics Meeting of the American Mathematical Society & the Mathematical Association of America*, Seattle, 6-9 January 2016. Retrieved from http://jointmathematicsmeetings.org/amsmtgs/2181_abstracts/1116-f1-1492.pdf

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education, 21*(5), 417-427.

Beswick, K., & Muir, T. (2013). Making connections: Lessons on the use of video in pre-service teacher education. *Mathematics Teacher Education and Development, 15*(2), 27-51.

Blomberg, G., Stürmer, K., & Seidel, T. (2011). How pre-service teachers observe teaching on video: Effects of viewers' teaching subjects and the subject of the video. *Teaching and Teacher Education, 27*, 1131–1140.

Borko, H., Carlson, J., Mangram, C., Anderson, R., Fong, A., Million, S., Mozenter, S., Villa, A.M. (2017). The role of video-based discussion in a model for preparing professional development leaders. *The International Journal of STEM Education*, *4*(1). DOI: 10.1186/s40594-017-0090-3.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*(2), 417-436.

Borko, H., Koellner, K. & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *The Journal of Mathematical Behavior, 33*, 149-167.

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice-based professional development programs. *ZDM – The International Journal of Mathematics Education, 43*(1), 175-187.

Boston, M., Bostic, J., Lesseig, K., & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. *Mathematics Teacher Educator*, *3*(2), 154-175.

Brophy, J. (Ed.). (2004). *Using Video in Teacher Education* (Advances in research on teaching, Vol. 10). Elsevier: Oxford, UK.

Brousseau, G. (1997), *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques* (1970-1990). New York: Springer.

Brown, J. (2018), Teacher noticing and teacher framing in teacher talk about a mathematics classroom. In F. Curtis (Ed.), *Proceedings of the British Society for Research into Learning Mathematics 38*(3), November 2018, pp.1-6.

Chapman, O. (2008). Mathematics teacher educators' learning from research on their instructional practices. In B. Jaworski & T. Wood (Eds.), *International Handbook of Mathematics Teacher Education: Vol. 4: The Mathematics Teacher Educator as a Developing Professional* (pp. 110–129). Rotterdam: Sense Publishers.

Chen, J.-C., Lin, F.-L., & Yang, K.-L. (2018). A novice mathematics teacher educator–researcher's evolution of tools designed for in-service mathematics teachers' professional development. *Journal of Mathematics Teacher Education, 21*(5), 517-539.

Childs, A., Hillier, J., Thornton, S., & Watson, A. (2014). Building capacity: developing a course for mathematics and science teacher educators. In K. Maaß, B. Barzel, G. Törner, D. Wernisch, E. Schäfer, & K. Reitz-Koncebovski (Eds.), *Educating the Educators: International Approaches to Scaling-up Professional Development in Mathematics and Science Education* (pp. 47–55). Munster: WTM

Choy, B. H. (2013). Productive mathematical noticing: what it is and why it matters. In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics Education: Yesterday, Today and Tomorrow, Proceedings of the 36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 186-193). Melbourne: MERGA.

Choy, B. H. (2014). Teachers' productive mathematical noticing during lesson preparation. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, Vol. 2, pp. 297-304. Vancouver, Canada: PME.

Coles, A., Horoks, J., & Chesnais, A. (2018). Theory and the Role of the Mathematics Teacher Educator: A Comparison of Teacher Education Sessions in France and UK using Video. *Annales de didactique et de sciences cognitives* (Special Issue English-French, pp. 119-144). Retrieved from https://mathinfo.unistra.fr/irem/publications/adsc/volumes/#c14903

Coles, A. (2013). Using video for professional development: The role of the discussion facilitator. *Journal of Mathematics Teacher Education*, *16*(3), 165-184.

Coles, A. (2014). Mathematics teachers learning with video: the role, for the didactician, of a heightened listening. *ZDM - The International Journal of Mathematics Education, 46*(2), 267-278.

Coles, A. (2019). Facilitating the use of video with teachers of mathematics: learning from staying with the detail. *International Journal of STEM Education, 6*(1), 1-12.

Downey, J. (2008). "It's not as easy as it looks": Preservice teachers' insights about teaching emerging from an innovative assignment in educational psychology. *Teaching Educational Psychology, 3*(1), 1-13.

Estapa, A.T., Amador, J., Kosko, Weston, T., de Araujo, Z., & Aming-Attal, R. (2018). Preservice teachers' articulated noticing through pedagogies of practice. *Journal of Mathematics Teacher Education, 21*(4), 387-415.

Gaudin, C., & Chalies, S. (2015). Video viewing in teacher education and professional development: a literature review, *Educational Research Review, 16*, 41-67.

Geiger, V., Muir, T., & Lamb, J. (2016). Video-stimulated recall as a catalyst for teacher professional learning. *Journal of Mathematics Teacher Education*, *19*(5), 457-475.

Fernandez, C., & Yoshida, M. (2004). *Lesson Study: A Japanese Approach to Improving Mathematics Teaching and Learning*. Lawrence Erlbaum Associates.

Hall, I. & Wright, D. (2007). *Literature Review of the Use of Video as a Resource for Professional Development of Mathematics Teachers*. Newcastle, UK: The Research Centre for Learning and Teaching.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2011). Analysing lecturer practice: The role of orientations and goals. *International Journal of Mathematical Education in Science and Technology, 42*(7), 975–984.

Hannah, J., Stewart, S., & Thomas, M. O. J. (2013). Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematical Education in Science and Technology, 44*(4), 475–489.

Herbst, P., Aaron, W., & Chieu, V. M. (2013). Lesson*Sketch*: An environment for teachers to examine mathematical practice and learn about its standards. In D. Polly (Ed.), *Common core mathematics standards and implementing digital technologies* (pp. 281–294). Hershey, PA: IGI Global.

Hill, H. C., Beisiegel, M., & Jacob, R. (2013). Professional development research: Consensus, crossroads, and challenges. *Educational Researcher*, *42*(9), 476-487.

Hill, H. C., Blazar, D., Humez, A., Litke, E., Beisiegel, M., Barmore, J., Chin, M., Corey, D., Roesler, S., Salzman, L. R., Lucas, R., Braslow, D. & Rabinowicz, S. (2013). *Examining high and low value-added mathematics instruction: Can expert observers tell the difference*. Retrieved from http://repository.upenn.edu/cgi/viewcontent.cgi?article=1002&context=cpre_workingpapers

Ho, W. K., Leong, Y. H., & Ho, F. H. (2015). The Impact of Online Video Suite on the Singapore Pre-service Teachers' Buying-In to Innovative Teaching of Factorisation via AlgeCards. In S. F. Ng (Ed.), *Cases of Mathematics Professional Development in East Asian Countries* (pp. 157-177). Singapore: Springer.

Hollingsworth, H., & Clark, D. (2017). Video as a tool for focusing teacher self-reflection: Supporting and provoking teacher learning. *Journal of Mathematics Teacher Education*, *20*(5), 457-475.

Hundley, M., Palmeri, A., Hostetler, A., Johnson, H., Dunleavy, T. K., & Self, E. A. (2018). Developmental trajectories, disciplinary practices, and sites of practice in novice teacher learning: A thing to be learned. In D. Polly, M. Putman, T. M. Petty, & A. J. Good (Eds.), *Innovative Practices in Teacher Preparation and Graduate-Level Teacher Education Programs* (pp. 153-180): IGI Global.

Jaworski, B. (1990). Video as a tool for teachers' professional development. *British Journal of In-Service Education, 16*(1), 60-65.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics, 54,*249–282.

Karsenty, R. (2016). *Preparing facilitators to conduct video-based professional development for mathematics teachers: Needs, experiences and challenges*. Paper presented at the 2nd International Conference on Educating the Educators, Freiburg, Germany, November 2016.

Karsenty, R. (2017). How do mathematics teachers learn from videotaped lessons of unknown peers? Exploring possible mechanisms that contribute to change in teachers' perspectives. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *Proceedings of the 9th Annual International Conference on Education and New Learning Technologies* (pp. 1718-1728). Barcelona, Spain: IATED Academy.

Karsenty, R. (2018). Professional development of mathematics teachers: Through the lens of the camera. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited Lectures from the 13th International Congress on Mathematical Education* (pp. 269-288). Hamburg: Springer.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education, 20*(5), 433-455.

Karsenty R., Arcavi, A., & Nurick, Y. (2015). Video-based peer discussions as sources for knowledge growth of secondary teachers. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education* (pp. 2825-2832). Prague, Czech Republic: CERME.

Karsenty, R., Peretz, Y., & Heyd-Metzuyanim, E. (2019). From judgmental evaluations to productive conversations: Mathematics teachers' shifts in communication within a video club. To appear in the *Proceedings of the 10th Congress of the European Society for Research in Mathematics Education* (CERME-10), Utrecht, the Netherlands: CERME.

Karsenty R., & Schwarts, G. (2016). *Enhancing reflective skills of secondary mathematics teachers through video-based peer discussions: The case of cross-cultural learning.* Paper presented in TSG-50 (Topic Study Group on in-service education and professional development of secondary mathematics teachers), the 13th International Congress on Mathematical Education (ICME-13), Hamburg, Germany: ICMI.

Karsenty, R., & Sherin, M. G. (2017). Video as a catalyst for mathematics teachers' professional growth. *Journal of Mathematics Teacher Education*, *20*, 409–413.

Kazemi, E., Franke, M. & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 11-30). Palmerston North, NZ: MERGA.

Kleinknecht, M., & Schneider, J. (2013). What do teachers think and feel when analyzing videos of themselves and other teachers teaching? *Teaching and Teacher Education, 33*, 13-23.

Koc, Y., Peker, D., & Osmanoglu, A. (2009). Supporting teacher professional development through online video case study discussions: An assemblage of preservice and inservice teachers and the case teacher. *Teaching and Teacher Education, 25*(8), 1158-1168.

Krainer, K. (2008). Reflecting the development of a mathematics teacher educator and his discipline. In B. Jaworski & T. Wood (Eds.), *International Handbook of Mathematics Teacher Education* (Vol. 4, pp. 177–199). Rotterdam: Sense Publishers.

Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. *Journal of Mathematics Teacher Education, 20*(6), 591-619.

Major, L., & Watson, S. (2018). Using video to support in-service teacher professional development: The state of the field, limitations and possibilities. *Technology, Pedagogy and Education*, *27*(1), 49-68.

Manouchehri, A. (2002). Developing teaching knowledge through peer discourse. *Teaching and Teacher Education, 18*, 715–737.

Marsh, B., & Mitchell, N. (2014). The role of video in teacher professional development. *Teacher Development*, *18*(3), 403-417.

Mason, J. (2002). *Researching Your Own Practice: The Discipline of Noticing*. London: Routledge.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin, & R. Zazkis (Eds.), *Learning Through Teaching Mathematics: Developing Teachers' Knowledge and Expertise in Practice* (pp. 23–48). Dordrecht, Netherlands: Springer.

Masingila, J. O., Olanoff, D., & Kimani, P. M. (2018). Mathematical knowledge for teaching teachers: knowledge used and developed by mathematics teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education, 21*(5), 429-450.

McGraw, R., Lynch, K., Koc, Y., Budak, A. & Brown, C.A. (2007). The multimedia case as a tool for professional development: an analysis of online and face-to-face interaction among mathematics pre-service teachers, in-service teachers, mathematicians, and mathematics teacher educators. *Journal of Mathematics Teacher Education, 10*(2). 95-121.

Muir, T. (2010). *Using Video-Stimulated Recall as a Tool for Reflecting on the Teaching of Mathematics.* Paper presented at the 33rd annual conference of the Mathematics Education Research Group of Australia, (pp. 438-445), Fremantle: MERGA.

Muir, T., Beswick, K., & Williamson, J. (2010). Up, close and personal: teachers' responses to an individualised professional learning opportunity. *Asia-Pacific Journal of Teacher Education, 38*(2), 129-146.

Oates, G. & Evans, T. (2017). Research mathematicians and mathematics educators: collaborating for professional development. In K. Patterson (Ed.), *Focus on Mathematics Education Research* (pp. 1-30). New York: NOVA Science Publishers.

Paterson, J., & Evans, T. (2013). Audience insights: feed forward in professional development. In D. King, B. Loch, & L. Rylands (Eds.), *Proceedings of Lighthouse Delta, the 9th Delta Conference of Teaching and Learning of Undergraduate Mathematics and Statistics: Through the Fog* (pp. 132–140). Kiama: Delta.

Paterson, J., Thomas, M. O. J., & Taylor, S. (2011). Decisions, decisions, decisions: What determines the path taken in lectures? *International Journal of Mathematical Education in Science and Technology*, *42*(7), 985-996.

Philipp, R. A., Ambrose, R., Lamb, L. L. C., Sowder, J. T., Schappelle, B. P ., Sowder, L., et al. (2007). Effects of early field experiences on mathematical content knowledge and beliefs of pre-service elementary school teachers: An experimental study. *Journal for Research in Mathematics Education, 38*(5), 438–476.

Robert, A., & Rogalski, J. (2002). Le système complexe et cohérent des pratiques des enseignants de mathématiques: une double approche. *Canadian Journal of Math, Science & Technology Education, 2*(4), 505-528.

Santagata, R. (2009). Designing video-based professional development for mathematics teachers in low-performing schools. *Journal of Teacher Education, 60*(1), 38-51.

Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: An empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education, 10*, 123–140.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.

Schoenfeld, A. H. (2008). On modelling teachers' in-the-moment decision-making. In A. H. Schoenfeld (Ed.), *A Study of Teaching: Multiple Lenses, Multiple Views* (Journal for Research in Mathematics Education Monograph 14, pp. 45–96). Reston, VA: NCTM.

Schoenfeld, A. H. (2010). *How We Theory of Goal-Oriented Decision Making and its Educational Applications*. NY: Routledge.

Schoenfeld, A. H. (2011). Noticing matters. A lot. Now what? In M. Sherin, V. Jacobs & R. Philipp (Eds.), *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes* (pp. 253-268). New York: Routledge.

Seago, N., Driscoll, M., & Jacobs, J. (2010). Transforming middle school geometry: Professional development materials that support the teaching and learning of similarity. *Middle Grades Research Journal*, *5*(4), 199-211.

Shepherd, C., & Hannafin, M. J. (2009). Beyond recollection: Re-examining preservice teacher practices using structured evidence, analysis, and reflection. *Journal of Technology and Teacher Education, 17*(2), 229e-251.

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. *Advances in Research on Teaching*, *10*, 1-28.

Sherin, M. G., Jacobs, V. R., &. Philipp, R. A. (Eds.). (2011). *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes*. New York: Routledge.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting Video Clips to Promote Mathematics Teachers' Discussion of Student Thinking. *Journal of Teacher Education*, *60*(3), 213–230.

Sherin, M. G., Russ, R. S., & Colestock, A. A. (2011). Accessing mathematics teachers' in-the-moment noticing. In M. Sherin, V. Jacobs & R. Philipp (Eds.), *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes* (pp. 79-94). New York: Routledge.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20-37.

Tekkumru-Kisa, M. & Stein, M. K. (2017). Designing, facilitating, and scaling-up video-based professional development: Supporting complex forms of teaching in science and mathematics. *International Journal of STEM Education*, *4*(27).

Tripp, T., & Rich, P. (2012). Using video to analyze one's own teaching. *British Journal of Educational Technology, 43*, 678–704.

van Es, E. A. (2012). Examining the development of a teacher learning community: The case of a video club. *Teaching and Teacher Education, 28*(2), 182-192.

van Es, E., & Sherin, M. (2008). Mathematics teachers learning to notice in the context of a video club. *Teaching and Teacher Education, 24*(2), 244–276.

van Es, E., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education, 65*(4), 340-356.

Yung, B. H. W., Yip, V. W. Y., Lai, C., & Lo, F. Y. (2010). *Towards a model of effective use of video for teacher professional development.* Paper presented at the International Seminar, Professional Reflections, National Science Learning Centre, York, UK.

Zazkis, R., & Mamolo, A. (2018). From disturbance to task design, or a story of a rectangular lake. *Journal of Mathematics Teacher Education, 21*(5), 501-516.

RESEARCH FORUM: RITUALS AND EXPLORATIONS IN MATHEMATICAL TEACHING AND LEARNING

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INTRODUCTION

The discourse of mathematics education research is replete with conceptual dyads such as “procedural vs. conceptual”, “individual vs. social”, “extrinsic vs. intrinsic motivation” and “mathematical dis/ability”. To these well-known dyads, a relatively new conceptual pair was introduced a decade ago: ritual vs. exploration. Originally, this pair has been conceived by Sfard (2008) as types of *routines*. Ritual routines were defined as routines "whose goal (closing condition) is alignment with others and social approval" (p. 301) while exploration routines were defined as routines "whose goal (closing condition) is production of an endorsed narrative" (p. 298). Later works, drawing on Sfard, defined ritual participation as mathematical performance for the sake of connecting with others or “people pleasing” (Heyd-Metzuyanim & Graven, 2016). In a PME Working Sessions in 2016, we examined the affordances and limitations of this dyad, and began asking questions about its connection with other prevalent dyads in the field. Recently, a special issue (SI) (Heyd-Metzuyanim & Graven, 2019) emerged from these discussions. The current Research Forum is intended to bring back to the PME community the product of this SI so as to spur further discussion and research around this dyad.

Sfard and Lavie (2005) initially coined the terms “rituals” and “explorations” based on a study of 4-5 year old children learning about numbers. Since then, the conceptual dyad has been found useful for description of Israeli middle-school learners (Heyd-Metzuyanim, 2015), South African elementary school learners (Heyd-Metzuyanim & Graven, 2016) and even instruction of pre-service teachers (Heyd-Metzuyanim, Tabach, & Nachlieli, 2016). Moreover, “ritual participation” has been paired with “ritual instruction” connecting learning and teaching practices (Heyd-Metzuyanim & Graven, 2016).

The growing number of studies looking into ritual and explorative participation has led to the emergence of several central questions. The first regards the relationship between the two types of participation as being consecutive or parallel. Originally, Sfard and Lavie (2005) suggested that ritual participation is an antecedent of explorations, often an inevitable practice that supports learning at the peripheral stage when participants do not have sufficient conceptual tools to follow the logic of the discourse and must

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 137-164). Pretoria, South Africa: PME.

rely on imitation. Despite the theoretical appeal of this conjecture, some evidence collected points to an alternative possibility – that students participating predominantly ritually advance in a parallel trajectory (often leading to failure) to those who participate more exploratively (Heyd-Metzuyanim, 2015).

Another question relates to the dichotomy of ritual and exploration. Is this dichotomy justified? Or is a take on ritual and exploration as a continuum more appropriate? And if a continuum is more appropriate, what are the characteristics of the ground that lies between ritual and exploration?

In this Research Forum, researchers from various geographical and educational contexts take up these questions, while forging new grounds for what can be studied using the ritual-exploration dyad. The first paper by Irit Lavie presents the latest developments in Sfard and her colleagues' (Lavie et al., 2018) conceptualisation of ritual routines and the process of "de-ritualisation" (the process by which ritual routines turn into exploration routines). Extending their long-held focus on young children's numerical discourse, Lavie will show, in her study of a young 2-3 year old, how these conceptual developments go some way in clarifying the "middle-grounds" between rituals and explorations.

Sally-Ann Robertson and Mellony Graven offer an alternative theoretical framework for looking at ritual participation, in the form of socio-linguistic theory. They examine a South African classroom where students' limited mastery of the language of instruction (English) provides a constant source of struggle for the teacher aiming for more exploratory talk in her classroom. Attacking directly the dichotomous nature of the ritual-exploration dyad, Olov Virrman takes us to the domain of university mathematics education. By relying on the commognitive theory, Viirman shows how biology students' participation in mathematical modelling activities are actually interweaved with rituals and explorations.

These three talks will form the first part of this Research Forum, relating to rituals and explorations in mathematical *learning*. The next three talks, taking place in the second part of the RF relate to the implications of this dyad for *teaching*. First, Talli Nachlieli and Michal Tabach examine the possible reasons for the prevalence of instructional practices that afford opportunities for ritual participation. Next, Einat Heyd-Metzuyanim takes us to the domain of teachers' learning to teach mathematics in professional development settings, showing how the conceptualization of learning as progressing from ritual to explorative participation fits the progression of two teachers learning to implement cognitively demanding tasks in a US middle school classroom. Finally, Jill Adler concludes with a commentary on the full set of papers, assessing the usefulness of studying rituals and explorations for understanding problems of practice, as well as pointing to some caveats that may be hidden in this conceptual dyad.

1st Session (90 minutes) – Rituals & Explorations in learning

This session will include an Introduction by Graven and Heyd-Metzuyanim, followed by three short lectures: the first by Lavie, the second by Robertson and the third by

Viirman (see papers below). The session will conclude with a 30 minutes discussion around the following question: *How does the ritual-exploration dyad relate to previous dyads (such as procedural-conceptual, traditional-reform)?* The discussion will be structured as follows: the question will be posed to the audience to think and reflect upon *before* the three short lectures. Following the short lectures, members of the audience will be asked to discuss in small groups their ideas around the question. These small-group discussions will be followed by a plenary discussion.

2nd Session (90 minutes) – Rituals & Explorations in Teaching

This session will open with a short supplementary introduction. Following that, three short lectures concerning rituals and explorations in teaching will be presented. The first, by Nachlieli and Tabach; the second by Heyd-Metzuyanim and the third – a commentary by Adler. Similar to the first session, this session will conclude with a 30 minutes discussion, around the following question: *How does our growing understanding of teaching that promotes ritual and explorative participation help for forwarding efforts to improve mathematics education in various contexts?"* The structure of the discussion will be similar to that of the first session.

Following are the short papers and the commentary, in the order they will presented.

ROUTINES - FROM RITUAL TO EXPLORATION

Irit Lavie

In an ongoing longitudinal study, we (Lavie & Sfard, under review; Lavie & Sfard, 2016) have been examining young children's early development of numerical discourse. During our years of analyzing our young participants' actions and articulations, we have considered and re-considered which lens would be best-suited to elucidate previously un-accounted for aspects of young children's initiation into the endorsed numerical discourse. Our exploration has lead us to suggest adopting the lens of *routine* development – repetition-generated patterns of actions – rather than the prevalent language of "concepts" and "skills development". By focusing on routines as our unit of analysis, we put forward the thesis that repetition is the gist of learning, and that learning mathematics, or learning in general, can be described as a process of *routinization* of our actions, i.e., modeling our present actions on prior relevant experiences, gradually *individualizing* socially endorsed routine performances. To clarify these claims, we first introduce our re-definition of the keyword routine and associating notions. We then distinguish between two type of routines, ritual and exploration, and introduce our view of this dyad as two opposite ends of a single continuum rather than a discrete differentiation. We then utilize this view to expand our description of learning as routinization.

Re-defining Routines

Our two decades of examining young children's' routine performances have provided us with countless observations of various reactions to carefully chosen settings, at times seemingly far removed from the socially endorsed expected reactions. When described in the literature, these were typically accompanied by an evaluation of a performance as "right" or "wrong" (as in Russac, 1978; Siegler & Svetina, 2006; Fuson, Secada & Hall, 1983), often times leading to a focus on the deficit of the child's' action. Focusing on what the child did do (rather than on what he did not) led us to acknowledge a key question: *why* did the child do that? Attempting to answer the question of why the child performed a specific routine - particularly one that we, as experts, would consider misaligned with the task presented to the child, led us to suggest that the child's interpretation of the given task needs to be further explicated. Revisiting our data, we began considering various possible task interpretations (Lavie & Sfard, 2016). A different interpretation of the task emerged as a plausible, reasonable, explanation for the child's' choice of procedure.

Actions are not performed in a vacuum, rather, they are performed as a response to some perceived need or request. The person's perception of the action-eliciting situation is what we refer to as *task situation*. We claim that one's interpretation of a given task is unique, and two people may be presented with the same task but interpret it (and therefore address it) differently. The task setter and the task performer can also hold different perceptions regarding the nature of the task. This understanding is particularly important when examining how a young child fulfils a task given to him by an adult. Perceiving it differently than what the adult had intended can result in the performance of a different task altogether.

Consider for example the following exchange between Milo, aged 2 years and 8 months old, and his mother, as she presented him with two bowls of his favorite snack.

	Speaker	What was said	What was done
76.	Mother:	Where is there more [pieces of] Bamba?	Milo holds two bowls, one with 5 pieces of Bamba and the other with 2
77.	Milo:	Here there is *a lot*	Puts down the bowl with the largest amount
78.	Milo:	and here there are two	Puts down the other bowl with the 2 pieces of Bamba

Milo's actions indicated he had interpreted his mother's question as a task situation calling for an action to be made. While his mother's task (representing here an expert in numerical discourse) was that of quantitative comparison, Milo's actions can imply several possible tasks he may have been actually looking to fulfil. One can be the task of merely choosing a bowl, expressed by first indicating one of the two and putting it down before the other. However, Milo could have been, just the same, fulfilling the task of attaching a number or quantity to each of the bowls, or a combination of these two tasks. The question we first need to establish is, therefore, what could have been Milo's task? Only subsequently can we examine why he chose a specific course of action to fulfil it.

Within a task situation, to select a course of action one must revisit past situations she considers similar to the current situation, considered as *precedents* for the current task situation. We refer to the set of all precedents one considers relevant to a current task situation as his *precedent search space*. Based on these, one must decide what he will now preserve from the previous actions he had taken. The preserved actions which the person re-enacts while performing the task is what we refer to as *procedures*, however these are only one part of the routine the person performs, while the other part is her (perception of the) task (Lavie & Sfard, 2016). Therefore, a routine is defined by these two aspects:

The task, as seen and executed by the performer *in a given task-situation*, is the set of all those features that are common to her present performance and to all the past performances that she views as precedents; to put it in "intentional" terms, it is the set of properties of the precedent events that the performer evidently tried to reproduce now.

The procedure is the prescription for action, possibly but not necessarily an algorithm, that fits all the precedent performances and can be seen as the one that guided also the performer's current actions in the given *task-situation*.

In the above example of Milo and his mother, the procedure Milo employed was that of attaching a number indicating the quantity of elements within each bowl and putting down each bowl upon declaring the number associated with it. However, the routine Milo was performing would vary, pending on the task he was seeking to accomplish. The routine of comparing quantities by attaching numbers and setting down first the bowl with the larger amount of items, is quite different than that of attaching numbers and setting down each bowl. Presenting Milo with a similar task situation and examining his responding performance can elucidate which is more probable. Thus, repeated observations are crucial in interpreting one's tasks and therefore one's routine performance.

Considering the task-procedure dyad, and how these relate to the child's prior experience, can provide an explanation of why different children react differently to the same given task: they might interpret the task differently, or consider different precedents as relevant. This also explains why the same child will occasionally perform different procedures to accomplish what we would consider a similar task (Lavie, Steiner & Sfard, 2018).

THE RITUAL-EXPLORATION CONTINUUM

Upon re-defining what a routine is, particularly considering one's task as a key element of his routine, we were encouraged to re-examine the initial classification of types of routines proposed in Sfard and Lavie (2005). Initially, we identified three types of routines as separate: deeds, rituals and explorations. However, the new insights regarding the role of the task implied a connection between the latter two, pointing to rituals and explorations being two extremes of a shared continuous spectrum differentiated by one's source of motivation or agency. In one extreme, because the performer wants to (exploration) and in the other extreme because the performer believes an accompanying expert in the discourse wants him to (ritual). These distinctions correlate with our initial differentiation between exploration and ritual, but imply that these are not merely discrete options, rather two extremes of a shared continuous spectrum, as one can be concurrently motivated by both of these (at least to some extent).

