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Editors:
Mellony Graven, Hamsa Venkat, Anthony A Essien and Pamela Vale

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TABLE OF CONTENTS
VOLUME 2

RESEARCH REPORTS (A – K)

CHANGES IN PRIMARY MATHEMATICS TEACHERS' RESPONSES TO STUDENTS' OFFERS 2-1
Lawan Abdulhamid

DISCOURSE IN CLASSROOMS OF PD PARTICIPANTS 2-9
Reema Alnizami, Anna Thorp, Paola Sztajn

TEACHERS' KNOWLEDGE DEVELOPMENT AFTER PARTICIPATION IN A COMMUNITY OF INQUIRY PROFESSIONAL DEVELOPMENT PROGRAM 2-17
Anabousy Ahlam, Tabach Michal

IMPLEMENTING INQUIRY-BASED LEARNING (IBL): OPPORTUNITIES AND CONSTRAINTS FOR BEGINNING SECONDARY MATHEMATICS TEACHERS 2-25
Judy Anderson, Una Cha

TAKING EACH OTHER'S POINT OF VIEW: A TEACHING EXPERIMENT IN COOPERATIVE GAME THEORY 2-33
Samuele Antonini

TEACHING FOR STRUCTURE AND GENERALITY: ASSESSING CHANGES IN TEACHERS MEDIATING PRIMARY MATHEMATICS..... 2-41
Mike Askew, Hamsa Venkat, Lawan Abdulhamid, Corin Mathews, Samantha Morrison, Viren Ramdhany, Herman Tshesane

BUILDING 3D SHAPES FROM SIDE VIEWS AND SHADOWS – AN INTERVIEW STUDY WITH PRIMARY SCHOOL STUDENTS 2-49
Daniela Assmus, Torsten Fritzlar

PRACTICALIZING PRINCIPLED KNOWLEDGE WITH TEACHERS TO DESIGN LANGUAGE-ORIENTED MATHEMATICS LESSONS: A DESIGN STUDY 2-57
Arthur Bakker, Farran Mackay, Jantien Smit, Ronald Keijzer

MATHEMATICAL KNOWLEDGE, INSTRUCTION AND LEARNING: TEACHING OF PRIMARY GRADE CHILDREN 2-65
Rakhi Banerjee

CHANGE IN POSING OPPORTUNITIES TO LEARN IN THE CONTEXT OF PROFESSIONAL DEVELOPMENT 2-73
Rinat Baor, Einat Heyd-Metzuyanim

LESSON STUDY IN PRIMARY PRE-SERVICE TEACHERS' EDUCATION: INFLUENCES ON BELIEFS ABOUT LESSON PLANNING AND CONDUCTION 2-81
Maria G. Bartolini Bussi, Silvia Funghi

A STUDY OF JAPANESE PRIMARY SCHOOL TEACHER PRACTICES DURING NERIAGE..... 2-89
Valérie Batteau

TEACHER INTERRUPTED: HOW MATHEMATICS GRADUATE TEACHING ASSISTANTS (DON'T) LEARN ABOUT TEACHING..... 2-97
Mary Beisiegel, Claire Gibbons, Alexis Rist

CHANGING TEACHER PRACTICES WHILE TEACHING WITH CHALLENGING TASKS 2-105
Janette Bobis, Ann Downton, Sally Hughes, Sharyn Livy, Melody McCormick, James Russo, Peter Sullivan

EYE-TRACKING SECONDARY SCHOOL STUDENTS' STRATEGIES WHEN INTERPRETING STATISTICAL GRAPHS……..... 2-113
Lonneke Boels, Arthur Bakker, Paul Drijvers

STUDENT TEACHERS' USE OF MEASURABLE PROPERTIES 2-121
Bruce Brown

VIMAS_NUM: MEASURING SITUATIONAL PERCPETION IN MATHEMATICS OF EARLY CHILDHOOD TEACHERS..... 2-129
Julia Bruns, Hedwig Gasteiger

INVESTIGATING MATHEMATICAL ERRORS AND IMPRECISIONS IN CONTENT AND LANGUAGE IN THE TEACHING OF ALGEBRA 2-137
Anne Cawley, April Ström, Vilma Mesa, Laura Watkins, Irene Duranczyk, Patrick Kimani

DIALOGUE AND SHARED COGNITION: STUDENT-STUDENT TALK DURING COLLABORATIVE PROBLEM SOLVING 2-145
Man Ching Esther Chan, David Clarke

INTEGRATING *DESMOS*: A CASE STUDY 2-153
Sean Chorney

A MATHEMATICS CURRICULUM IN THE ANTHROPOCENE 2-161
Alf Coles

RECONCILING TENSIONS BETWEEN LECTURING AND ACTIVE LEARNING IN PROFESSIONAL LEARNING COMMUNITIES 2-169
Jason Cooper, Boris Koichu

SUBJECTIVE APPROACH TO PROBABILITY FOR ACCESSING PROSPECTIVE TEACHERS' SPECIALIZED KNOWLEDGE 2-177
Rosa Di Bernardo, Maria Mellone, Ciro Minichini, Miguel Ribeiro

TEACHERS AND STANDARDIZED ASSESSMENTS IN MATHEMATICS: AN AFFECTIVE PERSPECTIVE 2-185
Pietro Di Martino, Giulia Signorini

THE CO-EMERGENCE OF VISUALISATION AND REASONING IN MATHEMATICAL PROBLEM SOLVING: AN ENACTIVIST INTERPRETATION 2-193
Beata Dongwi, Marc Schäfer

THE BODY OF/IN PROOF: EVIDENCE FROM GESTURE 2-201
Laurie D. Edwards

TEACHING TO CHANGE WAYS OF EXPERIENCING NUMBERS – AN INTERVENTION PROGRAM FOR ARITHMETIC LEARNING IN PRESCHOOL 2-209
Anna-Lena Ekdahl, Camilla Björklund, Ulla Runesson Kempe

ANALYSING LESSONS ON FRACTIONS IN THE MIDDLE PRIMARY GRADES: FOCUS ON THE TEACHER 2-217
George Ekol

TRIADIC DIALOGUE DURING CLASSROOM TALK IN THE CONTEXT OF PATTERN GENERALIZATION 2-224
Rabih El Mouhayar

EXPLORING THE IMPACT OF PRE-LECTURE QUIZZES IN A UNIVERSITY MATHEMATICS COURSE..... 2-232
Tanya Evans, Barbara Kensington-Miller, Julia Novak

DESIGNING CHALLENGING ONLINE MATHEMATICAL TASKS FOR INITIAL TEACHER EDUCATION: MOTIVATIONAL CONSIDERATIONS 2-240
Jill Fielding-Wells, Vince Geiger, Jodie Miller, Regina Bruder, Ulrike Roder, Iresha Ratnayake

THE INFLUENCE OF TEACHERS ON LEARNERS' MATHEMATICAL IDENTITIES 2-248
Aarifah Gardee, Karin Brodie

THE IMPACT OF MONTESSORI EDUCATION ON THE DEVELOPMENT OF EARLY NUMERICAL ABILITIES 2-256
Marie-Line Gardes, Marie-Caroline Croset, Philippine Courtier

INTEREST DEVELOPMENT AND SATISFACTION DURING THE TRANSITION FROM SCHOOL TO UNIVERSITY 2-264
Sebastian Geisler, Stefanie Rach

EXPLORING STUDENTS' REASONING ABOUT FRACTION MAGNITUDE…….. 2-272
Juan Manuel González-Forte, Ceneida Fernández, Jo Van Hoof , Wim Van Dooren

HOW ENGINEERS USE INTEGRALS: THE CASES OF MECHANICS OF MATERIALS AND ELECTROMAGNETISM 2-280
Alejandro S. González-Martín, Gisela Hernandes-Gomes

DESIGNING EFFECTIVE PROFESSIONAL LEARNING PROGRAMS FOR OUT-OF-FIELD MATHEMATICS TEACHERS 2-288
Merrilyn Goos, John O'Donoghue

INTERROGATING EQUITY AND PEDAGOGY: ACCESS TO MATHEMATICS IN AN INFORMAL LEARNING SPACE..... 2-296
Elena A. Contreras Gullickson, Lesa M. Covington Clarkson

DYNAMIC GEOMETRY CONSTRUCTION: EXPLORATION OF REFLECTIONAL SYMMETRY THROUGH SPATIAL PROGRAMMING IN ELEMENTARY SCHOOL..... 2-304
Victoria Guyevskey

LEARNERS' MATHEMATICAL MINDSETS AND ACHIEVEMENT..... 2-312
Lovejoy Comfort Gweshe, Karin Brodie

STUDENTS' GENDERED EXPERIENCES IN UNDERGRADUATE PROGRAMS IN UNIVERSITY MATHEMATICS DEPARTMENTS 2-320
Jennifer Hall, Travis Robinson, Jennifer Flegg, Jane Wilkinson

MAKING AND OBSERVING VISUAL REPRESENTATIONS DURING PROBLEM SOLVING: AN EYE TRACKING STUDY..... 2-328
Markku S. Hannula, Miika Toivanen

STUDENTS' USES OF ONLINE PERSONAL ELABORATED FEEDBACK.. 2-336
Raz Harel, Michal Yerushalmy

THE CASE FOR SELF-BASED METHODOLOGY IN MATHEMATICS TEACHER EDUCATION 2-344
Tracy Helliwell

VALIDATION OF A DEVELOPMENTAL MODEL OF PLACE VALUE CONCEPTS 2-352
Moritz Herzog, Annemarie Fritz

CONTRIBUTION OF ACADEMIC MATHEMATICS TO TEACHER LEARNING ABOUT THE ESSENCE OF MATHEMATICS……..……. 2-360
Anna Hoffmann, Ruhama Even

PREPARING IN-SERVICE TEACHERS FOR THE DIFFERENTIATED CLASSROOM 2-368
Lars Holzäpfel, Timo Leuders, Thomas Bardy

MENTAL COMPUTATION FLUENCY: ASSESSING FLEXIBILITY, EFFICIENCY AND ACCURACY 2-376
Sarah Hopkins, James Russo, Ann Downton

CULTURALLY DIVERSE STUDENTS' PERCEPTIONS OF MATHEMATICS IN A CHANGING CLASSROOM CONTEXT 2-384
Jodie Hunter, Roberta Hunter, Rachel Restani

WHAT IT MEANS TO DO MATHEMATICS: THE DISCURSIVE CONSTRUCTION OF IDENTITIES IN THE MATHEMATICS CLASSROOM 2-392
Jenni Ingram, Nick Andrews

ON THE LEARNING OF GROUP ISOMORPHISMS 2-400
Marios Ioannou

PRINCIPLES IN THE DESIGN OF TASKS TO SUPPORT PRE-SERVICE TEACHERS' NOTICING ENHANCEMENT……..... 2-408
Pedro Ivars, Ceneida Fernández, Salvador Llinares

PRE-SERVICE TEACHERS' NARRATIVES IN KINDERGARTEN TEACHER EDUCATION……..……. 2-416
Pedro Ivars, Ceneida Fernández, Miguel Ribeiro

PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING MATHEMATICS: WHAT MATTERS FOR PRESERVICE PRIMARY TEACHERS IN MALAWI?..... 2-424
Everton Jacinto, Arne Jakobsen

UNDERSTANDING OF WRITTEN SUBTRACTION ALGORITHMS: WHAT DOES THAT MEAN AND HOW CAN WE ANALYSE IT? 2-432
Solveig Jensen, Hedwig Gasteiger

NOTIONS, DEFINITIONS, AND COMPONENTS OF MATHEMATICAL CREATIVITY: AN OVERVIEW 2-440
Julia Joklitschke, Benjamin Rott, Maike Schindler

SECOND GRADERS' FIRST MEETING WITH VARIABLE NOTATION 2-448
Thomas Kaas

INDICATORS OF PROSPECTIVE MATHEMATICS TEACHERS' SUCCESS IN PROBLEM SOLVING: THE CASE OF CREATIVITY IN PROBLEM- POSING……...…… 2-456
Tuğrul Kar, Ercan Özdemir, Mehmet Fatih Öçal, Gürsel Güler, Ali Sabri İpek

EXPLORING RELATIONSHIPS BETWEEN NUMBER OF HOURS OF PROFESSIONAL DEVELOPMENT, MATHEMATICS KNOWLEDGE FOR TEACHING, AND INSTRUCTOR'S ABILITY TO MAKE SENSE OF PROCEDURES..... 2-464
Patrick Kimani, Laura Watkins, Rik Lamm, Irene Duranczyk, Vilma Mesa, Nidhi Kohli, April Ström

BRIDGING THE COGNITIVE GAP – STUDENTS' APPROACHES TO UNDERSTANDING THE PROOF CONSTRUCTION TASK 2-472
Katharina Kirsten

THE POWER OF THEIR IDEAS: HIGHLIGHTING TEACHERS' MATHEMATICAL IDEAS IN PROFESSIONAL DEVELOPMENT..... 2-480
Richard S. Kitchen

TEACHING GRAPHING FORMULAS BY HAND AS A MEANS TO PROMOTE STUDENTS' SYMBOL SENSE 2-488
Peter Kop, Fred Janssen, Paul Drijvers, Jan van Driel

TOWARDS COGNITIVE FUNCTIONS OF GESTURES – A CASE OF MATHEMATICS 2-496
Christina M. Krause, Alexander Salle

EXAMINING KNOWLEDGE DEMANDS FOR TEACHING CLASS INCLUSION OF QUADRILATERALS 2-504
Ruchi S. Kumar, Suchi Srinivas, Arindam Bose, Jeenath Rahaman, Saurabh Thakur, Arati Bapat

CLASSROOM SOCIAL CLIMATE IN THE CONTEXT OF MIDDLE SCHOOL GEOMETRY……..…… 2-511
Ana Kuzle, Dubravka Glasnović Gracin

RESEARCH REPORTS (A – K)

CHANGES IN PRIMARY MATHEMATICS TEACHERS' RESPONSES TO STUDENTS' OFFERS

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This paper contributes to research looking into changes in primary mathematics teachers' responses to students' offers, measured using an 'elaboration' framework. The framework was developed in a context of teacher practices characterised by absence of teaching that responds constructively to students' offers. Findings from the analysis of the teaching of one Grade 3 teacher across a two-year period revealed differences in interactions among her students, and her being more responsive to students' offers in the classroom. I argue that these observed differences are markers of changes towards incorporating practices that have been widely described in the literature as markers of responsive teaching quality.

INTRODUCTION

Eliciting and responding constructively to students' offers are high-leverage practices that have implications for students' access to the power of mathematics (Hallman-Thrasher, 2017; Hill et al., 2008; Mason, 2015). Such high-leverage practices are lacking in many primary mathematics classrooms in developing nations, and in South Africa in particular (Hoadley, 2006; Venkat & Naidoo, 2012). For example, in South Africa, classroom practices characterised by an absence of evaluative criteria that Hoadley (2006) have been described thus:

> The teacher engages in other work in her space and is not seen to look at what the learners are doing. She makes no comment on the work as it proceeds. No action is taken to ascertain what the learners are doing (p. 23).

A teacher's lack of interest in students' actions results in her students not knowing if what they are doing is mathematically correct, thus limiting their mathematical learning. Importantly, Hoadley noted that this absence of evaluative criteria represents a feature that has not been described as common in developed country contexts. In this context, for over 8 years, a longitudinal research and development project – Wits Maths Connect–Primary (WMC–P) – is developing and investigating interventions to improve the teaching and learning of primary mathematics in South Africa. In the course of this work it became necessary to have tools for examining differences in the quality of mathematics teaching, working from the base of non-responsive teaching described above, in order to understand the extent of improvement in teaching and its development.

Due to the specificity of the problems noted in developing nations, imported international theoretical frameworks (e.g. Hill and colleagues' Mathematical Quality of Instruction (MQI) (Hill et al., 2008); Rowland and colleagues' Knowledge Quartet

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(Rowland, Huckstep, & Thwaites, 2005)) largely result in deficit analyses, as they assume a baseline level of competence that is often not reached. Therefore, a language of description from a home-grown analysis was needed as a means to offer 'stages of implementation' (Schweisfurth, 2011) towards desired ends in relation to responsive teaching. An 'elaboration framework' was thus developed (Abdulhamid & Venkat, 2018) by paying close attention to the nature of teacher's responses in ways that moved away from deficit characterisations based on absences, to staging point characterisations directed towards improvement. The main question this paper thus addresses is:

- What changes over time in quality of primary mathematics teachers' responses to students' offers can be described through the lens of the elaboration framework?

THE ELABORATION FRAMEWORK

The lack of responsive teaching noted in South Africa led to the development of the 'elaboration' framework, which emerged from a grounded theory approach through analysis of 18 lessons taught by four primary mathematics teachers. Detail about the development of the framework has been written elsewhere (Abdulhamid & Venkat, 2018). Here an overview is provided to put the results presented later into context.

The framework provides a means to identify teachers' responses (and non-responses) to students' offerings in mathematics lessons and the extent to which these responses create opportunities for extending or deepening students' learning. The framework also allows us to chart and examine differences in responses over time, within four broad classroom situations where responsive teaching may be productive:

Breakdown – a situation of students offering incorrect mathematical answers or responses;

Sophistication – a situation with the potential to encourage more efficient use of mathematical representations and strategies;

Individuation – a situation where the teacher takes a group chorus correct mathematical offer and uses it to assess individuals' understanding; and

Collectivisation – a situation of opportunity for a teacher to 'unpack' an individual student's mathematical offer through sharing with whole class

A further crucial feature of the framework is hierarchies within the four situations that elaborate differences in the quality of teachers' responses. For example, in the case of breakdown situations, teacher responses that focus on students' offers are categorised into two types: (1) *teacher restating the students' offer and questioning its correctness*, and (2) *teacher probing students' offers with follow-up questions*. Fundamentally here, at level (1), we have acknowledgement of the incorrect offer, but no elaboration relating to how to go on to produce a correct offer, or to see why the given offer is incorrect – thus reinforcing a way of being with mathematics that is concerned primarily with the delivery of correct answers. The move, at level (2), is to probing reasons for the incorrect offer, and is thus geared towards mathematical processes as

well as their outcomes. Table 1 provides a summary of the hierarchical categories where the teacher responded to students' offerings, related to the focus of this paper.

Situations of elaborations	Hierarchy of categories of teacher responses
Breakdown	L1 – Restates students' offer and questions its correctness
	L2 – Probes students' offer with follow-up questions
Sophistication	L1 – Offers a more efficient strategy
	L2 – Elicits more efficient student offers
	L3 – Interrogates students' offers for efficiency
Individuation	L1 – Confirms chorus offers with individual students
	L2 – Interrogates chorus offers with individual students
Collectivisation	L1 – Confirms individual student's offer with whole class
	L2 – Interrogates individual student's offer with whole class

Table 1: Hierarchical categories of teacher responses within the elaboration framework

DATA SOURCES AND METHODOLOGY

To illustrate differences in the quality of teachers' responses to students' offers, I share data and analysis of one Grade 3 teacher, Thandi (pseudonym), teaching additive relations across a two-year period (2013 and 2014). Between the two years, I engaged with Thandi in a video-stimulated recall (VSR) interview. The aim of the interview was to both understand Thandi's rationales for classroom decisions, and to develop her mathematics knowledge for teaching through reflection on practice. Prior to the lesson observation, Thandi had attended a 1-year WMC-P 'maths for teaching' course in 2012. Thandi had more than 15 years of teaching experience.

I observed and video-recorded five lessons prepared and delivered by Thandi (2 in 2013 and 3 in 2014). Following the observations, I created verbatim transcripts that captured all the teacher talk, teacher–student interactions and descriptions of the tasks and representations that were produced and used by the teacher during the course of the lessons. The analysis began with identification of situations of elaboration, which form my unit of analysis. Each unit of analysis is initially examined as either the teacher providing elaboration (i.e. responding to the students' offerings) or not providing elaboration (i.e. ignoring or acknowledging students' offers and move on or pulling students' back to naïve strategies or representations). The incidents where elaborations are provided were then coded against the categories listed in Table 1 and allowed for an exploration of differences in hierarchies of responsive teaching.

ANALYSIS AND FINDINGS

I provide a narrative account of selected incidents based on data extracts, and some commentary relating to moves in quality of elaborations as highlighted in Table 2 below. These narratives qualitatively illustrate differences in responsive teaching that the elaboration framework allows me to theorise as changes in the quality of teacher's responses to students' offers.

Levels of responses	**Breakdown**		**Sophistication**		**Individuation**		**Collectivization**	
	2013	2014	2013	2014	2013	2014	2013	2014
Level 1	5 (71%)	4 (36%)	0 (0%)	5 (42%)	0 (0%)	4 (80%)	4 (100%)	2 (40%)
Level 2	2 (29%)	7 (64%)	0 (0%)	3 (25%)	0 (0%)	1 (20%)	0 (0%)	3 (60%)
Level 3	N/A	N/A	0 (0%)	4 (33%)	N/A	N/A	N/A	N/A

Table 2: Thandi's summary of quality of elaborations across 2013 and 2014 teaching

For breakdown situations, in 2013, 29% of incidents of elaborations were at level 2 in comparison with 64% in 2014. No sophistication and individuation elaborations were seen in 2013 lessons, while widespread elaborations in 2014 were seen with some moves to higher levels. For collectivization, in 2013 there was no incident of elaboration at higher level in comparison with 60% in 2014.

Breakdown-quality difference

In 2013 lesson 1, in the context of the task 25=30- _ involving using a number line to find the missing subtrahend, Thandi invited learners to work out the problem on the board. The following excerpt played out (L – student and T – Thandi):

285 L1: (*Learner points at 25 on the number line and indicates a backward gesture with her left hand and then pauses*)

286 T: Where do you go from twenty-five?

287 L1: Backward

288 T: She says we start at twenty-five and go back. Does the sum say 25 minus? No, it says 25 equals (*Teacher invites another learner*).

289 L2: (*Learner points at 25 and demonstrates a backward gesture*).

290 T: We are going backward, if we say twenty-five minus, then we move backwards. But our sum does not say that. It says twenty-five is thirty minus what? (*Teacher invites another learner*)

291 L3: (*Learner starts at 25 and demonstrates a forward jump to 30*)

292 T: What do we do next?

293 L3: We go back

294 T: Go and sit down (*teacher work out the problem herself*)

Challenges with directly modelling this problem type have been noted in the literature. In Carpenter, Fennema, Franke, and Levi's (1999) categorisation, missing subtrahend problems are harder to directly model as the number of jumps to make is not known. Further there is an extensive literature base on children interpreting the equals sign as a signal to operate, rather than to seek equivalence (Molina & Ambrose, 2008), making problems with the operation on the right hand side more complex through being less familiar.

Thandi's response does not recognise this complexity, as in Line 288, she began with a restating of the student's offer (*twenty-five and go back*), and she went onto link this offer with the problem '25 minus' and questioned whether this was correct in relation to the original question. Given this analysis, this incident was coded as '*restates student offer and questions its correctness*' – a level 1 category of the breakdown. The explicit rejection of the students' solution actions, without any further elaboration that potentially elicits a correct solution action, appeared to result in a situation where the mathematical object seemed not to emerge for many students.

In her 2014 lesson 1, in the context of a subtraction task 38-9, Thandi had earlier introduced adding and subtracting 'near 10' numbers by using 10 as a benchmark. She invited one student to work out the task on the board. The student drew an empty number line, and marked 38 towards the end of the line. She then made a backward jump of 10 and landed at 28. The following excerpt played out:

324	L:	Twenty- eight
325	T:	What do we do next? Yes?
326	L:	Minus one
327	T:	Minus one; she says minus one, if we say minus ten and minus one how much have we subtracted?
328	Class:	Eleven
329	T:	But, our problem says minus nine not minus eleven
330	L:	Plus one (*learner responds quickly*)

Thandi's response to the student's offer of 'minus 1' having already jumped back 10, involved establishing that the student's offer was actually taking away 11, not 9, and was coded as an incident of 'probing student offer with follow-up questions'. The literature suggests that this kind of response has more potential for extending student understanding than overt rejection of the offer (Brodie, 2007). In contrast to Thandi's 2013 instances of elaboration in breakdown situations where there was a prevalence of elaborations involving a restating of the student offer and acknowledging its incorrectness (71%), in 2014 she probed students' incorrect offers in 64% of her responses by establishing the possible consequence of student's solution actions without explicit rejection of the offer.

Sophistication-quality difference

In her 2013 lesson 1, upon completion of writing the 10s between numbers 0 – 100 on a number line, Thandi asked learners to point to the position of 25. One learner pointed at the mid-point between 20 and 30. Thandi accepted this offer and wrote down 25. She then wrote down the following task: 25+5 = ___. Students raised their hands and she invited one student (L) to work it out on the board using the number line. The following excerpt played out:

260 L: (*Learner starts at 25, already marked on the number line, and makes a single forward jump of 5 and lands at 30*) Thirty

262 T: Show us where we start and how we move. Draw the jumps

264 L: We start here and move five places (*learner uses ruler to show movement from twenty-five to thirty*)

265 T: Show us on the number line.

267 L: One, two, three, four (*uses chalk and makes four marks between twenty-five and thirty marks while counting*).

In the excerpt presented above, it was clear that the student involved could work out 25+5=_ by starting at 25 and making a single jump of 5. Thandi's response was coded as *pulling back* (within the 'provides no elaboration' category) given that the student demonstrated a single jump of 5, while Thandi asked for counting on in ones. Thandi did not comment on why she insisted on the student showing counting in ones in the VSR interview, suggesting that the pulling back was not part of her immediate frame of awareness. The move from counting in ones to flexible group counting is an important one in developing sophisticated strategies for addition and subtraction (Mcintosh, Reys, & Reys, 1992). This kind of 'unstructured' working in the context of work with structured resources like a number line has been described in prior work in South Africa (Venkat & Askew, 2012).

In her 2014 lesson 3, in the context of a similar addition task, 6+25 on a number line. Thandi invited one student to facilitate working out the sum on the board with the whole class. He drew an empty number line and marked 25 (in previous examples, there had been discussion about the efficiency of starting addition with the bigger number). Thus, my focus here, as in the previous incident, is on the ways in which she dealt with the need to count on. The student asked the class what number to add first. One learner offered 'plus 1'. He made a forward jump of 1 and wrote down 26. Another learner offered 'plus 1' again. He made another forward jump of 1 and wrote down 27. Another learner offered, 'plus 1'. At this moment, Thandi interrupted, and the following excerpt played out.

294 T: (*Teacher interrupts*). It has to be easy. It just has to be easy for us. So we take numbers that are going to make it easy for us to count. I am not saying this is wrong, because I know that you were going to get the answer, but I just want you to get your answers quickly and easily. Now we are going to do that. We said six plus twenty-five, isn't it?

297 C: Yes

298 T: Now let's look at twenty-five and say how many do we need to add to get to the next multiple of ten? Mpho?

299 L: Plus five

300 T: Plus five, five and five is ten, so twenty-five and five is…?

301 C: Thirty

In the interaction presented in the excerpt above, Thandi encouraged learners into flexible group counting by using ten as a benchmark (Mcintosh et al., 1992). This response was coded as an incident of provision of elaboration characterised by '*eliciting a more efficient strategy*'. This marked a contrast to what was seen in her 2013 teaching where pulling back was the only sophistication-related response seen. Thandi's 2014 elaboration actions were constituted by 58% at higher levels in the sophistication situation, and therefore indicated contrasts with teaching in South Africa characterised by limited progression to more flexible mathematics working (Ensor et al., 2009).

DISCUSSION

The finding that the directions of difference were broadly patterned towards 'higher' levels of elaboration within all the four situations in 2014 mirrored the findings in the broader dataset across all four teachers (Abdulhamid, 2016). This suggests that it is feasible to interpret these empirical differences as reflecting improvement in the teachers' responses to students' offers. This claim is further supported by the broad evidence of a strong 'plan-orientation' in 2013 – in which the teacher pushes for tasks to play out with focus on her intended objectives, with no awareness seen of the need to deviate from planned action (Rowland et al., 2005) or to establish balance between scripted planning and improvisation (Sawyer, 2004) in her teaching.

In Thandi's 2014 teaching, there was evidence of substantial engagement with students' thinking in responsive ways (Franke, Kazemi, & Battey, 2007). These differences suggest changes in her ways of being with mathematical knowledge (Coles & Scott, 2015) in teaching, greater interactions among her students, and being more responsive to students' contributions, a practice that has been widely described in the literature as a marker of responsive teaching quality (Hill et al., 2008; Sawyer, 2004).

CONCLUSION

The differences seen in the extent and quality of teacher responses to students' offers suggest positive changes in responsive teaching, which were made visible through the lens of the elaboration framework. Given the South African evidence of gaps in responsive teaching actions, exemplifying this nature and range of differences in teaching are important developmentally in relation to attempts to improve students' access to the power of mathematics.

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DISCOURSE IN CLASSROOMS OF PD PARTICIPANTS

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In this study we examined the mathematical discourse in the classrooms of three elementary teachers who participated in a professional development (PD) program designed to support teachers in promoting high-quality discourse during mathematics instruction. Analysis of classroom observations shows that the teacher with higher pre-PD mathematical knowledge grew in classroom discourse at a greater rate than the other two teachers with lower pre-PD mathematical knowledge.

INTRODUCTION

Orchestrating multi-directional mathematical discourse with students and among students in classrooms has benefits for students' mathematical learning (National Council of Teachers of Mathematics, 2014). Researchers have recommended practices that can enhance opportunities for mathematical discussions during instruction, such as: asking students questions that support them in thinking conceptually (Ghousseini, Beasley & Lord, 2017) and encouraging students to ask questions to each other and to the teacher (Boaler & Brodie, 2004); sharing authority over mathematical ideas with students (Tofel-Grehl, Callahan & Nadelson, 2017); and encouraging mathematical explanation that consists of mathematical argumentation beyond procedural explanation to help students conceptualize mathematics (Kazemi & Stipek, 2001). However, communications observed in mathematics classrooms in the US are mostly unidirectional—teacher to student—inhibiting students' interest in mathematics (Herbel-Eisenmann, Steele, & Cirillo, 2013). Despite significant efforts to promote multi-directional conversations in mathematics classrooms, orchestrating rich mathematical discourse continues to be difficult for many teachers (Gallimore, Hiebert, & Ermeling, 2014; Kazemi & Stipek, 2001). Given the importance of orchestrating high-quality mathematical discourse, professional development (PD) opportunities that effectively support teachers in enhancing their abilities to promote high-quality mathematical discourse in their classrooms are critical. To explore the value of participating in such PD initiatives, we examined observation data to capture changes in mathematical discourse that took place in classrooms of three teachers who participated in the Project All Included in Mathematics (Project AIM) PD program.

FRAMEWORK ON CLASSROOM DISCOURSE

A key framework for this study and for the design of Project AIM is the Mathematics Discourse Matrix (Sztajn, Heck & Malzahn, 2013). Based on literature on mathematics discourse (e.g., Hufferd-Ackles, Fuson & Sherin, 2004; Willey, 2010), the Matrix categorizes discourse into four types (correcting, eliciting, probing, and responsive) across four dimensions: questioning, explaining, listening, and modes of

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 9-16). Pretoria, South Africa: PME.

communication. The four discourse types in the Matrix can be seen to represent levels on a continuum of discourse richness from correcting (lower) to responsive discourse (higher). When a teacher initiates the communication and students respond, with authority residing solely on the teacher, this is considered *correcting* discourse. Breadth increases with *eliciting* discourse when more students participate in discourse, describing the *what* and *how* of their solutions. Higher in depth, *probing* discourse involves deeper mathematical explanation, where the teacher's discourse with students pushes for mathematical explanation and justification. At the higher end, *responsive* discourse is observed when eliciting and probing are maintained, as well as evidence of making mathematical connections and students taking ownership of their learning. It is considered that the different types of discourse may be appropriate for different purposes during instruction. However, if the dominating discourse during a lesson is correcting, the richness of the mathematics classroom discourse tends to decline.

RESEARCH QUESTION

This study explored the following question: How does discourse change in classrooms of PD participants whose early-observed discourse patterns are mostly unidirectional? More specifically, we conducted a retrospective analysis of change in mathematical discourse in classrooms of three teachers who participate in Project AIM.

METHODS

This investigation is part of a larger design research study. In PD design research, researchers design, implement, and analyse PD materials and activities for the purpose of helping teachers develop well-researched instructional practices while also generating knowledge and theory about PD design (Cobb, Jackson & Sharpe, 2017). The cycles of PD design and implementation include ongoing and retrospective analysis (Cobb, 2000). We report on a retrospective investigation of one implementation of Project AIM.

Context

Project AIM is a 40-hour, year-long PD program designed to support elementary teachers in promoting high-quality discourse during mathematics instruction. The PD consists of a three-day Summer institute and seven after-school sessions over the following school year. A main feature of the PD is the adaptation to mathematics of strategies typically used to support discourse during literacy instruction.

Through several implementations, Project AIM has continually generated value for teachers who participated in the PD, making it an appropriate context for retrospective analysis. For example, using the Learning Mathematics for Teaching (LMT) measure (Hill & Ball, 2004), the research team found significant increases from pre- to post-PD in participating teachers' mathematical knowledge for teaching (MKT). Data from a project questionnaire (Sztajn, Heck, Malzahn & Dick, under review) also indicated that participants increased in their perceived discourse-related practices from pre- to post-PD. Results from Project AIM over the years are summarized in Table 1.

Implementation	LMT	Questionnaire
2012-2013	Increase with medium effect size	Increase in all components of the questionnaire with effect sizes ranging medium to very large
2013-2014	Increase with medium effect size	Increase in 6 of 8 questionnaire components with effect sizes ranging medium to very large
2014-2015	Increase with relatively small effect size	Increase in 6 of 8 questionnaire components with effect sizes ranging small to large
2016-2017	Increase with large effect size	Increase in 5 of 8 questionnaire components with effect sizes ranging medium to large

Table 1: Project AIM Knowledge and Practice Results

During the 2012-2013 implementation, observation data were collected for 16 of 78 total participants. The sample was selected to be observed based on the levels of participants' responses to the pre-PD LMT measure and discourse-promoting practices questionnaire using a stratified sampling approach, which resulted in four strata combinations of teachers with higher and lower knowledge and practice levels. Mathematical discourse in the classrooms of this sample was observed two consecutive days at two time points—once early in the school year (Fall 2012) and again toward the end of the school year (Spring 2013).

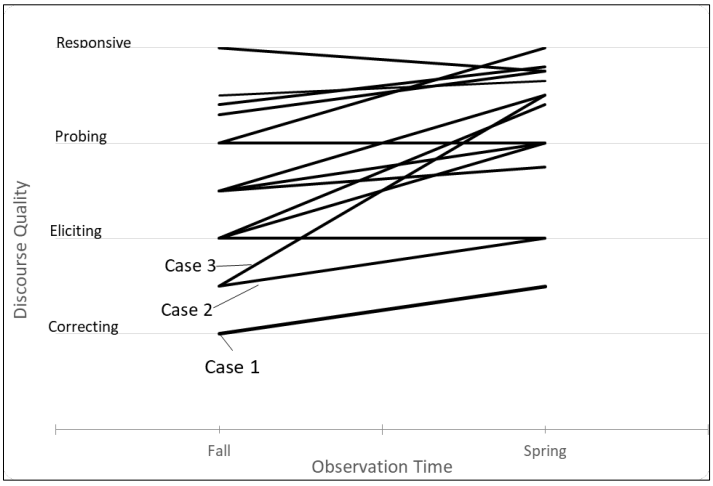

Figure 1: Change in overall discourse from Fall to Spring.

In a prior study, we conducted a retrospective analysis of the observation data for 15 of these teachers to further understand change in PD participants' classroom discourse (Alnizami, Thorp & Sztajn, in press). Data for the 16th teacher was dropped due to a

short time between the early and late data collection. Classroom observation protocols were coded using the Mathematics Discourse Matrix (discussed above). For each time point for each participant, a holistic discourse type was assigned, as seen in Figure 1. This study showed that teachers' pre-PD knowledge and practice levels did not define change in classroom discourse from Fall to Spring. Further, we found that mathematical discourse improved in most observed classrooms (Alnizami et al., in press).

Participants

For the present study, we investigated change in mathematical discourse by analyzing observation data, which is a data source suitable for learning about discourse quality (Desimone, 2009). Given our question about what happens in classrooms of teachers whose early observed discourse patterns are mostly unidirectional (i.e., correcting), we selected three of the 15 participants represented in Figure 1 for further investigation. These teachers' early discourse levels were the least rich (below eliciting) among the 15 teachers. Based on the pre-PD knowledge and practice measures, case 1 and 2 teachers scored low on the LMT, whereas case 3 teacher scored high on this measure. All three participants scored high on their self-assessments of their practice.

Data and Analysis

Twelve instructional lessons (four for each of the three teachers) were observed using a classroom observation protocol, resulting in a written description of the discourse that occurred during each lesson. The protocol specified that observers include examples and verbatim quotes from the lesson whenever feasible, which resulted in about eight-page long protocols for each timepoint.

Coding the classroom observation protocols was guided by the Mathematics Discourse Matrix (Sztajn et al., 2013). A pair of two consecutive lessons for a given teacher from the Fall or Spring time points was analyzed as one unit. For each pair of consecutive lessons, two authors determined the discourse type within each of three of the Matrix dimensions—questioning (teacher and students), explaining (teacher and students), and communication patterns (a component of the modes of communication dimension). Limitations imposed by reliance solely on field notes inhibited coding for discourse types on the listening dimension and the remaining elements of the modes of communication dimension. The two authors achieved more than 80% interrater reliability on the dimension coding. The two coders were not part of the project at the time of these lesson observations, hence they were not involved in the delivery of the PD or in the observation process.

FINDINGS

As illustrated in figure 1, while the overall discourse observed in two of the selected classrooms (cases 1 and 2) did not grow beyond eliciting, overall discourse in the classroom of the third teacher (case 3) improved beyond probing.

Case 1

There was no evidence of students asking questions of each other or of the teacher in either early or late observations of case 1 (Figure 2). Students' explanations declined in the level of discourse (from around eliciting to around correcting) from Fall to Spring. During the early observation, students' explanations consisted of providing their answers and how they found them in response to teacher's questions. For example, students shared their answers to a subtraction problem and explained how they solved the problem using methods such as base-ten blocks. During the late observation, students only provided short answers when asked questions by the teacher. For example, the teacher asked a question about representing a fraction, and the students answered with only *yes* and *no*. Changes in teacher questioning and teacher explanation are somewhat parallel. Specifically, the teacher's questions were below the eliciting level at both timepoints—they were mostly closed-ended questions that required short answers. The teacher's explanations also were below eliciting at both timepoints—she frequently explained step-by-step procedures.

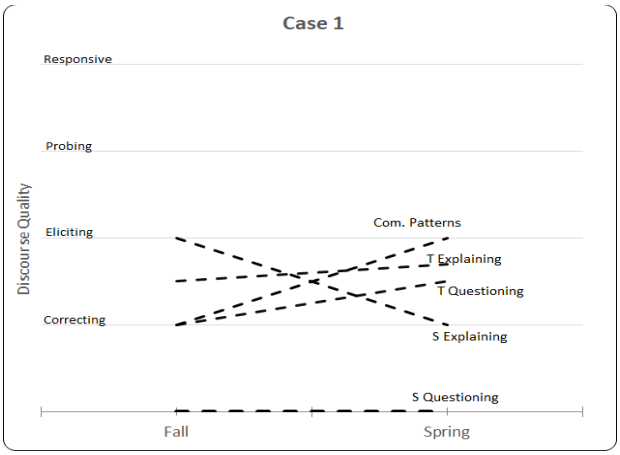

Figure 2: Change in discourse dimensions in classroom of case 1

Case 2

For case 2 (Figure 3), in both the Fall and Spring, teachers' questions of the students were mostly about how they found their answers (eliciting). The teachers' explanation increased some, but stayed within correcting. On the other hand, students' questioning and explaining were both above eliciting in the Spring. The higher level of increase for case 2 was on students' questioning. In the Fall, for example, students were given a stack of questions to ask their small-group members, but they were not observed asking questions on their own. In the Spring, some students asked discourse-rich questions of each other. For example, during a whole-class discussion, a student asked another student, *I still have a question about why you didn't cross out all of the tens?*

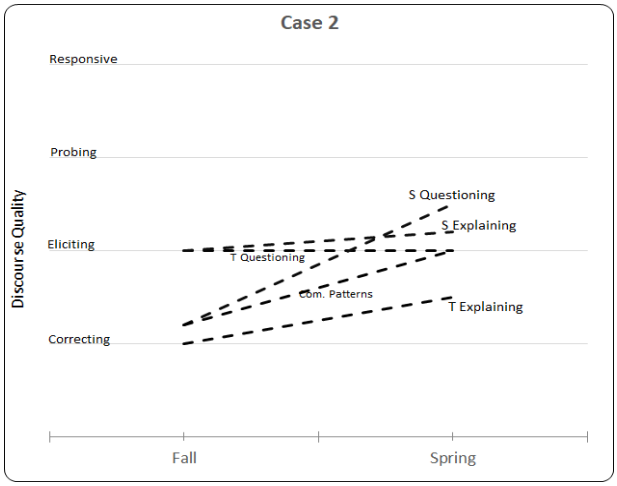

Figure 3: Change in discourse dimensions in classroom of case 2

Case 3

Finally, for case 3 (Figure 4), the discourse levels increased by at least one level on all of the five dimensions, with the greatest increase observed in student questioning, which was even the greatest increase in dimensions across all three cases. In the Fall, the only observed question that a student asked was posed to the teacher and sought to clarify a portion of the task—whether the portion of the problem that talked about getting 30 cents change was cents or money. In the Spring, students came up with their own questions to ask other students to compare between data sets that were collected by students.

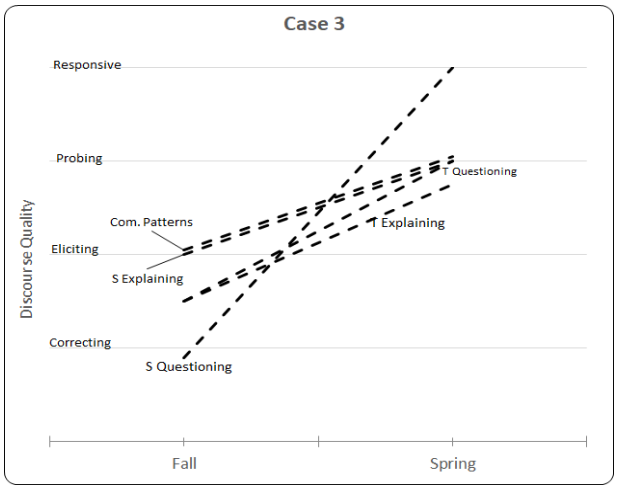

Figure 4: Change in discourse dimensions in classroom of case 3

Among all three cases, all the dimensions of discourse that were coded (teacher questioning, student questioning, teacher explaining, student explaining, communication patterns) increased or remained the same, except for students' explanations in the classroom of case 1. Change in discourse level for the communication-patterns dimension was comparable across the three teachers; and grew by one level. In the Spring, discourse levels for case 1 on the teacher components and communication patterns are those that are bringing the overall level up whereas the students' components are bringing the overall level down. On the other hand, for

cases 2 and 3, the students' components of discourse in Spring are bringing the overall levels up.

DISCUSSION

Change in discourse quality in mathematics classrooms requires change in both teacher and students' components. A teacher might be asking great questions, but if the teacher does not release some of the discourse authority to students, involving them in asking questions and explaining mathematical ideas, then the discourse will remain low. Discourse is not about just teachers' contribution to the discussion, rather, it is about what the teacher and the students contribute to classroom communications.

When taking into consideration teachers' pre-PD knowledge and practice levels, classrooms of the two teachers with lower knowledge (cases 1 and 2) did not grow beyond the starting level—although their self-reported data indicate higher pre-PD perceived practice levels. On the other hand, discourse in the classroom of case 3 (with higher knowledge score) grew beyond probing. Among these specific cases, we conjecture that teacher knowledge might have mattered to the observed changes in classroom discourse. This result is in line with prior findings indicating that teachers with more developed MKT find opportunities to engage in PD conversations in more meaningful ways (Wilson, Sztajn, Edgington, & Confrey, 2014). Future large-scale investigation is needed to examine change in discourse of teachers whose initial discourse levels are comparable to learn if, for those teachers, initial MKT matters for change in classroom discourse.

A note of caution is that the observations analysed here are not pre and post implementation of the PD; rather, the Fall observations were conducted relatively early in the school-year implementation stage of the PD and the Spring observations were conducted towards the end of the PD implementation. Participants had already completed the Summer PD institute when the early (Fall) observations were conducted. These results therefore need to be interpreted with caution.

Acknowledgments

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TEACHERS' KNOWLEDGE DEVELOPMENT AFTER PARTICIPATION IN A COMMUNITY OF INQUIRY PROFESSIONAL DEVELOPMENT PROGRAM

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Tel-Aviv University

The current study aimed to assess whether teachers' pedagogical technological knowledge (PTK) differed significantly after they participated in a professional development (PD) program based on Community of Inquiry (CoI) practices. It further sought to examine the effect of teachers' personal characteristics on the development of their PTK components. Forty-two middle school mathematics teachers participated in the study. Data collected using Thomas and Palmer's PTK questionnaire underwent statistical analysis. The results indicate that teachers' PTK components differed significantly after they participated in a CoI PD program, with the exception of the content knowledge component. Background variables had an impact on the development of some PTK components among the participants in the CoI PD program.

LITERATURE REVIEW

The Community of Inquiry (CoI) framework has been proposed for designing a PD program aiming at promoting mathematics teachers' knowledge related to technology integration (Thomas & Palmer, 2014). The present study seeks to examine teachers' PTK development in the context of such a PD program by means of two core themes: (1) teachers' knowledge and (2) PD program design. In the next section we examine the literature discussing these two themes in light of technology integration.

Mathematics teachers' knowledge

Shulman (1987) proposed a professional knowledge framework that incorporates seven domains of teaching knowledge. The category within this framework that revolutionized researchers' thinking was the pedagogical content knowledge (PCK) category, which links the knowledge bases of content and pedagogy. In particular, Shulman's PCK domain influenced teachers' knowledge frameworks for mathematics education. For example, Ball et al. (2008) proposed a model classified into six categories focusing on Mathematics Knowledge for Teaching (MKT): common content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, knowledge of the mathematical horizon, and knowledge of curriculum.

Shulman's PCK also influenced proposed theoretical frameworks for teachers' knowledge with respect to integrating technology into classroom practice. One of the most important of these theoretical frameworks is the technological-pedagogical content knowledge framework (TPACK), defined as the comprehensive body of knowledge and skills required for integrating technology in teaching (Koehler et al.,

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17-24). Pretoria, South Africa: PME.

2007), though it is not specific to mathematics education. The TPACK model describes the interactions between the three main domains of teachers' knowledge: content, pedagogy, and technology. These interactions result in new types of teachers' knowledge, namely PCK: technological content knowledge (TCK), technological pedagogical knowledge (TPK), and especially TPACK.

Thomas and Palmer (2014) proposed a theoretical framework in parallel to TPACK to describe teachers' knowledge with respect to integrating technology into mathematics classrooms—the pedagogical technology knowledge (PTK) framework. According to these researchers, several factors combine to produce PTK (Figure 1): MKT, which relates to pedagogical and mathematical content knowledge; technology instrumental genesis; and personal orientations. The present study utilizes this framework to measure PTK level and to examine whether this level differs significantly after participation in a PD program based on CoI design. In line with Thomas and Palmer (2014), personal orientation includes confidence and value of the use of technology.

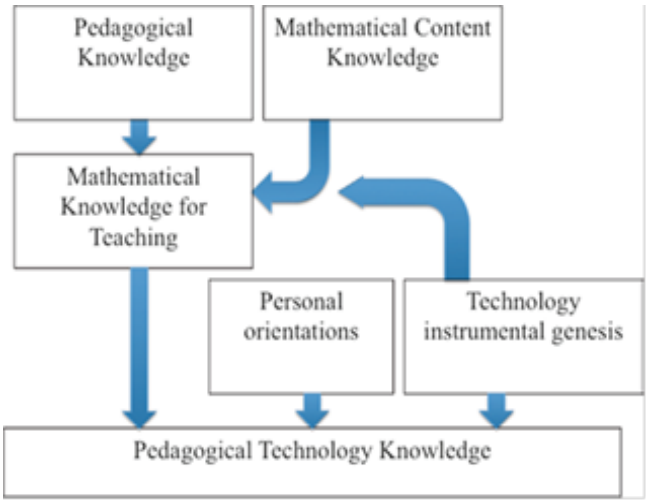

Figure 1: A model of the PTK framework (Thomas & Palmer, 2014)

Professional development program designs

In the absence of a "big" theory for teacher PD (Jaworski, 2006), researchers have attempted to identify frameworks for the professional development of mathematics teachers as well as types of PD programs. They identified two kinds of PD programs that influence learning and development among practising teachers: those that focus on content and process, and those that are strictly process-based (e.g., Simon, 2008). Programs that focus on content and process aim to promote mathematical and pedagogical knowledge, skills, and dispositions (ibid.). Process-only programs include, for example, the lesson study (LS) method developed in Japan. The LS method enables and encourages collaborative professional learning and sharing between teachers and their educators. Jaworski (2008) proposed a PD design based on inquiry that is parallel to the LS method and specifically geared for mathematics education. The inquiry takes place in an inquiry cycle (IC) of planning, acting and observing, reflection and analysis, and feedback.

Referring to PD programs aiming to promote mathematics teachers' integration of technology, Thomas and Palmer (2014) contended that a PD practice is best constructed around a supportive CoI that gives teachers the opportunity to observe, practice, and reflect on the use of digital technology in the classroom. They suggested organizing small heterogeneous groups of teachers in which each teacher, in turn, presents a prepared lesson incorporating technology. The lesson becomes the centre of

community discussion and reflection. In this way, the community takes advantage of those teachers who have a high level of PTK. The present study adopts this suggestion and implements the entire IC: plan, act and observe, reflect and analyse, feedback.

Several studies have investigated the PD of mathematics teachers within a CoI. For example, Jaworski (2008) describes the inquiry component in a development project in Norway titled Learning Communities in Mathematics (LCM): "inquiry was evident in the planning process, in ways in which teachers took workshop ideas back to schools and tried out ideas in classrooms and in the developing relationships between the participants as activity progressed" (p. 318). She also described the central role of this inquiry in sharing knowledge and expertise.

The current study continues the line of investigation from these previous studies while considering teachers' practice in the context of technology integration. The study adopts the suggestion of Thomas and Palmer (2014) and uses the IC to develop the PTK of mathematics teachers who work within a CoI.

Research questions

1. Do teachers' PTK scores differ significantly after they participate in a PD program based on a CoI framework?
2. Do background variables (seniority, previous technology integration level and employment status) affect changes in the PTK components from pre- to post-measurements among participants in the CoI PD program?

METHOD

The research was conducted during the academic year 2017-2018. The participants included 42 mathematics middle school teachers from several schools in average socioeconomic areas in Israel. Twenty-three of the participants were enrolled in a course titled Technology in Mathematics Education as part of their M.A degree in teaching mathematics. The rest were enrolled in a PD program aimed at increasing the level of technology integration in their classroom practices. The participants differed in their seniority. Twenty-one had been teaching for 0-10 years, while the other 21 had more than ten years of teaching experience. Moreover, the participants differed in their previous level of technology integration. Eight reported a low level of technology integration, 17 a medium level, and 17 a high level. Moreover, nine of the participants were ICT coordinators.

We used a PTK questionnaire as the data collection instrument. The questionnaire had two parts. The first part collected personal information, including seniority, employment status, and previous technology-integration level. The second part was composed of four scales: 1) personal orientation measuring two constructs—teacher's beliefs about the value of technology (26 items) and teacher's confidence in using technology to teach mathematics (7 items); 2) pedagogical knowledge (10 items); 3) technology instrumental genesis (5 items); and 4) content knowledge (6 items). Some of the scales (personal orientations, pedagogical knowledge, and technology

instrumental genesis) were borrowed from Thomas and Palmer (2014), while the content knowledge scale was developed by Hill, Schilling, and Ball (2004). Note that the scales by Thomas and Palmer were originally intended to examine teachers' confidence in using graphing calculators. In the present study, the word "technology" replaced "graphing calculators". Participants indicated their responses on a 5-point Likert scale, ranging from 1 (strongly disagree) to 5 (strongly agree). Because the scales had been translated, they underwent face validity testing. In addition, the reliability of each scale was analysed by computing its Cronbach's alpha based on the teachers' scores on the PTK questionnaire. These computations yielded Cronbach alphas ranging between .71 and .82, which are considered acceptable reliability scores.

In line with Thomas and Palmer (2014), the PTK levels for each teacher before and after the PD program were computed as the average of the following components: content knowledge, pedagogical knowledge, beliefs about the value of technology, confidence, and technology instrumental genesis. The first research question was analysed using paired-samples t-test. The second research question was analysed by two-way repeated measures ANOVA tests. To this end, two-way repeated measures ANOVAs were run with each of the background variables (seniority, previous technology-integration level, employment status) as a between-subjects factor and the PD program intervention as a within-subjects factor. The PD program intervention was represented in SPSS by a within-subject factor (time) of the two values: 1 for pre-intervention measurements and 2 for post-intervention measurements. Next, for the interaction analysis we ran post-hoc tests in SPSS with Bonferroni corrections, using the code 'EMMEANS=TABLES(A*B) compare(A) ADJ (Bonferroni)'. For example, in examining the interaction between PD program intervention and seniority we used the code: 'EMMEANS = TABLES (PD_time*seniority) compare (seniority) ADJ (Bonferroni)'.

FINDINGS

We first discuss the findings for the first research question and then those for the second question.

The effect of teachers' participation in a PD program based on CoI on their PTK level

To answer the first research question, we conducted a paired-samples t-test to compare the teachers' PTK and its components before and after PD program participation. Table 1 shows the means, standard deviations, and standard error means for the PTK components of the participating teachers before and after the PD program. The table indicates that the mean scores of the participating teachers after the PD program were higher than those before the PD program for all PTK components. To discover whether these differences are significant, we conducted a paired-samples t-test. Table 2 shows the results, indicating that the PD program yielded significantly higher means on all the components of the participating teachers' PTK, except for the content knowledge score.

Knowledge Component		M	SD	Std. Error M
Beliefs about the value of technology	Before	3.58	.50	.08
	After	4.05	.51	.08
Confidence	Before	3.85	.60	.09
	After	4.33	.50	.08
Pedagogical knowledge	Before	3.68	.51	.08
	After	4.01	.46	.07
Technology instrumental genesis	Before	3.69	.64	.10
	After	4.07	.63	.10
Content knowledge	Before	4.00	.44	.07
	After	4.04	.42	.07
PTK	Before	3.76	.38	.08
	After	4.12	.37	.09

Table 1: Means, standard deviations and standard error means for participating teachers' knowledge components (N=42)

Knowledge Component	Mean difference	SD	Std. Error M	95% Confidence Interval of the Difference		t	df
				Lower	Upper		
Beliefs about the value of technology	-.47	.45	.07	-.62	-.33	-6.75**	40
Confidence	-.49	.46	.07	-.63	-.34	-6.82**	40
Pedagogical knowledge	-.32	.38	.06	-.44	-.20	-5.41**	40
Technology instrumental genesis	-.38	.53	.08	-.55	-.21	-4.57**	40
Content knowledge	-.04	.16	.03	-.09	.01	-1.73	41
PTK	-.35	.25	.04	-.43	-.27	-9.17**	40

**p<.01

Table 2: Paired-samples t-test between participants' scores before and after PD program

Effect of interaction between PD program and background variables on participating teachers' PTK components

Each PTK component that exhibited different levels of the background variables (seniority, previous technology-integration level, employment status) was measured before and after the PD program. Two-way repeated measures ANOVAs were run with

each of the background variables as a between-subjects factor and the PD program intervention as a within-subjects factor.

The interaction between PD program intervention and the 'employment status' background variable did not yield significant results for any of the different PTK component scores (confidence: $F(1,39)=.86$, $p=.36$; pedagogic knowledge: $F(1,39)=.86$, $p=.36$; technology instrumental genesis: $F(1,39)=.77$, $p=.39$, beliefs about the value of technology: $F(1,39)=.84$, $p=.31$). Significant interactions between the PD program intervention and the other background variables are reported below.

Effect of interaction between PD program intervention and seniority on PTK components

A two-way repeated measures ANOVA was run, with seniority as a between-subjects factor and PD program intervention as a within-subjects factor. The results revealed a statistically significant effect of seniority on teachers' confidence ($F(1,39) = 10.11$, $p < .01$). Before the PD program, the confidence of teachers with seniority of ten years or less was significantly higher than that of teachers with seniority of more than ten years (mean difference=.91, $p<.001$). This mean difference decreased significantly after the PD program (mean difference=.48, $p<.001$).

In addition, the analysis revealed that the interaction between the PD program intervention and seniority had a significant effect on pedagogical knowledge ($F(1,39) = 4.23$, $p < .05$). Before the PD program, the pedagogical knowledge of teachers with more than ten years seniority was significantly higher than that of teachers with ten years or less seniority (mean difference=.33, $p<.05$). After the PD program, there were no significant differences between the participants' pedagogical knowledge (mean difference=.11, $p=.33$).

Moreover, the analysis revealed that the interaction between the PD program intervention and seniority had a significant effect on teachers' instrumental genesis ($F(1,39) = 9.04$, $p < .01$). Before the PD program, the instrumental genesis of teachers with ten years or less seniority was significantly higher than that of teachers with more than ten years seniority (mean difference=.80, $p<.001$). After the PD program, the mean difference still showed higher instrumental genesis among teachers with ten years or less seniority, but the difference had become lower and not significant (mean difference=.32, $p=.09$).

Effect of interaction between PD program intervention and previous technology-integration level

A two-way repeated measures ANOVA was run, with 'previous technology-integration level' as a between-subjects factor and PD program intervention as a within-subjects factor. The analysis revealed that the interaction between the PD program and previous technology-integration level ($F(2,38)=3.47$, $p<.05$) had a significant effect on confidence. Before the PD program, the confidence of teachers with a high level of previous technology integration was significantly higher than among those with a low technology-integration level (mean difference= .91, $p<.05$), and also significantly

higher than among teachers whose previous use was at an intermediate level (mean difference=.63, p<.05). After the PD program, the confidence of teachers with a high technology-integration level and of those with a low technology-integration level did not differ significantly (mean difference=.42, p=.18).

In addition, the analysis revealed that the interaction between previous technology-integration level and the PD program intervention (F(2,38)=4.64, p<.05) had a significant impact on pedagogic knowledge. Before the PD program, teachers with a high level of technology integration exhibited significantly higher pedagogic knowledge than teachers with an intermediate level of previous technology integration (mean difference=.45, p<.05) as well as higher pedagogic knowledge than teachers with a low level of technology integration, though the difference was not significant (mean difference=.02, p=1.00). After the PD program, teachers with a high previous technology integration level still showed an advantage in pedagogic knowledge over those with a low level (mean difference=.04, p=1.00) and those with an intermediate level of technology-integration (mean difference=.08, p=1.0), but this difference became lower and not significant.

DISCUSSION

The first goal of the current study was to examine whether teachers' PTK changed significantly after they participated in a CoI PD program. The statistical analysis showed significantly higher means for PTK and all of its components after participation in the PD program, except for the content knowledge component. These results support the suggestion of Thomas and Palmer (2014) to design a CoI PD program to develop teachers' PTK. The results seem to indicate that most PTK components can be influenced by experience related to technology integration, an interesting finding that needs verification through more extensive research.

The second goal of the current study was to examine whether the development of PTK components among participants in a CoI PD program was affected by background variables. The results for seniority revealed that teachers with more than ten years of experience exhibited a more significant increase in their confidence and their technology instrumental genesis than teachers with seniority of ten years or less. On the other hand, teachers with ten years or less seniority demonstrated a more significant increase in their pedagogical knowledge than teachers with more than ten years seniority. This finding can be explained by the sharing of knowledge among the CoI members, in accordance with Jaworski (2008), who described inquiry as centrally important in sharing knowledge to build new knowledge within the community. Moreover, the teachers' confidence developed based on the support in technology integration that those with less seniority provided to their more experienced peers (Thomas & Hong, 2013).

The results for the background variable of 'previous technology-integration level' indicated that teachers with a low level of previous technology-integration showed a significant increase in confidence as well as in their pedagogical knowledge regarding

technology integration. These results indicate that conforming to the IC can support teachers in their pedagogy, especially teachers with low PTK. This support is provided by sharing technological knowledge among the CoI members (Jaworski, 2008).

Finally, the findings of this study seem to indicate that PTK development was achieved by taking advantage of high PTK among teachers whose participation is valuable to the group (Thomas & Palmer, 2014).

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IMPLEMENTING INQUIRY-BASED LEARNING (IBL): OPPORTUNITIES AND CONSTRAINTS FOR BEGINNING SECONDARY MATHEMATICS TEACHERS

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The purpose of this study was to determine the factors impacting beginning secondary mathematics teachers' implementation of IBL. A survey of 29 novices in their first three years of teaching revealed most had a positive view about the benefits of IBL for students' learning yet struggled to implement IBL strategies effectively. While many respondents indicated they were supported in their schools and encouraged to try new strategies, they felt constrained by other factors including the lack of time and experience needed to prepare IBL lessons, and the crowded curriculum with mandated non-IBL assessment tasks. In-depth interviews with three teachers highlighted the impact of workload, particularly administrative tasks and extra-curricular activities, on beginning teachers' efforts to design and use inquiry approaches.

INTRODUCTION

Given the evidence of benefits of inquiry-based learning (IBL) strategies for students (e.g., Bruder & Prescott, 2013), teacher education programs in science and mathematics have been promoting the implementation of IBL for some time but there is little evidence of widespread implementation (e.g., Sullivan, Clarke, & Clarke, 2013). There is a rich history of research into IBL implementation, particularly for science teachers (e.g., Dorier & Garcıa, 2013) with findings suggesting the need for quality professional learning programs and ongoing mentoring for new teachers. In mathematics, rather than investigating IBL, research has focused more specifically on the implementation of problem solving and modelling perspectives (Maaß & Artigue, 2013) with recommendations to address teachers' beliefs as key to successful implementation (Anderson, 2014). While personal beliefs are an important consideration, context specific factors impact the development of each teacher's professional identity, and their pedagogical practices (Peressini, Borko, Romagnano, Knuth, & Willis, 2004). In mathematics education, the challenges for teachers to change their practice is well documented but there is limited research into studies of beginning mathematics teachers' use of IBL teaching approaches.

Because of recent changes in the Australian context including a new national curriculum (Anderson, 2014) and the promotion of integrated STEM education using inquiry approaches (e.g., National Council, 2015), this study sought to investigate the opportunities and constraints of implementing IBL for beginning teachers. The target group of teachers had all attended the same university education program and were

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taught by the first author of this paper. During their undergraduate program, the following definition was used to guide discussions and development of IBL tasks:

> The term ‘‘inquiry-based learning’’ generally refers to student-centered ways of teaching in which students raise questions, explore situations, and develop their own ways towards solutions (Maaß & Artigue, 2013, p. 780).

In the Australian context, IBL as defined here is not typical in secondary mathematics classrooms and is not evident in the types of tasks recommended in commonly used mathematics textbooks (Vincent & Stacey, 2008). So, for those schools who rely on a textbook as their guide for teaching, it is unlikely the students will be provided with IBL opportunities (Maaß & Artigue, 2013). This is potentially one constraint to the implementation of IBL practices but are there others? As beginning teachers struggle to familiarise themselves with school expectations, programs and assessment practices, with a new school culture, with a full teaching load, with managing student behaviour, and with accreditation requirements for teacher registration, do they have the time and support to develop IBL practices? Given the right support and contexts, are there opportunities which encourage and enrich beginning teachers’ efforts to implement IBL in their classrooms? These questions form the basis of the research reported in this paper.

LITERATURE REVIEW

Constraints to changes in practice for experienced teachers are well researched. Teachers frequently teach the way they were taught; hence changing pedagogies requires changes to deeply held beliefs about mathematics teaching and learning (Anderson, White & Sullivan, 2005). Such changes take time and effort but teachers also identify other issues constraining their implementation of new approaches to teaching and learning mathematics – these include the crowded curriculum, insufficient time for planning, few professional development opportunities (Anderson, 2014), lack of resources (Sullivan et al., 2013), assessment practices which focus on lower order skills and procedures, and students’ reluctance to engage with challenging tasks (Sullivan et al., 2013). These issues are all important and can create barriers to change but this situation is heightened for beginning teachers as they adjust to new work environments (Feiman-Nemser, 2001; Hudson, 2012; Peressini et al., 2004).

Beginning teachers fight to “sink or swim" (Feiman-Nemser, 2001, p. 1014) in their new profession and become tempted to use low maintenance practices that enable them to endure teaching rather than choose the more time-consuming IBL practices. As beginning teachers teach in potentially unfamiliar contexts, they are more likely to revert to the way they were taught with a focus on teaching facts and procedures through lower-order, repetitive tasks (Feiman-Nemser, 2001). In addition, beginning teachers are learning to manage behaviour in classrooms (Hudson, 2012), find useful resources (Sullivan et al., 2013), navigate an unfamiliar curriculum, and take on administrative work (O’Brien, Goddard, & Keeffe, 2008). O’Brien et al.’s longitudinal study into the issues of teacher retention and attrition identified beginning teachers’ additional workload from extracurricular and administrative duties such as marking

and writing reports as overwhelming. To counter these issues, support from mentors through school induction programs have been implemented but as Dorier and García (2013) suggest, beginning teachers may be exposed to observing traditional practices by mentor teachers, thus reinforcing the traditional methods they experienced in their own secondary mathematics education.

If we adopt a situative perspective, learning to teach evolves by participating in the different contexts of university program, field experience, and early school placements (Peressini et al., 2004). In their study of the learning trajectories of secondary mathematics teachers from undergraduate program into the first two years of teaching, Peressini et al. (2004) developed a conceptual framework with two related assumptions – learning is situated and "teachers' knowledge and beliefs interact with historical, social and political contexts to create situations in which learning to teach occurs" (p. 68). They argue for a process of "recontextualising resources and discourses in new situations" (p. 70) as beginning teachers develop their practice, and that discourse in the inquiry mathematics tradition is very different to the school mathematics tradition. Their framework informs this study as we used the task descriptions and inquiry discourse practices identified in their paper to design data collection and to gain a snapshot of beginning teachers' current IBL classroom practices.

THE RESEARCH DESIGN

To identify the factors which influenced beginning secondary mathematics teachers' implementation of IBL, particularly those that either provided opportunities or constraints, data were collected using a questionnaire and interviews. The questionnaire sought background information about the participants' workplace, their understanding and perceptions of IBL, their commonly used teaching practices, and the factors that have either supported or hindered the implementation of IBL in their current school contexts. Open-ended questions seeking information about influencing factors were coded and compared to responses from the interviews which had been transcribed verbatim. A convenience sampling strategy was used to recruit teachers at a secondary mathematics teachers' conference in the first half of 2018 – all 29 respondents were in their first three years of teaching and were alumni of the authors' university. Three questionnaire respondents volunteered to be interviewed by the second author. As background information, during the teachers' five–year double-degree program, research about the benefits of IBL was discussed (e.g., Bruder & Prescott, 2013). They had opportunities to try sample IBL tasks, compare these to more traditional textbook examples, and discuss curriculum implications of using inquiry approaches. It was an expectation that they would develop and implement IBL tasks during their field experience placements in schools, and this was accompanied by opportunities for reflection on their experiences.

RESULTS AND DISCUSSION

All 29 novices who participated in the study were teaching in their first school placement since graduating from the university. The sample included representatives

from the three sectors of secondary education in Australia (Department of Education [DoE] or public schools, Independent schools, and Catholic schools). Seventeen participants were in their first year of teaching, seven in their second year, and five in their third year. Interviewees included a female in her first year of teaching at a DoE school (X), a female in her third year of teaching at an independent boys' school (Y), and a mature-age male in his second year of teaching at a DoE school (Z). This section of the paper presents the results from both the questionnaire and interviews to connect key themes from both sources of data.

From a list of commonly used teaching practices, respondents were required to nominate the five practices they use most often in secondary mathematics classrooms. From their responses, the most commonly nominated teaching strategies (with the number of teachers who listed the strategy in brackets) included demonstration of procedures (19), whole class discussion (16), practice from textbooks (16), practice from worksheets (16), and individual practice (12). This set of strategies are typical of teacher-centred practices, which focus on developing mathematical skills, indicative of more traditional teaching (Bruder & Prescott, 2013), or the school mathematics tradition (Peressini et al., 2004), rather than student-centred approaches. The least frequently nominated practices included group work (8), real-world problems (5), open-ended questions (4), unfamiliar problem solving (4), student developed questions (1) and investigations (0), more usually associated with the inquiry mathematics tradition (Peressini et al., 2004) or student-centred pedagogies.