Aside from agency, or as a result of it, an exploration would differ from a ritualistic routine performance by the level of flexibility, bondedness, applicability, objectification of the discourse and substantiability. For example we consider a routine to be "bonded" if an output of a previous step taken, serves as input for a subsequent step (e.g., the result of counting the amount of items in two sets, leads to indicating the set with the larger amount of items as the set "where there are more" items in). A non-bonded routine performance (e.g., indicating the smaller amount as the second stage in the previous example in response to the question where are there more?") would more likely be indicative of a ritualistic performance, with the goal of imitating actions observed others perform, regardless of their role in completing the task.

Similarly, the nature of a routine performer's reply to a substantiation request can be indicative of its ritualistic-exploratory orientation. The more aligned the response would be with what can be considered an endorsed procedure description, the more indicative it would typically be of the performers' independent ability to assess its merits and its outcome, as opposed to relying on external feedback for these assessments.

While some routines are destined to stay rituals forever (i.e., brushing your teeth), mathematical routines, to be truly useful, must evolve into full-fledged explorations. Considering now the ritual-exploration dyad as two extreme ends of the same continuum, allows us to revisit our initial purpose of describing how learning mathematics can occur.

LEARNING AS ROUTINIZATION

In initial encounters with a new discourse, the learners can only participate in ritualized ways. Motivated mainly by social needs, learners will begin by performing routines through mimicking and adopting the performances of others they consider as experts in the discourse. Their task would typically be that of imitation, or one associated with past experiences that may be considered irrelevant by experts (i.e., interpreting a quantitative comparison task as the task of merely attaching numbers to quantities).

In further learning, the learner's routines are expected to undergo gradual de-ritualization until they eventually turn into full-fledged explorations. This happens as the learner re-encounters similar task situations (or situations he considers as similar) and his precedent reservoir gradually expands to include observations of others' actions, as well as his own, in a similar task situation. As his precedent reservoir expands, the search space relevant to the specific task situation will include more socially endorsed forms of action. The expanded search space will then afford the learner more flexibility in his choice of procedures, and gradually include more bonds between different procedures and within one sub procedure to the next. This will increase the routine's applicability and will typically be accompanied with more objectification of the discourse and the ability to substantiate the routine's result, thus indicating more exploratory routine performances.

However, a routine performance, as well as the process of de-ritualization, is idiosyncratic. It takes many forms and often requires an extended period of time. One's development of a routine is dependent on his individual interpretation of task situations, on personal experiences, and is an ongoing, convoluted, non-linear process. In implementing these insights, either for educational purposes or research related goals, one must be mindful that: (a) Identifying one's routine is an interpretive act, and observations alone may lead to partial or limited insight. (b) A single routine performance is not enough to identify one's search space or task. Repeated observations are crucial, and a vast inspection of repeated performances is advised. (c)

Choosing which routine to examine or teach is influential on the insight one can gain on one's learning.

Having said this - upon choosing a suitable germinal routine and affording the young learner repeated experiences with enacting it, examining his routine performance can illuminate various aspects of learning, and open the door for truly individualized learning. Repeated experiences are key in expanding young learners' search spaces. During these, one should be mindful of the child's task and how it may vary from his own. Initial attempts would inevitably be ritualistic, but it appears that that is how we learn. These ritualistic performances should not be considered hurdles but rather stepping stones, a vital intermedium to more exploratory routine performances.

EXPLORATORY MATHEMATICS TALK IN A SECOND LANGUAGE: A SOCIOLINGUISTIC PERSPECTIVE

Sally-Ann Robertson & Mellony Graven

Breaking away from rote-oriented practices towards more conceptually-oriented exploratory engagement is especially challenging when mathematics teaching takes place in a language that is different from teachers' and students' home languages (L1s). This is the case in most of South Africa's less affluent classrooms, including the one reported on here.

The guiding dyad we use is 'right answerism' as against 'exploratory talk' (Barnes, 2010). We use it to respond to the question: "What is the nature of the talk in the observed grade 4 mathematics lesson and how does it appear to enable or constrain mathematical meaning-making?" The lesson was on the relative sizes of different unit fractions. Even in optimal circumstances, helping students grasp the inverse order relationship in unit fractions is "not easy to attain" (Cortina, Visnovska & Zuniga, 2014, p. 81). This is made more difficult when, as was the case in this lesson, teaching takes place in a second language (L2). The teacher (Ms M) wanted students to demonstrate that they understood that ¼>1/8, the point of potential confusion being that 8>4. Our analysis of the lesson talk reveals a predominance of 'right answerism'. The absence of more exploratory forms of talk we see as being largely a consequence of the requirement that students learn mathematics in English (an L2 to both Ms M and the students). Their inadequate proficiencies in English profoundly constrained their opportunities to use it in exploratory ways "to think aloud … to talk their way into [mathematical] understanding" (Barnes, 2010, p. 9). Adler and Pillay (2017) note that the educational performance of students in South Africa's less affluent schools is "increasingly below the required level" (p. 13). Having to learn mathematics in an L2 has been directly implicated in these low levels of achievement in mathematics (Setati, Chitera & Essien, 2009).

Our work, set in a socio-cultural perspective (Vygotsky, 2012), uses insights from two educational linguists: Cummins and Gibbons. Cummins's work highlights important links between students' L1 proficiency and their developing proficiency in an L2; principally mastery of academic language in the L2. While conversational fluency in an L2 can develop within 6 months to 2 years, it takes considerably longer to develop proficiency in the more formal, academic registers associated with curriculum content (5 to 7, or even 10, years) (Cummins, 1994). Cummins's acronyms BICS (basic interpersonal communication skills) and CALP (cognitive academic language proficiency) are used hereafter in referring to these two language registers. Successful teaching, particularly in L2, requires mediation in navigating the 'mode continuum' from everyday, 'more spoken-*like*' ways of expressing ideas towards the more specialist, cognitively-demanding, and "written-*like"* subject-specific expression of ideas (Gibbons, 2006, p. 34). 'Semantic density' (Maton, 2011) makes subject-specific language more cognitively challenging as ever more meaning becomes condensed within particular words or phrases. So, for example, 'one whole cut into two equal parts' is less semantically dense and thus easier to conceptualise than '½' or '1 is the numerator and 2 is the denominator'.

When students face the challenge of *acquiring* an L2 while simultaneously *using* it to 'make meaning' of classroom encounters with mathematics it is helpful if teachers initially 'mesh' "everyday and subject-specific ways of meaning [so] building on [students'] prior knowledge and current language as a way of introducing them to new language" (Gibbons, 2009, p. 62). Commenting specifically on underachievement in South African rural and township schools, Cummins advocated expansion of "the instructional space to include students' and teachers' multilingual repertoires" (2015, p. 278). This acknowledgement of students' L1 as an important parallel resource underpins Cummins's linguistic interdependence hypothesis, whereby, through cross-lingual transfer, "conceptual knowledge developed in one language helps to make input in the other language comprehensible" (2000, p. 39). Hence, for example, students who have in their L1 the conceptual/ semantic grasp that 'the greater the number of equal pieces a whole is divided into, the smaller each piece will be' would not need to learn this general principle afresh in their L2. They would simply need new linguistic 'labels' for this concept in the L2. It makes sense therefore that there be a degree also of 'meshing' across the L1 and L2. The extent to which this can be realized depends on what model of bilingual education students experience. Many countries, South Africa included, advocate additive multilingualism, whereby students' L1s remain a resource to draw on throughout their schooling. In practice, however, most students in sub-Saharan Africa experience subtractive forms of bilingualism (see Heugh, 2005). At Ms M's school the language policy is 'straight for English'. Right from grade 1 English is the language of teaching and learning. Given that all of the students and a majority of the teachers are native speakers of isiXhosa this can only be described as bilingualism in its most subtractive form.

In using these sociolinguistic insights to analyse one of Ms M's grade 4 lessons on fractions, we illuminate aspects of the challenge she faced in attempting to move her students away from 'right answerism' towards more 'exploratory talk' around the relative sizes of unit fractions in and through an L2. (See Robertson and Graven (2018) for details relating to methodological decisions around our selection and analysis of this and other of Ms M's lessons.) The 50-minute lesson comprised 372 turns. We focused primarily on the first 158 turns, coding them into three categories:

- Talk embedded within students' lived-experience, and reflective of an 'everyday' BICS-/spoken-*like* register;
- Talk more closely aligned with an 'academic' CALP-/written-*like* classroom register;
- Talk containing both BICS- and CALP-like features, and which invites discussion to move in more 'exploratory' directions.

Both authors independently coded the lesson transcript data into these three categories, achieving a high degree of inter-rater correspondence. Table 1 provides a sample (Turns 1-57) of what this analytical coding revealed.

Turn	More BICS-*like,* context-embedded 'everyday' talk	More CALP-*like,* context-reduced 'classroom' talk
	Meshing of BICS-/CALP-like (and – *potentially* - 'exploratory') talk	
1-36	**T:** Thabo, we have done fractions for quite a long, long, long time [...]. I'm not going to draw anything on the chalkboard, but if I'm saying to you that there's this cake, do you like cake? **S:** Yes, Ma'am. **T:** Who doesn't like cake? Who doesn't? **S:** (grimace) **T:** You don't like it, Phumla? **S:** No, ma'am. [Turns 7-36: T continues polling students' preferences.]	
37-42		**T:** *(Writes ¼ on chalkboard.)* What do you call this fraction first? Thabo? **S:** One-fourth.

Turn	More BICS-*like,* context-embedded 'everyday' talk	Meshing of BICS-/CALP-like (and – *potentially* - 'exploratory') talk	More CALP-*like,* context-reduced 'classroom' talk
			T: One-fourth, or? **Ss:** One quarter. **T:** [...] And then [...] this one *(writes 1/8 on chalkboard.)* what do you call this one? **S:** One-eighth.
43-53		**T:** Thabo, would you rather have a quarter of a cake or an eighth of a cake? A quarter of a cake or an eighth of a cake? And Why? Which one? **S:** Quarter. **T:** A quarter? Why not an eighth? **S:** (silence) **T:** Why not an eighth? Because you say you prefer to have a quarter not an eighth. Why? Why? **S:** (silence) **T:** (Puts the question to the whole class.) Why? **Ss:** (no response) **T:** Which one? **S:** (no response) **T:** He [Thabo] does like to have a quarter, but he doesn't have an actual reason.	
54-57	**T:** Wazini, do you like cake or not? **S:** Yes, ma'am. **T:** You do? **S:** (nod)		

KEY: **T:** = Teacher; **S:** = individual student response; **Ss:** = chorused student response; [...] omitted text

Table 1: Movement along a BICS-/CALP-*like* continuum

As Table 1 shows, Ms. M's attempts at 'meshing' across and between conversational (BICS-type) and formal, academic (CALP-type) talk in *exploring* the 'which is bigger/smaller' questions produce only brief student responses, reflective of 'right answerism' (recalling, re-capping, and/or reiterating previously-established mathematical 'facts'). Her 'why?', 'which one?', 'why not? questions, designed to initiate some exploratory engagement, are met with silence. Her students do not take up her challenge to *explain* or *explore* in general terms why ¼>1/8. Her attempts at moving her students towards more formal, CALP-types of mathematical reasoning around the inverse order relationship in unit fractions thus appear to have been almost wholly stalled. We note, given the 5- to 7-, or even 10-year time-span perhaps needed for development of L2 CALP, and the fact of Ms M's students only being in the middle of their 4th year of English-only schooling, that moving along the 'mode continuum' required that they communicate in English at a level that may have been beyond them. We suggest that an 'additive', as opposed to the existing 'subtractive' form of bilingualism, would have better afforded these students opportunities to 'mesh' their L1 meaning-making resources in with their L2 encounters with mathematical concepts, so conducing to "more cognitively engaged [mathematical] learning" (Cummins, 2005, p. 13).

MATHEMATICAL RITUALS AND BIOLOGICAL EXPLORATIONS: UNDERGRADUATE BIOLOGY STUDENTS WORKING ON MATHEMATICAL MODELLING TASKS

Olov Viirman

This part of the research forum focuses on the interplay between ritual and exploration in the work of biology students engaged in Mathematical Modelling (MM) activity. We report on a project forming part of an ongoing collaboration between two Norwegian national Centres for Excellence in Higher Education, investigating the role of mathematics in university biology education. We summarize findings reported elsewhere (Viirman & Nardi, 2017; 2018; 2019), emphasizing the interplay between ritualized and exploratory routine use, and indicate where further analyses are heading.

BACKGROUND AND CONTEXT OF THE MM IN BIOLOGY PROJECT

Although much university mathematics teaching is aimed at students specialising in other fields of study, this aspect of university mathematics education is still under-researched. In particular, in biology, the increased importance of mathematical methods places new demands on the education of future biologists, causing some researchers to suggest a greater integration of mathematics and biology in the curriculum (e.g. Brewer & Smith, 2010). Research on the use of MM in university biology education indicates that engagement with MM activities can contribute to more positive attitudes towards, and self-perceived competence in, both biology and mathematics. The project we report here grew out of a wish to explore this conjecture further.

The project was conducted at a Norwegian university and its main part consisted of a sequence of four three-hour MM sessions with a group of 12 volunteering students (out of a cohort of approximately 100), concurrent with their mandatory first-semester mathematics course. This is a generalist course catering to students from several natural science programs, providing few opportunities for focusing on issues specific to biology. The research team comprised three mathematics education researchers and one mathematician. Sessions were taught by a research mathematician with extensive experience of MM and consisted of brief lectures introducing various aspects of MM, followed by group work on MM tasks set in a biological context. The aim of the sessions was not primarily teaching the students new mathematics, or even principles of MM, but rather letting them experience how MM can be relevant for addressing biological problems. The teaching was conducted in English, but most student group work and student contributions to group discussions were in Norwegian. All sessions were video and audio recorded. Before presenting our findings, we will give a brief outline of the sessions and of the theoretical constructs used in the analysis.

OVERVIEW OF THE MM SESSIONS

The first session began with an introduction to MM and to the modelling cycle (Blum et al., 2007), with an emphasis on the role of assumption building, that is, making simplifying or clarifying assumptions as a part of the translation of a problem in biology into the mathematical domain. Most of the session was spent on a task aimed at engaging students in assumption building, namely the *Roadkill Rabbits* problem, where students were asked to estimate the population density of rabbits based on observations of traffic intensity and the number of roadkill rabbits. The bulk of the second session concerned using MM to model change. The lecturer introduced a problem concerning *Yeast Growth* in a petri dish which, contrary to the very open *Roadkill Rabbits* problem, was broken into sub-problems that the students worked on for 10-15 minutes each, with whole-class summaries in between. The third session began with a further problem on the modelling of change, this time concerned with modelling the decay in the body of *Digoxin*, a drug used to treat heart disease. After short lectures on non-linear models and modelling using geometric similarity, the *Terror Bird* problem was introduced. In this problem, the students were asked to estimate the weight of an extinct species of bird, based on measurements of fossilized femur bones and present-day data on the relation between femur circumference and body weight in various bird species. Finally, the bulk of the fourth and final session was devoted to the *Rabbits and Foxes* problem, concerning the dynamics of the interaction between two populations, foxes and rabbits. Again, as in the first session, assumption building was central, but here students were provided with a number of assumptions based on which they were supposed to construct a model. For most of the sessions, the researchers engaged minimally in students' work, although they were always available for answering questions. However, for the last session a member of the research team joined each of the groups to provide support if needed.

RITUALS AND EXPLORATIONS IN MM ACTIVITIES FOR BIOLOGY

In our analyses, we build on commognitive theory (Sfard, 2008), particularly routine use and the relationship between rituals and explorations. As Lavie, Steiner, & Sfard (2018) note, rituals may morph into explorations as a learner's performance shifts from being process-oriented to becoming outcome-oriented. De-ritualisation may involve manifestations of flexibility and broadened applicability – or, in Lavie et al.'s terms, vertical (a new step in a procedure building on outcomes of previous steps) or horizontal bonding (a procedure being conducted in a number of alternative ways which generate the same output). In the summary of data analysis that follows, we investigate student discourse with an emphasis on the interplay between ritualized and exploratory participation in a cross-disciplinary context. It should be noted that we have interpreted "biological discourse" in a broad sense. Since participants were first-semester students, they had not actually studied much biology yet. Hence even though they often used arguments from biology in their reasoning about the problems, these relied as much on "colloquial" (Sfard, 2008, p.132) insights into (for example) the behavior of certain species as on more "literate" (p.132) elements of scientific biological discourse.

Our analyses so far have centered around two main areas of inquiry. First: how did the nature of the students' engagement develop when working on the two problems concerning the modelling of change - *Yeast Growth* and *Digoxin*? Second: How did the engagement with assumption building routines develop over the course of the sessions? Concerning the first area of inquiry, in Viirman & Nardi (2017), we showed how the scaffolded character of the *Yeast Growth* problem reduced student agency and invited ritualized engagement. At the same time, there were signs that the ritualized engagement with *Yeast Growth* may well have been a necessary step towards a more exploratory engagement with the *Digoxin* problem in the next session. In Viirman & Nardi (2018) we extended our analysis of students' engagement with the *Digoxin* problem, noting how it hinged upon exploratory engagement with an established graphing routine. The successful adaptation of this routine was based on an exploratory engagement with the biological discourse surrounding the problem.

Concerning the role of assumption building, in Viirman & Nardi (2019) we showed how students working on the *Roadkill Rabbits* problem in the first session, engaged with assumption building in a ritualized manner, through what we labelled "assumption as guesswork": when you need the value of some quantity in order to solve the problem, you pick a number that seems reasonable. Students then attempted to justify this guesswork through biological arguments. At the same time, they were dissatisfied with this way of approaching the problem, suggesting alternative methods of solving the problem empirically. Besides showing that, when working within biological discourse, the students were capable of exploratory engagement with the problem, this hints at what they considered a valid solution to a modelling problem. For the lecturer, solving the problem meant using mathematical relationships between variables to construct a model. For the students, however, a meaningful solution to the problem involved collecting empirical data, while the mathematical approach was seen as merely "guesswork". This conflict recurred in later sessions, for instance, when working on the *Rabbits and Foxes* problem. Where the lecturer and the researchers viewed the given assumptions as parts of the mathematization of the problem, for the students they were biological statements to be interpreted empirically. Still, working on this problem the students were able to engage with assumption building in a more exploratory manner. However, this exploratory engagement was mostly through biological rather than mathematical discourse, and students displayed difficulty moving between the two discourses. Indeed, there was evidence that their limited fluency with mathematical discourse, and lack of established mathematical routines, led them towards ritualized routine use. When engaged in algebraic routines, for instance, they were unable to move beyond talk of symbol manipulation towards considering the meaning of the symbols.

CONCLUDING REMARK AND WAYS FORWARD

In conclusion, we saw clear signs of exploratory engagement with the problems, but this engagement mainly took place within biological discourse, whereas the engagement with mathematical routines was mainly in the form of rituals, sometimes to the extent of interfering with the students' capacity to engage productively with the problems. At the same time, our findings support taking a more fluid perspective on students' routine use. The way the students engaged in biological discourse around the assumptions they constructed in, for instance, *Digoxin* and *Rabbits and Foxes* had elements of creativity that led us to characterize this engagement as exploratory. However, this creativity was hampered by ritualized mathematical routine use. At the same time, the ritualized engagement with assumption building or graph construction appears to have helped pave the way for more exploratory, creative engagement later.

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RITUAL-ENABLING OPPORTUNITIES-TO-LEARN IN MATHEMATICS CLASSROOMS

Talli Nachlieli & Michal Tabach

INTRODUCTION

Despite vast research calls for more reform oriented teaching, traditional teaching is still common worldwide. The traditional approach views teaching through a lens of knowledge transmission and therefore is often referred to as "teaching by telling" (Brown, 2003) or "pedagogy of control". In those classes, Mathematical knowledge is viewed as discrete, hierarchical, sequential, and fixed (Gregg, 1995) and is presented as a collection of facts and procedures. In contrast, reform-student-centered instruction places students at the center of classroom organization and respects their learning needs, strategies and styles (Brown, 2003). This approach is based on constructivist ideas that students learn by resolving problem situations that challenge their conceptual understandings. The teacher is a facilitator or coach who assists students who are seen as the primary architects of their learning (Gregg, 1995).

Provided that teachers aspire to perform their best practice, there seems to be tacit practical knowledge of teachers which requires further theoretical elaboration and could explain the resilience of traditional teaching. In the current study, we wish to take a different lens to look at teaching practices, and focus on teaching routines. In this study we ask: what could be gained by teaching that promotes ritual participation?

THEORETICAL BACKGROUND

Routines: rituals and explorations

Teaching and learning, as any other human-activity, include the performance of collectively-established routines (Sfard, 2008). The commognitive framework conceptualizes routines as discursive patterns that are repeated in similar situations (Sfard & Lavie, 2005). That is, when a participant considers a situation to be similar to one she previously participated in, she is likely to perform actions that could be considered the same. A routine consists of three parts: initiation, procedure and closure. The initiation and closure relate to the conditions under which a certain procedure is evoked and by whom, as well as the conditions under which a procedure is considered complete. The initiation and closure are the *when* of the routine. The procedure, which includes the process performed, is the *how* (Sfard, 2008).

Sfard and Lavie (2005) differentiate between two types of discursive routines: explorations and rituals. *Explorations* are routines whose success is *evaluated* by answering the question of whether a new narrative has been produced and endorsed (Lavie, Steiner, & Sfard, 2018). That is, the task of an exploration is to produce "historical facts" that are new to the learner or a new "truth" about mathematical objects. For example, a student that endorses the Pythagorean Theorem is endorsing a well-known historical mathematics narrative. Nonetheless, it is new to the learner. A student who learns exploratively focuses on the question: *What do I want to achieve?*

In contrast, someone who performs a ritual routine is concerned with the question of *how do I proceed?* or *how can I enact a specific procedure?* Often, rituals are routines performed for the sake of social rewards (Sfard & Lavie, 2005; Heyd-Metzuyanim & Graven, 2016). That is, the desire for interpersonal communication with an authority figure, underlies the performance of ritual routines. Usually, performing a ritual routine includes imitating someone else's former performance. The procedure is followed rigidly and the performer seldom tries to make independent decisions.

Rituals and explorations: teaching routines

As was stated in the Introduction, our purpose is to present a theoretical framework that would help us identify and explain teaching goals for which teaching for ritual participation would be advisable. For this purpose, we suggest two theoretical notions and operationalize them: (1) ritual-enabling Opportunities-To-Learn (OTLs) and (2) exploration-requiring OTLs.

The term *opportunities-to-learn* is defined by the National Research Council as "circumstances that allow students to engage in and spend time on academic tasks…" (p. 333). We suggest the term *ritual-enabling OTL* (rather than simply *ritual OTL*) to stress the fact that an OTL does not necessitate students' ritual performance. We therefore consider ritual-enabling OTLs as teachers' actions that provide students with tasks that could be successfully performed by rigid application of a procedure that had been previously learned. Yet, a task that could be considered to occasion students with ritual-enabling OTL, could actually be interpreted differently by different students.

In contrast, we suggest the term *exploration-requiring OTLs* as teachers' actions that provide students with tasks that could not be successfully solved by performing a ritual. Rather, a successful completion of the task can only be achieved by participating exploratively. That is, to produce mathematical narratives, an exploration-requiring OTL is necessary. Here, the student cannot meet the teacher's expectations by a direct application of a single procedure with which she is already acquainted; as a minimum, solving the task requires a new combination of known procedures, and such a combination can only be created by focusing on the expected outcome. This is likely to be the case, for instance, when the request is presented with words such as *what, why, find, explain* etc.

While it is clear that exploration-requiring OTLs are needed in the mathematics classrooms, we assume that ritual-enabling OTLs could serve certain teaching goals. Therefore, we ask: *what could be gained by ritual-enabling OTLs?*

METHODS

Data

We looked for lessons from different places in the world, as we sought to learn more about ritual-enabling OTLs worldwide. Hence, we analyzed all mathematics lessons taught in English that are part of the data corpus from the TIMSS 1999 Video Study (Hiebert et al., 2005) and were public in the TIMSS website. Our data comprise eleven eighth-grade lessons (including videos, transcripts, lesson-graphs, commentaries) from three countries: Australia, USA, and Hong-Kong SAR.

Data analysis

The analysis includes two phases: first, we identified all opportunities to learn in each of the lessons, as ritual-enabling or exploration-requiring. Second, we categorized all of the ritual-enabling OTLs according to the possible teaching goals they could achieve (for details see Nachlieli & Tabach, 2018).

FINDINGS

During a lesson, the teacher occasions her students with various OTLs, some ritual-enabling and others exploration-requiring. These OTLs often interweave: Some are nested within or followed by other OTLs. An *external* OTL is one in which other OTLs are nested. It is this external OTL that sets the boundaries within which students are offered OTLs. Therefore, it is the external OTL that determines the type of learning opportunity provided to the students. We found interplays between nested and external OTLs that may be typical of possible goals of ritual-enabling OTLs.

Ritual-enabling OTLs as preparation for explorations. The common thread through all examples found under this category is that the results of the ritual-enabling OTLs served as the entry point for possible exploration-requiring OTLs, the aim of which was to produce and endorse new narratives. That is, the teacher opened the activity with ritual-enabling OTLs that aimed at setting the ground for the exploration. An example is taken from a lesson about linear equations. The teacher aimed at exploring a new type of equation: *identity*. The students were first invited to solve two linear equations: $2x+4=x+6$, and $2x+10=2(x+5)$. Two students solved the equations on the board. The fluency with which the students solved the equations indicates that the procedure had been practiced previously. Therefore, we consider this part of the lesson a ritual-enabling OTL. The ritual-enabling OTL led the students to produce two narratives: $x=2$ and $0=0$, each of which became the focus of exploration. The first was used to address the explorative question *what a solution is*, while the second was used to introduce the mathematical object of *identity*.

In these cases, the stated goal is to endorse a new mathematical narrative and therefore the external OTL is exploration-requiring. Within this external routine the students were first provided ritual-enabling OTLs, followed by exploration-requiring OTLs.

Ritual-enabling OTLs as initial steps in entering into a new discourse. A discourse is considered new to the learner when new meta-rules or new objects are learned. In the episodes under this category, the teaching process is initiated by ritual-enabling OTLs, and may be followed by de-ritualizing OTLs towards exploration-requiring opportunities (Lavie, Steiner & Sfard, 2018). Such de-ritualization process could include decomposing of the exploration-requiring process to ritual-enabling OTLs so that the continuum of opportunities would enable students to enter the new discourse.

An example is taken from a lesson about exponents. After the teacher presented her students three exploration-requiring OTLs to produce three narratives for calculating exponents ($a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, and $(ab)^n = a^n b^n$), she tried to continue in the same vein, asking them to prove the equation $a^0=1$. However, this justification involves a change in how new narratives are endorsed. Instead of referring to the primary notion of exponents as repeated multiplication, the students are now required to derive the laws of the extended operation of *raising to a power*, from narratives that were previously formulated in the lesson and were true for the narrowly formulated operation of raising to a power. It seems that the teacher was not aware of the meta-shift in discourse required in this situation. The students tried to follow formerly-learned procedures and expand exponents as repeated multiplications. This turned to be an unsuccessful endeavour. The teacher was almost forced to provide her students ritual-enabling OTLs as a last resort by unpacking the initial task for them. She let the students answer short closed questions as a first step in leading them to what we consider to be a new discourse. We suggest that the students' difficulty stemmed from the meta-level transition they were expected to make: from using exponents in the discourse of natural numbers, as repeated multiplication, to using exponents in the discourse of integers, where repeated multiplication is no longer relevant.