To obtain information about beginning teachers' beliefs, and further information about their use of IBL, the questionnaire included an item with seven statements to elicit participants' level of agreement using a four-point Likert scale, from 'strongly disagree' to 'strongly agree'. Table 1 shows the seven items with level of disagreement (strongly disagree and disagree) and level of agreement (strongly agree and agree) combined.

Statements about Inquiry based learning (IBL)	SD & D	A & SA
IBL is effective in teaching students' mathematics	5	24
IBL is only effective with higher-achieving students	22	7
IBL has potential to motivate students	2	27
IBL only suits some students' learning preferences	3	26
I would like to implement more IBL practices in my classrooms	2	27
I already use IBL in some of my mathematics lessons	15	14
I need more support implementing IBL	2	27

Table 1. Beginning teachers' level of agreement with IBL statements

The data from these statements indicate beginning teachers' desire to implement more IBL practices in their classrooms (27). Most agreed IBL is an effective mathematics teaching method (24) and has the potential to motivate students (27). In terms of current

implementation of IBL, about half of the participants agreed they already use IBL in their lessons (14), but they would appreciate more support (27).

Identifying Constraints to IBL

From the questionnaire data, most teachers acknowledged the benefits of IBL but they wanted more support. so the first real constraint appears to be lack of support. However, during interviews, respondents indicated they felt supported if they wanted to use IBL but that having enough time was the real issue. Z said, "the school's very for it… it's just that we don't have the time".

To further explore potential constraints, the questionnaire asked teachers to list the constraints to their implementation of IBL. The identified factors were: preparation time (26), a crowded curriculum (25), not enough experience with IBL (24), struggle to balance other school responsibilities (21), IBL not included in assessment tasks (15) and poor student behavior (14). Those constraints listed by fewer teachers included adjusting to a new school environment (9), colleagues not using IBL (11), IBL not included in the recommended textbook (11), and mathematics department reluctant to change practices (13). In this item on the questionnaire, only 8 listed lack of support so it is possible some participants viewed lack of support as not enough time, or not enough resources. The three most frequently identified constraints were further explored during the interviews.

Lack of time - Since beginning teachers' attention is divided between administrative work and lesson preparation (O'Brien, Goddard & Keeffe, 2007), lack of time is a significant factor in constraining IBL. Because the nature of IBL is more complex, creating tasks and lessons which also align with curriculum requirements takes longer. Y has been teaching for fewer than two years at an independent boys' high school where she lives in the school's boarding house during school term. Her role as a teacher extends beyond regular teaching hours with additional responsibilities associated with being a Boarding House Tutor – "It's just that we don't have the time… there is not much time I can spend preparing the resources and setting it up as well". X detailed the multiple responsibilities she had as a teacher which hindered her from preparing enriching IBL lessons.

> So last term I helped organise props for the musical, and then end of last term, beginning of this term I've been co-coordinating our team for the Da Vinci Decathlon. And then there's marking exams/writing exams/writing reports… Not to mention (sigh) I haven't even started doing anything for my [teacher] accreditation yet. There are days where I literally just prep 10 minutes before class because I don't have the time

X raised the issue of the increased expectations required through the mandatory accreditation process. This additional requirement places an extra burden on new teachers. Data from both the questionnaire and interviews, highlighted the workload and stress placed on new teachers as they begin their profession with an expectation that they will 'hit the ground running' from the first day at school.

Crowded curriculum - It was apparent in the interviews that beginning teachers struggled to incorporate IBL in their lessons since their priority was to complete the required curriculum outcomes. Y stated:

> There's so much content to go through. If I were to use IBL in my classroom for one lesson, they will be a whole chapter behind, because I still need to do questions from the textbook to show that they have satisfied the requirements

Being concerned about curriculum coverage was noted in the Dorier and García (2013) study, where teachers who were focused on teaching for assessments and fulfilling curriculum requirements were not implementing problem-based tasks. If students are also focused on examination preparation and believe IBL is not as important as learning for the tests, this adds more pressure on teachers (Dorier & García, 2013). In one of the interviews Y commented "[the students] will care if it counts to their mark…". The influence of what is assessed is well documented (e.g., Sullivan et al., 2013), and this certainly appeared to be a factor for about half of the respondents in the questionnaire. Having the experience to manage a crowded curriculum and understanding how IBL can help students learn more than one concept at a time is an important skill for beginning teachers and one they feel ill-equipped to handle.

Lack of experience - The noisy classroom, typical of IBL environments, can be unsettling with new teachers feeling out of control. Allowing students the freedom to investigate their own problems and their own solution methods leads to discomfort (Anderson, 2014). It is challenging to provide the experience pre-service teachers need to establish appropriate strategies for managing inquiry discourse (Peressini et al., 2004) as well as access to suitable tasks. X noted:

> I think the main problem is having to find these activities. Like, if there was an online database which had an inquiry-based learning activity next to each syllabus dot point, ready with worksheets which you could just adapt for your own classroom then I think a lot more teachers would be implementing it in their classrooms. I personally think that the main problem is spending the time to find these activities, and good ones I mean, is just way too time consuming and we don't have the time.

During professional experience placements, university programs rely on experienced teachers and mentors to offer that support and allow the preservice teachers to 'experiment' with new pedagogies. This does not always occur and so for some, they enter the teaching profession without any real experience.

Identifying Opportunities for IBL

From participants' responses to the short answer question about opportunities which would allow them to implement IBL practices more effectively, two new themes emerged - school cultures and authentic practice. For a small number of participants, they were encouraged to use IBL practices by more experienced colleagues – for them the outcomes were rewarding and they could see the possibilities. Most of these teachers were in their third year of teaching which suggests they had settled into their school context and felt more confident to try new practices. While Y felt supported and

was encouraged to "give it a go", she was overloaded with administration and other co-curricular tasks and responsibilities. Some respondents acknowledged they were keen to try IBL practices but were worried about wasting valuable time – they wanted the opportunity to practice the skills needed to manage inquiry discourse without fear of judgement. Still others recognized that they were able to use IBL in some classrooms and not others because of the experience and readiness of the students to engage in inquiry by asking questions, exploring situations, and developing their own ways of solving problems. These comments provide evidence of Peressini et al.'s (2004) framework on a situative perspective to teacher learning but they raise further questions about the support and development required to prepare new secondary mathematics teachers for IBL practices.

CONCLUSION

The purpose of this study was to determine the factors which influence beginning mathematics teachers' implementation of IBL. Overall, most of the beginning mathematics teachers in this study had a positive view about IBL strategies and understood the benefits for students' learning yet struggled to implement these effectively. While many respondents indicated they were supported in their schools and encouraged to try new strategies, they felt constrained by other factors including the lack of time needed to prepare IBL lessons, the perception of a crowded curriculum, and lack of experience in creating and using IBL. Considering this, there were useful recommendations provided by study participants that require new approaches to teacher education and new approaches to managing beginning teachers' time and responsibilities in busy, demanding schools.

Suggested recommendations to increase the implementation of IBL strategies for beginning teachers included reducing administrative and extracurricular responsibilities for beginning teachers, collating IBL resources, tasks and lessons which are easily accessible and clearly connect to the content areas in the curriculum documents; identifying good examples of IBL projects which connect more than one concept or topic from the curriculum; considering new ways for pre-service mathematics teachers to gain experience of teaching using IBL practices before they enter the profession.

While this study involved a small number of participants from one university, the data provide useful information for revising program approaches, re-examining requirements during field experience, and supporting new teachers as they enter their first appointment in schools. Beginning teachers should not be overburdened with new roles and responsibilities and mentors of beginning teachers need to be able to model the types of IBL practices we believe better support student learning.

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TAKING EACH OTHER'S POINT OF VIEW: A TEACHING EXPERIMENT IN COOPERATIVE GAME THEORY

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This paper is drawn from a teaching experiment in the field of cooperative game theory where different approaches are possible and a criterion for the validity of a solution has to be formulated and justified in classroom. In particular, I propose an analysis of the dynamic of different points of view and of argumentation produced by the students in exploring and discussing a game theory problem. The analysis reveals that the task and the social negotiation under the guide of the teacher promote a rich and intriguing environment to develop a productive dialectic between different perspectives and different argumentation.

INTRODUCTION

Since the great seminal book of von Neumann and Morgenstern (1944), game theory has reached a wide dissemination and it has now relevant applications in many branches of social sciences, economics, biology, and other disciplines. Game theory is a field of mathematics dedicated to the study of mathematical models of interaction between rational decision-makers (called *players*) that has provided mathematical techniques for analyzing situations in which two or more decision-makers make decisions that influence the outcomes. The players can either form binding commitments or not. In the first case, the game is called cooperative; in the latter, it is called non-cooperative (see, for example, Osborne & Rubinstein, 1994).

There is a large consensus that games can play a relevant role in learning mathematics and many articles in mathematics education concerned games (Mousoulides & Sriraman, 2014) as an environment to promote genuine problem-solving activities (Martignone, 2007; Martignone and Sabena, 2014), activities of conjecturing (Soldano et al., 2019) and modelling (Steiner, 1988; Scholz, 2007).

The assumption of game theory in studying interactions and decision-making is that players make rational choices. However, some studies show that our mind and the complexity of situations do not assure that human beings behave according to the strategies developed in game theory (Simon, 1955, Camerer, 2003). From an educational point of view, I think that this gap between the mathematical models set up in game theory and the behaviors of human beings makes it interesting to investigate processes involved in exploring a game and in managing the different perspectives of the players.

In this paper, I propose an analysis of the dynamic of different perspectives emerging when students explore a cooperative game, with focus on the different argumentation and on the shift between various perspectives in an Italian classroom. The hypothesis

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is that cooperative game activity can promote a rich and intriguing environment to promote argumentation and to develop a dialectic between different perspectives.

THEORETICAL FRAMEWORK

The Italian National Curriculum Indications states that a middle school student should be able to construct reasoning "*formulating hypotheses, supporting his/her ideas and sharing each other's point of view*". (MIUR, 2012, p. 61, personal translation). The importance of the sharing of points of view and of argumentation is underlined from the first lines of these Indications: "*the mathematics contributes to develop the ability to communicate and discuss, to argue in a correct way, to understand each other's point of view and argumentation*" (MIUR, 2012, p. 60, personal translation).

These excerpts underline a link between argumentative competencies and the sharing of different points of view, and therefore, the social dimension of argumentation. Mathematics educators share that argumentative competencies should be promoted through mathematical activity in classroom including practice of production of conjectures and meta-mathematical knowledge about the acceptability and the rejection of claims (Yackel, 2001; Mariotti, 2006; Boero, 2011; Stylianides et al., 2016). The social dimension has a crucial role in negotiating criterion of validity of mathematical statements, solution of problems and, in general, mathematical knowledge:

> At school, the social dimension related to the community of mathematicians must be coordinated with the social dimension related to the classroom community. The crucial role of the teacher comes to the forefront, representing contemporaneously the guarantor of the mathematics community and the guarantor of the classroom community. (Mariotti, 2006, p. 188).

The social dimension in the development of argumentative competencies and the focus on the sharing of different points of view make crucial the role of the teacher both as cultural mediator (between the classroom and the mathematical community) and for managing the different perspectives of students within the classroom. The theoretical notion of Mathematical Discussion (Bartolini Bussi, 1996) is a suitable tool to design and to analyze teaching activities taking into account both these aspects:

> Mathematical Discussion is a polyphony of articulated voices on a mathematical object (e.g. a concept, a problem, a procedure, a structure, an idea or a belief about mathematics), that is one of the motives of the teaching-learning activity. The term voice is used after Wertsch (1991), following Bakhtin, to mean a form of speaking and thinking, which represents the perspective of an individual, i.e. his/her conceptual horizon, his/her intention and his/her view of the world. […] A form of mathematical discussion is the scientific debate that is introduced and orchestrated by the teacher on a common mathematical object in order to achieve a shared conclusion about the object that is debated upon (e.g. a solution of a problem). In this case the teacher utters a voice that represents the mathematical culture [...] (Bartolini Bussi, 1996, pp. 16-17)

In this paper, I present a teaching experiment in cooperative game theory, designed in a framework of mathematical discussion, with the goal to analyze the emerging and the

transition between different points of view and argumentation. In the following pages, I use the terms "perspective" ("perspective of an individual" in the quotation by Bartolini Bussi) and "points of view" (as in the National Indications) as synonymous.

METHODOLOGY

The teaching experiment involved four lessons of two hours in two classrooms, each with about 20 students, of a middle school (respectively grade 7th and 8th) of Northern Italy and consists in exploring problems of cooperative game theory. Nobody of the students have had some previous experience with game theory. Every problem was explored in small groups of 4 or 5 students and in a whole-class discussion. The data consists in video-recordings of the group activities and of the discussions. In this article, I analyze the whole-class discussion in one classroom (grade 7th) about the first task (for a mathematical analysis of this problem, see Patrone, 2006, p. 187):

> *Ada, Bea and Ciro are three musicians that have to play in an event. They can play alone, in duet or in trio. Their profits are the following: 100 euros for Ada if she plays alone, 150 euros for Bea if she plays alone, 180 euros to Ciro if he plays alone, 600 euros for all of them if they chose to play together. If they play in duet, the profits are the following: 400 euros for the couple Ada and Bea, 300 euros for Ada and Ciro, 420 euros for Bea and Ciro. Putting yourself in the shoes of the musicians, try to discuss and explain how Ada, Bea and Ciro could find an agreement. Justify your assertion!*

For example, if the three musicians agree to play together, they earn 600 euros that they have to share in some way. If Ada and Bea decide to play in duet, then Ciro can only play alone: Ada and Bea have to share 300 euros how they want and Ciro earns 180.

The problem has been chosen because, as in other problems from cooperative game theory, there is not, a priori, one procedure to determinate a solution and rather the notion itself of "solution" has to be constructed; in other words, it is necessary to understand that a criterion is needed to validate a proposal of division of the profit and then a criterion, among many possibilities, has to be identified and justified.

ANALYSIS OF THE DISCUSSION

After the small group work, the students present their proposals to the class. It is possible to identify three phases of the discussion of the whole class: discussion of the proposals of the small groups, game of points of view and construction of a model.

Phase 1: Discussion of the proposals of the small groups

Initially, all the groups propose a simple solution. For example, Roberto says:

> Roberto: if [the musicians] are friend, they share the profit in equal parts […] [They play together] because 600:3=200 euros that is greater than the profit they would have playing alone.

Roberto proposes that musicians, if they are friends, share the profit in equal parts but he also justifies the musicians' decision to play together through an argumentation

about the greater profit (I refer to argumentation like this with the label 'greater profit'). In considering the possibility that musicians are not friends, students propose other arguments:

Flavio: Ciro earns more [than the others when he plays alone] […]

Roberto: … then he could say 'why do we have to divide into three equal parts, if I am worth more, I want more' […]

Flavio: But even dividing into equal parts he earns more than playing alone

Teacher: Yes, sure…this is an observation, but he could say 'if I am worth more, I want more'…

The argumentation formulated together by Flavio and Roberto is based on the preserving of the order of the profit the musicians earn playing alone. I refer to this argumentation with the label 'monotonicity': if X earns more than Y when they play alone, then X has to earn more than Y when they play together and they share a common profit. When Flavio comes back to the greater profit as a criterion to accept a proposal, the teacher moves the focus on monotonicity to stimulate the students to take into consideration different points of view and argumentation. Roberto, as other students, on the base of monotonicity, proposes that the musicians play in trio and share the profit according the proportion of the profit they would earn playing alone (that are 100, 150 and 180 respectively for Ada, Bea and Ciro). Considering that $100+150+180=430$, Ada earns $100/430 \cdot 600 \simeq 139.53$, Bea earns $150/430 \cdot 600 \simeq 209.3$ and Ciro earns $180/430 \cdot 600 \simeq 251.17$. This proposal satisfies both the criterion of greater profit and the criterion of monotonicity. Now, the teacher focuses on the perspective of Ciro and then of all musicians:

Teacher: Is now Ciro happier? […] are all happier?

The students think the musicians are now happier if they are not friends. So the teacher asks a comparison between the two proposals (equal profit and proportional division):

Teacher: So we have two proposals… is there one better than the other?

Flavio: It depends on which way you look at it.

With the last two interventions, the teacher focuses on different perspectives in different levels: the three perspectives of the musicians and the perspective of the students about the proposals. Flavio answers that different perspectives are possible. Unfortunately, the teacher does not ask Flavio to explain his thinking in details.

Phase 2: Game of points of view

I cannot present here all the solutions emerged in classroom and I just report that the students proposed other ways to divide the profit, justified through the greater profit or the monotonicity criterion. During the discussion, when a student proposed that Ada and Bea play together, the teacher launches the game of standing in the shoes of the musicians in order to force the students to look at the situation from different perspectives and to promote the emerging of new argumentation and criteria:

Teacher: Ada and Bea are forming an agreement...[..]…is there anyone of you that want to do Ciro? [..] Vincenzo, you are Ciro, what do you do?

Vincenzo: Because I would earn more […] I propose Bea to come with me so that she earns more […]

The change of perspective regards not only to consider directly the points of view of the musicians, but also to move from accepting or not a proposal to consider the possibility of proposing a new agreement acceptable from someone because more convenient. For example, if Ada and Bea agree to play in duet and to share 400 euros in two equal parts, Ciro can propose Bea to play with him, offering 225 euros to Bea and 195 euros for him (the couple Ciro-Bea earn 420 euros and 420=225+195). This proposal is more convenient for both Ciro and Bea. Now, Ada can propose a new agreement with Bea, and so on.

The teacher's intervention in the discussion are like the following: "*okay* […] *and now I do Ciro*"; "*so let's try to do other proposals…* [to a student]*, do another proposal to convince the two of them to play with you*"; "*and then, what do you* [to a student] *do?*"*; "you* [to a student] *and Ada would earn…*"

The students act the part of the different musicians, as we can see from the language they use during the discussion: "*they* [Ada and Bea] *say no, we are not interested*"; "*so, I am Ciro and say…*"; "*to me* [she is acting the part of Bea] *is convenient…*"; "*I* [she acts as Ada] *propose her to play with me*"; "*to me* [she acts as Bea] *it is convenient to propose to play together because it is convenient for both*", and so on.

The focus on the musicians' perspectives, evident in the students' expressions ("I", "me", "you", etc. instead of "Ada", "Bea" and "Ciro"), moves the attention on what every musician thinks and, in particular, can do. Like in board games, every musician can make a move, that, in this case, is a proposal of an agreement more convenient for himself/herself and at least for another musician. The argumentations are now more refined than argumentation of greater profit, when the better profit was related only to the profit the musicians earn playing alone. During the discussion, the students play a show in which they put on the shoes of the musicians, negotiating an agreement and realizing that if a proposal is not accepted – then the negotiation is not stable - if one musician can propose a new agreement more convenient for himself/herself and for another musician. I call this criterion the 'stability of negotiation'.

Phase 3: construction of a model

The teacher shifts the attention to the identification of a criterion to validate the proposals. She is, now, the "voice" of mathematics, in particular of game theory, and she requires identification of mathematical relations to set up a model.

Teacher: Now listen to me… my question is… are there any reasonable conditions for which these proposals can be accepted?

After a discussion, in which a great tension between the criteria (greater profit, monotonicity, stability) appeared, the students selected the stability of negotiation as

their criterion of validity, with the awareness that other choices are possible. This leads to the construction of the model that I shortly summarize: Ada, Bea and Ciro can decide to play in trio if x(A), x(B), x(C) are respectively their final profits and:

1) $x(A) \geq 100$; $x(B) \geq 150$; $x(C) \geq 180$
2) $x(A)+x(B) \geq 400$; $x(A)+x(C) \geq 300$; $x(B)+x(C) \geq 420$
3) $x(A)+x(B)+x(C)=600$

When these relations are satisfied, every musician neither has a greater profit playing alone (because of the relations in 1) nor can make a proposal to play in duet with another one that is convenient for both (because the relations in 2). Therefore, the negotiation is 'stable'. Of course, different solutions are possible. For example, x(A)=170, x(B)=240, x(C)=190 or x(A)=120, x(B)=290, x(C)=190. The students realize that the previous proposals do not satisfy this criterion. For example, in the proportional division of the profit (139.53 to Ada, 209.3 to Bea and 251.17 to Ciro), $x(A)+x(B) =348.83< 400$, then for Ada and Bea is more convenient to play in duet. In fact, they would earn 400 euros and if they divide, for example, in 170 for Ada and 230 for Bea, both of them would earn more than what they get from proportional division.

DISCUSSION AND CONCLUSIONS

The cooperative game theory problems, as the game of the musicians, require the managing of the different points of view of the players. The problem has not one solution, rather the meaning itself of "a solution" has to be explored: then the task also requires to manage different points of view of the solvers. This has forced the students to suggest many possibilities supported with different argumentations.

The social interaction of students, orchestrated by the teacher, has been fundamental for the emerging of the different voices and solutions, to promote the construction of argumentation, and the listening and the comprehension of others argumentation and perspectives. During the discussion, the argumentations become criteria for accepting the proposals and the meaning of "a solution" is negotiated in the classroom. The students have negotiated different points of view on two levels: they have to negotiate a solution to a problem posed by the teacher about a negotiation of three characters (the three musicians). In other words, the discussion in classroom, with voices of the students and of the teacher, has as object the reflection about a game involving the voices of the players (the musicians).

The role of the teacher has been to orchestrate the different perspectives:

- stimulating the students to express, to explain and to argue their own point of view; to stand in the musicians' shoes; to listen, to take into consideration and to assume other's point of view; to shift from one to another perspective;
- promoting awareness of how the different proposals are justified on the base of different argumentation and criteria;

- uttering a voice that represents the mathematical culture, in particular regarding the knowledge of game theory, the logical aspects of argumentation and the epistemological aspects related to modelling (identification of criteria and translation in mathematical relations).

The results of this study are coherent with the entry "Mathematical Games in Learning and Teaching" in the Encyclopedia of Mathematics Education, where we read:

> […] mathematical games help the teaching and learning of mathematics through the advantage of providing meaningful situations to students and by increasing learning (independent and at different levels) through rich interaction between players. There are positive results, suggesting that the appropriate mathematics games might improve mathematics achievement. (Mousoulides & Sriraman 2014, p. 384)

Concluding, the cooperative game theory and the mathematical discussion can promote a rich and intriguing environment to develop a dialectic between different perspectives and different argumentation, until the construction of a model, confirming the hypothesis I have stated in the introduction. Moreover, the notion of mathematical discussion is particularly suitable to design and to analyze activities in game theory where different strategies are possible and the meaning of "solution" and the criteria of validation have to be negotiated.

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TEACHING FOR STRUCTURE AND GENERALITY: ASSESSING CHANGES IN TEACHERS MEDIATING PRIMARY MATHEMATICS

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From a sociocultural perspective that a teacher's use of mediational means is central to student learning, this paper presents an analysis of six teachers and their mediating, across a two-three year time gap. Drawing on the Mediating Primary Mathematics framework – developed to examine the type and quality of mediational means – we propose two composite assessments of quality of mediation – extent and depth –that indicate the extent to which teaching addresses mathematical structure and generality. The findings reveal a range of differences in these two assessments for each of the six teachers, but that all six teachers were more coherent in their use of mediational means in the later lesson than in the earlier one. These findings have implications for other schooling systems and researchers seeking to improve the quality of mathematics instruction.

INTRODUCTION

The multiplicity of factors that shape learning make it difficult to track forward in any direct way from teaching actions to learning outcomes. Yet studying changes in teaching that might better support mathematical learning remains, in many ways, a holy grail of mathematics education research, especially as research overviews continue to point to the quality of teaching as among the most influential factors impacting on learning outcomes (Coe Aloisi, Higgins & Major, 2014). Working, as we do, in disadvantaged contexts in a developing country, classrooms often provide the only sites of access to mathematical learning, and so understanding and improving the quality of teaching is important. With models of effective teaching from the 'global north' often assuming a baseline of quality in teaching yet to be established in less advantaged contexts, these models fail to capture nuanced features that mark significant changes in pedagogy. This paper presents findings from a context-sensitive framework for assessing quality of teaching – the Mediating Primary Mathematics (MPM) framework – that allows us to identify and track changes in teaching. The framework may thus be helpful to other researchers working in similar contexts of poor performance and imperatives to improve the quality of mathematics teaching.

BACKGROUND

Students' dependence on counting strategies not only well into the upper primary years but also into secondary schools has been widely written about in South Africa (Fleisch,

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2008). This continued dependence is fostered, on the one hand, by pedagogic practices in which numerical tasks are recurrently treated from scratch and counting is taken, by teachers and students alike, as the primary means of working out answers, and, on the other hand, by a marked absence of didactical practices oriented towards building a base of known results from which other results can be derived. In our work, we are keen to change this dependence on counting and to encourage teachers to begin to understand mathematics as a scientific discipline (Vygotsky, 1978) and adopt instructional practices that focus on mathematical structure and generality.

In addressing teaching for structure, we do not expect teachers to meet Freudenthal's (1983) high bar requirement of structure being the total network of basic and derived properties and actions that can be associated with an initial relationship like a + b = c. Instead, we take a more pedagogical position that learners' awareness of mathematical structure is as a network of basic and derived general relationships that expands over time and follow Brown's (2011) position that concepts are structures that emerge through noticing similarities and differences across tasks together with reasons that might underlie these relationships. Such awareness of structure is central to learners' engagement with mathematical generality, which we take to underpin the power of mathematics. We, thus, see teaching towards (awareness of) structure and generality as key in teaching in South Africa moving from the limitations noted above, but given the wide gap between this vision and current practices there was a need for a tool that can provide nuanced assessments of pedagogy that nudge in that direction.

To this end we developed the Mediating Primary Mathematics (MPM) framework for analyzing the quality of instruction in mathematics, the detailed background to which is written about elsewhere (Venkat & Askew, 2018). Here we present findings from the application of the framework and analyse earlier and later teaching of six teachers from six primary schools. This analysis allows us to characterize differences in the nature of mathematics teaching across a variety of topics and suggests that the differences can be interpreted as improvements in instruction in the direction of more attention to structure and generality. Key questions addressed are:

> Using the MPM framework, what differences, over time, in attention to mathematical structure and generality can be seen in these teachers' practices?
> Can these differences be interpreted as showing improvements in teaching?

THEORETICAL BACKGROUND: TEACHING AS MEDIATION

Prior writing details the theoretical and empirical concerns underpinning our work (Venkat & Askew, 2018); we therefore only overview that detail here. Our work is based in a sociocultural view of instruction as mediational and directed towards learning which focuses on mathematical structure and generality. Building on the classic subject–mediational means–object triad, we find it helpful to consider four, interconnected, strands of teacher mediation:

> tasks and associated examples;
> artefacts (physical equipment and manipulatives);

inscriptions (images, diagrams, written words and symbols produced within a lesson);

teacher talk/gesture (divided into three sub-strands):

- methods offered or accepted for generating/validating answers,
- mathematical connections made or accepted, and
- student-level connections offered and developed.

Tasks, and the examples accompanying these, form the base for mediation, which is enacted through coordinated use of artefacts, inscriptions and talk/gesture. These mediating means provided theoretical lenses for analysing lessons and led to the development of the MPM framework. Working with recordings of a number of baseline lessons in 2011/12 we parsed each lesson into a sequence of episodes and then analyzed the bundle of mediating means used within each episode, with particular attention to whether or not a teacher's mediation was likely to move students from a focus on simply succeeding at the task at hand, to beginning to develop a sense of mathematical structure and generality. Empirical contrasts from episodes observed were used, in conversation with the theoretical base, to construct levels of mediation within each of the strands. Iteratively moving between the framework and applying it to lesson episodes, led to a fine-tuning of the framework into its current form (Figure 1). With regard to instructional goals, Askew (2015, 2019) has examined ways in which teachers have used all four mediating means in ways that may or may not support moves to structure and generality. Here the framework is used to examine two aspects of the **quality** of mediation; extent and depth.

DATA SOURCES

Working with ten previously disadvantaged primary schools, we gathered baseline videos of mathematics lessons in 2011/12 with Grade 2 and 3 teachers. Returning to these schools to video lessons in 2014 there were six teachers from six different schools for whom we had both the earlier baseline and later lesson video-data: these six teachers form the sample reported on here. Of these six teachers, four taught in 'suburban' schools and two in 'township' schools. All of the teachers had taught in primary schools for more than five years, some for much longer. While all the schools served historically disadvantaged student populations, the township schools had larger school rolls and classes, and smaller classrooms than their suburban counterparts. Between 2011 and 2014 the project team worked on professional development in these six schools, more broadly focusing on supporting teachers in attending to the structures and generalities underpinning 'number sense', hence our interest in whether we could determine changes in teaching practices across the years.

METHODOLOGY FOR ANALYZING THE QUALITY OF INSTRUCTION

Analyzing the quality of instruction using the MPM framework required an approach that enabled us to look both within and across a lesson's episodes, to build a picture of

MEDIATING TASKS.EXAMPLES (Listed)			
MEDIATING ARTEFACTS			
0	1	2	3
No artefacts or artefacts that are problematic/ inappropriate	Unstructured artefacts used in unstructured ways (Bags of counters/tally marks)	Structured artefacts used in unstructured ways (Abaci, 100 squares, etc, used with unit counting, and without reference to structural properties)	Structured artefacts used in structured ways / unstructured artefacts used in structured ways (Abacus, 100 square/place value blocks/cards, number lines, etc used with reference to structure/relations)
MEDIATING INSCRIPTIONS			
0	1	2	3
No inscriptions or inscriptions that are problematic/ incorrect	Inscriptions that only record tasks or responses	Unstructured inscriptions (e.g. tally marks)	Structured inscriptions (e.g. tables of ordered bonds; structured/empty number lines; inscriptions underpinned by relations)
MEDIATING TALK & GESTURE			
Method for generating/ validating solutions			
0	1	2	3
No method or problematic generation/validation (e.g. mixing of knowns and unknowns)	Singular method/validation (provides a method that generates the immediate answer; enables production of answers in the immediate example space)	Localized method/validation (provides a method that can generate answers beyond the particular example space)	Generalized method or validation (provides a strategy/method that can be generalized to both other example spaces AND without restriction to a particular artefact / inscription)
Building mathematical connections			
0	1	2	3
Disconnected and/or incoherent treatment of examples OR oral recitation with no additional teacher talk	Every example treated from scratch	Teacher talk connects between examples or artefacts/inscriptions or episodes	Teacher talk makes vertical and horizontal (or multiple) connections between examples/ artefacts/ inscriptions/ episodes
Building learning connections: explanations and evaluations - of errors/ for efficiency/ with rationales			
0	1	2	3
Pull-back to naïve methods OR No evaluation of offers (correct or incorrect)	Accepts/evaluates offers Accepts strategies or offers a strategy OR Notes or questions incorrect offer	Advances or verifies offers. Builds on, acknowledges or offers a more sophisticated strategy OR addresses errors/misconceptions through some elaboration, e.g. 'Can it be ----?' 'Would – this be correct, or this?' Non-example offers	Advances **and** explains offers. Explains strategic choices for efficiency moves OR provides rationales in response to student offers related to common misconceptions OR provides rationale in anticipation of a common misconception

Figure 1: The MPM framework (Venkat & Askew, 2018, p.90)

instructional quality. To this end we overlaid onto the MPM framework the constructs of 'extent' and 'depth' of mediation for structure and generality. The 'extent' indicator provides an assessment of the extent of mediation for structure and generality *across* the lesson episodes, while the 'depth' indicator provides insight into the ways in which mediational means were deployed coherently and responsively *within* episodes, in relation both to the mathematical object of learning and to student offers.

The research team initially parsed lessons into episodes on the basis of tasks set and their related example spaces. Within episodes, we noted whether or not students made incorrect, and/or overtly inefficient, offers, as the absence of incorrect/inefficient offers suggested such episodes functioned largely as rehearsal or revision of prior learning. Low-level mediation in such episodes (no explicit instruction or simply recording or acknowledging results) was, thus, interpreted in relation to this inference of rehearsal/revision, and such episodes were not included in the lesson coding. Similarly, smoothly run chorused counting episodes and individual seatwork episodes that did not involve whole-class or small-group instruction were also omitted from the analysis. In contrast, evidence of incorrect/overtly inefficient answers indicated a need for responsive instruction that built connections with learning: to explain or remediate in some way, so these episodes were the focus of the quantitative coding reported on here.

Teachers' work in each episode was given a 0-3 score for each of the mediating strands, creating a mediation 'map' of the lesson. Initial coding was done in groups: individual team members coded teachers' earlier and later lesson videos, convened to discuss and agree on framework level summaries capturing the essence of mediation. Agreed MPM 'maps' of the earlier and later lessons were thus created for each teacher. For each of these maps, extent and depth judgments were made as two distinct but complementary ways of assessing mediating for structure and generality. Quantitatively, each cell of the MPM map could attain a maximum score of 3: multiplying this by the number of coded episodes and the teacher's actual scores across the episodes, produced an assessment of the extent of mediation across the strands of artefacts, inscriptions and the three aspects of talk/gesture. For example, if a lesson had four coded instructional episodes, there was a maximum score of 12 available for, say, use of artefacts. The summed coding score (S) for use of artefacts in relation to this total provided an S/12 fraction for artefact-based mediation. Given that the talk/gesture strands focused on three different dimensions of mediation, we produced proportional summaries for each of these strands in each lesson. However, artefact and inscription scores were averaged together to produce a single indication of the extent of use of the more structured, relatively 'permanent' mediational forms in each lesson. Comparisons of proportions for earlier and later lessons within each strand provides a lens for considering differences in the extent of mediation for structure and generality across teachers across years. Given the exploratory nature of this analysis, we set a relatively high bar (20% point difference) for considering difference in extent of mediation to be substantive. Depth arises from a 'vertical' reading of the coordination and coherence of mediation *within* episodes. For example, in a lesson with, say, six coded episodes, we looked

vertically for the number of episodes containing higher codes (2 or 3) across two or more of the artefact/inscriptions and talk strands, producing another proportional score.

This coding for both extent and depth allowed us to see differences between earlier and later teaching in terms of the teacher's mediation for structure and generality. Table 2 shows the 2012 and 2014 scores for one of our teachers Ms M. In the findings and analyses that follows, our focus is on patterns of shift towards greater attention to mathematical structure and generality that help us to understand the kinds of mediation underlying the differences in early number learning that we noted at the outset.

EXTENT	2012 (6 episodes)	2014 (5 episodes)
Artifacts & Inscriptions	4/18 & 5/18 (25%)	0/15 & 9/15 (30%)
Talk/gesture: generating solutions	3/18 (17%)	11/15 (73%)
Talk/gesture: mathematical connections	8/18 (44%)	13/15 (87%)
Talk/gesture: learning connections	5/18 (28%)	13/15 (87%)
DEPTH	2012 (6 episodes)	2014 (5 episodes)
Higher level mediation in two or more strands	1/6 episodes (17%)	5/5 episodes (100%)

Table 1: Scores of extent and depth in Ms M's lessons

FINDINGS AND ANALYSIS

Figure 2 presents the differences in the extent scores for each of the six teachers across the four strands of mediation. As can be seen, for two of the teachers, Ms M and Ms S, there was a substantially greater extent of mediation in their later lesson compared to their earlier lesson, with Ms M demonstrating the greatest difference. For Ms Mp there is some change from the early lesson to the later, but it is not substantial as for Ms M and Ms S. For Ms R and Ms D, the profile of extent is similar across both years, whilst Ms B's extent of mediation in 2014 looks somewhat weaker than in her 2011 lesson.

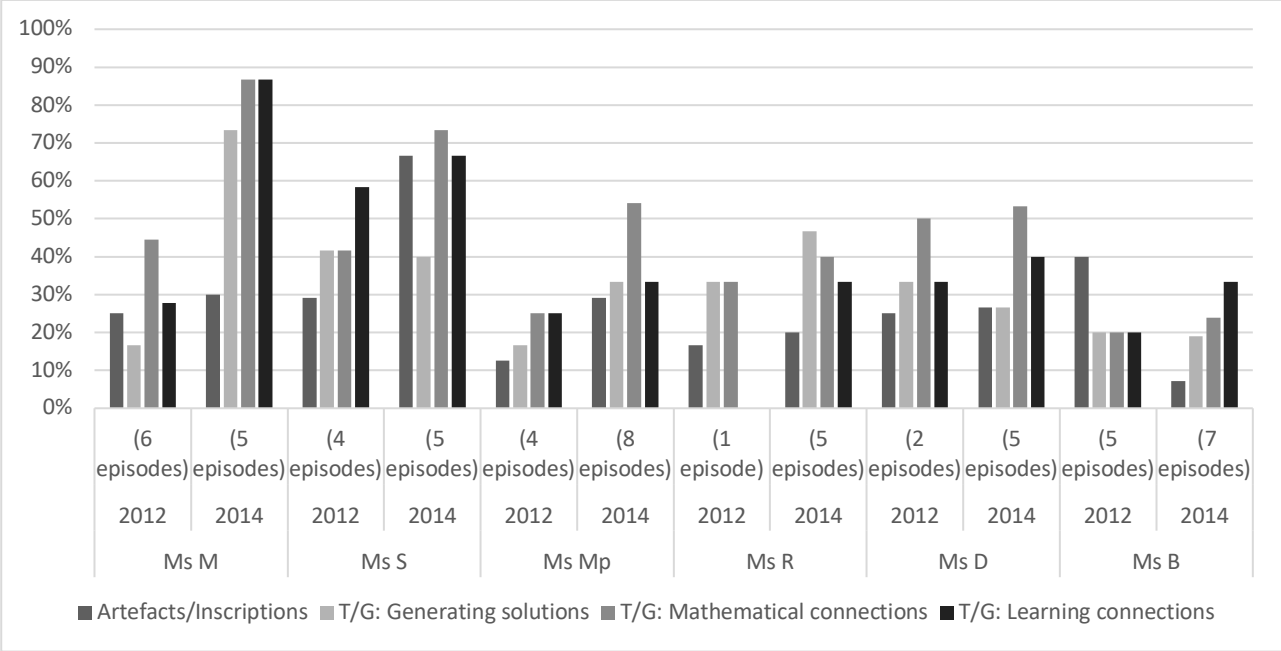

Figure 2: Differences in extent of mediation across teachers and years.

Using the 20% point difference, Ms M's teaching indicated differences meeting this criterion in 3 of the 4 categories (that is, across all of the talk/gesture strands). Ms S met this criterion in 2: arefacts/inscriptions and talk/gesture for mathematical connections with Ms Mp also meeting this criterion on talk/gesture for mathematical connections. The three other teachers did not cross this threshold on any of the strands.

Table 2 presents the assessment of depth for the teachers in each of the years. This shows that for all of the teachers, within the 2014 lesson, there were higher levels of mediational means being coordinated within the teaching episodes. All episodes in Ms M's 2014 lesson showed a coordinated depth of higher-level mediation for structure in two or more strands across all 5 episodes, with Ms S and Ms Mp achieving this in at least 50% of their episodes in 2014. The other teachers demonstrated less difference but given the lack of any coordination of mediating means in the earlier lesson, what increased scores they did achieve indicate a difference in their teaching.

	Ms M	Ms S	Ms Mp	Ms R	Ms D	Ms B
2012	17%	50%	0	0	0	0
2014	100%	60%	50%	40%	40%	29%

Table 2: Proportion of episodes with coordinated mediation at level two or three

DISCUSSION

On the empirical side, we see broad differences in terms of extent and depth of mediation for mathematical structure and generality employed across the 12 lessons within this analysis, with more extensive differences across some teachers than others. The theoretical derivation of the framework from Vygotskian notions of structured networks of scientific concepts allows the patterned direction of differences – higher scores in the later lessons – to be interpreted in terms of improved extent of mediation for structure and generality. The variation in the extent of mediation ranges from extensive change in extent across multiple strands – as in the case of Ms M – to lack of substantial change in extent of mediation within any of the strands – as in the case of Ms B. As Figure 2 shows, these six teachers essentially fell into two groups, those teachers who expanded their work with structured mediation, and those who have begun to introduce some structured mediation but not to the same extent.

A key commonality is the broad increase in depth of mediation for structure across five of the six teachers. This points to more episodes exhibiting a coordinated orchestration across mediational means. As our coding for depth involved looking for higher level mediation scores (2 or 3) that point to moves beyond simply a coherent lesson towards connection and generality, the increases in the depth indicator suggest, therefore, that connected teaching was more in view in the latter lessons than in the former ones. Of particular note is that substantial moves on this indicator were seen among teachers working from a low base of mediation for structure and generality, suggesting that a

focus on helping them to coordinate actions across mediational means may be a particularly useful direction to follow in primary mathematics teacher development.

Our sense is that the central commitment to mathematical goals of structure and generality are both important and pragmatic in a context where much recent writing has noted that the lack of attention to specialized knowledge may be part of the problem in relation to improving disciplinary instruction (Hugo & Wedekind, 2013). The explicitness of mathematical goals in the MPM framework – structure and generality – coupled with an openness on the pedagogic forms in which these goals are couched, means that the MPM framework provides a lens for exploring primary mathematics teaching in ways that are sensitive to the ground as well as being mathematically ambitious in seeking to expand the spaces of productive mathematical working.

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BUILDING 3D SHAPES FROM SIDE VIEWS AND SHADOWS – AN INTERVIEW STUDY WITH PRIMARY SCHOOL STUDENTS

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This interview study examines how 2nd and 4th graders use unit-sized cubes to build shapes from two given side views or two given shadows. Besides the reconstruction of the building processes in general, it will be investigated whether and to what extent building from shadows differs from building from side views. In addition, the study aims to empirically reconstruct difficulty factors for such tasks. First results show differences between shadows and side views with regard to the resulting shapes, but not with regard to the solution rates. The location of the highest cube tower was identified as an important difficulty factor. Seven different types of building processes could be differentiated.

THEORETICAL FRAMEWORK

The importance of visuospatial reasoning

After being neglected for a long time, visuospatial reasoning – which is also referred to for example as spatial thinking or visualisation – received a strong interest in the international mathematics education research over the last decade (see e.g. Sinclair, Bartolini Bussi, Villiers, & Jones, 2016 with many references). This is fully justified since many studies support a strong association between spatial and mathematics abilities or identify early spatial skills as predictors for mathematics achievement (e.g. Casey et al., 2015). Furthermore, spatial skills are related to academic performance in various STEM fields (e.g. Wai, Lubinski, & Benbow, 2009).

Although spatial skills are traditionally assumed to be innate and fixed, strong evidence from research suggests that spatial skills can be taught and improved (Uttal et al., 2013). Consequently, a stronger emphasis on visuospatial reasoning in mathematics teaching is demanded by numerous highly influential organizations such as NCTM and NRC (Davis et al., 2015).

How can visuospatial reasoning be described?

Thurstone (1938) described spatial ability as a primary intelligence factor. Since then, spatial skills have been the subject of detailed psychometric investigations and several models have been developed, for example by Linn and Peterson (1985). From a more didactic perspective, spatial skills were also investigated and modeled, e.g. by Maier (1999). However, there is no uniform understanding of processes or abilities in this field (Uttal et al., 2013).

According to the National Research Council (2006, p. 5), spatial reasoning "involves the location and movement of objects and ourselves, either mentally or physically, in

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49-56). Pretoria, South Africa: PME.

space." The extent to which it is possible and meaningful to separate from one another the involved processes and aspects of perception and imagination, of action and speech, is controversial. The approach of Davis et al. (2015, p. 5) seems promising: they are less concerned to find unambiguous definitions but rather aim at a "working knowledge that will support richer understandings of how children engage with / in their worlds." Instead of a somewhat reductionist approach of breaking spatial reasoning down into the smallest possible and therefore exactly describable skills, they try to look at the specific situation in its entirety. For its characterization, they use the typology developed by Uttal et al. (2013), which has proven itself in various ways and by which psychometric models and widespread spatial skill tests can be classified. This typology differentiates between *static* and *dynamic* situations as well as between *extrinsic* and *intrinsic* information to be processed. Extrinsic information refers to relations among objects or relations between objects and an overall framework or their environment, intrinsic information defines or describes an object. Building shapes from projections with unit-sized cubes could then be categorized as dynamic-extrinsic. It is dynamic because the cubes are moved and the shape is built step by step. It could be seen as extrinsic because the cubes have to be positioned in relation to one another and in correspondence to the given projections. But there are also intrinsic aspects if, for example, a student visualizes the shape as one object based on the projections. This exemplarily shows that spatial reasoning often involves several steps that move between and span categories (Davis et al., 2015).

Building 3D shapes from side views and shadows

In this study we investigate how primary students use unit-sized cubes to build shapes from two side views or two shadows. The projections are presented in upright frames, on suitably arranged cards or on a "double card" with two projections; Figure 1 shows eight examples.

Such activities are particularly appropriate to make important aspects of visuospatial reasoning accessible to research. Especially plane representations of space and the change between 2D and 3D are of enormous importance in everyday life. Previous studies have shown that building shapes from various orthogonal projections is the most difficult way of building from plane representations (compared, for example, to building from the isometric or from numeric coded views). The particularly high degree of difficulty results from the necessity to coordinate several projections (Gutiérrez, 1996). For successful processing flexibility of thinking (Hasdorf, 1976), as it is described from a psychological perspective, is therefore necessary. It is generally understood as the ability to switch from one aspect of consideration to another or to embed an issue in various contexts and to grasp the interrelatedness of facts and statements. When dealing with tasks, flexibility of thinking can take the following forms: restructuring the task or problem situation, reversing a train of thoughts, changing assumptions or criteria, grasping and applying interrelatedness, simultaneously considering several aspects. The coordination of both given projections for building a 3D shape can be seen as the latter form of flexible thinking.

In numerous teaching materials there are tasks in which objects are to be identified or produced from given side views or shadows. However, how primary students deal with these requirements has not yet been investigated. It is therefore unclear whether there are systematic differences with didactic relevance between tasks with shadows and those with side views. For example, students may have everyday experiences related to shadows and side views, which in different ways partly contradict the mapping rules of parallel projection, so that different abstraction processes have to be carried out. Further differences could arise from the fact that the projection planes are arranged once behind and once in front of the object.

Building shapes from projections can also be used to promote the development of corresponding abilities in mathematics teaching. From a didactic point of view, the different presentations of projections using upright frames (Fig. 1, tasks 1-4), suitably arranged cards (Fig. 1, tasks 5, 6) or double cards (Fig. 1, tasks 7, 8) appear particularly interesting because they could considerably vary the degree of the tasks' difficulty. Additionally, the use of upright frames and transparencies when building from side views could also support the understanding of such tasks while maintaining the leading idea of projection.

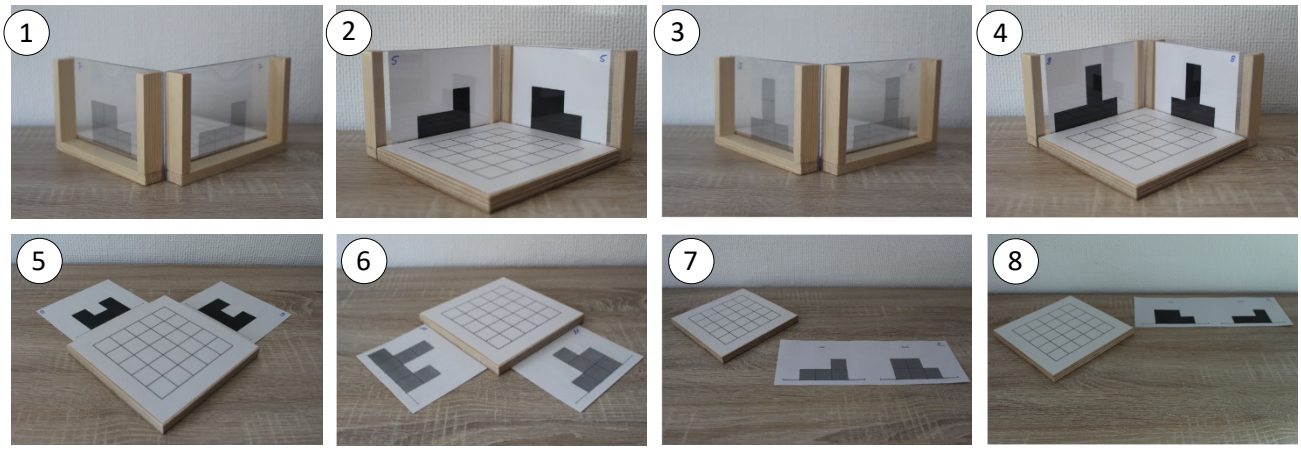


Figure 1: Tasks from the interviews

RESEARCH QUESTIONS AND METHODS

With this study we are investigating the following research questions:

1. How do primary school students build 3D shapes from given side views or shadows?
2. What are the differences in processes and products between building from side views and building from shadows?
3. What are the differences in processes and products between building from projections given in upright frames, on arranged cards or on double cards?
4. Which difficulty factors concerning projections can be reconstructed empirically?

Data is collected through clinical interviews that are videotaped for later analysis. Each participating student completes two approximately 25- to 40-minute interviews, in which at first prefabricated shapes have to be checked for whether they fit a given pair of projections and how they may be arranged. In the main interview phase, shapes made of unit-sized cubes must be built according to both projections. A quadratic 5×5-grid is provided for this purpose, its position makes it possible to work with two equivalent projections (front left and front right, respectively rear left and rear right). These are presented in upright frames, on suitably arranged cards and, finally, on double cards (cf. Fig. 1). In the first interview, either shadow projections or side views are given exclusively. The other type of presentation is used in the second interview, within its last phase the student builds alternating from shadows and side views. The tasks for shadows and side views correspond (by swapping the respective right and left projections; cf. tasks 1 and 2 from Fig. 1) in such a way that both projections could belong to the same shape.

The design of the tasks takes into account several non-independent aspects: Different pairs of projections (asymmetric, axially symmetric, and equal) should be given, the degree of difficulty of the tasks should vary and possible building processes should be particularly informative.

Up until now, the interview study involved 6 second and 14 fourth graders from two primary schools with different catchment areas concerning among others the socio-economic background. Prior to the interviews, the participating students worked on a compilation of tasks on spatial abilities (according to Thurstone) for which experience has already been gained from other studies. Two students each from the same school class and with comparable results formed a pair of test persons. In the first interview, one of the two students worked on tasks with shadows, the other on corresponding tasks with side views.

On the basis of the videotapes the building processes used by the students were reconstructed and the created shapes were described by numbered plans (cf. Fig. 2).

RESULTS

For reasons of space, we will not present the results of the interviews in detail, but only indicate a few aspects exemplarily.

Research question 4: As expected, the position of the highest (cube) tower (T) is a factor strongly influencing the task's difficulty. Thus, all tasks with the highest tower positioned inside (near the intersection line of the projection planes) were very successfully mastered by the participating students. This was shown, for example, in tasks 2 and 7 from Fig. 1, which were correctly solved by all students and 19 students respectively. On the other hand, task 1 was incorrectly processed by 9 students and also task 8 was successfully solved by only 8 students. Due to the highest tower being inside, the shape of task 2 or 7 can be built step by step from the far left to the far right or vice versa. Only the intersection column (T) has to be considered, which the students

did very easily. This approach is neither possible in task 1 nor in task 8, since the highest tower is positioned outside or in the middle with regard to the projections.

Also difficult (6-9 incorrect shapes) were tasks such as no. 6 in Fig. 1 (three different heights in both projections) and tasks on double cards (with the exception of no. 7).

Research question 2: As shown above, there were clear differences between the corresponding shadow and side view tasks (with swapped projections) when in one task T is inside and in the other T is outside. However, comparing the solution rates of tasks with the same characteristics (i.e. e.g. side view tasks and shadow tasks with T outside), there is no difference. The positions of the projection planes do not seem to have any influence in this respect. The built shapes, however, differ considerably from each other. Depending on details of the given projections (especially the position of T), many students strove to build the shape as close as possible to at least one projection plane. Thus, parts of the shapes built from side views are located further forward on the basic grid compared to the shapes in corresponding "shadow tasks". This is particularly evident when T is positioned inside. Here, most students built the shapes as close as possible to *both* projection planes. Exemplarily, Fig. 2 shows the most frequent shapes built in task 5 (Fig. 1) and the corresponding "side view task". In both cases, 12 out of 20 students built a shape according to Fig. 2.

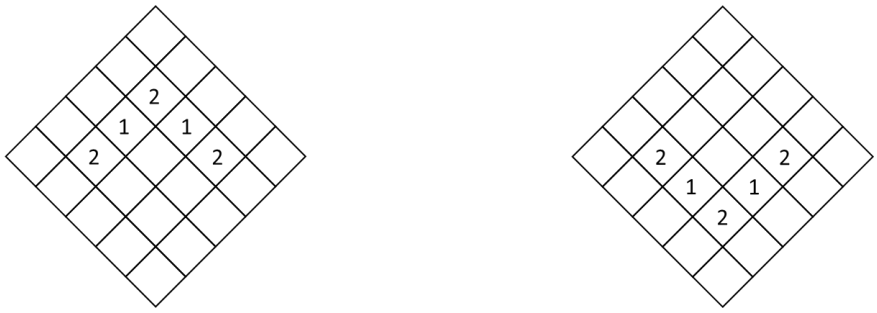


Figure 2: Shapes built from shadows in task 5 (left) and built from corresponding views (right)

Research question 1: Overall, the building processes of the students are very diverse. Nevertheless, some typical procedures can be identified. Using task 4 (Fig. 1) as an example, these are described in Fig. 3. Characteristic intermediate states are summarized in one picture each with the grey fields indicating the cubes added in this step. Usually these cubes are not added simultaneously – as the picture suggests – but individually, so that variations in the building procedure can lead to subcategories that are not shown here. So far, procedure 2 did not appear in task 4 but in other tasks and was transferred to task 4 for easier comprehension.

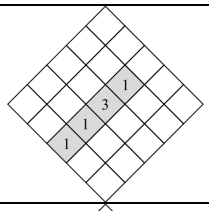
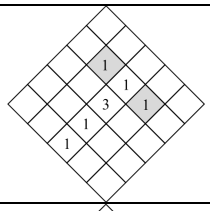
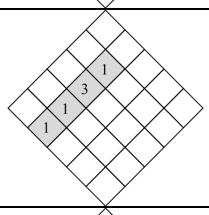
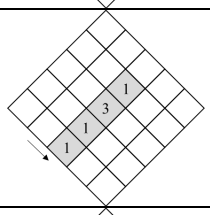
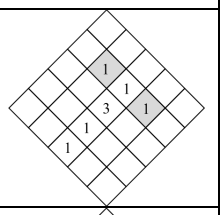
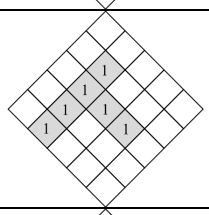
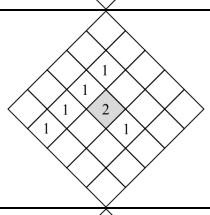
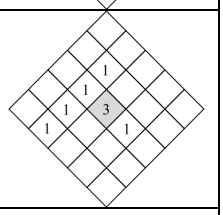
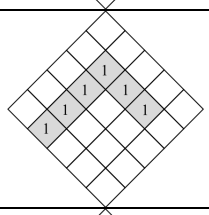
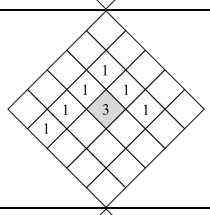
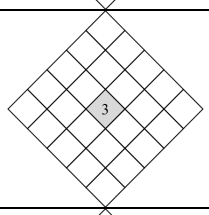
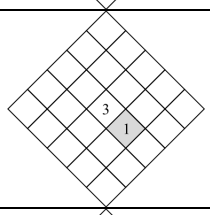
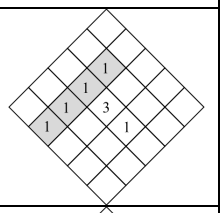
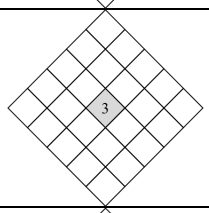
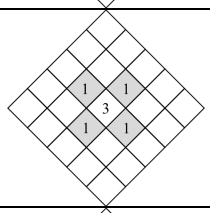
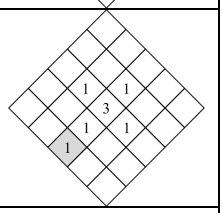
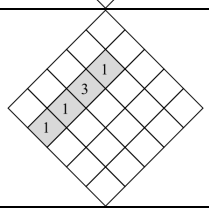
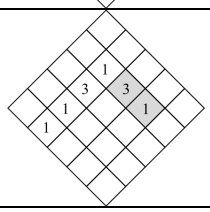
1.	Building a silhouette (a) and attaching missing cubes (b)	a) 	b) 	
2.	Building a silhouette (a), shifting (b), attaching missing cubes (c) (adapted)	a) 	b) 	c) 
3.	Layer by layer: Building the lower layer (a), building the 2nd layer (b), building the 3rd layer (c)	a) 	b) 	c) 
4.	Building the fringe near to projection planes (a), completing the tower (b)	a) 	b) 	
5.	Vertical layers one by one: Building a vertical layer (a), a second one (b), a third one (c)	a) 	b) 	c) 
6.	Concentric building around the highest tower: Building T (a), attaching the neighbour cubes (b)	a) 	b) 	c) 
7.	Building two connected silhouettes: Building the 1st silhouette (a), building the 2nd silhouette (b)	a) 	b) 	

Figure 3: Building processes in task 4

Since all tasks require the combination of two different projections, it seems obvious to start with one and then, after a change of perspective, add the missing cubes of the other silhouette (procedure 1, 2 and 7). Depending on the characteristics of the projections, this is more or less easy (see e.g. above: position of T). If the highest tower is not inside, as in the task considered here, the positioning of the cubes or silhouettes allows conclusions to be drawn about whether and when the two projections will be coordinated, i.e. whether and when flexibility of thinking emerges. While in procedure 1 both projections will be coordinated from the beginning and the first silhouette is immediately placed in the correct position, in procedures 2 and 7 the projections are

presumably considered one after the other. In procedure 2, the built silhouette is subsequently moved so that it matches the second projection. If a student uses procedure 7 – which leads to erroneous results for tasks with T not inside – it can be assumed that s/he does not coordinate the projections, but rather views them in isolation from each other. In very rare cases, a special form of procedure 7 occurs in which the two projections are reproduced completely independently of each other and both silhouettes do not contain any common cubes.

For procedures 5 and 6, it is necessary to consider both projections from the beginning. Also, for procedures 3 and 4 it can be assumed that the projections will be considered simultaneously early on.

In task 3 and 4, 15 (task 3) or 16 (task 4) procedures could be assigned to one of the above categories. Almost the half of the students (task 3: 7, task 4: 9) used procedure 1. Procedure 3 was used by 4 or 3, procedure 4 by 1 or 2 students, and procedures 5-7 once each. Thus, it is shown that the students coordinated both projections to a large extent from the beginning or at an early stage in the building process.

CONCLUSION AND OUTLOOK

Due to the small number of students, the results cannot be generalised. Nevertheless, important information can be gained about how primary school students build from two orthogonal projections. It does not seem to have any influence on the solution rate or the understanding of the task whether shadows or side views are given to build from. Based on this, both types of projections could be equally suitable for teaching. However, it should be considered that side views usually contain more information than shadows. (For shapes built with unit-sized cubes, the only difference is the number of “visible” cubes, which can be easily reconstructed.)

Differences in the positioning of the shapes indicate that objects (3D) and projections (2D) can be associated more easily if they are close to each other. This could explain the high solution rates of tasks with T inside, since in this case striving to build close to the projection supports the coordination of both given projections. The analysis of the building procedures provides indications on whether and when both projections will be coordinated, i.e. flexible thinking emerges. In further analyses, we will examine whether there are connections to the results of the preceded spatial abilities tasks. The planned expansion of the sample will allow the use of statistical methods for this purpose. Furthermore, a detailed analysis of the building procedures’ frequencies for all tasks is intended in future. Of particular interest here is whether there are differences between shadows and side views or task- or student-related accumulations of specific procedures. In addition, a larger sample enables us to investigate students’ age and sex as possible factors influencing solution rates, used procedures and flexibility of thinking.

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PRACTICALIZING PRINCIPLED KNOWLEDGE WITH TEACHERS TO DESIGN LANGUAGE-ORIENTED MATHEMATICS LESSONS: A DESIGN STUDY

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It has been argued that teachers need practical principled knowledge and that design research can help develop such knowledge. What has been underestimated, however, is how to make such know-how and know-why useful for teachers. To illustrate how principled knowledge can be "practicalized", we draw on a design study in which we developed a professional development program for primary school teachers (N = 5) who learned to design language-oriented mathematics lessons. The principled knowledge we used in the program stemmed from the literature on genre pedagogy, scaffolding, and hypothetical learning trajectories. We show how shifting to a simple template focusing on "domain text" rather than genre, and "reasoning steps" rather than genre features made the principled knowledge more practical for the teachers.

THE NEED FOR PRACTICAL PRINCIPLED KNOWLEDGE

On the one hand, knowledge generated in research is often not practical enough for designers and teachers to use productively in their educational practice. On the other hand, there is a need for principled knowledge that goes beyond local, situational, or contextual heuristics. It is for these reasons that Bereiter (2014) made a call for *practical principled knowledge* (PPK), knowledge that combines "practical know-how and scientific theory"; it offers practical guidance but also "meets standards of explanatory coherence" (p. 4). Research on what such PPK may look like is scarce (e.g., Kidron & Kali, 2017), and studies on how to develop PPK are even rarer. Bereiter suggests that design research could be extended to the creation of PPK, but this effort should not be underestimated: Janssen, Westbroek and Doyle (2015) argue that what researchers consider PPK is often not very practical in the eyes of teachers. To ensure practicality, we involved teachers in making adaptations to a professional development program. The research question addressed in the current paper is: What does "practicalizing" principled knowledge with and for teachers look like? An answer allows us to reflect on what supported this process.

The principled knowledge that we draw on comes from the literature on genre pedagogy (Hyland, 2004; Martin & Rose, 2008), scaffolding (Gibbons, 2002), and hypothetical learning trajectories (Simon, 1995). To create PPK that teachers consider practical, the methodological approach we use is that of design research. Note that both the theory of scaffolding informing our design approach and the methodological orientation of design research are inherently adaptive, allowing for continuous

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monitoring of the practicality of knowledge from the literature and responsiveness to teachers' needs.

DESIGN RESEARCH

Design research has grown out of the need to bridge the aforementioned theory–practice gap (Bereiter, 2014; Janssen et al., 2015), and to move beyond the typical emphasis of educational research on description, explanation, comparison, and evaluation (Bakker, 2018). It aims to realize and study education as it *could be* rather than as it was or currently is. The type of knowledge that design researchers are after is actionable knowledge about how something can be realized (e.g., achieving particular educational goals) or how particular problems can be solved. To do so, a design approach to mathematics education seems more appropriate than basic research (Wittmann, 1995). Design research adopts an iterative and adaptive stance by using and developing theory and using this to do real work (Cobb et al., 2003).

A useful methodological and design instrument, one often used within mathematics education, is Simon's (1995) notion of hypothetical learning trajectories (HLTs). An HLT specifies the starting point, the learning goals, learning activities, and hypotheses about how these learning activities help students achieve the desired goals; the hypotheses are based on practical experience and refined after scientific analysis. Being informed by educational research and practical experience, HLTs thus serve as an intermediary between theory and practice.

PRINCIPLED KNOWLEDGE DRAWN ON

To illustrate the process of practicalizing principled knowledge that is useful to teachers, we report on a design study that aimed to develop a teacher professional development program (PDP). The topic of the PDP was designing language-oriented mathematics lessons in primary education. The PDP was a sequel to an earlier design study that used genre pedagogy and other theoretical resources to scaffold students' mathematical language development in primary education (Smit et al., 2016). The earlier study (co-design with one experienced teacher) had delivered a set of strategies for scaffolding mathematical language, exemplary teaching materials, and theoretical insight into whole-class scaffolding and features of a genre of interpreting line graphs.

Genre pedagogy is a promising approach that explicitly addresses the language required for learning, in that it provides learners with metalinguistic knowledge about how (both spoken and written) language is structured and used to achieve particular communicative goals (e.g., describing or persuading) (Hyland, 2004). The notion of genre is typically associated with certain literary forms, for example a poem or a novel. In genre pedagogy, the concept of genre is particularly used for academic text types used throughout the curriculum. Commonly distinguished genres are report, explanation, procedure, discussion, recount and narrative (e.g., Derewianka, 1990), each with specific communicative goals. Genre pedagogy explicitly attends to how schematic structures help speakers or writers to accomplish their communicative goals within each specific genre. Furthermore, it centralizes how linguistic features (e.g.,

general academic language and subject-specific language) operate in a particular genre. As such, genre pedagogy supports learners in acquiring proficiency in school-bound genres (e.g., narratives, reports), with the ultimate aim of students' independence in these genres.

Informed by genre pedagogy, Smit et al. (2016) formulated linguistic and structure features needed for describing and interpreting line graphs. The linguistic features included, for example, the subject-specific vocabulary and phrases (e.g., the *graph rises*), and also the use of expression of gradations of steepness (as in "the graph descends *gradually*"), as well as general academic language to be employed when interpreting the graph (e.g., *the number* of people *increases*). The structure features comprised the stages of students' reasoning about graphs. For example, students are expected to identify all parts of the graph and underpin each interpretation (e.g., "his weight decreased quickly") with a description related to the course of the graph ("you can tell as the graph shows a steep fall"). Such explicit attention for linguistic and structure features of genres is assumed to help learners understand and participate in mathematical discourse.

The PDP design was informed by the theoretical idea of scaffolding—temporary adaptive support, which requires repeated diagnosis and responsiveness with the long-term goal the handover to independence (in our case independent design of language-oriented mathematics lessons).

METHODS

To develop the professional development program (PDP), we used design research and we intended to practice what we preach: being explicitly adaptive to learners' needs by constantly making predictions about the participants' learning (using HLTs), diagnosing their levels, and responding adaptively in line with the scaffolding idea. The *PDP* consisted of seven sessions (2.5 hours each), for which course materials were developed and adapted during the course of the program. The total number of hours spent by each participant, including preparing the sessions and completing assignments, was approximately 40 hours. The *participants* were five in-service primary teachers with a variety of backgrounds, years of teaching experience (a range of 10 to 25 years), and roles within the school (three mathematics specialists, one language specialist who did not have her own class, one general teacher). Four worked in regular primary schools, one in special education. Their students were of low to middle socio-economic status, attending Grades 3 to 6 (age 6–11). All five teachers entered the program voluntarily and were committed to become more knowledgeable in the enactment of language-oriented mathematics education.

Data collection consisted of participants' personal logbooks, our own HLTs, our own reflection documents written after each session, completed exercises by the participants and verbatim transcription of the interaction between the researcher-educator and participants from video recordings of each group session; and two semi-structured interviews of one of the teachers conducted by the researcher-educator. The teacher

was chosen based on the completeness of her logbooks and assignments, with the first interview between the fourth and fifth group session and the second after the final session. Audio recordings of both semi-structured interviews were made and transcribed verbatim.

Data analysis focused on the diagnoses of what teachers struggled with, which led to what we considered critical responses by the design team in the PDP, in particular the teacher educator. Diagnoses (D) were based on the reflection documents and were triangulated with the logbooks (LB) and video transcripts (VT). Incidental responses, less relevant ones such as organisational decisions, were left out of the webs (cf. Figure 1). A second researcher reviewed the filtering of responses to ensure consistency of the coding of critical vs. incidental ones; there was no disagreement. Figure 1 shows an example of a web about diagnoses and the team's responsive changes in the program. Next, the second researcher checked all diagnosis response relationships and triangulated them with the interview data.

EXAMPLES OF PRACTICALIZING PRINCIPLED KNOWLEDGE

In the first sessions, the researcher-educator explained the key ideas of genre pedagogy (principled knowledge from the literature). The notion of genre is rather broad (e.g., explanation, discussion), so she narrowed it down to what she then called pedagogical genres and showed concrete examples of such genres of interpreting line graphs, estimation, and expanded column method for subtraction. During the analysis of these genres, she drew the attention of the participants to the linguistic and structure genre features (structure features refer to the required ordering of the steps to be taken by students). While reviewing the first completed task of identifying a genre for a particular domain, the researcher-educator diagnosed that the participants were still struggling with the concept of genre. This diagnosis was corroborated when two participants contacted the researcher-educator to report that they could not grasp how to complete the homework assignment related to the estimation genre. The researcher-educator concluded that the notions of linguistic and structure features of genres were not well understood and that the term "genre" was too theoretical for the participants.

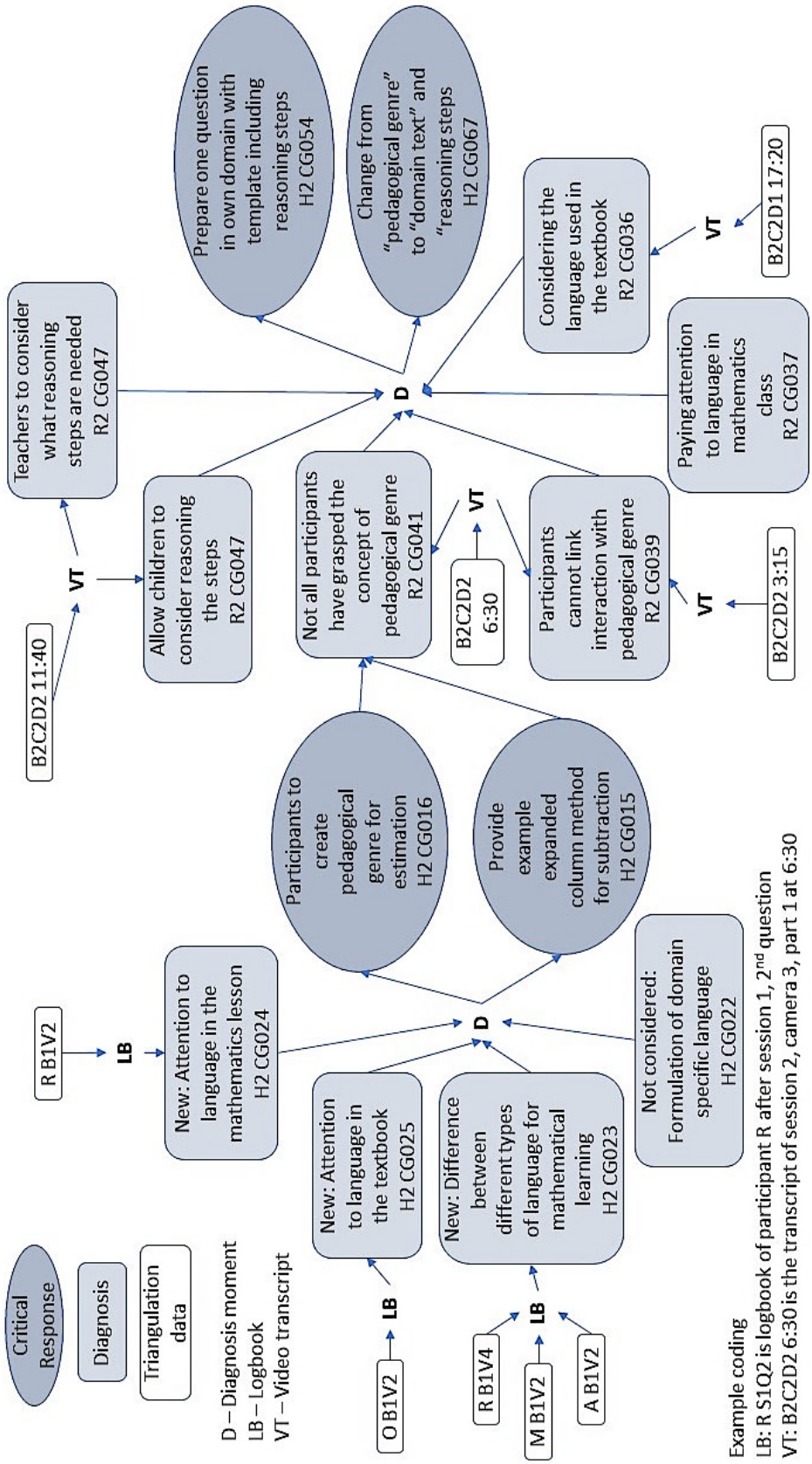

Figure 1: Example from the analysis showing the transition from pedagogical genre to "domain text."

In between two sessions, during a conversation with the language specialist (who brokered between the participants and the team), the researcher-educator explained the concept of genre in the context of language for mathematical learning as the text that includes the specific language and reasoning that is particular to that domain. It was in this conversation that the term "domain text" as a domain-specific prototypical text was first coined as an alternative to the more technical concept of genre: our first example of practicalizing.

To address the issue of structure features of genres within the context of mathematics, the researcher-educator's next response was to shift focus from identifying the structure of spoken or written mathematical text in a domain to identifying the "reasoning steps" needed to solve mathematical problems. This was regarded as crucial by both the participants and the researcher-educator, as each mathematical problem, even within one and the same domain, requires its own language to be discussed and resolved that is associated with the reasoning steps for that (type of) problem: our second example of practicalizing.

Based on participants' completed assignments, prior to the fourth session, the researcher-educator diagnosed that the participants needed support with identifying domain texts. As a response, in the fourth session the participants were set the assignment to use a domain text template that specifically included the identification of reasoning steps and the required language components for solving the mathematical problem at stake in order to prepare and enact a language-focused mathematics class. The new template included three steps:

1. Write the solution as a student should formulate it. This is the *domain text*.
2. What are the *reasoning steps* required to solve the problem?
3. What language does a student need to take these reasoning steps?

This template was also an attempt to capture the thinking behind HLTs, another notion that the participating kept struggling with, and that hence needed to be practicalized: our third example.

From teachers' homework and input during the last session there is some evidence that using the template worked well for the teachers. They made comments such as "reasoning steps stimulate thinking," "maybe we give too little attention to reasoning steps" and "normally language in the mathematics lesson is focused on the mathematical procedures, not the reasoning steps of a student." By the end of the sixth session, most of the participants showed some form of independence with respect to identifying reasoning steps during the session: "You get closer to the thinking of the children," and, with respect to language and reasoning steps, "[language and reasoning steps] support each other. You can see the thought process in the children."

In the final session, one participant gave a presentation on identifying language required for mathematical learning. During the presentation the reasoning steps were also addressed: "from A to Z, how you can get to the solution." The participant made

it clear that, in order to enable students to articulate how they are solving a problem, they must be equipped with the vocabulary and phrases required to describe their reasoning steps.

DISCUSSION

To continue the discussion between Bereiter (2014) on practical principled knowledge (PPK) and Janssen et al. (2015) on the practicality of PPK for designers and teachers, we ask in this paper what "practicalizing" principled knowledge with and for teachers may look like. We illustrate this process in a design study that aimed at a professional development program to help teachers (re)design language-oriented mathematics lessons. The principled knowledge that we used stemmed from the literature on genre pedagogy, scaffolding language, and HLTs.

Our illustrations show that technical key terms from the literature such as genre, including the structure and linguistic features of genres, proved confusing and impractical for teachers. Our scaffolding approach of repeated diagnosis and responsiveness in combination with the methodological orientation of design research ensured that we stayed in touch with the participating teachers. We collaboratively developed notions and a template that were much more practical for teachers to work with. The term "domain text" replaced the term genre, and we focused on "reasoning steps" rather than "structure features." Moreover, thinking through how students may formulate solutions of mathematics problems engaged teachers in HLT-type thinking without being intimidated by the background theories. Yet all of these developed notions were still connected to the scientific underpinning of genre pedagogy, scaffolding, and design-research thinking. No so-called "lethal mutations" (Brown & Campione, 1996, p. 291)—fatal changes contradictory to original intents—had taken place. Hence we think it is fair to speak of a process of practicalizing principled knowledge.

Admittedly, it is possible that teachers learned from struggling with the technical concepts. We do not know what the PDP had looked like if we had started with the more practical terminology from the start. Yet our study provides a proof of principle how a process of practicalizing can be elicited. We last speculate on the relevant mechanisms to allow for theoretical generalization. The methodological approach we took was design research, which is aligned in the sense that it deliberately aims to be adaptive to local circumstances, iteratively working towards what works best. In retrospect, we came to consider the teacher with whom the researcher-educator invented the idea of domain text as a broker between the other teachers and the design-research team. She was more experienced than her colleagues, which presumably helped in understanding the scientific literature enough to engage with the researcher-educator and take her colleagues' perspectives. We agree with Janssen et al. (2015) that PPK is not enough to ensure practicality; hence we argue that the need for practicalizing will always exist.

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MATHEMATICAL KNOWLEDGE, INSTRUCTION AND LEARNING: TEACHING OF PRIMARY GRADE CHILDREN

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The paper reports a small case from a study which aims to understand teacher's knowledge of mathematics, practices and mathematical instruction in the context of a developing country. The description and analysis of the case raises concerns about the opportunities that students, in such situations, get to learn mathematics and develop mathematical forms of thinking and reasoning, which has significant implications for their lives.

INTRODUCTION

The need for "good" (meaning qualified and trained) teachers is strongly felt across the globe and more so in developing countries like India. In India, many students drop out of school by the end of primary grades and for many grade 8 is the penultimate year in school. Often success in mathematics ends up being a crucial criterion for parents and students alike for deciding to continue or discontinue school education; its practical value for social mobility is un-questioned. The quality of education and teachers in the country is not uniform and one can surely link student learning outcomes to teacher's capacities to teach mathematics; such relationship is more clearly shown in Hill, Rowan and Ball (2005). In order to change the situation of access to and success in school education, there have been systemic attempts to change the curriculum for students as well as attempts to reform teacher education (including increased duration and curriculum change), in the last ten-fifteen years. The changes follow many of the changes and current thinking around the world. It is no doubt important to increase the duration of teacher preparation programmes, given the amount of knowledge that is needed to be acquired in order to teach. However, Shulman's (1986) idea of *Pedagogical Content Knowledge* (PCK), Ma's (1999) *Profound Understanding of Fundamental Mathematics* (PUFM) and Ball and her team's work on *Mathematical Knowledge for Teaching* (MKT) (Ball, Thames and Phelps, 2008) indicate the complexity of this knowledge. It is not easy to say whether this nuanced understanding about knowledge required to teach mathematics has found any place in the teacher education system (both pre and continuous professional development) programmes within India or in other places as well (e.g. Ball and Forzani, 2009; Adler, 2010).

In this paper, I analyse one teacher's knowledge, her classroom practices in the context of the mathematics that was taught and made available for learning in a primary grade. This is to serve as an illustration for the attempts being made through a larger study to capture knowledge, beliefs, practices and mathematics instruction of teachers in elementary grades (up to grade 8) and is in the preliminary stage of investigation. Given the complex scenario within which teachers are prepared to teach and conditions in

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which they teach, it has implications for imagining teacher education in the country and carrying out such studies.

THEORETICAL FRAMEWORK

In the context of countries which are struggling to meet the requirements of the society in health and education, teacher support, whenever available, is rather generic; familiarizing teachers to new expectations in terms of values and practices, exposition on general pedagogic strategies, including preparation and use of low-cost teaching learning materials, classroom management, sensitizing them to the diverse population of children and adolescents in school. While this may lead to some observable difference in schools, such support may not necessarily change teachers' teaching of mathematics or students' success in mathematics. It is not the general "feel-good" factor, but the nature and quality of mathematics instruction that perhaps matters most for students' subsequent life chances and therefore crucial for addressing issues of equity and fair chance for everyone (see also Adler, 2010 for similar comments). One part of the larger study was therefore designed to understand the knowledge of mathematics and teaching practices through mathematics instruction in the elementary grades classroom. The study builds on several such work in the past few years which have developed frameworks for understanding mathematics knowledge for teaching and tools for understanding quality of mathematics instruction in classrooms (e.g. Ball, Thames & Phelps, 2008; Learning mathematics for teaching project, 2011; Adler & Ronda, 2015; Rowland, 2013).

It has been a conscious decision in the study to understand teacher knowledge for teaching mathematics and their practice in the context of their teaching, as it is known that linking teacher knowledge outside the context of teaching with their practice is not easy (e.g. Chazan et al., 2003). Moreover, this is also critical for planning any systematic and focused intervention with the teachers, which may have some sustained impact on the teaching learning process.

On the one hand, the study aligns with the *Mathematics Knowledge for Teaching* (MKT) framework (Ball, Thames & Phelps, 2008), in its efforts to identify the presence or absence of knowledge of content (common, specialized and horizon content knowledge) and pedagogical content knowledge (knowledge of content and students, knowledge of content and teaching, knowledge of content and curriculum) in the observed lessons and during conversation with the teachers. Further, it explores teachers' understanding of the curriculum they teach, the usefulness of the textbooks or other materials they may use, the connections they see between ideas within and across grades, together with their own understanding of the content they taught during our observations and their reflection on their planning and practices inside the classrooms during interviews with them.

On the other hand, it tries to capture the nature and quality of mathematics in the instruction and learning opportunities that it provides in the lessons observed. Recognizing the importance of mathematics content in the classroom and engagement

of students in mathematical activity, a framework for the study was developed on the lines of Mathematical Discourse in Instruction (MDI) (Adler & Ronda, 2015), MQI (Learning mathematics for teaching project, 2011) and Knowledge Quartet (Rowland, 2013). The study is attempting to capture key aspects of instruction by the teacher during the classroom observations, focusing on: (a) transaction of the content, (b) teachers' awareness of common errors or misconceptions in the topic, (c) nature of questions in the classroom, (d) classroom organization, I material or resources used, (f) classroom norms and expectations from students, (g) learning opportunities to engage with mathematical thinking and reasoning and (h) equity and access issues.

Transaction of the content is the main body of instruction, where typically the teacher may move from basic terminologies, symbols, definitions to explaining a few worked out examples before moving to the exercises given in the text book. This part provides ample opportunities to observe teacher's knowledge and practices in the ways she communicates the meaning and purpose of the idea, exemplifies it, use of definitions and symbols, the choice of representation, explanations she provides for the concepts and procedures, and the way the task is put forth and utilized further. The way the transaction of the content unfolds in the classroom significantly determines (i) opportunities to learn mathematics for students (making sense of the concept, procedure or task, space for responding and justifying, space for making errors and learning from it, space for making connections between ideas) and (ii) the extent to which it provides equitable access for all students (explicitness of instructions and explanations, careful building of mathematical language and symbols, non-discriminating and inclusive practices of giving opportunity to all students to ask and respond to questions). Equity itself is intricately linked to opportunities to learn mathematics. For the purposes of the present paper, I will restrict the discussion to these aspects and not comment on the others.

METHODOLOGY

The study attempts to make detailed observation and analysis of teachers' knowledge, practices and beliefs as exemplified through their teaching in the classroom and their own reflections on various issues during interviews. As part of the study, teachers' classes are being observed and they are subsequently interviewed on a variety of issues. Tools for classroom observations and interviews with teachers have been developed. Criteria for classroom observation and interviews with teachers have been briefly discussed above. Since video/ audio recording are in general not acceptable by schools or teachers, detailed notes of classroom and often verbatim proceeds of the classrooms are maintained. After reading and re-reading it a few times, the classroom observation schedule is filled. Detailed notes of interviews/ conversations are also maintained.

There are no strict criteria for selection of schools (other than the medium of instruction which must be English). Schools are approached with the intent of working with at least 3-4 elementary grade mathematics teachers from the same school and their participation is entirely voluntary. Once agreed, each teacher is observed for 5-6 lessons (approximately 40 minutes each). The study is currently in its initial stages;

this paper reports a small piece from the ongoing work which focuses on teacher's knowledge, practices and mathematical instruction that promote or hinder mathematical learning. In the next section, I will analyse one of the seven teachers' classroom, who have been observed and interviewed till now, for the transaction of content and opportunities to learn mathematics that she provides in the classroom and her own reflections on them.

TEACHING ADDITION/ SUBTRACTION OF QUANTITITES

Ms T teaches grades 1 and 2 in a school catering to low income families, many of whom have migrated from other places in search of livelihood and better prospects. She is not trained as a teacher but as a computer professional. She has been teaching for two years now and was observed for six lessons altogether in the two grades. The administration makes efforts to develop the school and the capacities of the teachers using whatever internal and external resources are available and were quite willing to participate in this study. Ms T has also been a beneficiary of such efforts.

During the classes observed, Ms T taught addition and subtraction of measures of quantities [money (grade 1), length and capacity (grade 2)] on four occasions and reading time from the clock and shapes in the other two. So, I will focus on the former rather than the later to bring out the aspects of knowledge and practice and its affordances and limitations. After a few minutes of general discussion, energizing activities and quick review of the previous lesson or ideas necessary for the given lesson, Ms T went straight to give examples and explain the operation on the particular measure of quantities. The examples and the tasks across both the grades were similar: Add ₹32.64 and ₹29.37 (grade 1), Subtract 329 cm from 793 cm and Subtract 567 l 209 ml from 923 l 306 ml (grade 2) and so were the explanations. During independent problem solving of similar problems following her explanation, she emphasised speed with accuracy, which was rewarded by a star or appreciation.

Measures of quantities provided only a backdrop in which more practice of addition and subtraction were carried out. A couple of brief comments were made, in the entire duration, to give the students a sense of their utility in everyday life. Else the teacher wrote the question on the board and students were thereafter picked one by one by her to help solve the task bit by bit. As the class prepared to carry out the addition/ subtraction, the children decided which was the number to be written on top row and the number of "boxes" required, in the column arrangement. Figure 1 shows the work done on the blackboard for the two subtraction questions in grade 2. It shows how boxes were drawn to make space for writing the "carry-over" and the changed number post "borrowing". There was no apparent inaccuracy in the solution.

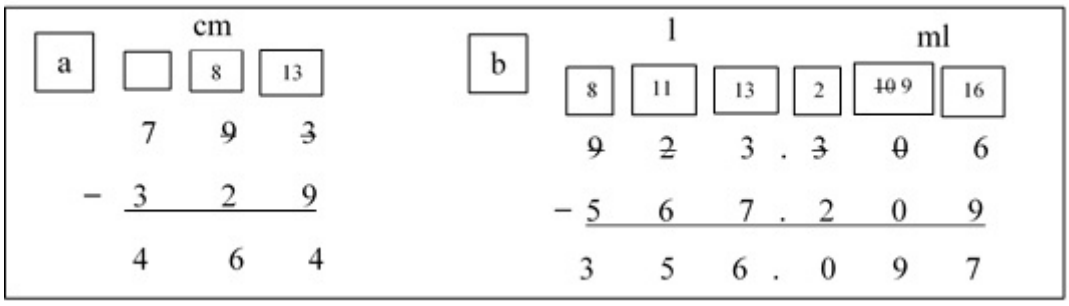

Figure 1: Blackboard work in Grade 2 for the two subtraction questions.

The conversation accompanying the blackboard work is significant, and have implications for opportunities to learn and reason about mathematics. It brings forth how the teacher makes an attempt to explain to grade 2 children the process of subtraction. This explanation might have been given earlier to students as well when they would have learnt subtraction with regrouping. In the transcript below, she explains why we need to regroup 9 tens (see Figure 1(a)) by modelling "borrowing" in the general sense (line 3) and then the children demonstrate a novel way of carrying out the subtraction on their fingers (lines 10-12). This particular technique does not leave any chances of making an error in computing single digit addition and subtraction, as there is no ambiguity related to which number to begin and end the counting, especially in the context of subtraction by counting-backward, but significantly limits opportunities to learn. It is simple and straightforward but I wonder what meaning it may have for the children.

6 T: I can subtract a big number from small number? If you have 3 chocolates, friend is asking 9, you can give?

2 S (chorus): No

3 T: You can borrow from others. [She gets two children S1 and S2 to the front of the room, having 3 and 9 chocolates respectively, representing ones and tens digits, standing in the same order from right to left]. I can borrow from S2 [who has 9]. Here only 3 is there [digit in the units place]. Can I subtract?

4 S (chorus): No. [One child is saying cut 3]

5 T: How much should you borrow?

6 S (a few): 1

7 T: 9 will be as it is? S3?

8 S3: 9 will become 8

9 T: 9 will become 8, we borrowed 1. I have 13 chocolates, can give 9.

10 S4: 13 in mind, 9 in finger.

11 T: Subtraction means *before* number.

12 S4: Before 13, …12, 11, 10, 9, 8, 7, 6, 5, 4 (folding the stretched out fingers one by one, almost sounding like a rhyme)

It is not clear here why one does not "borrow" 6 instead of 1, which will help equalize the units place of the minuend and the subtrahend, or how after having borrowed 1, the 3 becomes 13. In the next transcript, the teacher is explaining the solution for the question in Figure 1(b). An attempt is made to explain "borrowing across 0" (line 15-17), which is found to be difficult by many children. Once again children are not introduced to the important idea of regrouping or rate of composition or decomposition, crucial for developing a deeper understanding of numbers and number operations. Instead, another analogy is introduced in the form of human incapability of "climbing multiple steps in one go" (line 17). In the four classes related to addition and subtraction of quantities, there was no conversation about quantities or place value, a standard idea in such contexts. Also the decimal point is innocuously used in both grades 1 and 2, without any consideration to its meaning. Students are not formally introduced to it till they are in grade 4 in the country.

13 T: 6 in your mind, 9 in your finger. [The teacher initiates the student to compute]

14 S: Before 6, …5, 4, 3, 2, 1. So borrow [some stretched out fingers are yet to be folded, but the counting has reached 1, indicating the need to "borrow"]

15 T: I can borrow from 0, no?

16 S (chorus): No

17 T: Borrow from 3, we can climb 5 steps together?

6 S (chorus): No.

It is not the case that Ms T was completely unaware of the relationship between the digits in a number, as was revealed during the interview. She had weighed the options of possible representations (tally marks or based on ten-for-one exchange) and had decided to use the "fingers" method for grade 1 and 2 children to check errors as well as suitability for her students. She was also aware of the difficulties her students faced in relational problems like "___ is 12 less than 35", as also quickly finding sums of the numbers like, 7 and 8. She did not see these as connected to her teaching, where numbers were being repeatedly treated as concatenated digits, not just in the operations but also in the reading of numbers (seen in grade 1 where she read 32.64 as three two dot six four). She knew how to add and subtract two numbers but denied the possibility of doing it any way other than by the algorithm, including moving from the left to the right direction.

DISCUSSION AND CONCLUSION

Ms T seems to clearly lack Specialized Content Knowledge (SCK) for teaching mathematics (for example, in her inability to make connections between numeration system, numbers, why and when one will add or subtract quantities, or to evaluate mathematical suitability of an idea). She also lacked Pedagogical Content Knowledge (PCK), especially knowledge of students vis-à-vis the content (their informal knowledge, reasoning capacities, errors as a way to learn mathematics), and adequate knowledge of representations suitable for teaching multi-digit number operations. In

her case, it is difficult to say whether her trouble with teaching is also because of poor Common Content Knowledge (CCK) and identifying the boundaries between CCK and SCK. She was trying out ways which to her mind were clear enough as explanations or "catchy" for her students to remember, with complete disregard of the mathematics. Her explanations were not mathematical but were based on misplaced analogies. She used the common symbols and representations, with perhaps different meanings in the classroom. She used different combinations of numbers for her examples, all needing regrouping and including zeros, showing anticipation of student difficulties with particular combinations. Though the tasks were in themselves not rich, given her students' social and family background, they would surely have found interesting ways of computing them and contributed to their own learning. In the whole process, opportunities to develop ways of thinking about numbers or operations were severely constrained. Her instructions and explanations looked simple, however they are ambiguous, with multiple possibilities of interpretation. Some students had learnt to remember and use her rules successfully, which she appreciated, while some others may be looking for meaning and purpose and were baffled by her questions. In either case, it has serious implications for their future.

While the details of this particular teacher's knowledge and practices may be unique or different from others, literature abounds with such cases of teachers facing difficulties with teaching mathematics at all levels of schooling. The study at this point makes a contribution in the following ways: One, it is based in a context where it is not common to look at classrooms with a lens like this for developing an understanding of the continued poor performance of a large number of students, despite many interventions. It attempts to understand and analyse teachers' knowledge and practices independently and within the classroom context, focusing on important aspects of content transaction, opportunities to learn mathematics, and equitable access to the discourse for all. Two, an investigation of this kind is significant for the scores of students who study in similar schools across the country and may have little options and opportunities to engage with the subject outside the schools and success in the subject is key to their life chances. Three, the analysis indicates the need for focused intervention with teachers, and not techniques, like the use of fingers to increase accuracy of responses. This teacher and many other teachers like her could be supported in acquiring "mathematics for teaching" for the area of whole numbers by introducing them systematically to the structure of our numeration system (e.g. base 10, additive and multiplicative relationships), important aspects of learning oral and written numeration, conceptual structures required for learning numbers (single and multi-digit) and possible ways, importance of developing number sense and meaning of operations and students' informal experiences with and understanding of numbers.

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CHANGE IN POSING OPPORTUNITIES TO LEARN IN THE CONTEXT OF PROFESSIONAL DEVELOPMENT

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We apply the commognitive framework to examine changes in instructional practices of a teacher participating in the TEAMS (Teaching Exploratively for All Mathematics Students) professional development (PD) program. Specifically, we focus on the process that has often been named "lowering of cognitive demand". We conceptualize this lowering as "ritualization" of OTLs: transitioning from exploration-requiring Opportunities to Learn (OTLs) to ritual-enabling OTLs. Two lessons of one elementary school mathematics teacher who participated in the PD for two years are compared. Findings show a quantitative change in OTLs, as well as change in patterns of "ritualization"- transitions from exploration-requiring to ritual-enabling OTLs.

INTRODUCTION

Over the past several years, "powerful", cognitively demanding or "explorative" mathematics teaching has received widespread interest (Schoenfeld, 2014; Smith & Stein, 2011; Heyd-Metzuyanim, Smith, Bill, & Resnick, 2018). This type of instruction attempts to minimize the memorization of rules and procedures and to encourage the learner to struggle with cognitively demanding tasks, accompanied by discussions in which the students develop their mathematical thinking. Yet, despite the enormous amount of resources put into professional development for such cognitively demanding instruction, studies show that mathematics classrooms around the world often pose mainly ritual opportunities to learn (Nachlieli & Tabach, 2018). One of the main processes that may underlie this situation, especially in cases where the curriculum and the tasks afford explorations, is the phenomenon known as "lowering of cognitive demand" (Stein & Smith, 1998). Stein and Smith (1998; Smith & Stein, 2011) showed that this phenomenon is extremely widespread. Yet, the mechanisms underlying it are not yet sufficiently understood. In this study, we apply the commognitive framework (Sfard, 2008; Nachlieli & Tabach, 2018) to examine the discursive characteristics of the process of lowering cognitive demand, and how it can change over the course of a teacher engaging in professional development.

THEORETICAL BACKGROUND

The commognitive framework conceptualizes learning as a process by which learners move from enacting ritual routines, where procedures are imitated rigidly and performed for the sake of others, to explorative routines, where procedures are picked up flexibly for the sake of producing a certain mathematical narrative (Lavie et al., 2018). Teaching can offer students opportunities for enacting ritual routines or

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explorative routines. Nachlieli and Tabach (2018) defined the actions of the teacher that enable students to follow explorative vs. ritual routines as two distinct types of opportunities to learn (OTLs). Ritual enabling OTLs are teachers' actions that provide students with a task that can be successfully performed through rigid application of a procedure that had been previously learned. Exploration-requiring OTLs create a situation where students are required to produce mathematical narratives based on formerly established narratives and on their own authority. In exploration-requiring OTLs, there is no possibility of simply following previously learned procedures to satisfy the requirement of the task. Something new, even if very specific, needs to be conjured up by the student.

Multiple studies have shown that the practice of "ritualization", which we define as turning exploration-requiring OTLs to ritual-requiring OTLs, is wide-spread (McCloskey, 2014). Stein and Smith (1998) have termed it "lowering of cognitive demand", showing that often tasks that start out as posing multiple opportunities for explorations, end up as constrained to demanding only imitation of previously learned procedures. As a result of this observation, Smith and Stein (2011) came up with a PD program named "the five practices for orchestrating productive discussions" which is intended to help teachers avoid the lowering of cognitive demand. This program includes various teaching routines such as launching a task, assigning students to work in groups, calling them to the board to present their solutions and linking between them.

Previous studies have shown that although the "5 Practices" are often accepted with much enthusiasm by teachers, the realities of changing discursive practices in whole-classroom discussions are complex and not sufficiently understood (Heyd-Metzuyanim, et al., 2018). Their examination necessitates a discursive approach, which helps delineate the exact interactional processes that occur between the teacher and the student as this "lowering of cognitive demand" occurs. For this, the commognitive approach, which combines conceptual tools for looking at interactional and mathematical aspects of the discourse, is particularly useful.

According to Sfard (2008), routines are enacted in mathematics to produce endorsed narratives about mathematical objects or mathematical signs. Endorsed narratives are texts that are accepted as truths by the relevant community. Routines are identified by three distinct parts: initiation, procedure and closure.

Our research question is thus: how does the process of ritualization (turning from explorative-requiring to ritual-enabling OTLs), as identified in the teaching of one teacher, change in the lessons of that teacher after professional development?

RESEARCH METHODS

This study was based on a case of a teacher we shall call Simone. Simone was an experienced teacher (around 27 years of experience in teaching mathematics) and was usually teaching the higher grades of elementary school. She participated in the PD for two years, where this PD included around 60 hours of group instruction in a teachers'-district center, accompanied by 60 hours of individual work on lesson planning,

implementation and reflection. Parts of these individual-work hours were allocated for the coaching of the first author (who was also the PD instructor) with individual teachers who volunteered for the study. Simone received around 6 hours of such coaching, including lesson planning, observation of her lessons and a feedback session.

For the close analysis of the present study, Simone was chosen out of 30 teachers whom we have videotaped data on, since we got the impression, both by observations and by more macro-scale evaluation tools, that some aspects of her practice have changed. Yet despite this general impression, it was difficult to pinpoint what precisely had changed in Simone's teaching.

From the eight lessons that Simone taught and videotaped, two lessons were chosen for comparison - the first and the last. Both lessons took place with the same class, the first during December 2016 in Fifth Grade and the second during February 2018 in Sixth Grade. The tasks in both lessons were identified by us as cognitively demanding, requiring students to reason and form generalizations. The similarity between the tasks, their levels of cognitive demand, goals, visual mediators, and lengths created a good basis for comparison, which is essential for a micro-analysis.

Analysis

In order to identify exploration-requiring OTLs, the analysis was carried out in three stages. The first stage followed the method described in Nachlieli & Tabach (2018), and segmented the lesson to routines and sub-routines of OTLs. Generally an OTL opened up with a question or a prompt made by the teacher. Often, this question or prompt was rephrased by the teacher into another question, which consisted of a prompt for a sub-routine: a procedure that needs to be followed in order to produce the original routine. Routines were numbered using whole numbers (1, 2, etc.) while sub-routines and sub-sub-routines were numbered 1.1, 1.1.1 respectively.

The second stage of the analysis was intended to illuminate the nature of each of the narratives that the routines and sub-routines were intended to produce. This was imperative for determining the explorative vs. ritual status of the OTL, as will be detailed in the next section. For categorizing the narratives, we turned each of the teacher's prompts into an "expected narrative". An expected narrative is a narrative that is most likely to be received as a result of a teacher's question or prompt. For example, if the teacher asked "how many squares are there here?" and pointed to a picture with 4 squares, the expected narrative would be "there are 4 squares here". Since we do not have access to the teacher’s or students' expectations, we interpreted the expected narratives according to the context and to what would be reasonable for a student to imagine as an acceptable response. After coupling teachers' questions into "expected narratives", we turned to describe the students' replies as "received narratives". This produced two outcomes: one was a clear blue-print of the structure of the OTLs in terms of expected and received narratives. The second was a clear operationalization of ritual vs. explorative OTLs. This led to the third stage of analysis, which consisted of mapping OTLs to exploration-requiring vs. ritual-enabling. Exploration-requiring

OTLs were defined as teachers' prompts in which there were several (if not infinite) possible expected-narratives that could be derived from the prompt. Ritual-enabling OTLs were defined as prompts where the expected narrative was limited and well-defined. Exemplification of this analysis will be presented in the findings section.

FINDINGS

Our first finding concerns a quantitative comparison of the ritual and explorative sub-routines in each of the lessons (see Figure 1). This comparison shows change between the first and second lesson in the number of exploration-requiring and ritual-enabling OTLs. For the current comparison, we counted only OTLs occurring during the whole-classroom discussion. Therefore, the routine numbers (A4-A7, B4-B7) in Figure 1 start from 4. Whereas in the first lesson there were 18 exploration-requiring OTLs and 33 ritual-enabling OTLs, in the last lesson there were 20 exploration-requiring and only 10 ritual-enabling OTLs. We note that a high number of OTLs is not necessarily a good sign, since it often shows that the teacher asked many questions and did not give enough time for students to respond. Thus, a rise of the ratio of explorative to ritual OTLs (from 18:33 to 20:10), together with a lowering of the total number of OTLs (from51 to 30) shows a substantial change in the overall opportunities given to students to participate and contribute substantial mathematical narratives during the discussion.

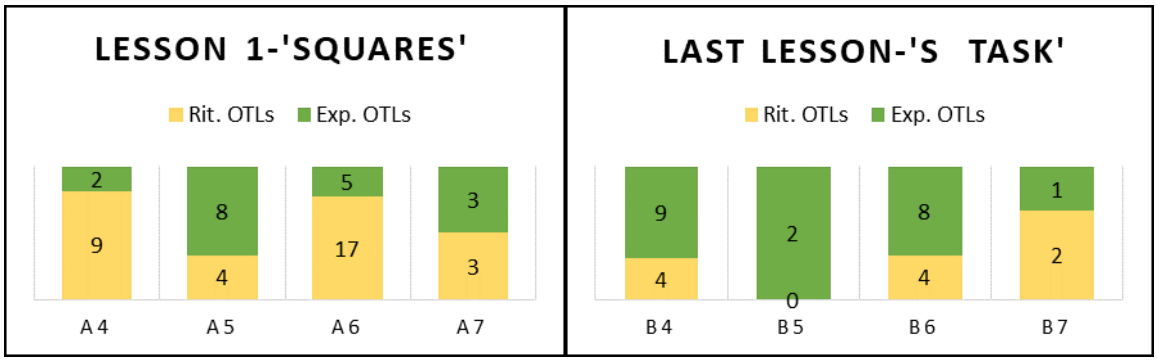


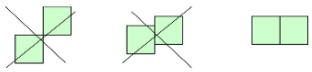
Figure 2: Exploration-Requiring and Ritual-Enabling OTLs in the two lessons

The transition from exploration-requiring to ritual-enabling OTLs

After mapping the OTLs, we examined all the situations in which the teacher opened up the routine or subroutine with an exploration-requiring OTL and followed this by switching to ritual-enabling OTLs. We found these situations to be generally characterized by the student failing to come up with one of the expected narratives that would be deemed as acceptable under the exploration-requiring OTL. This either happened when the students showed confusion with relation to the teachers' requests, or when they produced an answer that was not sufficiently clear (as perceived by the teacher) for other students to hear.

Squares and Perimeters

Build polygons from squares so that each square has at least one side shared with the neighboring square. For example:



Find all the possibilities for organizing 3 squares. What is the arrangement with the biggest perimeter? What is the arrangement with the smallest perimeter?

Find different polygons made with 4 squares … with 5 squares…. Continue to 6 squares and more.

Figure 3: the Squares and Perimeters task used in Lesson 1

For example, Table 1 shows the deterioration of an exploration-requiring OTL, where the teachers invited students to raise hypotheses regarding why they were asked to "continue on and on" with examining different square-configurations and their perimeters (see figure 2).

The expected narratives that could be produced as response to this prompt were various, including narratives about the arrangements producing the lowest and highest perimeters, which were probably what the teacher was after. The received answer, however, was rather limited. One student said "to find a rule". The teacher thus opened another explorative OTL (6.1) asking "how do we receive a shape with a large perimeter?" Although the possibilities for answering this question were more constrained, we still categorize this OTL as exploration-requiring since there are a variety of narratives that could be deemed as acceptable, all concerning the ways by which one could "receive the larger perimeter".

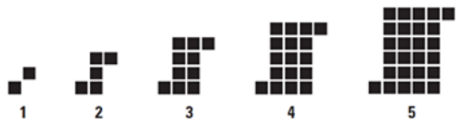
Narratives expected/ received in Lesson 1, Routine 6		
6. Expected: *a variety of narratives concerning the connection between the squares' arrangement and their perimeter*		
6. Received: (We were asked to continue on and on so that we find) a rule		
6.1 Expected: *variety of narratives regarding the relation between the shape and the largest perimeter*		
6.1 Received: inaccurate answer		
6.2 Expected: All arrangements (on the board) are built of 5 squares and their area is equal		
6.2 Received: Student remains confused		
6.2.1 Expected: All the shapes have the same area		
6.2.1 Received: The shapes are built from 5 squares		
Legend:		
Exploration-requiring	Ritual-enabling	Received

Table 1: Mapping OTLs according to Expected/Received narratives

Not having heard an acceptable answer to this question, the teacher quickly moved to an even more constrained question, this time, providing a ritual-enabling OTL (6.2): "What is common to all these arrangements I have here, that are built from 5 squares?". Now the only acceptable answers would be quite constrained, falling under the

narrative "they all have the same area". Since the students still did not produce the expected answers, the teacher continued posing this question in slightly different wordings (6.2.1); until she got the narrative that "all shapes are built from 5 squares". The above analysis was performed on all routines and sub-routines of the whole-classroom discussion. It revealed a more precise view of the ways by which the teacher changed her discursive practices around presenting OTLs from the first to the last lesson. Figure 4 shows a bird's-eye view of the change in ritual-enabling and exploration-requiring routines and sub-routines of two particular routines that we found the most amenable for comparison. The first routine is Ls1.routine 6, which was partially described above and in table 1. The second was a routine from the last lesson, which had a very similar function: Simone attempted to elicit from the students an explanation regarding how they had solved the S patterns problem (see figure 3).

The S Pattern task



Consider the pattern above: **a**. Find the next shape. **b**. Describe the general shape in the pattern.

Figure 4: The S task implemented in the last lesson

The routine occurred after one student had already presented his group's solution, which could be algebraically described as (n+1)(n-1)+2. The teacher, having monitored the students while working in groups, probably expected the group of girls she was inviting to the board to produce an alternative narrative which could be summarized as n*n+1. Yet, similarly to the first lesson, the students were unable to articulate their reasoning once invited to the board. This situation produced, in the first lesson, an immediate deterioration into a series of ritual OTLs. In the last lesson, however, as can be seen by the recurrence of exploration-requiring OTLs, the teacher went back and forth more flexibly between explorative-requiring and ritual-enabling OTLs. The ritual-enabling OTLs consisted of prompts encouraging the students to explain their solution to the 3rd structure (using the $3 \times 3 + 1$ calculation), yet immediately went back to requesting the students to explain their general solution, which provided an explorative-requiring OTL.

Importantly, the two teaching routines produced very different results, in terms of end narratives. Ls1.Routine 6 ended up in a narrative that was quite peripheral to the goal of the routine (*square cm are used to calculate area*). In fact, the whole lesson never produced a clear narrative regarding the connection between the shape of the squares and the perimeter. In contrast, Ls8.Routine 6 ended up with an acceptable mathematical narrative regarding the appropriateness of the girls' solution ($(n+1)(n-1)+2 = n \times n + 1$). Although this narrative, in its algebraic form, was beyond the reach of most of the classroom, it enabled the teacher to further explain the two forms of generalizations produced by the students.

First Lesson - 'Squares'	Last Lesson - 'S task'
6. expected: ***A variety of narratives concerning the connection between the squares' arrangement and their perimeter***	**6. expected:** ***Variety of additional options for calculating the S structure at any position***
6. received: (We were asked to continue on and on so that we find) a rule	6. received: $n \times n + 1$
6.1 expected: ***Variety of narratives regarding the relation between: an elongated rectangle and the largest perimeter; a smaller perimeter and a condensed arrangement***	**6.1 expected:** ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
6.1 received: Inaccurate answer	6.1 received: The student cannot explain
6.2 expected: All arrangements (on the board) are built of 5 squares and their area is equal	6.1.1 expected: (substitute the number n=3) $3X3 + 1$
6.2 received: Student remains confused	6.1.1 received: $3X3 + 1$
6.2.1 expected: All the shapes have the same area	**6.2 expected:** ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
6.2.1 received: The shapes are built from 5 squares	6.2 received: the student cannot explain
6.3 expected: We measure area with squares	6.2.1 expected: n=3 $3X3 + 1 = 10$
6.3 received: Sq. cm (without area)	6.2.1 received: $3X3 + 1 = 10$
6.4 expected: Sq. cm is a unit that measures area	6.2.2 expected: substitute the number n=20 in $n \times n + 1$
6.4 received: (Sq. cm is) Square cm	6.1.2 received: $20X20 + 1 = 401$
6.5 expected: the (Sq. cm) unit is used to measure area	**6.3 expected:** ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
6.5 received: We use this unit to calculate the number of squares	6.3 re: the student cannot explain
6.6 expected: To calculate area we use square cm (not to measure perimeter)	**6.3 expected:** ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
6.6 received: We use the unit (square cm) to calculate area	6.3 received: *We checked all the numbers*
	6.4 expected: ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
	6.4 received: The student cannot explain
	6.4.1 expected: In pattern 4 - the number of squares is 14
	6.4.1 received: In pattern 4 - the number of squares is 14
	6.5 expected: ***Variety of options for explaining*** $\boldsymbol{n \times n + 1}$
	6.5 received: $(n+1) \times (n-1) + 2 = n \times n + 1$

Exploration -requiring | Ritual-enabling | Received

Figure 5: "Birds eye" view of the patterns of Exploration-requiring and Ritual-Enabling OTLs in Simone's first and last lessons

We thus see a clear change in the ways by which the teacher handled situations that had originally, in her first lesson, produced deterioration into ritual-enabling OTLs. In the first lesson, the teacher's ritualization moves produced, gradually, partial narratives that became more and more peripheral to the main narrative that she wished to elicit from students. This main narrative was "the more elongated the shape of the squares, the bigger is the perimeter, while the more 'condensed' the shape is, the smaller the perimeter becomes". Instead of producing this narrative, her gradual production of OTLs that were more and more ritual-enabling, produced partial narratives such as "square cm measures area". In the second lesson, we did not see such "breaking up" of the main narrative into partial and meaningless sub-narratives. Instead, there was a back-and-forth movement between narratives about particular mathematical objects (such as shape 3 can be described by $3 \times 3 + 1$), geared towards describing a general shape.

DISCUSSION AND CONCLUSIONS

Change in teaching practices towards explorative instruction has been a notoriously difficult process to capture (Heyd-Metzuyanim et al., 2018). In the present study, we offer first steps to applying the commognitive framework, with its precise definitions of ritual and explorative routines, to capture this process. The commognitive approach has been widely used in the study of processes of mathematics learning in interaction (see review in Herbel-Eisenmann et al. 2017). Its usefulness for the study of teaching practices has only recently started to surface (Heyd-Metzuyanim et al., 2018; Nachlieli & Tabach, 2018). In the present study, we show this approach can illuminate the process often known as "lowering of cognitive demand" (Stein & Smith, 1998), as it happens in teacher-learner interactions. This has important advantages over approaches that simply show *that* such lowering occurs. Our study sheds light on *how* this process

occurs. We saw the teacher, as a reaction to children not producing expected narratives, "broke-up" the expected narratives into meaningless parts. We also saw that this ritualization process could change through a process of professional development process. In the case of Simone, the change occurred very gradually and over a long period. It remains to be seen, in future studies and PD interventions whether awareness of such discursive patterns of ritualization can help teachers make the transition towards more explorative instruction faster and more efficiently.

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LESSON STUDY IN PRIMARY PRE-SERVICE TEACHERS' EDUCATION: INFLUENCES ON BELIEFS ABOUT LESSON PLANNING AND CONDUCTION

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Lesson Study (LS) is a method of both pre-service and in-service teacher education that originates in Eastern countries but has been spread all over the world in the last decades. According to the framework of Cultural Transposition, we believe that introducing LS in cultural contexts that are so far from Eastern ones brings to light the differences between Eastern cultures and other cultures. In particular, the experience of LS can bring to light different cultural beliefs underlying the practice of teaching, and make teachers reflect on them. In this paper we analyse data collected from 5 Italian pre-service teachers after their experience of LS in Italian schools during their internship, concerning in particular their beliefs about the potential and the limits of a rigid or a flexible way of planning and conducting a math lesson.

INTRODUCTION

A Lesson Study (LS) cycle

> consists in preparation, actual class and class review sessions […] This process begins with finding and selecting materials relevant to the purpose of, and is then followed by refining the class design based on the actual needs of the students and tying all this information together into a lesson plan. The significance of LS is that all of these processes are performed in collaboration with other teachers. A classroom is then taught based on the teaching plan devised. The class is observed by many teachers, who are sometimes joined by university instructors and supervisors from the board of education, and a review session is held for all observers after the class. (Baba, 2007, p. 2)

LS is a model for mathematics teacher education and development used all over the world (Quaresma et al., 2018). It was initiated in Japan in the late 19th century (Isoda et al., 2007) and extended with adaptation in many others countries in Australasia, North and South America and Europe, up to the constitution in 2007 of the World Association for Lesson Studies (walsnet.org). Some key questions of the international reports concern the development of pre-service and in-service teachers' knowledge, with attention to teachers' learning. Less attention is paid to teachers' beliefs (for relevant exceptions see for example Inprasitha & Changsri, 2014).

According to the framework of Cultural Transposition (Mellone et al., 2018), we think that differences between the so called CHC (Confucian Heritage Culture), where LS originated, and Western culture (Li, 2012; Nisbett, 2003; see Phuong-Mai et al., 2005, for CHC definition) must be taken into account when LS is realized in Western countries; in particular, in order to conduct LS in Italian schools, we undertook a

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process of adaptation of the original structure of LS to make it more consistent with Italian culture and context (Bartolini Bussi et al., 2017). However, despite this attempt of adaptation, LS continues to be a method that is very far from Italian school culture, also because of its current scarce spread in Italian schools.

In Italy teachers rarely use very structured way of lesson planning. Italian scholastic context in fact is characterized by inclusiveness (Bartolini Bussi et al., 2017) (i.e. the same class can include children with learning difficulties, with physical and mental disabilities, children that are not Italian speakers, and so on), and a child-centered vision of teaching (i.e. teachers tend to believe that each child has his/her own specific characteristics and needs individualized support) (Li, 2012). Moreover, in Italy teachers are free to choose own teaching methods and the sequencing of the topics to be treated (Bartolini Bussi et al., 2017). Italian primary teachers usually spend many hours in the same class and teach in the same class for more than one year, and this makes them feel "less anxious about the short term effects of their teaching and encourages them to take care of and to observe long term processes" (Bartolini Bussi & Martignone, 2013, p. 3).

LS, instead, reflects a completely different way to conceive both the planning and the management of the lesson. The version of Lesson Plan we proposed to Italian teachers (for more details see Bartolini Bussi et al., 2017) consisted in a table to plan a lesson of exactly 60 minutes, requiring teachers to state explicit objectives and prerequisites of the lesson, and to describe and justify for each phase of the lesson the activity to be carried out, its educational aims, the time required for it, materials and focus of observation.

In this report we focus on the influence of this version of LS as introduced in the internship experience of Italian primary pre-service teachers. Data we present here are part of a larger study concerning primary pre-service teachers attending the University of Modena and Reggio Emilia, focused on their beliefs about mathematics teaching and the changes they undergo during their education (Funghi, 2019). This was the first university in Italy to introduce LS into the possible activities to be experienced during the internship of the last two years of the teacher education programme. For reasons of space, we focus on the discussion of one of the aspects emerged from data, i.e. *the potential and the limits of rigidity and flexibility in the way of planning and conducting mathematics lessons*.

THEORETICAL FRAMEWORK

Literature on beliefs: the case of cultural beliefs

The problem of identifying a shared definition of beliefs has been widely discussed in mathematics education, but nevertheless, there is no internationally accepted definition (Zhang & Morselli, in Goldin et al., 2016). Skott (2015) highlights that it is difficult to grasp all aspects of the 'belief' concept in a single explicit definition, and that it can be more useful to recognise the main properties of the concept itself, according to what has been discussed in the literature:

> The core of the beliefs concept may, then, be defined as subjectively true, value-laden mental constructs that are the relatively stable results of substantial prior experiences and that have significant impact on practice. (p. 6)

Abelson (1979) and Nespor (1987) have both argued that beliefs may originate from culture or personal experience. What seems to be missing in the literature is research on teacher beliefs originating from culture. We might adopt the term *cultural beliefs*, as mentioned by Bruner (1996) in his elaboration of "folk pedagogy" as "taken-for-granted practices that emerge from embedded cultural beliefs about how children learn and how teachers should teach" (p. 46).

The discussion about how to define "culture" is complex (e.g. Bishop, 1988, p. 4). We shall adopt a working definition that seems to encompass what is required for our purposes:

> [culture is] the system of shared beliefs, values, customs, behaviours, and artifacts that the members of society use to cope with their world and with one another, and that are transmitted from generation to generation through learning. (Bates & Plog, 1990, p. 7)

Within this frame, we propose to connect the definition of cultural beliefs to the "system of shared beliefs" mentioned in this definition of culture, adopting the following working definition:

> Cultural beliefs are, on the one hand, beliefs which are socially shared within a given culture and are considered as characterising this culture by its members, and, on the other hand, individual beliefs which are reconnectable and adherent – within a certain degree of re-elaboration and personalisation – to beliefs proper to the culture of the society of which the individual is a member. (Funghi, 2019)

This kind of beliefs are usually invisible to the eyes of the members of one culture, because they provide those shared and implicit meanings that allow them to think, communicate and share a common perspective on the world (Bruner, 1996). So, among the people coming from the same cultural context these meanings are taken for granted. Our claim is that when we deal with people, methods or tools coming from another cultural context, we are forced to reflect on those meanings we usually take as given, reaching a greater awareness about what we believe as members of a certain culture. Moreover, we suddenly realise that what we usually take for granted is not and that other meanings are possible. Our elaboration is consistent with the construct of Cultural Transposition (see Mellone et al., 2018).

Research questions

How did the experience of LS during the internship influence pre-service teachers' beliefs about the potential of a detailed and structured way of planning and conducting a math lesson?

Were their beliefs either consistent or contrasting the cultural beliefs shared by the Italian practicing teachers?

METHODOLOGY

For reasons of space, we focus on data collected from 5 pre-service teachers attending their 5th and last year of education, who participated to a LS cycle realized in a primary school within their internship. The participation to the interview was completely voluntary, so our sample is a convenience one. We refer to these pre-service teachers with the pseudonyms Ambra, Sofia, Rosanna, Renata and Daria. Each interviewee participated in a different LS project, conducted independently from one another and focused on different topics.

In order to study their beliefs we decided to collect narrative data, because according to Bruner (1990) narration is the means by which the human being makes sense of his own existence, organizes his own experience and his own memory – and therefore also his own beliefs (Nespor, 1987; Pajares, 1992). Moreover, it is well-known that when we deal with beliefs it is not only their content that matters, but also their organization and their mutual relationships (Green, 1971). Therefore, the collection and analysis of narrative material is particularly suitable to study phenomena intimately connected with the organization and personal processing of experience and memory, as in the case of beliefs and identity (Kaasila, 2007).

We chose semi-structured interview with open-ended questions as research instrument, because it "enables respondents to project their own ways of defining the world" (Cohen et al., 2007, p. 182). The main questions of the interview were established *a priori,* but the wording and the order of questions could vary, in order to let the interlocutors to focus on the aspects they retain more relevant (Furinghetti & Morselli, 2011) or even to focus on unexpected but interesting issues (Cohen et al., 2007). The interviews were conducted after the completion of the LS cycles by the interviewees. Many aspects of pre-service teachers' education were discussed, but for the aims of this paper we will limit our discussion to the part of the interview focused on the LS experience. It regarded mainly the following topics:

- a short description of the overall LS experience and of the planned lesson; reasons for their participation in a LS during the internship; difficulties found in lesson planning in relation to the class's specificity;
- interviewee's feelings in planning with in-service teachers; aspects that they liked or not of LS, and reasons for that;
- if they were willing to participate to other LS in the future and to use the Lesson Plan as a planning instrument for everyday lessons.

The interviews lasted between 30 and 80 minutes, depending on interviewees' will to talk; they were all audiotaped, and then fully transcribed.

Concerning the analysis of data, we refer to Lieblich et al. (1998), who describe different approaches to narrative analysis, depending on the two dichotomies *form vs content* (regarding the choice to focus the analysis on the form or the content of a narrative) and *holistic vs categorical* (concerning the unit of analysis, which can be the

whole text or single sentences extracted from the context, respectively). We adopted the content-categorical approach, since it is considered to be suitable to study a phenomenon common to a group of people (Kaasila, 2007).

RESULTS AND DISCUSSION

Our interviewees describe their first impression of the LS planning method as "mechanic" and rigid, difficult to be strictly followed in the complex setting of Italian classes. This aspect was highlighted in particular by Sofia, who claimed:

> *"A first, it [i.e. the LS method] seemed to me a little mechanical because it is given a specific timing for each activity within the lesson, and I think that in an Italian class giving a specific timing is not always easy, it depends on the class, on the situation and on the kind of activity, so I was a little doubtful about it".*

Ambra and Renata gave two different explanations for Italian teachers' difficulty in following a strict timing: Ambra highlighted the difficulty to meet the needs and the difficulties of every child within fixed times (*"If you keep on schedule […] you cannot spend other time on a new explanation of the same concept or searching for another way to make children understand it"*), whereas Renata commented that Italian teachers usually do not pay attention to the time they spend on a certain activity or topic (*"In my opinion the main difficulty that especially experienced teachers may meet […] is precisely this very detailed way of planning the lesson […] because many times teachers tend to improvise, or they are not used to monitor schedule and time spent on each part of an activity so much"*). These beliefs reflect Italian teachers' habits of planning and conducting lessons: the lesson is usually thought to be very flexible, since the teacher shapes its development depending on children's needs and reactions. This is due, in our opinion, both to inclusiveness and to the focus on long term goals of Italian school.

Another feature of LS very far from Italian teachers' habits consists in the LS implicit request to anticipate in detail the different possibilities of development of the lesson and children's reactions, as Renata noticed:

> *"What I really liked about LS is that when you plan the lesson you have to imagine all the answers children can give. [...] A lot of variables have to be taken into account. In my opinion, this is a very interesting feature of LS because usually teachers do not do that in their didactic practice".*

Concerning this issue, Rosanna emphasized also her initial feeling of discomfort related to the difficulty of knowing in advance students' reactions, since at first she could not understand its usefulness:

> *"At first, I was a bit bothered by the fact that I had to think about every small detail, to try to imagine everything children can say or do […] this idea of thinking about every 5 minutes of an hour of lesson [...] is so different from how we are used to plan lessons that I assure you that I did not understand what it was until I tried it".*

Therefore, our interviewees stated that they had beliefs that are consistent with cultural beliefs shared by Italian teachers in general, before they experienced LS.

Anyway, as we can see from the previous excerpts, LS constituted the opportunity to change their mind about the possibilities of other ways to plan and conduct a math lesson. All the interviewees realized that a certain strictness in the way of planning the lesson can be really useful, even if they gave different reasons for that. For example, Daria highlighted how the Lesson Plan makes the teacher reflect on his/her intentionality when he/she plans an activity:

> *"[I like] the idea of [expliciting] what do you do at a certain point, why you do it; for example, [when I write] 'I present materials to children' [I have to ask myself] «Why do I present it in this way? Because I know that children first touch something, then they understand that it is useful»; therefore, I think it is useful to keep on asking ourselves «Why I do it?», not just thinking «I do it because I already know that»".*

Sofia claimed that planning a precise timing for each part of the lesson avoids downtimes and fosters children's attention:

> *"Actually, I realized that the management of time is better when you plan in this way, I mean, there are no downtimes, so the time is exploited at its best. This is an important thing, also because children perceive that the lesson has been strictly planned, so that they themselves do not waste time".*

Rosanna and Renata, instead, observed that having a clear goal to reach within 60 minutes is very useful because it helps the teacher to manage better the time at disposal:

> Renata: *In my opinion, having a thread [of the lesson] is very useful, because it happens that during lessons I find myself at certain points when I do not know what to do, so I think that having a plan to follow can be really helpful.* Rosanna: [...] *having such a detailed timing is positive because within an hour, within a lesson, you reach a goal.*

Renata stated also that this way of planning avoids the teacher to get lost in the development of the lesson (*"In my opinion, this [i.e. the lack of a rigorous plan] constitutes actually one of teachers' problems, because they tend to get lost"*), and that anticipation of children's reactions and possible developments of the lesson is useful to exploit children's contributes at their best *("According to me [it is important] to give weight to children's answers [...] I noticed that, especially in the class where I am doing internship, the teacher tends to consider a lot children's responses, she stops, lets them reason, but after all... she does not use their answers to develop a reasoning [...], so I think LS can be useful to bring out [from children] some aspects related to the activity that is being done, in order to develop them [...] it made me realize that the teacher must consider pretty much everything that can come out from an activity"*).

These words testify that experiencing LS let our interviewees not only change their mind about the opportunities of using a rigid way of planning and conducting the lesson, but it made them reflect on the possibility that being flexible sometimes can be not necessarily the best way to cope with their needs. For example, both Renata and

Sofia in the excerpts above recognized that following a structured and rigid plan actually helps the teacher to exploit at best children's attention and children's contributes, which are elements of primary importance in a child-oriented view of teaching. So, LS revealed to be challenging but important in order to look critically to their unconscious beliefs concerning planning and conducting lessons that they absorbed from Italian context.

CONCLUSIONS

Our study shows the potential of LS as a means of becoming aware and challenging one's own beliefs. In the Italian case the LS could be - as was the case for our trainees - a good opportunity to review their beliefs about effective ways to design a lesson, making it possible to realize the potential of more structured design methods even within complex contexts, where the teacher needs to take on many important educational choices. In this sense, in some cases (e.g. Renata, last excerpt), they can take a distance from a shared practice observed during internship and connected to Italian teachers' cultural beliefs. The introduction of the LS in local contexts different from the original ones can be interesting both to explore the cultural beliefs typical of the original LS and to reflect on those local beliefs that seem ill-consistent with the LS: for instance, in the case of the Italian context, the belief to remain adherent to a child-centered teaching (hence to develop a flexible attitude) could be in conflict with the strict structure of LS – but as we saw this is not always the case. This potential is interesting as it applies to prospective teachers too, where training is in the foreground.

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A STUDY OF JAPANESE PRIMARY SCHOOL TEACHER PRACTICES DURING NERIAGE

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This research aims at analyzing teacher practices in a Japanese context with a focus on the neriage, a specific phase of structured problem solving lessons. We analyze teacher practices with the specific tools of the double didactical and ergonomical approach, during a neriage that takes place during a sequence of lessons on length in 3rd year of primary school. This research highlights some characteristics of practices during neriage promoted in the problem solving approach.

INTRODUCTION

Mathematics teaching in Japanese primary school has some specificities: the ordinary lessons are often in a structured problem solving lesson format (for example, Stigler & Hiebert, 1999; Takahashi, 2008). Pre-service and in-service often manage in Lesson Study, a format of teachers' professional development based on their collaborative works (for example, Miyakawa & Winsløw, 2009).

The structured problem solving lesson consists of several phases of which one is called *neriage,* a whole-class discussion in which students compare and discuss ideas, solutions or methods for solving the problem. *Neriage* means polishing up and is used by Japanese teachers and researchers in mathematics education (Shimizu, 1999). *Neriage* has a "dynamic and collaborative nature of a whole-class discussion during the lesson" (p. 110). According to Shimizu, the teacher's role is to orchestrate students' strategies and ideas, to highlight important mathematical ideas to reach the goals of the lesson, and to help students polish their solutions in order to learn mathematical content. During this phase, students struggle with the problem and should find their own way to solve it: this experience let them make links between their earlier knowledge and the new content that they are going to learn through *neriage* (Shimizu, 1999). For Japanese teachers, the *neriage* is considered as the heart of teaching mathematics through problem solving (Takahashi, 2008). Furthermore, the *neriage* is "critical for the success or failure of the lesson" (Shimizu, 1999, p. 110).

In a case study of three Swiss primary school teachers, we highlighted the difficulty for teachers to manage whole-class discussions with comparison of students' strategies, hierarchization of strategies, and to emphasize the knowledge or the method at stake in the problem (Batteau, 2018). This difficulty is well known in the French context also (for example, Charles-Pézard, Butlen & Masselot, 2012; Peltier-Barbier et al., 2004). In the Japanese context, this research proposes to focus on the teacher's practices during this specific phase, the *neriage*, because that is part of ordinary practices and considered as the heart of the lesson by Japanese teachers.

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We present some elements of the double didactical and ergonomical approach, the research question and, the methodology. Then, we present the analysis of the teacher practices during a *neriage* that takes place during a sequence of lessons on length in 3rd year of primary school (8-9 years old). The last part is a conclusion of this research.

THEORETICAL FRAMEWORK

Robert and Rogalski (2005) developed a framework based on a double viewpoint, one in the French didactic of mathematics and another in ergonomics with activity theory (Leontiev, 1975; Leplat, 1997). This framework allows to analyze practices in taking into account the complexity of teaching, both as an individual and a professional act. The term practice concerns speech, actions and, thoughts of teachers and also "all work done by that teacher, whether before, during, or after class time" (Robert & Hache, 2013, p. 25). Indeed, this framework aims at analyzing the relation between teachers' and students' activity in class, but also the constraints on teachers in the context of their profession. Thus, two closely linked elements are considered to analyze teachers' practices: students' activities and teachers' management of the class (Robert & Hache, 2013; Robert & Rogalski, 2005). Teacher practices are analyzed with two specific components of practices in the class, the organization of the tasks for the students, the *cognitive* component, and teachers' interactions with students, the *mediative* component (Robert & Hache, 2013; Robert & Rogalski, 2005).

> The cognitive component corresponds to a teacher's choices regarding content and tasks, including their organization, their quantity, their order, their inclusion within a curriculum beyond the class period, and plans for managing the class period. (Robert & Hache, 2013, p. 51)

The *mediative* component corresponds to the teacher's choices that

> may include improvisations, speech, student investment and participation, instructions, assistance to students in completing the tasks, identification of their work and the work of the teacher, validations, explanations of knowledge, etc. (Robert & Hache, 2013, p. 51)

To include the professional dimension in the practices' analysis, Robert and Hache (2013) add three other components of practices: *personal*, *social* and, *institutional*. The *personal* component describes how the teacher invests his/her leeway, what his/her representations (about mathematics, teaching of mathematics, his/her students) and, his/her mathematical knowledge are. The *social* component corresponds to the fact the teacher is not alone in his/her classroom, how he/she is enrolled in his/her school. The *institutional* component corresponds to constraints: schedule, official programs…

This article focuses on some aspects of the *cognitive* component of practices (the progress of the lesson, the choice of tasks) and on some aspects of the *mediative* component of practices: teacher interventions, validations, helps (collective, individual, with or not reduction of mathematical requirements, procedural or constructive) and, explanations of knowledge (contextualized to the task or decontextualized, new and old knowledge).

RESEARCH QUESTION

We analyze the *cognitive* and *mediative* components of practices in order to understand how a teacher manages *neriage*. The question is: what are specific tasks, interventions, validation of solutions, helps and explanation of knowledge managed by the teacher during the *neriage* in order to teach what is aimed in the activity?

Thus, the Lesson Study process is used as a favored access to analyze and to understand the Japanese teacher's practices during *neriage*. "Teaching is not a simple skill but rather a complex cultural activity that is highly determined by beliefs and habits that work partly outside the realm of consciousness" (Stigler & Hiebert, 1999, p. 67). So, we assume that Japanese teachers prepare, anticipate and, implement specific and culturally embedded practices in order to manage the *neriage* phase of structured problem solving lessons.

METHODOLOGY

Within a qualitative methodology we have been collecting data in a 3rd grade class (students of 8-9 years old) in a primary school during a sequence of 15 lessons, between 30 and 150 minutes each, on "feeling the length". The 3rd lesson is the *neriage* of personal strategies for the measure of the length of the corridor. The 8th lesson is a research lesson that takes place during a lesson study process inside the school. That means we collect the lesson plan of this research lesson and the report of the research lesson. The lesson plan includes teacher mathematics analysis about the sequence, not only for the research lesson. The corpus contents videos of 15 lessons and written data: blackboards of each lesson, lesson plan, report of the research lesson, textbook and teacher's guide. We analyze *cognitive* and *mediative* components of teacher practices from written data and from the *neriage* during the 3rd lesson, transcribed and translated.

The primary school is attached to the University of Education of Joetsu. It means that teachers are considered as experts and they do research, one of which lesson study. The teacher, Kazu, has twelve years of teaching experience.

The next part is some results of the analysis of the Kazu's practices in the double didactical and ergonomical approach.

ANALYSIS OF THE TEACHER'S PRACTICES

Cognitive component: Progress of the lessons

During the 1st lesson, Kazu presents the task: measure the length of the corridor in the school, students think about how they can realize this task. Kazu manages a whole-class discussion about their ideas. During the 2nd lesson, students measure the length of the corridor with personal strategies. This is the detailed progress of the 3rd lesson. At the beginning (1:00-6:21), each group of students give their results, the measured length of the corridor in meters and centimeters (in the table, see figure 1).

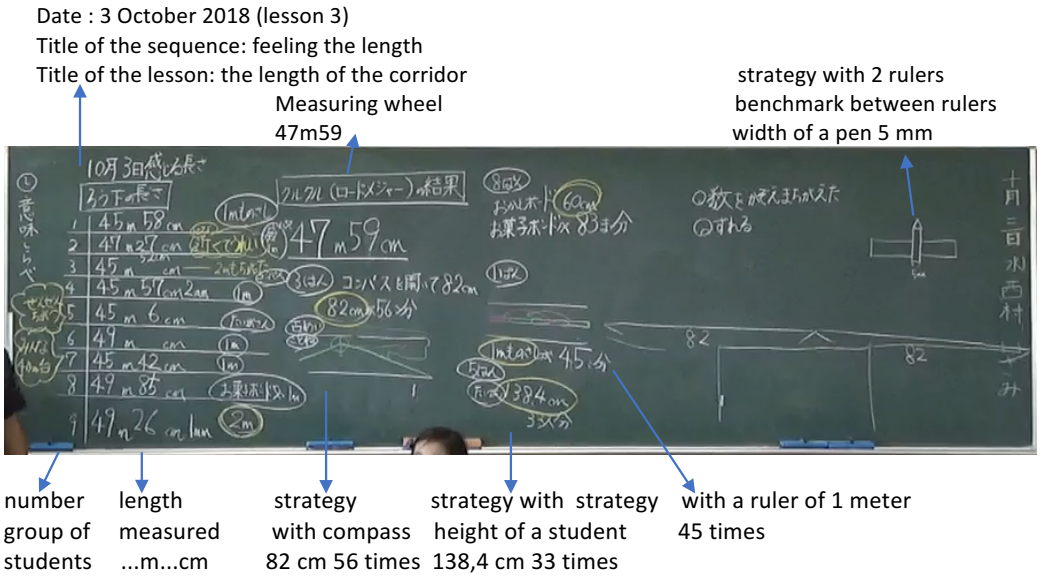

Figure 1: the blackboard at the end of the 3rd lesson

Kazu asks students to compare the different results. Then he presents a measuring wheel and its operation (8:55-11:39). One of the students measures the length of the corridor with the measuring wheel, observed by the others (11:39-19:39). Thus, they obtain the exact result of the length of the corridor, 47 m 59 cm (19:39-23:59). Then, Kazu manages the *neriage* phase (23:59-57:00).

	Kazu interventions during the *neriage*	Mathematical activity proposed by Kazu to students
23:59	What is the closest result to the exact length? How did students of the group n°2 measure the length? How did they feel?	Compare results with the exact result The group of students n°2 explains his strategy and what they feel
26:09	How did the other groups of students measure the length? Validate the students' strategy Justify measuring mistakes: why are the students' results different of the exact length?	Each group of students present and explain their strategy Students explain their measuring mistakes
49:28 - 57:00	What is the artefact used by each group of students to measure the length? What is the length of each chosen unit and the number of chosen units?	Each group gives the chosen artefact, the length of the chosen unit, the number of chosen units

Table 1: Interventions of Kazu and mathematical activity proposed by Kazu to students during the *neriage*

Kazu chooses one task, measuring the length of the corridor, during 5 lessons (almost 5 hours) even if the exact result is given at the beginning of the 3rd lesson.

***Mediative* component: Teacher's interventions**

The Kazu's interventions have the following characteristics: the importance according to various students' strategies and an "affect" dimension. During the *neriage*, students present five different strategies and it does not seem enough for Kazu. He said a lot of groups of students use the ruler of 1 meter to measure the length of the corridor. That is why during the next lesson, he asks again students to measure the same corridor with other personal strategies. At the end of the 4th lesson, students apply more than fifteen different strategies to measure the length of the corridor. This characteristic of his practices also affects the *cognitive* component (choice of tasks). Furthermore, during the *neriage*, Kazu asks several times to students what is their feeling about the task.