DISCUSSION

In the current study we suggested the notions of ritual-enabling and exploration-requiring OTLs to enrich our discourse on teaching. We also suggested that ritual-enabling OTLs may be a first, necessary, step towards object-level explorations and for meta-level shifts that allow entering a new discourse. Returning to our initial quandary of what teaching goals seem to be achieved *by ritual-enabling OTLs in mathematics classrooms*, we found that ritual-enabling OTLs serve several functions, for object-level and meta-level learning. We hypothesize that to lead to explorative mathematical discourse, teaching should include *both* ritual-enabling and exploration-requiring opportunities to learn. The proper mixture of those can only be fine-tuned by the teacher, over time, to meet the needs of her teaching goals.

FROM RITUAL TO EXPLORATIVE PARTICIPATION IN TEACHERS' PROFESSIONAL DEVELOPMENT

Einat Heyd-Metzuyanim

In this study, Sfard's (2008) theorizing of learning as a process of change from ritual to explorative participation is employed to theorize the learning trajectory of two middle school teachers attending a professional development (PD) program. The PD was designed around the 5 Practices for Orchestrating Productive Discussions and

Accountable Talk, in an Eastern district in the US. This, in the context of the notoriously problematic nature of capturing change in teachers' practice (Anderson, 1997), as well as the well-documented challenges in changing teaching practices from "traditional" to more "reform" or "dialogic" instruction (Spillane & Zeuli, 1999).

According to Sfard (2008), newcomers to any discourse are confined to ritual participation, produced through rigid imitation of the more expert participants, until they gain sufficient experience with the activity (or discourse) to set their own goals and conjure the means for obtaining them within the new practice. Thus ritual participation is process-oriented, that is, the learner is satisfied with having completed a certain set of actions (routines) that they have seen other, more skilful participants perform. The reasons for performing these routines remain concealed to the ritual participant, at least until he/she gets sufficient experience with them to observe their consequences and realize the various ways by which a certain goal can be achieved within the new discourse.

In the present study, our goal was to extend this theorizing of ritual and explorative participation to the domain of teaching practices. Of course, the discourse of mathematics and the discourse of teaching are inherently different. Whereas the mathematical discourse goes back more than 3000 years, and is characterized by "mathematical truths" that have been unquestioned for generations, the pedagogical discourse (including narratives about "best practice") is far from having such consensus. Still, there are merits in applying lessons learned from the close scrutiny of mathematical learning processes to learning processes of teaching. This, since the theory of ritual-towards-explorative participation can aid in characterizing trajectories of learning of teaching practices, as well as offer some implications for design of effective PD environments.

The 5 Practices / Accountable Talk PD

The 5 Practices (5Ps) is a set of practices for orchestrating productive discussions around tasks that are characterized as "cognitively demanding" (Stein, Engle, Smith, & Hughes, 2008). The goal of the 5Ps is to make the orchestration of classroom discussions more manageable by moderating the amount of improvisation required by teachers during a lesson. In particular, the practices emphasize the importance of task selection, anticipating students' responses, selecting solutions to be presented in the discussion, sequencing these solutions thoughtfully, and linking between the different solutions in the whole-classroom discussion. Accountable Talk® (AT), which was the second basis for the studied PD program, provides teachers with a set of specific *talk moves* that hold students accountable to the learning community (e.g., “Can someone add on to what was said?”), to knowledge (e.g., “Can you give me an example?”), and to rigorous thinking (e.g., “What do you mean by…?").

The PD project providing the context to this research took place in a large urban district in the eastern United States. It was led by Margaret Smith and Victoria Bill, researchers and teacher educators from the Institute For Learning at University of Pittsburgh (https://ifl.pitt.edu/) and included 5 PD sessions during the 2014-2015 school year (total 31 hours) in addition to school-based support (i.e., one-on-one coaching, professional learning communities provided by trained mathematics coaches). Out of an average number of 40 teachers who attended the PD, 8 teachers participated in the larger study. For the present case study, we chose for closer analysis two teachers (pseudonymed Ms. Mathews & Ms. Wetherill) who co-taught one 6th grade classroom in a particularly low-achieving school, and were observed to be particularly engaged and devoted to the PD. Our research question was: How can the process of change that the two teachers underwent be characterized along the aspects that characterize ritual vs. explorative participation, namely: imitation vs. self-directedness, rigidity vs. flexibility, and logical incoherence vs. well-justified actions?

METHODS

Data included four cycles of lesson video-recordings (in September, December, February and April), including pre- and post-lesson interviews as well as recordings of all PD sessions. All lessons and interviews were fully transcribed. The analysis of teachers' ritual vs. explorative participation consisted of in-depth examination of the teachers' pedagogical discourse, as well as their actions during the lesson. Ritual engagement in the teaching practices was operationalized as rigid imitation of practices observed in the PD session and incoherence between teaching actions and underlying goals, whereas explorative engagement was characterized by flexibility and finer attunement of the practices to the context and to students' needs.

FINDINGS

Ms. Mathews and Ms. Wetherill (M&W from hereon) had a unique teaching arrangement, one which probably assisted them in implementing the ideas of the PD. Ms. Wetherill is a special education teacher, while Ms. M. is a general education teacher. Their 6th grade classroom had 24 students, 11 of whom were diagnosed as having special education needs. Unlike most US classrooms, where the special education teacher teaches "her" students separately from the rest of the class, M & W co-planned and co-taught all their lessons in that classroom. M & W's process of learning will be exemplified here through short descriptions of two lessons, one at the beginning and the second in the middle of the PD.

Lesson 1

For the 1st lesson, M & W chose the following problem:

> *A publishing company is looking for new employees to type novels that will soon be published. The publishing company wants to find someone who can type at least 45 words per minute. Dominique discovered she can type at a constant rate of 704 words in 16 minutes. Does Dominique type fast enough to qualify for the job? Explain why or why not.*

This problem offers multiple solution strategies, including constructing a ratio table and computing ratio against the given unit rate. These multiple routines can lead to the discussion of different solution paths, in addition to exploration of the "ratio" and "unit rate" concepts. Thus, the task held the potential for a productive classroom discussion. Yet the ways in which Ms. Mathews described the task selection indicated ritual participation in the practice of task selection. The main justification for choosing the task was it being in a certain place in the curriculum, not the mathematical goals to be achieved by it; another justification was that it would "prompt discussion". However, nowhere in the interview did M & W explicate what the discussion would be about. With relation to the practice of anticipating students' responses, Ms. Wetherill confessed: "we don't really know what they're going to do, which kind of, makes us a little nervous".

During the lesson, M & W started by reading the task aloud, together with the students. They then let the students work on the problem alone for 2-3 minutes after which they asked students to work on it in groups. This classroom routine of "launch the task, work individually, work in groups" was performed precisely as the recommended lesson format of the 5Ps. During the individual and group work time, M & W walked around the classroom, monitoring students' work while looking at their "monitoring sheets" (clipped on to their clipboards) – an artifact promoted by the PD to assist teachers in tracking the various anticipated and enacted solution paths. Their actions resembled quite precisely those of the PD leaders during the preceding PD session. Once the class was called for the whole-classroom discussion, M & W's actions continued to show rigid imitation of practices discussed (and observed) in the PD session, without coherence with the logic of these practices. For example, Ms. Mathews "took a poll" to see which groups thought Dominique (the typist) could get the job, yet despite the students having different opinions, she did not ask anyone to justify or argue their claims. Instead, M & W invited three students to present their solutions at the board. These presentations were heavily scaffolded by the teachers and in fact functioned as a proxy for the teachers' explanations of different solution paths to the problem.

Lesson 2

As October and November went by, M & W reported on trying out a "PD task" approximately every two weeks. Along the way, they discovered some things that worked better for their classroom than others. One of these discoveries was that their students react better to problems that were connected to their teachers' (personal) lives. Importantly, this message was never relayed in the PD. It was thus a creative insight that M & W came to independently. In line with this insight, Ms. Mathews prepared for the 2nd lesson a video with her own children playing a cards game, after which she asked her students to "help" her kids solve a problem: the three kids ended the game with the scores -1, -2 and -4 respectively. The students were asked to state who won the game and justify their answers.

This time, in the pre-lesson interview, M & W explained the choice (and design) of the task by relating to their students' current difficulties with understanding integers. There was thus coherence between the choice of the task and the goals of the lesson. In addition, there was flexibility in M & W's selection of the task, seen both in the change of the numbers (the numbers given in the textbook task, on which this task was based, were 0, -6 and -7. Ms. Mathews changed them to -1, -2 and -4) and in a change of the contextual aspects of the task (by video recording Ms. Mathews' children to enact the story of playing the game). In particular, the change in numbers was justified by the fact that M &W expected their students to identify 0 too easily as the winning number, indicating a growing familiarity with their students' specific understandings of integers.

The lesson enactment and orchestration of discussion were, similarly to the choice and design of the task, better attuned to the abilities and needs of the students. In a nutshell, the discussion revolved around the disagreement between one student, who claimed (-4) to be the winning number, and other students who thought it was (-1). The teachers used the Accountable Talk moves in this discussion to flexibly and sensitively call on students whom they thought, based on their previous monitoring of group discussions, could provide arguments for and against these claims. The discussion was thus genuine, students seemed very engaged, and meaningful thought processes seemed to be taking place during the lesson, rather than presentations of previously learned procedures. As such, the teachers' engagement with the practices of the 5Ps and AT was more explorative – namely logically coherent, flexible and well attuned to the goal of affording students opportunities to construct their own understandings.

DISCUSSION

The development from ritual to explorative participation in the 5Ps and AT could be seen in the case of M & W both in task selection (including anticipation of students' responses) and in the practice of orchestrating mathematical discussions. Viewing learning-of-teaching within a theory that conceptualizes learning as a process of becoming a participant in a certain discourse or activity, complexifies intuitive and commonplace notions about teachers' needing to first "believe" and "understand" how and why to teach in a certain way, before they can actually do it. Moreover, the reports about ineffectiveness of PD (e.g. Santagata et al., 2011; Spillane & Zeuli, 1999), and the common complaints that teachers do not implement the kind of teaching they were inducted into in their PD courses – may have their roots in the slowness and fragility of this kind of learning. It may also be due to the fact that not much is yet known on how to assist the movement from ritual to explorative participation. This latter issue is certainly a topic worthy of future research.

Acknowledgments

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LEARNING ABOUT MATHEMATICS TEACHING AND LEARNING FROM STUDYING RITUALS AND RITUALIZATION? SOME COMMENTS

Jill Adler

Does the ritual-exploration dyad introduced by Lavie here, and then developed in various ways in the following four papers offer new insights or advances in our field? My answer is yes, and importantly so. The research reported in each of the five papers helps us understand how and why routines - repetitive patterned actions - some of which are imitative, and thus akin to rituals, have significance in and for learning and doing mathematics, and learning and doing mathematics teaching across empirical sites and educational contexts. The papers provide important narratives that run contrary to the frequent and typically decontextualized bad press given to rituals and other so-called traditional forms of mathematical activity. As Lavie shows, more flexible, applicable, agentic participation in mathematics *begins with rituals as necessary initial forms of participation* that can, through a process of de-ritualization, enable entry into new discourses. An orientation to changing participation, to developing a new discourse through both ritual and explorative routines helps challenge persistent dichotomising in our field between procedural and conceptual orientations to mathematics. Investigations into mathematics teaching and learning through careful considerations of routines, in the context of the tasks that give rise to these, and rituals more specifically, can indeed take our field forward. Moreover, as a unit of analysis for research in mathematics education, routines have applicability: the research reported in the papers operationalises ritual and explorative routines in relation to different research question(s) and different empirical fields. I comment briefly here on how research in the other four papers here illuminate key issues for the research forum.

The empirical site for Viirman's study is tertiary learning and it occurs in the context of two disciplinary discourses - mathematics and biology. The tracing of ritual and explorative routines in the intersection between discourses is indeed a productive endeavour, revealing fluidity across routines and disciplinary boundaries in complex ways. Their results challenge oppositional binaries like relational and instrumental understanding, including the ritual-explorative conceptual dyad.

That rituals and explorations are not in opposition, and significantly, that rituals are important in the learning process, is reinforced by Nachlieli & Tabach's novel study. They investigated *ritual-enabling* and *exploration-requiring* opportunities to learn (OTLs) in and across the publicly available lessons (video-records, transcripts etc) in English from three countries in the 1999 TIMSS study (Australia, Hong Kong and the US). By juxtaposing the profiles they develop for each lesson, they provide evidence first of the prevalence of ritual-enabling routines; and then how ritual-enabling OTLs can serve as necessary starting points for exploration-requiring OTLs. They argue strongly for teaching to include both acknowledging there are no clear cut guidelines for this.

With this comes a challenge to the oft stated dichotomy in mathematics education - between traditional and reform teaching. Heyd-Metzuyanim takes up this challenge. She shows how a study of ritual and explorative routines in a professional development setting illuminates teachers learning the new discourses and practices of teaching. She shows a similar trajectory from ritual to exploration of two US teachers learning through their participation in PD over time. They demonstrated take up of desired practices that began with forms of imitation and moved to forms evidencing more agency and flexibility with the promoted practices. This study brings to the fore the fragility of learning new teaching practices and how the journey to explorative teaching might never be completed. This is sobering food for thought across much of the literature on mathematics PD that bemoans the equivocal impact of reform oriented PD on the quality of learning and teaching mathematics in general, and teachers' instructional practices in particular. Related here is a vexing question for those of us that work in PD and researching its take-up or teachers' learning from this: how we describe change or learning about teaching without a deficit description of the initial practice. For while the imitative towards more flexible explorative forms can be expressed in terms of their presences, it seems impossible to avoid a prior description of their absence (and so in terms of deficits) in participating teachers current or initial practices.

That the journey from ritual to exploration is complex, fragile and perhaps never completed is given further substance in Robertson's study of mathematics instructional talk in a Grade 4 classroom in South Africa. This study confronts the constraints teachers face when they are teaching in a bi/multilingual context of limited material resources, where in addition, students have limited linguistic resources in English, the language of instruction. It is also a context where traditional forms of teaching are associated with widespread failure. Working with a different conceptual dyad drawn from socio-linguistics – right-answerism vs exploratory talk – they open up just how much more complex is this journey in a multilingual and 'developing country' context. This study appears to move in parallel with the extensive research on teaching and learning mathematics in multilingual settings. This research has also drawn on sociolinguistics and illuminated the complex journey from informal talk in students main language (everyday, local or situated talk if you like) to formal talk and writing of mathematics in English (e.g. Setati & Adler, 2001). There are multiple tensions for teachers and teaching in this work with interesting recent research by Barwell positing an additional binary – situational-distal sources of meaning (Barwell, 2018). Barwell studied students' repertoires of meaning in a bilingual setting in Canada, showing the multiple ways students draw on sources of meaning, both situational and distal. This research reinforces the potential of orientations to interpreting practice that do not dichotomise analytic binaries, but explore their inter-relation.

Together these papers provoke interesting questions about the ritual-exploration dyad itself as it has been applied across empirical contexts, and then how this relates to the other dyads in our field that I have pointed to above.

References and reading materials for the audience

Adler, J. (2019). Learning about mathematics teaching and learning from studying rituals and ritualization? A commentary. *Educational Studies in Mathematics*, *101*(2), 291-299.

Adler, J., & Pillay, V. (2017). Mathematics education in South Africa. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 9-24). NY: Routledge.

Anderson, S. E. (1997). Understanding teacher change: Revisiting the concerns based adoption model. *Curriculum Inquiry*, *27*(3), 331–367. https://doi.org/10.2307/1180105

Barnes, D. (2010). Why talk is important. *English teaching: Practice and critique, 9*(2), 7-10.

Barwell, R. (2018). Repertoires of three second language learners of mathematics: distal and situational sources of meaning. *Research in Mathematics Education*. doi:10.1080/14794802.2018.1490664

Blum, W., Galbraith, P. L., Henn, H.-W., & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education: The 14th ICMI study*. New York: Springer.

Brewer, C., & Smith, D. (Eds.). (2010). *Vision and change in undergraduate biology education: A call to action*. Washington DC: AAAS.

Brown, K. L. (2003). From teacher-centered to learner-centered curriculum: Improving learning in diverse classrooms. *Education, 124*(1), 49-55.

Cortina, J. L., Visnovska, J., & Zuniga, C. (2014). Unit fractions in the context of proportionality: Supporting students' reasoning about the inverse order relationship. *Mathematics Education Research Journal*, *26*, 79–99.

Cummins, J. (1994). The acquisition of English as a second language. In K. Spangenberg-Urnschat & R. Pritchard (Eds.), *Kids come in all languages: Reading instruction for ESL students* (pp. 36-62). Newark: International Reading Association.

Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire*. Clevedon: Multilingual Matters.

Cummins, J. (2005). Teaching for cross-language transfer in dual language education: Possibilities and pitfalls. In *TESOL Symposium on dual language education: Teaching and learning two languages in the EFL setting* (pp. 1-18).

Cummins, J. (2015). How to reverse a legacy of exclusion? Bilingual and immersion programs. Identifying high-impact educational responses. *Language and Education*, *29*(3), 272-279.

Fuson, K. C., Secada, W. G., & Hall, J. W. (1983). Matching, counting, and conservation of numerical equivalence. *Child Development*, *54(1)*, 91-97.

Gibbons, P. (2006). *Bridging discourses in the ESL classroom: Students, teachers and researchers*. London: Continuum.

Gibbons, P. (2009). *English students, academic literacy, and thinking: Learning in the challenge zone*. Portsmouth: Heinemann.

Gregg, J. (1995). The tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education*, 442-466.

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, *70*, 49–70.

Heugh, K. (2005). Mother-tongue education is best. *HSRC Review*, *3*(3), 6-7.

Heyd-Metzuyanim, E. (2015). Vicious cycles of identifying and mathematizing – a case study of the development of mathematical failure. *Journal of the Learning Sciences*, *24*(4), 504–549.

Heyd-Metzuyanim, E., & Graven, M. (2016). Between people-pleasing and mathematizing - South African learners ’ struggle for numeracy. *Educational Studies in Mathematics, 91*(3), 349–373.

Heyd-Metzuyanim, E., Tabach, M., & Nachlieli, T. (2016). Opportunities for learning given to prospective mathematics teachers: between ritual and explorative instruction. *Journal of Mathematics Teacher Education*, *19*, 547–574.

Heyd-Metzuyanim, E., Smith, M., Bill, V., & Resnick, L. B. (2018). From ritual to explorative participation in discourse-rich instructional practices: a case study of teacher learning through professional development. *Educational Studies in Mathematics*, *101*(2) 273-289.

Heyd-Metzuyanim, E., & Graven, M. (2019). Rituals and explorations in mathematical teaching and learning: introduction to the special issue. *Educational Studies in Mathematics*, *101*(2), 141–151.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., ... & Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis*, *27*(2), 111-132.

Lavie, I., & Sfard, A. (2016). How children individualize numerical routines – elements of a discursive theory in making (in Hebrew). *Studies in Mathematics Education* (Iyunim BeHinuch Matematic), 22-68 ,4.

Lavie, I., Steiner, A., & Sfard, A. (2018). Routines we live by: from ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153-176.

Maton, K. (2011). Theories and things: The semantics of disciplinarity. In F. Christie & K. Maton, (Eds.), *Disciplinarity: Systemic functional and sociological perspectives* (pp. 62-84). London; Continuum.

Nachlieli, T., & Tabach, M. (2018). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253-271.

Robertson, S. A., & Graven, M. (2018). Exploratory mathematics talk in a second language: a sociolinguistic perspective. *Educational Studies in Mathematics*. *101*(2), 215-232.

Russac, R. (1978). The relation between two strategies of cardinal number: Correspondence and counting. *Child Development, Vol. 49, No. 3*, 728-735.

Santagata, R., Kersting, N., Givvin, K. B., & Stigler, J. W. (2011). Problem implementation as a lever for change: An experimental study of the effects of a professional development program on students' mathematics learning. *Journal of Research on Educational Effectiveness*, *4*(1), 1–24.

Setati, M., & Adler, J. (2001). Between languages and discourses: Code-switching practices in primary mathematics classrooms in South Africa. *Educational Studies in Mathematics., 43*, 243-269.

Setati, M., Chitera, N., & Essien, A. (2009). Research on multilingualism in mathematics education in South Africa: 2000–2007. *African Journal of Research in Mathematics, Science and Technology Education*, *13*(sup1), 65-80.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? — Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237–309.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York, NY: Cambridge University Press.

Siegler, R. S. and Svetina, M. (2006), What leads children to adopt new strategies? A microgenetic/crossectional study of class inclusion. *Child Development*, 77: 997-1015. doi:10.1111/j.1467-8624.2006.00915.x

Spillane, J. P., & Zeuli, J. S. (1999). Reform and teaching: Exploring patterns of practice in the context of national and state mathematics reforms. *Educational Evaluation and Policy Analysis*, *21*(1), 1–27.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*, 313–340.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), Learning and teaching mathematics: An international perspective (pp. 5-28). Hove, England: Psychology Press/Erlbaum (UK) Taylor & Francis.

Viirman, O., & Nardi, E. (2017). From ritual to exploration: The evolution of Biology students' mathematical discourse through Mathematical Modelling activities. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Conference of European Research in Mathematics Education* (pp. 2274-2281). Dublin, Ireland.

Viirman, O., & Nardi, E. (2018). Ritualised and exploratory graphing routines in mathematical modelling: The *Digoxin* task. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education, Vol. 4* (pp. 363-370). Umeå, Sweden: PME.

Viirman, O., & Nardi, E. (2019). Negotiating different disciplinary discourses: Biology students' ritualized and exploratory participation in Mathematical Modelling activities. *Educational Studies in Mathematics*, *101*(2), 233-252.

Vygotsky, L. S. (2012). Thought and language. Cambridge, MA: MIT Press.

WORKING GROUPS

STEM AND SOCIAL JUSTICE EDUCATION: INITIATING INTERNATIONAL DISCUSSIONS

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This working group will build on two agendas which have attracted attention at recent PME conferences – STEM education and teaching mathematics for social justice. In 2018 Anderson and Li convened a PME Working Group (Anderson & Li, 2018) to explore the role of mathematics in integrated STEM education and effective strategies for integrated STEM education policies and practices. An outcome of this 2018 WG is a Springer volume *Integrated Approaches to STEM Education: An International Perspective*, with chapters submitted to this volume currently under review. A second agenda is the area of teaching mathematics for social justice. PME research reports by Nicol, Bragg, Radzimki, He and Yaro (2017) and Anderson and Kreisler (2018) focused on the need for mathematics education to embrace social justice approaches to address diversity and equity issues in the classroom but also to engage students in what Gutstein (2006) drawing from Freire (1976/2000) refers to as reading (interpreting) and writing (transforming) the world with mathematics. In both RR sessions participants expressed keen interest in continuing the conversation focused on mathematics and social justice. A question discussed during Nicol and colleagues' RR was the need for teachers and academics to have not only a deep understanding of mathematics but also other disciplines in order to support students' inquiry of mathematics within social issues. For example, mathematics problems developed to explore the impacts of hunger and poverty could require exploring connections to climate change, world population growth, and food security technologies.

Mathematics education for social justice therefore opens-up connections across disciplines providing opportunities for mathematical inquiry through STEM fields. Conversely, the emerging field of integrated STEM education can benefit from a critical perspective. A social justice approach to STEM education has potential to move justifications for STEM education beyond career and economic revitalization to also include critical engagement, that is, using STEM fields to understand and respond to local and global issues. We see the importance of this proposed working group as providing a unique opportunity for academics interested in these fields to come together to explore possible synergies and collaborations. This working group meeting will, therefore, provide a platform for international scholars to explore the potential of exploring integrated STEM education and social justice mathematics education. What do approaches such as critical mathematics education (Skovsmose, 2012) offer integrated STEM perspectives? How can a social justice approach to STEM education open-up critical arenas for curriculum projects and research across international

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 167-168). Pretoria, South Africa: PME.

contexts? Our discussions will draw upon research and practice in the fields of science, technology, engineering and mathematics and the challenges of educational practices that embrace social justice perspectives. The working group discussions will consider the themes of developing a conceptual basis for teaching and learning mathematics in integrated STEM education for social justice, reviewing research that spans STEM and social justice fields, and as this is a relatively new area of research, developing possible collaborative research agendas across cultures and international contexts.

PLAN FOR WORKING GROUP SESSIONS 1 AND 2

30 mins	**Session 1** – Brief introduction – overview of STEM perspectives from previous two conferences, Judy Anderson and Yeping Li, and overview of social justice perspectives, Cynthia Nicol
45 mins	Participants share STEM education perspectives and social justice perspectives to develop shared understandings of research globally and to raise questions for research agendas
15 mins	Summary of common (existing and/or emerging) themes, topic areas, questions and making connections with scholars with similar interests
30 mins	**Session 2** – Summary of Session 1. Short presentations on research projects from representatives in the working group
45 mins	Discussing possible future research collaborations and publications for the next PME conference in Thailand
15 mins	Developing a final set of outcomes for the discussion group and strategies for continued communication and collaboration

Table 1: Outline of the two sessions

References

Anderson, J., & Kriesler, A. (2018). *Making maths matter: Engaging students from low SES schools through social justice contexts.* Paper presented at the 42nd Annual Conference of the International Group for the Psychology of Mathematics Education. Umeå: PME.

Anderson, J., & Li, Y. (2018). *Integrating mathematics in STEM education: an international perspective.* Paper presented at the 42nd Annual Conference of the International Group for the Psychology of Mathematics Education. Umeå: PME.

Freire, P. (1970/2000). *Pedagogy of the oppressed*. New York, NY: Bloomsbury.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Towards a pedagogy for social justice.* New York, NY: Routledge.

Nicol, C., Bragg, L. A., Radzimki, V., He, T., & Yaro, K. (2017). *Teaching mathematics for justice: Pedagogies of discomfort, contradictions and dialogue*. Paper presented at the 41st Conference of the International Group for the Psychology of Mathematics Education. Singapore: PME.

Skovsmose, O. (2012). Towards a critical mathematics education research programme? In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education* (pp. 343-368). Netherlands: Sense.

MATHEMATICAL THINKING

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THEORETICAL BACKGROUND

Learning mathematics is as much a process of learning to actively participate in the activity of doing mathematics, as it is one of formal reproduction. For this reason, teachers need to be able to facilitate their learners' mathematical engagement in such a way that learners come to participate in a culture and practice of authentic mathematical activity (Keng & Kian, 2010). To do this, teachers themselves need to have experienced and participated in the processes and practices that constitute authentic mathematical activity. The development of mathematical thinking is an essential part of the process and not the product of mathematical activity (Mason, Burton & Stacey, 2010). This working group will focus on the development of an analytical framework for mathematical thinking and the possibility of using such a framework for more effectively embedding mathematical thinking processes in the teaching of mathematics in initial teacher education (ITE) programmes. We have begun to characterise active process elements that can be identified in the process of mathematical activity – ways of engagement that we notice as people do mathematics. These elements occur in combination with other elements of mathematical thinking but none of them occur individually. They may be categorised as actions and dispositions although we will focus exclusively on actions as part of this working group session. The proposed analytical framework organizes these elements into 4 possible categories each corresponding to a different orientation of mathematical engagement. These are analytical distinctions, as the process of mathematical thinking involves creative balancing and moving between these different orientations and elements. The orientations are 1) Playful engagement to develop, or search for, mathematical insight, 2) Represent and use mathematics; 3) Develop mathematical productions and 4) Reason and reflect. The characteristic elements of each orientation will be presented and discussed in more detail within the working group session.

GOAL AND ORGANIZATION OF THE WORKING GROUP

The goal of this working group is to present, critique and refine an analytical framework for mathematical thinking and to formulate ways to deepen prospective teacher's consideration of mathematical thinking processes in classrooms through the use of this, or similar analytical frameworks. The framework, which includes different orientations of mathematical engagement, is informed by various literature in the field

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of mathematical thinking, processes and practices (Devlin, 2012; Katz 2014; and Schoenfeld, 2017).