Another characteristic of Kazu's interventions is that he asks students to explain their strategy with details and he writes it on the blackboard for each strategy (Figure 1): the different measured lengths, the used artefact (ruler of 1 meter, compass, height of a student…), the length of the chosen unity (1 meter for the ruler, 82 cm for the compass, 138,4 cm for the height…), the number of times the chosen unity is used, some diagrams of strategies (with the compass and with the rulers).

Kazu also asks to compare the different results between us ("Can you say that everyone is on the 40 meters' range?") and with the exact result measure with the measuring wheel ("Which is the closest?" or "Was it accurate as it was?").

From the exact result, Kazu asks students to explain why they did not find the exact result. It means the reasons of their mistakes to justify the difference between their results and the exact result. Thus, he asks students to have a reflexive attitude about their own strategy.

Kazu's interventions also prepare the next phase: the *matome*, the summing up by the teacher, what is aimed in the task. There is not written mathematical expression during this *neriage*, but the written information and Kazu's interventions prepare the mathematical expression of the *matome* that takes place during the next lesson. The *matome* is: "the whole length = the length of the chosen unity × the number of chosen units". For example, the group 3 uses a compass for a blackboard to measure the length of the corridor. One of the students repeats and explains the strategy on the blackboard (Figure 1). The compass spacing is 82 cm.

1	Teacher:	I mean that this is 82 centimeters. So, how about the second time? So?
2	S1:	Again like this, again at 82 centimeters, 82 centimeters plus 82
3		centimeters, what is it? 164. So, again with such feeling, we will
4		measure more and more. […]
5	S2:	What did you calculate?
6	S1:	As Miki, you calculate as a calculator, for example, 82 plus 164, and so
7		on.

8 Teacher: If so, how many 82 centimeters? […]

9 S3: 56 times.

Kazu asks students how many times they used the compass' spacing of 82 cm. He uses the idea of multiplication as an iterative addition from the S1 speech. During the next lesson, he writes on the blackboard: 82 cm 56 times, $82_{cm} \times 56_{times} = 4592_{cm} = 45_{m}92_{cm}$.

Mediative component: Validation

Every result proposed by students (see table of the Figure 1) is different, Kazu asks students, "Is there anything correct in this?" In order to validate the different students' results, he presents a measuring wheel and its operation. Thus, one of the students measures the length of the corridor with the measuring wheel, observed by the others. The validation of the results of the task is done by students themselves, and not by Kazu. There is also a validation of students' strategy by Kazu when students explain that they measure two or three times the length of the corridor, he answers, "It will be a good result if you do it three times, do not you?"

Mediative component: Helps

Kazu does not propose helps to students during the *neriage* because every group of students already finds a result for the length of the corridor. And when a group of students explain that they encountered a difficulty, he asks all students how they can find a solution. For example, the group 3 did zigzag when they measured the corridor, so they have to follow a straight line in the middle of the corridor to measure it.

Mediative component: Explanation of knowledge

The objective of this sequence of fifteen lessons is to feel long lengths, to discover a new unit of length: the kilometer, to manipulate long lengths (addition, comparison). Students already know to convert centimeters in meters and centimeters, to manipulate lengths in meters, centimeters and, millimeters. During this lesson, Kazu converts centimeters in meters and centimeters when he reads the length on the measuring wheel, for example. The explanation of knowledge concerns some conversions already known. During this *neriage*, we do not find new knowledge: it means students using old knowledge to execute the task and to explain it. Kazu writes on the blackboard each separated mathematical elements of the previous mathematical expression: the whole length, the length of the chosen unity and, the number of chosen units. The knowledge during the *neriage* is contextualized to the task. The Kazu's interventions during the *neriage* prepare the new knowledge, summing up in the *matome* that is decontextualized to the task.

CONCLUSION

This analysis of *cognitive* and *mediative* components of practices during a *neriage* highlights some characteristics of practices promoted by Japanese problem solving approach. Kazu uses a same task during five lessons even if students already find the exact result of the length of the corridor. A characteristic of Japanese problem solving approach is that the lessons do not end even if each student find the solution of the

problem and the heart of the lesson begins after students come up with solutions (Takahashi, 2008).

In the *mediative* component of Kazu's practices, the "affect" dimension and the importance of various strategies can be explained by one of the objectives of problem solving teaching: create interest in mathematics and stimulate creative mathematical activity (Takahashi, 2006). During the *neriage*, Kazu compares students' results, asks students to explain their own strategy and, to adopt reflexive attitude about it. The explanation and the comparison of strategies are also characteristics of practices promoted by Japanese problem solving approach. In the written lesson plan, he compares different students' strategies: it is difficult to have a precise measure with a long chosen unity and it is difficult to use a small chosen unity to measure a long length. Kazu analyzed the given task, anticipated the mathematical expression of *matome* and his interventions during the *neriage* necessarily to the mathematical expression. The *neriage* is indeed the critical phase in which students use their own knowledge to explain how they executed the task whereas the teacher aims to teach the new knowledge and methods from students' strategies.

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TEACHER INTERRUPTED: HOW MATHEMATICS GRADUATE TEACHING ASSISTANTS (DON'T) LEARN ABOUT TEACHING

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Despite their impact on undergraduate learners, few studies have investigated mathematics graduate teaching assistants' (MGTAs) needs for professional development as they learn about teaching. In this study, we surveyed and interviewed MGTAs longitudinally as they progressed through their graduate programs. With surveys and interviews, we aimed to capture changes in MGTAs' views of mathematics and teaching, whether they felt that they received adequate support, and what other support they feel they needed to grow as teachers. Using two phases of thematic analysis, we found several issues that interrupted MGTAs' progress as teachers.

INTRODUCTION

Researchers have shown that mathematics teaching in post-secondary contexts has a considerable impact on learners. In particular, lecture-based teaching frequently results in lower success rates in mathematics courses and lower retention in Science, Technology, Engineering, and Mathematics (STEM) degree programs. In comparison, active learning experiences contribute to better student outcomes, such as pass rates, higher retention, and improved self-efficacy (Chen, 2013; Flick, Sadri, Morrell, Wainwright, & Schepige, 2009; Freeman et al., 2014; Laursen, Hassi, Kogan, & Weston, 2014; 2012; Saxe & Braddy, 2015). Despite this research base, mathematics graduate teaching assistants (MGTAs), who generally represent future instructors and professors of mathematics, are not provided with substantive professional development experiences that would teach them how to engage learners in active mathematical learning experiences. As a result, MGTAs' teaching remains rooted in lectures and undergraduate students in STEM disciplines do not experience the proven benefits of evidence-based teaching practices (e.g., Miller et al., 2018; Stains et al., 2018).

MGTAs, who often find work as faculty members in mathematics departments, exert a significant impact on undergraduate learners' trajectories in STEM fields (Belnap & Allred, 2009; Ellis, 2014), and can negatively impact students' enrolment and choice of major in STEM fields (Bettinger & Long, 2004). Yet, experts in MGTA professional development have not reached consensus on the breadth and scope of programs that prepare MGTAs to teach. Professional development programs that aim to teach MGTAs about teaching vary significantly across institutions, both in the approach to teaching and the amount of time spent. In addition, few programs extend beyond MGTAs' first year in graduate school (Deshler, Hauk, & Speer, 2015; Ellis, 2014; Harris, Froman, & Surles, 2009; Kung & Speer, 2009; McGivney-Burrelle, DeFranco, Vinsonhaler, & Santucci, 2001).

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Miller and colleagues (2018) completed a review of the literature of professional development for MGTAs so that they could characterize the types of research conducted regarding MGTAs' growth as teachers. They identified 26 peer-reviewed articles published since 2005 that investigate MGTAs' teaching development, only 17 of which attend to growth. Thus, the authors concluded that "GTAs' growth as teachers is a largely unexamined practice" (Miller et al., 2018, p. 2) and suggest that this area of study would benefit from longitudinal studies that make explicit a model of growth.

With this in mind, the purpose of this study is to investigate MGTAs' development as teachers, what and how they learn about teaching, and changes in their thinking about teaching and learning longitudinally as they progress through their degree programs. The research questions that guide this study are: (1) What are the developmental stages for teaching that MGTAs go through over the course of their graduate programs? (2) What features of their graduate school and teaching experiences support or hinder their learning about teaching and their development as teachers?

THEORETICAL FRAMEWORK

Because research has not yet addressed MGTAs growth as teachers, we looked to the K-12 literature, where researchers have studied schoolteachers' experiences in order to gain an understanding of teachers' growth over time. Katz (1972) described four developmental stages, which include: (1) *survival* of the first year of teaching, with particular focus on classroom management and the routines of classrooms and schools; (2) *consolidation*, in which teachers begin to understand which skills they have mastered, and what tasks they still need to master; (3) a period of *renewal*, when teachers become tired of their routines and start to think of how things might happen differently; and (4) reaching *maturity*, where teachers think more broadly about the contexts of schools and students' learning (p. 52-53).

CONTEXT AND METHODS OF THE STUDY

At the beginning of the academic years in 2015-2018, participants were recruited from the mathematics department at a large, doctorate-granting institution. Approximately 5,000 undergraduate students enrol in courses such as Pre-calculus, Differential, Integral or Vector Calculus, Business Calculus, or Differential Equations each year. Most of these courses are structured as three hours of lecture with 150-250 students per class and are taught by an instructor. MGTAs are generally assigned to run recitations (60-80 minute workshops each week) of smaller groups of students from the large lecture sections. MGTAs are not assigned to courses based on knowledge, skill, or experience; their assignments to courses mostly depend upon scheduling.

When new MGTAs first arrive to this graduate program in mathematics, they receive 2 ½ days of professional development for their teaching assignment, with a primary focus on how to support active learning and student engagement in mathematics during recitations. In the first term of their graduate program, they attend a seminar for one hour each week that addresses teaching-related concerns such as grading papers, student conduct issues, and lesson planning. In the summer after their first year, they

have the opportunity to teach their own course, then they return to the main MGTA duty of leading recitations. Only informal mentoring happens before and during the summer sessions and into the MGTAs' subsequent years.

We developed two beginning-of-the-academic-year surveys, one for new and one for experienced MGTAs, and protocols for mid-year and end-of-year interviews. Surveys are used at the beginning of the year because of logistical issues. They include open-ended questions that inquire about MGTAs' thoughts about teaching and learning mathematics, how they would describe a well-taught mathematics lesson, and what influenced the way they think about teaching. Mid- and end-of-year interviews allow a deeper investigation of MGTAs' teaching practices, their most recent teaching experiences, whether they feel that they are receiving adequate support, and what other support they feel they need to grow as teachers. The intention of the study is to survey and interview participants for the duration of their graduate programs to study their development over time. Table 1 illustrates participation in the study.

Recruitment Year	Number of Participants
2015-2016	11 new participants: 4 first year, 2 second year, 4 third year, 1 fourth year
2016-2017	11 continuing participants; 6 new participants: 4 first year, 1 third year, 1 fourth year
2017-2018	11 continuing participants; 10 new participants: 8 first year, 1 third year, 1 fifth year
2018-2019	15 continuing participants; No new participants

Table 1: Study Participants

Our research team analysed participants' responses to survey and interview questions in two rounds of coding using thematic analysis (Braun & Clarke, 2006). Thematic analysis has six stages which include: (1) familiarization with the data; (2) coding interesting features of the data in a systematic way and collating data that is appropriate for each code; (3) possibly combining codes into themes and collect data for each them; (4) reviewing the themes and supporting data for each theme; (5) continuing to analyse the themes, generating a clear definition for each; and (6) producing the report of the themes with selected data to provide evidence of each theme. In the first round of coding, we applied a deductive approach (e.g., using a pre-existing coding frame) where we looked for instances of the participants' experiences that could be elucidated with Katz's (1972) four-stage model of teacher development. In the second round of coding, we used an inductive approach that focused on codes we developed through the first round of analysis. These codes included what had an *impact on their views about teaching* (graduate course work, their students, the instructor they are assigned to, the course they are assigned to, their previous experiences as learners, office hours, their MGTA peers, and the resources they use for teaching), *issues of identity* (being a

teaching assistant versus being an instructor, resignation), and *teaching* (descriptions of their teaching practices, the transitions they have made in their teaching, what changes they would make to their teaching, and what is important for their teaching).

FINDINGS

In our first round of analysis, we found Katz's (1972) framework to be a useful lens to view MGTAs' development as teachers. We observed most of the first-year MGTAs talking about *surviving* the first year. The quote below is an exemplar of this:

> By that point the quarter [midterm exams], it was just getting really hectic and I wasn't able to plan as much as I usually like to plan for courses. Sometimes I was looking at the material for about two hours before I started that day, whereas usually I like to look at it the day before or during the weekend or something. And so sometimes, though, the classes that I went to where I was kind of doing it on the fly, where I was literally looking at it like an hour or two before class.

We observed some second-year MGTAs in the *consolidation* stage. This participant is describing his approach to teaching a new course:

> I think virtually the same. They would work for a little bit. It was 15 minutes. So, depending on the class, if it's longer, maybe we'll do it a different way. With 15 minutes, we can take a little bit more, take some time at the beginning. They work on something together, then back together as a class, and then quiz.

Later-year MGTAs sometimes spoke differently about their teaching experiences. For example, a third-year MGTA summarized her transition from the *survival* stage to the *renewal* stage in the following way:

> I think previously, I was more focusing on, "I just want to survive my first teaching experiences." So, now that this is my fourth time teaching, I feel a little bit more comfortable trying to incorporate more active learning in my classroom, and trying non-traditional techniques whereas previously, when I taught, for example, my first time teaching my own class and I taught Calculus, I did mostly lecture because I just wanted to do what I felt most comfortable with – what I felt I could be successful at.

Another participant we saw as being in the renewal stage spoke about it in the following way: "In the few classes that I've taught, I've tried to little-by-little implement more and more activity-based learning and group-based learning and different things."

Teacher Interrupted

Despite seeing some progress, we found that only a few MGTAs spoke about teaching in a way Katz would describe as *renewal*, and only one or two of the MGTAs spoke of teaching in ways that Katz would categorize as *maturity*. Many MGTAs appeared to be stuck in the *consolidation* stage. We also found that MGTAs do not pass through Katz's developmental stages linearly. In fact, they sometimes return to the survival stage. We then conducted a second round of analysis to see what experiences prevented the MGTAs from moving through the developmental stages. We looked for what might have an impact on the MGTAs in terms of learning about teaching and moving toward the active learning strategies espoused in the 2 ½ day professional development

program. We expected that most, if not all, of the participants' work as teaching assistants would consist of leading recitations for lower-division mathematics courses and that there would be some consistency in the courses they were assigned to. However, when we looked at what work they were assigned to term-by-term and year-by-year, there was significant variation not just in the courses they were assigned to (algebra vs. differential equations), but also in what work they did (e.g., grading, assisting on research studies, assisting in online courses). This is an important finding because this variation has an impact on how the participants have the opportunity develop as teachers. Below, we describe the issues that we interpreted as interrupting the MGTAs' teaching and learning about teaching.

Course assignments and expectations had significant impacts on the MGTAs. More specifically, how the instructors designed courses, the assignments, and exams had a disruptive impact on the MGTAs. This first-year MGTA describes what it was like:

> From term to term, depending on the instructor mostly – not necessarily the class, but the instructor – their expectations of the TA are different in terms of what they want you to do. When I designed my materials, I knew where the solution was, what the problem was, what kind of outcome was expected from the students, what kind of mistakes I will see. So, the grading was faster in my opinion. But when everything is given to me, I have to first solve it and then figure out all of those details in the goal. And then I start to grade and then I encounter things that I didn't expect.

Because of the inconsistent ways they were assigned to courses, the MGTAs did not have the opportunity to work with the same group of students from one academic term to the next. A participant spoke of what it was like to 'lose' his first group of students:

> I had all these students, I knew all their names, I was, they like me, I like them. We were really excited about this stuff and we'd been together for ten weeks and now that was all over and I had to do it all over again. And you know there's so much energy, like, that had gone into the last class and so much had come out of it. And now I was just like you know 95 more students who I don't know and I have to do this all over again. And I was, so to a certain extent I just, it was a lot harder to care that term. I was losing the students that I had so much already invested in.

Their role as graduate teaching assistants had an impact on the MGTAs' teaching, leaving them powerless to teach in the ways they wanted:

> But I have no power over what happens in the recitation hour. There is a quiz that was written up, there's an activity that was written up. […] So I'm not grumpy about that, necessarily. It's the instructor's course. That's fine. But it means I'm not choosing any of the problems, or anything like that. So what can I do? And they're working in groups, for the most part. And their group is working well. Somebody knows how to do the problem already, and they don't really need me except for the details. In that sense, I am interchangeable with any grad student. [...] I guess in a TA setting, I have such little power to actually impact their learning.

Another participant had a similar remark about feeling powerless to incorporate different teaching strategies:

> Whereas a TA, you know, you're exactly that. You're an assistant. So, kind of ... you're in the in between. So, you don't have that much power, I guess, but some form of that to try incorporate [active learning] into teaching would be really interesting, but you know, I guess I don't really know exactly how.

Some MGTAs' work assignments took them out of the classroom. In particular, two of the participants who had worked as teaching assistants in the first three years of their graduate program became research assistants with no teaching duties. They both remarked that they had not had the opportunity to grow in their thinking about teaching nor in their teaching practices. One of them noted their desire to gain more teaching experience, despite being assigned to research duties:

> I think the biggest thing is I just want more experience teaching because I still – because it's something that I can theorize and I can plan, but I just kind of know that's one of those things that I feel like I won't be able to intentionally get everything out of the class that I want until I've spend time teaching the class.

Another participant had a different type of interruption in his teaching experience when he was given a grading assignment: "I hadn't TA'ed for almost six months, because I didn't teach in the summer. I wasn't in the classroom with somebody for the whole summer and for the fall. It was like six months of not being a TA, so I was like, 'Oh, man. I almost have to relearn this'."

Despite the lack of attention to their teaching, we noticed that many of the participants want to learn and grow as teachers, even into the fourth and fifth year of their graduate programs. In general, they voiced a sense of familiarity with teaching and a desire to learn more about teaching, to grow their teaching practices beyond lecturing:

> Because I've already gone through three years now teaching. So I'm already comfortable with coming up to class and writing things down and grading things in a reasonable enough fashion and in good time. But, yeah, it would be really great to be able to like just take it another step further.

But some participants also noted that they were not sure what that would look like:

> That would be kind of nice to know how to make that work efficiently, and as a TA I don't know how to do that, or how to take this to another level. I don't know. I don't even know what that would look like. … Something that where people are ... I mean, in my head I kind of have ... imagine like a play where it's people are engaged in having fun with it, but also being challenged and respected.

IMPLICATIONS

Researchers recommend that MGTA professional development programs be informed by research findings (Miller et al., 2018) and should take into account the needs of MGTAs at different stages of their development (Park, 2004). Additionally, DeFranco and McGivney-Burrelle (2001) noted that professional development programs should be "viewed as ongoing professional development experiences that support [MGTAs] through the long and complex process of changing their teaching practices" (p. 688). We observed that MGTAs in their assignments as teaching assistants were unable to

change the way they teach for several reasons. We recommend that professional development providers take into account the impacts described in this paper (e.g., varied and inconsistent course and work assignments, the powerless role of teaching assistants). We also recommend that professional development programs extend far beyond MGTAs' first year in graduate school, possibly into their third and later years, as they reach the renewal stage (Katz, 1972).

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CHANGING TEACHER PRACTICES WHILE TEACHING WITH CHALLENGING TASKS

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In this paper we report on changes to teachers' practices as a result of their participation in a research-based program of professional learning focused on challenging tasks. Seventy early years teachers responded to a survey at the end of a year-long program asking them to nominate and describe the teaching practices that changed most as a result of their involvement in the program. The majority of teachers nominated each of 'Allowing students time to struggle' and 'to share their thinking' as the two practices that changed most. Qualitative responses are used to interpret the nature of and reasons for these reported changes in practice.

INTRODUCTION AND BACKGROUND LITERATURE

It is widely acknowledged that "student learning is greatest in classrooms where the tasks consistently encourage higher-level student thinking and reasoning" (National Council of Teachers of Mathematics, 2014, p.17). Stacey (2003) suggested that a key difference between high-performing and other countries on international comparative tests of mathematics is their capacity to engage their students in higher-level thinking via cognitively demanding tasks. In a study designed to encourage Australian teachers to use challenging experiences in their mathematics classrooms, Sullivan, Borcek, Walker, and Rennie (2016) found that implementation was more successful when teachers adopted specific practices, including (1) increased time for students to struggle on tasks without, or prior to, instruction, and (2) increased use of carefully orchestrated dialogue during lessons that emphasised higher-level student thinking and reasoning. Following these findings, Sullivan et al. (2016) hypothesised that learning would be further enhanced if purposeful follow-up challenging tasks were posed to consolidate student learning.

The potential of purposefully designed sequences of connected challenging tasks was the focus of the professional learning (PL) project at the centre of the research reported in this paper. In particular, we report on changes to teachers' practices as a result of their participation in a PL research project focused on sequences of challenging tasks.

Challenge, struggle and student thinking

The tasks and lessons that are likely to foster higher levels of cognitive activity in students are regularly termed 'challenging'. They are considered challenging because the answers are not immediately obvious to students and therefore provide them with opportunities for prolonged thinking, reasoning, problem solving and risk taking (Sullivan et al., 2016). However, Stein, Grover and Henningsen (1996) argue that the

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possibilities afforded by challenging tasks are dependent upon the willingness of teachers to implement such tasks, and critically, to maintain high levels of cognitive demand as work on tasks progresses even when students are struggling.

Struggle involves students expending effort while experiencing some level of confusion while attempting to resolve problems where the solution strategy or answer is not apparent. Hiebert and Grouws (2007) suggest that struggle is particularly important for students to develop conceptual understanding. They claim that if students struggle in order to make sense of mathematics they are more likely to remember the mathematics. At the same time, students are more likely to develop a disposition for persistence and an underlying growth mindset. They argue that providing students with opportunities to engage in struggle is an essential aspect of effective classroom practice. In a study involving 36 lesson observations of six different middle school teachers, Warshauer (2015) found that the way teachers responded to students struggling was critical in determining opportunities for learning. The most successful teachers responded with practices that maintained a balance between the level of challenge and the degree of support provided as the task unfurled.

Unfortunately, research indicates that many teachers are hesitant to integrate cognitively demanding tasks into their classrooms (Cheeseman, Clarke, Roche & Wilson, 2013). This finding raises concerns about some teachers' capacities to activate high-level thinking in their students or to promote sustained periods of student struggle. Prompted by concerns that students were not being provided with opportunities to regularly experience cognitively demanding tasks and evidence confirming the potential benefits afforded by such experiences, a year-long project involving teacher PL focused on challenging tasks and the associated pedagogy was designed. Following recommendations by Sullivan et al. (2016) regarding the potential benefits of follow-up tasks, the PL focused on sequences of connected challenging tasks as opposed to isolated tasks with little coherence to longer-term learning.

THEORETICAL FRAMEWORK OF THE CURRENT STUDY

The PL design, data collection and analysis were informed by Clark and Peterson's (1986) framework. This framework suggests that teachers' knowledge, dispositions, and the opportunities and constraints they anticipate experiencing will influence their intentions to act and their classroom actions. Therefore, we conjectured that teachers would be more willing to implement challenging tasks if they were better informed of their benefits, had ready access to research-developed sequences of tasks and were provided with practical classroom support during implementation. In the study reported here, the elements of the framework referring to teachers' pedagogy and classroom actions are particularly relevant.

The current study addressed the research questions: (1) What teaching practices do teachers report changed most as a result of participating in a program of professional learning focused on challenging tasks? (2) How do teachers characterise the changes to these practices?

METHODOLOGY

Participants

Participants included 102 teachers of F-2 students (approximately 5-8 years of age) and teacher leaders from 21 schools who took part in the PL. Seventy participants responded to the survey, 68 of whom implemented at least some of the suggested sequences in their classroom. Of these 68, 27 (40%) had up to four years teaching experience, 11 (16%) had 5-9 years, 30 (44%) had 10 or more years of experience.

Instrument and Procedure

At the conclusion of the year-long program, teachers were invited to complete an on-line survey while attending the final PL day. The survey consisted of 24 items. The first four items collected background data for each participant and their school, including years of teaching experience and their role at the school. The remaining items required mostly qualitative responses to gain insight into teachers' views of various aspects of the program's implementation, including the nature of the support they received to implement sequences, the effectiveness of specific learning sequences and the challenges they faced implementing the changes in their classrooms. The findings from these aspects, and data collected by observations and interviews, are the subject of a range of papers currently in production.

In this paper, we focus on teacher responses to survey items asking them to report on the classroom practices that they considered changed most over the course of the PL. In Item 13 teachers were asked to select three teaching practices from a list of nine that they considered to have changed most as a result of their participation in the program. The list of nine practices were selected based on the initial aims of the PL, the research-based learning and teaching principles underlying the design of the program, and the practices emphasised during PL days and school-based site visits by the research team. These practices are represented in Figure 1 and summarised in Table 1. In Item 14 teachers had to briefly explain what one changed practice 'looked like' in their mathematics classroom. Fifty-five participants responded to this item.

The entire survey took participants approximately 25 minutes to complete.

Data Analysis

Teacher responses to Item 13 were collated to find the total number of teachers selecting each practice as one of the three practices they considered to have changed most. A deductive analysis process was used to code each response to Item 14. The nine practices constituted the initial codes as it was anticipated that responses would link to one of these. It was revealed early in the analysis that a few responses incorporated aspects of at least two practices from the list. This meant that several responses were dual-coded to practice-based themes. For example, the following response was allocated to both the *Allowing students to share their thinking* and to the *Questioning of students* themes:

> The biggest change in my classroom practice was to the questioning and sharing of thinking all through the lesson – not just at the end of the lesson.

Responses describing the two prompt practices (*Use of extending prompts* and *Use of enabling prompts*) were collapsed into one (*Use of extending and/or enabling prompts*) since only one respondent mentioned extending prompts and this was done in conjunction with enabling prompts. One response was omitted from the analysis because it came from a non-teaching staff member confirming that changes in *all* the practices had been observed but no detail as to what those changes looked like was provided. While only one of the remaining 54 qualitative responses to Item 14 was coded as *Routinely use connected sequences of tasks*, it was decided to retain this code for reasons explained later. Final analysis involved seven practice-based themes.

RESULTS AND DISCUSSION

Figure 1 presents the findings from Item 13. It shows that 51 out of 68 (75%) teachers considered *Allowing students time to struggle on maths tasks* as a practice that had changed most as a result of their involvement in the PL. The next most commonly selected practice was *Allowing students to share their thinking* with 36 teachers selecting this practice as one of their three from the list provided. The remainder of responses were evenly dispersed across six other practices. Only eight teachers selected *Routinely use connected sequences of tasks.*

Teacher responses to Item 14 provided insight into the nature and extent of changes to each of these practices. They also provided some possible reasons as to why changes to certain practices were more commonly cited by teachers than others. Table 1 provides a summary of the number of responses coded to each of the eight practices. Understandably, the number of responses detailing changes to each practice recorded for Item 14 roughly corresponded to the number of teachers selecting them as one of the three practices that changed most. Due to space limitations, reporting of qualitative responses focuses on just three practices – the two practices most commonly chosen by teachers (*Allowing students time to struggle* and *Allowing students to share their thinking)* and the least common (*Routinely use connected sequences of tasks)*.

Allowing students time to struggle on maths tasks was the second most frequently selected practice teachers considered to have changed since the start of the PL – 21 teachers opted to elaborate upon it in Item 14. Approximately one-third of comments indicated that allowing students to struggle with mathematics without teacher intervention was something "new" to their pedagogy, that felt "uncomfortable", "challenging" and "hard at the beginning" because they had the urge to step in "right away" to "save and help my students". This finding connects to those by Cheeseman et al. (2013) that a possible reason for teachers' reluctance to implement challenging tasks is their discomfort with student struggle. According to Clark and Peterson's framework, teachers' intentions and practices in the classroom are influenced by constraints (feeling uncomfortable with struggle), that could have been circumvented by opportunities (support to implement challenging tasks) and influenced by

knowledge of mathematics and pedagogy (provided by the PL). As suggested by Sullivan et al. (2016), teachers need to anticipate constraints such as student negative reactions to struggle and be prepared to address them via appropriate practices. While approximately equal emphasis was given to the nine practices throughout the PL, the large number of teachers who acknowledged that the notion of student struggle in mathematics as potentially beneficial was new to them, is a possible reason why so many selected this practice as one that changed most. A quarter of responses explicitly referred to struggle as something that both teachers and students gradually learnt to "embrace" as an integral "part of their learning".

> Allowing students to struggle was something new to me. The students found it difficult at first that I was allowing them to struggle and they wanted my assistance. Over time this changed, and they became used to struggling first and then gaining clarity at the conclusion when we went through tasks and possible solutions.

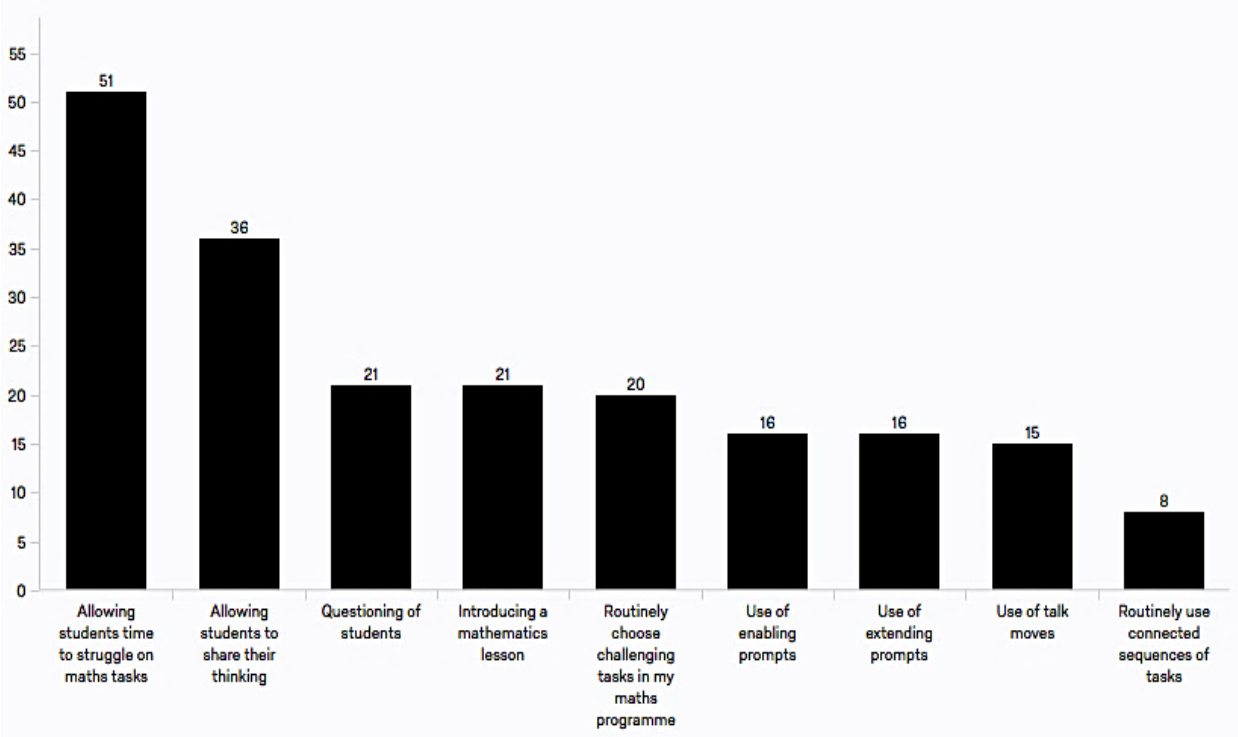

Figure 1: Number of teachers (n=68) selecting each practice that they considered changed most. Each teacher selected three practices (204 practices in total)

Teachers expressed a willingness to trial and continue allowing students struggle time in their classrooms because they could "see the value" in terms of students' increased "independence" and "persistence" in solving challenging mathematics problems. A range of strategies for introducing struggle time into their classrooms were described, but the inherent message was to allow students "to struggle first" and "struggle more" often. As illustrated in the following quote, teachers typically accompanied struggle time with student-led discussion and questioning:

> We would show the wording of the problem, let them have time to read it and try and work out what it means, before discussing. Let them lead the discussion. Let the students ask questions for clarification IF they want to.

Increased "talking" and "more discussion" also characterised teachers' elaborations of how the practice of *Allowing students to share their thinking* changed over the course of the PL. More than half the teachers who responded to Item 13 nominated this practice as one that changed most. It was also the most commonly chosen practice for teachers to elaborate upon. Similar to the practice of allowing time for struggle, teachers noted the difficulties associated with getting students to talk about their thinking early in the implementation of the program. This difficulty is an indication that it was not a common practice prior to the PL:

> Allowing students to share there thinking was difficult at first as they would often just give and answer. Now my students are able to give there answer but explain how they got it or what they did to find a solution. More talk within the classroom and lots more discussion.

Practice	Number of responses coded
Allowing students to share their thinking	21
Allowing students time to struggle on maths tasks	14
Introducing a mathematics lesson	9
Use of talk moves	7
Use of enabling and/or extending prompts	5
Routinely choose challenging tasks in my mathematics program	3
Questioning of students	2
Routinely use connected sequences of tasks	1

Table 1: Summary of the analysis of teachers' responses (n=54) describing change in their practices. Note that some responses were coded to more than one practice

One teacher revealed that it was "harder for the teacher than the students during…thinking time" because of the sometimes "uncomfortable" silence. All responses referring to this practice included clearly articulated strategies for its implementation, as demonstrated by the following two teacher quotes:

> I strategically select students to share their thinking based on the least efficient strategies to the most efficient strategy used in that task, so children can see a range of strategies and learn all the different ways to think and solve a task. Students are to then go and borrow one of the strategies they listened to and give it a go during a consolidating task.

> We have in depth discussions where we show students work on the TV and students are given opportunities to challenge or agree with other students.

When describing the nature of student sharing, teachers referred to "high level thinking", "reasoning", "collaborative…problem solving" and "strategy" discussions. One teacher explained that students were "not talking about the answer, as much as discussing how they got there and what their logic was". It was apparent from teacher

descriptions that not only was there a great deal more sharing of student thinking occurring in their classrooms, but that students were being encouraged to think about mathematics at much deeper levels than before. The Clark and Peterson (1986) framework guiding our research emphasised connections between teacher beliefs, knowledge and their practices. In this case, teachers' beliefs about the importance of students explaining their reasoning gained via the PL were presumably reflected in the increased time they allocated to students talking about and sharing their thinking.

The practice selected least often by teachers was *Routinely use connected sequences of tasks*. This finding was surprising given that a focus of the PL sessions was on the importance of providing students with sequences of connected and challenging tasks as opposed to isolated challenging tasks with little coherence to longer-term trajectories of learning mathematics. As part of the PL, teachers were provided with suggested sequences of learning experiences for a range of content areas that were aligned to the mandatory curriculum. Instructional materials containing pedagogical advice to guide them in their implementation of the sequences were also provided along with occasional in-class support during site visits from a member of the research team or their school-based mathematics instructional leaders. This finding could reflect where the majority of teachers were 'at' in terms of their professional learning and their capacity for adopting new practices associated with challenging tasks. Teachers' responses indicated that it had taken effort and time for many of them and their students to feel comfortable with struggle and challenge in the mathematics classroom. It is possible that teachers need much more time than one year to embed such practices into their instructional routines before gradually working-up to the implementation of whole sequences of challenging tasks. This proposition is supported by the fact that more than half the teachers who selected this practice as one of their most changed, also reported prior involvement in PL involving challenging tasks. Further support is provided by the teacher who chose to elaborate upon this practice in Item 14:

> We have always included a challenging task as one of our weekly lessons from Foundation to Year 6. This year we were able to plan sequences of lessons with the same challenging approach. We had to let go of … a teacher modelled and teacher centre lesson phase. This has been the most difficult step to convince teachers to adopt.

This comment indicates that challenging tasks were already an accepted part of practice for teachers at this school. The notion of connected sequences of challenging tasks was therefore less daunting for them and a logical next step in the PL for these teachers.

CONCLUSION

The research reported above was part of a large PL project exploring the potential of purposefully designed sequences of connected challenging tasks. Reported changes to teachers' practices as a result of their participation in the PL was the focus of this paper. In terms of the first research question, teachers reported changes to each of the nine practices focused upon in the PL, but the majority of teachers reported most change in just two practices – *Allowing students time to struggle* and *Allowing students to share their thinking.* Regarding research question 2, teachers characterised the changes to

these practices in terms of student and teachers gradually accepting them as the norm in mathematics lessons. Both practices were linked to increased amounts of time for student talk and questioning. The surprising lack of change surrounding the use of sequences of challenging tasks is an indication that PL needs to consider teacher prior learning and be structured in stages with some aspects potentially needing to be learned before others. In the current PL, teachers indicated that a familiarity with challenging tasks preceded their readiness to adopt whole sequences of lessons involving such tasks.

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EYE-TRACKING SECONDARY SCHOOL STUDENTS' STRATEGIES WHEN INTERPRETING STATISTICAL GRAPHS

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Students persistently misinterpret histograms. Based on a literature review we conjectured that the confusion of histograms with case-value plots was leading to inappropriate interpretation strategies. Hence, the question for this research is: what are the most common strategies for secondary school students when estimating the mean in histograms and case-value plots? In a task with twelve graphs (histograms or case-value plots) we measured students' eye movements (N=10, grade 10–11. The most common strategies students use are: a case-value plot interpretation strategy and a computational strategy both applied to histograms. Furthermore, some students reported strategies that are not in line with their gaze data nor their estimated mean.

BACKGROUND

For more than two decades misinterpretations with histograms have been reported in the literature (e.g., Friel & Bright, 1995; Whitaker & Jacobbe, 2017) as well as attempts to improve students' understandings of histograms through interventions (e.g., Meletiou-Mavrotheris & Lee, 2005). An extensive review of the literature on histograms showed numerous misinterpretations that are widespread amongst students from every school level as well as countries (Boels, Bakker, Van Dooren, & Drijvers, 2019). In addition, many people are not clear on what a histogram is. In the context of this paper we use the following criteria for a histogram (Boels, et al., 2019): (1) it is a graph with bars; (2) the data of only one statistical variable are presented; (3) these data are measured at ratio measurement level; (4) the vertical axis displays frequency.

The persistence of misinterpreting histograms was our rationale for searching for underlying difficulties. In a literature review we found two difficulties: the lack of understanding the big statistical ideas data and distribution (Boels et al., 2019). The big idea data refers amongst others to the measurement level of the data. Distribution refers amongst others to how the data are distributed (e.g., variability or spread) and how this is depicted in a graphical representation (shape). The literature suggests that a histogram is confused with a case-value plot (e.g., Cooper & Shore, 2010; Garfield & Ben-Zvi, 2007). A case-value plot is a kind of look-a-like of the histogram as it shares the same salient features with a histogram. These salient features are: two axes, numbers along the vertical axis and—in our construction—the same number of bars and the variable weight in kilograms. Nevertheless, on the level of the big ideas data and distribution a histogram is very distinct from a case-value plot. For example, a case-value plot depicts two variables (here: name and weight, see Figure 1) whereas a histogram represents only one (here: weight). Furthermore, in a case-value plot each bar is one measurement while in a histogram each bar usually stands for several

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measurements. As a result, in a case-value plot the heights of the bars are the measured value whereas in a histogram the positions of the bars on the horizontal axis are the measured values. In addition, the variability of the data in a histogram is the horizontal spread weighted by the frequency whereas the variability of the data in a case-value plot is given by the vertical difference in the heights of the bars.

a. Mean in histogram (here: 3.3 kg) b. Mean in case-value plot (here: 3.7 kg)

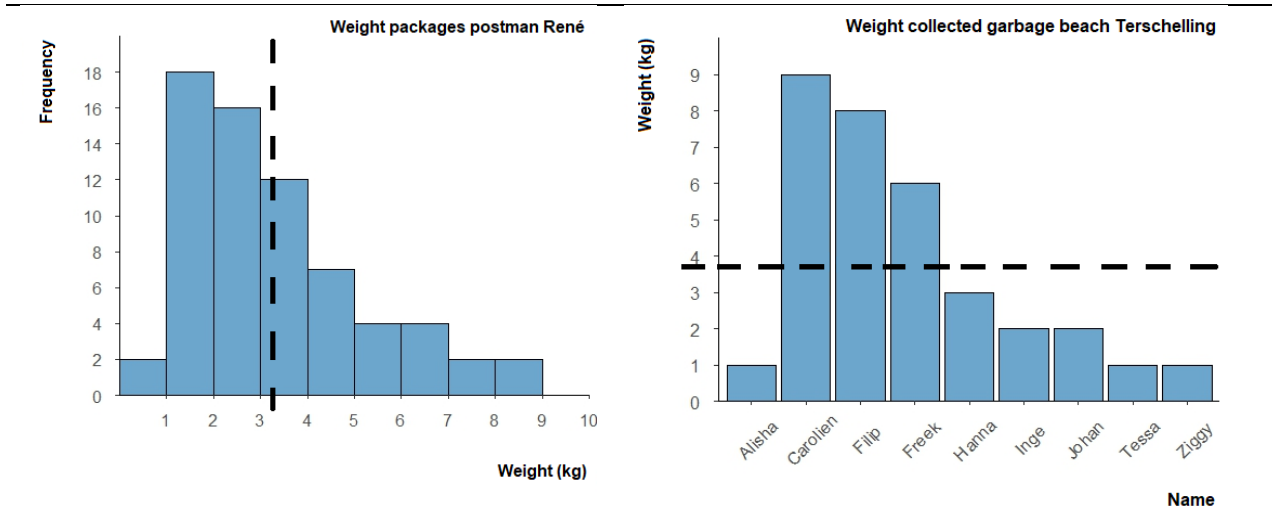

Figure 1: Measures of centre are depicted differently in a histogram (a, Item 1) from a case-value plot (b, Item 4).

Although the confusion of histograms with case-value plots can explain a substantial part of the misinterpretations, this does not shed any light on how this confusion arises. In a first exploratory study, we therefore studied the interpretations strategies that students used to answer questions about histograms and case-value plots (Boels, Ebbes, Bakker, van Dooren, & Drijvers, 2018). In our first study we were searching for persistent misinterpretations after learning about histograms. We therefore included university students only. In preparing that study, we were concerned that we would not find any misinterpretations at all as we thought that we had made the differences between the case-value plots and the histograms too obvious, see Figure 1. For example, we had clearly placed labels next to the axes showing that the requested information (average weight) could be found on either the horizontal axis (histogram) or the vertical axis (case-value plot). We nevertheless found several students applying a case-value plot interpretation strategy onto a histogram (e.g., a horizontal looking pattern and reading of the numbers on the vertical axis thus the frequency instead of the measured values).

In an ideal world, students would enter the university without misinterpretations, as interpreting histograms is part of the Dutch secondary school curriculum. In order to track down where these misinterpretations come from, we need to study secondary school students' interpretations of histograms. These students have learned about histograms in grade 9 or 10 (usually at age 14–16). Hence, the aim of our current study

is to identify the most common strategies of secondary school students when interpreting histograms and case-value plots.

In searching for a suitable task for interpreting histograms, we noted that many researchers ask for the variability depicted in a histogram (e.g., Garfield & Ben-Zvi, 2007). Nevertheless, we decided to ask for estimating the mean from a histogram. The first reason for asking for the mean is that finding the mean in a histogram can be regarded as a prerequisite for using measures of variability. As measures of centre are depicted differently in histograms from case-value plots, the same holds true for measures of variability. Moreover, variability can be considered as variation around the mean. The second reason for using the mean instead of measures of variability is that secondary school students are very familiar with the arithmetic mean (as this is, for example, used for grading almost all summative assessment tests in secondary schools in the Netherlands) and not so much with measures of variability. Hence, the question for this research is: what are the most common strategies for secondary school students when estimating the mean in histograms and case-value plots?

METHOD

To answer our research question we constructed twelve items with histograms or case-value plots. Six of these items were either single histograms (three) or case-value plots (three). The question for these items was to estimate the mean weight of the packages of a postman (histograms) or garbage collected on a beach (case-value plots). These questions were open ended. Three of the other six items concerned two histograms each and the other three two case-value plots each. These six items held the question in which graph the estimated mean was bigger. The three answer options were: the graph on the left, the graph on the right or both graphs approximately the same. All histograms had a look-a-like case-value plot that shared the same salient features with the histograms such as number of bars and range of the weight scale, see for example Figure 1 (Item 1 and Item 4). Two items with the same salient features never followed one another and there were no more than two items with the same graph type in succession. The graph construction was in line with the recommendations to use stimuli that differ systematically on relevant features but are similar for irrelevant features (Orquin & Holmqvist, 2017). To avoid priming (Lashley, 1951) we started with a single histogram in the first two items as we expected that students misinterpreting histograms would apply a case-value plot interpretation strategy onto a histogram (see data analysis for an explanation on the strategies) and not so much the other way around. The twelve items were followed by another thirteen items that were part of an exploratory study not reported here.

Two questionnaires were used to gather data on the students' background (e.g., their grades for mathematics) as well as their precognition on—for example—the meaning of frequency and arithmetic mean. By asking first to estimate the mean of five assessment scores (each between one and ten) and then to calculate this mean, the difference between an estimation and a calculation was subtly stressed as we did not want students to make precise calculations.

Figure 2: Setup of the equipment (the person in the photo was not a participant)

Eye-tracking in combination with retrospective thinking aloud with students' eye movements as a cue was used. We could have collected only the answers for these twelve items (e.g., Whitaker & Jacobbe, 2017). Although this method is fruitful for discovering how widespread the misinterpretations are as well as for getting indications of the type of misinterpretations students have, it does not tell us in detail which strategies students use to answer these items. In addition, we considered other methods, such as combining the assessment items with a thinking aloud protocols or with retrospective interviews, both without the use of eye-tracking. We rejected all these options for several reasons. Firstly, when people need to explain their strategy during thinking aloud, this not only slows down their work on the task, but also changes their cognitive processes (Ericsson, 2006). Secondly, retrospective interviews are often unreliable but can be improved when participants' eye movements are used as a cue (van Gog & Jarodzka, 2013). As the eye-mind hypothesis does not always hold (Anderson, Bothell, & Douglass, 2004; Schindler & Lilienthal, 2019) a retrospective interview is needed to link the eye movements data to the students' strategy.

Ten secondary school students of one pre-university level school (grades 10–11, mean age: 16 years; range 15–18 year) participated in the study reported here which is part of a larger study with secondary school students and their STEM teachers. Students participated voluntarily and consent was signed by the students and in most cases also by their parents (required for students aged 15 or less). All participants received a small gift (mathematical puzzle) for their participation. The first author recruited the students, carried out the research and conducted the interviews at the participants' school. To stay within the time frame of one lesson of this school the time per student was limited to 45 minutes. In this time frame all work with the participant was done, which included filling in two questionnaires, answering the tasks described above and being interviewed with participants' own gaze data as a cue. The eye-tracking part took approximately 10–15 minutes per student, leaving about 20–25 minutes for the interview.

A Tobii Pro X2-60 eye-tracker with a 60Hz sampling rate was used, mounted with adhesive mounting brackets on a laptop with a 13-inch display. The Tobii Pro Studio 3.4.5 software recorded in real time where people were looking on the screen using harmless infrared light to detect the gaze. We also used a chin rest to gain a better quality of the eye-tracking data, see Figure 2. For recording the retrospective interviews a Rode NT-USB studio microphone was used. Students were tested in a one-on-one setting in a room in their own school. Before starting the task, a calibration procedure

was used with nine points on the screen. The calibration was followed by a validation procedure with four points on the screen. Between all items a fixation cross was used for validation as well as making sure that all eye movements start at the right-hand side of the screen which was a blank area in the next screen with the item.

a. Case-value plot strategy applied onto a histogram (Item 1)	b. Case-value plot strategy applied onto a case-value plot (Item 4)

The colours indicate where students' gaze was less (green), medium (yellow) and most (red). This student stated that the mean weight was approximately ten (a, histogram) and five (b, case-value plot). The arrow points at fixations around frequency ten in the histogram. Note that there are no fixations on the horizontal weight axis in the histogram although the label weight on the horizontal axis was read by this student.

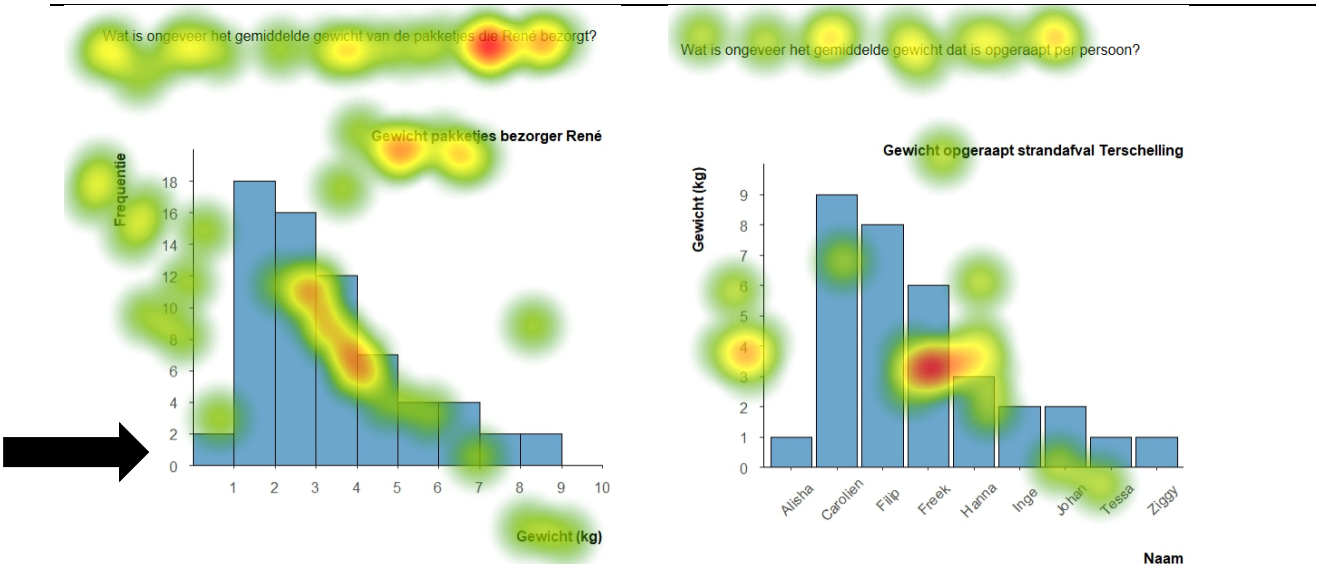

Figure 3: Heatmap of a student applying a case-value plot interpretation strategy onto a histogram (a) and a case-value plot (b).

DATA ANALYSIS

We analysed the answers given by the students for each item. Furthermore, from our previous studies we conjectured that students would use two interpretation strategies: a case-value plot interpretation strategy and a histogram interpretation strategy (Boels, Bakker, & Drijvers, 2019; Boels et al., 2018). We define a case-value plot interpretation strategy as a strategy in which students display a horizontal looking pattern and read the requested mean at the vertical axis, see Figure 3. During the interview, these students report that they take the middle of the bars or make the bars "the same" [height]. Sometimes students move their hands in a horizontal line when they explain what they did. If that happened, the interviewer reported this aloud in the interview. We defined a histogram interpretation strategy is as a strategy in which students display a vertical looking pattern and read the requested mean at the horizontal axis. During the interview, these students—for example—reported that they balance the graph. Sometimes they might also point to a balancing point at the horizontal axis. The interviews and gaze data were qualitatively coded with open, axial and selective

coding (Corbin & Strauss, 1990). From these data we inferred the strategies that the students used. We also used heat maps, showing students' gaze data in gradients from green via yellow to red (see Figure 3) and gaze plots showing the order of students' gaze data.

RESULTS

For the single histogram items students correctly answered on average 0.5 item out of 3 items which was much less than the average number of correct answers for single case-value plots, see Table 1.

3 items; single histograms	3 items; double histograms	3 items; single case-value plots	3 items; double case-value plots
0.5 [0–2]	0.7 [0–3]	1.5 [1–2]	2.3 [2–3]

Table 1 Average number and [range] of correct answers per student.

a. Computational strategy applied onto a histogram (Item 1)	b. Computation strategy applied onto a case-value plot (Item 4)
This student stated that the mean weight was approximately six (a, histogram) and three plus one third (b, case-value plot). Note the many fixations on both axis and the going back and forth of the fixations from axis to the graph area indicating the computational strategy.	

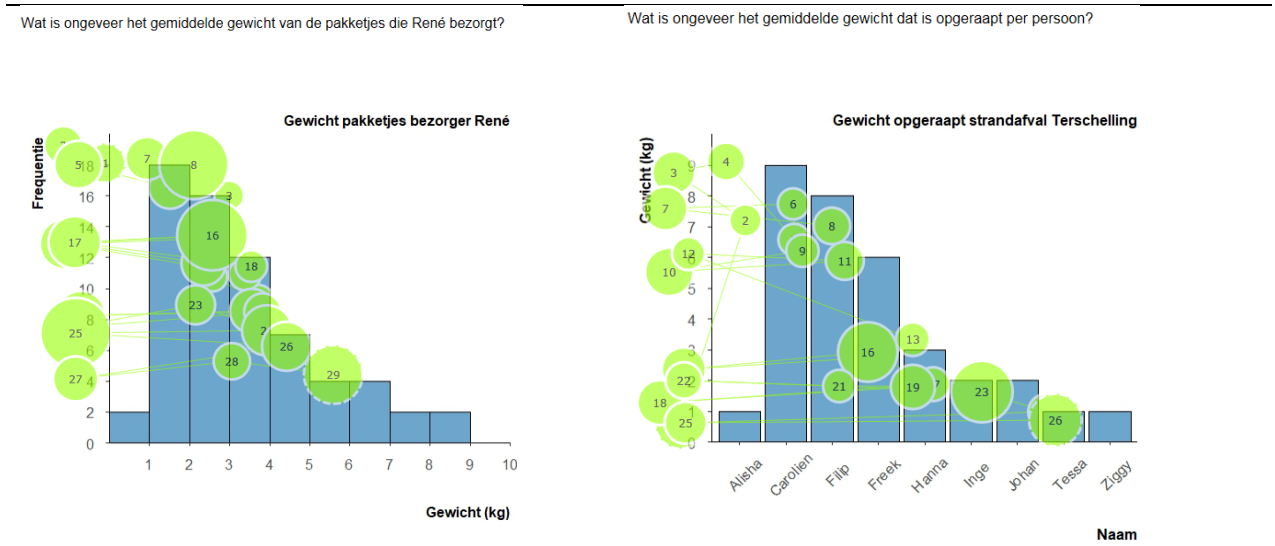

Figure 4: Plot of a student's gaze. This student uses a computational strategy.

In the qualitative analysis of our data we found the two interpretation strategies described in the data analysis as well as a new strategy that we call a computational strategy. In this strategy a student computes a total (either a total of the frequency or a total of the frequency times measured value) and divides this by either the number of the bars or sometimes the total frequency. In the gaze data the looking pattern appears as jumping from bar to bar, for example along the horizontal axis (counting the number of bars, see Figure 4).

Furthermore, some participants report strategies and answers that did not match the answer given earlier during the online eye-tracking session. For example, the participant whose strategy is shown in Figure 3 reported a lower answer than ten in his retrospective interview and said to have looked on the horizontal axis to find this answer. This is not in line with the answer that was given during the online eye-tracking session (ten) nor with the gaze data as there are no gazes on the horizontal axis. The validation data suggest that gaze data are accurate so this is not due to any off-set nor drift, indicating that this participant did not look at the horizontal axis at all. Although the axis could have been seen in a peripheral view, this is unlikely for reading off numbers.

CONCLUSIONS AND DISCUSSION

The first conclusion is that these ten students used two strategies most frequently: a case-value plot interpretation strategy applied onto a histogram and a computational strategy. The case-value plot interpretation strategy is in line with many findings in the literature (e.g., Cooper & Shore, 2010). The computational strategy was found in our literature review but is rarely reported (Ismail & Chan, 2015).

The second conclusion is that several students reported strategies as well as answers in their retrospective interview that did not match the answers given during the online measurement and nor their gaze data. This indicates that a retrospective interview (a thinking aloud protocol) might not always be reliable for secondary school students. Gaze data are measured online (meaning: during the performance of the task) and are therefore a useful complement in finding students' strategies.

The third conclusion is that even students who read titles and axis labels misinterpret histograms. We therefore speculate that it will not suffice to learn students to (better) read the labels of a graph. We conjecture that underlying students inappropriate use of strategies is that they do not have understood the big ideas of data and distribution and specific how these effect measures of centre and shape in different types of graphs.

Acknowledgement

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STUDENT TEACHERS' USE OF MEASURABLE PROPERTIES

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Measurement is an important part of the elementary school curriculum. But weaknesses in prospective teachers' knowledge of the concepts of 'attribute' and 'measurement' have been identified. Drawing on the modelling perspective, this research investigates how a cohort of South African student teachers identify and distinguish particular properties as measurable quantities, in their use of mathematics to make sense of the world. Difficulties with measurement and the identification of properties are noted. Also, a possible influence of appropriate property identification on the precision of students' arguments is noted, suggesting that the concept of 'property identification' should be considered alongside 'attribute' and 'measurement' in curriculum design.

INTRODUCTION

Elementary school mathematics provides the basic foundation on which all higher mathematics is built. But more than this, it provides basic conceptual and operational tools that all people could use to help make sense of the world and of their activity in the world (OECD, 2017). In their elementary school learning, children engage with two of the worlds of mathematics, the conceptual-embodied world and the proceptual-symbolic world (Tall, 2008). Even though mathematical, these worlds are grounded in everyday experience: of objects and their properties in the conceptual-embodied world and of actions and their products in the proceptual-symbolic world (Tall, 2006). This grounding links mathematics to children's experience and so allows them to make mathematical sense of their world and experiential sense of their mathematics. As well as conceptual insight, mathematics brings precision, operational efficiency and strategic power (Kilpatrick, Swafford and Swindell, 2001; Milgram, 2007; Schoenfeld, 2007) to the way we make sense of the world. Making mathematical sense of the world is generally included in school curricula under the topic of measurement (Passelaigue & Munier, 2015). But it is also important when formulating situations mathematically to solve contextual problems or develop mathematical models (OECD, 2017).

Passelaigue and Munier (2015) discuss the importance of developing a grounded understanding of the concepts of 'attribute' and 'measure' if we wish primary school children to master the learning of measurement generally required in the Primary School curricula. In their research into elementary school teacher's specialized content knowledge (Ball, Thames & Phelps 2008), they identify and discuss some of the difficulties experienced by student teachers in distinguishing and relating these two concepts. This paper provides further preliminary findings to support their results and provides further insight into some of the misconceptions they identify, by including

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elements of horizontal mathematization in the analysis of how students formulate situations mathematically.

From the perspective of modelling, the process of horizontal mathematization – generating a mathematical model to represent salient features of a problem situation is identified as an important stumbling block to many students (Jupri & Drijvers, 2016; Verschaffel & De Corte, 1997). Identifying particular properties of things in the problem situation as both corresponding to measurable attributes and appropriate for solving the problem, is an important element of horizontal mathematization (Van den Heuvel-Panhuizen, & Drijvers, 2013). Measuring for mathematical sense making in elementary school mathematics may thus be seen as including three layers: First, identifying particular properties in a situation that are relevant to the problem at hand and relate appropriately to measurable attributes. Second, understanding general attributes (such as length, area, speed, and so on) that may be compared and measured. And third, being able to measure such general attributes (using particular instruments and units) and so quantify these particular properties.

This paper discusses the manner in which a cohort of South African pre-service education students, studying to become Foundation Phase (grade R–3) teachers, engage with each of these three layers of sense making. It discusses the responses of students to assessment questions that involve measurement and/or the identification and use of quantities to solve situated problems. It is evident that, even though a pre-requisite for acceptance on the programme was at least a pass in grade 12 mathematical literacy, many students were not proficient with making precise mathematical sense of questions, by identifying and quantifying relevant properties that could be mathematicised.

THE RESEARCH PROJECT

This research forms part of a curriculum development / design research project that is based on the premise that mathematics provides a means for making sense of the world and that phenomena in the world provide means for making sense of mathematics. The aim of the research is to develop a curriculum for the preservice education of primary school teachers, that will generate a deep engagement with both common and specialized mathematical content knowledge (Ball, et. al, 2008) important for elementary school teaching. The aim is to provide an authentic experience of the process of making sense of and with mathematics and so inducting student teachers into mathematics as an activity (Van den Heuvel-Panhuizen, & Drijvers, 2013), to enable them to foster and develop this sense making practice in their teaching of primary school mathematics to children.

This paper presents a number of the insights generated from the early, exploratory (and predominantly qualitative) phase of the project, focusing on those that relate to measurement and the quantification of properties identified in situations being modelled. The qualitative generation and justification of conjectures based on this

exploratory research, is discussed in this paper. The second phase of the project will include a quantitative investigation of this conjecture.

Student responses to four assessment questions were analysed and three will be discussed in this presentation. Two questions involved measuring an irregular area. These similar questions were set in the early and final assessments and so this data provides some indication of the possible learning of the students. The third question required students to relate time, distance and speed.

RESULTS

Questions 1 & 2: Area

These questions involved measuring an irregular area using two different drawn shapes as units and then querying the relationship between the two different measurement units and the corresponding measurement values. This paper will focus on the second part of these questions.

The second, comparative, part of these questions asked students to work with two different units, to get two different measurement values and then to identify how the unit area comparison and the measurement value comparison related to each other. Student responses were classified using a number of different criteria. First, comparing the two units of area, was performed in 4 different ways: ***rational, descriptive, linear*** and ***shape***. For 'rational', students compared the area of the units, generating a rational (multiplicative) relationship between the two. A qualitative comparison (such as only identifying one as bigger) was classified a 'descriptive'. 'Linear' comparisons involved comparing the units along one, or possibly two linear dimensions. Finally, responses that only compared the shape of the two units were classed under 'shape'.

	Rational	Descriptive	Linear	Shape	None / unclear
Percentage (Q1)	36	28	5	10	21
Percentage (Q2)	58	17	3	5	19

Table 1: Comparisons of unit areas

The second criteria for analysing student comparisons, related to the comparison of the two measured values. Five different response types were identified: ***Rational, additive, descriptive, unit area, conflated.*** The first three types compared the measured values, but in different ways: 'rational' responses provided a rational (multiplicative) relationship, 'additive' responses provided an additive (difference) relationship and 'descriptive' provided only a qualitative comparison. The other two response types showed a lack of distinction between the measured value and the unit area: responses classed as 'unit area' compared the area of the units, while those classed as 'conflated'

showed a conflation of the measured value with the area attribute, for example, a change in value was discussed as a change in area.

	Rational	Additive	Descriptive	Unit area	Conflated	None / unclear
Percentage (Q1)	2	3	21	40	0	35
Percentage (Q2)	21	3	9	23	6	38

Table 2: Comparisons of measurement values

Only 26% (Q1) and then 33% (Q2) of the students identified the measured values as salient for this question, distinguishing these from the unit areas – with 2% (Q1) and 21% (Q2) comparing these rationally. A sizeable percentage of students, 40% (Q1) decreasing to 23% (Q2), interpreted this question as requiring a comparison of the areas of the two units and a similar proportion provided no, or unclear answers. This suggests that, even after the course, approximately two thirds of these students did not effectively distinguish between the attribute of area and the area measure.

The third criteria involved relating the two comparisons. Three classes were identified in this analysis: ***Appropriate, descriptive*** and ***no relating.*** Responses that showed (implicitly or explicitly) the reciprocal relationship between the two comparisons, were classed as 'appropriate'. Those providing a purely qualitative comparison were termed 'descriptive' and responses that did not relate the area and value comparisons were classified as 'no relating'. Many of these responses were a repeat of one or both of the comparisons).

	Appropriate	Descriptive	No relating	None / unclear
Percentage (Q1)	7	10	45	38
Percentage (Q2)	21	8	26	45

Table 3: Relating comparisons

It is interesting that the 21% who compared the measured values appropriately (table 3), also identified and used the appropriate reciprocal relationship between value and unit area. However, even at the end of the course, the majority of students – 83% (Q1) and 71% (Q2) – did not clearly relate the unit and measured value comparisons. It appears from these results that the students do show some progression of learning, but that by the end of the course, the majority of students did not display stable conceptual mastery of the process of area measurement in this situation.

An interesting observation emerged about student responses to the second area question. In this question the first unit was a square and the second was an isosceles right angled triangle, that could be constructed by subdividing the square along the two

diagonals. This triangle was similar to, but half the area of, the triangle formed by bisecting the square by a single diagonal. Even though the two units were drawn to scale, it was remarkable that 36% of the students interpreted the second unit as the triangle that formed half the square. This may have been a slip, where the similarity in shape was sufficient to convince the students that this choice was good even though the area was not conserved. But even so, the dominance of shape over conservation again suggests the lack of stability, even fragility, of these students' understanding of area measurement.

Question 3: Distance, speed and time

The third question required students to relate time, distance and speed for two people travelling the same route in different ways. The question asked the students to show that two people who sharing a bicycle equally (one riding then walking, the other walking then riding) when they both ride at the same speed and walk at the same speed, results in them both taking the same length of time for the journey. Responding appropriately to this question requires at least a qualitative conceptual understanding of the general attributes of distance, speed and time and the relationships between them. But it does not require measurement to quantify these attributes.

It was apparent in student responses that all the students who explicitly mentioned distance, speed or time, did so in such a way that indicated at least a qualitative understanding of this attribute. Three different ways of mentioning this attribute were noted. These were a descriptive/comparative mention, a special case example of a possible measurement, and a more general discussion that showed the students awareness that this attribute could be quantified.

	Quantifiable (Q)	Special case (S)	Descriptive (Desc)	None mentioned
Distance (%)	74	9	2	15
Time (%)	61	8	18	14
Speed (%)	44	5	11	41

Table 4: Students demonstrated understanding of each attribute

In the case of distance and time, over 80% of the students showed at least a qualitative understanding of the attribute. When it came to speed, 59% of the students explicitly showed such an understanding.

The second response characteristic that was analysed was the nature of the argument provided by the students. Five categories of argument were identified: ***Explicit, Implicit / incomplete, Special case, Descriptive*** and ***None / inappropriate***. 'Descriptive' arguments were those that included no more than descriptive comparisons of any of the attributes relevant to the problem. 'Implicit / incomplete' arguments drew on the relationship between the attributes, but did not explicitly detail either this relationship, or the observation that two journeys over the same distance at

the same speed would take the same time. 'Special case' arguments presented specific values for at least some of the attributes and drew the general conclusion from this. 'Explicit' arguments explicitly mentioned either the relationship between attributes, or the observation that two journeys over the same distance at the same speed would take the same time. Every explicit response provided was also complete and correct and so these characteristics were not separated in this analysis.

	Explicit	Special case	Implicit / incomplete	Descriptive	None / inappropriate
Percentage	6	6	29	44	16

Table 5: Argument types

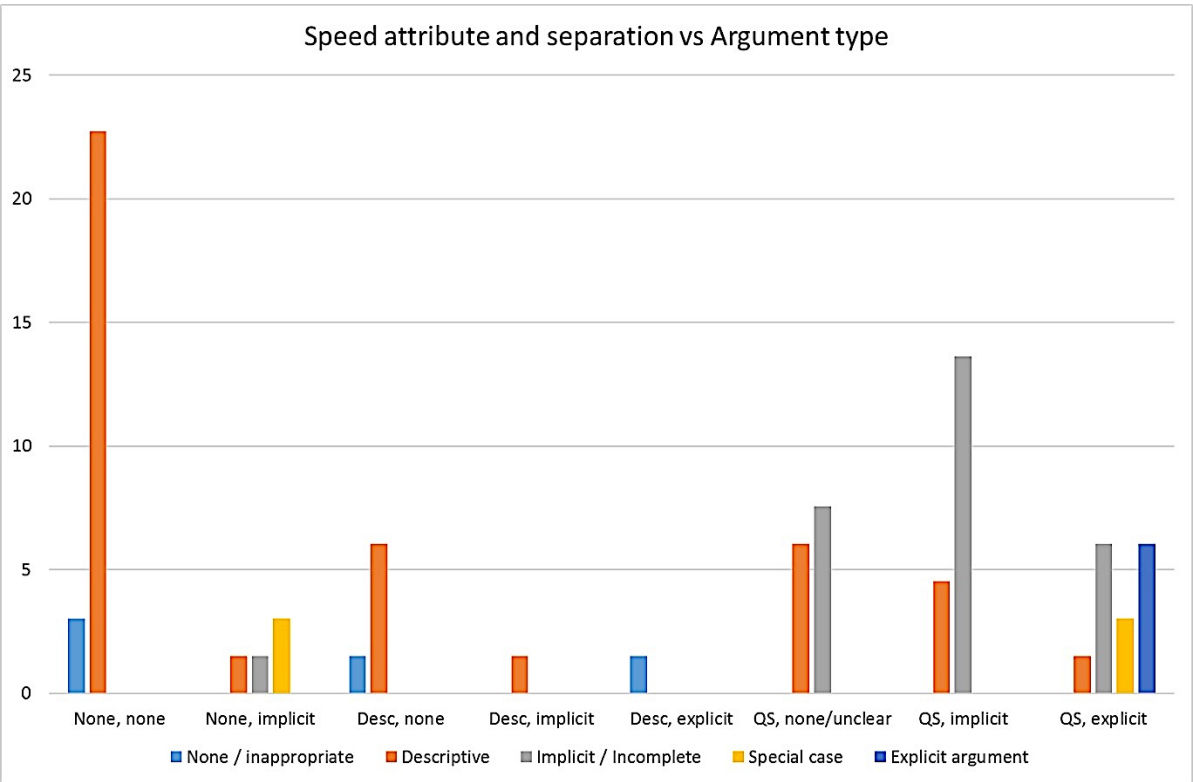
It was noted that the students' understanding of speed showed some relationship to the argument they provided in response to this question – only 3 students (5%) who did not mention speed, provided more than a general and often vague discussion of possibilities.

A third response characteristic was also identified as well related to the arguments provided by students. This was whether or not students were able to identify the particular properties that possessed each of these attributes and were relevant to this problem: the attributes of the person riding, the attributes of the person walking, the attributes of the friend riding, and the attributes of the friend walking. The following categories relating to this separation, were identified: ***No response, no separation, unclear, implicit*** and ***explicit.*** For 'unclear separation', some attempt was made to identify different properties, but this identification was not clear. An argument showed some appropriate awareness of the separate properties, even though they were not explicitly identified was classed as 'implicit'. An 'explicit' argument showed some explicit identification and relating of the particular properties corresponding to at least two of the attributes involved.

	Explicit	Implicit	Unclear	No separation	No response
Percentage	23	23	11	36	9

Table 6: Separation of particular properties vs Presentation of speed

The identification of particular properties possessing the different attributes also appeared closely related to more precise and less descriptive arguments. For this reason, these proportions are cross tabulated against the students' presentation of the speed attribute and related to the students' argument form, as shown in graph 8. It appears from this graph that two factors that may have contributed to the precision of the argument: Firstly, the understanding of speed as a quantifiable attribute related to distance and time, and secondly the identification of specific properties possessing these attributes, that were appropriate to the problem.



Graph 7: Comparison of argument types

CONCLUSION

This research supports the findings of Passelaigue and Munier (2015) that many prospective elementary school teachers do not distinguish appropriately between the concepts of 'attribute' and 'measurement', when using mathematics to make sense of the world.

It also suggests a third concept that could be included for consideration in order to extend our insight into students understanding of this process. This is the concept of 'properties' – particular properties of objects or actions that possess a general measurable attribute, for it appeared that the capacity to identify and distinguish between two different properties that corresponded to the same general attribute and then relate the differing values of the attribute, was a contributing factor to the precision of students' description and argument when using mathematics to make sense of a contextual situation. It may be this precision that students are considering in their incorrect responses to Passelaigue and Munier (2015), that measurement makes an attribute of an object precise, rather than providing an approximate value of the precise attribute.

Even in the simple case of the cardinality of a set of numbers, it was possible for students to identify different properties as corresponding to this attribute. This suggests the possibility of confusion due to the standard use of the term 'number' to refer to digits in South African schools.

These observations suggest that students need a greater exposure to this three layered process of measurement for sense making, together with the development of the

capacity to conceptualize, discuss and work with these analytical distinctions in their mathematical practice and their teaching.

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VIMAS_NUM: MEASURING SITUATIONAL PERCPETION IN MATHEMATICS OF EARLY CHILDHOOD TEACHERS

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Situation-specific skills (situational perception, interpretation, decision-making) as a structural relation between teachers' knowledge and their actual teaching is gaining more and more importance in educational research. However, there are multiple theoretical approaches and a lack of standardized instruments, especially for early childhood teachers. The project Vimas_num addresses this gap by developing a) a coherent and substantive definition of the construct 'situational perception' for early childhood teachers and b) a video-based instrument to measure this construct. But although we followed a proved development approach with several cycles, our results show that our operationalization is not yet successful. This gives reason to discuss the complexity of the construct and especially its measurement via standardized test instruments.

INTRODUCTION: SITUATION-SPECIFIC SKILLS AS PART OF TEACHER COMPETENCE

Competence models describe teachers' professional competence in their structure (Gasteiger & Benz, 2018a; Jenßen et al., 2015, Lindmeier, 2011). Concerning this competence structure different authors currently promote approaches that conceptualise competence as a continuum integrating (math-related) knowledge and beliefs on the one side, skills in actual teaching (performance) on the other side and situation-specific skills as a structural relation in between (Blömeke, Gustafson, & Shavleson, 2015; Gasteiger & Benz, 2018a). Theoretically, it is assumed that these

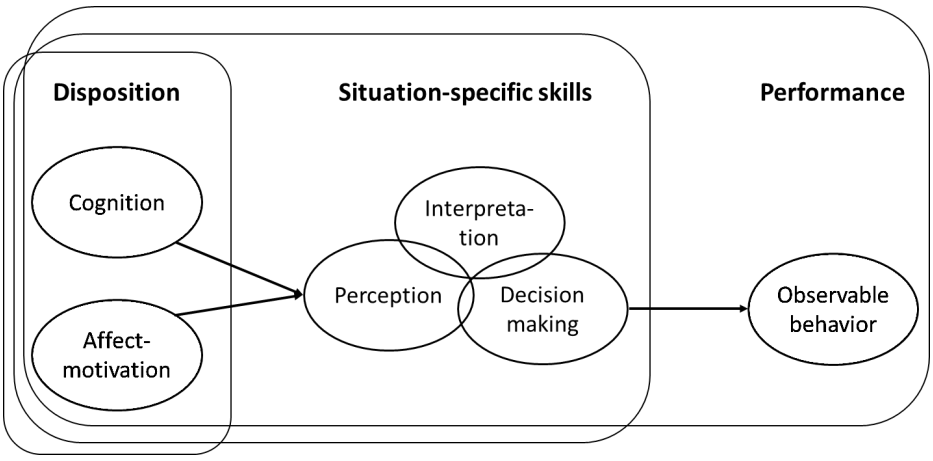

Figure 6: Competence as a continuum (Blömeke et al., 2015, p. 7)

situation-specific skills function as a bridge between disposition and performance (see

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Figure 1). The situation-specific skills are further differentiated in situational perception, interpretation and decision-making.

To get an overview over the state of research on teachers' situation-specific skills in mathematics, Stahnke and colleagues (2016) conducted a systematic review. They found 60 different studies focusing on teachers' perception, interpretation, and decision-making. In their analysis Stahnke et al. (2016) found meaningful differences in the theoretical foundation as well as in the research methods used. They state that the studies use multiple terms to describe the same theoretical aspects or vice versa that the same terms were used to describe different theoretical aspects. Concerning the research methods, they found a variety of research instruments used, but only 10 studies collected data with a standardized test instrument. Additionally, only two studies integrated a sample of early childhood teachers. These results indicate a lack of standardized instruments measuring situation-specific skills but also a research gap concerning the situations specific-skills of early childhood teachers. Additionally, the findings call for a clear definition of the construct of situation-specific skills as a whole but also of situational perception, interpretation and decision-making in detail in order to soundly operationalize each of the given constructs. This led us to the question: 1) How could early childhood teachers' situation-specific skills be conceptualized? To answer this question, we firstly outline the context of early childhood teaching and summarize the theoretical foundation on early childhood teachers' situation skills. We will focus on early childhood teachers' situational perception, as perception seems to play a central role in early childhood teachers' competence structure (Dunekacke, 2016; Gasteiger & Benz, 2018a). This, as well as the lack of standardized research measurements led us to our second question: 2) How can early childhood teachers' situational perception be measured? To answer this question, we present our attempt to develop a standardized measurement approach by outlining the steps of the development process and their results.

SITUATION-SPECIFIC SKILLS IN THE EARLY CHILDHOOD CONTEXT

Lately, several authors have highlighted early childhood teachers' mathematics-related competence and its relevance to children's mathematical learning (e. g. Dunekacke, Jenßen, & Blömeke, 2015; Gasteiger, 2014). The job of early childhood teachers differs from the job of school teachers as early mathematics education often takes place in natural learning settings and has a less formal character than learning in school (Gasteiger, 2014; van Oers, 2010). To guarantee that children benefit from these more informal mathematical learning situations, early childhood teachers do not only need theoretical based knowledge but additionally sound situation-specific skills. The conceptualization of these situation-specific skills is based on the demands of early mathematics education. In this context, *situational perception* is described as identifying mathematics in children's play and recognising everyday situations with mathematical potential (Björklund & Barendregt, 2016; Gasteiger & Benz, 2018a). *Interpretation* is related to the mathematical development of the child and *decision-making* focuses on the planning of mathematical activities for children as well as the

spontaneous act of offering support in a natural learning situation (Gasteiger & Benz, 2018a).

In a typical kindergarten situation with mathematical learning potential an early childhood teacher has to activate his/her situation-specific skills based on his/her knowledge. To use the learning potential of this situation, the early childhood teacher firstly has to interpret this situation as one relevant for mathematical learning (van Oers, 2010). This interpretation in turn depends on the mathematical aspects the early childhood teacher perceives in the given situation. If the early childhood teacher cannot perceive the mathematical potential of the situation or only parts of it, (s)he will most likely not use the full potential of the situation to support the mathematical learning of a child (see also Björklund & Barendregt, 2016; Gasteiger & Benz, 2018b). Following this line of argument, we do assume that situational perception is based on theoretical knowledge and that sound skills in perceiving mathematical concepts in everyday situations in kindergarten can be seen as a precondition not only for interpretation and decision-making but also for good mathematics teaching in early childhood.

Some empirical results also lead to the assumption that early childhood teachers' situational perception plays a central role in early mathematics education: Dunekacke and colleagues (2015) find relations between professional knowledge and mathematics-related perception and decision making. Additionally, Gasteiger and Benz (2018b) show qualitative results indicating an impact of situational perception on early childhood teachers' pedagogical and didactical actions. Lee (2017), however, found no statistical relation between perception and interpreting. These contradictory results concerning the relevance of situational perception could, as also Stahnke and colleagues (2016) argued, be based on either different theoretical foundations or different methodological approaches or both, which requires a closer look at the situational perception of early childhood teachers.

EARLY CHILDHOOD TEACHERS' SITUATIONAL PERCEPTION CONCERNING MATHEMATICS

Situational perception was picked out as a field of research by Goodwin (1994) under the keyword professional vision. He defined professional vision as "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606) and reasons that "the ability to see relevant entities is lodged not in the individual mind but instead within a community of competent practitioners" (p. 626). Van Es and Sherin (2002) adapted this concept of professional vision to the context of mathematics education. They called this facet of teacher competence 'noticing' and defined it as "learning to identify what is noteworthy about particular situations" (p. 573) which "involves using what one knows about the context to reason about a situation" (p. 574).

In addition, at the field of early childhood education situational perception is discussed from a diagnostical point of view and described by the term observation or pedagogical documentation (e.g. Heiskanen, Alasuutari & Vehkakoski, 2018; Schulz, 2015). The

focus of this line of research lies on teachers' skills in and methods of observation and pedagogical documentation of children's development (e.g. Knauf, 2015).

The concept of situational perception of early childhood teachers concerning mathematics can be traced back to aspects of both described research fields: professional vision/noticing in mathematics education and observation in early childhood education. Definitions of situational perception of early childhood teachers in mathematics stress, however, that situational perception is more than observing children's mathematical development: Gasteiger and Benz (2018a) define situational perception as recognising the mathematical relevance of everyday situations, Dunekacke (2016) as the identification of surface characteristics (as mathematical themes) and more sophisticated characteristics (as the level of development of children). Both definitions have in common that they do not only focus on the learning process of the child but also include the perception of the mathematical potential of different situations in a kindergarten setting.

RESEARCH INTEREST

Recently, more and more empirical research focuses on early childhood teachers' situational perception. Björklund and Barendregt (2016) assume based on survey data of Swedish early childhood teachers that teachers "seem to be quite perceptive of their environment and the mathematics that may be recognized within it" (p. 370) – except if it comes to situated mathematics learning which is especially important in early mathematics education. Additionally, Lee (2017) found differences in early childhood teachers' situational perception related to years of teaching experiences as well as educational background. Bruns and colleagues (in press) compared the situational perception of early childhood students with and without mathematical background and found that students with mathematical background identified more mathematical aspects in given video-situations. Although these results give a first impression of early childhood teachers' situational perception in mathematics, empirical research is still at the beginning. This is partly related to a lack of instruments to measure this comparably new and complex construct: Only the study of Dunekacke et al. (2015) uses a standardized test instrument, while the other studies use interview instruments (e. g. Lee, 2017) or qualitative survey data (e. g. Björklund & Barendregt, 2016). To get a more detailed picture of early childhood teachers' situational perception in mathematics, standardized instruments that can be used in quantitative studies are needed. The development of such an instrument is the aim of the project *Vimas_num*.

METHODS

The development of the test-instrument followed Wilson's (2005) *Four Building Block* approach (see Figure 7): In a first step, the construct map defines the content of the construct 'early childhood teachers' situational perception concerning mathematics' and its characteristics concerning the underlying continuum 'not able to perceive mathematics' to 'expert in situational perception'. Afterwards, we designed items that seemed suitable to measure this construct. The outcome space is the basis for the scoring of these items. Finally, the measurement model relates the outcome space to the construct and helps us to interpret the item scores in relation to the construct.

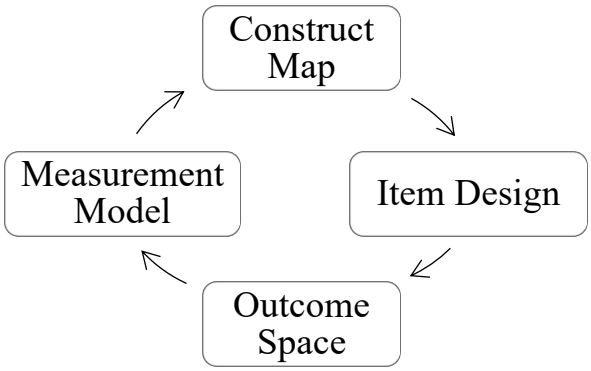

Figure 7: Construct Modeling: The "Four Building Block" Approach (Wilson, 2005)

Test instrument

As described, early childhood teachers' situational perception concerning mathematics is defined as recognising mathematical aspects in everyday (play) situations (Duneckacke, 2016; Gasteiger & Benz, 2018a). Hence, items measuring this construct need to be based on concrete situations. To realize this, different authors propose a video-based measurement approach (e.g. Blömeke, 2013; Dunekacke, 2016; Lindmeier, 2013).

Decide for each aspect, if it describes the mathematical content of the situation concerning set and numbers correctly.

	Yes	No	I did not know the answer
(1) Comparison of sets		X	
(2) Seeing 5 items at a glance		X	
(3) Ordering numbers		X	
(4) Numbers as representation of stets	X		

Figure 8: Example of the items 1.1 to 1.4, the cross indicates the right answer

Following this proposal, we developed four video vignettes showing typical mathematical activities of children in a real kindergarten setting. To these vignettes, we collected open answers via a written questionnaire from n = 54 pre-service and n = 52 in-service teachers to the question: Which mathematical aspects do you see in the given situation? After several considerations concerning the item design and a pilot study with multiple-choice items, we developed - based on the open answers - four complex multiple-choice items to each video. For each item participants have to decide whether the given mathematical aspect is part of the situation. Figure 3 shows item examples.

Sample and item analysis

The 16 video-based items (four to each video) were piloted with a sample of N = 91 early childhood teachers. The teachers had visited a professional development course on early mathematics education. They were average 41.58 years old (*SD* = 10.66, *Min.* = 21, *Max.* = 62) and had work experience from 1 to 43 years (*M* = 16.13, *SD* = 10.13). 83 of the participants were female, 8 participants were male.

To examine item quality, we used descriptive analysis and checked if the 16 items fit to the 1PL Rasch-model (software package eRm, Mair, Hatzinger & Maier, 2018).

RESULTS

Looking at the descriptive results, we found two items that were solved by nearly the whole sample: Item 1.4 was solved by 93.0% (*SD* = .25) and item 4.4 by 99.0% of the sample (*SD* = .10). Therefore, both items were excluded from further analysis. The 14 items left and included in the modelling showed solution rates from 21.0% to 89.0%.

To check, if the items left fit to the 1PL Rasch-model, we used the Wald test, which examines whether the items are unidimensional, and mean square fit statistics. Looking at the results of the Wald-test, seven items seem to be problematic: one item to video 1 and 4, two items to video 3 and three items to video 2. Concerning the mean square fit statistics, no further items showed a misfit. However, five of the remaining six items are quite easy and calculation of Cronbach's α fails due to negative correlation between the six items, indicating that the operationalization of the construct is not yet satisfying.

DISCUSSION

As stated there is a need for standardized test instruments to measure early childhood teachers' situational perception concerning mathematics. Within our project, we made a proposal for a clearly defined construct of early childhood teachers' situational perception. Based on this construct, we made a first attempt to operationalize early childhood teachers' situational perception. However, the developed items did not prove to be a reliable measurement, although we followed a proved development approach with several cycles and data analysis showed a fit of six items to the Rasch-model. This shows the complexity of the construct *situational perception* and its measurement.

Looking at the problems of the measurement model, three reasons could be identified: One reason could be the small sample of items and participants. Another reason could

be, that the items developed are quite easy and therefore do not discriminate well between the participants. As Wu et al. (2016, p. 152) put it: "if all items are equally "bad" ([…] low discrimination power), the items will still show good fit, because they have equal discrimination". Additionally, our analysis did not consider that four items each were nested within a situation. One possible approach to overcome these problems would be to group the items of each video to one item and use a partial credit Rasch-model instead of the 1PL Rasch-model. This model needs, however, a larger data basis – considering the sample size but also the number of items.

The next steps will therefore be a careful analysis of the problematic items by using inter alia a qualitative think aloud study. Based on these results, we will revise our existing items and develop further vignettes and accompanying items. The success of these steps will be examined by another pilot study with a larger sample of participants.

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INVESTIGATING MATHEMATICAL ERRORS AND IMPRECISIONS IN CONTENT AND LANGUAGE IN THE TEACHING OF ALGEBRA

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The purpose of this paper is to describe one characteristic of instruction focused on mathematical errors and imprecisions in content and language that instructors make while teaching algebra in community colleges within the United States. Findings from our study reveal that instructors' mathematical errors and imprecisions relative to language and notation are present during instruction, which have the potential to lead to a lack of clarity and student misconceptions.

INTRODUCTION

For the past decade, many states in the U.S. have implemented the *Common Core State Standards for Mathematical Practice* (SMPs) in elementary and secondary education (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). One of these practices includes *SMP6: Attend to Precision* which states that students should learn to communicate precisely and accurately in both their writing and verbalization of mathematical ideas. As a companion publication to support the implementation of the *Common Core* in the U.S., the National Council of Teachers of Mathematics published *Principles to Actions* (NCTM, 2014) to illuminate the Mathematical Teaching Practices (MTPs) that instructors should weave into every lesson in order to build students' mathematical practices. One of these practices includes *MTP3: Use and Connect Mathematical Representations* where instructors are expected to "engage students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 24). In order to illuminate mathematical connections and help students attend to precision, instructors need to use mathematically precise language and notation when making their thinking public in order for students to develop appropriate and robust ways of knowing mathematics. Otherwise, the potential for a lack of clarity of the mathematics can persist and ultimately, student misconceptions of the content may develop.

This study is situated in U.S. community colleges (tertiary-type B institutions), which offer courses in the first two years of post-secondary education (OECD, 2017). These institutions provide advanced mathematics courses (such as Calculus and Differential Equations), as well as the prerequisite algebra courses needed for all science, technology, engineering, and mathematics degrees. Community colleges serve almost

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half of all U.S. post-secondary students (Snyder, de Brey, & Dillow, 2018), yet little research has been conducted to understand the quality of instruction in these colleges. With the emphasis in secondary education on attending to precision, as well as using and connecting mathematical representations, it is important to understand the ways in which content and language precision is (or is not) enacted in community colleges.

THEORETICAL FRAMEWORK WITH SUPPORTING LITERATURE

Consistent with Cohen, Raudenbush, and Ball (2003), we believe *instruction* is defined as the interaction among faculty, students, and content that is embedded in particular contexts and evolves over time. We contend that this belief about instruction acknowledges the nature of knowledge exchanges between the instructor and students, and ultimately recognizes that communication among these individuals is critical for developing an understanding of the content. Evidence in secondary mathematics suggests that, under certain conditions, students do learn more mathematics and that understanding the conditions for which teaching causes learning is critical for improving education (Hiebert & Grouws, 2006). Specifically, we believe that attending to the nature of communication exchanges with a lens towards precision and connections can illuminate the conditions that allow students to learn.

One strategy for teaching mathematics is for instructors to leverage students' mathematical errors. Borasi (1996) wrote that making errors in mathematics can serve as productive sources for learning opportunities. Although using student errors to leverage a learning opportunity is shown to have instructional benefits, un-identified instructor errors or imprecisions cannot productively advance student learning. Kalder (2012) stated "When students hear correct language being used consistently and are asked to do so themselves, their understanding of mathematical concepts and procedures will no doubt improve" (p. 91). This area of interest serves as the basis for this investigation which specifically attends to the utterances and written notation provided by the instructor while teaching algebra. We assume that the nature of communication exchanges has a significant impact on students' development of key mathematical ideas, and that attending to precision when communicating mathematics is essential to students' learning of the content.

A common practice in teaching mathematics is to use everyday language when describing and explaining ideas. For example, instructors may use phrases such as "plug in" when describing the process of evaluating a function at a given input value. Another example is the use of "cancel" when instructors are describing the process of dividing common factors. The use of pronouns (e.g., it, this, that) to refer to various mathematical expressions and objects is also a common practice, yet an overuse of such pronouns when speaking mathematically can often obscure the interpretation and mathematical meaning for students. Adams (2003) suggests that instructors can facilitate students' recognition and use of technical language instead of relying on informal language to illustrate mathematical concepts and meanings to everyday words and "by explicitly evaluating students' ability to use technical language appropriately" (p. 149). In light of this movement towards mathematical precision, we were curious

how instructors at the post-secondary level attended to the mathematical precision envisioned for students. Thus, this paper addresses the following research question: *In what ways do errors and imprecisions in content and language surface during the teaching of mathematics in community college algebra courses?*

METHODS

The wider focus of this study is on the various aspects of mathematics instruction at community colleges, captured through video-recorded instructional episodes. Using the video coding rubric, *EQIPM: Evaluating the Quality of Instruction in Post-secondary Mathematics* (Cawley et al., 2018), we chose one characteristic of instruction, *Mathematical Errors and Imprecisions in Content or Language* (MEICL), that was intended to capture instructional events that are mathematically incorrect or that demonstrate problematic uses of mathematical ideas, language, or notation. This characteristic applies to the work and utterances of the instructor in the course. The EQIPM builds upon the work in secondary mathematics by Hill et al. (2008) and Litke (2015) and extends their work to the community colleges.

During the Fall of 2017, trained observers video-recorded 2-3 whole-class sessions from 44 instructors covering mathematical topics such as linear, rational, or exponential equations and functions. The instructors, who volunteered to be observed, were located at six diverse community colleges, which represent a range of institution size, degree of urbanicity, and region (Southwest, Midwest, and Central in the U.S.), and student background. In total, 143 class sessions were video-recorded with over 190 hours of instruction. The EQIPM rubric was used by 17 trained coders to rate the characteristics of instruction that appeared in each 7.5-minute segment (n = 1,576 coded segments of which 1,236 are full 7.5-minute segments) of every class session (n = 143). A rating of 1 to 5 representing the quality of instruction was assigned by coders for each segment. Along with assigning a rating, coders provided supporting evidence for each rating they assigned by documenting specific times and a brief explanation of the quality of the characteristic based on the guidelines described in EQIPM. For the MEICL characteristic, a rating of 1 indicated that no errors or imprecisions were made or that errors or imprecisions were clearly corrected within the segment. A rating of 3 indicated that content errors and/or imprecisions occur in the segment or that errors and/or imprecisions obscure the mathematics, but for only part of the segment. A rating of 5 indicated that content errors and/or imprecisions occur in most or all of the segment, errors and/or imprecisions obscure the mathematics of the whole segment, or errors and/or imprecisions disrupt the opportunity for students to make sense of the content. Ratings of 2 or 4 were reserved for "in between" cases. For example, a 4 was assigned when the error and/or imprecisions were more egregious than that of a rating of 3, but were not quite as persistent as those that received a rating of 5. We chose to specifically look at this characteristic since this particular code did not load with any of the other characteristics in EQIPM when a factor analysis was performed. Furthermore, a similar instructional characteristic from the Hill et al. (2008) and Litke

(2015) rubric did not load in their factor analysis, suggesting the need to take a closer look at the evidence of this characteristic.

The MEICL characteristic specifically focuses on the errors and imprecisions made by the instructor. Coders ignored intentional errors or imprecisions made by the instructor or errors that were corrected within the segment. For this paper, we focus on three areas within this rating: imprecision in language, imprecision in notation, and mathematical errors that muddle the understanding of the content. Imprecision in language include use of colloquial language to describe mathematical terms (e.g., cancel, top, bottom, over). Imprecisions in language can also include use of pronouns that makes it unclear as to what the instructor is referencing (e.g., this, that, move it over there). Notation includes conventional mathematical symbols (such as +, -, =) or symbols for fractions and decimals, square roots, angle notation, functions, exponents, etc. Imprecisions in notation might include inaccurate use of the equal sign, parentheses, or division symbol. Imprecision in notation can also include lack of accurate labelling in problems (e.g., not labelling important aspects within a graph such as the x- and y-axes or the function). Finally, mathematical errors are considered explicit mathematical mistakes that the instructor may commit. Errors could be instances where the instructor does not take into account all possible contexts (e.g., horizontal asymptotes are never crossed by the function, anything to the 0 power is 1), or contradicts given definitions or assumptions.

For this paper, we isolated segments that were rated as a 4 or 5 for the MEICL characteristic. A total of 35 segments received a rating of 4 (n = 31) or 5 (n = 4), from 17 instructors. Of those 17 instructors, four instructors had three or more segments that received a rating of 4 or 5. The content areas that received these ratings were rational and exponential equations or functions. Once we isolated the segments that received a 4 or 5 rating, we analysed the coder comments provided for each rating and identified the four instructors with three or more segments in this group. From here, we reviewed the video segments to further analyse relative to the MEICL characteristic to identify the context and potential subtleties associated with the rating given by the coder.

FINDINGS

The following discussion is organized based on imprecisions in language, imprecisions in notation, and mathematical errors presented during instruction. Among the 35 segments rated as a 4 or 5, *imprecisions in language* was the most prevalent. One of the most common instances included use of colloquial language or use of pronouns. For example, when teaching about how to translate between logarithmic and exponential notation, instructor 41322 used imprecise words and phrases to refer to mathematical statements or procedures, such as “on the bottom... wanted to move it up..." when referring to manipulations of fractions or referred to the argument in a logarithm as a “guy” (e.g., "this guy equal to this here..."). He used the phrase “cancel” throughout his description of logarithms to indicate principles such as $\log_b b^a = a$ where he explained that $\log_b$ and the argument b would “cancel out.”

During instructor 41022's lesson on how to simplify a complex fraction, the instructor's imprecise language left two students confused. At a certain point in the problem, students were expected to simplify the fraction by dividing common factors. The following solution was provided for the students:

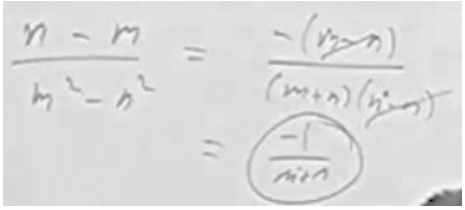

Figure 1: Board work for instructor 41022.

The instructor described his work that led to the simplified response of $\frac{-1}{m+n}$ by pointing to the factors $m-n$ and $n-m$ and saying "these almost cancel, but the negative is backwards." One student asked the instructor to clarify how he manipulated the numerator from $n-m$ to get $-(m-n)$. The instructor replied, "If we factor out a negative...all that does is just toggle the signs...this minus is left over" (pointing to the -1 in the numerator). The way that the instructor described his solution was unclear as he continuously referred to "factoring out a negative" rather than indicating that he was factoring out a negative one. He used the term "minus" and "negative" interchangeably and did not describe the mathematical result of distributing the negative into the factor to show that it was equivalent to the original. After a few minutes, another student indicated that she still did not understand how the instructor "got the one out" of the factor in the numerator. The instructor simply proceeded to explain his work in the same way as he originally offered to students. This explanation had many imprecisions in language through the use of pronouns, as well as imprecise mathematical terms, that clearly confused at least two students in this segment and his further explanation potentially did little to clarify the mathematics for the students.

Imprecisions in notation were also apparent in many of the segments that were rated a 4 or 5, though not as notable as imprecisions in language. In particular, there seemed to be imprecisions in writing function notation. For example, instructor 11423 spent time discussing five different transformations of the function $h(x)=\frac{1}{x}$. While he wrote $h(x)=\frac{1}{x}+1$ on the whiteboard, he also wrote the additional transformed functions without function notation or an equals sign to indicate an equation. He simply wrote the following expressions on the whiteboard: $\frac{1}{x}-1$, $\frac{1}{x-1}$, $\frac{1}{x+1}$, $\frac{1}{x-1}+1$. In this example, the instructor described each of these as being rational functions, however, did not use any function notation or an equals sign to indicate an equation as opposed to an expression. Similarly, instructor 10322 did not use precise notation when describing composition of functions. In an example of a function *f* composed with *g*, he opted to denote the composition as "fog" (in his written and verbal explanation) without attending to precision in notating the composition as $f(g(x))$ or $(f\circ g)(x)$.

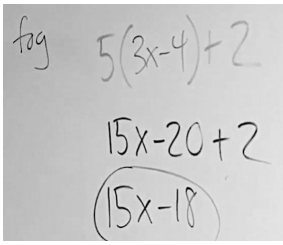

Figure 2: Board work for instructor 10322.

He also did not use an equals sign to indicate an equation, thus the idea of a functional relationship among inputs and outputs was non-existent (see Figure 2). In this example, his work implies that the result of a composition is an expression, which loses the critical idea that a function relates an input to its output, and that when composing functions, a new equivalent function is created.

Instructor 10322 also demonstrated imprecision in notation when he sketched graphs, ignoring important features such as labelling axes, using function notation, as well as indicating that the function continued beyond the scope of the sketch by using arrows at the ends of the function. Figure 3 shows what the instructor wrote on the whiteboard to illustrate the graph of the functions $f(x) = 2^x$ and $g(x) = 2^{x+2}$.

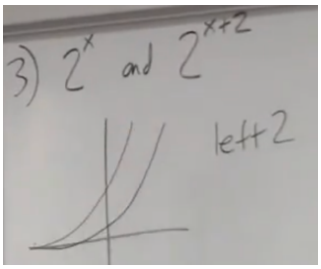

Figure 3: Imprecision in notation when sketching the graph of two functions.

In this example, neither of the functions are labelled in the graph, which may lead to confusion when identifying each function and their subsequent transformation. Anytime the instructor 10322 sketched graphs of specific functions, he consistently did not write the equation using function notation, indicating that he was sketching an expression which is erroneous. For example, previously in the lesson he showed the graph of 5^x without indicating a functional relationship between input and output values. The mathematical error in writing the function only as an expression removes the necessity of having an output, *y*, for the function with input of *x*. This instructor's work undermined the topic of functions, specifically notation, which is a challenging topic for students (Breidenbach, Dubinsky, Hawks, & Nichols, 1992).

Finally, we saw examples of *mathematical errors* in the work that instructors wrote on the whiteboard. A mathematical error came when instructor 11432 graphed the function $f(x) = \frac{x-1}{x^2-4}$. Using a table of values created by a graphing calculator, the instructor sketched the following on the whiteboard shown in Figure 4 (left graph):

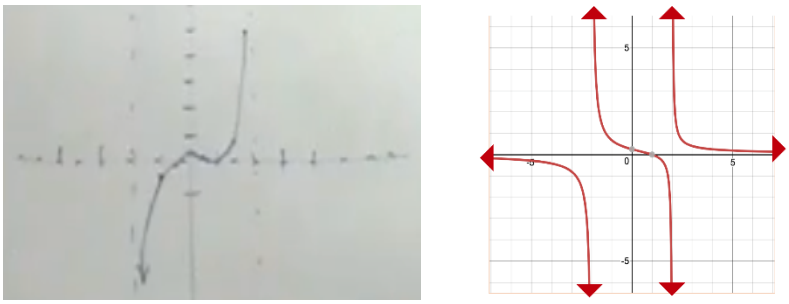

Figure 4: Instructor's graph of $f(x) = \frac{x-1}{x^2-4}$ on the left, correct graph on the right.

The graph has certain key pieces of information, including the two vertical asymptotes and the horizontal asymptote. However, the graph is incomplete; there are two sections of the graph missing and the function is incorrectly plotted: the function is positive when $-2 < x < 1$ and negative when $1 < x < 2$. While the instructor did indicate in a subsequent segment that the source of the error was that he had entered the equation incorrectly in the calculator, he did not correct the graph written on the whiteboard.

DISCUSSION

The current work demonstrates examples of imprecisions in language, imprecisions in notation, and mathematical errors that were detected in the video coding of instructional episodes. In particular, we noticed that many important mathematical ideas were reduced to singular words or pronouns that lost the rigor and meaning of the mathematics being discussed. The mathematical notation, specifically examples of graphing and using function notation, demonstrated what appeared to be shortcuts for writing the information on the board and had the potential to contribute to a lack of clarity of the mathematics for students.

We believe that making mathematical *errors* is a key component to learning, and that therefore, decomposing mathematical errors are an important process to better understanding mathematical content (Borasi, 1996). However, mathematical errors are only useful if they are recognized in the moment and those involved can engage in fruitful discussion of the error. In our case, we observed mathematical errors that were generated by the instructor and not resolved within the segment, which have the potential for students to develop misconceptions of the mathematics.

We see the mathematical *imprecisions* as the more subtle and challenging aspect of teaching. Instructors are often required to cover a vast amount of content in a short period of time, which may result in a lack of precision. However, when multiple imprecisions occur consistently and without connections to the appropriate mathematical language and notation, we believe that this practice can potentially contribute to students' development of mathematical misconceptions and lack of clarity that may further obscure students' understanding of the material.

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DIALOGUE AND SHARED COGNITION: STUDENT-STUDENT TALK DURING COLLABORATIVE PROBLEM SOLVING

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Student-student talk has been found to play an important role in facilitating classroom learning in mathematics across cultures. This report examines the socio-cognitive activity associated with meaning negotiation in the mathematics classroom. Terms such as co-cognition, inter-cognition, and intra-cognition are operationalised using data from student collaborative problem solving activity in a laboratory classroom to explicate the socio-cognitive aspect of meaning negotiation within a mathematics classroom setting. We argue that awareness of these different forms of socio-cognitive activity may have useful pedagogical implications for facilitating productive classroom dialogue.

BACKGROUND

The notion of shared cognition conceptualises cognition as a collective and social process beyond the individual. In the mathematics classroom setting, shared cognition can be seen as occurring during meaning negotiation between teacher-student and student-student interactions as students are initiated into the discourse of the mathematics classroom. In this report, we examine the socio-cognitive activity associated with individuals' participation in meaning negotiation through collaborative problem solving. In particular, we examine the socio-cognitive aspect of meaning negotiation by identifying indicators of shared cognition. The connection between these indicators of shared cognition and dialogic processes is discussed.

Socially shared cognition

Vygotsky's (1978) work on the sociocultural nature of learning generated high interest in the role that a more able other (e.g., a caregiver or a peer) might play in facilitating the learning of individual children, while also highlighting the cultural historical context of both knowledge and learning. This sociocultural view of learning directed researchers' attention to the importance of both the context in which learning occurs and the primacy of the culturally-framed social interactions through which learning occurs. Contemporary theories such as Distributed Cognition (Hutchins, 2006) and the Social Brain hypothesis (Dunbar, 1998) and terms such as *shared cognition* (Resnick, Levine, & Teasley, 1991) and *interthinking* (Littleton & Mercer, 2013; Mercer, 2000) continue the expansion of the notion of cognition from a strictly individualistic process to a collective process.

Rather than viewing social and cognitive perspectives of learning as dichotomous, the social constructivist view integrates both perspectives in conceptualising the learning process. In their effort to resolve the social and cognitive divide in psychological

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theory, Resnick and her colleagues (1991) used the term *shared cognition* to describe "cognition that is not bounded by the individual brain or mind" (Resnick et al., 1991, p. 1). Drawing from the work of Vygotsky (1978) and Mead (1934), Resnick asserted that the kinds of interpretative processes available to individuals can be shaped by social experience. Instead of focusing on how individuals think about social phenomena, social processes can be treated as integral to cognition, and examined in terms of the way in which people jointly construct knowledge through processes shaped by particular social purposes and interactions.

Classroom learning as meaning negotiation

In terms of our conceptualisation of classroom learning, the process of learning can be seen as the construction of knowledge by students through their interaction and participation within the classroom setting. In associating learning with participation in a community of practice, Lave and Wenger (1991) asserted that "participation is always based on situated negotiation and renegotiation of meaning in the world" (p. 51). They used the term *situated learning* to emphasise the relational character of knowledge and learning, the negotiated character of meaning, and the problem-driven nature of learning activity (p. 33). Clarke (2001) suggested that the presumptions of meaning are community, purpose, and situation, since "it is futile to discuss the meaning of a word or term in isolation from the discourse community of which the speaker claims membership, from the purpose of the speaker, or from the specific situation in which the word was spoken" (p. 36). The negotiation of meaning may be concerned with the substantive mathematical content that is the pretext for the social gathering called "a mathematics class" or it may be concerned with establishing a set of social obligations and responsibilities, without which neither a class nor a collaborative group will run smoothly.

Clarke (2001) posited that for negotiation to take place, there needs to be something that is unresolved and uncertain. The inclination of humans to want to achieve consensus or intersubjectivity, where *intersubjectivity* is described as "a mutual or taken-as-shared understanding of an object or an event" (Cobb & Bauersfeld, 1995, p. 295), can be seen as a driving force for the negotiation of meaning by the actors within the classroom setting. Cobb and Bauersfeld (1995) defined the negotiation of meaning as "the interactive accomplishment of intersubjectivity" (p. 295). While acknowledging the importance of intersubjectivity in the meaning negotiation process, Clarke (2001) reasoned that as meaning negotiation is mediated by language (or some form of conventionalised communicative process), which presumes intersubjectivity, some level of existing intersubjectivity is required for the negotiation to occur at all. In this conception, a level of student-student and student-teacher intersubjectivity is a prerequisite to the negotiative process, so that negotiation may proceed by the incremental refinement of intersubjectivity. This suggests that intersubjectivity is not only a goal but also a pre-condition for the negotiation of meaning within a classroom setting.

In order to further examine the shared cognitive process occurring through meaning negotiation within the mathematics classroom setting, we particularly focus our attention on the socio-cognitive aspect of meaning negotiation by identifying indicators of shared cognition. We employ the term *socio-cognitive* (Van Dijk, 2008) to both acknowledge the fundamentally social nature of human cognition (Mercer, 2016) and also to contest the image of a cognitively bounded individual evoked by reference to individual cognition or cognitive activity without explicit acknowledge of the fundamentally social nature of human cognition. We refer to the notion of *co-cognition* (as employed by Heal, 1998) and propose the terms *inter-cognition* and *intra-cognition* to distinguish different kinds of socio-cognitive activity involved during collaborative problem solving. These terms are operationalised using data from the Social Unit of Learning project, which involved student collaborative problem solving activity in a laboratory classroom.

THE STUDY

The Social Unit of Learning project was conducted in a laboratory classroom situated within the Melbourne Graduate School of Education at the University of Melbourne, Australia. One class of Year 7 students (26 students) provide the focus for this report. The class participated in a 60-minute session in the laboratory classroom involving three separate problem solving tasks that required them to produce written solutions. The students attempted the first task individually (10 minutes), the second task in pairs (15 minutes), and the third task in groups of four to six students (20 minutes). The tasks have the characteristics of allowing students to express their thinking through multiple modes (e.g., verbal, graphical, textual, etc.) and approach the task using different strategies or prioritise different forms of knowledge or experience.

The resulting data collected in the project included: all written material produced by the students; instructional material used by the teacher; video footage of all of the students during the session; video footage of the teacher tracked throughout the session; transcripts of teacher and student speech based on the video recording; and recording and transcripts of pre- and post-session teacher interviews.

CO-COGNITION, INTER-COGNITION, AND INTRA-COGNITION

The term *co-cognition* has been used by Heal (1998) to describe a person's capacity to mentally simulate the thoughts of another, akin to the theory of mind (Flavell, 1999). The term can be contrasted with *meta-cognition*, which refers to thinking about one's own thinking (Wilson & Clarke, 2004). In examining the data from the Social Unit of Learning project, we found instances of co-cognition when students explicitly tried to understand each other's perspective. For example, one of the pair tasks that was given to the students was specified as follows:

> The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer.

Students Anna and Pandit (pseudonyms; both female) were solving the task and Anna had drawn five circles on the working out sheet denoting the five people in the household as specified in the task. She wrote "25" above to the middle of the circle, circled the middle circle, and drew a line linking the number and the middle circle.

The pair then had the following conversation (the number denotes speaker turn and // denotes overlapping speech):

47 Anna: Twenty-five.

48 Pandit: Why are you saying that dude's 25? They don't have to be 25.

49 Anna: It - it - this one is 25 because that's the average.

50 Pandit: Average doesn't have to - doesn't mean that one guy has to be 25.

51 Anna: Oh okay, okay. That makes sense then.

52 Pandit: Altogether it's 125 because like ...

53 Anna: Yeah, yeah, yeah.

54 Pandit: And ...

55 Anna: Now, I get it. I thought that was //just 25.

56 Pandit: //Yeah, yeah. So one dude's 13. That means the other four is 112.

57 Anna: What do you mean? No. It can't - they can't all be like so equal.

58 Pandit: They're not. Oh my God. Look, so 25's one guy, right. No. It's like for, you know, average means like ...

59 Anna: I know, I know.

60 Pandit: Yeah. So 25 times five is the total, right?

61 Anna: Yeah. I know.

62 Pandit: So everyone's 125. And one guy is 13.

63 Anna: I know, one guy. So ...

The conversation between Pandit and Anna suggests that the students had different conceptions of average. For Anna, as shown in her writing, she imagined average to be the middle number among a group of numbers (Turn 25 "… this one is 25 because that's the average"). Pandit did not agree with Anna (Turn 48 Pandit: "They don't have to be 25.") and tried to explain to her that the specification of the average meant that the total of the five numbers had to be 25 times 5 (Turn 52 "Altogether it's 125 because like ...). Pandit's explanation was cut off by Anna's response (Turn 53 "Yeah, yeah, yeah."). When Pandit went further to suggest that they could subtract the Year 7 student's age (13 years) from the total of 125 (Turn 56 "So one dude's 13. That means the other four is 112."), Anna appeared to think that Pandit was implying that the other four people had to be of the same age (Turn 57 "No. It can't - they can't all be like so equal."). Her disagreement with Pandit prompted Pandit to trace back Anna's thinking (Turn 58 Pandit: "Look, so 25's one guy, right. …") and to try to explain to her understanding of Anna's thinking (Turn 58 Pandit: "It's like for, you know, average means like ..."). Although Pandit's attempt was cut short by Anna's response, Pandit's utterance "it's like for you" was an obvious indication that Pandit was trying to reflect

Anna's thinking back to her in order to resolve their disagreement. Pandit's attempt to explain to Anna her own thinking seems to indicate the socio-cognitive process that Heal (1998) considered as co-cognition. Anna also appeared to attempt to understand Pandit's thinking, as evident in her response "What do you mean?" (Turn 57) when she did not follow Pandit's statement "That means the other four is 112" (Turn 56).

Co-cognition can be contrasted with what we have called *inter-cognition*, which invokes the notion of extending or building upon the thoughts of another without the level of empathic insight presumed by Heal's (1998) term. We propose that inter-cognition occurs where the expressed thought of one individual stimulates or provides a platform or scaffold for the (further) thought of an interacting individual. There is a fundamental temporality to inter-cognition that is not required for co-cognition. For example, below is the conversation between Audrey and Katie (both female) after they have worked out their solution to the same task that Anna and Pandit were attempting. The conversation took place as they were in the process of writing out their final solution, with Katie doing the writing.

83 Katie: Let's write the explanation now. Why did we choose these ages?

84 Audrey: We chose these ages as we wanted a variety (laughs).

85 Katie: We wanted …

86 Audrey: Because they were all just - ah, I just stabbed myself with a pen. No. Does this have to be …

87 Katie and Audrey: (Laughter)

88 Katie: Forty-five, forty-five doesn't make a variety.

89 Audrey: Just say because we wanted a variety of ages. We know this is correct as … as we have used addition to add them all.

90 Katie: We …

91 Audrey: We used addition to …

92 Katie: No. We can't say it's correct because there could be many answers.

93 Audrey: Oh we know this is one of the many answers.

94 Katie: We know... (Laughs).

95 Katie: … the answers. As…

96 Audrey: As we have used addition to add these five numbers up.

97 Katie: No. We used all of them, divide everything, times.

98 Audrey: Subtraction. Division.

99 Katie: Multiplication.

100 Audrey: And multiplication to make sure our answer is precise.

101 Katie: Pretty sure I spelled that wrong (laughs).

102 Audrey: Pretty sure that's an "I".

103 Katie: You spell it.

104 Audrey: To make our answer as precise as it can be.

The above excerpt suggested a coordinated way in which Audrey and Katie tried to justify their solution in writing. Katie raised the question "Why did we choose these ages?" (Turn 83), and Audrey responded ("We chose these ages as we wanted a variety…"). Katie pointed out that the choice of two ages of 45 years does not constitute a "variety" (Turn 88), and Audrey responded by assuring her that the wording was fine and Katie followed by writing the sentence down. Audrey then tried to justify their answer by mentioning that they thought that their answer was correct, but Katie disagreed (Turn 92 "We can't say it's correct because there could be many answers"), and Audrey echoed Katie's point (Turn 93 "Oh we know this is one of the many answers"). There is a coordinated pattern in the way Katie and Audrey operate, where one makes a suggestion and the other person responds by extending or building on the person's suggestion. Even when there was a disagreement, the disagreement was resolved quickly, either by one person maintaining her stance (e.g., Turn 89 Audrey: "Just say because we wanted a variety of ages"), or adjusting her stance by matching the stance of the other person (e.g., Turn 93 Audrey: "Oh we know this is one of the many answers"). We suggest that the way in which Katie and Audrey expressed their thoughts and stimulated each other's thinking for further thought appears to characterise a different type of socio-cognitive activity to which we called inter-cognition.

Contrasting with inter-cognition, *intra-cognition* can be seen when two or more people regulate their own thinking as a cognising unit. We suggest that intra-cognition can be signalled by the prominence of "we" as the agentic pronoun. Statements that we would characterise as intra-cognitive make reference to, evaluate, or regulate the spoken contributions of all (both) participants in the student talk. In such intra-cognitive exchanges, it is the combined thoughts of the dyad that are the subject of the conversation. The excerpts of Anna and Pandit and of Audrey and Katie provide a useful contrast in illustrating intra-cognition. The frequent use of "you" and "I" in the conversation between Anna and Pandit (e.g., Turn 48 Pandit: "Why are <u>you</u> saying…" [Emphasis added]; Turn 59 Anna: "<u>I</u> know. <u>I</u> know.") suggests that the two students were not thinking of themselves as a single unit. Audrey and Katie, on the other hand, used the pronoun "we" a lot more frequently compared to the former pair in the excerpt (e.g., Turn 83 Katie: "Why did <u>we</u> choose these ages?"; Turn 100 Audrey: "… to make sure <u>our</u> answer is precise…"). The use of the pronoun "I" and "you" appeared in statements regarding individual actions (e.g., Turn 86 Audrey: "ah, <u>I</u> just stabbed myself with a pen"; Turn 101 Katie: "Pretty sure <u>I</u> spelled that wrong.") and signalled delegation of individual responsibility (Turn 103 Katie: "<u>You</u> spell it."). The use of "we" both accords the dyad status as a social unit and as the principal cognising entity.

SHARING COGNITION WITH DIALOGIC PROCESSES

By distinguishing student-student talk in terms of co-cognitive, inter-cognitive, and intra-cognitive, we are highlighting the socio-cognitive aspect of the meaning negotiation process, rather than the character of the reasoning evident in the dyadic interaction according to Mercer's categorisation (Mercer, 2000). We postulate that

these different kinds of socio-cognitive activity are crucial for students to achieve intersubjectivity during collaborative work as students each think about the other person's thinking (co-cognitive); build on each other's thinking (inter-cognitive); and regulate each other's thinking as a single cognising unit (intra-cognitive). Our categorisation focuses less on the nature of the argumentation and reasoning, and more on the level of intersubjectivity achieved during student-student talk through indicators of shared cognition.

In terms of teaching implications, we share the same educational aspiration with dialogic perspectives in terms of encouraging students' active participation in their own learning. We believe the socio-cognitive aspect of meaning negotiation that we highlighted suggests possible ways in which students could reflect on and regulate their shared thinking with others. Guiding questions that may facilitate student monitoring and reflection of their interactions with others during collaborative work may include: "Do I understand the other person's perspective?" (co-cognitive); "Are we building on each other's ideas?" (inter-cognitive); and "Are we thinking as a team or as separate individuals?" (intra-cognitive). These questions should complement classrooms that have already adopted a dialogic teaching approach which encourages communication, questioning, inquiry, reasoning, collaboration, and student empowerment (Alexander, 2018; Lefstein & Snell, 2014).

CONCLUSION

Through examining the socio-cognitive aspect of meaning negotiation within a mathematics classroom setting, we have found many useful connections between dialogic perspectives and our socio-constructivist view of classroom learning. Distinguishing between different kinds of socio-cognitive activity during collaborative problem solving in terms of co-cognitive, inter-cognitive, and intra-cognitive, foregrounds the level of intersubjectivity involved during the meaning negotiation process. We see the perspective of classroom learning presented in this report as complementary to the dialogic perspectives and encourages further thoughts and inquiry into the processes of teaching and learning within the classroom setting.

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INTEGRATING *DESMOS*: A CASE STUDY

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In this study I look at a high-school teacher's use of an online graphing calculator software, Desmos, in his high school classroom. Using Ruthven's Structuring Features of Classroom Practice framework to frame reported data and to identify expertise. The case study focuses on how the expertise of the teacher is constructed and enacted. In this research report, I highlight how the teacher, without formal training, uses Desmos to create a rich learning environment. The strategies reported in this study will also be helpful and potentially supportive of future integration of digital technology.

INTRODUCTION

When learning mathematics, digital technology can be a powerful learning tool (Sinclair, 2014). Engaging in mathematical tasks that draw on the dynamic and interactive affordances of digital tools can provide students rich and profound insight into mathematics (Kaput, 1992). Well established digital tools such as calculators, graphing and scientific, are commonly integrated in high school mathematics classes but they have limited affordances and are being replaced by newer digital tools.

Despite the potential benefits of newer technologies, however, integration in mathematics teaching is rare (Bretscher, 2014). In this paper, I focus on the role teachers can play. Teachers play an important and critical role in how digital technology is integrated (Charalambous & Hill, 2012). Teachers, however, are challenged significantly. Training and support of teachers in using digital technology is not common. Curriculum documents are frequently not prescriptive in naming a particular software for teachers to use. This makes the choices for teachers more difficult. One tool that addresses some of these challenges is *Desmos*. *Desmos* is an online graphing calculator that also provides a rich learning environment for students. *Desmos* has become popular among high school teachers in many jurisdictions across North America. The question this paper explores is what does *Desmos* provide for teachers and subsequently to their students that improves mathematics teaching and learning.

Using Ruthven's Structuring Features of Classroom Practice framework (2009), I look for innovations of the integration of *Desmos* by the teacher in this study. Since there is not a lot of research in teacher integration of digital technology this paper contributes to this gap. The results of this study can be helpful in planning professional development and in addressing curricular issues. This study provides teachers and researchers an opportunity to see what pedagogical approaches emerge in using *Desmos*.

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FRAMEWORK

Ruthven's Structuring Features of Classroom Practice framework (SFCP) (2009) offers a way to analyze teacher's implementations of digital technologies. The structuring framework lays a base by which to analyze digital technology. Ruthven shifts away from discussing teacher's knowledge of technology as a significant component of good implementation such as the TPACK framework which has been described as something that "misses the fundamentally new" (Pimm, 2014).

Instead, Ruthven widens the unit of analysis to include the material and technical environment, as well as teacher practices, as central aspects to good implementation. That is, it is a robust framework because it moves away from analysing isolated events and static knowledge and attempts to outline a set of categories that first define characteristics of a teaching situation and then follows up with dynamic practices of teaching that changes and develops over time. Ruthven calls this dynamic teaching aspect "expertise" (2014). An example of expertise may include how teachers respond to various situations that arise in the classroom. Below is an outline of Ruthven's framework. One finds the five categories in the left column, the "defining characteristics" in the middle column which can be thought of as the nouns such as physical set-up or concretized intentions like "templates for classroom action". It is within the parameters set in this column that the potential for action is established. The third column describes the "examples of craft knowledge". This column can be seen as the actions. It is particularly this second column that draws attention to what teachers do in the context of the first column.

Ruthven (2014) describes his framework as a "top level" construct and says "further studies are now required in which data collection (as well as analysis) is guided by the conceptual framework, so that it can be subjected to fuller … elaboration and refinement" (p. 388). Bozhurt and Ruthven (2017) argue that to contribute to the framework a "systematic inventory of such types of knowledge" (p. 326) need to be generated. One way this study contributes to this inventory is to analyze "new structures of interaction involving student, teacher and machine and the appropriate (re)specification of role" (p. 391).

Structuring feature	Defining characterisation	Examples of associated craft knowledge related to incorporation of digital technologies
Working environment	Physical surroundings where lessons take place, general technical infrastructure available, layout of facilities, and associated organisation of people, tools and materials	Organising, displaying and annotating materials Capturing or converting student productions into suitable digital form. Organising and managing student access to, and use of, equipment and other tools and materials Managing new types of transition between lesson stages (including movement of students)
Resource system	Collection of didactical tools and materials in use, and coordination of use towards subject activity and curricular goals	Establishing appropriate techniques and norms for use of new tools to support subject activity Managing the double instrumentation in which old technologies remain in use alongside new Coordinating the use and interpretation of tools
Activity structure	Templates for classroom action and interaction which frame the contributions of teacher and students to particular types of lesson segment	Employing activity templates organised around predict-test-explain sequences to capitalise on the availability of rapid feedback Establishing new structures of interaction involving students, teacher and machine and the appropriate (re)specifications of role
Curriculum script	Loosely ordered model of goals, resources, actions and expectancies for teaching a curricular topic including likely difficulties and alternative paths	Choosing or devising curricular tasks that exploit new tools, and developing ways of staging such tasks and managing patterns of student response Recognising and responding to ways in which technologies may help/hinder specific processes and objectives involved in learning a topic
Time economy	Frame within which the time available for class activity is managed effectively so as to convert it into components of "didactic time" contributing directly to desired student learning	Managing modes of use of tools so as to reduce the "time cost" of investment in student familiarisation with them or to increase the "rate of return" in terms of student learning Fine-tuning working environment, resource system, activity structure and curriculum script to optimise the return on time investment in terms of student learning

Table 1: Components of the structuring features framework (Ruthven, 2014, p. 387)

Desmos offers capabilities beyond the traditional graphing calculator. How might a teacher take advantage of these functionalities? And specifically, what obstacles and challenges do they encounter? What strategies, framed in this study as expertise through the SFCP framework, have they found to mitigate the challenges?

CONTEXT

Description of *Desmos*

Desmos is an online web-based calculator (www.Desmos.com) that has calculating, graphing and geometric functionalities. In many North American jurisdictions, *Desmos* has replaced handheld graphing calculators because it is free (although an internet connection is mandatory), it has more accuracy, and more functionality. It can be used on a computer, iPad, tablet, or smart phone.

Desmos also offers a set of guided learning activities written by a team of teachers, educators and researchers. The activities are organized in the form of slides presenting students with interactive prompts that ask them to interpret and predict. An affordance of the *Desmos* activities include the functionality to list all students' predictions on all students' screens so that not only are students aware of their peers' predictions but also, they have the opportunity for refining their own prediction based on what their classmates have predicted. In many activities, *Desmos* prompts students for a revised personal prediction after classmates' predictions are listed. Teachers have control over the activities by the function "pace". The pacing function sets a limit to how far a student can progress in an activity. A teacher can put a limit on slide seven, for example, which may allow the teacher to "discuss the screen" (Ruthven, 2014, p.385).

Case study

A case study was undertaken to observe the implementation of *Desmos*. This was an exploratory, non-interventive, observational study that reports what was observed through the lens of Ruthven's framework. This study was carried out in a public school in the province of British Columbia, Canada. Multiple teachers were observed over many lessons. This research reports on one teacher, Andrew, and a single lesson in which he implemented *Desmos*. The class episode presented highlights a good example of integrating *Desmos* in terms of showing the expertise of the teacher.

Data was collected through video recordings of the class. There was only one video camera which was set up at the back of the room to capture the activities in all areas of the classroom. Teachers were interviewed after each lesson using semi-structured interview questions. The observations presented below were guided by Ruthven's framework. Each category in Ruthven's framework became a base from which observations were grounded.

Observations

Andrew has been a teacher for almost 20 years and had been using *Desmos* for seven years. He had recently completed her master's degree in mathematics education. For this class reported below, Andrew progressed through the "Racing Dots" activity (https://teacher.desmos.com/activitybuilder/custom/56d139907e51c4ed1014b51f).
The racing dots activity has eight slides. The first slide depicts two moving dots that start from different positions and move in the same direction at different rates; it also

prompts students to predict where the two dots will meet (Figure 1). In terms of content, Andrew wanted to introduce students to linear systems. The students had used *Desmos* before but had not engaged with any of the activities. They had also not been introduced to linear systems but were knowledgeable about lines. Slide 2 listed all the student predictions on all the screens.

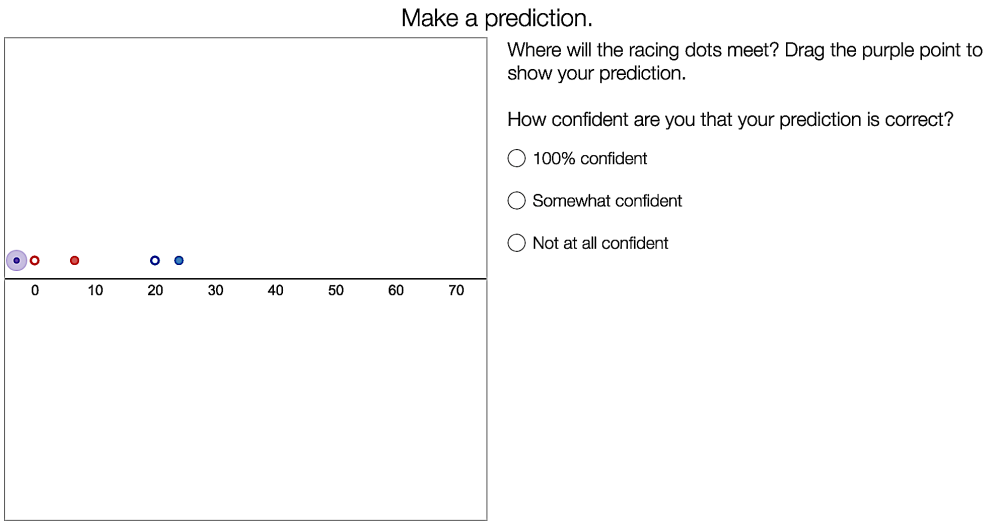

Figure 1: Screen 1 of Racing dots activity

Slides three and four prompt students to reflect on how they made the original reflections in slide 1 and 2 asking in slide four, what "tools and methods" they used to help clarify their prediction. Using that prompt Andrew asked students to go to the whiteboards in group of three and express their "tools and methods". Later in the interview, it was learned that his intention was that the students would create graphs, tables and equations which would match the last four slides of the Racing Dots activity.

Working environment and resource system

The episode reported here was in Andrew's regular classroom and the classroom was already organized to have a lot of resources and tools available for students. Andrew was inspired by the "Thinking Classrooms" model (Liljedahl, 2016) that he learned about in his master's program that he had completed two years earlier. This model does away with private notes and uses the whiteboard as a public place for students to solve problems and communicate. In previous classes, students were frequently working at the whiteboards on a mathematical task, or they were sitting at their small roundtable in groups without writing paper either discussing or writing on the table top with a whiteboard pen. Andrew's working environment was already organized to be resource rich and organized to be public, collaborative and with access to multiple tools. In terms of Ruthven's "working environment" and "Resource System", the introduction of *Desmos* on the front projector screen and on the iPads was not a significant change in

terms of access and/or transition. Students were already used to using whatever tools were available to do mathematics.

Activity structure

The most significant observable action Andrew enacted was his decentralization of the new digital technology. His use of the "Racing Dots" activity alongside previously used technologies was what allowed his students not only to forge relationships between different mathematical representations, but also between themselves and the mathematical tools. The "Racing Dots" activity was organized by the *Desmos* team but Andrew added his own expertise in using the teacher function "pace" which allowed him to freeze the activity at slide four. This pacing function stopped students from working ahead, so it paced the class so they went through the slides at a similar time scale and it also provided Andrew the opportunity to ask students to go to the whiteboard to express what they were thinking. He asked students to go to the whiteboards and represent the racing dots in a way "that made sense to them". The *Desmos* activity had not shown any traditional graphs or tables yet and Andrew was hoping to elicit these models from his students before these representations were revealed in slides five to eight. To model a system and explore the different representations inherent in that system aligns with Kaput's notion of "dynamic linking" (Kaput, 1992).

Students worked for 20 minutes using the whiteboard, but also used their traditional calculators, and at times worked at their tables. At the end of the 20 minutes Andrew went to the whiteboard and commented on student's representations. At the whiteboard, some of his comments were: "Well this group just continued that pattern and made a table"; "We have a much more visual table of values over here, careful detail of the two dots"; "And then we have graphs, couple groups actually just to graph the lines, pretty good estimate, off by a smidge." He concluded with "Really happy with the variety of solutions, I think this covers every single one of them." Once back at the front machine which was projecting *Desmos* on the screen, he went through slides five through eight legitimizing student's representations on the board by highlighting the same *Desmos* representations in the "Racing Dots" activity. At each slide he spent time talking about the representation, what it affords and the reason why one might choose one representation over another.

Curriculum script

Prediction in math education has been shown to support pattern recognition and improve their understanding of a concept (Buendía & Cordero, 2005). Screen one asks students to predict when the dots will meet (Figure 1). The second screen collates all the predictions from students in the class and list them out (Figure 2). Screen three provided the students to refine predictions. In class, Andrew said, "Were trying to predict, I think it's really interesting to see the range of predictions. What information

do we need to know [to help us with our predictions]?" He goes through each response and comments on each, "This group is thinking about graphs", "This group is using a table". After going through five predictions, he says "Interesting how nobody commented about starting points."

Andrew was covering linear systems. The contents of the linear system included solving graphically, solving algebraically, connecting ordered pair with meaning of an algebraic solution, solving problems in situational context (BC Ministry of Education, 2015). In terms of broader curricular competencies, Andrew was also focusing on "Represent mathematical ideas in concrete, pictorial, and symbolic forms" (BC Ministry of Education, 2015). Andrew's work at the whiteboard was used as a way to convince his students that there were multiple ways of expressing a relationship.

Time economy

Time economy, in terms of time, Andrew spent one full class on the Racing Dots activity. This lesson was slow and deliberate but also the most effective in terms of student engagement and clarity of message. Andrew seemed to have enough time to have a rich lesson. Time did not seem to be an issue for Andrew. He did not seem to focus on time–he focused on connections.

ANALYSIS AND CONCLUSION

Answering the research question, we see Andrew performing a number of moves that utilize the affordances of *Desmos*. One of the most important aspect of expertise to identify is that Andrew crafted his lesson not only on the content of linear systems but also the affirmation of students' responses and contributions basing those on the affordances of *Desmos*. Through Ruthven's framework we are able to notice how Andrew framed the access to digital technology in an environment that already valued and practiced multiple tools and access points. The digital technology was not centrally placed but was placed alongside a well-developed system of tools. The class culture was then able to uptake of *Desmos* without a significant change of organization.

In terms of activity, Andrew used the already established slide-show activity of Racing Dots in *Desmos*. He also applied his expertise by freezing one screen and prompting students to express their own understanding on the whiteboard so as to elicit a conversation about representation and meaning. Andrew was implementing multiple representations to strengthen student conceptual understanding. But he also did this as a way to support students' own understandings. When Andrew asked the students to express their thinking on the whiteboard, he used their familiarity with previous forms of representations as a way to confirm their own contribution to enable and encourage their agency. He balanced new technology with old and found a balance and a way to develop "dynamic linking" between old technology and the new.

Andrew's strategies resulted from his years of experience of trying things out. His 21 years of experience and using *Desmos* since 2011 made his practice seem fluent. Although this research report has not focused on students, the broader overall study has found that student responses to this and other activities have been overwhelmingly favourable in terms of both engagement and disposition.

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A MATHEMATICS CURRICULUM IN THE ANTHROPOCENE

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Mathematics education as a field has had little interaction with issues of environmental sustainability, yet the world faces unprecedented global and societal challenges. Human intervention has led to the designation that we have now entered a new era of geological time, the Anthropocene. The label 'Anthropocene', for some, signals the shift from hopes of 'saving nature' and 'solving' problems, to living with crises and problems as our new and permanent condition. It seems we know and yet do not know, that how we live and therefore how we educate, must change. This theoretical report asks what recognition of the Anthropocene means for mathematics curricula. I suggest there is an opportunity to bring the mathematics curriculum close to the concerns of communities and that this is unavoidably political work.

INTRODUCTION

The world faces unprecedented global and societal challenges. The extent and range of potentially imminent crises threatens the sustainability of human and many other forms of life on the plant. The aim of this theoretical report is to consider what recognition of such a context could mean for mathematics curricula. The first section is a brief review of past work linking mathematics education to issues of sustainability, a strand of thinking that goes back several decades. The more recent concept of the Anthropocene is then introduced, along with the post-human perspective it suggests. A final section considers possible responses.

MATHEMATICS EDUCATION AND SUSTAINABILITY

Given the scale of challenges facing the world, there is perhaps a surprising sense of the work of mathematics education continuing as normal. A recent special issue of the Philosophy of Mathematics Education Journal ('Mathematics Education and the Living World: Responses to Ecological Crisis', November 2017) is one of the largest single collections of writing relating mathematics education to issues of ecology and sustainability, and this issue comprises an Introduction and nine articles. In that Introduction, Boylan and Coles (2017) identify four strands of past work within mathematics education related to 'the living world', namely: critical mathematics education (Skovmose, 1994); an emancipatory perspective on mathematical modelling, which links to ethnomathematics (Barbosa, 2006; D'Ambrosio, 1999); sustainable mathematics education (Renert, 2011); and, work linking mathematics education and climate change (Barwell, 2013). Boylan and Coles (2017) suggest that Barwell's work represents a shift away from a focus, shared across the first three strands, on desired or ideal methods of classroom organisation, within a broadly emancipatory agenda, with

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a common link to the work of Friere (1970). Such a shift is mirrored by implications that have been drawn from a recognition of the Anthropocene, discussed in the next section.