We, the proposers of this working group, plan to devote the first 90-minute working group session to the presentation, critique and refinement of the analytical framework. This will involve presentations form the working group participants (national and international) followed by small group discussions related to the critique and refinement of the framework. We will reconvene and consider feedback from the groups based on the input and their discussions. We then hope to finalise an updated version of the framework. The second session will focus on the development of tools that incorporate this framework for foregrounding, teaching for, and assessing, mathematical thinking in ITE programmes. We will draw from the reporting of Goos (2018) on interesting approaches to assessing reasoning and different forms of assessment that can be found in other research studies. This session will also include discussion of the research experience of Kullberg, Runesson and Mårtensson (2014) on the design and implementation of tasks. The two session summaries and participants inputs will be used in the preparation of a Research Report for submission to PME44.

References

Devlin, K.J. (2012). *Introduction to mathematical thinking*. Palo Alto: Keith Devlin.

Goos, M. (2018). *A review of research on understanding and promoting students' mathematical thinking published in Educational Studies in Mathematics 2014 – 2018.* First PME Regional Conference. Chile: South America.

Katz, J. (2014). *Developing Mathematical Thinking: A guide to rethinking the mathematics* Maryland: Rowan and Littlefield.

Keng, K.N. & Kiam, L.H. (2010). Learning Mathematical Concepts through Authentic Learning. In L. Sparrow, B. Kissane & C. Hurst (Eds), *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

Kullberg, A., Runesson, U., & Mårtensson, P. (2014). Different possibilities to learn from the same task. *PNA, 8*(4): 139-150.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking Mathematically* (2nd Edition). Essex, England: Pearson Education Limited.

Schoenfeld, A. H. (2007). What is Mathematical Proficiency and How can it be assessed? In Schoenfeld, A. H, (Ed.). *Assessing Mathematical Proficiency* (pp. 59-73). Cambridge: Cambridge University Press.

EARLY YEARS TEACHER EDUCATION NUMBER SENSE CURRICULA

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This working group focuses on mathematics curricula of initial teacher education programmes concerned with preparing teachers for teaching mathematics in the early years of elementary schooling (Foundation Phase). The aim of the working group is to engage PME participants in discussions regarding the knowledge required by prospective teachers for developing Foundation Phase learners' number sense. Participants will be invited to consider a proposed Foundation Phase teacher education number sense curriculum and to interrogate the theoretical underpinnings of the proposed curriculum.

THEORETICAL BACKGROUND

Number sense, initially described as an intuitive feel for numbers or a friendliness towards numbers (Howden, 1989; Dehaene, 2011), is viewed as necessary for the development of both an understanding of school mathematics, and the mathematics required to function effectively in everyday life (Graven, Venkat, Westaway, & Tshesane, 2013). Number sense in its basic form is a component of what cognitive scientists refer to as *core domain knowledge* (Spelke & Kinzler, 2007), is bound up with intuitive perceptions of and actions on aggregates, and is learned without much conscious effort. School mathematics proper is a species of *noncore domain knowledge* and requires explicit teaching and learning (Spelke & Kinzler, 2007). Gelman (2009, p. 247) describes a domain of knowledge as "a set of coherent principles that form a structure and contains domain-specific entities […] that can combine to form other entities within the domain". The point made is that the teaching of school mathematics in the foundation years is often such that it does little to effect the growth of number sense beyond core domain conceptions of aggregates and operations on aggregates. Further, teacher education often fails to sensitise preservice teachers to the distinction drawn here between domains of knowledge and of the implications of such a distinction for the pedagogy of school mathematics. This working group thus, attempts to address this challenge, by developing an understanding of the knowledge prospective Foundation Phase teachers require in order to develop learners' number sense.

GOALS

The central theme is mathematics teacher education curricula, specifically, the knowledge required to prepare Foundation Phase preservice teachers to develop their learners' number sense. We intend to draw on the expertise of participants to provide feedback on a proposed Foundation Phase teacher education number sense curriculum.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 171-172). Pretoria, South Africa: PME.

We plan to invite collaboration with international partners in preparing a Research Report for submission to PME44 on the implementation of number sense curricula.

PLANNED ACTIVITIES

Session 1

- Introduce the aims of the working group and the two sessions.
- Contextualise the focus on number sense and Foundation Phase preservice teacher education.
- Present a theoretical basis for the working group drawing on number sense literature.
- Plenary discussion on number sense literature.
- In four groups, participants will engage in a discussion on the knowledge prospective Foundation phase teachers require to develop learners' number sense.
- Plenary presentation and discussion on each groups' knowledge requirements.

Session 2

- Review the work from session 1.
- Present a proposed Foundation Phase teacher education number sense curriculum.
- In four groups, participants will engage in commenting on a proposed Foundation Phase teacher education number sense curriculum.
- Plenary presentation on the feedback of the proposed number sense curriculum.
- Summation of the discussion and an invitation to international partners who wish to engage in collaborative research.

References

Dehaene, S. (2011). *The Number Sense* (2nd Edition). New York: Oxford University Press.

Gelman, R. (2009). Learning in core and noncore domains. In L. Tommasi, M.A. Peterson & L. Nadel (Eds.), *Cognitive biology: Evolutionary and development perspectives on mind, brain, and behavior*. Cambridge, MA: MIT Press.

Graven, M., Venkat, H., Westaway, L., & Tshesane, H. (2013). Place value without number sense: exploring the need for mental mathematical skills assessment within the Annual National Assessments. *South African Journal of Childhood Education*, *3*(2), 131-143.

Howden, H. (1989). Teaching number sense. *Arithmetic Teacher, 36*(6), 6-11.

Spelke, E. S., & Kinzler, K. D. (2007). Core knowledge. *Developmental Science*, *10*(1), 89–96.

MATHEMATICAL LEARNING DISABILITIES: A CHALLENGE FOR MATHEMATICS EDUCATION

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CONTEXT

In recent times, research interest in learning difficulties has increased around the globe. Some of them are still subject to little research (Lewis & Fisher, 2016). This is the case of Mathematical Learning Disabilities (MLD) which are the source of raising educational and social inequalities. Research regarding MLD is carried out in different fields, with various theoretical backgrounds, research hypothesis and aims (Lewis & Fischer, 2016; Scherer & al., 2016): cognitive sciences, neuroscience, psychology, mathematics education. There is not a clear scientific consensus about MLD definition and diagnosis. Moreover, the links between these different fields of research are not enough developed and they should be improved. Our team - called RITEAM (see riteam.ch) - claims that specific studies should be structured and developed in mathematics education regarding MLD in order to improve the identification and the remediation of MLD in an educational context (Dias & Ouvrier-Buffet, 2018). In particular, that implies a better knowledge of the existing research dealing with MLD.

At PME 42, we already proposed a WG with the main aim of identifying current and future research interests about MLD in math education (Ouvrier-Buffet & al., 2018). For that, we proposed a short survey about MLD to collect information regarding different countries. This work allowed us to make a first picture of the international practices about MLD and to confirm the necessity continuing those research in math education. Moreover, we have identified specific keywords about MLD used in different countries. Following the results from the WG of PME42, we have started a literature review on MLD which will be a starting point for the WG of PME43.

PLANNED ACTIVITIES

This WG is conceived in the continuity of the WG of PME42. It aims at the creation and dissemination of a survey getting a full picture of the international educative practices about MLD. The international community of PME will be fundamental for the adaptation of the questions to different cultural contexts. This survey will allow us to compare educative practices in different countries where policies can be pretty different (Scherer & al., 2016) and to federate collaborative research about remediation and teachers training.

The objective of the first session will be a discussion around the MLD question and a first draft of the survey. The session will start with a presentation of the research group RITEAM, the results and the issues of the previous WG (PME42), the first results of

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 173-174). Pretoria, South Africa: PME.

the ongoing literature review and the objectives of the current WG (15min). Then, we will share fundamental bibliographic references about MLD thanks to the inscription to a Mendeley public group created for the previous WG (15min). The participants will be divided in groups and a topic related to MLD will be assigned at each group (for example, how schools deal with MLD students in their country). They will be driven in a discussion about this topic with the objective to identify some crucial questions for the survey (1h).

The objective of the second session will be the presentation and amelioration of the survey conceived in session one and its public release via google forms (1h15). During this time, each group will explain the questions they created. Other participants can intervene with remarks to adjust and improve the questions proposed. When the survey will be stabilised, the participants will be asked to share the survey with some of their contact (15min).

OUTCOME

Through the survey creation, the working session will bring an international contribution on research about MLD. This is essential given the huge cultural differences across countries. The participants will also be provided with fundamental bibliographic texts.

References

Dias, T., & Ouvrier-Buffet, C. (2018). Perspectives de recherches sur les difficultés d'apprentissage en mathématiques. *Revue de Mathématiques pour l'école (RMé), 229*, 47-53.

Lewis, K.E. & Fisher, M.B. (2016). Taking stock of 40 years of research on mathematical learning disability: methodological issues and future direction. *Journal for Research in Mathematics Education, 47(4)*, 338-371.

Ouvrier-Buffet, C., Robotti, E., Dias, T., and Gardes, M.-L. (2018). Mathematical Learning Disabilities: a challenge for mathematics education. In Bergqvist, E., Österholm, M., Granberg, C., & Sumpter, L. (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 207-208). Umeå, Sweden: PME.

Scherer, P., Beswick, K., DeBlois, L., Healy, L., & Opitz, E. M. (2017). Assistance of Students with Mathematical Learning Difficulties—How Can Research Support Practice?—A Summary. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp. 249-259). Cham: Springer.

CONCEPTUALISING THE EXPERTISE OF THE MATHEMATICS TEACHER EDUCATOR

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BACKGROUND

The aim for this working group is to further the development of research within the domain of the mathematics teacher educator (MTE), specifically to move beyond descriptions of MTE knowledge towards ways of conceptualising and researching the *expertise* of the MTE. The phenomenon of *expertise* of the MTE is not easily defined. We follow Beswick and Goos (2018) by using the label MTE as "anyone engaged in the education or development of teachers of mathematics" (p. 418) and recognise that MTE, as a role, encompasses a diverse set of practices within mathematics teacher education. Recently, within mathematics teacher education, there has been increasing interest in the development of theories that can account for what and how MTEs learn; for example, the publication of a recent special issue of the *Journal of Mathematics Teacher Education* where some scholars have extended existing models of mathematics teacher knowledge as a way of describing the knowledge of the MTE (e.g., Leikin, Zazkis & Meller, 2018). This working group builds on foundations from previous PME working sessions that have been centred around MTEs (Goos, Chapman, Brown, & Novotna, 2011; Beswick, Goos, & Chapman, 2014) and looks to extend existing conceptualisations of the expertise of MTEs beyond descriptions of MTE knowledge (e.g., Appova & Taylor, 2019).

AIMS OF WORKING GROUP

- To begin to theorise the expertise of MTEs by exploring personal stories, experiences and a variety of frameworks.
- To formulate researchable questions.
- To explore and develop potential methodologies that support these questions.

OUTLINE OF SESSIONS

Session1

- Introductions and sharing of experiences that inform ways of thinking about expertise and asking: what specific expertise do we have as MTEs?; how might we differentiate between MTE expertise and other domains of expertise, for example, teacher educator expertise more broadly?; The two presenters will begin by sharing their stories as MTEs (of practice, context, research). Attendees will then be invited to share their own stories leading to a discussion of themes, commonalities, differences, etc.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 175-176). Pretoria, South Africa: PME.

- Brief presentation of existing ways that the expertise of MTEs has been described (considering what has not yet been asked or answered) moving to some suggestions for possible new approaches to and conceptualisations for describing MTE expertise that offer alternatives to expertise as knowledge.
- Discussion in groups with a focus on the development of frameworks that support the interaction between the practice of MTEs and conceptualisations of MTE expertise.

Session 2:

- Building off session 1, groups discuss: Creating a set of researchable questions in the area of MTE learning/expertise and consider i) which of these questions would you like to/be able to research? ii) what theoretical frameworks could you use/develop to support this research? iii) what methodologies are appropriate for these questions? iv) what would data consist of?
- Each group to share responses and then discuss and agree on next steps for future collaborations, including consideration of a possible joint output for participants such as a special issue for the *Journal of Mathematics Teacher Education*.

References

Appova, A., & Taylor, C. (2019). Expert mathematics teacher educators' purposes and practices for providing prospective teachers opportunities to develop pedagogical content knowledge in content courses. *Journal of Mathematics Teacher Education, 22*(2), 179-204

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education, 21*, 417-427.

Beswick, K., Goos, M., & Chapman, O. (2014). Mathematics Teacher Educators' Knowledge. Working Session. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, p. 254). Vancouver, Canada: PME.

Goos, M., Chapman, O., Brown, L., & Novotna, J. (2011). The learning and development of mathematics teacher educator-researchers. Working Session 5. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 173). Ankara, Turkey: PME.

Leikin, R., Zazkis, R., & Meller, M. (2018). Research mathematicians as teacher educators: focusing on mathematics for secondary mathematics teachers. *Journal of Mathematics Teacher Education, 21*, 451-473.

SUPPORTING (PRESERVICE) TEACHERS' LEARNING TO USE CURRICULUM RESOURCES PRODUCTIVELY

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In this new working group, we focus on ways to support (preservice) teachers' learning to use curriculum resources (a set of resources for daily instruction including student texts, teachers' guide, and digital resources) productively. Our goal is to develop research plans to investigate this topic, based on presentations of examples and issues to consider, and discussions with working group participants on what productive resource use means and how we study it, especially in different cultural contexts.

THE TOPIC, GOAL, AND STRATEGY

The goal of this new working group is to make a set of actionable research plans to investigate ways to support (preservice) teachers' learning to use curriculum resources productively. We will start with two pilot studies to have some specific contexts to think about productive resource use. We will also present issues to consider (such as types of resources and cultural contexts) in investigating the topic. Based on the presentations, the participants will discuss what it means to use resources productively and how we study it, especially in different cultural contexts. The participants will also be invited to share their experiences and efforts in this topic. Finally, we will develop research plans for the organizers and participants to work on in the coming year(s). We plan to present the results in a subsequent conference; we may need to continue the working group another year depending on the plans and the results of the work.

THEORETICAL BACKGROUND

In many countries, mathematics curriculum resources (a set of resources for daily instruction including student texts and teachers' guide) serve as a main tool for instruction. Brown's (2009) notion of Pedagogical Design Capacity (PDC)—"individual teachers' ability to perceive and mobilize existing resources" (p. 29)—highlights the importance of teachers' productive resource use. In this context, the two pilot studies draw on different frameworks to support teachers to develop such capacity. We will also relate PDC to digital resources as they come with new challenges related to effective use (e.g., Pepin, Choppin, Ruthven, & Sinclair, 2017).

Frameworks in two examples

One example is based on tasks involved in productive resources use, such as *identifying the mathematical points of individual lessons*, and *recognizing affordances and constraints of the resources they use* (Kim, in press). These tasks are incorporated in activities to support preservice teachers' learning of productive resource use in a methods course. To examine the productiveness of those teachers' resource use,

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 177-178). Pretoria, South Africa: PME.

Sleep's (2009) conceptualization of teaching toward the mathematical point is utilized. Sleep identified specific strategies and issues in the task, which serve as a useful tool to discern what teachers do well and what they need to work on in classroom teaching.

Another example is engaging prospective secondary teachers (PSTs) in the acts of offloading, adaptation, and improvisation (Brown, 2009) of mathematics lessons incorporating a variety of digital resources (e.g., Desmos). PSTs work within different participatory structures (individually, with the instructor, or with the classroom community) in the design, refinement, and enactment of a lesson (*design cycle*). Working group participants will examine data from design cycles to develop analytical frameworks rooted in PDC (Brown, 2009) to measure productive designs and consider how the design cycle can be revised to increase the likelihood of productive designs.

WG ORGANIZATION AND PARTICIPANT ENGAGEMENT

In the first session, the two pilot studies described above (20 minutes each) will be shared to provide the setting for discussions (50 minutes) concerning what it means to use resources productively, what this looks like from the perspective of different cultural contexts, and how we can investigate it within and across different cultural contexts. In the second session, we will use 30 minutes to facilitate discussions around issues to consider in investigating this topic, such as curriculum resources as cultural artifacts (e.g., U.S. standards-based textbooks), the broader educational contexts in which they reside, and challenges related to incorporating digital resources. The remaining 60 minutes will be used to consider the broader implications of this work and to assist participants in developing their own research plans to investigate the productive use of curriculum resources. We will also set up a collaborative research agenda. The participants will be also invited to conduct collaborative research with the organizers, contribute to the subsequent working group if organized again, or present their results at future PME conferences.

References

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). New York: Routledge.

Kim, O. K. (in press). Teacher capacity for productive use of existing resources.

Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: foundations for change. *ZDM, 49*(5), 645-661.

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal, 49*(5), 935–970.

INTERNATIONAL PERSPECTIVES ON EVOLUTION OF RESEARCH ON TEACHING MATHEMATICS

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In this working group, the international group of researchers will explore current issues related to the evolution of research on teaching mathematics. The goal is to examine the current state of presage-process-product research in mathematics with respect to conceptualization, instrumentation, and design, and to explore the likely direction of further developments. In the past twenty years, researchers used a wide range of conceptual and theoretical frameworks in an effort to advance knowledge in presage-process-product research in mathematics education. The discussion of theoretical and methodological challenges associated with developing a domain, instrumentation, and design and analysis of this aforementioned research will be included for participants in the WG activities led by a diverse group of researchers.

OUTLINE OF THEORETICAL BACKGROUND

Researchers used different conceptual frameworks in presage-process-product research in mathematics education when discussing the relationships between 1) activities teachers do outside of the classroom, such as planning, assessment, etc.; 2) activities teachers do inside of the classroom such as presenting a lesson, asking questions, reacting to students' answers, etc.; 3) student learning activities in the classroom; and 4) student learning outcomes measured after the teaching (e.g., Blömeke, Busse, Kaiser, König & Suhl, 2016; Liljedahl, 2016; Martinovic & Manizade, 2018; Medley, 1987; Ribeiro, Mellone & Jakobsen, 2016). This WG based on the aforementioned theoretical frameworks will discuss evolution of such research in math education.

WORKING GROUP AS A NEW INITIATIVE

The purpose of this WG is to provide a platform for sharing international perspectives on aforementioned variables in the context of presage-process-product research in mathematics. Our goal is to start a dialogue amongst researchers in order to provide a critical review of currently existing presage-process-product research and discuss strengths and limitations associated with conceptualization, developing a domain, instrumentation, and research design. By doing so, our intent is to identify future paths for research on effective mathematics teaching in the digital era.

SESSION 1: AIMS AND THEORETICAL FRAMEWORKS

Opening presentation (Makonye, 10 min.): Different purposes and conceptualizations in current presage-process-product research in mathematics education. *Discussion*

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 179-180). Pretoria, South Africa: PME.

opening (Manizade, 20 min.): Overview of current research connecting the following variables: 1) pre-existing teacher characteristics, and math teachers' competencies, knowledge, and skills; 2) math teachers' activities outside of the classroom (planning, assessment, etc.); 3) teacher-led activities during math instruction; 4) student learning activities in mathematics; and 5) student learning outcomes. *Discussion* (50 min.): All invited researchers and WG participants will engage in discussion of relationships between aforementioned variables identified by the researchers in the current presage-process-product research in mathematics education. *Summary* (Jakobsen, 10 min.): The intent of the summary discussion is to create a diagram describing connections amongst aforementioned variables.

SESSION 2: THEORETICAL AND METHODOLOGICAL CHALLENGES

Opening presentation (Ribeiro, 10 min.): Overview of current research on variables that can affect student learning but are not under direct control of the teacher (e.g. individual student characteristics; internal context variables; external context variables such as support systems, technology, materials, etc.; and teacher experiences). *Discussion opening* (Mellone, 15 min.): Discussion of theoretical and methodological challenges associated with conceptualization, instrumentation, and research design of current presage-process-product research in mathematics education. *Discussion* (55 min.): All invited WG participants will engage in discussion of research with respect to conceptualization, instrumentation, design, and potential direction of further developments of research. *Closing discussion* (Manizade, 10 min.): Summary of research discussed in the WG and Springer book proposal.

References

Blömeke, S., Busse, A., Kaiser, G., König, J., & Suhl, U. (2016). The relation between content-specific and general teacher knowledge and skills. *Teaching and Teacher Education*, *56*, 35-46.

Liljedahl, P. (2016). Building Thinking Classrooms: Conditions for problem-solving. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and Solving Mathematical Problems* (pp. 361–386). Switzerland: Springer International Publishing.

Martinovic, D., & Manizade, A. G. (2018). The challenges in the assessment of knowledge for teaching geometry. *ZDM*, *50*(4), 613-629.

Medley, D. M. (1987). Evolution of research on teaching. In M. Dunkin (Ed.), *The international encyclopedia of teaching and teacher education* (pp. 105–113). New York: Pergamon.

Ribeiro, M., Mellone, M., & Jakobsen, A. (2016). Interpretation students' non-standard reasoning: Insights for mathematics teacher education. *For the Learning of Mathematics*, *36*(2), 8-13.

INTERNATIONAL PERSPECTIVES ON PROOF AND PROVING

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This working group is a new initiative, bringing together research on proof and proving and international comparison. The aim is to foster research on proof and proving from an international perspective, and to continue at the next two PME annual conferences.

The past two decades has seen a strong increase in research into proof and proving in mathematics education. Much of this has been conducted in single national and cultural contexts. This means that it is not clear whether the results are transferable, or indeed if the assumptions on which the studies are based are valid elsewhere. There are four areas in which international comparisons could shed light on the teaching and learning of proof and proving: curriculum, teaching resources (including textbooks), student achievement, and teaching practices (the latter encompassing teacher knowledge and teacher education).

Very little information exists about the role of proof in curricula internationally. Existing curricular comparisons do not focus on proof, and the few comparisons that have focussed on proof have compared only a few contexts (e.g., Hemmi et al., 2013). Similarly, while teachers might be guided by textbooks or curriculum documents, existing research (e.g., Fujita, & Jones, 2013; Miyakawa, 2012) has compared such practices in only a few contexts. In international comparisons of student achievement (e.g., TIMSS and PISA), very little is known about how students' skills to construct and interpret proofs varies internationally. Results from a TIMSS 1995 item suggests that the variance might be quite wide (Reid, 2015). Comparisons of classrooms practices in teaching proof exist (e.g., Knipping, 2001), but, given the difficulty of conducting such research, are limited to a few contexts.

The Working group focuses on a series of research questions such as:

- To what extent, and to which students, is proof taught in different countries and what goals for teaching proof apply in different countries/cultural contexts?
- How do teaching resources for teaching proof differ between contexts?
- How capable in proof and proving are students in different contexts?
- What teaching practices for proof and proving occur in different contexts and how is teacher knowledge and teacher education developed?

Outcomes are intended to include a reference list of existing research on proof and proving from an international perspective, a preliminary report on the role of proof in national/regional curricula, textbooks, student achievement and teaching practices from the working group participants, and the organising of networks of researchers

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 181-182). Pretoria, South Africa: PME.

interested in collaborating on comparative research focussed on curricula, textbooks, student achievement or teaching practices. The latter can begin at the working group meeting and continue after it.

STRUCTURE OF THE SESSIONS

Involvement of participants is essential to the functioning of the group. Over the two sessions the following activities are planned:

- Introduction to the working group, research questions, and the focus areas of curricula, textbooks, student achievement and teaching practices.
- Introductions of participants and identification of overlapping experiences in various national contexts.
- Formation of context specific subgroups, to prepare initial sketches of the relative importance of textbooks and curriculum documents in each context, the role of proof in the curriculum and textbooks, and existing research on proof and curricula, textbooks, student achievement and teaching practices in each context.
- Brainstorming of collaborative projects focussed on specific research questions and focus areas.
- Formation of project specific subgroups, to prepare initial sketches of the project and to outline plans for the coming year. Participants can choose to be involved in a project as leaders, participants, or informants, according to their degree of interest

References

Corleis, A., Schwarz, B., Kaiser, G., & Leung, I. K. (2008). Content and pedagogical content knowledge in argumentation and proof of future teachers. *ZDM, 40*(5), 813-832.

Fujita, T., & Jones, K. (2013). Interpretations of National Curricula: the case of geometry in textbooks from England and Japan. *ZDM*, *45*(5), 671-683.

Hemmi, K., et al. (2013). Analysing proof-related competences in Estonian, Finnish and Swedish mathematics curricula. *Journal of Curriculum Studies, 45*(3), 354-378.

Knipping, C. (2001) Towards a comparative analysis of proof teaching. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of PME25* (Vol. 3, pp. 249–256). Utrecht, Netherlands: PME.

Miyakawa, T. (2012). Proof in geometry: a comparative analysis of French and Japanese textbooks. In T-Y. Tso (Eds.), *Proceedings of PME36* (Vol. 3, pp. 225–232). Taipei, Taiwan: PME.

Reid, D. (2015). Student understanding of proof and proving: Is international comparison possible? In K. Beswick, T. Muir & J. Fielding-Wells (Eds.), *Proceedings of the PME39* (Vol. 4, pp. 65–72). Hobart, Australia: PME

LEVERAGING TECHNOLOGY TO IMPROVE ACCESS TO MATHEMATICS – CHALLENGES AND RESPONSE

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Over the past decade, technology is increasingly becoming a part of most mathematics classrooms in the developing world context. While there have been several studies and reviews regarding use of technology in the mathematics classroom, these have largely been situated in the context of the developed world. Understanding the nature of challenges faced in using technology in developing countries - with a diversity of cultures, languages, caste, class, ethnicity and policies thereof, is certainly of emergent interest to the PME community. We propose an initiative to bring together a Working Group that will focus on the challenges of design and adoption of digital technologies for improving access to mathematics learning for socio-economically disadvantaged students in the context of the developing world.

Researchers have discussed factors determining the success of a digital technology in the mathematics classroom (e.g. Drijvers, 2015). - the design of the digital technology and of the actual tasks within, the educational context of the learner, and the role of the teacher being some crucial ones. This perspective will broadly guide the discussions in our group, and an important goal of the group would be to explore the question: 'What are some critical factors that enable successful integration of technology for improving access in the mathematics classroom?'

The working of this group will be structured around our experiences in designing and implementing technology-enabled learning modules for the Connected Learning Initiative (CLIx) – a collaboration between Massachusetts Institute of Technology (USA), Tata Institute of Social Sciences (Mumbai) and Tata Trusts (Mumbai). We will share data and insights from the study, under three themes – namely, 1) the design of the digital resources, 2) the learners' context and language, and 3) classroom adoption by the teacher. Each theme would be initiated by a presentation by the researcher, followed by a discussion where other participants would share viewpoints and experiences from their own context. The whole group discussions would focus on identifying common challenges across contexts and developing future research themes aligned to these.

In the first theme, the group will examine challenges at the three inter-related levels of design (Drijvers, 2015) - the digital technology, the actual learning activities, and the lessons in general. We will also present, for discussion, our rationale for designing a learning game as a response to some of these challenges. Some larger questions about the principles of design that we struggled with would be put up for discussion – for

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 183-184). Pretoria, South Africa: PME.

instance, 'should we design for the present, or the future, and what guides such an alignment?'

Our second theme will focus on challenges relating to the learner's context and language. In an environment where technology is integrated for its meaningful application, functional forms of language of technology, textbook language, language of teaching and learning and home language frame students' thinking, communication and comprehension. The group will discuss the challenges students face negotiating these languages during meaning-making and strategy design during game playing and tackling tasks.

Under the theme of classroom adoption, presentation and discussion will focus on how to support building of teachers' beliefs and knowledge (content, pedagogical, technical) for using tech based interventions while simultaneously supporting development of practices to allow student exploration and expression of mathematical ideas so that tech resources are used in a meaningful and integrated way.

Session 1: Introduction, S. Srinivas, A. Bose and R. Kumar (5') / Group discussion - 'What are the challenges of using digital technologies in math classrooms in the developing world?' (15') / Theme 1 Presentation, S. Srinivas: *Designing technology-enabled learning modules for Govt. schools in India – challenges and response (15')* / Group Discussion - Theme 1 (20') / Theme 2 Presentation, A. Bose: *Interaction of multilingualism and multiculturalism in task strategy and meaning making* (15') / Group Discussion - Theme 2 (20')
Session 2: Summing up of Session 1 (5') / Theme 3 Presentation, R. Kumar: *Technology-enabled mathematics learning – challenges in classroom adoption* (15') / Group Discussion – Theme 3 (20') / Planning next steps: possible collaborations and future research topics (45') / Summing up (5')

Table 1: Outline of the two sessions

References

Drijvers, P. (2015). Digital technology in mathematics education: why it works (or doesn't). In S.J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 135- 151). Dordrecht: Springer International Publishing.

Srinivas, S., Khanna, S., Rahaman, J., & Kumar, V. (2016, December). Designing a Game-Based Learning Environment to Foster Geometric Thinking. In the Conference Proceedings of the *IEEE Eighth International Conference on Technology for Education* (pp. 72-79). doi: 10.1109/T4E.2016.023

TASK DESIGN FOR EARLY ALGEBRA

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The focus of this newly established working group is on the complexities of designing tasks for engaging children in algebraic thinking. The working group aims to engage PME participants in interrogating the multiple ways in which robust task design supports teachers in facilitating children's learning, within the topic of early algebra. The facilitators of the working group will present relevant theory from the distinct research fields of task design and early algebra, and participants will be invited to explore how insights from task design may be made manifest to address the specific needs of children engaging with algebraic thinking in elementary school.

THEORETICAL BACKGROUND

For the purpose of the Working Group, we define a task as information that prompts students' work, including representations, context, questions and instructions (Sullivan, Clarke & Clarke, 2013). In referring to tasks and the design of tasks, we will remain cognisant of the disparities that arise between the intended use of tasks and the enacted use of tasks in classrooms by teachers (Sullivan, Knott & Yang, 2015). In aiming to establish a *domain-specific frame* for task design, we anticipate that such a frame would encompass explanatory materials for teachers to support interpretation, and mathematical knowledge for teaching (Kieran, Doorman & Ohtani, 2015; Sullivan, et al., 2015).

Kieran, Pang, Schifter and Ng (2016) identify essential elements of algebraic lessons as including the language of generalisation, the search for structure, and thinking analytically about indeterminate amounts (Radford, 2014). Seeking and describing structure in algebraic ways requires teaching approaches that encourage discussion, justification, conjecturing and exploration. Sullivan, et al. (2015) emphasise the potential of tasks to either (a) facilitate discovery within specific mathematical content, or (b) identify to learners the target content at the beginning of the lesson, thus removing the potential for discovery learning. Tasks presented for use by teachers in early algebra lessons should play a dual role in providing a catalyst for children's thinking, while also motivating teachers to facilitate children in thinking deeply about relationships and change.

GOAL AND ORGANIZATION OF THE WORKING GROUP

In the ICME-13 Topical Survey of Early Algebra, Kieran, et al. (2016) highlight that many early algebra interventions in classrooms involve *researchers* teaching the content to children. The authors emphasise the need for resources to be developed to support teachers' independence in teaching early algebra content. In addition, where

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resources are available for teachers, the implemented curriculum may too easily diverge from the intended curriculum, when teacher understandings, beliefs and approaches are not consistent with the perspective of the task designers.

The goal of this working group is to define a framework of task design principles to provide structure for the design, development and implementation of early algebra tasks. We will draw upon the framework of Sullivan et al. (2015) in delineating the mathematics of the tasks, the pedagogies and the student learning (p. 84). We, the proposers of this working group, intend to present theoretical foundations for task design and for early algebraic thinking to participants during a preliminary session. Thereafter, we intend to draw on the expertise of participants to formulate specific design principles for tasks that will contribute to our understanding of the interplay between the two fields of task design and early algebra. We plan to devote the first 90-minute working group session to drafting a framework through small group focus on task design for (a) generalised arithmetic and (b) functional thinking. For the second session we will share the output from the first session and invite participants to further develop the framework by incorporating considerations relating to teacher interpretation. We anticipate that the working group will apply existing theory to formulate design frameworks, and also develop theory in identifying specific tasks, (to include representations, foci of questioning, and anticipated interpretations) within a theoretical developmental pathway. We anticipate that the working group facilitators will be joined by participants in preparing a Research Report for submission to PME44.

References

Kieran, C., Doorman, M. & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson & M. Obtain (Eds.), *Task design in mathematics education, an ICMI study 22* (pp. 19-81). Cham, Switzerland: Springer.

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). Early Algebra: Research into its Nature, its Learning, its Teaching. In G. Kaiser, (Ed.), *ICME-13 Topical Surveys.* Dordrecht: Springer Open.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington: National Academies Press.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal, 26,* 257–277.

Sullivan, P., Knott, L., & Yang, Y. (2015). The relationships between task design, anticipated pedagogies, and student learning. In A. Watson & M. Obtain (Eds.), *Task design in mathematics education, an ICMI study 22* (pp. 83-114). Cham, Switzerland: Springer.

Sullivan, P., Clarke, D., & Clarke, B. (2012). *Teaching with tasks for effective mathematics learning*. New York: Springer Science & Business Media.

SEMINAR

WRITING AND PUBLISHING JOURNAL ARTICLES

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INTRODUCTION AND GENERAL GOALS

More and more researchers in mathematics education worldwide indicate that there is an increasing expectation that they publish their research findings in international scientific journals with a high impact factor. Still, this is not an easy endeavor. Even experienced researchers often struggle to get their work published. The good news is that much can be learned.

As coordinators of this session, we start from our position as, respectively, the editor-in-chief and associate editor of one of the A* journals in the field of mathematics education: *Educational Studies in Mathematics* (Nivens & Otten, 2017; Törner & Arzarello, 2013; Williams & Leathan, 2017). Based on these editorial experiences, as well as our experiences in writing papers for various international journals, we set up a seminar with a twofold goal. Each goal will be addressed in a separate session.

As a first goal we intend to share advice and experiences regarding how to write a coherent and attractive empirical research article that may convince both the editor and reviewers that publication is worthwhile. Second, we want to provide insight into the actual publication process and the various steps involved, including submission, review, various rounds of revision, and finally the publication process. Many aspects will apply more generally, but concrete examples will stem from *Educational Studies in Mathematics*.

TASKS AND ACTIVITIES

Session 1: Writing

On the basis of examples (preferably participants' papers if they send own paper well in advance), we will discuss the key ingredients of empirical research articles: problem, solution direction, knowledge gap, research aim, research question, key concepts, methodological approach, results, and discussion. Such ingredients need to form the argumentative "skeleton" of the paper (e.g., Bakker, 2018, Chapter 7). We will provide tips and tricks that are relevant to the writing of each of these parts, discuss common issues, et cetera.

The second topic to be discussed is how to write a succinct introduction that captures the first part of such chain of reasoning. Participants will identify the chain of reasoning used in one paper and discuss it with a peer. The third topic is overall coherence. We present a number of guidelines on how to promote the coherence, cohesion, and consistency of an article. Fourth, we pay attention to the formulation of research questions—which we consider they heart of any article. Throughout the session, there

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will be space for questions of participants related to article writing, as well as for discussion among participants on the topics that were considered.

Session 2: Publishing

In the second session, we will provide the participants a peek behind the curtains of the review and publication process. We go through this process step by step, starting with preparing the manuscript for submission, all the way to publication.

First, we will pay attention to the importance of choosing the right journal for a particular piece of research (Nivens & Otten, 2017; Törner & Arzarello, 2013; Williams & Leatham, 2017), and the risk of incurring a reject-without-review decision because a submission is out of scope for the journal. Second, we will look at what a submission looks like, and the formatting guidelines (e.g., APA6). Third, we will clarify the typical aspects of papers that editors as well as reviewers look for in order to judge the soundness of the paper and the potential contribution to the field and to the journal. Fourth, we will clarify how the selection of reviewers typically takes place, and how the evaluations provided by reviewers are used in coming to a final decision. Fifth, we will explain the expectations and good practices in the cases when authors get to revise their manuscript, and illustrate how good response letters are written. This will be done by means of example response letters. Also in this session, there will be room for questions.

References

APA6 (2010). *Publication Manual of the American Psychological Association.* https://www.apastyle.org/manual

Bakker, A. (2018). *Design research in education: A practical guide for early career researchers.* London, UK: Routledge.

Nivens, R. A., & Otten, S. (2017). Assessing journal quality in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 348–368.

Törner, G., & Arzarello, F. (2013). Grading mathematics education research journals. *Mitteilungen der Gesellschaft für Didaktik der Mathematik, 39*(95), 31–34.

Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 369–396.

COLLOQUIUM

DEVELOPING AWARENESS: LEADERSHIP IN A DEVELOPMENT PROJECT

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In this colloquium we investigate the impact of experiences on a variety of mathematics education sector personnel who participated in a five year development project intended to build capacity for rural and remote teachers to access and participate in ongoing professional learning. We frame the conversations around developing as leaders (National Council of Supervisors of Mathematics, 2008). Papers focus on the impact of experiences on the participants (including lead mathematics teachers, mathematics educators from teacher colleges, district academic officers, school quality assurance officers and adult educators) as they learned to lead: mathematics learning; teaching colleagues in professional development; and, large development projects. The papers reflect and interpret on those experiences in light of John Mason's (1998) notions of shifts in attention and levels of awareness. The first paper investigates the participants' development of awareness of mathematics; the second paper focuses on teachers developing to lead colleagues in professional learning; and the third paper explores leadership in terms of the project itself. This colloquium will contribute to people's understanding of collaborative international development work intended to enhance mathematics education in rural and remote communities in the "global south."

Presenters:

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Elaine Simmt, University of Alberta & Andrew Binde, University of Dodoma, Tanzania

References

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*(3), 243-267. https://doi.org/10.1023/A:100997371747

National Council of Supervisors of Mathematics. (2008). *PRIME Leadership Framework: Principles and Indicators for Mathematics Education Leaders*. Bloomington, IN: Solution Tree.

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INTERNATIONAL DEVELOPMENT: ATTENDING TO AWARENESS IN LEADERSHIP

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Leadership of a mathematics education development project is studied through the lens of Mason's (1998) structures of awareness. The paper illustrates awareness-in-action in both the preparatory and emergent decision making throughout the course of the project conducted in Tanzania by an international team funded by Global Affairs Canada. The paper contributes to the literature on management and leadership which could be used in future international development projects. In this way it is a first step towards awareness-in-counsel in leadership focused on international mathematics education development projects.

INTRODUCTION

Leadership occurs in many forms and is critical within many different facets of mathematics education. Leadership studies include foci on instructional leadership, professional learning communities, curriculum reform and teacher education broadly. However, little research is found in mathematics education research journals about leadership of international development. A capacity building project conceived and implemented in Tanzania provides us with an opportunity to investigate project leadership using John Mason's (1998) constructs of shifts in attention and levels of awareness.

BACKGROUND

Financial Aid and Development

Tanzania is a country that receives significant international aid from funding agencies. Although much of the aid goes directly to the government as block funding, there are millions of dollars that are project-based. A case in point—the Global Affairs Canada (GAC) is currently supporting the *Teacher Education Support Project* in Tanzania for approximately 53 million USD. *Hisabati ni maisha* / *mathematics is living* became the local name for a Canadian funded project to build capacity for mathematics teaching implemented between 2012 and 2018.

The long history of aid to the "global south" has been criticized in terms of influence, effectiveness and sustainability (Freedman, 2000). Smaller projects such as *Hisabati ni maisha* are critiqued as largely ineffective and unsustainable (Smillie, 2000). Leadership and management are two of the many factors that are attributed to the successes and failures of such projects. But how does leadership (and management) contribute to the outcomes of a project specific to mathematics teacher development and what can we learn about our roles as leaders and educators?

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Many development projects that use aid funds are implemented by agencies and NGOs that do not have research as a primary mandate. In contrast the *hisabati ni maisha* project was implemented by mathematics education researchers / teacher educators from two Canadian and one Tanzanian university; hence a research (and educator) perspective permeated the project from its inception through its completion and post-project as we now return to it as a case for researching project leadership. The project has provided us with an opportunity to ask: **What are the planned and lived experiences of a development project? What do those experiences teach us about leadership for collaborative international educational endeavours?** This paper highlights the role of leadership in the execution of a development project to build capacity for mathematics education in rural and remote communities in Tanzania.

Management and Leadership

We recognize that in mathematics education, management and leadership are exercised at many levels, the classroom, the school, and the broader community (National Council of Supervisors of Mathematics, 2008). However, leadership of an international development project is not day to day work for the vast majority of leaders in mathematics education, but it becomes a possibility for mathematics educators as they broaden the scope of their work beyond their own school systems.

Some may argue management and leadership are not easily separated. This is because often the two roles are within the same person, they have their own features but are often complementary. Project management is commonly understood as attending to the work that needs to be done to ensure that a project progresses effectively and efficiently through to completion. Leadership on the other hand is focused on the vision to promote, create opportunities, address challenges and elaborate goals of the project (Hoyle & Wallace, 2005; Coleman & Glover, 2010). Leiding (2004, p. 36) borrows from Benning, 1999) to contrast managers with leaders (Table 1).

Manager	Leader
administrates	innovates
maintains	develops
accepts reality	investigates reality
focuses on systems and structures	inspires trust
relies on control	focuses on people
short-range view	asks what and why
has eye on bottom line	has eyes on horizon
initiates	originates
accepts status quo	challenges status quo

Table 1: Role differences between manager and leader

Leadership role plays a vital role in facilitating various project factors that contribute to project performance (Turner and Müller, 2005), although some researchers (Anantatmula, 2010) challenge that leadership style and competence are not directly related to project success. The emphasis on providing vision, motivation of personnel of different specializations, and the ability to cope with change are argued as key features of leadership (Kotter, 1999). Leadership is vital in defining the vision and the mission of a project and establishing trust among the project team and participants. Leaders attend to project complexity (Turner and Muller, 2005). Each project is unique in terms of goals, context and time. *Hisabti ni maisha* was not an exception in terms of its complexity and the issues that emerged throughout the project. Uncertainty and unknowns were ever present despite the good project design.

All large projects require a project management plan and a leadership team. However, the planned or intended project, just like the planned curriculum, must be implemented and lived; and in the living of the project, challenges and opportunities will arise. Those challenges and opportunities require the awareness and wisdom of the leaders to make decisions in action. In this paper, we primarily explore leadership actions that result in "setting the values and the vision" of a project rather than managing the "day to day enactment of the vision" (Coleman & Glover, 2010, p.3).

THEORETICAL FRAME

In this paper we use Mason's shifts in attention and levels of awareness that he has formulated by working with teachers to explore the education of ourselves as leaders. "Knowing what" and "knowing that", Mason (1998) argues are not enough for the mathematics teacher to successfully teach mathematics. He asserts that the mathematics teacher needs to "know to act" in order to be successful. He asks how do we educate teachers to "know to act." His conjecture is that:

> [T]eachers need more than just knowledge in its traditional sense. They need an awareness of being educated, that is, awareness that attention is structured, awareness of the structure of their own awareness, and awareness of what they are stressing and ignoring while speaking to students" (p. 251).

With preparation and deliberate reflections on acting mathematically, a teacher's attention can shift to the discipline itself and on acting in pedagogically. With that shift of attention the teacher is able to develop forms of awareness that enhance both teaching and teaching teachers. "Awarenesses-in-action are the sensitivities to certain situations which provoke and enable action" (p. 258). When teachers shift their attention and become aware of their awarenesses-in-action they are more sensitized to the learner. "Awareness-in-discipline" arises when people become aware of the awareness-in-action which enable them to function within the domain (p. 258). Finally, "awareness of awarenesses-in-discipline provides access to sensitivities which enable us to be distanced from the act of directing the actions of others, in order to provoke them into becoming aware of their own awarenesses-in-action and awarenesses-in-discipline" (261). Mason calls this, "awareness-in-counsel drawing on the sense of a counsel as a source of collective wisdom, separated from immediate action and transcending specific disciplines" (p. 261).

In this paper we seek to observe and reflect on our actions as leaders with the goal of illustrating leadership awareness-in-action, thereby offering an opportunity to enhance our awareness of awareness-in-action (awareness-in-leadership discipline) and moving towards the possibility of awareness-in counsel where we reach out to the mathematics education community with new understandings of educating for leadership in multi-national development endeavours.

METHOD

We draw from the five year GAC funded project, *Hisabati ni maisah*. That project involved activities leading to three distinct goals. The first goal was to create a network of approximately 50 decision makers who would contribute to policy recommendations related to in-service primary school mathematics teacher professional learning. The second goal was to enhance the capacity of persons in positions to work with primary mathematics teachers to offer support for their ongoing professional development. Some 60 mathematics lead teachers, mathematics teacher educators, school quality assurance officers, district academic officers and adult educators in Tanzania were offered 11 weeks of training (referred to as short courses) over 4 years and the opportunity to develop and facilitate teacher in-service activities for some 430 primary school teachers in rural and remote communities in 9 districts within the country. Finally, 245 community leaders and parents were offered two workshops to help raise their awareness about the value of mathematics education for children and how they could support the mathematics education of children in their community.

In this paper we focus on data related to the project team leadership and management, rather than on leadership developed among participants within project activities, although it will become clear with other research papers proposed for PME 2019 (Swai & Glanfield; Mgombelo) that educating awareness of both teaching and leading were also elements of this project. We offer short accounts of the planned and enacted leadership of the project from archived documents (project implementation plan, working documents and semi-annual project reports) and personal reflections. Following Mason (2002) we first reflected on accounts “of” (that is accounts stripped of motivations and reasons for the actions) various challenges, opportunities and complexities of the project. However, the limited space in this paper means we offer accounts that include motivations and justifications for actions (accounts “for”) as we investigate leadership as awareness. We begin with a description of the personnel and leadership structures within the project and follow those with the accounts.

ORGANISATION AND MANAGEMENT STRUCTURE

The project was co-led by a Canadian coalition of three mathematics teacher education researchers, two from the University of Alberta and one from Brock University and a Tanzanian mathematics teacher educator and researcher from the University of Dodoma. Two project managers were hired: one in Canada and the other in Tanzania. A project accountant was hired in Tanzania. There were both a steering committee and an advisory board for the project.

Project Co-directors

The Canadian co-director was nominally responsible for providing directorship for all project activities; ensuring project was implemented in line with contribution agreement, as well as other sub-contracts and directly supervise Canadian based project staff. Similarly, the Tanzania project co-director provided directorship for all project activities, managed local support for the project and supervised TZ based project staff.

Project Managers

The CND project manager prepared all project reports in accordance with contribution agreement; ensured financial management of the project was accurate and aligned with the contribution agreement; and provided support to ensure timely implementation of the project activities. The TZ project manager prepared contributions for project reports; ensured financial management of the local activities; and planned all logistics for local activities to ensure timeliness.

Steering Committee

The coalition members and the two project managers constituted the steering committee. This committee provided the practical leadership for the project. Members of it were responsible for: visioning, planning, implementing and monitoring the project in its entirety. It was a collaborative committee and used a consensus model for decision making. The steering committee, project accountant, and graduate student assistants met weekly via video conferencing. As well the steering committee held formal face to face semi-annual meetings to ensure the project would meet planned project milestones and come to a successful conclusion.

LEADERSHIP AS PLANNED AND EMERGENT

We offer three accounts from the project. Together these accounts and the interpretations we offer provide an opportunity to shift our attention from awareness-in-action to awareness of awareness-in-action or awareness-in-discipline and to each of these as we seek awareness of counsel.

Establishing presence of "yet another project"

A number of challenges exist when a team creates an opportunity to work with not yet known colleagues on the "other side of the world." The Canadian team members having entered into a collaboration with the local country director, worked with him to establish an appropriate advisory board. The advisory board advised the steering committee about the local context, related activities in the country, advocated for the project, and provided practical and strategic advice on the activities of the project.

One of the most significant contributions of the advisory board (as we observe in hindsight) was the generation of the project slogan, *hisabati ni maisha* / mathematics is living, at its first meeting. It came out of collaborative brainstorming and discussion after a member of the advisory board shared a story of a very small but successful project whose slogan on t-shirts served to raise awareness of their issue. The value of a slogan was far from the awareness of the steering committee in the planning phase and in its initial activities. As leaders looking forward in the project we had a vision for the project and we trusted the advice of the advisory board. When offered the idea of a slogan our (we speak collectively here because we were acting together with one voice) attention shifted to this possibility.

As managers we began branding our work with this slogan, posters, t-shirts, curricular materials and so on. But it an action of one of the Canadian teacher educators that solidified the slogan. One day in the short course classroom when the facilitator wanted to get the attention of the participants, she called out *hisabati* (mathematics) spontaneously someone shouted back *ni maisha* (is living). She called back *ni maisha* and the participants collectively responded *hisabati*. This refrain (and the slogan) served as a touchstone for the diversity of participants in the project. It became their call to action, to enhance mathematics teaching and learning in their communities. Collective awareness-in-action guided both the discussion and the adoption of the slogan. We leave unpacked the chanting of the slogan and the way in which the news of the project spread in the mathematics education community in some parts of Tanzania.

The challenge of working with high level officials

The initial participant list for the decision maker network. included many high-level district and ministry officials (GAC final report, 2018). Due to competing demands on their time, this group was unable to participate fully in the activities planned for them that required 2 – 3 consecutive days. Frequently, officials would step out of a session to take a call or leave all the session completely to return to their place of work to attend to a pressing need. This generated some distress among members of the steering committee, but it was clear that they could not demand participation. The third time the network met the participants were to conduct a focus group composed of Ward Education Officers (WEC) and Head Teachers (HT) to gather information about the needs of rural and remote teachers. It was during this focus group that the project leaders observed the benefits of the voices and perspectives of the WECs and HTs in relation to the network goals. The steering committee realized that not only did the WECs and HTs have a direct impact on the implementation of policy at the local ward and school levels but they were also instrumental in the organization and planning activities for teachers. Noticing this led to invitations to the WECs and HTs from each of the 27 project wards to join the network as permanent members. By shifting attention from the policy makers to the policy implementers the steering committee's awareness-in-action resulted in a much more dynamic and attentive group of participants for the network.

The challenge of maintaining interest and securing a commitment

Short courses were planned as the main element of the project to educate those people who work most closely with teachers because those people were well positioned academically, professionally and geographically to offer ongoing professional development in their local rural and remote communities. The project vision was to meet with this group for two weeks once a year for 3 years. After the first short course, the team recognized that meeting the group once a year with little contact between meetings would not sustain their interest or commitment to the project. Questioning that reality led us to three strategies (ones that were put in place immediately). The first was to reschedule the meetings to one week in duration twice a year; the second strategy was to give the participants a task to work on between the sessions. We asked the teachers in the group to try out a mathematics activity in their classrooms and bring the results of that back with them to the next meeting. At the next session the teachers shared student work with other members of the short course. Subsequently, (at a future session), we asked the groups of participants to propose an activity for the next gap between sessions. They were to write proposals which we reviewed and provided small micro-grants to carry out the work. This pattern of week long sessions followed by participant generated activities back in the schools became entrenched in the project. With our bi-annual steering committee meetings we were able to reflect on the leadership decisions we made in action developing awareness of our awareness-in-action. Our shift in attention from how do we keep people engaged between sessions to what are the implications of this decision suggest we were developing what Mason refers to as awareness-in-discipline.

CONCLUSION

These three examples illustrate leadership in an international development project. As Mason (1998) proposed shifts in attention and noticing our levels of awareness leads to growth, in our case, as leaders. There is a need for attention on leadership in mathematics education, not only teacher leadership which is critical but in understanding leadership in the context of international collaborations and projects. The few accounts offered in this paper are just a few of the many accounts of leadership-in-action that we could point to in *hisabati ni maisha*.

References

Anantatmula, V. S. (2010). Project manager leadership role in improving project performance. *Engineering Management Journal of System Management, 22*, 13-22.

Benning, W. (1999). The leadership advantage. *Leader to Leader, 12*, 18-23.

Coleman, M. & Glover, D. (2010). *Educational leadership and management: Developing insights and skills*. UK: McGraw Hill.

Freedman, J. (2000). Introduction: Aid at the forks. In J. Freedman (Ed.), *Transforming development: Foreign aid for a changing world*, (pp. 3 -11). Toronto, Canada: University of Toronto Press.

Hoyle, E. & Wallace, M. (2005). *Educational leadership: Ambiguity, professionals and managerialism*. London, UK: Sage.

Jugdev, K. & Müller, R. (2005). A retrospective look at our evolving understanding of project success. *Project Management Journal, 36*(4), 19–31.

Kotter, J. (2001). What leaders really do. *Best of HBR,* 3-11.

Leiding, D. (2004). *Managers Make the Difference: Managing vs. Leading in Our Schools*. Lanham, Maryland: Scarecrow Education.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*(3), 243-267. https://doi.org/10.1023/A:100997371747

Mason, J. (2002). *Researching your own practice: the discipline of noticing*. UK: Routledge.

National Council of Supervisors of Mathematics. (2008). *PRIME Leadership Framework: Principles and Indicators for Mathematics Education Leaders*. Bloomington, IN: Solution Tree.

Smillie, I. (2000). NGOs: Crisis and opportunity in the new world order. In J. Freedman (Ed.), *Transforming Development: Foreign Aid for a Changing World* (pp. 114 – 134). Toronto, Canada: University of Toronto Press.

Turner, J. R. & Müller, R. (2005). The project manager's leadership style as a success factor on projects: a literature review. *Project Management Journal, 36*(2), 49–61.

DEVELOPING AWARENESS IN MATHEMATICS AND RESOURCEFULNESS THROUGH CONCEPT STUDY: BUILDING CAPACITY FOR MATHEMATICS TEACHER EDUCATION

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The paper draws from research on a development project aimed at building capacity for mathematics teaching in rural and remote communities in Tanzania. The international development project was guided by the question: How can universities contribute to effective and sustainable solutions to deep problems in complex contexts? Specifically, the paper explores participants' development of awareness in mathematics in a professional learning setting. Using the lens of Mason's work on awareness, the paper will offer a discussion about how the project utilized concept study to develop participants' awareness in mathematics and resourcefulness.

INTRODUCTION

Recent research in mathematics education points to the need for professional learning activities to take into account the development of teachers' mathematics knowledge needed for teaching. However, there is little agreement on what mathematics teachers need to know and how this mathematics for teaching could be developed. On one end of the spectrum, there is an assumption in the research that the mathematics teachers need to know is what can be offered from their undergraduate or college mathematics courses. This assumption has been challenged by evidence in research on the other end of the spectrum that indicates that many teachers still struggle when teaching school mathematics for understanding even though their knowledge of mathematics may be adequate (Cooney, 2002; Kinach, 2004; Adler & Davis, 2006). Following this evidence, researchers have continued to articulate mathematics knowledge for teaching moving beyond college courses to its articulation in the work of teaching (Ball & Bass, 2002; Davis & Simmt, 2005; Adler & Davis, 2006).