One future-focused article from the Special Issue (Gutiérrez, 2017) argued the need for a radical re-thinking of the subject of mathematics itself and proposed a new term for it, Mathematx (pronounced 'Mathematesh'). The argument invites us to rethink mathematics to support the development of new relationships between humans, mathematics and the planet. The 'x' in 'mathematx' links to the use of 'x' in critical/emancipatory movements and the 'x' as a political disruption (following Malcolm X) as well as an 'x' found in the Nahuatl language. The notion of Mathematx draws on principles and beliefs of reciprocity, linked to indigenous cultures in the Mayan concept of 'In Lak'ech' and Nahuatl (Aztec) term 'Nepantla'. The themes of re-thinking the subject matter of mathematics and linking to political concerns are connected, by others, to theorising around the Anthropocene, which is the focus of the next section.

THE ANTHROPOCENE AND POST-HUMANISM

That the world faces unprecedented global and societal challenges caused by human intervention is linked to the designation that we have entered a new epoch of geological time, the Anthropocene (Finney, 2014). The label 'Anthropocene' (Crutzen & Stoermer, 2000) and has now been formally adopted (Subcommission on Quaternary Stratigraphy, 2016). One technical aspect of the designation is the recognition that there will be a visible change in rock strata being laid down at this time, with the label 'anthropos' pointing to the human cause, from plastic to radiation to CO_2 to erosion. Collective human action has taken on a geological scale.

The label 'Anthropocene' is a cause of dispute (Haraway, 2015), and concerns have been raised about the analytical coherence of invoking the category of a single species in thinking about global challenges such as climate change (Malm & Hornborg, 2014). There is also controversy over what we mean by 'human'. Colebrook (2016) sees the 'Anthropocene' as the first truly post-human concept. With the concept of the Anthropocene, we are imagining a world without humans and recognising that our impact will still be able to be 'read' in the rock strata after other visible traces of civilisation disappear. The concept is post-human in the sense of indicating a perspective that is not an irreducibly human one.

Post-humanism has been taken up within mathematics education in moves to re-think the role of the body and the materiality of mathematical concepts (de Freitas & Sinclair, 2014). There is an overtly political dimension to these moves, for example in exposing mathematics itself to be far from neutral and value-free.

Within a post-human reading (e.g., Hynes, 2016), the term 'Anthropocene' has been appropriated to signal the shift from hopes of 'saving nature' and 'solving' problems, to living with crises and problems as our new and permanent condition (Purdy, 2015): 'Paradox, partiality, and the mixed-up character of everything have come after the grasp at wholeness that began the ecological age.' (p. 227). Living with crisis rather than solving problems, while perhaps new and challenging in some parts of the West, is something known for generations in parts of the world and within particular communities.

Latour (2017) suggests one reason for a general inaction in the face of climate change is that the ecological war (to reduce emissions, etc) took place thirty years ago, without us even realising. And this was a war we lost. He would reject the framing of 'crisis' because of the sense that crises are things we can overcome; the situation we are in, is trying to live with the after-effects (on a likely timescale of millenia) of having missed the time when manmade geological planetary changes could have been averted.

In rejecting the idea that there are problems that can be overcome, the Anthropocene has marked a shift from the early environmental movement, when invocations of 'saving nature' were perhaps powerful calls to action. Coles (2017) drew a parallel between environmental hopes of nature, as something to be saved, and mathematics education reforms that hold up ideal images of classrooms or, say, ideas from the 1960s of 'mathematics for all'. If the Anthropocene suggests hopes of 'saving nature' are now self-defeating, could the same be said of holding up ideal reform mathematics classroom practices? Skovmose's (2011) conjecture is relevant here, that "*90% of research in mathematics education concentrates on the 10%, the most affluent classroom environments in the world, while 10% of the research addresses the remaining 90% of the classrooms*" (p.18, italics in original). And Skovsmose goes on to point out that some sites of teaching and learning mathematics in the "90%" may not share any prototypical features associated with classrooms. A concern of critical mathematics education, which feels highly significant in discussions of the implications of the Anthropocene, is not to repeat 'the bias that is established through discourse centred around the prototypical mathematics classroom' (2011, p.20).

Some suggestions of appropriate values for a curriculum in our current age include the acceptance of paradox (de Freitas & Sinclair, 2014); a focus on ethics (Boylan, 2016); and, education about risk and uncertainty (Stinson & Bullock, 2012). Latour (2017) points to the paradoxical way in which it seems we both know and do not know about the drastic changes happening around us. Somehow, we know this and yet at the same time, in communities where climate change has yet to have long-term impact at least, we manage to not know this in continuing our daily lives as we ever have done.

The Anthropocene has perhaps shone a light on the way the problems that face the world are of seemingly incomprehensible complexity; it is these problems that might offer a route in to considerations of paradox, ethics and uncertainty. In the next section, I consider what this could mean for mathematics curricula.

A CURRICULUM CLOSE TO COMMUNITY CONCERNS

The preceding discussion makes the case that there will be an on-going need, in relation to teaching and learning mathematics, *to think differently*. The uncertainty and paradox surrounding our current time make it hard to predict what kinds of new habits of mind (Cuoco, Goldenberg & Mark, 1996) might be needed for a mathematics education in the Anthropocene, but they surely cannot be the habits we now display collectively. There is perhaps a need for learning about the habit of forming new habits.

However, the complexity of our situation could also provide an opportunity, since if we start questioning, then we can begin with almost any real-world context or problem, no matter how local, and we will soon be faced with issues that relate to global challenges. This suggests that a place to begin in considering a mathematics curriculum for the future might be the very particular concerns of the community within which any school finds itself.

I will draw on two contrasting examples of what such work might look like. These two are chosen for no other reason that the fact that I am involved in efforts to instigate work in both places along lines I will describe. One community is in a relatively affluent city in the South-West of England and the other community is in a relatively poor rural community in Mexico. Both contexts relate to pollution. Landigrand et al. (2017) reported that pollution is the largest cause of disease and premature death in the world. Communities, especially children, from low and middle-income countries are the most vulnerable. Feldman and Nation (2015) argue that education responses are essential not only to understand the implications of our actions on ecosystem degradation processes but also in constructing new responses, as we are forced to adapt to the new climatic, economic, technological and societal conditions of the world.

Air-quality in the City of Bath, England

Air-quality has been in the news in England, with a report in 2017 indicating that 37 out of 43 monitoring zones reported levels of Nitrogen Dioxide that were non-compliant with EU rules (DEFRA, 2017). The health impacts of air pollution range from increased likelihood of miscarriages in pregnancies, increased likelihood of dementia and impairment of cognitive functions for young children and they are quantified in terms of numbers of early deaths. Some primary schools in Bath are sited next to main roads where nitrogen dioxide levels frequently exceed safe and legal limits; considering air quality is not part of the curriculum in these schools. I am personally involved in this issue as a parent, as well as an educator.

Water pollution in the Atoyac River, Mexico

The Atoyac river is Mexico is classified as "heavily polluted" (Lopez-Vargas, et al., 2018, p.640). Around 1.5 million people live in communities that are settled on the banks of the river. Tests on children living near the river "show early biological effects that might lead to health problems in their adult life" (p. 640). The perception of people living in affected communities is that rates of leukemia among children are at levels unknown to previous generations. And yet, a consideration of the river, its pollutants, their causes, appropriate safety responses and so on, do not appear in the curriculum of schools within affected communities (Arellano-Aguilar, personal communication). My personal involvement is through an on-going research collaboration with mathematics educators in Mexico.

How might a mathematics curriculum respond to concerns?

It is not obvious how to bring issues of local, but pressing, concern into the curriculum and what teaching and learning mathematics might look like if that happened. However, it is clear that considering issues such as air and water pollution leads quickly into political territory – and questions of how pollution is allowed to happen and who is responsible. In February 2018, an environmental law organisation 'ClientEarth' won a case against in the UK which determined that the government's failure to enforce action by local authorities against air pollution was unlawful (https://www.bbc.co.uk/news/science-environment-43141467). In Mexico, it is known that the main cause of pollution in the Ayotac River is liquid industrial discharge; Lopez-Vargas, et al., report health authorities have no monitoring programme for the effects on the population. It would seem that local authorities quickly come into view and, from there, national governments and international standards, when considering questions of pollution. Questions around risk and uncertainty and ethics are likely to arise. Some barriers to bringing community concerns into the classroom appear to be common across contexts. Having set them out, below, I consider possible responses.

(a) How do mathematics teachers find out about the concerns of the community?

(b) How might accessible data be made available, or collectable, in school?

(c) Where are the spaces in the mathematics curriculum to address community concerns?

(d) What are the skills needed for mathematics teachers to bring community concerns into their classrooms?

(e) How might charges of politicisation of the mathematics curriculum be met?

One way it is possible to address (a) and (b) is bringing people together. There is a need for networks, involving some or all of the following groups: community leaders, scientists, teachers, school leaders, teacher educators, policy makers, non-governmental organisations. However, access to communities is not always straightforward. The group of educators in Mexico I am connected with is currently setting up just such a network and the experiences and skills of non-governmental

organisations working in affected communities is critical to gaining the engagement of community leaders and schools. With regard to the City of Bath, similar kinds of network are needed and I am involved in work to set one up involving scientists, parents, local politicians and school leaders.

In terms of the spaces within the mathematics curriculum (c), it was clear at a recent ICMI Study Conference (ICMI Study 24, on School Mathematics Curriculum Reforms) that a phenomenon occurring in many countries is a move towards a competency-based curriculum or at least a curriculum that includes a competency component. In such a context bringing issues, such as pollution, into the mathematics classroom is perhaps easier to justify. Any focus on pollution will inevitably involve data collection and so the topics of statistics and probability also provide obvious opportunities. Conducting data collection within a school may need the involvement of Science teachers and potentially outside funding; data collection provides an opportunity for inter-disciplinary work, with all the complexities as well as affordances this entails.

There are potentially new skills needed, of teachers, to bring an issue such as air quality or water pollution, into the mathematics curriculum (d). In the Mexican context, as well as network meetings to learn from teachers and schools, there are plans for teacher workshops to feedback some of the learning from the network and to support teachers to raise new issues in their classrooms. Skills of running discussion are particularly significant and skills of handling potentially emotionally charged debate. Teachers have expressed concerns related to a potential lack of confidence about the inter-connection of issues and where debate may lead.

Finally, an important consideration is how to deal with the charge that issues such as pollution are not appropriate ones for a mathematics classroom (e). In England, since the introduction of a National Curriculum in the late 1980s, there has been a strong sense in which a classroom is not a place for teachers to express political opinion. In considering a response, it is important to recognise that debates around pollution can never take place within a vacuum. The causes of pollution inevitably link to industrial production and there are strong forces wanting to maintain the status quo and, for instance, not question the assumption of economic growth or, more radically, which actively question the scientific basis of global warming. I have come to the conviction, in the face of the evidence of human effects on the planet recognised in the naming of the Anthropocene, that going along with an ostensibly non-political stance in mathematics education, is to take up a political position that is about maintaining a fiction that life can go as normal.

DISCUSSION

This article aimed to consider issues relevant to mathematics education in the light of a recognition of human influence on planetary processes and a sense we have moved into a new geological era, the Anthropocene. In conversations with many mathematics educators in recent months I have asked the question of how ‘what we do’ might take

account of current crises. I have sensed a recognition of the question and an awareness that none of us have answers. And indeed, a post-human reading of the concept of the Anthropocene moves away from a sense of 'growth' (Morgan, 2016) and setting up specific and defined 'ends' for education or the search for determined answers.

One of the tensions of teaching is that while we are preparing students for a future world that is uncertain and unknowable, our teaching must take place now. The thinking reviewed and used here, linked to the Anthropocene, suggests a role for a mathematics curriculum close to the concerns of communities, that centrally considers uncertainty, risk, ethics and paradox. This is local and uncertain work itself; and there is a need for sharing of experiences across contexts, nationally and internationally. It may be that we can learn from each others' experiences of, for example, establishing networks. Some stories of successful, or unsuccessful, work in schools and communities may provide important ideas for others. Latour (2017) suggests it is time for academics to reply to skeptics and climate change deniers with the cry "Go tell your masters that the scientists are on the warpath" (p.30, quoting from Pierre Daubigny's play, *Gaia global circus*). We may never know where or what this warpath is, but this report argues it is time, as mathematics educators, we join it.

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RECONCILING TENSIONS BETWEEN LECTURING AND ACTIVE LEARNING IN PROFESSIONAL LEARNING COMMUNITIES

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Professional learning communities (PLCs) are considered an effective vehicle for teacher professional development, yet their emphasis on discussions-based learning practices may create tension with the expectation for growth of content knowledge. We have been leading a PLC of practicing and prospective heads of school mathematics departments, in which this tension was particularly salient. We investigate ways in which lectures and workshops conducted by content-experts can support the development of desirable PLC characteristics, rather than being at odds with them. Findings suggest that the tension can be reconciled by means of ongoing debriefings with a focus group comprised of the PLC participants, contributing to the careful design of community activity surrounding the expert-provided lectures.

INTRODUCTION

Learning in Professional Learning Communities (PLCs) has been found to be particularly effective for the professional growth of teachers in general (Shulman, 1997; Vescio & Adams, 2015), and of mathematics teacher leaders in particular (Koellner, Jacobs, & Borko, 2011). These researchers highlight 5 key characteristics of learning in professional communities (see table 1), which we use as a theoretical framework for the design and analysis of our work with heads of middle-school mathematics departments.

C1	*Generative* content to yield new understandings and to support future learning
C2	*Active* learning
C3	*Reflective* dialog about practice (teaching, learning, leading teachers)
C4	*Collaboration* for collective work, scaffolding and supporting each other
C5	*Making practice public*, learning from others' success and failure

Table 1: Characteristics of learning in profession learning communities

Though these principles have been described and discussed in research, fleshing them out and applying them in particular contexts remains a challenge. In the context of our work with heads of departments (HoDs), a central challenge has had to do with participants' agency. Participants have expectations based on prior experience with PD, which tends to be "based on a deficit model of linear teacher learning, as if learning is done to achieve an end result or fill a gap in a teacher's development" (Vescio & Adams, 2015, p. 277, based on Webster-Wright, 2009). The responsibility for filling

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such gaps is on the PD leader, not on the participants, thus conventional PD tends not to harness teachers as agents in their learning. In particular, it may not be consistent with the characteristics of *active* (C2), *reflective* (C3) and *collaborative* (C4) learning.

In our vision, school mathematics departments should function as PLCs led by the head of department (HoD). To this end, the HoDs themselves participate in municipal PLCs, comprising other HoDs in the same city and led by mathematics educators and researchers, in order to familiarize themselves with this kind of learning, and thus to acquire knowledge and skills for this strategic role. Both the content and the process of learning in these PLCs aims to provide a model for HoDs to draw on in leading their local communities. Thus, another challenge arises from "the essential tension between professional development geared to learning new pedagogical practices and that devoted to deepening teachers' subject matter knowledge" (Grossman, Wineburg, & Woolworth, 2001, p. 942). In our context, the new pedagogical practice for HoDs is leading a school-based PLC (i.e. heading the department as a learning community in ways consonant with the 5 principles of PLC), while the subject matter knowledge pertains to HoDs both as mathematics teachers (i.e. mathematical and pedagogical content knowledge) and as teacher leaders (i.e. leadership skills).

Yet another challenge of PLCs is that some of the learning characteristics rely on the emergence of community norms. Notably, collaborative learning (C4) and making practice public (C5) rely on the development of trust and the suspension of judgment. Grossman et al. (2001) have described phases in the life cycle of PLCs, whereby some characteristics of PLC learning are more likely to emerge as the community matures.

Our research is guided by the aim of reconciling HoDs' expectations to learn content knowledge from experts with the characteristics of learning in professional communities. We wish to characterize PD activities that provide content knowledge for HoDs, while at the same time supporting learning processes of a PLC. To this end, we inquire *how content-oriented PD activities evolved with respect to the five characteristics of learning in professional communities, and with respect to the roles of learners as more or less independent agents of their learning*.

SETTING AND METHODOLOGY

As in many educational contexts around the world, heads of departments (HoDs) in Israel are usually practicing mathematics teachers with many years of teaching experience and some leadership skills. Their responsibilities include coordinating pacing across classes, organizing the preparation of shared exams, student placement, and others. We undertook a municipal PD for middle-school (grades 7-9) mathematics HoDs, commissioned and funded by the Israeli Ministry of Education (MoE) and supported by a private foundation. The community on which we are reporting was co-led by both authors of this report (henceforth L1 and L2), and comprised 16 teachers from two neighboring cities, half of whom were practicing HoDs and half of whom were preparing for this position. The scope of the PD was 60 hours per year for two years. We are reporting on the first year of activity (2017-2018), during which 11

meetings took place at the municipal center for teachers' professional development (4-6 hours each). Meetings were organized around topics that were chosen based on the leaders' preliminary views on knowledge required for heading mathematics departments, and on expectation questionnaires filled out by the participants. There was a high level of agreement on relevant topics among the participating teachers and the program leaders, which included strategies for helping low achievers, issues of assessment (summative and formative, conventional and alternative), integrating technology, curriculum design, implication of research, and a collection of leadership issues pertaining to interactions with teachers in the department.

After the fourth meeting of the community, we set up a focus group, which comprised two teachers: T1 – a practicing HoD – and T2 – an experienced teacher who also held a coaching position from the MoE – along with the two community leaders (the authors of this report, L1 and L2). Debriefing-meetings (approximately 30 minutes each) were held after all meetings of the community commencing with the fourth. One of the goals of these meetings was practical – to develop and fine-tune appropriate formats for the PLC activity (ourselves adhering to C3 – *reflective dialog about practice*). Accordingly, we held the debriefings guided by three questions: *1. Which activities did work? 2. Which could we have done without? 3. What could we have done differently?* Discussions of the third question gradually led to formats that were increasingly appropriate for the community of HoDs. This focus group also served a methodological role in providing authentic data on how learners perceived the appropriateness and the relevance of learning activities. The analysis in our current report relies on the focus-group and the PLC meetings, which were audio-recorded and transcribed by the authors. Six activity formats emerged and evolved – expert-lecture, mathematical workshop, teacher-led discussion of a practical dilemma, report on routine activity of the school community of mathematics teachers, report on enactment of PD activity within the school community, and final assignment. Space permits us to present and analyze one such activity. We have chosen the expert-lecture, whose gradual adaptation aligned with many of the key characteristics of PLCs. The emergence of the modified format took place in meetings 4, 5, 7 and 10, as summarized in table 2.

Each of the 4 meetings is described as follows: 1. The main activity – its rationale and relevance for PLC, and how it drew on insights gained in prior focus group discussions; 2. Selected utterances from the meeting transcripts; 3. Based on the transcripts - a discussion of adaptations of the expert-lecture format that were proposed for future meetings in order to increase its relevance. Each adaptation is analyzed along two dimensions: the characteristics of PLC learning that are addressed, and the nature of HoDs' agency. The need for brevity does not permit us to present lengthy transcripts, hence we draw on the tradition of *narrative inquiry* (Clandinin & Caine, 2008), which is "marked by its emphasis on relational engagement between researcher and research participants" (p. 542). Recognizing this relationship as a key aspect of the research, and not as bias in an ostensibly objective report, we present selective data with minor editing to tell a story in which we played a significant part.

Meeting	Emergent ideas for expert-lecture format
4	**Send advance questions to lecturer**. Teachers send pertinent examples from personal experience Involve department teachers in the pre-lecture assignment.
5	Increase HoD compliance (superintendent pressure, allot PD time)
7	**Discuss implications for teaching and practical applications** (possibly preparing lectures with relatively unequivocal implications).
10	Cultivate community norm of keeping learning "on track"

Table 2: Evolution of lecture format (ideas in bold were subsequently implemented)

ANALYSIS AND FINDINGS

Meeting 4

The main activity in this meeting – a lecture on textbook analysis – drew on the leaders' prior conception of relevance for the PD, assuming that one role of HoDs is textbook selection. This content is *generative* (C1) in suggesting criteria for HoDs ongoing selection of learning resources. However, it was not perceived by teachers as particularly relevant, because schools are required to use textbooks for at least 3 years so that they can be handed down and re-used, thus selection takes place infrequently. Furthermore, the lecturer presented a complex multifaceted view of textbook analysis, while the HoDs were expecting prescriptive findings for making decisions. This instigated a search in the focus group for alternative notions and modes of relevance.

3 L2: We thought the topic of how to select a textbook in an informed manner was a must. Now I'm not so sure.

14 L1: It's relevant even if you can't replace the book, if you become critical towards it and recognize the need to supplement it with external material.

19 T2: So the question needs to be different. Not just how to choose it, but what I do with it. Something else significant needs to happen, every teacher should bring some insight, something he takes part in. Not just listening passively.

40 L1: I want to propose a model – priming. Before the lecture on textbooks, think of three reasons I'm satisfied with my textbook and three why I'm not. Share [with the community] and send to the lecturer as background.

L1 suggested ways in which the topic could be generative (C1), not only for selection of future textbooks, but also for instigating *reflection* on current practices (C3), while T2 stressed the importance of *active* learning (C2). In response to this, L1's proposal addressed C2 (*active* learning), and C3 (*reflection* on practice) in a priming activity before the lecture.

Meeting 5

The 5th meeting included a lecture on low-achieving students. The rationale for this lecture drew on the intersection of the community leaders' prior conception of relevant content and on the HoDs expectations as elicited in questionnaires. Applying the adaptations proposed in the previous debriefing, we asked the HoDs to share "either a question that you would like to be addressed or a related idea or thought, e.g. an assignment you gave to low achieving students, or an interesting answer from a low achiever". Three out of the 16 HoDs responded, requesting that the lecture touch on the following topics: 1. Motivating students with mediocre ability; 2. Common misconceptions and strategies for addressing them. The third input was a ruse on the negative implications of insisting that chronically low achievers must learn the standard curriculum. The first two issues were touched on in the lecture. T1 and T2 commented in the debriefing on the low level of response to the invitation to influence the lecture.

2	T2:	The priming, it didn't work! We might need to involve the superintendent.
4	L1:	Maybe we can make use of the time in the PD sessions [instead of at home]

The level of response is probably due to the early stage in the development of the community – the norms of collaboration (C4) and sharing practice (C5) in a safe and supportive environment were just emerging. The focus group discussed two ways to enhance compliance. However, involving the superintendent would not have been consonant with the communal norms we were aiming to achieve (C4), nor were we comfortable with the idea of restricting all the teachers' activity to the PD hours, feeling that the principle of *active* learning (C4) implies at least some work outside the PD meetings. No new adaptations of the lecture format were proposed in this debriefing.

Meeting 7

The 7th meeting included a lecture, given by L2 on the contributions of research in the field of mathematics education. The topic of the lecture was in the intersection of the leaders' conception of relevant content and the HoDs expectations.

2	T1:	You lost them after 30 or 40 minutes. The idea is collegial learning, that everyone shares something of himself and then we discuss it.
19	T2:	I think the lecture should have been summarized differently, and not only at the end…. The research should have been processed and adapted, and time allotted to discuss: does it have significance for teaching? What would you do with it as a HoD? And share this [with the community].
28	L2:	I tried to stop after every piece of research and ask, from the perspective of HoDs and teachers, how it can be applied.
31	T2:	But you didn't give it space. And *you* brought the questions.
34	L1:	We say that implications [of research] are complex, not unequivocal. This is not the answer they want to hear.
39	L2:	These are low expectations from HoDs.

40 L1: No, they're *different* expectations. We can try to do episodes, describe one research and then give 10 minutes in groups to suggest practical implications for practice, and hold a discussion.

76 T2: [Or we can offer] two questions [for the HoDs]: Is the phenomenon familiar to you from your school, and what is its significance in teaching.

While previous suggestions focused on teacher activity before and after the lecture, this meeting elicited an idea of how to involve teachers during the lecture. It calls for *collective* learning (C4), recognizing that different community members have distinct types of expertise. Furthermore, the adaptation is *generative* (C1) in the sense that familiarity with research can have implications in the future, it involves teachers *actively* (C2), it encourages a *reflective* stance on practice (C3) in connecting research to teachers' practical experience, and in so doing makes their practice *public* (C5). T2 stressed the importance of learner agency in posing questions about application.

Meeting 10

The 10th meeting included a lecture by a teacher educator on her experience with community learning in a different PD. The lecture was followed by a discussion led by T2, driven by the question of how to connect the presented experience with the participants' experiences. Learning from a case in this meeting can be seen as modeling C5 (making practice public), and was consistent with expectations that the PD introduce other educational contexts. The discussion touched on the expert-lecture format in general, with an eye towards planning the following year.

38 L2: We should tell lecturers that their lecture will be followed by a discussion on how to take it to school.

41 T2: Also need to develop within the community a more critical culture… If you didn't get what you wanted [from the lecturer] – ask. The lecturer may have had things to tell us that we didn't ask about.

Here T2 recognized the importance of community norms that can keep the learning on track. *Active* learning (C2) does not only imply that teachers participate in the learning actively, but also that they may have an active role as agents in directing its course.

DISCUSSION

We elaborate in this section on the emergence over time of two aspects of the lecture activity – alignment with the characteristics of community learning and learner agency, as summarized in table 3.

Meeting	Characteristics of adaptations	Agency in proposal
4	C1, C2, C3	
5	N/A	*Instructor* agency in enforcement
7	C1, C2, C3, C4, C5	*Learner* agency in raising questions
10	C2	*Learner* guiding and directing lecture

Table 3: Learning characteristics and agency in proposed adaptations of format

The emergence of community learning characteristics

In three of the four focus group discussions, new ideas were suggested for aligning the lecture activity with desired characteristics of community learning. Presenting and discussing research (meetings 4 and 7) could be seen as generative (C1), both in its findings, and in the reflective stance that it encourages. All suggestions emphasized the importance of active learning (C2), in raising questions, in considering applications and implications, in guiding the lecturer, and generally not just learning passively from the expert. Adaptations in meetings 4 and 7 suggested how learners could reflect (C3) both on their practice and on their learning, and meeting 7 suggested how teachers could collaborate (C4) through sharing their work, and make their practice public (C5) through sharing their experience.

The emergence of learner agency

While meeting 5 raised some ideas that tend to deny learners agency (extrinsic enforcement of active participation), meetings 7 and 10 suggested how learners could act as independent agents in lectures, through question-raising throughout the lecture.

Active participation before, during and after the lecture

Our attempt to align the PD with principles of PLCs began with a point in time at the beginning of the PD, when learners were invited to fill and submit expectations questionnaires. In responding thoughtfully, they were acting as agents of their learning by influencing the selection of content topics. Over the course of the PD, many other points in time were shown to be appropriate for community learning: Before lectures (advance questions and experiences to lecturer), during lectures (keeping the lecturer on track, asking questions), and after the lecture (discussing implications for practice).

The role of the focus group in the process

Though the principles of learning in a professional community were known to us in advance, fleshing them out in a particular context was not a straightforward task. The focus-group debriefing served a crucial role in the process, as an instance of what Shulman has advocated as "strategies … that permit both intention and chance to be represented in their collision…" (1997). Setting up the focus group was a *strategy*, its *intention* was refinement of learning activities, which collided with the *chance* enactment of the individuals comprising the focus group. We consider the product – refinements of the lecture-format – to be valuable findings; in combination with the *process* by which these findings emerged they become *generative* for the community.

We also note that the focus group has played a role in upscaling; in the second year of the project T2 became one of the leaders of the community.

Relevance for school PLCs led by heads of departments

Preparing HoDs as leaders of local school-based PLCs has been an explicit goal of the PD from its outset. Thus, it would have been natural to expect the lecture format to evolve in ways that attend to its role in these local communities. The fact that this did

not happen at the outset could be anticipated in view of Grossman et al.'s accounts of the development of communities over time. Now that we have entered the second year of the program, and the community has matured, interactions with school-based PLCs are an explicit focus of our work, and will be reported in due course.

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SUBJECTIVE APPROACH TO PROBABILITY FOR ACCESSING PROSPECTIVE TEACHERS' SPECIALIZED KNOWLEDGE

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In this paper, we present an inquiry regarding developing specialized knowledge in prospective primary teachers (PTs) in the context of probability. In particular, we discuss the knowledge involved and required in the context of a task focusing on the subjective approach to probability. The implementation of such a task allowed us to investigate PTs' criteria to establish the degree of confidence that a given event occurs and investigate if these criteria are linked with PTs' specialized knowledge.

INTRODUCTION

In recent years, there has been growing interest in the topic of probability both in mathematics curricula around the world and in mathematics education research. Indeed, probability plays a crucial role in people's mathematical literacy (e.g., OECD, 2016), whereas probabilistic reasoning is at the root of many daily decision problems as well as in scientific issues. Even if the development of awareness about probability assessments is something culturally and socially relevant, the research reveals both pupils' and teachers' difficulties in this topic (e.g., Batanero, 2015). Research shows teacher education plays a core role in filling such gaps and improving practice (e.g., Linares & Krainer, 2006; Boyd, Grossman, Lankford, Loeb & Wyckoff, 2009).

In particular, the different philosophical conceptions of probability (classical, frequentist and subjectivist) provide different systems of concepts and procedures that serve to analyze uncertain situations. Teachers should be aware of these conceptions because they influence students' reasoning when confronted with chance situations. In that sense, it is crucial to think of new ways/approaches to develop teachers' knowledge and awareness about probability after their initial education program. In pursuing such an aim, we have taken the subjectivist approach of probability as a starting point (de Finetti, 1931). According to this perspective, probability is no longer a physical objective property, but rather a subjective judgement conditioned by a person's information and knowledge. This approach captures the psychological basis of probability and can be meaningful and consistently linked with the other approaches.

In this scenario, we designed a particular task with the aim of general inquiry and to develop PTs' knowledge. The task proposes PTs to discuss particular betting contexts, some of them linked to everyday decisional situations that can be also read through the mathematical lens of Probability. The task design and the analysis were conducted based on the model of Mathematical Teachers' Specialized Knowledge – MTSK (Carrillo et al., 2018), since we also consider all the teachers' knowledge to be specialized. We particularly focused on the sub-domain of the MTSK model named

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Knowledge of the Topic (KoT) to address the following research questions: 1) Can the subjective perspective allow us to assess PTs' specialized knowledge and awareness about probability? and 2) Does PTs' KoT allow them to consciously links between the different approaches of classical, frequentist and subjective probability?

THEORETICAL FRAMEWORK

Although probability is present in most of the national mathematics curriculum in primary school all around the world, research reveals that teachers have fragmented knowledge in the probability domain (e.g., Batanero, Godino, & Roa, 2004). Indeed, mathematics teachers frequently lack specific preparation in statistics and probability education. For example, in Italy, until some years ago, prospective secondary teachers with a Master's Degree in Mathematics were not obliged to receive specific training in probability. The situation is even worse for primary teachers, most of who have not had basic training in probability; this problem is common to many countries. The research showed that this lack of knowledge creates serious difficulties for teachers to link their intuitions and daily knowledge with the formal knowledge they are going to teach (Batanero, Godino, & Roa, 2004).

The complexity of the topic is also related that the Probability Theory has different philosophical approaches in its inner and actual formalization. Some of them (classical and frequentist approaches) look at the probability as an objective property of an event or an element of a class, while another (subjectivist approach) conceives it to be related to a person's judgment. The classical view defines the probability of an event as a fraction between the number of favourable cases of a particular event and the number of all possible cases (i.e., a priori probabilistic study), while the frequentist approach sees the probability of an event as the limit of the frequency relative of that event when the number of experiments goes to infinity (i.e., a posteriori probabilistic study).

A different perspective is, instead, followed within the subjective interpretation of the probability of an event that was conceived with the intention of discerning the different formal approaches of probability and their psychological basis. In particular, de Finetti defines the subjective probability of an event E as the price p that an individual, being the banker in a gambling game, is available to pay if E occurs (de Finetti, 1931). The subjective approach of probability is guided by the idea that in some cases, the judgment of the degree of confidence about events' occurrences can be accessed through measurement procedures.

Batanero, Godino, and Roa (2004) argue that it is necessary to provide teachers with a deeper preparation that allows them to develop an awareness of the different philosophical approaches to probability. Assuming that teachers' knowledge influences practice and thus, the students' learning processes (e.g., Nye, Konstantopoulos, & Hedges, 2004; Zakaryan & Ribeiro, 2018), it becomes necessary to research and deepen the content of probability knowledge. Considering the specialized nature of the work of teaching, we refer to the MTSK conceptualization (Carrillo et al., 2018), which considers such specialization in an explicit manner both

in mathematical and pedagogical content knowledge and built around the beliefs. In this vision, in the topic of probability, it's important to obtain a deeper understand on the content of specialized knowledge a teacher needs to have and in which sub-domains such knowledge can be located (Carrillo et al., 2018). It is crucial, for example, that teachers know that there are different, consistent approaches and definitions of probability, each one suitable to read and grasp different aspects of a situation.

In this study, we focus on the KoT dimension of the MTSK that is divided into: i) Definitions, properties and foundations; ii) Procedures; iii) Registers of representation, and iv) Phenomenology and applications. This sub-domain, in the case of probability, concerns: the intra-conceptual connection between the different definitions of Probability linked with the different philosophical approaches (KoT – Definitions); the related procedures like for example statistical analyses or evaluations based on combinatorial calculus (KoT – Procedures); number representations as fractions or percentages, and graphical representations like drawings, tables, histograms, etc. (KoT – Registers of representation); and the evaluation of the conditions under which a bet would be made and the recognition of the suitable definitions to read and grasp features of a situation (KoT – Phenomenology and applications).

In this scenario, it is vital to let the teachers explore and reflect about how the degrees of confidence attributed to different events can be based on personal sensations, but also carried out and formalized through measurement processes concerning statistical analyses or evaluations based on combinatorial calculus. This kind of work can be developed with PTs by designing suitable tasks, implemented through the orchestration of collective discussions among teachers. Indeed, the research suggests that the *mathematical discussion* is potentially a valuable tool for developing pupils' learning (Bussi, 1998), as well as to promoting teachers' specialized knowledge (e.g., Levin, 1995). In the case of probability, a mathematical discussion that is suitably orchestrated can allow PTs to develop an awareness of the differences and links between subjectivist, classic, and frequentist approaches to probability.

CONTEXT AND METHOD

This research is part of a wider project aimed at investigating teachers' specialized knowledge in different mathematical domains; here we focus on the field of probability. Probability, with the name "Data and Predictions" is one of the four founding areas of the National Italian Guidelines (MIUR, 2012) and there are different attempts throughout the country to support teachers in including this topic in their mathematics education proposals.

Here we report on data collected during November 2018 in the context of the Master's Degree Course for kindergarten and primary PTs at the University Suor Orsola Benincasa of Naples, Italy. The data collected regards a task implemented during two classes at the end of the Mathematics Education course. This course is placed in the third year of PTs' five-year university path, after other two Mathematics exams (in these exams there are no specific contents regarding probability). Our sample is

characterized by having poor mathematical formal knowledge about probability, mainly stemming from their school experiences.

For this reason, the PTs required extra time to think on the task, so we opted to give them the task prior to the two classes dedicated to discussing it. This way, the PTs could work on it individually or in groups at home before class. This choice was made in order to give them extra time to reflect upon and search for ways to complete the task and to promote a more fruitful in-class discussion (Bartolini, 1998; Levin, 1995), as the focus is on developing PTs specialized knowledge and not on identifying the gaps in such knowledge. Indeed, during the class, attended by about 60 PTs, we asked the PTs to share with us and with their colleagues their reasonings and productions, used by the educators as starting point for orchestrating the mathematical discussion.

With the goal to allow to more meaningfully assess PTs knowledge of classical and frequentist probability and develop specialized knowledge, we have designed the following task (Fig. 1), starting from the context of betting, to promote a subjective view of probability.

Imagine to have to bet on one of the two events E_1 and E_2 described in the following different contexts, which kind of information you would look for in order to decide on which one to bet?

1) Roll of two non-rigged dice

E_1= 7 as sum of the two numbers on dice faces; E_2= 3 as sum of the two numbers on dice faces

2) Weather forecast for tomorrow in Naples

E_1= it rains; E_2=it is sunny

3) Weather forecast for January the first (2019) in Naples

E_1= it rains; E_2=it is sunny

4) In a coffee shop the person sit to the table closed to mine

E_1= he/she will order a glass of wine ; E_2=he/she will order a beer

Figure 1: The teacher education task

This task allows us to assess which information PTs were looking for to choose their betting and, if in possession of this information, how they decide on the betting. We expected that PTs would look for information by referring to knowledge constructed during the school years or from their life experience. Indeed, with such task, we intend to investigate PTs' criteria to establish the degree of confidence that a given event occurs and how these criteria are linked with KoT. In particular, this task let us assess the degree of awareness with which they use their KoT in probability. Moreover, this task was designed to promote discussion in a deeper way about the different approaches to decision problems by linking them to measurement processes of probability (whenever possible), while discussing different strategies where it is not possible.

Data collection concerns the group discussion audio recordings and the photos of the writings made by PTs on the whiteboard during the discussion. Due to the space limitation, we focus on two excerpts of the discussion in which, by using the MTSK lens, we will show how the PTs try to invest their KoT, even if in same cases very poorly, in the proposed betting contexts. In particular, we discuss how the stimuli and provocations posed by the task create PTs' reactions that we, as educators, could use

in a mathematical discussion in order to support PTs' development of specialized knowledge. Our goal was to make PTs link their criteria to establish the degree of confidence that a given event occurs and the different approaches to probability meant exactly as measurement tool of the degree of confidence that certain events occur.

ANALYSIS

In the first excerpt of discussion, we are going to analyze a student showing her strategy to the first question of the task. In the context of the roll of two non-rigged dice, we asked them to imagine which kind of information they would look for in order to decide to bet on E_1= 7 as sum of the two numbers on dice faces or E_2= 3 as sum of the two numbers on dice faces.

Francesca: I would bet on seven because I have thought that there are more possibilities that seven will occur than three. Seven can be obtained by six plus one, five plus two and four plus three, while three can be obtain only with one combination, that is two plus one. There are three combinations, that means there are more probabilities in comparison to one combination. After this reasoning, I was intrigued and to verify it, I performed the calculus [she went to the board writing the production in Fig. 2]. Practically, three are the combination, while twelve are the possibilities since six are the faces of a dice and six plus six is twelve, so we have twelve possibilities to arrive to the right combination.

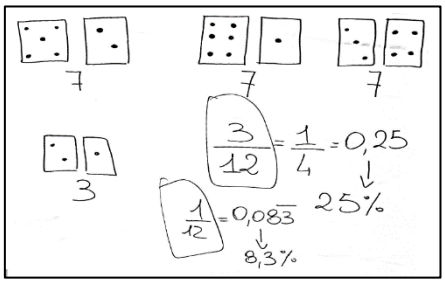

Figure 2: The whiteboard's production of Francesca

Educator: In which sense twelve are the possibilities to arrive to the right combination?

Francesca: Because twelve are the total faces of the two dice.

In this part of the discussion, it is interesting to see how the context of betting seems to naturally orient Francesca to root her criteria in the combinations that, from her point of view, are the ones that give the sum of seven and three. She made a common mistake when finding the amount of favorable events for seven (only three) and for three (only one), and her criteria to compare the degree of reliability of the two events is comparing the amount of possible combinations for seven (three) with the amount for obtain three (one). When comparing the degree of confidence, she consistently perform the comparison of the two numerators 3 and 1 with the same denominator expressing the total number of combinations, assuming the classical probability view point as a natural way to order the degree of confidence, even if she is not aware of such fact.

Francesca perceived a sort of dissociation between criteria and calculus, as can be seen in her words "to verify it, I performed the calculus", obviously linked to her belief about mathematics at the core of the MTSK model. Indeed her "calculus" seems quite forced, and her error in considering as total events the sum of the faces of the two dice "Because twelve are the total faces of the two dice", reveals a loss of sense making when applying the formula (KoT – Procedures). However, looking beyond the mistakes, a relevant aspect here concerns her attempt to elaborate an intra-conceptual connection between subjective probability and classical probability (KoT). Moreover, the use of multiple representations properly linked to each other in her white board answer–drawings, fractions, decimal representations, percentages (KoT – Representations), together with the arguments provided verbally, reveal a deep search for meaning undertaken by Francesca.

Continuing the discussion, other PTs' comments contribute to clarifying the mistakes in Francesca's reasoning and to meaningfully build a shared recognition that it is more convenient to bet on 7 because it can occur in six combinations that are (1;6), (2;5), (3;4), (4;3), (5;2) and (6;1) on 36 possible combinations of the six faces on the two dice, while 3 only in two combinations (1;2) and (2;1) on 36.

The next excerpt is related to the fourth question of the task. In this question, we asked PTs to imagine which kind of information they would look for in order to decide, in a coffee shop context, to bet that the person sit in the table closed to mine E_1=will order a glass of wine or E_2=will order a beer, aiming at discussing which can be the possible measure strategies of an open context such this one.

Elisa:	I could look how he is dressed, or the kind of coffee shop, but I have also thought that I could ask to the barman what the people order most in that place: a glass of wine or beer.
Educator:	Could you link it to the strategies that we have already discussed?
Elisa:	Yes, as for the context 3 in which we would see in the previous fifty years the weather on the first of January in Naples, also here we are going to search the frequencies of the clients who order a glass of wine or beer.

Elisa connects the subjective request of the task to the frequentist probability, navigating in the intra-conceptual connections between subjective and frequentist approach to Probability (KoT – definitions). The fact that there are no explicit references to a formal definition of frequentist probability brings to front the use of a naive idea of such probabilistic view elaborated from personal previous experiences, where the questions posed on the task may assume a core role (shaping the content of KoT – Phenomenology and applications).

CONCLUSION

Teacher's specialized knowledge must be wider and deeper than that of the student (Zakaryan & Ribeiro, 2018) and, therefore, because of its attention to the psychological aspects on which the concept of probability is based, we assume that the subjectivist point of view (de Finetti, 1931) can be a useful direction in the teacher education field.

In particular because, in agreement with Batanero, Godino and Roa (2004), the subjectivist approach allows one to build knowledge from the information one has, and thus can potentially create connections between intuitions and formal approaches to probability.

In the present study we have explored the possibility to inquiry and develop prospective teachers KoT within an approach to probability in the context of betting around events whose degree of consistent has been established according different criteria. We saw in the analysis of the two excerpts of discussion about the task, that PTs' in a certain sense implement the classical and frequentist approaches to probability, even if in the first excerpt such implementation was grounded in problematic aspects of PTs knowledge (mistakes) and without any explicit references to the formal definitions. This fact reveals PTs lack of awareness in linking the kind of information they want to access for decide the betting and the way in which they want to use them with the classical and frequentist approaches to probability. Nevertheless the implementation of the task allowed us to see that PTs use naive idea of such probabilistic views, elaborated from own previous experiences and school knowledge, and invest them in the task with a sense making process, where the questions posed on the task assumes a crucial role (shaping the content of KoT – Phenomenology and applications).

In this sense the subjective approach evoked by betting contexts, like for example the ones described in task presented in this paper, potentially creates a space of reflections and discussion in which the educators can promote PTs' KoT development and awareness about the links between the different approaches to probability. Possible future research paths could be aimed to analyse and deepen the potentialities of the mathematical discussion promoted in problem solving contexts on betting, in terms PTs' development of KoT and awareness between the different probability approaches.

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TEACHERS AND STANDARDIZED ASSESSMENTS IN MATHEMATICS: AN AFFECTIVE PERSPECTIVE

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Standardized assessments in mathematics have an increasing relevance in the educational debate and, often, they heavily affect educational policies. Specifically, the framework and the items of standardized assessments suggest what is considered relevant as an outcome of mathematics education at a certain school level. The strength and the quality of the educational impact of standardized assessments seem to depend heavily on teachers' affective reactions to standardized assessment; however, studies focused on this issue are very rare: what are teachers' attitudes towards the standardized assessments and their effects? In this frame, we carried out a large qualitative research to investigate teachers' attitudes in the Italian context.

INTRODUCTION

National standardized tests have been officially introduced in several countries on the wave of the most famous international programmes for student assessments (PISA and TIMSS). The framework and the items of the standardized assessments suggest what is considered relevant as outcomes of mathematics education at a certain school level. Therefore, more or less in an explicit way, standardized assessments intend to impact directly not only the educational reform promoted by politicians (Breakspear, 2012), but also, at the classroom level, teachers' educational choices. The test results are often used to assess the general quality of an educational system and this has often triggered a dispiriting horse race between countries, but also between schools at a national level. Nowadays, the growing relevance of standardized assessments in mathematics in the educational debate is a fact (Kanes, Morgan & Tsatsaroni, 2014). Several studies from different traditions in educational research have focused on the reliability of standardized assessments' results, discussing what such assessments really assess, to which degree they may be viewed as didactically consistent with official curricula (Bodin, 2005), and discussing their equity (Boaler, 2003). Another line of research focuses on students' performance, analyzing, in particular, students' errors (Wijaya et al., 2014) and interpreting factors affecting students' performance (Papanastasiou, 2000). Even though it has been shown that teacher affect heavily influences instruction and learning (Jacobson & Kilpatrick, 2015), it is curious to observe that the majority of these studies has a cognitive and epistemological perspective and little regard is paid to the variable 'teachers': only recently, Di Martino and Baccaglini-Frank (2017) introduced and discussed the concept of "developmental potential" of standardized tests, seen as the educational opportunities for teachers offered by a critical approach to tests.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 185-192). Pretoria, South Africa: PME.

In this frame, we carried out a large qualitative research to investigate teachers' attitudes towards the Italian test promoted by the National Institute for the Assessment of the Educational and Instructional System (INVALSI). In this paper, we focus on the following research questions: what are teachers' emotions towards the INVALSI test and what are the declared reasons to justify these feelings? Are there significant differences between teachers of different school levels?

METHOD AND RATIONALE

The context

Every year INVALSI develops and administers in May a census test for grades 2, 5, 8, 10. The number of items and the time granted vary depending on the school level. INVALSI shares the PISA framework and it designs the test items according to the official Italian National Standards. Despite this, there is a harsh debate because there is a unique test for grade 10, that is not differentiated for scientific high schools and professional institutes. The results do not affect the students' marks except for at grade 8, where each student's mark is entered (at least up to 2018) as part of the exam marking. In July, INVALSI sends back to the schools a quantitative report with the average score of each school class, and the comparison of the average scores of other schools with similar characteristics (numbers of students, social environment, etc.).

The collection of data

The choices of the research instruments and how to use them are not neutral. We developed our research within the more recent trend on affective factors in mathematics education. In particular, we refer to the shift from a normative approach – aimed to explain affective phenomena through general rules based on a cause–effect scheme – to an interpretive one – aimed to interpret the phenomena, "making sense of the world" (Di Martino & Zan, 2015). A movement towards the development and use of qualitative methods (essays, diaries, written open questionnaires, oral interviews) emerges in research on affect (and more in general in mathematics education). We developed our research within this frame.

In the first stage of our research we developed an online questionnaire and promoted the teachers' participation (on a voluntary and anonymous basis) with the help of the Italian regional education offices. In the last part of the questionnaire, participants had the possibility of sharing their e-mail address to participate to a non-anonymous semi-structured interview. This interview was developed to zoom into certain issues that emerged from the questionnaire answers. We were aware that we would not get a statistical sample in this way; however, our goal was to describe, interpret and understand a phenomenon and not to, in some sense, *measure* it. Therefore, we believe that having a convenience sample is not a limit. Participation exceeded all expectations: we collected 1964 questionnaire replies (very well distributed among the three school levels, see Table 1) and 798 participants agreed to participate in the second interview-stage. On one hand, this exceptional participation confirmed the teachers' interest in making their voices heard about this topic; on the other hand, it raised the problem of

the analysis of such a large amount of qualitative data, which needed to be very structured.

Conference Year	Number of questionnaire replies
Primary school (1-5)	635
Middle school (6-8)	643
High school (9-13)	681
Others	8

Table 1: Distribution of questionnaire replies among school levels.

The online questionnaire includes 28 questions divided into screens: background information (4), emotions (4), view on INVALSI items (5), perceived goals of the national assessment (3), strengths and weaknesses of the evaluation system (4), relationship between test and didactical practices (4), view on the evaluation system (4). Excluding the section about background information, the questions were mostly (15 out of 24) open questions; indeed, this approach allows to "catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data" (Cohen et al., 2007, p. 249). Psychologically central beliefs and emotions of the respondent emerge, differently from what happens with the traditional scales, where the respondent has only to express a degree of agreement with respect to items chosen by others, which may be not relevant for him/her.

The analysis of data

We approached the data within the social constructionist paradigm of the grounded theory. In their original work, Glaser and Strauss (1967) introduce grounded theory as an inductive approach of research (theory and focus have to emerge from data) that is at odds with "grand" theory, where data are used to verify a pre-determined theory. The radical grounded theory conflicts with the current view that researcher's knowledge, interests, values, attitudes, emotions and beliefs not only strongly affect the research in all its development, but constitute an added value. Strauss himself understand the limits of the original and radical idea of grounded theory, stating: "the final theory that is constructed through grounded in data is a representation of both participant and researcher" (Strauss & Corbin, 1998, p. 35). This consideration led to the development of different versions of the grounded theory, including the social constructionist one:

> *Rather than assuming that theory emerges from data, constructionists assume that researchers construct categories of the data* [...] *Social constructionists disavow the idea that researchers can or will begin their studies without prior knowledge and theories about their topics. Rather than being a tabula rasa, constructionists advocate recognizing prior knowledge and theoretical preconceptions and subjecting them to rigorous scrutiny.* (Charmaz, 2008, p. 402-404)

As for the present study, the context is quite well known by researchers: the INVALSI test provoked a spirited debate since its introduction; teachers and students often stand up against them and during the test days strikes are not rare. The main purpose of the present study was to understand the phenomenon and its causes. In order to do that, one issue was finding ways to reduce the copious amounts of data into manageable and comprehensible proportions. This was done through a process of coding that constantly took shape (the codes initially introduced were often refined during the analysis) and that ended when *theoretical saturation* was reached:

> *In constant comparison the researcher compares the new data with existing data and categories, so that the categories achieve a perfect fit with the data. New and emergent categories are developed in order to be able to incorporate and accommodate data in a good fit.* (Cohen et al., 2007, p. 493)

This coding process allowed to detect frequencies (which codes were occurring most commonly) and patterns (which codes occurred together).

RESULTS AND DISCUSSION

In this paper we will discuss the analysis of the answers to Question Q5 ("*What emotion comes to your mind when you think of INVALSI?*") and Q6 ("*What are the reasons for this emotion?*"), designed to investigate teachers' emotions and the declared reasons for these emotions. The tag cloud in Figure 1 summarises the wide range of labels used and the number of their occurrences in the answers to Q5. '*Anxiety*' is the label with the largest number of occurrences (240: more than 12 per cent of the large sample), followed by '*curiosity*' (182). More in general, the quantitative analysis of the data confirms a clear prevalence of labels that describe negative emotions towards the test (the 57% of the total). There are also very interesting qualitative differences: we can recognize a category of recurring emotions (*curiosity, interest, useless, boredom*) that, in some sense, expresses the judgment about the relevance of the test; but, in the negative case, we can also recognize a category that describes a strong emotional involvement (*anxiety, stress, anger, frustration, apprehension*).

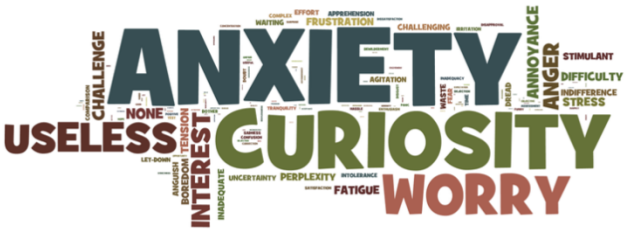

Figure 1: Tag cloud for Q5

The analysis of the answers to Q6, developed using the methodology described above, allowed to recognize three main and strictly interrelated categories of reasons for expressed emotions. These categories are related to judgements about the *adequacy, use* and *educational effects* of this kind of standardized assessment.

Adequacy

The adequacy is assessed at different levels, in terms of: mathematical content; item typology; general test structure and setting; *equity* of the test.

Mathematical content. Negative emotions are associated to the belief that INVALSI mathematics item content differs too heavily from the implemented curriculum ("*The arguments do not fit the actual curricula*", "*Items include questions about probability and statistics that are addressed only peripherally in primary school*"). The point is that INVALSI items are actually linked to the official curriculum (for example, probability and statistics are included in the Standards for primary school). The gap between Standards and implemented curriculum appears very clearly: in particular, at the primary school level, where the absence of a final examination probably gives greater freedom in the choice of the implemented curriculum.

Item typology. INVALSI items challenge students to solve stimulating non-routine problems: here again, the gap emerges between the problems proposed in the textbooks, and ultimately with the implemented curriculum. This fact is used as a reason to justify both positive and negative emotions. The positive emotions related to this fact implicitly highlight the negative opinion about the *curricular* problems ("*I like the types of problem used by INVALSI because they are different from the one used in classroom: they also seem closer to students' context of life*"). Vice versa, reasons for negative emotions highlight the belief that a *good* standardized assessment should adapt the items to the traditionally implemented curriculum ("*They need to design a test that is in line with what teachers do in classrooms everyday*") or to the textbooks' style ("*The problems in the test are completely different from those contained in the textbooks*") rather than to the Standards.

General test structure and setting. This is a recurrent issue to justify negative emotions towards the test: multiple choice tests are considered inadequate for assessing mathematics competence; primary school teachers label the children's lack of opportunity to ask the teacher for help as unnatural; the allotted time period is considered inadequate for coping with challenging items. Interestingly, we found 267 occurrences of the time factor in the analysis of Q6: 258 of them are related to a negative emotion and only 9 of them to a positive emotion! In the latter case, teachers underline that the setting is similar to other selections that children will have to face in their lives and for this reason it can be formative.

Equity of the test. A number of criticisms accuse the test and its scoring system as being unequal because they are intended to assess all students equally, without taking into account students' social backgrounds and their starting points. In particular, secondary teachers underline the difference between students from high schools and professional

institutes, while primary and middle school teachers report the difficulties, also at an affective level, for students with special needs.

Use

A widespread perception emerges about the test results being used to assess teachers' efficacy. While official documents make it explicit that the INVALSI test will not be used to evaluate teacher (as it is, instead, in other countries), it is true that, locally, teachers with the worst results are often put under pressure by their principals. In this case, it is very interesting to look at the different reactions of primary and middle school teachers on one side, and secondary teachers on the other. The former uniformly criticize the fact that they are being indirectly assessed; the latter accept the possibility of being assessed, but complain about the students' scarce motivation to perform well and about the absence of differentiations ("*I think it is not right that results obtained from different classes, different schools, different regions, are used to draw conclusions about the validity of a teacher*").

Educational effects

This category is strictly related to the previous one: teachers believe that the tests have educational effects, especially because their use goes beyond simply assessing students' mathematical competence. In some sense, there is the widespread belief that the introduction of the test has really affected teachers' practice. While some secondary school teachers appreciate that INVALSI *forces* the teachers of the lower school levels to cover all the contents in the Standards and to propose non routine mathematical problems, in general primary school teachers criticize the educational effects of the test. They feel the pressure of having to cover all the topics included in the curriculum, regardless of the specific needs of their classes ("*You should not try to standardize teaching, you should encourage teachers to take into account the specific needs of their classroom*"): they believe that this system forces to privilege quantity rather than quality in education. On the other hand, the main criticism advanced by middle and secondary school teachers is the risk of promoting a method of education focused on preparing students for a standardized test, the so called teaching to test approach ("*The tests are likely to interfere excessively with the teaching, inducing an ad hoc training*").

CONCLUSION

The quantitative analysis of the data collected in our study highlights two facts. On one hand, the exceptional voluntary participation proves the teachers' interest towards the issue of standardized assessment of students' mathematical competence. On the other hand, the supposed prevalence of negative feelings towards the national standardized assessment of students' mathematical competence is confirmed. However, what is really important is the qualitative analysis of the different reasons for the teachers' declared emotions. What emerges is a complex picture that includes positions of

principle against the standardized assessments and their uses, but also more specific criticism towards the design of the test. In this contribution, we discussed only a small part of the data collected in the whole research project, but, for example, the data collected with the closed question Q25 ("*What would you do with the INVALSI test?*") show that *only* the 27% of respondents stated the desire to suppress the standardized assessment: the 60% of the sample would maintain the test, recognizing its role, but asking for significant amendments.

At the end of our study, we are even more convinced it is fundamental to consider teachers' affect and to listen teachers' voices about standardized assessment: teachers are one of the key players in the educational context and their active involvement is needed to improve the tests and to exploit their informational and developmental potential (Di Martino & Baccaglini-Frank, 2017). As Ponte et al. argue:

> The reasons for studying the views and attitudes of teachers are grounded in the assumption that these have a significant influence on their thinking and actions. Views and attitudes act as a sort of filter. (Ponte et al., 1994, p. 347)

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THE CO-EMERGENCE OF VISUALISATION AND REASONING IN MATHEMATICAL PROBLEM SOLVING: AN ENACTIVIST INTERPRETATION

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This paper reports on a small part of a Namibian case study that focussed on the relationship between visualisation and reasoning when selected young learners solved geometry word problems in small groups. Rooted in an enactivist perspective, the study sought to analyses and theorise this relationship through the enactivist mediums of co-emergence and structural coupling. The broader aim of the research was to contribute to a deeper understanding of how visualisation processes are integral components of reasoning in the solving of geometry word problems.

THEORETICAL FRAMEWORK

Enactivism as defined by Begg (2013) is "a way of understanding how all living organisms including human beings, organize themselves, and interact with their environments" (p. 18). According to Maturana and Poerksen (2004), an enactivist perspective questions how the elements of a system work together to form that system. In this paper, we report on how selected Grade 11 learners worked together in small groups to solve word problems and reach a collective solution. The analysis of the relationship between reasoning and visualisation was rooted in notions of co-emergence in our study.

Co-emergence as interpreted by Li, Clark and Winchester (2010, p. 407) refers to a situation whereby a change of both, a living system and its surrounding environment depends on the interaction between the system and its environment. When the system and the environment interact, they become structurally coupled. This means that the mutual interaction of the organism and the environment causes changes and transformations in both (Khan, Francis, & Davis, 2015).

Begg (2013, p. 82) claims that from an enactivist perspective, humans and the world are inseparable: they co-emerge. Brown (2015) concurs that "we are co-emergent and where there is a coordination of actions, like in a classroom, or a collaborative group in a research project, a culture of practices emerges that is good-enough (effective action) to get done what needs to be done" (p. 188). However, while co-emergence suggests that the system and the environment interact, Li et al. (2010, p. 407) caution that it does not guarantee greater or lesser adaptation on the part of either to each other. Students bring forth a world; they emerge with it, but it is their structures that bring them forth (Proulx, 2008, p. 22). This inseparability of body, mind and environment is known as embodied cognition (Alibali & Nathan, 2012; Antel, 2009; Wilson, 2009).

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In our study, enactivism acted as a linking mediator to understand the bringing together of visualisation and reasoning processes. It is the lens through which the co-emergence of visualisation and reasoning was observed and analysed when the research participants solved word problems in small collaborative groups. Khan et al. (2015, p. 272) enlighten that enactivism is attentive to the coupling of organisms and their environments, action as cognition, and sensorimotor coordination. Therefore, the enactivist notion of *"learning in action"* resonates strongly with this study as opposed to the embodied cognition notion of "*learning from action"* (p. 272).

RESEARCH SITE, PARTICIPANTS AND METHODS

The research was conducted in a well-resourced secondary school in the south of Namibia with a cohort of purposefully selected Grade 11 learners. The aim of the study was twofold: in the first phase, to identify research participants who preferred the use of visual imagery as opposed to algebraic methods of problem solving. In the second phase, to observe and analyse the relationship between visualisation and reasoning processes while learners solved geometry word problems. Initially a cohort of 17 mixed gender and mixed ability learners were selected for the first phase and then eight participants were selected for the second phase. For each phase we used a different version of the Enacted Visualisation Geometric Reasoning Tasks (EVGRT) worksheet as our data gathering instrument.

Enacted Visualisation Geometric Reasoning Tasks (EVGRT)

The EVGRT Worksheet 1 (EVGRT W1) was implemented in the first phase. It consisted of 10 tasks that were solved by all the 17 participants in a one-on-one task-based interview environment. The EVGRT W2 was implemented in the second phase. It consisted of 5 tasks that were solved in small collaborative groups by the eight participants.

The tasks for both worksheets were all of such a nature that they encouraged the participants to make use of their own visual representations. In the task based interviews the participants were asked by the researcher to explain their visual representations and articulate their mathematising and their mathematical reasoning.

The task-based interviews were both voice and video recorded. The data was transcribed and our line-by-line analysis was guided by two analytical tools; one to seek for evidence for visualisation and the other for reasoning processes. As part of the validation process, both EVGRT worksheets and analytical frameworks were piloted and refined.

THE NATURE OF VISUALISATION AS A WORD PROBLEMS SOLVING STRATEGY

According to Arcavi (2003) mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena, relies heavily on visualisation in its different forms and at different levels, far beyond the obviously

visual field of geometry, and spatial visualisation (pp. 216–217). Arcavi's (2003) defines visualisation as follows:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding. (p. 217)

In addition to embracing this definition, we adapted Presmeg's (1986) categories of visual imagery to construct analytical frameworks that enabled us to understand in some depth the nuances of the visualisation and reasoning processes that were evident as our participants were observed solving the tasks in the EVGRT worksheets. Visual imagery as defined by Hegarty and Kozhevnikov (1999) is the ability to form mental representations of the appearance of objects and to manipulate these representations in the mind. The five categories of visual imagery (5Vs) that Presmeg (1986) suggested are defined as follows:

Concrete pictorial imagery (CPI) – this refers to the concrete image(s) of an actual situation formulated in a person's mind – i.e., a picture in the mind, drawn on paper or described verbally.

Pattern imagery (PI) – this refers to the type of imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme. The essential feature of pattern imagery is that it is pattern-like and stripped of concrete detail (Presmeg, 1986).

Memory imagery (MI) – this refers to the ability to visualise an image that one has seen somewhere before. This too includes a history of recurrent occurrences.

Kinaesthetic imagery (KI) – this is the kind of imagery that involves muscular activity. A kinaesthetic visualiser wants to feel and touch.

Dynamic Imagery (DI) – this imagery involves processes of transforming shapes i.e. redrawing given or initially own-drawn figures with the aim of solving a problem.

The notion of using one's body to solve word problems both individually and in a social setting was particularly significant to our study of enacted and embodied visualisation, hence the inclusion of KI in our analytical framework. We observed that the participants engaged their bodies substantially to solve problems in EVGRT W1. Alibali and Nathan (2012) describe the activity of using one's body in problem-solving in terms of "gestures". They assert that "gestures are often taken as evidence that the body is involved in thinking and speaking about the ideas expressed in those gestures. That is, gestures are taken as evidence that the knowledge itself is embodied" (p. 248).

Since it is difficult in this very brief paper to report on major aspects of a study which resulted from extensive qualitative data, we present some of the data from both phases of data collection. We start with a brief discussion of data from the first phase when the participants attempted to solve Task 9 of EVGRT W1 (Figure 1).

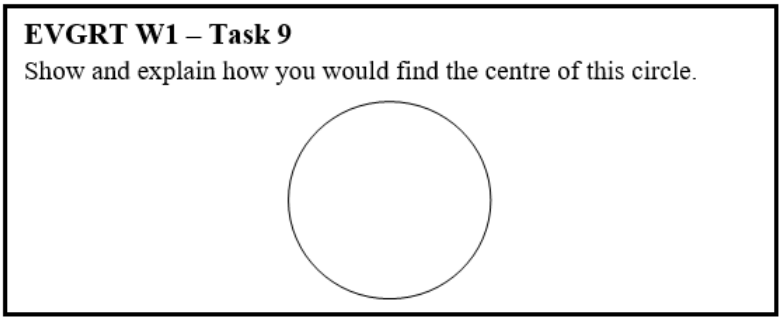
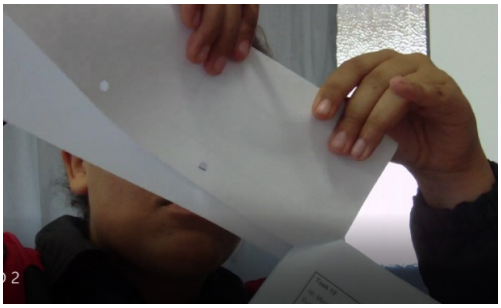
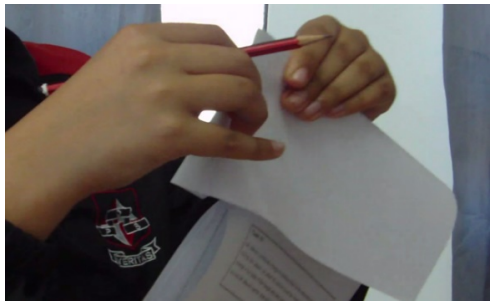


Figure 1: EVGRT W1 Task 9

A number of our research participants were natural visualisers but one of the participants, Millie, stood out in how she applied visualisation processes to solve most of the tasks of the EVGRT W1. She interacted well throughout the task-based interview and portrayed strong relations between her use of visual imageries and reasoning processes. For Task 9 (Figure 1), Millie applied a combination of visualisation processes as part of her problem solving strategy. These co-emerged with various forms of reasoning processes. While others reached for a ruler and other geometry instruments to attempt to find the centre of the circle, Millie used CPI combined with KI and DI to figure out the centre of the given circle (Figure 2). Once Millie asked whether she could fold the paper, she did not utter a word. She worked silently, engaged her body and made facial expressions which she claimed to be part of her thinking process until the final step (Figure 2 (d)) when she exclaimed, "*there is the centre! Perfect!*" Antel (2009) claims that giving considerations to ways in which cognition is rooted in bodily actions will contribute to learners' successful development into active, thinking adults (p. 30).

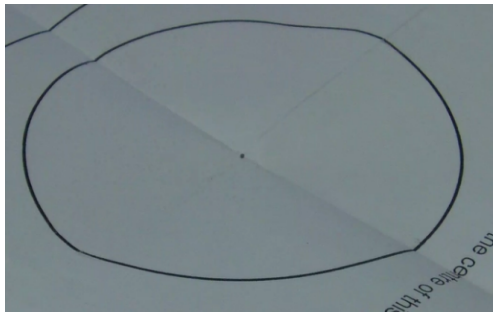


(a) "*can I fold the paper*?"



(b) Measures and folds the paper silently

(c) Folds the paper into two symmetrical semi-circles



(d) "*There is the centre! Perfect!*"

Figure 2: Millie's visualisation processes of EVGRT W1 Task 9

We needed those participants who preferred visual methods over algebraic methods to proceed with to the second phase of data collection. It was during this second phase were we observed closely the participants' mathematical reasoning in relation to their already noted visualisation processes enacted in the first phase of data collection. But, before we look at that relationship, we ought to comprehend how literature defines mathematical reasoning.

MATHEMATICAL REASONING

Mathematical reasoning as defined by Brodie (2010, p. v) refers to a process that involves forming and communicating a path between one idea or concept and the next. It is a means to sense-making of and in mathematical activity. Further, "only through making sense of the mathematics can we truly move to sense-making as a worthwhile everyday life activity" (Brodie, 2010, p. 59). In our observations we sought evidence of reasoning alongside visualisation processes as the participants solved the individual tasks of the EVGRT W2. We considered four reasoning processes (4RPs) viz. explanation, justification, argumentation and generalisation for the purpose of our study. These are defined as follows:

Explanation (RPE) refers to the classification aspects of one's mathematical thinking that he/she thinks might not be readily apparent to others.

Justification (RPJ) is defined by Staples, Bartlo and Thanheiser (2012) as "an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (p. 448).

Argumentation (RPA) is defined by Lithner (2000) as the "substantiation, the part of reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate" (p. 166).

Generalisation (RPG) refers to the process of identifying operators and the sequence of operations that are common among specific cases and extends them to the general case (Swafford & Langrall, 2000).

It was during the second phase of data collection that the participants' reasoning processes were analysed and presented in relation to their 5VIs. Building on Millie's visualisation processes discussed earlier, we present some of the data from the second phase in the form of one of the vignettes that we generated from her interactions with Rauna when they worked in their small group. The vignette presented in this paper serves to illustrate the co-emergence and structural coupling of visualisation and reasoning as a result of the participants' interaction during a focus group task-based interview.

THE CO-EMERGENCE OF VISUALISATION AND REASONING

In this vignette, we discuss the reasoning process of argumentation (RPA) with reference to the 5VIs when Millie and Rauna embraced RPA to solve Task 5 of EVGRT W2 (Figure 3).

EVGRT W2 – Task 5
In a cube with sides of length 10 cm, denote one vertex by the letter V. Find the sum of the total distances from V to each of the other vertices of the cube.