Yet even with the elaboration of teachers' mathematics knowledge for teaching and articulation of its specificity to practice there is still no consensus on what aspects of mathematics for teaching knowledge are crucial for teachers to act in the moment-to-moment nature of teaching and how these crucial aspects could be developed in professional learning settings. Some studies argue for explicit mathematics for teaching knowledge that can be assessed through observations, interviews or tests (e.g. Ball, Hill, & Bass, 2005). In contrast, other studies including the research reported in this paper, argue that the mathematics knowledge for teaching is implicit or tacit and it is that which is called to mind in the moment-by-moment of teaching (Mason & Davis, 2013). It is a kind of knowing that requires awareness (Mason & Spence, 1999). This paper utilizes the lens of Mason's work on awareness (Mason, 1998) to explore

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participants' development of awareness in mathematics in a professional learning setting. Specifically, it draws from research on a development project aimed at building capacity for mathematics teaching in rural and remote communities in Tanzania (Simmt, Binde, Glanfield, & Mgombelo, 2011). The international development project was guided by the question: How can universities contribute to effective and sustainable solutions to deep problems in complex contexts? Knowing that the project was to be offered in highly constrained and under-resourced contexts we had to think about availability and use of resources as well as sustainability of the teaching practices. The paper offers a discussion about how the project utilized concept study (Davis & Rennert, 2013) to develop participants' awareness in mathematics and resourcefulness.

CONTEXT

The development project which serves as a context for the research reported in this paper, involved a Canadian Coalition of two universities and local partner - a university in Tanzania (Simmt, Binde, Glanfield, & Mgombelo, 2011). The project focused on three areas of activities: 1) Teacher Education Policy where network meetings, consisting of different categories of stakeholders were involved in reviewing current policy documents as well as collectively formulating recommendations for gender-sensitive teacher development practices and policies focusing on mathematics teaching in primary schools in rural and remote communities. 2) Mathematics Teacher Development where short courses covering various mathematics content and pedagogical strategies. Short course participants then used the skills they learned to develop and deliver professional development modules for teachers. 3) Community Participation, which involved developing, and conducting workshops on enhancing community mathematical literacy and awareness of the importance of mathematics education for girls and boys. This research report focuses on the experiences of participants in (2) the short courses. The purpose of the short courses was to build capacity for leadership in mathematics teacher education in rural and remote communities in Tanzania by increasing the knowledge of gender-sensitive (more inclusive) instruction strategies and evidence based professional education for lead teachers, teacher educators, educational decision makers, quality assurance officers and adult educators through short courses. The development project delivered five short courses including four - two-week long short courses and one - one-week long short course.

FRAMEWORK

The framework draws from the work of Mason and Spence (1999) on "knowing-to act," as a kind of knowing that requires awareness, and Mason's (1998) work on levels of awareness in mathematics teacher education. Mason and Spence conceptualize teachers' knowing of the mathematics needed for teaching, from a dynamic, active and evolving perspective of knowing-to act as opposed to the static, passive and possessive perspective of accumulated knowledge. Furthermore, Mason's forms of awareness provide a way of conceptualizing the mathematics needed for teaching as both a development of sensitivity to mathematics (awareness in discipline) and being able to support this development from a learner's perspective.

Building on Gattegno's (1970) idea of awareness as that which enables powers that have been integrated into one's functioning to be employed, Mason describes three forms of awareness: awareness-in-action, which involves a human being's powers of construal and of acting in the material world; awareness-in-discipline, which is awareness of awareness-in-action, emerging when awareness-in-action is brought into explicit awareness and formalized; and finally *awareness in counsel*, which is awareness of awareness-in-discipline and involves enabling others to work on their awareness-in-discipline. To put this into a mathematics perspective, awareness-in-action might be exemplified by an act of counting numbers (one, two, three) without being aware of the underlying notions such as one-to-one correspondence. Awareness-in-discipline emerges when one becomes aware of this one-to-one correspondence in counting. Finally, awareness-in-counsel emerges when one is able to let others develop their awareness of counting as one, two, three, as well as develop their awareness of the notion of one-to-one correspondence. As Mason (1998) notes, "Awareness of awareness-in-discipline provides access to sensitivities which enable us to be distanced from the act of directing the actions of the others in order to provoke them into becoming aware of their own awareness in action and awareness-in-discipline" (p. 261).

Mason and Davis (2013) contend that awareness is not something that can be measured but rather "it is a portmanteau for 'being in the situation', sensitized and responsive to what emerges, informed by personal experience of mathematics, learning, and teaching" (p.188). In order to develop awareness, teachers need to work at developing their being through engaging in mathematical thinking for themselves and with other like-minded colleagues. This paper discusses how concept study strategies were used to develop participants awareness of mathematics and resourcefulness in short courses aimed at building capacity for mathematics teaching.

METHOD

The development project delivered short courses to teacher educators, educational decision makers, quality assurance officers, adult educators and lead primary school teachers, the same participants were invited to each of the short courses for the entirety of the project. 52 participants, with 29% female participation were involved in the short courses. Data used for this research report came from evaluation of the workshops/short courses conducted for participants using pre and post questionnaires with both closed and open-ended questions. The pre and post questionnaires were analysed and presented in semi-annual reports. In addition to the pre and post questionnaires we collected artefacts from group work in the form of notes on chart papers from mathematics tasks and group discussions during the workshops, transcripts from interviews and facilitator field notes.

FINDINGS AND DISCUSSION

In this section, I present and discuss the findings through accounts of participants' development of awareness of mathematics and resourcefulness as they emerged in the sessions. Concept study strategies served as mechanisms to focus participants' attention as they worked collaboratively and engaged in examination and elaboration of mathematical understandings of a range of concepts from primary school mathematics. In addition to the accounts offered participants' views of their experiences in the short courses are also used in this analysis.

Using Student generated examples: developing awareness of conceptual understanding and procedural fluency in addition and subtraction

In one of activities of the short course that focused on number sense participants were divided into two groups: Members of one group were asked to write an addition equation with numbers between 100 and 1000 and members of the other group were asked to write a subtraction equation with numbers between 100 and 1000. Then participants were asked to sort the questions from easy to hard and discuss what makes one question easy and another question hard. Next, participants were asked to examine how other people in the opposite group ordered the questions, explain why they thought each example was ordered as it was, and then discuss their responses with group members.

One of remarkable discussion that emerged from this site was focused on a response in which one group identified and ordered 1000-999 (written vertically, as column) as difficult because of the number of "borrowing/trades" you have to do in order subtract. In the discussion on participants noted (and most others agreed) if the question had been written horizontally rather than vertically it would be a very easy question. It is tempting to dismiss this action as just lack of conceptual understanding. But for us the question is about what aspect of mathematics was called to mind in the moment and how can we direct attention to the conceptual aspects of the question. "'knowing' is not so much about *having* as it is about *doing*" (Mason & Davis 2013, p. 187). Using learner generated examples provided a site for participants to reflect on knowing in the moment and make meaning of other people's understanding. The awareness of how the focus on "borrowing/trades" method of subtracting in mathematics teaching prevented the conceptual understanding of the operation. It seems the participants' focus was on the form of the question rather than on the operation itself. When faced with a subtraction question written in vertical form, the participants called on trading or borrowing to answer the question. When presented with a different form (horizontal) of the same operation on the same values the participants seemed to recognize this question as a difference and immediately responded without any written or expressed procedure, that it was "one". The concept study strategy to focus on a piece of primary school mathematics created the possibility of shifting the participants' attention from responding to a question based on its form to seeing that both questions are about "difference" and that form need not be the driver of a particular procedure but rather one could first attend to what meaning does the question seek. This in turn allowed the participants develop their awareness of conceptual aspects of subtraction

Mobilizing students' powers of mind (generalizing, specializing, etc., Mason, Burton & Stacey, 2010) in teaching algebra.

In another of the activities in the short course participants were asked to work in small to groups to find a solution for the dot problem (Fig. 1); that is to determine how many dots at time 't'.

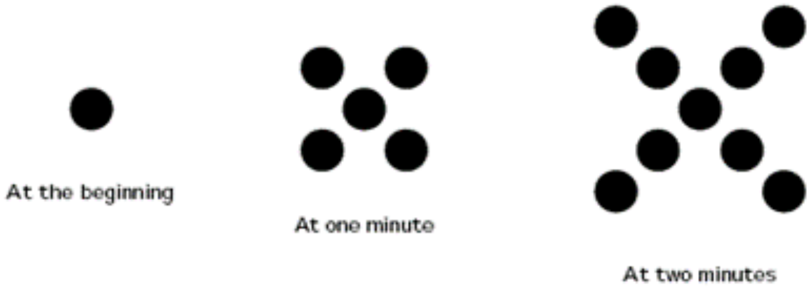

Figure 1: The dot problem

Participants were prompted to avoid the temptation to rush into finding a formula, and instead start with visualizing or seeing the pattern by examining the physical or geometric regularities of the growing pattern; How are the stages of the growing dots formed? Can you find a way of counting the number of dots in the general (t minutes)? Participants were asked to keep in mind that there are many ways of seeing/visualizing the pattern and finding a way of counting the dots and were asked to show their way using the picture of the growing dots. They were encouraged to then move from expressing the rule from a picture to words and finally to symbols. Participants found it interesting to see how there were different ways of seeing the dots growing. For example, some saw a growing square and others saw an "X" etc. Getting participants to say something of what they see, and to listen to what others say they see has potential of developing their awareness of possible interpretations or ways of seeing, and sometimes of detecting similarities that can emerge as generalities (Mason et al., 2010). Developing such awareness of the discipline is critical for the teacher as a step towards developing awareness of counsel.

Teacher as a resource: developing awareness in resourcefulness

In designing and implementing the short courses the development project considered the fact that it was working in limited and under-resourced contexts. Therefore, the courses aimed at shifting participants' thinking of resources in mathematics teaching by educating the awareness of resources beyond the common-sense understanding of resources as material objects to become aware of human and cultural resources such as language and time (Adler, 2000). Further, the courses aimed at increasing participants' awareness of what resources are to how resources function as an extension of the mathematics teacher in the teaching and learning process (Adler 2000).

A great resource in mathematics education especially in the context of African mathematics education is the late Prof Paulus Gerdes, a renowned scholar on ethnomathematics. Sensitive to context Prof Gerdes was invited to give keynote presentation for the opening of the project. He was inspirational, taking photos on his 8-hour journey to Dodoma from Dar es Salaam and sharing their own mathematics with the Tanzanian mathematics educators. Dr Gerdes set the stage for rethinking Tanzania as poverty–stricken to resource-full, challenging us to look for the plentifulness of resources available to mathematics teachers and learners. To do that teachers need a shift in thinking about resources (Adler, 2000). The project began by drawing awareness to this in the first short course "The Teacher as a Resource". With this short course participants engaged in activities that promoted the use of resources such as songs, games and poems for teaching number. One participant (a lead teacher) with a deep interest in poetry saw an opportunity to use poems as resource for teaching mathematics. That awareness led to a regular activity within the short courses where, with the help of other participants this teacher created poems to express their learning. Through creating the poems participants' awareness developed from awareness of discipline to awareness in counsel as the participants began to attend to their learning as the basis from which they could teach others.

Participants' views about their experiences

Data analysis from the interviews and pre and post questionnaire indicate the potential of concept study in developing awareness of mathematics and resourcefulness. Participants reported to have developed the awareness of mathematics concepts as well as awareness in resourcefulness. For example, on participants reflected on growth in mathematical understanding:

> We have also been able to go through different concepts and clearly understood them such as algebra, geometry and also ratios. All these concepts have been really helpful to us and have changed the way we used to do teach mathematics at our respective work places (participant interview).

And in terms of resourcefulness, one participant noted:

> we have learned about the use of different tools which are useful in building a lasting memory to students. The use of these tools enables students to clearly understand the subject matter. Also, we learned about tools which are easily available within their environment, which means even teachers from the rural areas can access them with no cost. Which means in the project we also learned about how to minimize cost by focusing on using what is available around our environment (participant interview).

In post questionnaires, participants indicated that they have developed awareness of both mathematics and resourcefulness as well as awareness in counsel. For example one participant noted, "this short course has changed my way of thinking about math, I never thought of the use of games and aids could help my students understand the concepts easily" and another noted, "I learnt a lot about addition, subtraction, fractions, counting which will lead me to facilitate mathematics professional development courses for teachers in rural and remote communities".

CONCLUSION

Despite the progress in mathematics education research regarding the elaboration of teachers' mathematics knowledge for teaching and the articulation of its specificity to practice there is still little agreement on what aspects of mathematics for teaching knowledge are crucial for teachers to act in the moment-to-moment of their work. This paper contributed to the research by highlighting awareness of discipline as a crucial aspect of mathematics for teaching and how it could be developed through in professional learning setting through concept study.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3,* 205-224

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching mathematics teacher education. *Journal for Research in Mathematics Education, 37,* 270–296.

Ball, D., & Bass, H. (2002, May). *Toward a practice-based theory of mathematical knowledge for teaching.* Plenary lecture at the 26th annual meeting of the Canadian Mathematics Educators Study Group, Kingston, Ontario, Canada.

Ball, D. L., Hill, H.C, & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(1), pp. 14-17, 20-22, 43-46. http://hdl.handle.net/2027.42/65072

Cooney, T. J. (2002, November). *Mathematics teacher education in rural communities: Developing a foundation for action.* Paper presented at the ACCLAIM Research Symposium. McArthur, OH: Ravenwood Castle,

Davis, B., & Renert, M. (2013). Profound understanding of emergent mathematics: broadening the construct of teachers' disciplinary knowledge. *Educational Studies in Mathematics, 82*(2), 245-265. doi:10.1007/s10649-012-9424-8

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics teaching and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137–167.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*(3), 293–319.

Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge and Kegan Paul.

Kinach, B. M. (2002). Understanding and learning-to-explain by representing mathematics: Epistemological dilemmas facing teacher educators in the secondary mathematics "methods" course. *Journal of Mathematics Teacher Education, 5,* 153–186.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1,* 243–267.

Mason, J. & Davis, B. (2013). The Importance of Teachers' Mathematical Awareness for In-the-Moment Pedagogy. *Canadian Journal of Science, Mathematics and Technology Education*, *13*(2), 182–197, 2013

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd extended ed.). Harlow, England: Prentice Hall.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135–161.

Simmt, E., Binde, A., Glanfield, F., & Mgombelo, J. (2011). *Capacity development for mathematics teaching in rural and remote communities in Tanzania*. Ottawa: Global Affairs Canada.

AWARENESS-IN-ACTION: MATHEMATICS TEACHER LEADERSHIP IN TANZANIA

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This paper explores the teacher leadership practices of mathematics teacher leaders (MTLs) through the lens of Mason's (1998) notions of awareness-in-action and awareness of awareness-in-action (or awareness-in-discipline). The study employed a multisite case study design. Data were collected through in-depth interviews. The study revealed five practices that teacher leaders employed to influence change in mathematics education in their districts. The practices include talking less and listening more, cascading teacher leadership, leading within and beyond the district, embracing the familiar, and leading learning for all.

INTRODUCTION

Our study explored teacher leadership practices of mathematics teacher leaders (MTLs) as they engaged in the facilitation of primary school mathematics teachers professional learning in Tanzania. The MTLs were primary school mathematics teachers themselves who were teaching in rural and remote communities. The MTLs participated in a series of professional learning sessions that were a part of a development project funded by the Government of Canada (Simmt et al., 2011). One outcome of the project was raising awareness among mathematics teachers regarding their position and capacity for enacting change to revitalize mathematics education in their communities. In particular, the project engaged mathematics teachers in the process of becoming mathematics teacher leaders capable of facilitating and promoting teacher professional growth as well as contributing to untangle challenges facing mathematics education in their districts and beyond. These project goals were consistent with the recommendation by the National Council of Teachers of Mathematics (NCTM) (2000) that "[t]here is an urgent and growing need for mathematics teacher-leaders ... who can assist with the improvement of mathematics education" (p. 375) in schools. The essentiality of such leaders lies in their position, expertise, and experiences of helping students to develop an understanding of school mathematics.

Our study involved five MTLs who, during the time of this research, were engaging in enacting teacher leadership practices in their respective districts. Our research was framed by the question: *What leadership practices do mathematics teacher leaders undertake to improve mathematics education?*

ANCHORING THE WORK: TEACHER LEADERSHIP

Teacher leadership has gained widespread attention in the teacher professional learning literature since education reform initiatives of the 1980s. Since then, the process is widely viewed as crucial for teachers to participate in improvement reforms grounded

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to improve practice in schools (Balka, Hull, & Miles, 2010; NCTM, 2000). As such, the literature has been challenging classroom teachers to become "leaders ... who can make a difference in schools and schooling now and in the future" (Lieberman & Miller, 2004, p. 11). Teacher leadership is built on the idea of ensuring that teachers participate in the process of influencing the development of professional capital among other teachers.

Sanocki (2013) defined teacher leaders as teachers who are classroom teachers; aware of their position; confident; take action; promote teacher professional growth; and are change agents. In working as leaders, mathematics teachers are expected to "maximize their own improvement and be able to make effective judgments using all their capabilities and judgments" (Hargreaves & Fullan, 2012, p. 3). Equally, teacher leaders have "capacity and power to participate in change efforts that traditionally either have been tacitly assumed by them or deliberately defined by others" (Webb, Neumann, Jones, 2004, p. 255). In such a context, therefore, possibilities exist for them to design conditions powerful in addressing dilemmas that face their teacher colleagues in their professional learning journey.

The notion of a primary school mathematics teacher becoming and serving as a leader is innovative within Tanzania. In 2015, Hardman et. al. found the education system of Tanzania embraced a traditional view of a teacher. As a consequence, Mhando (2012) found most of the teachers in Tanzania did not engage in improvement processes because they had "little innovation in the process of developing knowledge" (p. 163). Central to the development project (Simmt, et. al., 2011) was the acknowledgment that mathematics teachers are not likely to participate in owning their professional learning nor contributing in improving mathematics education unless they develop an awareness of the role they could play (DuFour et al., 2008; Loucks-Horsley et al., 2010; Jackson & Allender, 2016). In participating in the project, the teachers raised their awareness on different themes, including school mathematics, gender-sensitive pedagogy, and professional development models such as lesson study and concept study. As the teachers engaged in enacting teacher leadership, we became interested in understanding their practices as leaders. In Mason's (1998) words, we became interested in the teachers' awareness-in-action and their awareness of awareness-in-action (or awareness-in-discipline; the discipline of mathematics teacher leadership).

METHODS AND THEORETICAL FRAMING

This multisite case study involved five MTLs (Ivan, Abba, Perry, Kenzo, and Shinje) with a variety of years of teaching experience. Ivan and Abba had less than 10 years' experience; Perry and Kenzo between 20 – 30 years' experience; and Shinje more than 30 years' experience). Each teacher leader was a case as they were working in different school districts, experiencing different working conditions. However, they shared common characteristics, that is, they all participated in the development project activities.

We used one-hour focused interviews to explore the MTL's teacher leadership practices in enhancing mathematics education in their districts. We used Miles and Huberman's (1994) data analysis procedures to analyse data collected from each case. We read and reread transcripts of each case to identify themes connected to the research question and coloured chunks of phrases, sentences, and paragraphs of transcripts to classify and categorize data into groups, facilitating understanding of leadership practices through Mason's lens of awareness-in-action and awareness of awareness-in-action. We drew conclusions based on the meanings that emerged from the displayed data and paid close attention to the practices that each teacher leader reported undertaking in his/her district.

RESULTS

Our analysis revealed five leadership practices undertaken by each case to improve practice in their districts and within the country.

Perry: Talking Less and Listening More

From the onset of her teacher leadership work, Perry reported using a novel practice to promote the professional growth of mathematics teachers when she facilitated professional learning. She started her work by negotiating roles for herself and for the teachers. As she expanded:

> I decided to be a person who talks less and listen more … to hear what teachers do but also, I wanted to motivate them. I didn't want to do like what I was doing in my own classroom with my students or do what most facilitators are doing here.

The statement showcases Perry's determination in rendering the control of the professional learning to the mathematics teachers. By talking less, as she said, the intent was to encourage teachers to own their learning but also to become active players of their professional learning. With such practice of leadership, Perry invited teachers to openly talk about what they know, do, and experience in their schools and classrooms. This was a novel approach as mathematics teachers in Tanzania are used to being listeners and being talked to in professional learning sessions. Perry was confident that "those teachers have so much to say about the work they [have been] do[ing] in their classes." As such, she predicated professional learning on experiences of teachers who are more informed of what they need to know to improve practice in their classrooms. McCormack, Gore, and Thomas (2006) would support Perry's take on inviting teachers to talk more than her even though she was a leader of the learning sessions. Consistently, the scholars suggest teacher leaders create spaces for teachers to "voice their most pressing issues and concerns, examine prior knowledge in the light of new understanding and construct new knowledge through the processes of reflection [and] dialogue" (p. 99).

Shinje: Cascading Teacher Leadership

Shinje acknowledged his teacher leadership work as being something that should not be confined to professional learning sessions but should be extended to include teachers that did not participate in the professional learning sessions. Through such an up-surging understanding, Shinje became motivated to make teacher leadership an integral part of a school improvement agenda in the learning of mathematics. As he expanded:

> Since the day I became aware that this work is for all teachers, I have been working to spread it to my fellow teachers in my school. I have already introduced it to teachers, and I'm working with them, as their mentor, to help them realize what and how they can contribute towards school improvement. For me, this work is important for every teacher. We need to have teacher leaders who can bring changes in our school.

The account shows the process in which Shinje engaged in persuading his colleagues to think about how they can tap opportunities associated with teacher leadership in order to improve their school and students' mathematics learning. With such a pioneering spirit, Shinje was successful in inviting some of his teacher colleagues to see themselves as teacher leaders by establishing small projects to initiate changes in the teaching of mathematics. This strategy of using what might work in relation to the nature of the context is consistent with DuFour and colleagues' (2010) suggestion that MTLs should customize strategies they learn from other contexts to fit their specific contexts of their work.

Ivan: Leading Within and Beyond the District

In addition to facilitating the professional learning of teachers in his district, Ivan reported being involved in other educational undertakings that had reaches beyond the district. Ivan reported participating actively in several events held outside the district. One of the events was his participation in a national meeting of the Congress of Mathematicians and Mathematics Teachers of Tanzania. Ivan was invited to share his experience of leading teacher learning with participants at the Congress. Ivan recalled spending a whole day working with mathematics teachers from around the country, demonstrating different pedagogical strategies that can help them to situate effective instructional practices in their classrooms. As a follow-up to what he shared, as he described, they, then, engaged in a discussion about how they can teach mathematics in ways that could help students learn the subject given conditions of their own classrooms and schools. Such conversations motivated Ivan to continue to reach out to many teachers through the networks he established during the Congress. These networks fuelled Ivan's desire to prepare a workbook, containing exemplary lessons for facilitating the teaching of primary school mathematics. The idea behind such a pursuit is, as he described, to "help teachers to see how other teachers prepare and teach math lessons in their classrooms."

Kenzo: Embracing the Familiar

Kenzo was not well-informed about how local materials could be used to teach mathematics prior to participating in the development project. Based on what he learned, Kenzo described how he exposed mathematics teachers to using local learning materials for teaching mathematics:

> I prepared materials in advance. I also asked teachers to think about materials that are found in our places. Then, I asked them … how they can be used to teach math to students. I remember seeing some of them shocked as they didn't think materials can be resources for learning. After we worked with stones, leaves, and other materials, they started to see the connections and the importance of those things.

It is clear that incorporating the materials was an essential first step in making such materials commonplace not just in professional learning sessions but also in classrooms. Also, making teachers understand how to use the materials with their students was a completing step in making the materials an integral part of mathematics pedagogy.

Kenzo's use of local materials enriched the professional learning of mathematics teachers and helped those mathematics teachers to develop an awareness of the application and power of local materials in enriching the mathematics learning of their students. As Kenzo noted, the teachers promised to use learning materials in their teaching as a way of situating useful mathematics learning in classrooms.

Abba: Leading Learning for All

Abba was motivated by her past experiences of learning mathematics in a boys-dominated context during the time of her primary and secondary education in her teacher leadership practices:

> My past experiences as a girl in math class are still with me. I always remember to have struggled to learn math. Because of that, I now pay attention to girls' math learning. But also, to supporting my fellow female teachers in our group for them to effectively learn how to teach math in their classrooms without any challenge. I work hard to help them ... I collaborate closely with them to address challenges that they face during sessions.

The plot speaks to the personal desires of some MTLs to make a difference in how practices are conventionally understood or implemented. Abba was determined to make sure that female teachers, like male teachers, find themselves supported to promote their professional growth. To actualize her determination, she had to confront historical narratives that, for quite a long time, favoured the development of male mathematics teachers at the expense of the professional growth of female colleagues. Such an engagement highlights Abba's courage to overturn community and/or district practices that were considered a way of implementing teacher learning. Fullan and Quinn (2016) would commend Abba for her audacity as it enunciates a dedication to "deep learning for all … regardless of background or circumstance [and] commitment to the moral imperative of education for all" (p. 17).

DISCUSSION

The practices suggest that teacher leadership of mathematics education can vary greatly from one mathematics teacher leader to another mathematics teacher leader. As emerged in this study, the MTLs' decisions on the practices were highly influenced by what teacher leadership scholars, Katzenmeyer and Moller (2009), called "the complexity of the context factors" (p. 134). Despite the variations, aspects of the leadership practices by some MTLs were to achieve similar goals. On the one side, most of the teacher leaders (Perry, Abba, and Kenzo) led to promote effective professional learning of other mathematics teachers. Katzenmeyer and Moller (2009 would call these MTLs, on their words, "leaders of change" (p. 6). Such a name reflects their commitment of the teacher leaders in taking responsibilities for helping fellow mathematics teachers to advance professional knowledge and skills. On the other side, two MTLs (Shinje and Ivan) took leadership of mathematics education to a broader context, tailoring it to their own schools (Shinje) and to the entire nation (Ivan), working to influence change on the provision of mathematics education in schools. However, the variation we observed suggests the MTLs' awareness of what can be done to improve practice within and beyond their districts.

Two dimensions are quite characteristic of the MTLs who participated in our study. On the first, all the MTLs had participated in the professional learning sessions designed and implemented to empower mathematics teachers to develop an awareness of what it means and takes to become a mathematics teacher leader in a district. As such, they were expected to build on such awareness to effectively engage in taking actions for the betterment of student mathematics learning in their schools and beyond. The second aspect speaks to the issue of finding connections between teacher leadership and mathematics education within districts and beyond. The MTLs demonstrated taking initiatives to introduce novel practices depending on what they were individually experiencing in the districts regarding teaching and learning of mathematics in classrooms and teacher learning sessions.

We became aware that the development project sessions were the precursor of the subsequent actions that the MTLs assumed to improve mathematics education in their localities and beyond. That is, the sessions helped the teacher leaders to heighten their awareness of their position and capacity in contributing to the efforts of enhancing mathematics teaching and learning. Consistent with what we have learned in this study, Mason (1998) notes that awareness has the potential to "provoke and enable action" (p. 23). We are certain that the sessions opened the eyes and minds of the MTLs to become change agents, becoming innovative and committed to take actions geared to improve the teaching and learning of mathematics.

CONCLUSION

Our study is explicit about how the MTLs differently engaged in figuring out ways and strategies relevant for making a difference in mathematics education. They were not willing to embrace conventional practices that were characteristic of mathematics education in their districts. Instead, the MTLs were determined to navigate the journey of influencing change to enhance meaningful learning of mathematics among children. Such an observation demonstrates the potential of empowering mathematics teachers to heighten their awareness and capacity to participate in inspiring change for effective mathematics education in schools and suggests that the MTLs' leadership practices suggest their awareness of awareness-in-action and gives rise to a discipline of mathematics teacher leadership.

References

Balka, D. S., Hull, T. H., & Miles, R. H. (2010). *A guide to mathematics leadership: Sequencing instructional change.* Thousand, Oaks: CA: Corwin, A Sage Company.

DuFour, R., & DuFour, R. (2008). *Revisiting professional learning communities at Work™: New insights for improving schools*. Bloomington: Solution Tree Press.

DuFour, R., & DuFour, R. (2010). The role of professional learning communities in advancing 21st century skills. In J. A. Bellanca (Ed.), *21st century skills: Rethinking how students learn* (pp. 77-95). Bloomington: Solution Tree Press.

Fairman, J. C., & Mackenzie, S. V. (2012). Spheres of teacher leadership action for learning. *Professional Development in Education*, *38*(2), 229-246.

Fullan, M., & Quinn, J. (2016). Coherence making. *School Administrator, 30*.

Hardman, F., Hardman, J., Dachi, H., Elliott, L., Ihebuzor, N., Ntekim, M., & Tibuhinda, A. (2015). Implementing school-based teacher development in Tanzania. *Professional Development in Education*, *41*(4), 602-623.

Hargreaves, A., & Fullan, M. (2012). *Professional capital: Transforming teaching in every school*. New York: Teachers College Press.

Jackson, Y., & Allender, D. (2016). Waiting to excel: Mediating 21st Century teaching through the pedagogy of confidence. In J. A. Bellanca (Ed.), *Connecting the dots: Teacher effectiveness and deeper professional learning* (pp. 279-296). Bloomington: Solution Tree Press.

Katzenmeyer, M., & Moller, G. (2009). *Awakening the sleeping giant: Helping teachers develop as leaders*. Thousand Oaks, CA: Corwin Press.

Lieberman, A., & Miller, L. (2004). *Teacher leadership*. San Francisco: John Wiley & Sons.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2009). *Designing professional development for teachers of science and mathematics*. Thousand Oaks, CA: Corwin Press.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, *1*(3), 243-267.

McCormack, A., Gore, J., & Thomas, K. (2006). Early career teacher professional learning. *Asia-Pacific Journal of Teacher Education*, *34*(1), 95-113.

Mhando, E. S. (2012). *Reflective teacher: Essays on Education.* Morogoro: Elimu Reflective Networks.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. Thousand Oak, CA: Sage Publications.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Sanocki, S. J. (2013). *The process of how teachers become teacher leaders and how teacher leadership becomes distributed within a school: A grounded theory research study*. Unpublished dissertation, Western Michigan University.

Simmt, E., Binde, A., Glanfield, F., & Mgombelo, J. (2011). *Capacity development for mathematics teaching in rural and remote communities in Tanzania*. Ottawa: Global Affairs Canada.

Webb, P. T., Neumann, M., & Jones, L. C. (2004). Politics, school improvement, and social justice: A triadic model of teacher leadership. *The Educational Forum*, *68*(3), pp. 254-262.

NATIONAL PRESENTATION

SOUTHERN AFRICA REGIONAL PRESENTATION

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The regional presentation gives an overview of mathematics education in Africa then focuses on three examples drawn from southern Africa. The report has five sections: an introduction, overview of mathematics education in Africa, mathematics education in Botswana, mathematics education in Mozambique, mathematics education in South Africa, and finally some concluding remarks.

INTRODUCTION

Mercy Kazima, Joanna O. Masingila

We have the pleasure of introducing the regional presentation for southern Africa, with some input from the rest of Africa. The format of presentation will start with the general overview of mathematics education in Africa. In this presentation, Nouzha El-Yacoubi, current president of the African Mathematical Union (AMU), discusses the important role of mathematics in education in Africa and how the African Union recognises that this is crucial for Africa's development. She then discusses the current situation of mathematics education in Africa and provides examples of programmes that have been launched to support the teaching of mathematics in Africa. We then move to country presentations as examples of mathematics education in Southern Africa. We focus on three countries – Botswana, Mozambique, and South Africa – that were selected to represent the diversity of southern Africa. The presentation on Botswana, by Sesutho Kesianye, discusses the school system, mathematics teacher education, and an intervention that is implemented across the country with the aim of improving the teaching and learning of mathematics in Botswana. The second country presentation is on Mozambique and presented by Marcos Cherinda. He starts with a historical overview of mathematics education in Mozambique from the colonial period to the present. He then focuses on the post-independence era and discusses the current school system and mathematics teacher education. Finally, he discusses one example of an initiative in mathematics education that is encouraging mathematics and mathematics education in Mozambique. The third and final country presentation is on South Africa by Lyn Webb. She takes a different approach and focuses on the works of mathematics education chairs and numeracy chairs in South Africa, all with a common agenda of improving the teaching and learning of mathematics in South Africa. Through the discussion of the various chairs' work, she informs the reader about the objectives and achievements of each chair's project. All have made significant impact on the

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 221-240). Pretoria, South Africa: PME.

development and improvement of mathematics education in South Africa, the region, and beyond.

OVERVIEW OF MATHEMATICS EDUCATION IN AFRICA

Nouzha El Yacoubi

Introduction

In this new millennium, the role of education moved towards forging connections between knowledge development and its application to the workplace. The identified key factors to be a successful worker were critical thinking skills, problem-solving, and creativity. Therefore, innovations and new trends were necessary to fulfil the expectation that an educated person of this 21st Century should be able to access, interpret and possess information, to understand science and use technology. Mathematics, being an essential asset for addressing challenges like innovation and competitiveness in Science and Technology, which are major for economic growth and national development, changes were then useful in the curriculum content, the instructional process, and teacher–student relationships, allowing students to learn mathematics more meaningfully and more flexibly, and to make connections between mathematical concepts and their applications in real life, problem solving, etc. In particular, Information, Communication and Technology (ICT) stood to make the greatest impact on education as a whole and more specifically on Science, Mathematics and Technology.

In the developed countries, various innovative and emerged trends were introduced on moving away from a teacher-centered approach to one that is more student-centered. The teacher should be a coach and supporter, and learners are empowered to take ownership of their own learning processes. It was important to provide sizeable means in the technological equipment of schools, universities, and training for the teachers and educational personnel, to involve parents and other segments of the society. Unfortunately, Africa is still lagging far behind, in particular some Sub-Saharan Africa countries. The continent is at a comparative disadvantage with regard to overall development because of low investment in Science, Technology and Innovation (STI), and adoption of a short-term view of human development. The lack of Science and Mathematics knowledge has been identified by the African Union as one of the outstanding challenges Africa needs to solve: “Teaching is arguably the strongest school-level determinant of student learning and achievement. It is therefore important to pay attention to teacher quality and, by extension, to teacher preparation and the continuous development of teachers” (USAID, 2011).

The current situation of Mathematics education in Africa

According to some recent reports[1], African countries are broadly similar in key issues, institutional and national conditions that help or hinder the mathematics development, except for few countries, where some progress has been registered.

For example, in many African countries, teachers have little or no pre-service preparation before starting to teach. After beginning to teach, they have few or even no opportunities to participate in in-service professional development activities. Such teachers with so little preparation and a lack of support during the practice of their profession, fall short of acceptable professional standards in their function.

Some African governments have committed themselves to ensure the needed number of teachers and improve their quality for the renewal of the school. They have invested heavily in initial teacher training, in different ways according to their membership in the English, French or order educational system, but many problems have occurred in creating teacher training institutions and also in attracting interesting and motivated students.

Even for African regions that have succeeded in preparing mathematics teachers in the needed number for their countries (like in North Africa), little research has been done to assess whether they are producing teachers with knowledge, skills and abilities enabling them to form a foundation of schools of a good standard.

I emphasize that since 1990, reforms, Decennial and Emergency plans and special programs have been launched in African countries in search of innovative strategies for tackling the numerous problems facing education. These have included: societal and cultural barriers, funding, infrastructure, school environment, teacher qualification, quality assurance and permanent assessment, efficiency and relevance in managing and applying new strategies, introducing innovations and adopting new trends.

After more than a quarter of a century, the following impediments are still to be faced:

- The shortage, in particular in some Sub-Saharan countries, of well-trained teachers, lecturers and researchers;
- The availability and capacity of a well-trained and motivating teaching staff remains the largest obstacle, even in countries where governments put resources into this capacity building;
- The professional development of mathematics teachers is hindered by a lack of qualified and adequate mentors;
- The lack of relevant reform that is stimulating, responsive to the local context, and comprehensive in addressing issues of curriculum, assessment, pedagogy and progression;
- The poor accompaniment and lack of creativity in implementing innovations in mathematical sciences education;
- Deficient and outdated infrastructure, instrumentation, and teaching materials;
- Insufficient funding and budget mismanagement; and
- The brain drain of competent and well-trained African mathematics teachers and researchers.

In particular, the teaching of mathematics has not yet reached the minimum skills required to teach mathematics for this new century, consisting of a deep understanding of mathematics (content), teaching and learning (pedagogy), and the use of ICT (technology). Improving mathematics outcomes in Africa is a necessity that imposes changes, innovations and new trends in teaching and learning mathematics. Most African mathematics teachers need first to upgrade their mastery of mathematics since, in the best situation, they enrolled for initial training, whereas in some African countries mathematics teachers are recruited without any initial training and often without any diploma in mathematics.

To remedy such a situation, Mathematics teacher Continuous Professional Development (MCPD) is necessary. Teachers should be retrained to get a better understanding of recent reforms for effective integration into their profession, to be initiated to educational methodologies and tools useful for the teaching/learning related to the using of ICTs in mathematics courses. Some actions have been undertaken in this direction such as the GENIE Program *"GENéralisation des technologies d' Information et de communication dans l'Enseignement"* launched in Morocco in 2009. These sessions have been the most regular and successful ones in the MCPD programs in Morocco (El Yacoubi, 2015).

Other programs were launched through Africa:

In 1998, Kenya launched its own program to Strengthen Mathematics and Science Education (SMASE). It was seen as promising approach to address the challenges facing mathematics and science education.

In 2001, some 11 countries from Eastern, Central and Southern Africa met in Nairobi to create a platform around which they would create synergy in addressing mathematics and science challenge. Countries from West Africa joined this initiative in 2003, and the SMASE in Africa has been created in order to be opened up for all African countries, it becomes SMASE-Africa.

SMASE-Africa brought together about 35 countries. Its secretariat was hosted at the Centre for Mathematics, Science and Technology Education in Africa (CEMASTEA) in Nairobi.

In 2004, the Association for the Development of Education in Africa (ADEA), Japan International Cooperation Agency (JICA) and the Ministry of Education, Science and Technology (MOEST) of Kenya partnered to form the Working Group on Mathematics and Science Education (WGMSE). The WGSME has been converted into Inter-Country Quality Node on Mathematics and Science Education (ICQN-MSE) in 2014 with the goal to encourage countries facing similar challenges to come together with strategic partners who have expertise in a specific field and promote capacity building, networking, analytical work, information dissemination, and advocacy.

As for North Africa, the region benefited from the financial and technical support of five development agencies: the French Development Agency (afd), the African

Development Bank (ADB), the World Bank (WB), the European Investment Bank (EIB) and the European commission (EC).

International Support

Since 1996, international institutions and foundations have been launching projects, supporting the education development in Africa, and assisting African governments to reduce the gap in innovative and emerging trends in mathematics education. Improving the quality of education, the teacher training, the school administration and management have been identified as key fields where it is important and urgent to act.

These projects provided scientific and financial support to equip universities first and progressively high, secondary and primary schools with ICT materials, funding teachers and educational personnel training, creating pilot (experimental) classrooms as well as the Centre for Mathematics Science and Technology Education in Africa (CEMASTEA). They aimed to encourage initiatives in some African countries that seek to enhance teachers' pedagogical content knowledge and improve students understanding and motivation, particularly in science and mathematics, and also to positively enhance both teachers' and learners' attitudes towards mathematics and sciences.

At the tertiary level, several actions have been undertaken, aiming to foster and nurture the development of high-quality mathematical research in Africa. It is clear that all the investigators had a common goal to support the best African mathematical talent by creating a more adequate environment allowing them to better realize themselves and progress in a credible mathematics research, such as:

- The International Science program (ISP) launched in 2002;
- The African Institute for Mathematics Sciences (AIMS) created first in Cape Town in 2003, which is a Pan-African network of centres of excellence enabling Africa's talented students to become innovators driving the continent's scientific, educational and economic self-sufficiency. The AIMS-NEI (AIMS Next Einstein) has succeeded to extend the experience of the Cape Town AIMS to other AIMS in other African Countries: Senegal, Ghana, Tanzania, Cameroon, and recently in April 2019 in Rwanda.
- The African Mathematics Millennium Science Initiative (AMMSI) launched in 2004, and the Mentoring African Research in Mathematics (MARN) in 2005 in collaboration with the London Mathematical society.
- The African Mathematical Schools (AMS) was initiated by African Mathematical Union (AMU) in collaboration with The Centre for Pure and Applied Mathematics (CIMPA) in 2009.
- The Pan African Centre for Mathematics (PACM) is a collaborative project established in May 2010 between the University of Dar es Salaam, Tanzania and Stockholm University in Sweden.

- The African Union Project: Pan African University (PAU): The African Union (AU) decided to focus particular attention to intensify the scientific and technological development and innovation in the African continent and launched scientific projects in the five regions among them, one is devoted to mathematics: "Africa Mathematics Project" which was launched in 2012 with the support of the Simons Foundation.

Other projects were launched at the tertiary level, with special goals consisting of a joint venture between some African University and two or three universities out of Africa. That was to enhance international exposure of staff through conference seminars and partnerships.

Conclusion

In this new century, the innovative teaching and learning practice should be a major thrust for fostering creative learning and teaching. It should focus on improving learners' critical thinking skills, synthesizing decision-making, promoting students' construction of knowledge, applying knowledge and information to new situations by using their own creativity, and giving students the ability to communicate that knowledge with others.

Creativity in education is not just an opportunity; it is a necessity for both students and teachers. Teachers have to attract student interest and attention in a new way. They should act as catalysts, letting students ask questions, solve problems, investigate, analyze, and develop new knowledge. Thus, teachers need to be skilled in the specific process necessary to cultivate learner-centered environments and changing the focus from teaching to learning.

References

Nouzha El Yacoubi: "New Reforms to Improve Professional Development of Mathematics Teachers in Morocco". *Int J Edu Sci, 8*(1-ii), 199-205.

Working Group on Mathematics and Science Education (WGMSE). Retrieved: http://www.adeanet.org/en/working-groups/mathematics-and-science-education

ADEA and JICA support quality science and mathematics education in Africa (2016) Retrieved: https://www.afdb.org/en/news-and-events/article/adea-and-jica-support-quality-science-and-mathematics-education-in-africa-16212/

Kisangi Albert and Ateng' Ogwel: "Enhancing Teachers' Capacity Through In-Service Education and Training: A Reflection On Projects For Strengthening Mathematics And Science Education. Retrieved: http://www.academia.edu/419166/

African Development Bank. 2009. National Education Emergency Support Program. <http://www.afdb.org/

Agence Francophone pour le Développement 2009. Education Sector Policy letter, Ministry of Nation- al Education, Higher Education, Professional Training and Scientific Research. <http://www.afdb.org>.

European Commission 2008. Mission d'assistance pour la définition d'un système de formation continue des enseignants du Ministère de l'Education Nationale. From <http://www2.men.gov.ma/cf/documents/fc/etude_cee_enseignants/rapport intermédiare. doc>.

BOTSWANA MATHEMATICS EDUCATION

Sesutho Koketso Kesianye

Introduction

Mathematics education in Botswana is presented as a practice from the two perspectives of how it is encompassed within the school system and how it is provided for through mathematics teacher education. The school system is intended to provide a picture of the stages of education at which mathematics is taught and an indication of the degree of importance bestowed to the subject by the country. The perspective of the mathematics teacher education is intended to portray the different paths of becoming a teacher and/or an educator in the subject, together with the general qualifications commonly possessed by those tasked with the teaching of mathematics.

Mathematics education is undisputedly a cornerstone of the economic development of any society in modern years. Botswana is being considered here as a case example of the entire southern African countries with respect to the practice and developments in mathematics education. Since most countries within this region have economies of comparable stature and the fact that they share various cross-national aspirations, particularly in the area of economic advancement through science and technology innovation, which includes mathematics education (SADC Protocol on Science Technology and Innovation, 2008), it is befitting to learn from each other, hence this presentation. Most countries within this region are still developing, although at different rates, but the fact of the matter is that they all belong to the same category of developing countries that have challenges of resources and low performance in mathematics in comparison with developed countries as evidenced by recent international studies like the Trends in Mathematics Studies (TIMSS), TEDs-M and SEACMEQ (formerly SACMEQ).

The School System

The Botswana school system is comprised of pre-primary, primary and secondary levels. The pre-primary school level has been recently introduced in pilot schools to offer pupils pre-school education in government schools at no cost as opposed to what is happening in private schools (ETSSP, 2015-2020). Pupils enrol in the programme for a period of one year to be initiated into the school culture. On completion of pre-primary education, learners are admitted for Standard 1 of the Lower Primary school level that runs for 4 years, together with those who did not participate in the pilot exercise and those coming from private pre-schools, if any. They then sit for school-based assessment tests at the end of their Standard 4 to check their progress and other instructional related factors or issues. This enables teachers to make informed decisions about learners' progress before they move to the last stage of Upper Primary education, which runs for three years. The length of primary education is seven years after which learners sit for the Primary School Leaving Examination (PSLE) that is conducted nationally by the Botswana Examination Council (BEC). All learners progress to a three-year junior secondary school stage of the secondary school education. On completion of this stage, they sit for the Junior Certificate Examination (JCE) that is administered nationally by the BEC. All learners at pre-primary, primary and junior secondary learn mathematics as a core subject. The JCE is used for purposes of selection for senior secondary schooling and placement at various mathematics syllabi within this stage of secondary school education. The senior secondary school stage takes two years and on completion they sit for either the Core or Extended mathematics syllabi within the Botswana General School Certificate of Education (BGSCE) administered by the BEC nationally. In addition, and according to prior selection on entry to senior secondary schooling, learners study additional mathematics and statistics that are administered by Cambridge Education internationally.

Mathematics is a subject taught at all levels within the school education. It is a subject that occupies prominence in the school curriculum to an extent that it features as one of the core subjects at both the junior and senior secondary school levels (Botswana General Certificate in Secondary Education Mathematics Syllabus, 1999; Republic of Botswana Junior Certificate Examination Mathematics Syllabus, 2010). It is rightfully viewed as a subject of great importance, something that is supported internationally as seen by the prominence of the subject in any education system. Therefore, it is appropriate to examine the nature of preparation of those mandated to teach the subject to understand their caliber and suitability to produce good results for such an important contributor to the economic development of the country.

Mathematics teacher education structure and nature

Mathematics teacher education in Botswana is offered through two main routes. The first route is that of Colleges of Education, which are affiliated to the University of Botswana (UB). The Colleges of Education offer Diploma in Education Certificates. At these colleges, students studying to become mathematics teachers undertake mathematics as a major subject with an option to study other science related subjects. The mathematics major components are Calculus, Mechanics, Statistics, and Professional Studies, as well as other support disciples. This is to ensure that students are grounded in both the subject matter knowledge (SMK) and pedagogical content knowledge (PCK) together with educational foundations content. Furthermore, some students study mathematics as a minor subject whereby their major subjects would be other science subjects. For those taking mathematics as a minor subject, they are study mainly pure mathematics as a way of exposing them to mathematics content and also study professional studies, which enables them to teach mathematics at the junior secondary school level, where it is necessary. The Colleges of Education are affiliated to UB and as such their programmes are moderated by UB.

The second route is that of university education, in which the University of Botswana (UB) has been the main education provider before the establishment of other universities recently. Mathematics education at UB is attained through two routes. The first being the Bachelor of Education (BEd) where learners specialize in mathematics education and graduate at the end of the four-year programme. In their BEd mathematics component, learners study mathematics subject content from the Faculty of Science, methodology courses from the Department of Mathematics and Science Education (DMSE) and general education courses from the Department of Educational Foundations (EDF), as well as other support disciples. For Bachelor of Science (BSc) graduates in mathematics, they are allowed to enroll in a Post Graduate Diploma in Education (PGDE) programme that runs for one academic year. In this programme, learners study general education and methodology courses. The graduates of both the BEd and PGDE programmes are prepared to teach at secondary school level, particularly at senior secondary schools. These programmes undergo reviews by internationally identified reviewers with the aim achieving quality assurance and relevance in the global village. The reviewers have for each and every review commended DMSE for the high caliber of both the academic and support staff who are tasked with the preparation of teachers, in particular, mathematics ones. Currently, mathematics educators are comprised of one (1) Associate Professor, three (3) Senior lecturers who are also doctorate holders, and three (3) Master's in Education (mathematics) lecturers. This is an indication of the fact that mathematics teachers are prepared by qualified educators within DMSE.

On completion of these mathematics preparation programmes, graduates start teaching in secondary schools and continue to participate in professional activities offered by DMSE-INSET and the in-service department within the Ministry of Basic Education. It is through these activities that mathematics teachers participate in mathematics initiatives that are meant for provision of the improvement of the teaching and learning. One such an initiative in recent years is the Strengthening of Mathematics and Science Education Africa (SMASE) (Africa Report on the COMSTEDA-15) whose mission was "To promote effective classroom practices in primary and secondary mathematics, science and technology education through research, fostering relevant policies, networking, collaboration, advocacy and teacher capacity development" (pp iv). The mission and objectives of the intervention seem to have been a good development towards changing the low quality status quo of performance in the involved disciplines, mathematics included. However, the outcome of the intervention seems not to have produced expected outcomes. This could be a result of how the intervention was implemented, considering that key stakeholders in the preparation of teachers were not taken on board despite their expertise on how teachers are to be prepared.

References

SADC Protocol on Science Technology and Innovation (2008).

Strengthening of Mathematics and Science Education Africa Report on the COMSTEDA-15 (2017).

Republic of Botswana. (2010). Junior Certificate Examination Mathematics Syllabus.

Republic of Botswana. (1999). Botswana General Certificate in Secondary Education Mathematics Syllabus.

MATHEMATICS EDUCATION IN MOZAMBIQUE: AN OVERVIEW OF THE POST-INDEPENDENCE PERIOD

Marcos Cherinda

Mathematics Education in Mozambique – An Overview

Mathematics education in Mozambique can be described and analyzed in four different phases of the history of country, namely (a) in the feudal times, (b) in the Portuguese colonial period, (c) during the National Liberation Struggle, and (d) since the National Independence, in 1975, up to the present. However, the presentation will be limited to the period after the Independence. For these purposes, the presentation is basically supported by Gerdes' papers from the early 1980s and by the Mozambican dynamic transformations given in the education sector up to recent times.

The development of mathematics education in Mozambique it is indeed embedded in the development of the whole education sector in the country. Like in all other sectors, the revolution of 1975 brought deep changes in the education system. The Mozambican people, who had been denied the right to education, then were stimulated to go to school, to acquire scientific knowledge. The illiteracy rate in Mozambique at the time of Independence was 93% — one of the highest in the world. The school net was limited and the number of teachers, at all levels, was very limited. Colonial education did not reach the Mozambican masses.

As a result of that stimulation for acquiring scientific knowledge, immediately after Independence there was an educational explosion. The enrolment rate changed very drastically after 1975. In 1973, there were 589,000 children in primary schools. In 1978, there were 1,419,000 out of a total population of about 11,000,000 at that time. There were 33,000 pupils in secondary schools in 1974. In 1978, there were 82,000. During the First National Literacy Campaign in 1978, 130,000 were taught in the priority centres (sectors of collective production such as factories, state enterprises and agricultural cooperatives, etc.). For the Second National Literacy Campaign in 1980 the organisers planned to teach 300,000 workers (Gerdes, 1981).

This situation led to great challenges for the government, particularly for the education authorities. Despite all efforts of the government, together with the community's initiatives in opening new schools and training new teachers, Mozambique is still facing difficulties in education since that time. The economic growth and the investment made in the education sector do not align with the rapid increase of the population. In recent years, the education sector receives around 20% of the country's annual budget; however, more than 90% of the amount goes to personnel salaries and, therefore, very little investment is made in education infrastructure and equipment.

The efforts continue and besides some negative impacts, there are some signs of improvement. According to the Report on Holistic Situation on Teachers in Mozambique (2017), in recent years there has been a significant downsizing of the number of pupils per class, which reflects an increasing of the number of classrooms and of teachers. The pupil-teacher ratio in the lower level primary education went from a national average of about 80 pupils in 2006 to 60 pupils in 2016.

The School System

In 1983 – eight years after Independence – the government approved the Law of National Education System (SNE), whose main objective was to train people in the construction of a socialist society in Mozambique. This Law was replaced in 1992 by a new one following the approval of the 1990 Constitution, which introduced regulations based on pluralism of opinion and reaffirmed in the 2004 Constitution that develops and deepens the fundamental principles of the Mozambican State.

Thus, within the framework of the social organization of the Mozambican State, among other aspects, the possibility was opened of participation of other entities in the provision of education services materialized by the 1992 Law. In fact, the 1992 Law is currently misaligned with reality for a number of reasons, including:

- The maintenance of the structure of primary education of two grades (lower and upper) inherited from the 1983 Law, where the first (grade 1 to 5) is taught by one teacher per class and second (grade 6 and 7) by several teachers per class, causing limited access to the second grade due to insufficient teachers.
- The age of seven-year primary school finalists, aged twelve to thirteen, is not conducive to entry into basic technical-vocational courses, as pupils complete entry into the labor market on the one hand. On the other hand, the output profile of these graduates does not meet the demands of the labor market.
- The definition of grade 7 as a requirement for admission to the Institute for Teacher Training of primary education, at a time when more than 80% of the candidates who enter these courses finished grade 12.

In summary, the new Law of the National System of Education advocates for a basic education of 9 to 10 grades, aligning and harmonizing with the international conventions, of which Mozambique is a subscriber. Furthermore, it enshrines different global and regional agendas, like the 1997 SADC Protocol on Education and Training; the Agenda 2030 on Sustainable Development Goals, which advocates quality education; and the Agenda 2063 of the African Union.

With the new Law, the SNE aims to achieve the following objectives:

- To readjust the SNE to the current socio-political and economic context, ensuring equitable, inclusive and sustainable education for all citizens that responds to the demands of Mozambican society;
- To reorganize the structure of the current seven-grade primary education,

subdivided into two grades which refers to the plurality system in the second grade, for a six-grade primary education, with a continuous curricular plan, taught in a single-teacher regime;

- To move from grade 7 to the secondary level because of its complexity;
- Provide full primary education in all primary schools in the country, allowing all children to complete this level of education in time
- To establish a basic education of 9 years of schooling.

Therefore, the new Law aims to modernize and adjust the structure and functioning of the National Education System to the challenges of the present and the future. It is necessary and urgent to *promote an inclusive, effective and efficient Educational System that guarantees the acquisition of the required skills at the level of knowledge, skills and attitudes that respond to the human development needs*" (PQG 2015-2019, 61) and training the citizen to meet the challenges of the 21st century.

Mathematics Teacher Education

Taking a historical retrospective on overall teacher education – viewing the nationalization of education as a significant achievement of the Independence – there was an urgent need to fill the large gap left by the Portuguese teachers that left the country *en masse*. The educational explosion, as mentioned previously, aggravated that gap. An immediate solution was found, just as during the Liberation War, where students in higher grade started to teach those in lower grades. In this way, thousands of students with high mathematical achievement and good behavior but with no professional teacher training at all were appointed as teachers.

The appointed teachers were receiving "attachment" courses during the breaks of the school calendar. At the same time, the training of teachers began straightaway through intensive courses. In 1975, 10 Primary School Teacher Training Centers (the later called Instituto de Formação de Professores-IFP) were created, one in each province. Their courses had a duration, which expanded from 6 months at the start, to 12 months in 1980. The sixth year of schooling was the level necessary for entry (Gerdes, 1981). Late, in 1977 a large number of finalist secondary school students received from the former Presidente Samora Machel "the call of the country" to carry out different professional courses, including mathematics teacher education.