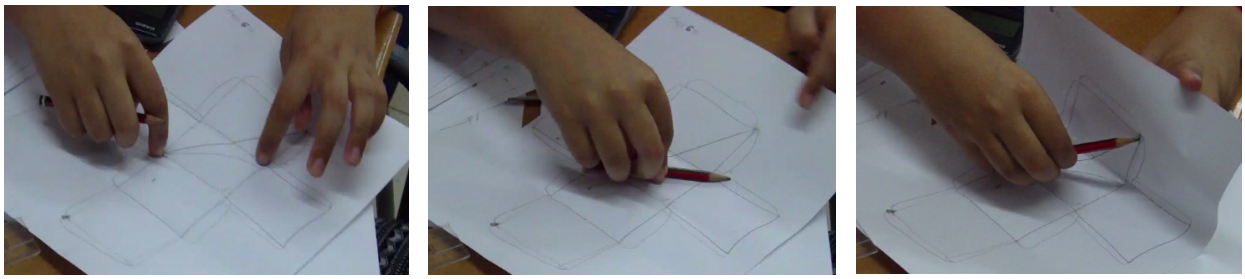
Figure 3: EVGRT W2 Task 5

Rauna studied the cube that she held and rotated between her fingers as she spoke of "*going through the cube*". When the others could not comprehend her thoughts, she sketched a net of a cube (Figure 4 (a)) as a way of increasing understanding. She argued that the diagonal of the cube equalled the length of its face, which was 10cm plus the diagonal of the face, which was 14.1cm (Figure 4 (b) and (c)).

(a) *"if we have to do this"* (b) *"the distance here in the middle"* (c) *"is the same distance as if we go through"*

Figure 4: Rauna's argumentation for the fifth task

Although Rauna's argument in Figure 4 above was not accurate, she inadvertently convinced Millie to accept it as a result of her own visualisation and reasoning skills. Figure 5 illustrates that mathematical arguments do not have to be accurate in order to convince someone. Here all it took was some of Rauna's argumentative skills merged with visual imagery to convince Millie. Initially Millie was convinced, see Figure 5, as she imitates Rauna's argument.



(a) *"she was saying, this distance…"* (b) *"…is fourteen point one"* (c) *"if you fold it like this, it is still 14.1"*

Figure 5: Millie convinced by Rauna's argument

Both Rauna and Millie employed an amalgamation of visual imageries and reasoning processes during this process of collaborative argumentation. Although Millie was initially convinced by Rauna's arguments, she eventually refuted her claims as the picture in her mind became clearer and as she visualised and reasoned more on the task. From an enactivist point of view, we concluded that there was a structural congruence between the participants' interaction and their environment (Maturana & Varela, 1998,

p. 95). Rauna, Millie and the environment had undergone transformations – they became structurally coupled (Maturana & Varela, 1998, p. 102). The result of this coupling was the co-emergence of visualisation and reasoning processes.

SIGNIFICANCE FOR MATHEMATICS TEACHERS AND EDUCATORS

From the above observation snapshot and in the context of this study, we are able to conclude that learners involved in word problem solving not only employ a variety of visualisation processes, but also incorporate those in their mathematical reasoning. The extent to which these processes are employed is a matter of preference. This has implications for mathematics teachers and the roll-out of their lessons.

This study taught us the importance of task-based conversations which teachers could use to communicate with their learners in order to understand their mathematising better and to anticipate possible misconceptions. In our case, Rauna's misconception was so entrenched in her thinking that she initially failed to tell the difference between the length of the diagonal on the face with that within the cube. She justified her reasoning by unconsciously folding the net to reach for the opposite vertex (Figure 4). Nevertheless, she argued well, and others were convinced – which was the overall focus of this study. Despite the misconceptions, what became important to us were the visualisation and reasoning processes that emanated from the participants' interactions.

Although studies on mathematical reasoning, visualisation and problem solving are widespread in various part of the world, our use of enactivism to theorise the interaction and interplay between visualisation and reasoning is a new contribution that we are glad to explore further.

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THE BODY OF/IN PROOF: EVIDENCE FROM GESTURE

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Doctoral students in mathematics took part in clinical interviews during which they were asked about their experiences with teaching, learning and doing proof. The analysis here focuses on the discourse, both spoken and gestural, used by these students in talking about and carrying out proofs, and on an analysis of this discourse from the framework of embodied cognition. Specifically a comparison is made with gestures used in everyday discourse related to "if-then" statements.

INTRODUCTION

The purpose of this paper is to examine mathematical proof and logical reasoning from the perspective of embodied cognition (Edwards, 2011; Varela, Thompson, & Rosch, 1991), using data collected from clinical interviews with 12 doctoral students in mathematics. In recent decades, researchers have investigated how the body in implicated in mathematical teaching and learning, challenging the paradigm that cognition is amodal and abstract, based solely "in the head." In addition, attention to embodiment has broadened the focus within mathematics education research beyond written symbols, images, and oral speech to include modalities such as gesture and other bodily movements (Edwards, Ferrara, & Moore-Russo, 2014; Hall & Nemirovsky, 2012).

The analysis presented here is based on the principle of cognitive continuity; that is, the theory that there are not two or more different kinds of thinking, for example, abstract proof and everyday thinking, but instead, even "advanced" mathematical thinking involving proof and logic are grounded in basic human experiences. As Johnson states, "we do not have two kinds of logic, one for spatial-bodily concepts and a wholly different one for abstract concepts. There is no disembodied logic at all. Instead, we recruit body-based, image-schematic logic to perform abstract reasoning" (Johnson, 2012, p. 181).

Following Hanna (1990), we take proof to be:

> [A] finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last sentence is the one to be proved. (Hanna, 1990, p. 6)

The specific focus here is on logical statements that take the form of "if-then" statements; these statements can be seen as the building blocks of proofs. The central research question is whether the physical gestures that accompany these "if-then" statements when talking about proof are similar to those accompanying "if-then"

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statements in non-mathematical contexts. If so, then this would provide support for the notion of cognitive continuity between these two contexts.

THEORETICAL CONTEXT AND RELATED RESEARCH

Within an embodied cognition framework, mathematics is not seen as a transcendental, formal collection of rules and patterns, unrelated to everyday thinking and experience, but instead, as a human intellectual product, one which develops both historically as a discipline over time, and ontologically as it is constructed by an individual learner. It is socially-constructed, but not in an arbitrary way, being both constrained and enabled by the biological capabilities and physical situatedness of human beings. Embodiment does not negate the influence of social interaction and culture; rather it grounds it in shared biological constants (Hall & Nemirovsky, 2012; Nuñéz, Edwards, & Matos, 1999). As stated by Hall and Nemirovsky (2012), "We think of concepts (in mathematics but also in other domains) as forms of modal engagement in which bodies incorporate and express culture" (p. 212).

Prior research on proof

Prior research has examined the learning and teaching of proof (a selection of recent work can be found in Lin, Hsieh, Hanna & deVilliers, 2009). The current research builds on this foundation, particularly in seeing proof as a form of socially constructed knowledge and a specific form of discourse (Balacheff, 1991; Sfard, 2001). The current analysis adds the lens of embodiment and gesture studies in analyzing this discourse.

Prior research on conditional statements

Logical deductions or "if-then" statements belong to a linguistic category known as conditionals, specifically the type called epistemic conditionals (because they reference a reasoning process). Two examples are: "If the car is in the driveway, he must be home" and "If x is even, then x/2 is an integer" (Dancygeir & Sweetser, 2005, p. 17). These kinds of conditionals involve what Danceygeir and Sweetser call a "metaphoric 'compulsion'" (p. 20) in which the speaker is "forced" to draw the given conclusion, either based on inductive reasoning ("the car is almost always in the driveway when he is home") or deductive logic (the mathematical definition of "even"). The embodied metaphorical basis for construing proof in terms of the "compulsion" found in physical causality is discussed in Edwards (2017).

In addition to a linguistic analysis of conditional statements, recent research by Sweetser has examined gestures associated with orally-stated conditionals. In a study involving 402 video clips of talk shows, Sweetser and Smith (2015) found that conditionals were generally accompanied by a particular hand motion, specifically a movement along a transverse axis through gesture space, starting on the speaker's left and moving toward the speaker's right. The current analysis examined the gestures of

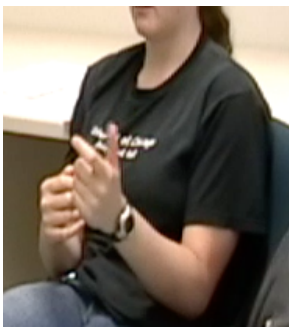
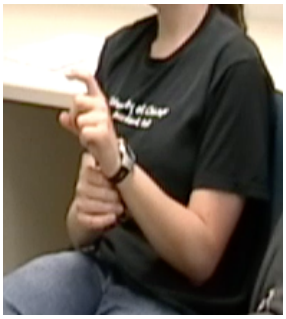
mathematical doctoral students to see whether they also reflected this characteristic motion when orally stating epistemic conditionals. If so, then this would constitute evidence of the continuity between everyday uses of conditionals and their use in mathematical proof.

METHOD

This qualitative study was based on a 90-minute clinical interview with 12 doctoral students in mathematics at a research university in the United States. The students participated in pairs in the interview and problem solving session, which was videotaped. The first portion of the interview consisted of general questions about their mathematics learning, teaching, and ideas about proof. The second part lasted 40 minutes, during which the students were asked to prove a conjecture together, writing their work on the blackboard. During the third part, the students were asked to evaluate a visual "proof." The illustrations presented here are drawn from the first portion of the interview.

RESULTS

The analysis of the doctoral students' gestures when making conditional statements did indeed reveal the presence of the same left-to-right transverse gesture previously identified in non-mathematical contexts. The example shown in Figure 1 illustrates three instances of this gesture form. In this example, the epistemic conditional that the student is expressing can be summarized as follows: "If you have a scalar function and a vector function, then the rule for finding their product is the same as the rule for finding the product of two scalar functions."

AC: Well, I guess, so, the other day they were trying to prove that, um, if you have some <u>scalar function of T</u> Int: Uh huh AC: —and some <u>vector function of T,</u>	 Figure 1a  Figure 1b	Left hand starts in horizontal C-shape ("bracket") facing upward on left side of body Left to right motion with left hand along transverse axis, ending in middle of body, with C-shape turning vertical

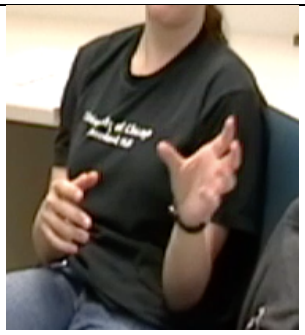
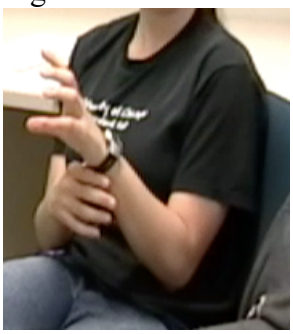
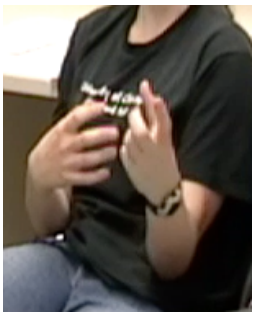
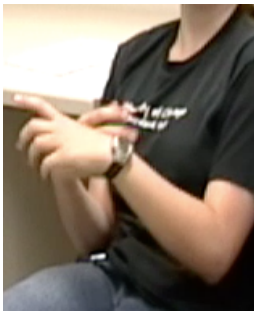
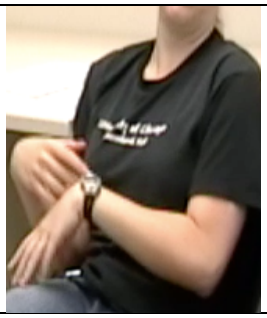
Int: Uh huh AC: —that the derivative of their product...	 Figure 1c  Figure 1d	Left to right motion with left hand along transverse axis, with left hand open and facing outwards. Left hand begins on left side of body and ends in middle of body.
is the same...	 Figure 1e  Figure 1f	Rapid left to right motion with left hand along transverse axis. Left hand starts in loose horizontal C-shape ("bracket") facing upward on left side of body and ends in pointing gesture to the right.
AC: ...product rule essentially that you know from just, you Int (talking over): Uh huh.		A complex motion in which the left hand begins by pointing downward, then is moved in a circle twice around the right hand while saying "you know," ending up open and facing the speaker

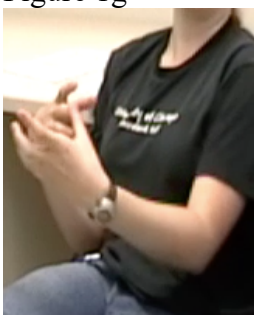
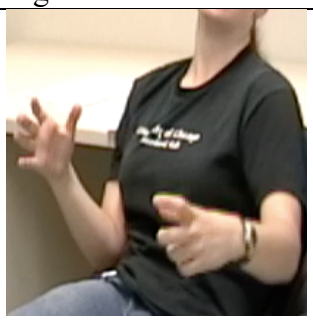
	Figure 1g  Figure 1h	
AC: know from like <u>scalar functions</u>	 Figure 1i	Left hand moves to right and finishes in horizontal C-shape ("bracket") on left side of body. This is the same shape and location as when the phrase "scalar function" was initially uttered.

Figure 1: Student's discourse about scalar functions

Note: Underlined speech indicated the stroke or emphasized portion of the gesture

The sequence of gestures accompanying the student's speech is very rich, taking into account characteristics including hand shape and orientation, hand location, and movement of the hands through space. Consistent with other conditionals used in non-mathematical contexts, the sequence includes left-to-right motion along the transverse axis; in fact, this transverse motion occurs three different times, as shown in the pairs of figures above:

- Figure 1a – b: A relatively small left-to-right motion of the left hand, as AC begins by saying, "If you have some scalar function of T and some vector function of T." This sequence also includes a change in orientation of the left hand; when holding it on the left, AC uses an upward-opening (horizontal) C-shape as if "bracketing" or "holding" a scalar function. As she moves her hand to the right, she rotates her wrist so that when she says, "vector function," the C-shape is now vertical. She thus uses both hand shape and hand location to gesturally distinguish the two different kinds of functions.
- Figure 1c – d: A wider left-to-right motion of the left hand, as AC says, "the derivative of their product." In this case, the hand shape stays the same throughout, open and facing outward.
- Figure 1e – f: After saying "derivative of their product," AC pauses briefly, then makes a very rapid left-to-right motion of her left hand while saying, "is the same," starting with a horizontal C-shape and ending with a right-facing point.

As can be seen above, in addition to an overall left-to-right movement that occurs three times during the sequence, gestures are also used to mark or indicate specific

mathematical objects, in a scheme that Calbris (2008) calls "two-entity opposition." Two-entity opposition occurs when either two locations in space or the two hands are used to denote or "mark" two related but distinct entities. In Figure 1, this happens when AC uses a horizontal "bracket" held to her left when saying "scalar functions" and then a vertical bracket held to her right when saying "vector functions." The terms "derivative" and "product" have the same hand shape but are marked by left and right hand locations, indicating two-entity opposition.

The discourse segment ends with AC discussing a "product rule" while using an iterative circular gesture during a pause in speech (possibly searching for her next words), and then comparing it to the rule for scalar functions. Interestingly, the final gesture of the sequence, associated with the words "scalar function" has an identical shape and location as the gesture used the first time the words were uttered. This is an example of using specific hand shapes and locations in gesture space to "hold" a referent in discourse (Calbris, 2008; McNeill, 1992).

DISCUSSION

Calbris (2008) has stated that in gesture space, the transverse axis can represent logico-temporal concepts, such as cause and effect, or before and after:

> A path in space or time is depicted by a left-to-right movement. But give that body symmetry allows this axis to account for splitting in two as well as two-entity oppositions, it can be used to oppose past and future, or precedence and successor, by locating the past on the left side and the future on the right side. (Calbris, 2008, p. 43)

In the current case, and in the research by Sweetser and Smith (2015), the transverse axis is used to indicate the premise followed by the conclusion of a conditional, "if-then" statement.

The transverse axis of the body has been also called "the axis of reading and writing, pointing to the right in the Western world" (Calbris, 2008, p. 28). In this case, the motion of AC's gestures is consistent both with the placement of the "cause" (premise) on the left and the "effect" (conclusion) on the right, as well as the left-to-right order in which premise and conclusion are generally written in English. In the example given above, the left-to-right motion along the transverse axis is thus consistent both with how "if-then" statements are written in English, and with prior research and theory identifying this gestural motion with logical and conditional statements.

Taken in conjunction with related research (Edwards, 2010, 2011, 2017), we would argue that the examples above provide further evidence that proof and its building blocks, statements of logical deduction, are not abstract elements of disembodied rationality. Instead, we argue, these sophisticated forms of discourse make use of metaphorical mappings related to motion and are supported by conceptual metaphors grounded in physical experiences.

Mathematical proof is thus seen as a specialized cultural product and a specific form of discourse, with particular constraints that distinguish it from everyday speech and

make it more powerful for the purpose of exploring structure and patterns. Yet the form that this discourse takes is not arbitrary, but rather is grounded in embodied human experience. As shown above, there exists a continuity between the gestural grounding for the logical conditionals used in proof and those used in non-mathematical contexts. This kind of analysis is relevant to mathematics education because the conceptual sources that students draw from in constructing new mathematical knowledge may not correspond to the more sophisticated intra-mathematical sources that their instructors use (c.f., Núñez, Edwards & Matos, 1999). For example, students who are beginning to learn about formal logic often "import" expectations about conditionals from everyday speech, assuming that "if A, then B" implies "if not A, then not B" (Evans, Newstead, & Byrne, 1993). A better understanding of the cognitive roots of mathematical thinking may help in developing better instruction in such situations.

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TEACHING TO CHANGE WAYS OF EXPERIENCING NUMBERS – AN INTERVENTION PROGRAM FOR ARITHMETIC LEARNING IN PRESCHOOL

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This paper reports on an eight months long intervention program with eight five-year-olds in Swedish preschool. Four main activities were designed to enable the children to discern part-part-whole relations of the first ten numbers. The aim of this paper is to present how progress in children's arithmetical skills are associated with the activities they have encountered in the intervention program. Learning outcomes based on pre-, post- and delayed interviews show that the participating children made distinct progress in the way they experience numbers, with long-term effects on their arithmetic skills. In this paper we discuss the analysis of what was taught and what was learnt in commensurable terms.

BACKGROUND

This study directs attention to a theoretically designed mathematics intervention program and how this is reflected in the children's learning outcomes. There are many kinds of interventions aimed at young children's mathematical development, but few make it an issue to focus on intervention implementation related to learning outcomes, probably due to the complexity of studying the relation between what is taught and what is learnt (Domitrovich et al. 2010; Durlak 2010; Hedges, 2018). This is addressed in our study by focusing on 5-year-old children's learning outcomes related to the implementation of a certain way of teaching about numbers.

INTERVENTIONS IN EARLY MATHEMATICS

Several international studies have identified positive effects of intervention studies in early childhood mathematics education (e.g. Jordan, Kaplan, Ramineni & Locuniak, 2009; Sarama, Clements, Wolfe & Spitler, 2012). Interventions in early childhood usually show growth in children's knowledge and skills and low-attainers often gain the most in the process (Mononen et al., 2014). The vast majority of interventions in preschool constitute a battery of activities to be conducted during a certain period of time by the preschool teachers. Most intervention outcomes are assessed by standardized tests of mathematical abilities and arithmetic skills in particular. These generally provide results that children manage to solve more advanced arithmetic tasks after an intervention, but research also reports a severe fade-out effect in longer terms (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). Generally, what stands out in the

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 209-216). Pretoria, South Africa: PME.

intervention programs conducted with preschool children is that they test children's increase in producing correct answers to numerical tasks or abilities to use certain strategies as a measure for successful interventions. However, there are also qualitative analyses of how children understand some of the objects of interest in the interventions, which broadens the picture of what an intervention actually develops among the children. Mulligan and Mitchelmore's studies (2013) are examples where the outcomes are tested by children's ways of expressing structural awareness in drawings, not the frequency of correct answers alone.

A strong focus on skills and abilities to use arithmetic strategies leaves important aspects in the background that might influence on the stability of learnt skills. The aim of this paper is therefore to present how progress in 5-year-old children's arithmetical skills are associated with the activities children have encountered in an intervention program. The specific research question is: How can the development in children's arithmetic skills be associated with what was afforded in the activities? In our study we implement a theoretically driven approach (Variation theory of learning) to teaching basic arithmetic to young children that takes its departing point in how children experience numbers, rather than their skills to solve arithmetic tasks correctly.

THEORETICAL FRAME

In this paper we study the enacted object of learning (the implemented program) related to the lived object of learning (what the children actually learnt) based on Variation theory that describes these objects of learning in commensurable terms. Variation theory proposes that looking at children's learning as a function of what is possible for a child to experience in a given situation, could open for studying what in an intervention theoretically and empirically influences learning outcomes. Seeing learning as a change in ways of experiencing a phenomenon, is a fundamental principle in variation theory (Marton, 2015). Learning something is due to experienced aspects of what is to be learnt (the object of learning) that have not previously been discerned, and thereby perceiving the world in a more differentiated way. Discernment presupposes an experienced variation of the aspect in question. These principles – discernment and variation – can be applied when designing tasks and activities to draw learners' attention on essential structures and make learners aware of relationships of numbers. Subsequently, from these principles, it is possible to analyse what is afforded to learn, also. In this way, learning and teaching are described in commensurable terms; as afforded variation (the enacted object of learning) and as experienced variation (lived object of learning). We would suggest these theoretical notions give a coherent foundation possible to apply to data for relating what is taught and learnt.

THE STUDY

This paper focuses on how the implementation of designed activities following theoretical principles inflicts changes in children's ways of experiencing numbers.

For this particular paper, one preschool group (8 of 65 children participating in the program, part of a larger project financed by the National Research Council, grant no 2014-1791) was selected for qualitative investigation of the teaching and learning processes they have taken part in. This group was chosen due to their collective low results on the pre-interview and the substantial amount of available observation data from the intervention activities.

The intervention

Teachers and researchers collaborated in planning, enacting and reflecting on number activities during a period of eight months. The teachers conducted each activity several times in their preschool groups and video-recorded some of their enactments as basis for assessment, analysis and further development of the activities. We aimed for the children to develop their ways of experiencing numbers in that they would handle numbers in arithmetic situations in proficient ways, which includes knowledge and awareness of part-part-whole relations and other necessary aspects of numbers (Marton, 2015; Baroody, 2016).

Four main activities constituted the program, all designed to enable the children to discern in particular the part-part-whole relations of numbers by enacting the activities through certain patterns of variation. *The statement game*: The children show the number of dots on a dice with their fingers, using both hands. The teacher encourages them to show the number in different ways. *The snake game*: A string with ten beads, grouped as five in one color and five in another color. The children first present the whole number (ten) with fingers. Some beads are hidden, the children show the visible number of beads with a finger pattern. By looking at their unfolded/folded fingers they 'see' the missing part. *Finger patters*: The children identify finger patterns shown by the teacher (>5) with a number word. The teacher asks how many more/less fingers are needed to represent a different number. *Context problems*: Number stories to which the children model the problems on their fingers.

Analysis

In total 28 video-recorded activities and task-based individual interviews from three occasions were analysed. The eight children were interviewed before and immediately after the intervention and one year after the program ended. The interviews constituted task-based arithmetical problems within the number range 1-10. The same tasks were given at all three occasions. For this particular analysis, a total of 64 answers/observations of 8 tasks were analysed. The children's answers have been analysed both quantitatively and qualitatively. The latter is in focus for this paper, in which we have interpreted children's ways of handling numbers in finding a solution to the arithmetic tasks. These acts and utterances are interpreted as expressions of their different ways of experiencing numbers.

The intervention activities were analysed in terms of the afforded aspects of numbers and each interview was categorized according to the expressed ways of experiencing numbers. Variation theory offered a theoretical framework for interpreting this relation between what was taught and what was learnt in commensurable terms.

RESULTS

First, we present what the activities in the intervention program afforded the children to learn: finger patterns illustrating numbers, numbers as composite sets, part-part-whole relations of numbers, and the commutativity principle and how this was enacted in the activities. The children's learning outcomes are thereafter presented in qualitative terms and related to the afforded learning objects in order to answer the research question what was taught and what was learnt.

Finger patterns were promoted as means for representing number relations in all activities. However, for modelling the part-part-whole relationship on fingers and seeing them as composite sets of units (not as single units) the children first needed to discern that a specific number can be represented on fingers as different finger-pattern sets (six can be represented as a whole hand and one thumb but also as three fingers on one hand and three on the other).

The statement game enabled the children to see that there were different ways of modelling the same number, as the teacher compared children's ways of showing numbers by asking: 'same way?', 'different ways?', 'in how many different ways?' The activity thus enabled, when enacted in this way, the children to discern numbers as composed by different parts. Furthermore, the game opened up for finger patterns to be used to represent numbers, both parts and whole, as related to each other.

The finger patterns worked as a tool to open up for different ways of representing numbers but also to structure number relations in a sensual way. This was further developed to enable the children to experience the relation between parts and whole, simultaneously, which was enacted primarily in the snake-game. When finger patterns represent two sets of units (folded and unfolded fingers) the usefulness of structuring the part-part-whole relation on the ten fingers emerged when the children were asked to identify the visual part, show it with their fingers and then identify the hidden part. Some children looked at their knuckles, others turned their hand around – the finger pattern thus became a tool for identifying missing parts, without having to count single units.

One important aspect of learning arithmetic skills is to recognize *numbers as composite sets.* This was introduced in the Statement game but further elaborated in the following activities. The teachers offered an alternative to counting single units (beads or fingers) and rather 'see' the pattern as a composed set that can be part of larger sets or

partitioned into smaller sets. The dimension of variation that was opened up through the activities was the use of subitizing strategies (recognizing a set without counting single units) and the possibility to extend the subitizing range (conceptual subitizing), usually using the whole hand as 'five' and extending the set by 'seeing' for example 'the five and two more'. This was enacted as the teachers contrasted the procedure of counting single units with seeing collections of items as composite sets of units, such as the five beads in the same color on the snake-game string or the 'whole hand' constituting five as a composite set that does not need to be counted. Extending the children's conceptual subitizing range by recognizing finger patterns that include the 'undivided five' was also coming through in the finger pattern activity when the teacher showed a finger pattern including five (e.g. 7 or 8) and promoted them to discern 'the five in the eight'.

Children are in the activities given opportunities to discern that parts (collections of items) can be decomposed into smaller sets of units or as one composite set and some single units. The semi-decimal structure (two hands with five fingers or a bead string with two groups of five) further supports the discernment of collections slightly above five.

When the *part-part-whole relation* is discerned it is possible to see that by knowing the whole and one part, the unknown part can be identified. This was foregrounded in the snake-game, and in some number stories (context problems) where the whole was known and one part missing. Alternatively, by knowing two parts, the whole can be identified. In the snake-game the teacher after having identified the missing part often emphasized that the two parts (collections of beads or for example bears in the story) can be composed to the same whole number again.

Operating on the relation between two parts in a specific part-part-whole relation can open up for discerning a *commutative pattern*. Especially in the context problem activity the principle of commutativity was made visible when presented in a sequence, such as in the first story two bears were playing and six bears joined in, in the next story six were playing and two more were joining.

Also the snake-game offered the children to discern the commutative principle, within the same example: if there are two beads visible and eight hidden it does not matter if the finger pattern consists of eight folded fingers and two unfolded, or eight unfolded and two folded, it constitutes the same part-part-whole relation. This dimension of variation strengthens the children's discernment of the part-part-whole relation of number and thus brings fore the flexibility that commutativity provides to arithmetic operations.

Learning outcomes

Analysis shows that the children participating in the intervention programme developed skills that made them produce significantly more correct answers to arithmetic

tasks after the intervention and also in a delayed test one year after the intervention (reported in Kullberg et al., 2018). In this particular study we direct attention towards the qualitative changes in the ways the children experience numbers, that in accordance with our theoretical framework determines what the children can do with numbers in arithmetic tasks. Arithmetic skills are in this sense based on what aspects of numbers children experience when trying to solve an arithmetic problem.

The observations show that most children in the first interview do not experience a relation between parts and whole, or the ordinal and cardinal aspect of numbers simultaneously and thus cannot coordinate what to do with the given numbers in a task (40 of 64 observations). The first interview only shows 4 observations (of 64) where the children make efforts to structure numbers or present an answer as a known fact while in the second interview (after the intervention) all but six observations show evidence of structuring or using known facts to solve the tasks. The interviews after the intervention thus reveal that the children developed their ways of seeing numbers in that they experience arithmetic tasks as structures constituted of numbers in a part-part-whole relation.

An example of how a child solved the arithmetic task: "You have 10 candies, ate 6 of them. How many are left?" in the first interview:

Child: Maybe five.
Interviewer: How do you know it is five?
Child: Because five comes after six.

The child answers with two consecutive numbers in the counting sequence. Our interpretation is that she experiences the ordinal aspect of number, but not necessarily the cardinal aspect, which is necessary to see the relation between the parts and the whole, and cannot find the missing number.

The second interview is showing a remarkable change in that almost all observations are categorized as expressions of children experiencing a part-part-whole relation in numbers, which helps their finding solutions, or they at least attempt to solve the arithmetic tasks. At the second interview the child solved the same task as follows:

> The child puts her ten unfolded fingers on the table, folds all five fingers on her left hand and the thumb on her right hand at the same time, looks at her structured finger pattern and answers "four".

Only one observation in the second interview shows a child handling the numbers in the tasks by counting single units. This means that after the intervention program the children did not count single units on their fingers or by counting words when they solved the arithmetic tasks. The vast majority have developed their ways of seeing numbers as structured or even as known facts after participating in the intervention.

CONCLUSION

What seems critical for learning arithmetic skills is that the children through the intervention activities were given opportunities to experience the necessary ordinal and cardinal aspects of numbers, and what is even more important (according to the field of research, e.g. Fuson, 1982; Davydov & Andronov, 1981), to coordinate these aspects in the arithmetic problem-solving process. In the different activities in our program the children were constantly encouraged to reason about number sets (illustrated as finger patters) and relate numbers to other numbers, thus opening up the necessary dimensions of variation. However, a mere focus on ordinality and cardinality may induce counting procedures as ways to determine sets, this was intentionally avoided in the program. Instead, a structural approach was promoted by using children's presumed subitizing abilities and extending the subitizing range by using familiar sets such as the whole hand constituting 'five' as one benchmark. The outcomes of the intervention programme clearly show that children did embrace this structural approach, since they did not rely on counting single units in the post- and delayed interviews.

In summary, the children have been taught to experience numbers as a structure of parts and whole and that it is possible to solve arithmetic tasks by using this structure of part-part-whole relation. They do this primarily by using their fingers to create finger patterns. Some children learned number facts and were able to retrieve from memory the part-part-whole relation and use decomposition strategies when applicable and consequently did not need to use their fingers to the same extent. Those children who did not experience numbers as known facts made use of finger patterns to create the necessary structure and then 'see' the missing part or the whole. They did this without counting single units.

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ANALYSING LESSONS ON FRACTIONS IN THE MIDDLE PRIMARY GRADES: FOCUS ON THE TEACHER

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The paper draws on Vygotsky's sociocultural historical theories of learning, elaborated by the 'mediating primary mathematics' (MPM) theoretical framework, to analyse lessons in the middle primary grades 3-5 on the topic of fractions. Data analysis suggests that teachers in this sample focused primarily on the concepts of addition of simple fractions, but barely moved learners from single concepts to the more general concepts of fractions. Moreover, data analysis revealed the need for teacher training in integrating available corporeal objects in developing the meaning of the concepts of fractions for the learners before introducing formal definitions. The paper extends the application of the MPM analytical framework to specific topics in the primary mathematics curriculum.

INTRODUCTION

Concerns have been raised in the recent past by many stakeholders, especially in the Eastern and southern African regions, about the low learning outcomes in mathematics at the primary school level. However, these concerns, as most of them always do, come as reactions to the outcomes (end product), rather than to the processes of learning that go on in the classrooms before national assessments. For example, a study conducted by Uwezo (2015) in East Africa revealed that some learners in grade 7 in some countries in East Africa, were unable to solve simple mathematics tasks meant for learners in grade 4. This finding caused much debate in the region. Rather than focus on the outcomes of learning, this study attempts to understand the processes of learning in a typical primary mathematics classroom in Uganda. Two concerns motivate this study. First, in general, primary school mathematics is taught as one subject, yet individual teachers may have challenges with particular topics in mathematics. Second, research studies on specific topics in mathematics are rare, making it difficult to address the topic-specific needs of teachers. The topic of fractions was chosen in this study based on the in-service primary mathematics teachers' responses to a questionnaire, which showed that fractions was their favourite topic and division was the most challenging topic to teach. Given the low learning outcomes reported earlier, the current study aimed at interrogating the topic of fractions to uncover the extent to which teachers taught fractions and what were the areas that could be improved through interventions. Hence, the purpose of this study is to establish the patterns the lessons on fractions are likely to present and to point out common weaknesses that teachers can pay attention to through on-going professional development. The specific research questions are:

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1. What patterns are likely to show in the teachers' lessons on fractions in the middle primary grades?
2. Which aspects of the lessons might require specific intervention?

Our assumption is that topic-specific intervention will increase teacher confidence and will improve learner performance in mathematics at the primary level.

THEORETICAL FRAMEWORK

The current study is informed by the Vygotskian sociocultural theory of learning, elaborated by Venkat and Askew (2018), which views the teacher's classroom activities and teaching strategies collectively, as mediation for the learners to become aware of the concepts of mathematics. The teacher mediates the mathematics concepts using different means available, such as tools, hereafter referred to collectively as 'artefacts'; inscriptions; and talk, including gestures; all these in a planned, or structured way. For example, the teacher draws a circle and divides it into equal parts, say six. He then uses the parts to introduce the concepts of numerator, denominator and fraction. In this paper, we modify and extend Venkat and Askew's (2018) 'mediating primary mathematics' (MPM) framework to the analysis of teachers' classroom lessons on the topic of fractions.

Mediating primary mathematics (MPM) framework

Venkat and Askew's (2018) mediating primary mathematics (MPM) framework is founded on the Vygotskian social cultural historical theories of learning (Vygotsky, 1978), which, as stated in the previous paragraph, assume that learners become aware of the mathematics concepts through the mediation role of the teacher. In implementing the mediation role, a teacher uses different tasks and examples in his or her lessons. Venkat and Askew (2018) group these tasks and examples given to the learners into three interconnected "strands" or categories of teacher mediation, namely, mediation with artefacts; mediation with inscriptions; and mediation with teacher talk/and gestures. Tasks or examples that come through the teacher's use of artefacts, inscriptions, or through talk and gestures, all form the 'bundle' for mediation, which I name mediation bundle (MB) in this paper (cf Arzarello's, 2006, semiotic bundle). In this paper, the MB also provides the units of analysis of the teacher's lessons on fractions. We do not adopt Arzarello's semiotic bundle, because, in the MPM analytical framework, inscriptions, gestures, and artefacts are analysed individually in their units.

Artefacts:-These include physical objects and manipulatives. Examples are a cardboard drawing of a circle divided into equal parts and small square blocks that the teacher can use to mediate the concepts of fractions.

Inscriptions:-These include images, diagrams, written words and symbols, produced by the teacher during the lesson. Examples include a number line model, an area model of fractions, and a fraction as a rational number, for example, 4/5.

Teacher talk/gestures: Venkat and Askew (2018) propose three "sub-strands" of teacher talk/gestures, namely, *validating/generating solutions*; *building mathematical*

connections; and *building learning connections*, respectively. The first sub-strand is about the methods the teacher uses to solve mathematics problems and also to provide feedback to the learners on a given task in class; the second sub-strand is about building connections between mathematical ideas; and the third sub-strand extends connections beyond mathematical ideas to explanations and evaluations of concepts. For example, a teacher may ask a learner to evaluate qualitatively how far half (1/2) a kilometre is. Some innovative learners may use known land marks in the school, such as 'five times the length of a football field' to estimate half a kilometre distance.

The analysis of teacher mediation of fractions using MPM

Table 1 provides a framework for analysing teacher mediation of fractions adapted from Venkat and Askew (2018). Whereas Venkat and Askew (2018) adopt quantitative measures 0, 1, 2, and 3 of teacher mediation in each of the three mediation strands (ie, artefacts, inscriptions, and talk/gestures), we prefer qualitative representations, for example, A_0, A_1, A_2, and A_3, with respective cell indicators. For example, cell A_0 represents teacher mediation that involves no artefacts; and cell A_1 represents inappropriate artefacts used inappropriately. All other cells in the "MPM map" (Venkat & Askew, 2018) have specific indicators. We have adopted a qualitative approach to support our knowledge claims in this study. Unlike Venkat and Askew (2018), we do not claim to "measure" teacher mediation; rather we attempt to understand or to contextualise what is going on in the classroom lessons on fractions.

MEDIATING TASKS/EXAMPLES			
MEDIATING WITH ARTEFACTS			
A_0	**A_1**	**A_2**	**A_3**
No artefacts or artefacts that are not appropriate for tasks	Inappropriate artefacts used in in-correct ways	Appropriate Structured artefacts used in unstructured ways	Structured artefacts used in structured ways
MEDIATING WITH INSCRIPTIONS			
R_0	**R_1**	**R_2**	**R_3**
Inscriptions that are problematic or show no clear links/examples with the concepts of fractions	Inscriptions that focus on only single concepts or examples for the learners	Inscriptions that are extended beyond the targeted concepts or examples	Inscriptions that link more general concepts., e.g., a fraction is an example of a rational number
MEDIATING WITH TALK/GESTURES			
Method for validating/generating solutions			
S_0	**S_1**	**S_2**	**S_3**
Disconnected and /or incoherent method of generating examples	Strategy/method that produces immediate answer; or aims at specific responses answers from learners	Strategy/method that can generate answers around specific examples)	Strategy/method that can be generalised to other examples without restrictions

Building mathematical connections			
C_0	**C_1**	**C_2**	**C_3**
Disconnected and incoherent examples	Standalone examples	Talk connects between examples/inscriptions or artefacts	Talk incorporates multiple connections between examples/artefacts
Building learning connections			
E_0	**E_1**	**E_2**	**E_3**
No evaluation of learners' contributions (correct or incorrect)	Accepts/evaluates learners' inputs; offers strategy, feedback to learners' contributions	Builds on learners' contributions and helps them clarify their answers;	Advances learners' contributions and explains common misconceptions.

Table 1: The modified MPM framework Venkat & Askew (2018)

METHODOLOGY

The study design is a single case study. Participants in the study are 10 primary school teachers purposely sampled from a larger sample of 28 in-service primary school mathematics teachers from Uganda; 21 were undertaking a bachelor of education (B.Ed.) degree up-grading programme in a university in Uganda; and 7 were fulltime teachers in one of the top performing private primary schools close to the said university. The 10 teachers were chosen because they had indicated in a separate survey that fractions was their favourite topic. All the 10 teachers in this study were male. In fact, in the entire sample (F=6, M=22), no female indicated that fractions was her favourite topic.

DATA ANALYSIS

Although Arzarello (2006) places inscriptions in the "semiotic bundle" which includes gestures, glances, drawings, and other extra-linguistic expressions, in this study we segregate inscription and analyse it as a unit, given the huge role it plays in teachers' mediations. Under the inscription strand there are four sub-strands marked R_0 to R_3; each with indicators as in Table 1. Due to page restrictions, only three inscriptions of teachers T1, T2, and T3 are presented and discussed in this paper. In the data, the *[italics]* sign is used to clarify the teacher's inscriptions, whereas [...] represents sentences that have been shortened to save space, but without losing the original meaning. The symbol *[number]* refers to the line numbering on the teacher inscriptions. For example, in the T1 inscriptions below, [3] refers to line number 3, having key words such as "same denominators" and "numerators".

T1

1 I involved grade 3 class in a discussion of the previous lesson […] about
2 addition of fractions with the same denominator. When fractions are having
3 the same denominators, the denominator is maintained and the numerators are
4 added, for example (i) $1/4 + 3/4 = (1+3)/4 = 4/4. = 1$. Then adding fractions

5 with different denominators, e.g. $1/2+1/3$. Change them [denominators] into
6. equivalent fractions, e.g. $1/2=(1\times2)/(2\times2)=2/4=3/6=4/8=5/10$ and
7 $1/3=2/6=3/9$ to make them have the same. Choose fractions with the same
8 denominators then add $3/6+2/6=5/6$. I asked pupils [to] form equivalent
9 fractions of: $2/3$, $1/4$, and $1/2$. Then I guided them on how to add 1/4 + 1/2
10 by choosing those fractions with the same denominators.

Three key concepts that appear in teacher T1's mediation with inscriptions are the "same denominators" [2] "different denominators" [5], and "equivalent factions" [6], [9] respectively. However, the concepts are applied in different examples that do not build on or link with one another. Thus, the teacher T1 mediation with inscriptions focuses on single concepts or examples synonymous with cell R1 in Table 1.

T2

1 I began the lesson by defining a fraction. Then I told them [learners about the]
2 types of fractions, ie, simple (proper), improper, decimal and recurring and non-
3 recurring factions respectively. […] Simple fraction are fractions where the
4 numerator is smaller than the denominator. […] simple fractions are of two
5 types, those having the same denominator. e.g. 2/5 + 3/5 or […] those having
6 different denominators, e.g. 2/3+3/5. Teaching addition of [fractions having] the
7 same the denominator, e.g. 1/3+2/3, you just add the upper part and take 3 as the
8 denominator without looking for the least common multiple (LCM). Then those
9 having different denominators, e.g. 2/3 x 3/5= [(2x5) + (3x3)]/ (3x5). Look for
10 the [sum of the product of] diagonal 1 and diagonal 2, divided by the product of
11 the denominators. After that I gave them some [tasks] to try on their own and I
12 moved around in class helping [learners] and correcting their mistakes [ ].

Teacher T2's mediation with inscriptions includes some concepts such as proper, improper, decimal, recurring [2] and non-recurring fractions [3], but T3 limits his examples only to simple fractions. Even with simple fractions, the inscriptions focus on basic rules of getting to the answer rather showing the learner the founding ideas behind such techniques. T2's own admission in line [12] suggests that some of his learners still did not understand the techniques he used in his lesson. It seems that more illustrations with artefacts and more examples of each of the concepts mentioned above would have supported learners more in understanding the concepts and applying them. T2's mediation inscriptions focus on simple fractions indicating category R1 in Table 1.

T3

1 I started the lesson by giving the definition of a fraction as 'a part of a whole
2 number' or a number that has a numerator and the denominator […]. There are

3 different types of fractions. Can you give some […] types of fractions that you
4 know? Most learners were able to give the different types of fractions, but could
5 not explain what each of them mean. The teacher later explained […] proper
6 fraction as fraction [with] the numerator smaller than the denominator
7 e.g. 1/2, 2/6, 1/7'. Improper fraction has the numerator bigger than the
8 denominator, e.g. 7/4, 9/2, 4/3. Mixed fraction has a whole number and a
9 proper fraction, e.g. $2(1/7)$. Later, I told [learners] that when adding fractions
10 with the same denominator, we add the numerators and retain only one
11 denominator, e.g. 3/7 +2/7=(3+2)/7=5/7; 5/9+3/9=(5+3)/9=8/9 […]. I ask
12 learners to add 2/11+5/11. Most of the learners were able to add the given
13 fraction correctly as 2/11+5/11=(2+5)/11=7/11 […].

Unlike T1 and T2, T3 attempts a definition of fraction, albeit in an abstract way [1]. Abstract, in the sense that concept of 'part of a whole number' if presented as such with no artefact to mediate them, would definitely challenge the learners' comprehension. I expected T3 to demonstrate to the learners the concept of 'whole' by using a physical object, such as the drawing of a circle divided into equal segments. Each segment of the circle can then represent 'part of the whole'. Such an initial representation might mediate the definition in [1] that T3 "started the lesson" with. Moreover, T3 also provides some working definitions of "proper", "improper" and "mixed", fractions. However, without enough illustrations, save for the skeletal examples [lines 7, 8, 9, 11, 13] that accompany the abstract definitions, it is debateable if T3's learners understood the meaning of the terms. Moreover, the examples are based on only one concept of simple fractions. Hence T3's mediation with inscriptions, again, like T1 and T2, do not extend beyond cell R1 in Table 1.

DISCUSSIONS AND CONCLUSIONS

There are many ways of representing fractions (Venkat & Bowie, 2017; Yee, 2006:115; Watanabe, 2002; Van de Walle, 2004; Carthcart, et al., 2005). The current study focused on teacher inscriptions while delivering lessons on the topic of fractions. On the first research question, the analysis of data revealed that, although some teachers in this sample were aware of the general concepts of fractions, their examples to the learners focused on, and were limited to the addition of simple fractions. In respect of mediating primary mathematics framework, the teachers' inscriptions focused on only single concepts with no clear links to other concepts. It may be that the teachers' choice of fractions as a favourite topic was based on the assumption of simple addition, which also showed in their inscriptions. On the second research question, our data revealed that teachers' examples did not build on learners' experiences with corporeal objects in their immediate environments, to develop the learners' understanding of fractions. Data analysis revealed the need for teacher training in integrating available local objects in developing the meaning of the concepts of fractions before introducing formal definitions. Concepts such as improper fractions, recurring and non-recurring

fractions, and decimal fractions were mentioned in passing, without fully developing their meanings and providing many examples. The modified MPM framework enabled us to focus on the teachers' inscriptions on fractions and unpack their assertions that fractions was their favourite topic. The study revealed that teachers in this study focused primarily on the concepts of addition of simple fractions, but barely moved learners from single concepts to the more general concepts of fractions.

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TRIADIC DIALOGUE DURING CLASSROOM TALK IN THE CONTEXT OF PATTERN GENERALIZATION

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The purpose of this study is to explore knowledge types and forms of interaction during classroom talk in the context of pattern generalization. Ten sessions in two classrooms in grade 7 were video-taped and the videotapes were transcribed verbatim. Episodes corresponding to levels of generalization were analyzed. The findings showed that the types of knowledge were mediated by different levels of generalization. Furthermore, classroom talk predominantly used procedural knowledge in episodes corresponding to different levels of generalization except for episodes that involved noticing commonalities in the pattern. Conceptual knowledge type was predominantly used in those episodes. The findings also showed that triadic dialogue was the dominant discourse structure in classroom talk in which the teacher initiated a question, which was followed by a student's response and then the teacher provided feedback, evaluation or elaboration based on student's response.

BACKGROUND

Classroom talk is central in learning and teaching of mathematics. One would expect that teachers might have instructional strategies to involve students in classroom talk in a variety of manners. However, findings in previous studies (e.g. Mercer, 1995; Mortimer & Scott, 2003; Lemke, 1990) report that classroom talk involves a turn-taking structure that follows an initiation-response-feedback (I-R-F) pattern of interaction. In such a case, the teacher initiates a question, the student responds and the teacher provides feedback. Lemke (1990) called this pattern of interaction the triadic dialogue (TD). According to findings in the literature (e.g. Alexander, 2000), the TD is the dominant style of classroom talk. Those findings lead us to believe that TD helps teachers to have control over the development of mathematical lesson and the participation of students; however, TD has been critiqued because it leads to teacher-traditional instructional approaches and it limits student independence (Lemke, 1990). In alignment with Lemke's (1990) findings, Mercer (1995) found that the teacher in a TD has the majority of turns in which s/he leads the classroom talk. In contrast, the students have few opportunities to influence the classroom talk.

Although there are limitation and critiques of the I-R-F instructional approach, not all researchers in mathematics education agree that the TD does not lead to meaningful learning. For example, Nassaji and Wells (2000) report that the I-R-F structure was the dominant dialogue within an inquiry-style teaching. Other studies (e.g. Wells, 1999; Wertsch, 1998) found that dialogic discourse within TD involve students in contributing ideas, which may influence classroom discussion. Variations in the TD instructional approach and the sequence of speaking turns is important during

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classroom talk since a teacher utterance or a student utterance in itself is not meaningful because meaning making is made through relations between utterances (Barwell, 2016). I suggest that variations in TD need to be further studied in mathematics classroom talk specifically in teaching and learning of pattern generalization(PG) where language plays a fundamental role (Radford, 2003).

The purpose of the present study is to understand the nature of classroom talk in terms of knowledge types in TD during generalization of figural patterns. There is a relative lack of research in mathematics education on discourse quality, classroom talk and the source of utterance (teacher or student) during PG.

LEVELS OF GENERALIZATION

In a series of studies, Radford (e.g. 2003; 2008; 2010a) has focused on studying students' levels of generalization (LoG) in PG tasks. This line of investigation was to establish the hierarchical nature of the LoG and to characterize them operationally using students' embodied and discursive activity as level indicators. This line of research uses a semiotic-cultural approach to distinguish between students' processes of objectification - that is, the ways in which the learners make visible new relationships and objects that they refer to in their generalizing activity. This approach involves an analysis of semiotic means of objectification – that is, gestures, body, posture, and other embodied signs and resources (e.g. language, writing, speech, tools, analogies, etc.) in presymbolic and symbolic generalizations. The findings in those studies report that there is evidence in support of the hierarchical nature of LoG and that students can be assigned a LoG based on their use of semiotic means of objectification.

Radford's taxonomy is formed of five LoG: Abduction, arithmetic, factual, contextual, and symbolic generalization levels. Abduction and arithmetic generalizations (AG) are two non-algebraic levels of generalization. More specifically, abduction involves noticing a commonality in the context of generalization whereas AG entails extending a commonality to subsequent terms of a pattern without the ability to come up with a rule that determines any term of the pattern (Radford, 2010b). Thus, AG is limited in determining a particular term of the pattern based on making use of a noticed regularity in the structure of the pattern.

Factual (FG) and contextual generalizations (CG) are two types of algebraic, but presymbolic, generalization. The latter reveals a higher LoG in comparison with the former. FG applies to generalization of objects at the same concrete level (e.g. numbers). In contrast, CG goes beyond the realm of specific steps of the pattern and it deals with generic objects (e.g. "the figure", "the top row"). The semiotic means of objectification for FG mainly involve bodily gestures in particular steps of the pattern. More specifically, FG involves various types of semiotic means of objectification: spatial positional linguistic terms (e.g. "the next"); adverbs (e.g. always); rhythm and

movement (e.g. 1+2, 2+3, 3+4, 4+5 …), etc. In contrast, CG involves discourse that addresses generic objects. Symbolic generalization (SG) is the third and highest level of algebraic generalization. It involves expressing the generalization through alphanumeric symbols and bypassing the positioning problem to produce nonspatially, based symbolizations.

RATIONALE OF THE STUDY

Researchers in mathematics education argue that there is a need for further investigation into classroom discourse due to its fundamental impact on students' learning. More specifically, there is a need to do further research on TD due to variation in its forms and functions (Mortimer & Scott 2003; Nassaji & Wells 2000). For instance, Mortimer and Scott (2003) note that the TD entails different communicative approaches that varies between dialogic and authoritative approaches. Findings from previous studies (e.g. Salloum & BouJaoude, 2017) lead us to believe that there are different types of patterns of discourse, which influence the quality of teaching and learning within the TD.

This study aims to analyze mathematical classroom talk in the context of PG for several reasons. PG is well researched. It is seen as a route to develop students' algebraic reasoning (e.g. Radford, 2008). To date, the focus of previous studies analyzed how learners develop their LoG as they participate in group discussion in the context of PG (e.g. Radford, 2003; 2008). Other studies focused on students' strategies (e.g. El Mouhayar, 2018b) or on teachers' ability to notice students' reasoning in PG (e.g. El Mouhayar, 2018a). The present study extends previous research on classroom talk in the context of PG.

RESEARCH QUESTIONS

What types of knowledge does classroom talk use in promoting LoG?
What patterns of interaction exist during episodes corresponding to LoG?

METHOD

Setting and Participants

The data analyzed for this study were collected in a grade 7 classroom in a moderately sized private school in a suburb in greater Beirut area serving middle socio-economic background students as judged by the school tuition level. This school was selected since it had a good reputation (due to high percentage of success in the national brevet / grade 9 exam) and because private schools serve 60% of Lebanese students. Furthermore, this school followed the national mathematics curriculum, which was adopted by most of the private as well as public schools. On the other hand, it was difficult to involve a mathematics teacher from public schools in this study due to complicated bureaucratic regulations that govern such participation. The mathematics

teacher in the selected classroom was highly regarded in the school because of his hard work, care that his students develop meaningful understanding in mathematics and because of his good relationship with the students.

Growing figural patterns was new for the students in grade 7. In previous years, the students had some experience in repetitive patterns. Tasks that involve the students to find the number of squares in near (e.g. Figure 9) or far generalizations (e.g. Figure 100) are new for the students. They had no experience in generalizing figural patterns similar to the one included in Figure 1.

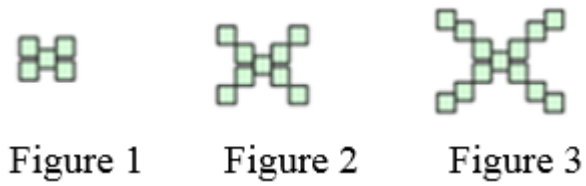

Figure 1: An example of a growing figural pattern

Classroom mathematical activities were designed in a particular structure as follows: (1) Either small groups, formed of two students, carried out the mathematical activities and then this was followed by classroom talk organized by the teacher or (2) classroom talk controlled by the teacher carried out the activities without group or individual work. The first structure has characteristics of student-centered activity since it allows the students to share their ideas, analyze the pattern within small groups, and then revise their strategies within classroom talk. However, the second structure allowed as well for dialogic classroom talk in which students shared their reasoning and strategies with the teacher within whole classroom discussion.

Data collection and analysis

Ten sessions in two classrooms in grade 7 were video-taped such that five sessions took place in each section. A criterion for selecting grade 7 was that algebra is first introduced in grade 7 and because PG is implicit in algebraic reasoning in general and, particularly, in concepts such as function and variable.

To get an in-depth characterization of classroom talk, videotapes were transcribed verbatim, and time was coded each 2 minutes. Six levels of analysis were used to analyze video transcripts. The first level of analysis identified pattern task and session. The purpose of this level of analysis was to identify the learning outcomes of the session and the pattern task. The second level of analysis divided each session into events corresponding to different generalization types (e.g. determining near or far generalizations). The third level of analysis divided each of those events into episodes corresponding to different LoG. An episode is a section where classroom talk occurred within one LoG, in a particular event, during specific session and dealing with a

particular pattern task. More than one episode within the same event and corresponding to the same LoG may occur if classroom talk involved more than one student in generalizing a pattern in different manners and based on different perspectives in noticing the pattern. The fourth level of analysis divided episodes into either classroom talk or pair work. Only episodes corresponding to Radford's levels of generalization and to class talk were considered for analysis. The fifth level of analysis focused on the teacher's and students' meaningful mathematical utterances during classroom talk within episodes corresponding to LoG. Each utterance was coded based on knowledge type categories (Table 1) using revised Bloom's taxonomy (Krathwohl, 2002). Categories of knowledge types included factual, procedural and conceptual. Metacognitive knowledge type was omitted from the analysis because the frequency of utterances corresponding to this type was relatively small.

Knowledge type	Definition
Factual knowledge	The basic elements that students must know to be acquainted with a discipline or solve a problem in it.
Conceptual knowledge	The interrelationships among the elements within a larger structure that enable them to function together.
Procedural knowledge	How to do something, methods of inquiry, and criteria for using skills, algorithms, techniques, and methods.
Metacognitive knowledge	Knowledge of cognition in general as well as awareness and knowledge of one's own cognition.

Table 1: Definition of knowledge types (adapted from Krathwohl, 2002, p. 214)

In order to discern patterns of interaction and to analyze the use of TD in different LoG, each utterance was coded based on an interaction type. The analysis was based on a scheme (Mortimer & Scott, 2003) in which five forms of interaction are identified: Initiation (I) usually via a question posed by the teacher; Response (R) from the student; Evaluation (E) normally by the teacher; Feedback (F) from the teacher and based on student's response; Elaboration (El) of the student's response.

Two researchers coded the data independently and several meetings between them followed where discrepancies in coding data were negotiated until consensus was reached. The coded data were subjected to a series of quantitative analyses. First, interaction types were determined to look at patterns of TD. Second, a cross tabulation of type of knowledge by LoG was done to explore the possibility of significant differences in knowledge type across LoG. The data were also subjected to qualitative analysis because our aim was to discern trends in ways the TD was used and how the classroom talk moved among knowledge types in representative episodes.

FINDINGS

Knowledge types

Findings of classifying utterances by their interlocutor (student or teacher) (Table 2) show that the teacher talk dominated classroom talk: 62.7% (844 utterances) of the total meaningful mathematical utterances resulted from teacher talk whereas 37.3% (503 utterances) of the total meaningful mathematical utterances resulted from students. The findings also show that the frequency of utterances corresponding to different knowledge types varied across generalization level episodes. For example, Table 2 shows that classroom talk (teacher and/or students' utterances) in the abduction level episodes were mostly of the conceptual knowledge type followed by procedural and then factual. In other episodes, classroom talk utterances were mostly at the procedural knowledge type, followed by conceptual and then factual.

	Type of knowledge							
	Factual		Conceptual		Procedural		Total	
Classroom talk	#	%	#	%	#	%	#	%
Abduction	30	11.9*	151	59.7*	72	28.5*	253	100.0
AG	30	6.8	83	18.8*	329	74.4*	442	100.0
FG	23	10.6	30	13.8*	164	75.6*	217	100.0
CG	2	1.1*	23	12.7*	156	86.2*	181	100.0
SG	14	5.5	42	16.5*	198	78.0*	254	100.0
Total	99	7.3	329	24.4	919	68.2	1347	100.0
Teacher talk								
Abduction	18	10.5	104	60.8*	49	28.7*	171	100.0
AG	22	7.8	51	18.1*	208	74.0*	281	100.0
FG	13	10.6	17	13.8*	93	75.6*	123	100.0
CG	1	1.0*	12	12.0*	87	87.0*	100	100.0
SG	10	5.9	31	18.3*	128	75.7*	169	100.0
Total	64	7.6	215	25.5	565	66.9	844	100.0
Student talk								
Abduction	12	14.6*	47	57.3*	23	28.0*	82	100.0
AG	8	5.0	32	19.9	121	75.2	161	100.0
FG	10	10.6	13	13.8*	71	75.5	94	100.0
CG	1	1.2*	11	13.6*	69	85.2*	81	100.0
SG	4	4.7	11	12.9*	70	82.4*	85	100.0
Total	35	7.0	114	22.7	354	70.4	503	100.0

* Adjusted residual >|2|

Table 2: Frequency and percentage of knowledge types of utterances in different LoG

A chi-square showed significant differences between utterances of classroom talk in the distribution of knowledge in different generalization level episodes ($\chi2$ (8) = 257.191, p = 0.0). Similarly, a chi-square on the data classified by the teacher and the students showed significant differences between utterances of the teacher ($\chi2$ (8) = 165.482, p = 0.0) and between utterances of the students ($\chi2$ (8) = 93.861, p = 0.0).

Interaction types

The analysis showed that TD was the most dominant interaction practice during classroom talk in episodes corresponding to different LoG. More specifically, the I-R-E, I-R-F and I-R-F-R-F- patterns of interaction was very common in those episodes. The teacher used TD to elicit students' responses in order to promote different LoG. Patterns of interaction showed that the teacher initiated classroom talk, one student selected by the teacher responded and then the teacher provided feedback. After that, either the teacher initiated a new question or the same student responded to teacher's feedback. At the end of an episode, classroom talk ended with teacher's feedback or evaluation. The extract below shows this aspect of classroom talk. It corresponds to an episode at the abduction level, which took place during session 1 and which addressed the growing pattern represented in Figure 1.

Teacher:	Look at this pattern. Try to figure out what is changing? What is fixed?
Maggie:	Plus four in each figure.
Teacher:	We are adding four squares in each figure.
Teacher:	Any more propositions?
Mohamad:	In figure three it will be plus four. In each figure, we added plus four to the figure before it.
Teacher:	Start from the beginning. From figure one to figure two what is happening?
Mohamad:	Plus four.
Teacher:	We are adding four. From figure two to figure three?
Mohamad:	Plus four.
Teacher:	Plus four. That is what Maggie said. We are adding four each time.

From this representative extract, it can be seen that the teacher used TD to elicit students' responses in order to promote the abduction LoG. TD was coded as follows: I – R – F – I – R – F– R – F – R – F. Classroom talk also moved among conceptual and procedural knowledge types. The teacher initiated classroom talk with a conceptual question: "What is changing? What is fixed?" to involve students in noticing commonalities. Classroom talk then moved to procedural knowledge about how the pattern is increasing. The episode ends by teacher's feedback in rephrasing what the students have noticed in the pattern "We are adding four each time".

DISCUSSION

One major finding in this study is that types of knowledge were mediated by LoG. Another finding showed that classroom talk predominantly used procedural knowledge in episodes corresponding to different levels of generalization except for episodes at the abductive level in which conceptual knowledge was the most frequent knowledge type. Qualitative analysis showed that episodes at the abductive level occurred at the beginning of the sessions in which classroom talk started with conceptual utterance question asked by the teacher (e.g. "What is changing?"; "What is the constant?"; etc.)

to involve the students in noticing commonalities. In conclusion, the findings in this study suggest that when TD is used appropriately, it becomes proper for construction of meaning. Thus, developing I-R-F at the conceptual knowledge type in classroom talk becomes a valuable competency for math teachers, where they can manage classroom talk as facilitators.

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EXPLORING THE IMPACT OF PRE-LECTURE QUIZZES IN A UNIVERSITY MATHEMATICS COURSE

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Our work contributes to the fast-growing field of blended-learning, a modality combining face-to-face instruction with online activities to support learning. Pre-lecture quizzes, comprising two multiple-choice questions focusing on the main learning outcomes, were introduced for a whole semester. At the completion of the first iteration of design research, our findings suggest that this relatively small change in course instruction can improve both efficiency and effectiveness of learning, affecting the majority of students. The design principle that we developed and tested in practice is generalisable and transferable to other educational settings. Central to those identified heuristics are the mechanisms that increase frequency of learners' engagement and the quality of engagement by enabling extrinsic motivational drivers.

BACKGROUND

Blended learning, the integration of face-to-face and online instruction, is being widely adopted as the 'new normal' in course delivery across higher education. In mathematics classes, this new mode of instruction is commonly seen at all levels, yet the extent to which it is effective raises important questions about its pedagogical merit and the responsibility of instructors with its evaluation. Engelbrecht & Harding (2005; 2009) have captured the scope of Internet teaching and learning of undergraduate mathematics prior to 2009 and laid a foundation for systematic approaches for research investigations. Borba, Askar, Engelbrecht, Gadanidis, Llinares, and Aguilar (2016), summarise findings from many projects supporting blended learning and state: "Studies show that the online component of blended learning increases student agency (allowing them to control both instructional pacing and sequence), reduces distractions that are typical in classrooms or lecture halls, increases time-on-task, and improves student performance" (p.603). However, Nardi and Knuth (2017, p.272) in reference to a study by Roy, Inglis and Alcock (2017), which serves as a cautionary tale about instructional interventions based on technology, argue that "evaluating the effectiveness of instructional resources needs to be done in far more robust and systematic ways than merely relying on students' self-reported learning outcomes and satisfaction with the resources."

Driven by the advances of emerging technologies, the higher education sector is forced to move with the times and look for ways to become sustainable in a technological era. As a result, new instructional design theories that combine the body of knowledge from distance education (e.g. Moore, 2013) with the face-to-face instructional principles (e.g. Merrill, 2002) have been developed. We position our study globally, oriented by

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Keller's (2008) motivational theory that currently dominates instructional design with respect to learner motivation. In particular, we refer to the basic principles that characterise learner motivation with the emphasis on effectiveness, efficiency and engagement. Oriented in this way, we present a conceptual framework, which we call the *Course Transaction space* (*CT-space*), comprising three dimensions as illustrated in Figure 1:

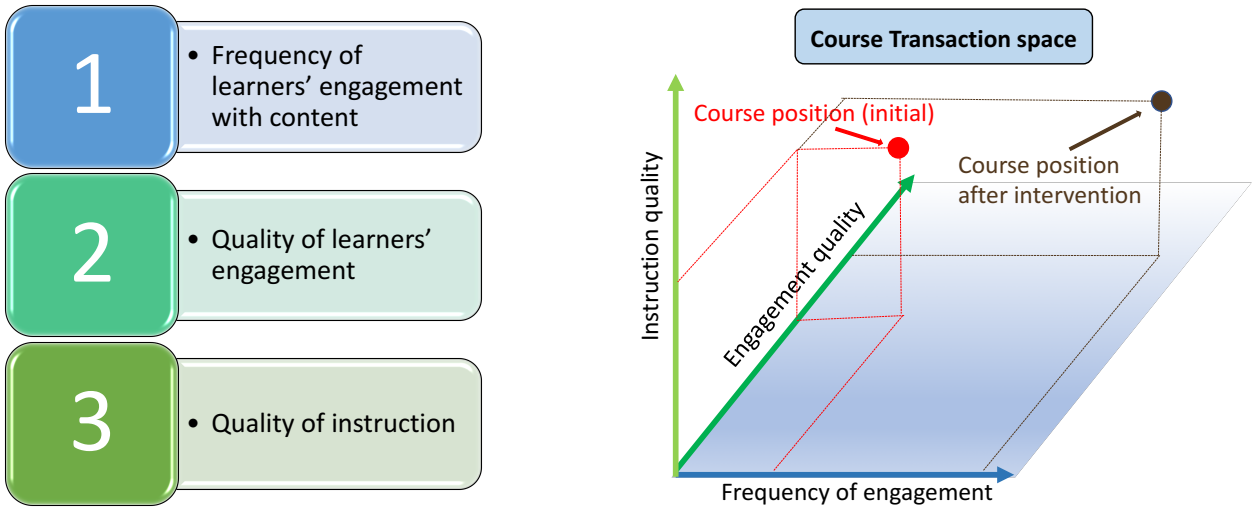

Figure 1: Course Transaction space: three dimensions of educational exchange

The effectiveness and efficiency of the educational exchange depends on the three variables and is affected not only by the *distance* from the origin, but also by the position of the course in the *CT-space.* Understood this way, mathematics courses in a tertiary programme can be represented as a sequence of points in the *CT-space.* It can be used to gauge the *distances* between courses following an intervention, bringing to the fore potential difficulties that may arise for students with the quality or frequency of their engagement with the content and the quality of the instruction received.

In this research report, we use our framework to describe and analyse an online intervention in an undergraduate mathematics course, involving pre-lecture quizzes, and demonstrate the impact this intervention had on student learning.

METHOD

Methodology and research site

The study was conducted as a design research project, which is typically based upon interacting cycles of research and development. This approach continues to gain prominence as an effective methodology in the mathematics education research community (e.g. Artigue, 2015; Cobb, Confrey, deSessa, Lehrer & Schauble, 2003; Goodchild, 2008, 2014; Prediger, Gravemeijer & Confrey, 2015; Schoenfeld, 2007) with the aim to "yield useful knowledge (tied to design) that is sensitive to context and yet general enough to use in new situations" (Bakker, 2018, p.47). Such knowledge is often summarised in the form of design principles, which are updated in the iterative process. This is different from experimental research designs in which some variables

are manipulated and effects on other variables are measured. Design researchers are concerned with designing and studying new forms of instruction with a goal to generate theories about the process of learning. Central to this is the identification of mechanisms that support that learning, which are practice-oriented, pragmatic, and realistically valid. In essence, the design principle tries to generalise the findings that come out from the case in such a way that it can be transferable to another situation.

The study was conducted at a large research university in New Zealand in an undergraduate mathematics course covering Calculus II, Linear Algebra II and introduction to Ordinary Differential Equations, serving the needs of students majoring in a variety of disciplines. For a large proportion of non-mathematics majors taking this course, lack of interest in the subject contributes to low intrinsic motivation, skewed attitudes and deficient engagement with the course. An additional challenge is the size of the course: the enrolment numbers are usually in the range of 350-550 students per semester. The course is delivered over 12 teaching weeks and has the following structure: 3 one-hour lectures per week (transmission style instruction) and one-hour weekly tutorial (student-centred utilising active-learning strategies in groups of 25-30 students in a room).

Another important characteristic of the course is that, due to a mandatory policy, all lectures are video-recorded and made available to students on the same day. After this new policy was rolled out, the attendance of live lectures dropped down significantly. In many cases, the attendance rates fell below 30%.

Intervention: design principles.