Thus, the Secondary School Teacher Training Courses were opened at the Eduardo Mondlane University in Maputo, training teachers to teach mathematics who after graduation were equally distributed over the provinces. Note that there was no training institute for secondary school teachers in Mozambique before liberation. In 1986, a specialized institute for teacher training – the Instituto Superior Pedagógico (ISP) – was created to prepare teacher of all levels and subsystems of the National System of Education. This institute became well known worldwide by its outstanding research of the emerging mathematics and mathematics education area – the ethnomathematics. In 1996, under the rectorate of Professor Paulus Gerdes, ISP was upgraded to the category of university – Universidade Pedagógica (UP). Very rapidly UP expanded its branches to all provinces, training teachers of all secondary education subjects, including TVET. The primary education teachers continue to be trained at IFPs.

Teachers may continue their training at various levels and in-service. UP and other institutions provide Distance Education, offering courses also specialized teacher training courses. Being at school, groups of teachers are organized by years in primary schools and by subjects in secondary schools, in which the teachers prepare their lessons together, discuss their difficulties and sit in on one another's lessons. Thus, the scientific and pedagogical weaknesses of the untrained teachers are partly compensated by the collective work of preparing lessons and sharing of experiences with colleagues that are more qualified (Gerdes, 1981). This practice continues until today in all our schools as recommendation of the Ministry of Education and Human Development.

Examples of Initiatives in Mathematics Education in Mozambique

Looking at the history of mathematics education in Mozambique, there is no doubt that the world's ethnomathematics movement came to know one of its major contributors in the development of the field, the Mozambican Ethnomathematics Research Project, founded and led by the late Professor Paulus Gerdes (1952-2014). The ethnomathematics studies, including doctoral theses, carried out since the 1980s in Mozambique produced many interesting results, both for mathematics and mathematics education.

One of the greatest challenges to mathematics education that African countries are confronted with is the problem of "low levels" of achievement in mathematics. The fear of mathematics is widely diffused. Many children (and teachers too?) find mathematics as a subject quite strange and useless, imported from outside Africa (Gerdes, 1991).

In order to respond to that challenge, ethnomathematicians argue that (African) heritage traditions and mathematics practices must be "integrated" or "embedded" in the school curriculum. It is in the context of seeking a response to this challenge to mathematical education in Africa that ethnomathematical research has begun in Mozambique (Cherinda, 2012) and has been stimulating many students, teachers and mathematics educators for further research.

Conclusion

Mozambique still faces many challenges in the education sector, as well as in other social and economic areas. Mathematics education can improve within the general structural development that the country is beginning to cultivate. The threats of climate change are also a major challenge that should be addressed to scientific research, challenging, on the other hand, the sociocultural self-confidence of students and teachers to find meaning in the role of mathematics education (in Mozambique).

References

Gerdes, P. (1991). Etnomatemática – Cultura, Matemática, Educação: Colectânea de Textos 1979-1991. *Instituto Superior Pedagógico*, Maputo

Gerdes, P. (1981). Changing mathematics education in Mozambique. *Educational Studies in Mathematics, 12*, 455-477

Cherinda, M. (2012). Gaining Cultural Self-Confidence: A Reflection on the Ethical Dimension of Ethnomathematics in the Mozambican Educational Context. *New International Studies in Applied Ethics*, 7

Governo de Moçambique (2015). *Plano Quinquenal do Governo - PQG 2015-2019*. Maputo

SOME INITIATIVES IN MATHEMATICS EDUCATION IN SOUTH AFRICA

Lyn Webb

It is well documented that learners in South Africa scored dismal results in TIMSS and PISA in the past. Statistics show that this situation improved marginally until 2011, then the upward trend slowed (Spaull, 2019). However, at PME we wish to celebrate the success stories, the reasons academics research and implement - and hope. What is happening in projects in South Africa that could improve mathematics education?

Mathematics Education Chairs Initiative (MECI)

(Information extracted from MECI Public Communication Learning Brief, 2018)

The MECI was founded with a vision to strengthen mathematics education by fostering collaboration between government, private sector, universities and other civil society organisations. The initiative has been funded by the National Research Foundation (NRF) and the First Rand Foundation in partnership with Anglo-American Chairman's Fund (AACF). There are currently four University Chairs in the second phase of this initiative – two Mathematics Education Chairs focusing on secondary schools Prof Jill Adler (University of Witwatersrand) and Prof Cyril Julie (University of Western Cape) and two Numeracy Chairs (Prof Mellony Graven (Rhodes University) and Prof Hamsa Venkat (Wits University) concentrating on primary schools. It is impossible to document the entirety of the projects and successes of the MECI, but a few examples of the dissemination of knowledge can demonstrate the diversity and impact of the project.

Wits Maths Connect Project – Secondary

This project is part of Prof Jill Adler's secondary Mathematics Education Chair

- Transition Maths 1 Course (TM1) – for teachers of Grades 8, 9 and 10 mathematics, focusing on developing their mathematics knowledge for teaching and their pedagogical skills in the transition from Grade 9 to 10 mathematics. The course has been attended by over 150 teachers from approximately 80 schools in 7 districts in the Gauteng province of South Africa. There is strong evidence of statistically significant gains in teachers' mathematics knowledge for teaching over the year of the course.

- The Learning Gains studies investigated the impact of teachers' participation in the TM1 course on learner attainment over one academic year. Different kinds of comparison groups have provided counter-factual evidence against which to measure the gains of the "TM-learners". For example, in 2018 the Grade 9 and 10 learners of teachers who had done TM1 in 2016 were compared with the learners of teachers who had done the course in 2017 as well as a comparison group who had no previous contact with the WMCS project. The "2016 group" made substantially more gains than the "2017 group" and the comparison group. The gains of the 2016 group were statistically significant with effect sizes for Grade 9 and 10 groups of d=0.68 and d=0.50 respectively.
- The Mathematics Discourse in Instruction (MDI) analytic framework for describing teaching (Adler & Ronda, 2015) and its sister Mathematics Teaching Framework (MTF) for working on teaching with teachers were developed grounded in the realities of mathematics teaching and learning in secondary schools in the project. A range of qualitative studies evidence teachers' diverse take-up of aspects the MDI-MTF in their practices, e.g. Ntow & Adler (2019); Ronda & Adler (2019 – PME RR)
- A model of "Lesson Study", adapted to suit to conditions in project schools, was explored as complementary to TM1 and systematically researched in 2016. The MDI framework structured lesson planning and reflection. Exemplification emerged as a key pivot in this collaborative work, together with the significance of the mutual roles of teachers and researchers in improving the lesson (Adler & Alshwaikh, 2019)
- Ongoing qualitative studies of teachers' "take-up" from TM1 particularly show engagement with and valuing of the mathematics learned and of the teaching practices experienced.
- Ongoing research from the Learning Gains data set is focusing on learner errors in introductory algebra, with particular focus on negative numbers and equations.

Wits Connect Project – Primary

This project is part of Prof Hamsa Venkat's South African Numeracy Chair.

- A Primary Mathematics Knowledge for Teaching course is offered over 20 days in three iterations to 112 Gauteng teachers. Statistics showed a 12-14% mean gain between pre-tests and post-tests. The number of teachers participating in the course from the project schools increased year on year, as did the mean gain. A 10-day abridged version of this course offered to 100 Free State teachers produced mean gains of 8.4% across two cycles.
- "I Hate Maths" Public Seminars are run by Prof Mike Askew. High numbers and positive response to these seminars indicate the need for platforms for primary teachers focused on engagement with content, practices and norms of

mathematics – a shift from a teacher discourse of "I can't do; I won't do" to "I can do; I will do".

- A Mathematics Subject Advisors' "Coaching for Development Course" is a 16-day course offered over 2 years focused on a combination of improving primary mathematics knowledge for teaching and supporting primary mathematics teaching development in schools.
- A Structuring Number Starters Project is an intervention model consisting of three annual workshops with grades 1- 3 teachers with follow-up in-class observation and coaching showed gains in early number performance of grades 1-3 learners drawn from the 10 project schools. Improvements in learning outcomes over time are reported in this PME Conference. Shifts in teaching towards more connected instruction focused on structure and generality that underpin the learning gains have been theorised in the 'Mediating Primary Mathematics' framework (Venkat & Askew, 2018) and operationalised in a range of research papers.
- An Additive/Multiplicative Word Problems Project is an intervention model involving a carefully designed sequence of lessons (minimum 4 lessons), and including pre- and post-tests across grades 1-7 has shown improved learner performance in the four operations. Online tools are available for all include multiplicative reasoning materials for grades 1-7; structuring number starters project materials and a Wits Maths circles booklet for supporting primary mathematics teaching and learning. (https://www.wits.ac.za/wits-maths-connect/wits-maths-connect-primary).

South African Numeracy Chair Project (SANCP)

This project is part of Prof Mellony Graven's South African Numeracy Chair. Some of the initiatives introduced by the SANC team in Grahamstown, now Makhanda, and environs include:

- Early Number Fun programme for Grade R teachers as well as Early Numeracy Inquiry Community of Leader Educators (eNICLE) programme for Grade 1 and 2 teachers and Grade 3 and 4 teachers (NICLE)
- After School Maths Clubs for grades 1 to 6 as a fun programme for learning mathematics have impacted learners and teachers and expanded to 4 provinces. Pre-and post-tests indicate a gain of 21%
- Family Math Story Time Programme for Grade R parents, learners and teachers on reading early number stories.
- Number Talks are 5 to 15-minute conversations around problems that learners solve mentally, which encourage sense-making. This follows research conducted in the USA by Jo Boaler.
- Diagnostic Assessments and Teaching Support Materials Tests and teaching

materials are being developed in collaboration with the Wits Chair team and DBE Grade 3 teachers and learners.

- Independent learner resources ('homework' drive for Gr 3-4 learners) are provided in the form of 'write-in' booklets to learners to work on at their own pace to support development of basic numeracy fluency
- Provision of single or double page spreads of mathematical activities with parent guidance for publication in local press.
- STEAM Camps Aimed at Grade 4 and 5 club learners and their maths club leaders, the camps have a dual focus on taking maths out of the classroom and into the real world and connecting mathematics to home, science and environmental awareness issues.

All activities are freely available on the internet at https://www.ru.ac.za/sanc/resources/

The Local Evidence-Driven Improvement of Mathematics Teaching and Learning Initiative (LEDIMTALI)

This initiative is part of Prof Cyril Julie's Mathematics Education Chair. LEDIMTALI focuses on the development of the practice of teaching for the specific enhancement of achievement in high-stakes school mathematics examinations. It is based on "best evidence" emanating from systematic reviews of research on strategies and tactics that improve achievement and ways of offering continuous professional development to teachers. The programme was taken by teachers and subject advisors and targets grades 7 – 12. The project focuses on supporting school Mathematics Departments as a whole and as a professional learning community.

- Analysis of the classroom teaching of participating teachers demonstrated shifts towards incorporating those aspects of teaching in their classroom that are known to impact positively on learner achievement in mathematics.
- Participating teachers, who were all qualified, addressed their mathematics knowledge gaps by declaring these and engaging in appropriate mathematical tasks; and then reflected on their learning through self-evaluations, as expected of professionals.
- Classroom tested school mathematics toolkits are available at https://ledimtali.wixsite.com/ledimtali/uploads

PrimTed project

The Primary Teacher Education Project (PrimTed) is a national initiative that has been developed through the Teaching and Learning Development Capacity Improvement Programme (TLDCIP) which is being funded and implemented through a partnership between the Department of Higher Education and Training and the European Union. The PrimTed Project is under the co-ordination of Nick Taylor.

PrimTed is a four-year collaborative initiative bringing together representatives from Higher Education Institutions in South Africa, to work on key common standards, materials and assessment approaches for teachers in initial teacher education (ITE) to be better prepared for the teaching of mathematics and language/literacy in the primary school. The project is a combination of different streams – "Number and Algebra", "Measurement and Geometry" and "Mathematical Thinking" as well as a work stream on language and literacy and work integrated learning (WIL).

The content streams are in the process of designing and sharing standards, materials, assessment tools and research relating to their focal areas. It is envisaged that by the end of the project there will be coherent curriculum frameworks and materials that can be utilised in initial teacher education programmes in order to create a platform of common understanding and standards at universities in South Africa.

At present collaborative teams are working across institutions in South Africa. Publications in accredited journals track the progress of the project.

Early Grade Mathematics Teaching

For the last two years, the Department of Basic Education has been working on the Early Grade Mathematics Project, which aims to evaluate and build evidence on effective interventions to improve the teaching and learning of early grade mathematics. A scoping study was completed largely through an evidence mapping process that included 161 studies related to early grade Mathematics teaching or learning.

The results of the scoping study included recommendations for a pilot study in order to test the effectiveness of using coaches to catalyse Communities of Practice (CoP) amongst Foundation Phase teachers within schools, in such a way that the CoPs could be sustained over time and persist even once support from the coach has ceased. A second suggested pilot study recommended testing the effectiveness of supporting Foundation Phase teachers to teach mathematics using a multilingual approach, involving translanguaging practices and multi-bilingual teaching and learning resources. Piloting is due to commence soon.

Funda Wande/Bala Wande and the Magic Classroom Collective
https://www.youtube.com/watch?v=dXmWoabS66o;
https://www.mandelainstitute.org.za/our-work/

In the early grades an emphasis on language and numeracy is intertwined so it is not surprising that literacy initiatives are closely followed by numeracy initiatives – and vice versa.

Funda Wande: Reading for Meaning is an initiative funded by the Allan Gray Orbis Foundation Endowment and donor partners, but which works closely with government agencies as well as Rhodes University. The aim is to ensure all children have an opportunity to master reading for meaning in their home language while in the

Foundation Phase. Stemming from this project is Bala Wande: Calculating with confidence which will be piloted in the Eastern Cape.

The Magic Classroom Collective in the Eastern Cape is guided by the Nelson Mandela Institute for Education and Rural Development at the University of Fort Hare. The Magic Classroom Collective provides a complete set of teaching materials for each classroom in the project. The teachers are supported by classroom –based instructional coaches. In addition the Institute, together with the Faculty of Education at Fort Hare, is developing a bilingual isiXhosa-English four-year BEd qualification for Foundation Phase teachers.

South African Schooling: The enigma of Inequality (2019)

A new publication, edited by Nic Spaull and Jonathan Jansen, will be published shortly by Springer. It contains chapters written by influential policy makers and researchers. It says far more, far more broadly, and also in much more detail than a short presentation at PME could achieve.

The problems remain, but there are initiatives that are making inroads in the sphere of mathematics education in South Africa.

Acknowledgements

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References

Adler, J & Alshwaikh, J. (2019) A case of lesson study in South Africa. In R. Huang, A. Takahashi, & J. da Ponte (Eds.), *Theory and practices of lesson study in mathematics: An international perspective*. Dordrecht: Springer.

Adler, J & Ronda, E. (2015). A framework for describing Mathematics Discourse in Instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237-254. DOI:10.1080/10288457.2015.1089677)

Ntow, F. D., & Adler, J. (2019). Identity resources and mathematics teaching identity: an exploratory study. *ZDM*, 1-14.

Spaull, N. (2019). Equity: A Price Too High to Pay? In N. Spaull & J. Jansen (Eds), *South African Schooling: The enigma of inequality* (pp. 1-19). Dordrecht: Springer.

Venkat, H. & Askew, M. (2018). Mediating primary mathematics: theory, concepts and a framework for studying practice. *Educational Studies in Mathematics*. https://link.springer.com/content/pdf/10.1007%2Fs10649-017-9776-1.pdf

CONCLUDING REMARKS

Mercy Kazima and Joanna O. Masingila

The presentations have given a picture of mathematics education in Africa with a focus on southern Africa, and specifically the countries of Botswana, Mozambique and South Africa. The school systems and teacher education in the rest of southern Africa are not very different from the cases of Botswana and Mozambique. Furthermore, the issues experienced in these country examples are common across the region and not necessarily unique to these countries. All countries recognise the importance of mathematics to economic development of the region. Therefore, all countries have some initiatives to improve the teaching and learning of mathematics in schools. The work of mathematics education and numeracy chairs in South Africa deserve to be applauded. The chairs' projects have provided research and development in mathematics education that has moved the field forward as well as encourage the teaching and learning of mathematics. The work has not only impacted South Africa as a country, but also the region and beyond Africa.

Finally, it is important to mention that Southern Africa has an association called the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE) which supports mathematics education in the region as well as science and technology education. The work of SAARMSTE includes an annual conference, an affiliated journal – the African Journal of Research in Mathematics Science and Technology Education (AJRMSTE), an annual research school and capacity building in Mathematics, Science and Technology Education research in southern Africa.

REGIONAL CONFERENCE REPORT

REPORT OF THE FIRST PME REGIONAL CONFERENCE: SOUTH AMERICA

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From November 14 to 16, 2018, the first PME Regional Conference was held in Rancagua, Chile. The conference was aimed at reaching and gathering South American researchers. This report provides details about the organisation of the conference, its participants, as well as the scientific programme. We end with an overview of the major issues that were addressed during the discussion sessions held with the conference participants.

INTRODUCTION

The International Group for the Psychology of Mathematics Education (PME) decided in 2016 to support the organization of a series of regional conferences. The goal of these PME regional conferences is to help scientists from regions which are currently underrepresented within PME, and in which comparable initiatives are hard to start without external funding, to start up a conference initiative in their region with the goals to:

- support the development of a regional research community that pursues the goals of PME (network building), and in this way,
- attract researchers from the region to actively participate in future PME conferences (international networking) and support them in preparing high quality PME contributions.

The first call for proposals was done in 2017, and after a positive vote at the Annual General Meeting at PME41 in Singapore, led to the organization of a South American PME Regional Conference in November 14-16, 2018. The conference chairs were David Gómez (Universidad de O'Higgins, Chile) and Wim Van Dooren (University of Leuven, Belgium).

The venue of the conference was the Universidad de O'Higgins, a newly created Chilean University located in Rancagua, Chile, not far from the capital Santiago. The organisation of the conference was further supported by the International Group for the Psychology of Mathematics Education, the Center for Advanced Research in Education (CIAE) of the Universidad de Chile, and the Chilean Society of Research in Mathematics Education (SOCHIEM).

CONFERENCE THEME

The theme that was chosen for the conference was "Understanding and promoting students' mathematical thinking". By choosing this theme, we wanted to emphasise the role of mathematics education research in helping educators to foster mathematical

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 243-248). Pretoria, South Africa: PME.

thinking in their classrooms, dealing not only with how mathematics education can be made more effective but also more inclusive and equitable.

The theme was intentionally kept broad, allowing the participants to reflect on these crucial goals in mathematics education starting from the diversity of perspectives and research traditions that is the hallmark of PME. While it primarily focused on students, the theme has long reaching relations to, for instance, teacher training and professional development, and cognitive as well as sociocultural approaches to learning.

PARTICIPANTS

The Regional Conference brought together 61 researchers and students. The majority of them came from South-American countries such as Chile, Brazil, Colombia, and Peru. In addition, a limited number of participants from outside the region (most of them PME members) took part in the conference too. They came from the USA, Mexico, Belgium, Germany, Spain, Ireland, and Denmark. The relatively small number of participants made it possible to have intensive interactions, build up research collaborations, and exchange ideas in informal settings.

Thanks to the support provided by IGPME, regional researchers got a reduced registration fee and a sizable number of them received financial support for their travel and accommodation.

PROGRAMME OVERVIEW

The scientific programme was to a large extent shaped in line with the programme of “regular” PME conferences. It included plenary lectures, Research Report sessions, Oral Communication sessions, and Poster Presentations.

In addition, three “PME sessions” allowed regional researchers to get acquainted with PME as an organization, its goals, scientific activities, and mechanisms to support participation from underrepresented regions.

Finally, two group Discussion Sessions allowed regional and non-regional researchers to share their views on the particular topics, contribution, and challenges of South American mathematics education research, as well as exploring avenues for a closer integration of the South American and PME communities.

Plenary lectures

Three persons were invited to give a plenary lecture at the regional conference: An established member of PME from outside the South American region, an established PME member from within South America, and a South American researcher who is not a member of the PME community.

In her plenary lecture, Merrilyn Goos (Ireland) looked back at the five years that she has served as the editor in chief of one of the most important journals in the mathematics education community, Educational Studies in Mathematics. She provided a review of the work that has been published in this journal in relation to the conference theme: understanding and promoting students’ mathematical thinking. She started by looking into the conceptualisations of what can be considered "mathematical thinking" in the research literature, selected curriculum documents, and international assessment programs such as PISA. Using this framework, she analysed the occurrences and the nature of the work related to students' mathematical thinking. The review suggested some salient features regarding the nature of the research conducted and the areas in the world in which it is conducted as well as on the educational level of participants, research aims, theoretical perspectives, and methodological approaches. She ended by pointing out some future research directions and opportunities.

The second plenary lecture was given by Marcia Pinto (Brazil). In her lecture, the sense-making of students who are conducting mathematical activities was the central theme. Her focus was on the sense-making strategies students employ when learning mathematics, and how this evolves over time. Marcia pointed out how in the last decades there has been a movement from the motivation of informing alternatives to classroom teaching (in the nineties) towards the development of quantitative studies to inform educational policies. In her lecture, she explored this scenario, and included a discussion on a new view on abstraction that conceives specific cognitive processes underlying mathematical concept construction interrelated with a particular strategy of sense-making.

The third plenary lecture was given by María Victoria Martínez (Chile). Her lecture focused on the potentials of classroom observation as a method to enrich student thinking about mathematics. She presented the design of an observation protocol that can be used as a tool to observe mathematics classes. It can also be employed in working on teacher feedback, as well as in teacher professional development. She presented an application of such classroom observations and the use of such a tool in work with mathematics teachers in Chilean public schools, and results from work in a bilateral project with a Mexican team. The construction of the observation protocol was explained in detail, as well as a study of its validation process, thereby detailing the various steps and giving extensive examples.

Submitted contributions – RR, OC and PP

The usual PME presentation formats, where participants can submit contributions, were also present in the programme. The way in which these were administered in the programme (timing, chairing, etc.) was identical to the way in which this is done in regular PME conferences.

In total, there were presentations of 12 research reports, 26 oral communications, and 12 poster presentations. The written outcomes of these presentations are published in the proceedings of the regional conference, which are available at http://www.uoh.cl/pme-regional/ and at the IGPME website.

The reviewing of these submitted contributions was coordinated by the International Programme Committee, which consisted of the two conference chairs (David M. Gomez of the Universidad de O'Higgins, Chile, and Wim Van Dooren of the University of Leuven Belgium) and two additional members (Manuel Goizueta of the

Pontifical Catholic University of Valparaíso, Chile, and Stefan Ufer of the Ludwig Maximilians University Munich, Germany).

The International Programme Committee received a total of 31 submitted Research Reports. These were sent out for external review in the usual double-blind peer review format. Each submission was reviewed by two or more peer reviewers who volunteered to conduct reviews for this conference. They are also experienced reviewers in the regular PME conferences. Five of the submitted Research Reports were accepted immediately for presentation at the conference. Another group of 10 submissions were given the chance to conduct revisions and resubmit a new version that was re-reviewed by one of the former reviewers. This is a deviation from the normal reviewing procedure at PME conferences, considered specifically for PME Regional conferences. After this round of revisions, the final number of accepted Research Reports increased to 13. Of those not accepted as Research Reports (after two rounds), 10 were invited to be re-submitted as Oral Communications and 5 as Poster Presentations.

The number of submitted Oral Communications proposals was 39. These proposals were reviewed by the International Programme Committee. Twenty-four of these proposals were accepted for presentation. Of those not accepted as Oral Communications, 3 were invited bot e re-submitted as Poster Presentations. In the end, considering re-submissions of Research Reports as Oral Communications, 26 Oral Communications were presented at the conference.

Finally, there were 10 Poster Presentation proposals submitted to the conference. These were also reviewed by the International Programme Committee. Seven of these proposals were accepted for presentation. In the end, considering re-submissions of Research Reports and Oral Communications as Poster Presentations, 12 Poster Presentations took place at the Regional Conference.

PME sessions

A session format that is not included in the regular PME conferences are so-called “PME sessions”. These sessions are specifically intended to allow regional researchers to get acquainted with PME as an organization, its goals, scientific activities, and mechanisms to support participation from underrepresented regions. Three such sessions were organised.

The first PME session was specifically intended to familiarise the participants with IGPME as an organisation and PME as a conference. We clarified the goals of PME, the way in which the organisation is run, the key figures in it, how membership can be achieved, and the various possibilities that members have to take part in the policy of the organisation. Also the role of other central persons such as the ombudsman and the presubmission support coordinator was clarified. Next, the general concept and structure of PME conferences was clarified, including the various conference formats. Finally, extensive time was spent to talk about the spirit behind PME conferences, the scientific and social community that PME stands for, and how one can engage in this community.

In the second session, specific information was given about the various conference formats, what they are, what their goal is, and points of attention in the submission and reviewing procedure were clarified. Most of the time was devoted to discuss Research Reports, Colloquia, Oral Communications, and Poster Sessions, but we also clarified how one can get engaged in Research Forums and Working Groups, for instance. In the second part of this session, we provided detailed information on the Skemp fund regulations and procedures, and pointed out how participants could benefit from this funding opportunity.

In the third session, we focused on the scientific quality of Research Reports. Starting from providing participants a "peek behind the screens" on the way in which Research Reports are evaluated by means of a peer review procedure and subsequently handled by the International Programme Committee, we clarified the major criteria that Research Reports need to meet, frequent reasons why research reports are accepted or rejected, and ways to improve chance for acceptance.

Discussion sessions

Finally, two time slots were dedicated to so-called group discussion sessions. Unlike the format with a similar name in previous regular PME conferences, the object of the ongoing discussions was not a specific topic in the area of psychology of mathematics education. Rather, the object of discussion was the regional conference and its organization itself. With the entire group of participants, we explored a wide range of issues, keeping in mind the general goals of the PME Regional Conferences that were given earlier in this report.

In a first session, we discussed mathematics education research in South America in a general sense. A discussion took place regarding what is specific to South American mathematics education research as compared to the rest of the world, what the strengths of this research are and what can others learn from it. We also explored the specific challenges that occur in South American mathematics education research. The issues discussed related to education and the educational systems in South American countries, educational policy, and organization. Further issues related to challenges in terms of research, funding, facilities to conduct research, and possibilities and hindrances in establishing collaborations. Finally, we discussed about the dissemination of mathematics education research in South America, both to a wider audience as well as to the international research community.

A second session was devoted to IGPME as an organization, PME conferences, and their relation to mathematics education research in South America. We started from the observation that many South American countries are underrepresented in the PME community in terms of numbers of participants, submissions, and presentations, as well as in terms of PME conferences being held in South America. In the discussion, we explored possible reasons for these observations, as well as possible ways in which more people from South American countries could be more engaged in PME conferences. We made an inventory of considerations and hindrances of conference attendees to come to the forthcoming PME conferences.

A final topic of discussion was the organization of the regional conference itself. A discussion was held about the way in which the conference was announced and disseminated, about the way submissions were solicited, and on the submission and reviewing procedure. We had a critical look at the conference programme and its various parts, as well as the practical organization.

An inventory of the major lessons from these discussion sessions is made, and the implications for the policy and organization of future PME conferences and future Regional PME conferences will be shared with the International Committee of IGPME.

ACKNOWLEDGMENTS

We want to thank all sponsors that made the organisation of the first PME Regional Conference possible, including the International Group for the Psychology of Mathematics Education, the Center for Advanced Research in Education (CIAE) of the Universidad de Chile, and the Chilean Society of Research in Mathematics Education (SOCHIEM). We are also grateful to the host institution Universidad de O'Higgins, the local organising committee, the added members of the International Programme Committee, and all PME members who acted as reviewers. And finally, we want to send a warm thank you to all the participants for attending, and for making this into a productive and fruitful conference.

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