Guided by our *CT-space* framework, we designed an intervention targeting increases along the two dimensions, frequency of engagement and its quality, to primarily focus on the learners. A bank of multiple-choice questions was developed and delivered as online quizzes, using Canvas, the Learning Management System adopted by the university. The students were expected to take the quizzes prior to attending each lecture, starting from week two of the semester so as not to disadvantage students who enrolled late. Each quiz was designed to assess the two main learning outcomes from the previous lecture. The decision about what learning outcomes to assess was made by the first author, who at the time of the intervention had been teaching on the course for 7 semesters. The students were allowed two attempts at completing each quiz, containing two multiple-choice questions, and their highest score was recorded. Each question was randomly selected from the bank of questions that was created containing 2-3 versions of the same question (for example, different numerical values). The marking scheme awarded one mark for each correctly answered question, so that the maximum mark for a quiz was 2 and the minimum was 0. The time limit was set for 30 minutes to provide enough time for students to revise the material while taking each quiz. Figure 2 shows an example of a question:

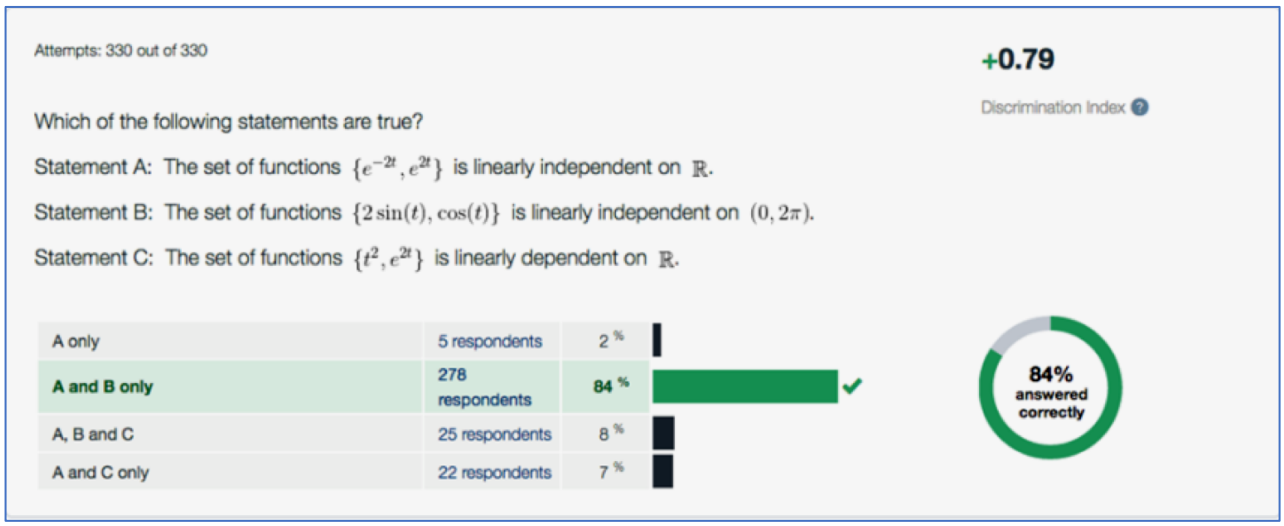

Figure 2: Quiz question: instructor's view of a learning analytics tool

The contribution of quizzes was allocated 7% of the final grade for the course. Out of 32 quizzes, the best 28 scores were recorded to address any practical issues. Out of 393 students, only two asked to reopen quizzes for them after they missed the deadline due to personal circumstances.

Data analysis

The data collected was analysed using a triangulation technique that facilitates validation of data through cross verification from more than two sources. We report on the data collected from the Learning Management System (providing grades and learning analytics); student survey (via Likert-scale questionnaire selecting their choices from: not at all, a little, somewhat or significantly) at the end of the trial semester; and a focus group interview with 5 students. Due to limited space, we are unable to report on data from the video-stimulated recall and reflection from an instructor, and an interview of a tutor (teaching assistant facilitating tutorials).

RESULTS AND DISCUSSION

The most notable result was the increased rate of engagement. The completion data for all 32 quizzes was in the range 81.2% - 96.45% with monotonically decreasing pattern: Quiz 1 was completed by 96.45% of enrolled students, Quiz 14 – by 92.13%, Quiz 28 – by 87.3%. We had hoped for some good engagement with the quizzes initially, but had not anticipated this level of engagement to be so high and remain continually high as the semester pressure tightened. We were further surprised that the high proportion of students who had achieved their best 28 quiz scores to be counted towards the final grade, still attempted the last four quizzes ranging from 87% down to 81.2% for the last Quiz 32. It is noteworthy to reiterate that the incentive for students to attempt a quiz was only 0.25% contribution to their final grade, which manifested as an effective extrinsic motivation. In the past, the course assessment consisted of 3 assignments, 10 tutorials, a test and an exam. In the trial semester, we added 32 quizzes keeping all other assessment components unchanged.

To provide evidence about the quality of the students' engagement, the second dimension in the *CT-space* that we were interested in, we report on the data from the questionnaire (paper-based, conducted in class at the end of the semester, response rate: 98% with 140 individual responses). We were interested in what students do in the process of preparation to take a quiz, and, most importantly, if the first attempt is marked as a wrong answer (the online system automatically provides instant feedback alerting the student). Following an incorrect answer, the student can try again with plenty of time to refer to their notes and other resources as the quiz is set for 30 minutes to answer the two short questions. Over 88% of students reported that they would spend some time studying before taking the second attempt. In response to the question "How much did the quizzes contribute to your understanding of the course material?" over 81% of students reported a somewhat/significant impact.

Analysis of the focus group transcript provided further insight about the students' perspectives of the value of the quizzes:

> Student 1: I think that, for some subjects, it is so easy to just turn up to lectures and to just turn up to tutorials and do the work and then if you forget about it until the next week, then that's fine. But the quizzes in some way – you get home in the evenings and you like 'Oh, I've got to do my quiz'. So, you map stuff out… so you are more inclined to do things. I would probably do stuff anyway but, yeah, it motivates me a bit more.
>
> Moderator to student 2: How about you?
>
> Student 2: I think it helps a lot for a subject like maths because it helps you do practice every day, so it makes you practice every day.

Unanimously, at the end of the focus group, all five participants agreed that incorporation of online quizzes was a good innovation and that it should be used in other mathematics courses.

Unexpected fluctuations along the third dimension in the *CT-space* (Instruction quality) led to iterative revision of the initial design principle. At the design stage, we envisioned that before the start of every lecture an instructor would access the data from the learning analytics tool to see the quiz results. In line with the Just-In-Time Teaching approach (Novak, Gavrin, Christian, & Patterson, 1999), we expected that the feedback would allow an instructor to identify the level of student preparedness on each topic and any misconceptions that students might bring to class. We assumed that receiving such insights before class would enable the teacher to adapt the lecture as needed. In particular, we expected that at the beginning of each lecture the teacher would address poorly answered questions from the pre-lecture quiz by revising the corresponding material from the previous lecture.

In reality, unexpectedly, a different line of events was unfolding. A large majority of students not only chose to engage with this new resource, but they were answering both questions correctly. Figure 2 shows an example of a learning analytics tool that was accessed by instructors before the start of each lecture, illustrating a typical rate of

correct responses. This, we observed, brought a change in teaching behaviour. The analysis of lecture recordings from the trial semester revealed a significant reduction of time spent on revision by the instructor as compared to previous semesters. This change in teaching behaviour can be explained as a consequence of the insights gained from learning analytics. The instructors, as evidenced by their actions, did not need to spend time on revision freeing up valuable teaching time during the lectures, possibly for less-rushed explanations, more examples, etc. More research is needed to understand the implications of the aforementioned change on the quality of instruction. However, what we observed in the trial semester was that a large majority of the students came to the lectures prepared – they had revised the material from the previous lecture and passed the assessment.

The improved engagement with the content is also supported by evidence from the comparison of grades distribution in the trial semester with the previous semester of the same course. While admitting the limitations of this approach, we note many similarities in the delivery of the courses: the same course coordinator, the same instructor (for half of semester), the same course book and all other materials, identical split into lecture topics and the same external assessor. The role of the assessor is to benchmark and calibrate the test and exam (contributing 20% and 60% respectively to final grade) against previous semesters. A deliberate attempt was made by the assessor to ensure that the exam was at least as hard as in the previous semester. Figure 3 demonstrates the shift in course grades across the grade bands: from reduction in the proportion of fails to an increase (by almost identical amount) in A-band grades. The observed difference is significant at p-value <0.05 (Chi-square test: $\chi^2(3)=10.57$, $p=0.014$).

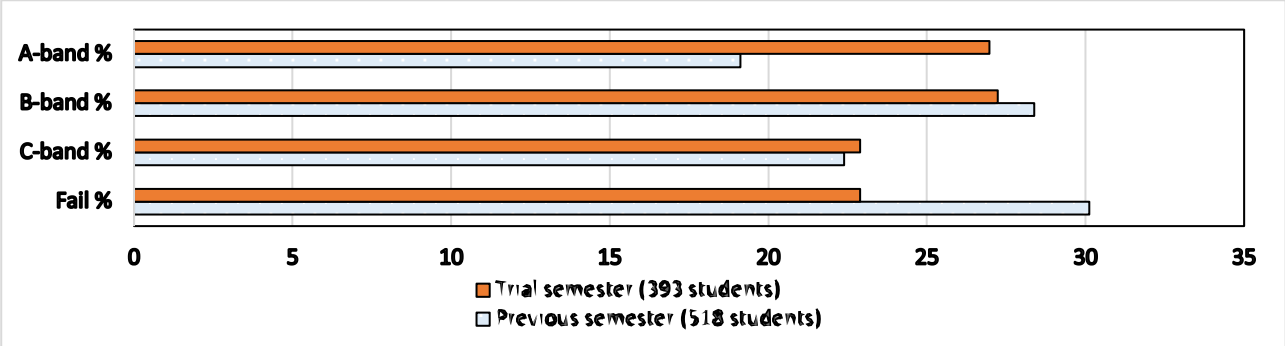

Figure 3: Final grades comparison from the trial semester against previous semester.

CONCLUDING REMARKS

At the completion of the first cycle of incorporation of pre-lecture quizzes, our findings suggest that this relatively small change in course instruction, can improve the course 'coordinates' in the Course Transaction space leading to more efficient and effective educational exchange, affecting the majority of students. What we noted in our own reflections was the simplicity and enjoyment of the quizzes for the students, which we had not factored as a driver. The effect, we further noted was also motivational, supporting Keller's (2008) theory, and particularly encouraging for those less able as

"[m]otivation to learn is promoted when learners anticipate and experience satisfying outcomes to a learning task" (p.177). Having instant feedback and a small reward of 0.25% was galvanising, but the surprising effect was the increased attendance at lectures. We are looking at further research in these areas.

The design principle of the quizzes that we developed and tested in practice is generalisable and transferable to other educational settings as heuristics for instructional design. Central to the design are the mechanisms that increase the frequency of learners' engagement and the quality of engagement by enabling extrinsic motivational drivers. For students, the presence of quizzes as motivational stimulus leads to heighten sense of accountability for their actions in making choices to attend or skip a live lecture, and their intent to learn during the lecture.

The difficulty of design research in terms of evaluating instructional interventions is well recognised. Stylianides and Stylianides (2013) proposed three dimensions of evaluation of classroom interventions: how amenable to scaling up, how practicable for curricular integration, and how capable of producing long-lasting effects. Evaluated this way, our intervention, arguably, can be deemed effective: it is amenable to scaling up with a relatively small development investment, as the number of students utilising online quizzes is unlimited; it is practicable for incorporation into existing curricular structures at any level as most modern Learning Management Systems provide capability for setting up online quizzes. To determine long-lasting effects is difficult - more research cycles are needed with more nuanced theoretical and practical considerations.

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DESIGNING CHALLENGING ONLINE MATHEMATICAL TASKS FOR INITIAL TEACHER EDUCATION: MOTIVATIONAL CONSIDERATIONS

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Strengthening the personal mathematical capabilities of initial teacher education students is an international priority. In this paper, we report on an aspect of a study that investigated the potential of challenging online tasks to meet this goal. Here, we focus on the analysis of survey data in the form of student feedback on trialled tasks from the perspective of motivation theory. Findings of this aspect of the study are reported along with implications for the design of challenging online mathematical tasks and their implementation in teacher education programs.

BACKGROUND

Engaging learners in tasks that provide genuine challenge has been argued as essential for mathematical development (Sullivan, 2011) and identified as vital in school-focused curriculum documents internationally (e.g. Australian Curriculum Mathematics, 2017; US Common Core Standards, 2010). Despite such support, the implementation of programs that maintain a focus on challenging tasks in mathematics has been problematic.

This paper reports on an aspect of an international collaboration between Australian and German researchers through the project, *Designing Challenging Online Mathematics Tasks*. The purpose of the project was to develop principles of design for tasks implemented in programs aimed at Initial Teacher Education students (ITEs) preparing to teach mathematics. The challenging dimension of the tasks targeted the development of ITEs' personal mathematical capability. It was also intended that tasks would provide ITEs with insight into the nature of such problems, thus developing the confidence necessary for implementation within their own future classrooms. The project also aimed to accommodate the opportunities now available via student online learning systems, acknowledging the rapid movement of initial teacher education programs toward online or *blended* delivery.

The project was initiated by developing principles of design for challenging mathematical tasks through a synthesis of literature. These principles were subject to iterations of revision focused through the generation of initial tasks which were then trialled with ITEs in both Australia and Germany. The synthesis of literature focused drew on: task design in digital environments (Geiger, 2017); the notion of mathematical challenge (Sullivan, 2011); a map of pedagogical opportunities and

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mathematical analysis software (Pierce & Stacey, 2010); and, the quality of online learning environments (Bruder & Sonnberger, 2008). Through the process of iterative revisions, the principles of design were identified as: engagement, transparency, accessibility, challenge and feedforward/feedback. In this paper, we focus on the principle of engagement, particularly the connection between motivational influences and mathematical challenge in addressing the following research question: What motivation influences are evident in primary ITE's engagement with challenging online tasks?

CONNECTING ENGAGEMENT, MOTIVATION AND TASK DESIGN

Engagement is underpinned by aspects of motivation (Martin, 2012), as motivation is the process which serves to energise, direct or sustain activity, with engagement being the outward, observable outcome of this process (Schunk & Mullen, 2012). Motivation is key to engagement and learning achievement, as it focusses the attention of the learner towards specific goals and dictates the energy, effort and persistence that the learner will contribute to engaging in an activity. While energy and engagement are sometimes observable, the motivational influences that act upon engagement are harder to determine (Skilling, Bobis & Martin, 2015) but essential as drivers of engagement.

In the specific context of mathematics education, engagement and motivation are essential considerations in the design of mathematics tasks (Gresalfi & Barab, 2011) as tasks must be designed with an eye as to how they will be valued by students (Hulleman, Durik, Schweigert, & Harackiewicz, 2008). Such valuing has been linked to notions of purpose and utility (Ainley, Pratt & Hansen, 2006). In this view, a task which has *purpose* is "one that has a meaningful outcome for the pupil" (Ainley et al., 2006, p.29), while a task with *utility* addresses the ways in which the mathematical ideas inherent in the task are useful (as distinct from predominantly procedural).

THEORETICAL FRAMEWORK

Expectancy-Value Theory (EVT) provides insight into the engagement potential of a task by identifying individual's motivational influences. EVT proposes that the extent to which individuals *expect to achieve success (expectancy)* and their *subjective valuing of a task (value)* strongly influence their motivation and therefore engagement (Eccles & Wigfield, 2002). Under EVT, *expectancy* of achieving success is determined by prior experiences that shape self-efficacy and self-concept and, in turn, anticipation of progress and challenge. *Subjective valuing of a task* is influenced by *interest or enjoyment value*, *attainment value*, *utility value* and *relative cost* of undertaking the task. *Interest value* is differentiated into individual (personal) interest and situational interest. Individual interest being directed towards a specific object or activity and enduring over time (e.g., Hidi & Renninger, 2006), while *situational interest* is evoked by specific features of an activity, such as those that are personally relevant. *Attainment value* is considered the extent to which achieving on a task supports an individual's perception of themselves, for example, as a person, student or mathematician. *Utility value is* the potential of a task to further some future goal, such as a teaching capacity.

More specific to mathematics tasks, the notion of *utility value* relates to tasks in which the mathematical ideas embedded are deemed useful (Ainley et al., 2006). Finally, *relative cost* addresses aspects of a task that may be potentially demotivating, such as the perceived effort required for success. An individual's responses to these aspects of motivation – *expectation of success* and *subjective task valuing*, guide the amount of energy and focussed, sustained attention that is then given to a task. Thus, EVT provides a framework to structure insight into motivational influences which impact pre-service teacher engagement with challenging online mathematical tasks.

METHODOLOGY

Participants

This paper focuses on the survey responses of 43 Australian ITEs and follow-up focus group interview with three members of this group who were undertaking the second mathematics course in a four-year degree leading to a Bachelor of Education (Primary and Early Childhood). The course was concerned with developing content knowledge in mathematics for teaching primary and early childhood contexts. In this instance, the tasks were used as part of a graded assessment portfolio.

Online Challenging Tasks

Three tasks were designed in alignment with the five principles for online challenging mathematics tasks (engagement, transparency, accessibility, challenge and feedforward/feedback) and implemented with the cohort of ITEs. A brief description of these tasks follows:

Task 1: BBQ Task. Students were asked to compare the price of different BBQ packs (both non-vegetarian and vegetarian), for an upcoming social event.

Task 2: CO2 Emissions. Students were asked to make conclusions based on data represented in an interactive graph (generated through www.gapminder.org) about carbon dioxide emissions from different countries over time.

Task 3: Precious Cargo. Students were asked to decide the number and type of bottles that would fit in a triangular prism shaped box used to transport wine.

Data collection

Data sources utilised in this paper include pre-post surveys and a small group, semi-structured interview. Data were collected anonymously using a student generated two letter, two number code (for example, AB12).

Pre-post surveys: A pre-post survey was developed in alignment with the previously identified principles of design for challenging online mathematical tasks. Survey questions included both 4-point Likert scaled items and open-ended questions. Pre-surveys (14 statements/questions) were administered one week prior to undertaking the online challenging tasks. The post-survey (34 statements/questions) was administered at the end of the semester after students had competed the online tasks. Included in the pre-surveys were questions designed to ascertain ITEs' beliefs and expectations

regarding challenging online mathematics tasks. The post-survey was designed to match the pre-survey with additional questions aimed at identifying pre-service teachers views on the specific tasks they had undertaken in their course.

Interviews: Semi-structured interviews were conducted with a focus group of three volunteer Australian students via online video conferencing. Interviews were based on questions that were aligned with the principles of task design. The interview lasted for approximately 90 minutes and was audio-recorded and transcribed for data analysis.

Data analysis

Survey data: Duplicate entries were removed where students commenced the survey, withdrew, and then returned to re-do at a later time. The responses to the statements were converted to percentages and examined via graphical representation.

Interview transcripts: Transcripts were analysed through an iterative process using content analysis. This enabled a search for patterns and themes based on a theoretical lens anchored in EVT. Using this grounded methodological approach (Strauss & Corbin, 1998), distinctive themes related to ITEs' *expectation of success* and *subjective task valuing* were identified.

RESULTS

The EVT framework (Eccles & Wigfield, 2002) was employed as the theoretical construct to categorise ITE's responses to pre-, post-surveys and semi-structured interviews in relation to *expectancy of success* and *subjective task valuing*.

Expectancy of success

ITEs' expectancy of success was addressed through the inclusion of two statements on the pre-survey to which students had the option to respond with Strongly Agree (SA), Agree (A), Disagree (D), or Strongly Disagree (SD). The first of these was, "I expect to make good progress with the challenging tasks". The responses overwhelmingly indicated that students anticipated success with 41 of 43 responses indicating agreement (SA or A) with this statement. Forty-two of the 43 respondents also indicated agreement with the statement, "I expect the tasks to be challenging", suggesting that while acknowledging the challenge in the task, they still felt that progress would occur. One of the two students who thought they may not make good progress expressed a desire for *support and clues* when they became *stuck* [EL22]. The other student who anticipated a lack of progress also indicated that the tasks would not be challenging, casting some doubt on the consistency of their responses.

Subjective task valuing

In this section, we present an analysis of survey and interview responses related to the subjective task valuing component for the EVT framework. The four survey questions relevant to the interest aspect of the subject task valuing component and a graphical representation of ITEs' responses (Figure 1) follow:

1. The online mathematics tasks were interesting. [Variable: Task Overall]
2. The context of the online tasks (e.g., BBQ) was interesting enough to work on until completion: [Variable: Task Context]
3. I found the mathematics content (e.g., geometry, statistics) for tasks [all named] to be interesting: [Variable: Maths Content]
4. The challenging aspects of the online task kept me interested: [Variable: Task Challenge]

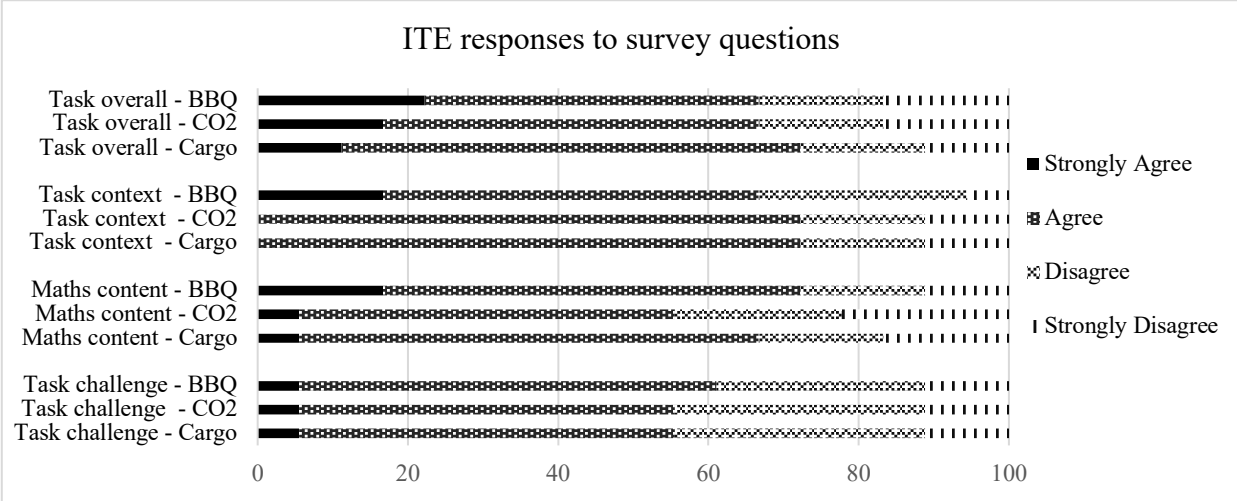

Figure 1: ITE student responses to above questions on the post-survey

Interest or enjoyment value: While the BBQ task had the least total agreement with regards to being interesting enough to work through to completion, it also had the highest 'strong agreement' and was the only task to have students comment on their enjoyment of the context: *The BBQ task…was interesting to see the price differences among different places* [JU27] and *I enjoyed the BBQ task as I enjoyed the investigation of finding different meat prices and found the results really interesting* [TI10]. Relevance to current issues was an additional aspect noted with one student identifying the CO_2 Emissions Task as appealing *because I am interested in global problem CO_2 [as it is] relevant for climate change* [CU24].

The process of mathematical thinking was also identified as a reason a problem was interesting, for example, *The precious cargo task [because it] involved the most mathematical reasoning* [JO14]. There were also students for whom the challenge inherent in the tasks was appealing*: I believed that the CO_2 Emissions task was the best because it was the most challenging* [MI16] and *Precious Cargo…challenged me the most and allowed me to be creative with my answers* [CA37]. While task challenge is included here as a positive aspect leading to reported interest, this was not universal with some students feeling the challenge was excessive. The negative aspects are reported below under relative cost.

Attainment value: The value of obtaining success on tasks was strongly influenced by the assessable portfolio. Hence, ITEs were motivated by an extrinsic aspect (good grades, academic success). This aspect of attainment value was reflected in student

comments when asked what they found difficult in addressing the open-ended aspect of tasks, for example, *I was often unsure if my mathematics working was going in the right direction [TI10];* and, *I struggled with the open-endedness* [TI10]. While these are all difficulties typically encountered when individuals attempt to solve complex problems (Makar & Fielding Wells, 2011), students concern about success can be exacerbated when assessment performance is a factor. This can be of particular concern when students believe mathematics is a subject in which uncertainty should not be present *–my favourite thing about maths…is that it does have a right and wrong answer… I don't like that uncertainty, I really do like things black and white* [SH01].

Utility value: Student perceptions of the utility value of the tasks and the mathematics inherent in the tasks were not highly prevalent in their responses, however, two aspects related to the utility of the mathematics tasks were identified: the potential use of tasks as a future teacher, and the use of task to enhance ITEs' personal mathematical knowledge, for example, *Opportunities for me to use different strategies to find the volume and area of 2-D shapes and 3-D shapes* [JT32]).

Relative cost of undertaking the task: Many of the students expressed surprise at the effort and time required to engage with challenging tasks. For some this was in the application of aspects of mathematics, for example, *I found the geometry extremely hard* [LO26]), while for others it was the lack of direction they felt when faced with an open-ended task, for example, *I wasn't sure how much information to provide* [KI16].

DISCUSSION AND IMPLICATIONS

In utilising the EVT framework (Eccles & Wigfield, 2002) for analytical purposes, we could see that while ITEs largely expected to achieve success on the challenging tasks (*expectancy*), their motivation to engage with the tasks varied considerably according to their own perceptions of mathematics and mathematical learning (*subjective task valuing*). As such, *subjective task valuing* (*interest, attainment, utility and relative cost*) is of interest in this study and is predominantly addressed in the discussion.

On being questioned, ITE's equated *interest* predominantly with the life-related context of each task but to a lesser extent with the specific mathematical content and degree of mathematical sophistication. Some of the context interest was derived from the *situational interest* of the context but more so existing *individual interests* (Hidi & Renninger, 2006) which are unlikely to be known to the task designer. This suggests to the task designer that there are multiple aspects of the task that may have potential to pique interest but that the degree of challenge and the mathematics inherent are potentially the most critical to consider in designing and scaffolding tasks.

Some ITEs indicated they enjoyed the challenge and the opportunities for reasoning and creativity associated with open-ended questions. Others, however, argued the absence of explicit direction as to the mathematics and mathematical processes to be used was problematic. These latter responses suggest existing views of mathematics as predominantly memorisation and application of procedures and thus the tasks presented conflicted with these views. As such, the benefit of undertaking the tasks

(*purpose*) and the usefulness of the mathematical approaches (*utility*) was not evident to many of the ITEs (Ainley et al., 2006). A further factor impacting on motivation relates to *utility value* as defined by Eccles and Wigfield (2002) – while some ITEs valued the activities for future classroom use, others could not see how the tasks could be used either in terms of context or mathematical content. ITEs may not readily make connections between the underlying intentions of challenging tasks and this may need to be made explicit.

If ITEs are to develop competence and a positive disposition towards implementing challenging problems in their future classrooms, then these aspects of challenging mathematics problems that appear to serve a demotivating influence must be addressed. Accordingly, teacher education programs must: 1) challenge the notion that the nature of mathematics is purely procedural; 2) provide scaffolded opportunities for ITEs to experience open-ended problems as a learner; and, 3) make explicit the utility value and purpose of challenging tasks to ITEs.

This study has provided evidence that the implementation of challenging mathematical tasks with ITEs is complex, suggesting that tasks must be carefully designed, scaffolded and their use justified with the ITE cohort if motivation to engage deeply with such tasks is to be cultivated and maintained. A major issue in achieving this goal, however is impoverished attitudes and beliefs about the nature of mathematics as a discipline held by some ITEs. These attitudes and beliefs must be addressed if ITE's are to be motivated to implement challenging mathematical tasks in their own practice.

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THE INFLUENCE OF TEACHERS ON LEARNERS' MATHEMATICAL IDENTITIES

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In this paper, we develop a framework to define and research learners' mathematical identities, as both constructed by learners and influenced by their teachers. We show how five teachers offered learners different social identities through the kinds of access they provided to learning mathematics. Two teachers offered all learners opportunities to become full members of their classroom communities, and three teachers marginalised those whom they thought could not do mathematics. The offered social identities strongly influenced how learners constructed their mathematical identities and how they learned mathematics.

INTRODUCTION

Over the last two decades, the concept of identity has received significant attention in the field of mathematics education and is useful in theorizing learning (Radovic, Black, Williams & Salas, 2018; Sfard & Prusak, 2005). In recent years, researchers are developing operationalized definitions of identity and Radovic et al. (2018) describe three ways in which identity is defined in research: social or subjective; enacted or representational; and transformative or stable. In some research, identity is conceptualised as a social becoming, being constructed in social contexts, while in other research, identity is conceptualised as subjective, emphasising the individual self and personal experiences. Some researchers define identity as enacted and related to action, while others define identity as representational, mediated and constructed through discourse. Most researchers emphasise the malleable and transformative nature of identity, while few view identity as static. In this paper, we begin by presenting a framework to define identity, which moves between the social, subjective, enacted, representational and transformative features of mathematical identity. We use our definition to understand how five secondary school mathematics teachers influenced learners' mathematical identities.

THEORETICAL FRAMEWORK

Drawing on the work of Marks and O'Mahoney (2014), we define learners' mathematical identities as being constituted by relationships between learners' personal identities, social identities and agency. Personal identities are the subjective dimension of identity, involving learners' understandings of self, as informed by their interests, aspirations, experiences and motivations. Social identities are the social dimension of identity, involving the ways learners position themselves and would like to be perceived by others in their classroom communities. Agency refers to learners' choices in participating in mathematics and developing their identities. Learners'

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mathematical identities are influenced by the social identities offered to them by their parents, teachers and peers, which influence the opportunities provided to learners to participate, learn and become members of the classroom community (Fig 1).

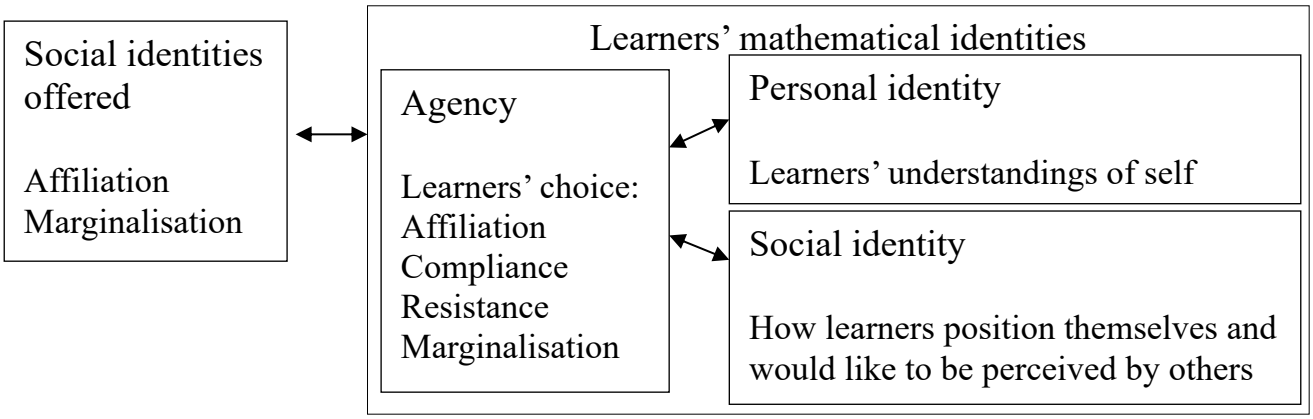

Figure 1: Identities offered and constructed

Teachers can offer learners identities of affiliation, by supporting them to learn mathematics and become full members of their classroom community. Some teachers offer learners identities of partial affiliation, by inviting participation and supporting learners to some extent. Teachers can offer learners identities of marginalisation by limiting learner opportunities to learn mathematics and become full members of the classroom community.

While teachers offer learners certain social identities, learners exercise agency and construct their identities in different ways, influenced by their own personal and social identities. Learners can construct identities of affiliation when offered opportunities to affiliate, or identities of marginalisation when marginalised by teachers. Learners can also exercise agency by developing their identities differently from what is offered, in compliance with or resistance to the offered social identity. Compliant learners often aim to achieve peripheral membership in the classroom by participating peripherally rather than fully when offered identities of affiliation, or may participate with others and not with their teachers when offered identities of marginalisation. Resistant learners may not want to be identified as members of the classroom community when offered identities of affiliation, or may want to achieve full membership when offered identities of marginalisation.

Experiences of exercising agency concomitantly influence learners' personal identities, as they learn more about themselves in terms of their interests, aspirations and motivations, and their social identities, as they evaluate their membership in the community. This means that learners' identities are constantly being developed as they learn and participate (or not) in their classroom communities. As learners enact their identities through interacting with teachers, teachers can continue offering them the same social identities as previously offered or may offer learners different social identities from those offered in the past. Different teachers may also offer different identities to the same learners. Learners may enact and reproduce the same identities

constructed previously or may adopt different identities, as influenced by their experiences with new teachers in different classrooms, highlighting the malleable nature of identity.

Different mathematical identities can be enacted as learners participate with others in the classroom (Radovic et al., 2018). As Wenger (1998) argues, identity is related to learning because as learners participate, they are able to further affiliate themselves with their classroom community. Identity is also representational and constructed through learner narratives (Radovic et al., 2018). Narratives, according to Sfard and Prusak (2005), are a collection of socially shaped stories, which influence action when being reifying, endorsable and significant. Reifying narratives contain verbs and adverbs that stress the repetition of actions. Narratives are endorsable when the storyteller declares the authenticity of narrative, and narratives are significant when a change in story is likely to lead to a change in feelings about an identified person. Observing classroom interactions and accessing narratives enable understandings of the identities offered to and constructed by learners. We now show how our framework is useful to research learners' mathematical identities.

METHODS

This study took place in a well-resourced, technical school in Johannesburg over a period of two years in the third school term (August - September): two weeks in 2015 (Grade 9) and four weeks in 2016 (Grade 10). Five mathematics teachers participated in the study, two in the first year: Mr Moyo and Mr Molefe (Gr 9), and three in the second year: Mr Sithole, Mr Ndlovu and Mr Ncube (Gr 10) (all names in this paper are pseudonyms). Nineteen learners participated in the study, eight in the first year, with an additional eleven in the second year. From the initial eight participants in the first year, six continued to participate in the second year, while two transferred to another school.

All of the teachers who participated in the study had academic degrees to teach mathematics. Mr Moyo was newly qualified and had joined the school at the end of the second term (May), Mr Molefe had five years of experience teaching mathematics, Mr Sithole had thirty-two years, Mr Ndlovu, seven years and Mr Ncube, five years. The eight learners who initially participated in the study were selected by their grade 9 teachers using the following criteria: one high performing learner, one low performing learner and the person sitting next to them, which enabled us to observe teacher and learner interactions with learners displaying different levels of achievement. In grade 10, we continued to observe the six learners from Grade 9 and selected an additional eleven learners, who were seated next to the initial participants. We asked the Grade 10 teachers whether each learner was a low, average or high achieving learner, which enabled us to understand how teacher ideas about learner performance influenced teacher and learner interactions.

Data were collected in the form of videotaped lessons, field notes, photographed learner notebooks and multiple audiotaped semi-structured interviews with learner and

teacher participants. A total of 480 minutes of lessons were analysed for each teacher. All video and audio recordings were transcribed, and data were analysed qualitatively. From the videotaped lessons, we coded the kinds of social identities offered: affiliation (A), partial affiliation (PA) or marginalisation (M), by considering the interactions between teachers and learners. Observing classroom interactions also enabled us to understand how learners exercised agency and enacted their identities as affiliation (A), compliance (C), resistance (R) or marginalisation (M). We analysed the semi-structured interviews by identifying narrative excerpts to understand the representational component of identity, using Sfard and Prusak's (2005) criteria of narratives needing to be reifying, endorsable and significant.

FINDINGS

An overview of our findings

Our analysis showed that two teachers offered all learners identities of affiliation with their classroom communities, while the other three offered learners differential identities depending on learners' achievement in assessments. We found that learners developed their identities in affiliation with, compliance with or marginalisation from the social identities offered. None of the learners resisted the offered social identities, highlighting that learners' mathematical identities are strongly influenced by the social identities offered by teachers. Table 1 shows the social identities offered to learners by the teachers and the identities constructed by learners.

Two teachers, Mr Moyo (Gr 9) and Mr Ncube (Gr 10), offered all learners social identities of affiliation. They engaged with and assisted all learners, including struggling learners, and treated learners and their contributions respectfully. All four learners in Mr Moyo's Grade 9 classroom developed their identities in affiliation with the social identity offered. In Mr Ncube's classroom, three learners developed their identities in affiliation and three developed their identities in compliance with the social identity of affiliation offered.

The other three teachers, Mr Molefe (Gr 9), Mr Sithole (Gr10) and Mr Ndlovu (Gr10) offered learners differential social identities based on learners' achievement. High achieving learners were offered social identities of affiliation - the teachers engaged with them, treated them respectfully and were willing to help them. All the high achieving learners besides Fred, developed their identities in affiliation with the offered identity of affiliation. Low achieving learners were offered social identities of marginalisation - the teachers often ignored them, provided them with limited assistance and in some situations, treated them disrespectfully. Some of the marginalised learners developed their identities in compliance with the offered social identity by participating peripherally, while others developed their identities in marginalisation from the classroom community.

Two learners, Sunny and Jack in Mr Molefe and Mr Sithole's classrooms, whose marks were inconsistent, were offered social identities of partial affiliation - the teachers were willing to assist them to a certain extent and ignored them at other times. Sunny was

identified by Mr Molefe as a "naughty" learner, who was at times the highest achiever in the classroom, and at other times, did not perform as well. Jack was identified as a learner who had "mathematical intelligence", and whose marks were sometimes average and at other times lower. Both learners enacted identities of compliance. Lane, who was identified as a quiet and "slow" learner, whose marks were sometimes average and at other times very low, was marginalised by Mr Sithole and responded with compliance.

Grade	Teachers	Learners' achievement	Learners	Identity offered	Identity constructed
9	Mr Moyo	High achievers	Shane & Jack	A	A
		Average achiever	Lane	A	A
		Low achiever	Senzo	A	A
10	Mr Ncube	High achiever	Thandi	A	A
		Average achiever	Amy	A	A
		Average achiever	Thabo	A	C
		Low achiever	Mike	A	A
		Low achievers	Jimmy & Tess	A	C
9	Mr Molefe	High achiever	Sindiswa	A	A
		Inconsistent	Sunny	PA	C
		Low achievers	Jennifer & Tess	M	C
10	Mr Sithole	High achievers	Phillip, Shane, Lebo & Karabo	A	A
		Inconsistent	Jack	PA	C
		Inconsistent	Lane	M	C
		Low achievers	Tommy & Senzo	M	M
10	Mr Ndlovu	High achiever	Edward	A	A
		High achiever	Fred	A	C
		Low achiever	Sunny	M	M

Table 1: Identities offered and constructed

Over the two-year period, learners' mathematical identities changed, some significantly, others negligibly. Jack, Lane, Senzo and Sunny enacted different identities and provided different narratives of their identities in grade 9 and 10, with their teachers usually being the main reason why they no longer enjoyed mathematics in grade 10. Shane and Tess enacted similar identities in both grade 9 and 10 and both learners' stories of themselves and ways of participating changed negligibly.

The social identities offered

The teachers' narratives of who could do mathematics influenced how they interacted with learners and the kinds of social identities offered to learners. For Mr Moyo and Mr Ncube, all learners could learn mathematics. Mr Moyo said that he envisioned a classroom where all learners participated. He believed that struggling learners could learn mathematics and said that he tried to motivate them. Mr Ncube also thought that all learners could learn mathematics. He said, "mathematics is like any other subject meant for everyone to do, but it requires for some to go an extra mile than some", meaning that some learners had to work harder than others to understand concepts. He said he encouraged learners to participate and that he was willing to help struggling learners.

While providing all learners access to learning mathematics, both teachers still saw ability as a characteristic, which enabled some learners to do better than others. For Mr Moyo, some learners achieved well because they were "fast thinkers" because they practised mathematics and had "high IQs". He said that slower learners could also succeed when given more time and that "someone who is just intelligent but who does not practise, that person does not become good in maths". For Mr Ncube, high achieving learners were "gifted" and understood concepts by doing few examples, whereas others may have to practise more to achieve the same level of understanding. Even though both teachers discussed ability, they both stressed the importance of effort in enabling all learners to learn mathematics.

For Mr Molefe, Mr Sithole and Mr Ndlovu, ability was key to learning mathematics. These teachers did not think that all learners could learn mathematics, did not assist all learners and offered learners differential identities. For Mr Molefe, learners who did not understand mathematics had "deeper psychological problems". He said, "yes there are incompetencies in maths but those incompetencies are a result or consequence of these psychological factors". He said he was not "competent" to treat psychological problems and he usually referred these learners to the school counsellor. For Mr Sithole, competence in mathematics was a "gift" from "God" and there was "nothing" he could do for struggling learners so he concentrated on high achieving learners. Mr Ndlovu also attributed mathematical competence to how "God" created people.

> Maths was not made for everybody…because when God probably created people, we know in science there are right people and there are left people. There are people that are good in numbers. There are people who are not good in numbers.

He said that he did not help struggling learners and would advise them to get help from another teacher or drop mathematics as a subject in school.

Learners' mathematical identities

Learners identified teacher practices as the most important factor that influenced their learning of and identification with mathematics. Learners who affiliated when offered social identities of affiliation said they enjoyed learning mathematics and identified themselves as successful or growing learners of mathematics, who participated as full

members of the classroom community. For example, Senzo explained why he affiliated with the social identity offered by Mr Moyo in grade 9.

> I am starting to enjoy maths because of Mr Moyo and I realised that it is important for me and my life, like my future. And because at home I revise everything every day, every after school and morning before I come to school.

However, in grade 10, Senzo's identity changed substantively. Senzo, like other learners who developed their identities in marginalisation, said that he did not enjoy learning mathematics and participated in limited ways, because of Mr Sithole.

> I am demotivated because Mr Sithole is demotivating... He gives you a sum, you try it then he comes and checks and tells you that is rubbish. You know that is demotivating. You know, he doesn't really care.

Learners who complied with the social identities offered participated peripherally. Fred, like other compliant learners, was not entirely comfortable to participate. He said that he did not always participate even though he enjoyed learning mathematics. He explained "well, maybe I am scared that I will get such a comment which will… sometimes teachers talk a lot of things that hurt you inside". He said that Mr Ndlovu did not give learners many opportunities to practice mathematics and that Mr Ndlovu was "full of anger". Even though Mr Ndlovu provided opportunities for Fred to affiliate, Fred chose to become a peripheral member because of his discomfort, indicating that learners' experiences of their teachers' practices shaped how they developed their identities.

DISCUSSION AND CONCLUSION

In this paper, we proposed a framework to define and research learners' mathematical identities. Drawing on Rodovic et all's. (2018) model of the three ways in which identity is defined in research, our framework shows how identity construction is social, as learners' social identities informed how they positioned themselves and were positioned by others in the classroom; and subjective, as learners' personal identities informed how they identified themselves as learners of mathematics.

We showed how teachers' narratives of who could learn mathematics influenced the social identities offered to learners. Two teachers, who thought that all learners could learn mathematics, offered all learners social identities of affiliation, by emphasising effort. While these two teachers stressed the importance of learner effort, they also attributed success to ability. Research points to the dangers of attributing learner success to ability, with some researchers showing how such teacher views result in inequitable teacher practices and exclusion of learners (e.g. Hodgen & Marks, 2009). The dangers of attributing success to ability is highlighted further in our findings, as three teachers offered learners differential social identities based on their ability to perform well in assessments. Sfard (2009) cautions against the use of marks as labels for competence, as these labels may become an obstacle for learners to construct their identities positively.

We showed how learners' personal and social identities were enacted, as learners agentively participated in the classroom. We also showed how identities were representational and constructed through learner narratives, in terms of how learners viewed themselves personally as learners, and socially, as members of their classroom community (Rodovic et al., 2018). We showed that all learners either identified or complied with the social identities offered, with the teacher being the most important reason for learners' identification with mathematics. Learners who affiliated with the offered identities of affiliation were comfortable to participate in the classroom because of their teachers' practices. Their teachers treated them respectfully and provided them with opportunities to affiliate. Learners who complied with or developed their identities in marginalisation from the offered social identities were not comfortable to participate in and become full members of their classroom community, as influenced by their experiences of their teachers' practices. None of the learners resisted any of the social identities offered. Based on these findings, we conclude that learners' mathematical identities are strongly influenced by their experiences of the social identities offered to them by their teachers.

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THE IMPACT OF MONTESSORI EDUCATION ON THE DEVELOPMENT OF EARLY NUMERICAL ABILITIES

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Studies on Montessori education's impact on development are rare, especially in France. In this report, we present a study that aimed to compare the development of preschoolers from Montessori Education and Conventional Education in France in a wide range of numerical skills using a longitudinal randomized controlled design. Based on a didactical analysis, we made the hypotheses that children would succeed in different numerical tasks according to the type of education they receive. Children were tested at the beginning of their first pre-school year and again a year and a half later. Significant differences between the two educations were identified. These results are discussed through a didactical analysis of teaching practices.

INTRODUCTION

Montessori education was developed in the first half of the 20th century by an Italian doctor, Maria Montessori. She extended her sensorimotor work on children with mental disabilities to children who were ordinarily developing and from low-income backgrounds. From her observations, Maria Montessori proposed principles of child development. She then based her philosophy and teaching method on it. First, she described that the child is born with an "absorbing mind" (Montessori, 1988) and a motivation to learn. Then the child goes through "sensitive periods" (Montessori, 1966) where he is particularly attracted by some specific learning. He is receptive to assimilate them easily and quickly. Finally, she argued that not all children progress at the same rate. It is therefore necessary to allow the latter to exploit their sensitive periods at the appropriate moment.

In the classroom, these principles involve several specificities concerning the environment offered to children and the posture of the teacher. The classes are mixed in ages (3 to 6 y-o children in preschool). Children organize their own daily schedule by choosing their activities, how long they spend working on them, who they work with and where they sit. They can repeat each activity as many times as they wish during their three years in the classroom. The materials used in these activities provide corrective feedback and are unique in the classroom. They are displayed on small shelves in the classroom, grouped into curricular areas of Practical Life, Sensorial, Language, Mathematics and Geometry, Geography, Biology, Music, and Visual Arts (Montessori, 1967, 1989a, 1989b) and organized according to their level of difficulty.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 256-263). Pretoria, South Africa: PME.

The teacher presents the activities to children and makes sure pupils only choose activities whose level of difficulty is appropriate. The lessons are given almost exclusively in small groups or one-on-one.

Studies of Montessori education's impact on development are scarce (Lillard, 2012) and the majority have been carried out in the USA (Marshall, 2017). Among the few studies that exist, some show better outcomes of Montessori than other teaching methods (e.g. Lillard & Else-Quest, 2006; Besançon & Lubart, 2008; Lillard et al., 2017), and others show similar or even worse outcomes (e.g. Ansari & Winsler, 2014; Lopota, Wallace & Finn, 2005). These inconsistent results could be due to methodological shortcomings such as variations in the quality of Montessori education implementation, imperfect control groups or small sample sizes (Lillard, 2012; Lillard et al., 2017). In the present study, we compared the development of a wide range of numerical skills in preschoolers from Montessori education (ME) and conventional education (CE) using a longitudinal randomized controlled design. Moreover, the study aimed to discuss possible explanations of our findings using a didactic analysis of materials and practices.

THEORETICAL BACKGROUND

To our knowledge, only 3 studies have specifically compared the math performance of preschoolers in ME to that of preschoolers in CE and their results are mixed (Lillard & Else-Quest, 2006; Laski, Vasilyeva & Schiffman, 2016; Wexley, Guidubaldi & Kehle, 1974). In 1974, Wexley and colleagues compared 3 to 5 year-old disadvantaged children in a Montessori program and a day care conventional program matched in age, sex, race, socio-economic status, number of parents living in the household and number of years in preschool. They also compared these two groups to two control groups of children without preschool education, from disadvantaged and advantaged neighborhoods. They used the arithmetic test from the Wechsler Preschool Scale of Intelligence (as well as other tasks assessing other aspects of cognitive development). This study showed that students in ME performed better in arithmetic than the disadvantaged control group but did not differ from the children in the day care program and the advantaged control group. In 2006, Lillard and Else-Quest conducted an evaluation of the Montessori method using the school lottery system. Children whose parents entered the lottery and were accepted constituted the ME group. Those not accepted, thus joining other education systems, were assigned to the control group. Overall, 112 American children from 5 and 12-year old age groups were compared on a range of cognitive measures. For the math measure, they used the Applied Problems task of the Woodcock-Johnson 3 Test Battery. A significant group difference in favor of the Montessori group was found for the 5-year olds but not for 12-year olds. More recently, Laski and colleagues (2016) showed also similar results. They conducted a longitudinal evaluation of knowledge of the decimal system and arithmetic among 150

children in early elementary school from Montessori and non-Montessori schools. At Time 1 (T1), children in pre-school or in first grade were tested with a base-10 block task and an addition task. Then two years later, in 2nd or 3rd grade, they were tested with five place-value problems and ten arithmetic problems. The authors found that at T1, children from Montessori schools performed better in the base-10 block task in kindergarten but not in first grade. Moreover, no program differences were seen in place-value understanding in second and third grades and no program differences were found at any grade level in arithmetic accuracy or strategy use.

To conclude, these studies seem to show a superiority of ME over CE for math skills in preschool, but no difference in primary schools. However, these findings must be interpreted with caution because they are few, they have several methodological limitations (e.g. no randomized assignment of children into experimental and control groups and/or no baseline scores) and they only included limited measures of mathematical skills.

In French conventional preschool education, the math program aim is to enable each child to understand that a number can express both a quantity and a rank in a list. Number learning is based on the understanding of quantity, its verbal and written codification, the acquisition of the oral sequence of numbers and the use of enumeration, and precedes the introduction of the numeral system (MEN, 2015). No specific learning activities are yet proposed on place value understanding as groupings and exchanges activities, or recognizing the correspondence between the position of the digit and the number of groupings (Margolinas & Wozniak, 2012, 112). Moreover, there is no specific work on operations. On the contrary, in ME, mathematics activities aim at introducing the decimal numeration (verbal and written) and calculation. Children are first exposed to quantities and symbols separately and then quickly to the association between quantities and symbols. Next, the decimal system is presented. Finally, children who have already mastered these previous activities learn the four operations and the fractions (Montessori, 1934).

In terms of the key end-of-preschool competencies (according to the French standards), we compared activities in math textbooks used in these two institutions: a common textbook (Duprey, Duprey, & Sautenet, 2016) to model conventional education and a text from Montessori (Montessori, 1934) devoted to the presentation of all materials and description of the mathematical concepts to describe the expected competencies in ME. This comparative analysis of ME and CE from their respective math textbooks allowed us to identify seven types of tasks addressed in these two institutions (Croset & Gardes, in press): Count Verbally (T1), Recognize the numerosity of a given set (T2), Create a set given a specific numerosity (T3), Solve simple non-symbolic arithmetic problems (T4), Create a set of the same numerosity as another distant set (T5), Compare the numerosities of two sets (T6), Recognize number symbols (T7). We

found that some of those tasks are often proposed to children in CE and less in ME (i.e. T4 and T5), or the contrary (i.e. T3, T7 and T1), and some are performed equally in ME and CE (i.e. T2 and T6). We will compare the evolution of performance of children in ME and CE in these tasks between the first year and second year of preschool using a test that we designed for this purpose.

Research hypotheses

According to our didactical analysis (Croset & Gardes, in press), we therefore expect differences of three orders:

- Solving simple non-symbolic arithmetic problems (T4), creating a set of the same numerosity as another distant set (T5) should be performed better by children in CE than children in ME.
- Creating a set given a specific numerosity (T3), recognizing number symbols (T7) and counting verbally (T1) should be performed better by children in ME than children in CE
- No group difference is expected for the tasks: recognizing the numerosity as a given set (T2) and comparing the numerosities of two sets (T6).

METHODOLOGY

We assessed 76 French preschoolers from the same public school: 32 children in ME (17 girls) and 44 controls (21 girls) in CE. The school, located in a low-socioeconomic neighborhood has several classrooms serving children from 3 to 6. The school has two types of classroom environments: three Montessori multi-aged classrooms and six conventional single-aged classrooms. Children were randomly assigned to these environments by the teachers during registration at the school. They were first tested at the mean age of 3.37, at the beginning of their first year of preschool (Y1) and then 1.5 years later (Y2), at the mean age of 4.62. They were tested individually with the same tasks both times.

Based on our didactical analysis (Croset & Gardes, in press), we designed a test to assess the seven task types described above. The test is called the Mathematical Didactic Diagnosis battery (MDD) and is based on tokens that children manipulate. For tasks 2 to 6, we proposed different subtests with increasing difficulty levels. For example, for T2, children have first to recognize the numerosity of a set of 3 tokens, then of 7 and finally, of 11 tokens. On these three tasks, a child's score for T2 can vary from 0 to 3 depending on their success. In total, children were tested on 27 subtests.

RESULTS

For each tasks type, scores were analyzed using Year (Y1, Y2) x Education (ME, CE) frequentist and Bayesian repeated-measure ANOVAs. We found significant and positive evidence of higher progress for children in Montessori education in two tasks:

the simple addition and subtraction task (T4), and the number symbols recognition task (T7), in which they had to transcode visual symbols to number words. There was no significant interaction in the other tasks. All results are summed in Table 1.

Tasks	ANOVA Results		
T1	$F(1,74) = 1.51$	$p = 0.22$	$BF_{10} = 0.45$
T2	$F(1,72) = 0.04$	$p = 0.85$	$BF_{10} = 0.16$
T3	$F(1,74) = 3.31$	$p = 0.07$	$BF_{10} = 0.85$
T4	$F(1,74) = 5.78$	$p = 0.02$	$BF_{10} = 4.22$
T5	$F(1,73) = 0.30$	$p = 0.59$	$BF_{10} = 0.21$
T6	$F(1,74) = 0.22$	$p = 0.64$	$BF_{10} = 0.24$
T7	$F(1,74) = 7.85$	$p = 0.006$	$BF_{10} = 15.43$

Table 1: Results

Our first hypothesis is invalidated: children are not better on T4 and T5 in CE than in ME. It is, in fact, the contrary: ME children showed a significant higher progress on solving simple non-symbolic arithmetic problems (T4, see Figure 1). Children succeed as well in the T5 tasks as CE children, contrary to our *a priori* hypothesis, based on teachers' practices. Our second hypothesis is partially validated: children in ME recognize much more number symbols (T7, see Figure 2) than in CE. Children with CE count as far away (T1) and create set given a specific numerosity (T3) as well as children from Montessori Education. Finally, our third hypothesis is validated: children recognize (T2) and compare numerosities (T6) as much with CE as with ME.

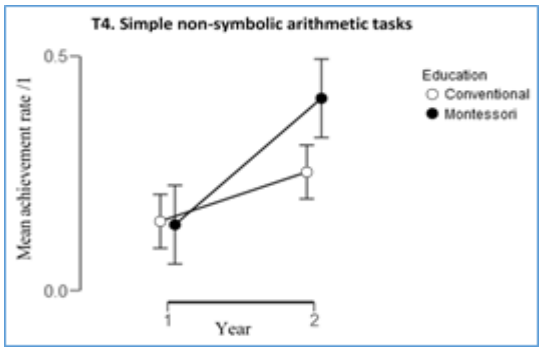

Figure 1: Simple non-symbolic arithmetic tasks (T4)

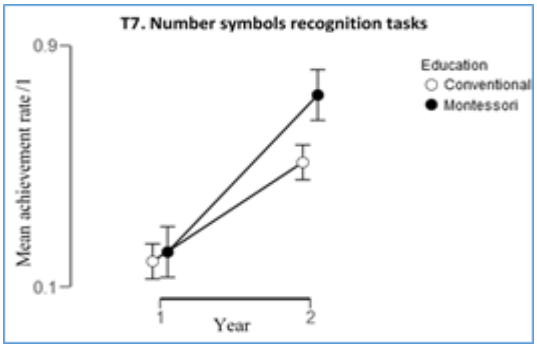

Figure 2: Number symbols recognition tasks (T7)

One limitation of our study is that we tested children in their first and second year of preschool but not during the third year. In addition, success in the most difficult subtests of the MDD is actually expected during the third year of preschool. However, our didactical analysis showed that numerical abilities expected in ME go beyond those expected in CE (Croset & Gardes, in press). For this reason, we created a new exploratory variable "V" that sums the scores of the subtests that are specifically expected to be mastered in the final grade level in French school. This variable sums the scores of 12 subtests. We hypothesized that children in ME would outperform children in CE in this measure. We indeed found a significant interaction between Year and Education in favor of ME ($F(1,74) = 11.31$, $p = 0.001$, $BF_{10} = 69.61$). ME scores are on average 17% better on specific tasks of higher level than CE scores (Figure 3).

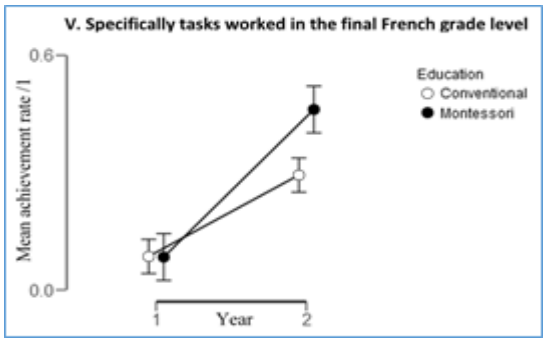

Figure 3: Added variable V

DISCUSSION & CONCLUSION

Our previous didactical analysis of ME and CE textbooks, in which we analyzed and quantified the different tasks proposed in both education systems, allowed us to identify keys differences that could explain partially our results. First, the number of numerical tasks proposed in CE is more important than in ME because CE offers more varied tasks for the same learning objective. For example, pupils in CE are confronted with 26 different activities on the recognition of the numerosity of a given set (T2) and will rarely have the opportunity to repeat the same tasks. On the contrary, pupils in ME encounter only 4 different counting tasks but they will repeat these tasks at least until the teacher considers them to be mastered. Second, the tasks in ME are very basic, i.e.

there is little problem solving. Numbers are always explicitly appointed as the "object" of the study. On the contrary, more complex tasks are proposed in CE, in which numbers appear as a possible procedure (tools) but other procedures could be used to solve the problem. For instance, the cardinality matching task (T5), a complex task, is common in CE while absent in ME. Finally, the time dedicated to developing numerical skills is much shorter in ME, 6 months than in CE, 3 years. We can therefore say that CE favors a variety of tasks, including complex tasks, with the aim of enabling students to transfer their knowledge in various contexts and over a long time. On the contrary, ME seeks to entrench knowledge in specific and limited contexts.

Despite these observations, children in ME seem to be as good, if not better than, children in CE. Especially, children in ME exceed the expectations of the second year of preschool. The results seem to show that ME students manage to transfer their knowledge outside the contexts studied. We can therefore infer that a limited number of well-chosen tasks could contribute to a more efficient numerical development for disadvantaged preschoolers.

On a shorter learning time, with targeted and basic tasks that are repeated until their perfect mastery, children in ME progress as well as students with French CE. This remains true on tasks that are not explicitly taught in ME, which exhibits some ability to transfer in Montessori students. At a time when the French government is wondering about the target of mathematics education (Villani & Torossian, 2018), this type of analysis seems particularly important.

Moreover, this study shows that crossing didactical and psychological approaches can enrich the results of the research question. The didactical analysis enables the generation of more specific hypotheses to be tested and the methodology in cognitive psychology enables the evaluation of these hypotheses. The results are then analysed using statistical tools and finally discussed from both points of view.

However, these results are preliminary: Future goals are (1) to increase sample size, (2) to evaluate the impact of ME on the development of math skills from age 3 to 6 and not just from age 3 to $4^{1/2}$ (3) to extend the analyses to other types of skills (e.g., language, social), (4) to explain differences using in-depth didactical analyses.

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INTEREST DEVELOPMENT AND SATISFACTION DURING THE TRANSITION FROM SCHOOL TO UNIVERSITY

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High dropout rates in mathematics reveal students' problems during the transition from school to university mathematics. As the subject mathematics differs between school and university, we use specific questionnaires, which distinguish interest in school and in university mathematics. The aim of this contribution is to analyse the role and development of students' interest in school and university mathematics during the transition. Our results indicate that interest in university mathematics is positive associated with students' study satisfaction, while interest in school mathematics seems to be disadvantageous. Moreover, we identify different interest profiles of students that may hinder or improve learning processes. Based on these results, we discuss possibilities to support students during this challenging transition.

INTRODUCTION

The transition from school to university mathematics is a challenging process for many students. This is illustrated by high dropout rates in mathematics – especially during the first terms at university (Chen, 2013).

According to theories of person-environment fit (e.g. Swanson & Fouad, 1999) students' personal characteristics – such as interest, knowledge, and learning behaviour – must fit to the learning environment at university to ensure a successful transition. A sufficient fit determines students' achievements as well as their own satisfaction whereas an insufficient fit probably leads to less achievement and demotivation. Haak (2017) proposes two possible ways in which students can react on insufficient fit: They can either adapt their personal characteristics or they can decide to dropout. Therefore, not only students' characteristics at the beginning of their studies, but also the development of these characteristics during the first semester is relevant for a successful transition. Indeed, students who drop out differ from those who continue their studies concerning their subject interest more at the time of the dropout, than at the beginning of their studies (Schiefele, Streblow, & Brinkmann, 2007).

In this contribution, we discuss the role of different interest facets and their development during the first term at university for students' study satisfaction. Following Blüthmann (2012), we understand study satisfaction as an evaluative rating building on affective experiences and cognitive comparisons.

INTEREST IN MATHEMATICS

Interest is considered to play a crucial role in learning processes. Krapp (2007) describes interest as a special relation between a person and an object. Individual

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interest is rather stable over time and comprises emotional- (feelings of enjoyment while in interest based activities) and value-related (subjective esteem for the objects of interest) characteristics. Subject unspecific studies report a positive impact of students' individual interest on study satisfaction (Schiefele & Jacob-Ebbinghaus, 2006; Blüthmann, 2012). However, mathematics related studies investigating the relation of interest and study satisfaction are scarce but necessary – as pointed out in the following.

General interest in mathematics

Since studies focused on mathematics provide inconsistent results, the role of students' interest for a successful transition is ambiguous: Rach and Heinze (2017) found neither significant effects of interest in mathematics on achievement during the first term nor on students' exam-attendance. Geisler and Rolka (2018) reported that students with low interest in mathematics are more likely not to attend their exams, which – according to Baars and Arnold (2014) – is a useful indicator of dropout.

Even though interest plays a crucial role for students' choice to study mathematics (Di Martino & Gregorio, 2018), students' interest in mathematics seems to decrease during the transition (Sonnert & Sadler, 2015). Daskalogianni and Simpson (2002) refer this decline to unfulfilled expectations and beliefs that do not fit to the mathematics encountered at university. However, Liebendörfer and Schukajlow (2017) found no differences between students' interest in mathematics at the beginning and the end of the first term at university.

Interest in school and university mathematics

The inconsistent results concerning the relation between interest in mathematics and study success as well as concerning the interest development may be grounded in the instruments, which are used to measure interest in mathematics. Most of the questionnaires use statements such as "I like mathematics" which students are asked to rate. However, it might not be clear which kind of mathematics is meant here. Thus, it has been widely discussed that the nature of mathematics changes significantly during the transition to university. At school, mathematics is often applied to solve real-world problems and schematic calculations play an important role. Furthermore, new concepts are introduced with many examples and aim on a rather intuitive understanding. Mathematics at university is much more abstract and formal: real-world problems and applications do not play an important role in studies with a major in mathematics. Both lectures – real analysis and linear algebra – which mathematics freshmen usually attend during their first year at university, are focused on proofs and formal definitions (Witzke, 2015; Halverscheid & Pustelnik, 2013).

Regarding these changes in the nature of mathematics, some authors have argued that by a distinction between interest in school mathematics and interest in university mathematics, the role of interest in learning processes could be analysed in detail (Liebendörfer & Hochmuth, 2013; Ufer, Rach & Kosiol, 2017). Otherwise, it is uncertain to which kind of mathematics students refer when talking about interest. Ufer

et al. (2017) developed questionnaires that distinguish between students' interest in school mathematics and in university mathematics. The developed scales address students' relationship with mathematics as experienced in the two institutions. Kosiol, Rach, and Ufer (2018) report a positive relation between interest in university mathematics and students' study satisfaction, while interest in school mathematics seems to be negatively associated with study satisfaction. This is in line with theoretical assumptions: If one is interested in university mathematics, it is plausible that one is satisfied with one's studies which mainly contain university mathematics. In their qualitative interview study, Liebendörfer and Hochmuth (2013) report decreasing interest in university mathematics during the first year at university.

RESEARCH QUESTIONS

The focus of our study is the development of students' individual interest facets in mathematics during the transition to university and its relation to study satisfaction. Precisely, we want to answer the following questions:

1) In which way do students' interest facets (at the beginning and in the middle of the first term) and their study satisfaction relate to each other?

Based on recent studies (Koisol et al., 2018) and theoretical assumptions, we expect positive relations between interest in university mathematics and study satisfaction and negative relations between interest in school mathematics and satisfaction (H1).

2) In which way do the specific interest facets develop during the first term at university?

Following Liebendörfer and Hochmuth (2013), we expect a decline of interest in university mathematics (H2). We do not have any special hypothesis concerning the development of interest in school mathematics.

To gain a deeper insight in students' interest development and the effects on study satisfaction, we want to shed light on the following (more explorative) questions:

3) In which way is it possible to identify different profiles of interest development?
4) To what extent do students with different profiles of interest development differ concerning their study satisfaction?

METHODS

We used two questionnaires in the lectures real analysis and linear algebra. Both lectures take place during the first term at university. In order to control for students' prior achievement, we asked students to report their school qualification grade. Besides, students stated their interest in school and in university mathematics within the first questionnaire during the second week of the term (T1). Six weeks later (T2), the second questionnaire captured students' satisfaction with their studies of mathematics and again their interest in school and in university mathematics. Students rated all statements on a five-point likert scale (1=totally disagree, 5=totally agree). All used scales had at least satisfying reliabilities (Table 1).

132 mathematics freshmen (in a Bachelor mathematics program or in a teacher education program) voluntarily filled out the first questionnaire (T1), with 89 students completing the first (T1) and the second questionnaire (T2). A MANOVA revealed that students who participated in both measurement points (T1 & T2) report significant higher interest in university mathematics than those who participated only in T1 (η^2=.08; p<.01). Therefore, the sample of 89 students is probably positive selected.

Variable	Source	No. of items	α(T1)/ α(T2)	Example Item
Interest in School Mathematics	Ufer et al., 2017	5	0.71/ 0.79	In school, mathematics was very important for me.
Interest in University Mathematics	Ufer et al., 2017	5	0.87/ 0.89	The kind of mathematics that is done at university is fun for me.
Study Satisfaction	Schiefele & Jacob-Ebbinghaus, 2006	4	- /0.82	All in all, I'm satisfied with my studies of mathematics.

Table 1: Instruments used in the questionnaires (α = Cronbach's α)

RESULTS

We used hierarchical linear regressions (method inclusion) to assess the relations between students' interest in mathematics and their study satisfaction. Model 1 only consists of students' school qualification grade. Model 2 consists of the interest facets at T1, while the school qualification grade was still controlled. In Model 3, the interest at T2 was added. Since we expected significant correlations between students' interest at both measurement points, we screened for multicollinearity. For all performed regressions, multicollinearity was no problem (VIF<2.85; Tolerance>.35).

Table 2 summarizes the results of the linear regressions. Model 1 shows that students' school qualification grade is no significant predictor of the study satisfaction. The inclusion of the interest facets at T1 (Model 2) increases the explained variance significantly (R^2=.47) and reveals a strong positive relation between students' interest in university mathematics and their study satisfaction (β=.67; p<.001). Model 3 can explain 62% of the variance in the study satisfaction. Confirming H1, students' interest in university mathematics at T2 is still positively related to their study satisfaction (β=.58; p<.001) while interest in school mathematics is a weak negative predictor (β=-.22; p<.05).

Predictor	Model 1	Model 2	Model 3
School Qualification Grade	-.12	-.09	-.06
Interest School T1		.01	.13
Interest University T1		**.67*** **	.20

Interest School T2			**-.22***
Interest University T2			**.58*****
R^2	.14	**.47*****	**.62*****

Table 2: Results (standardized beta-coefficients) of the hierarchical linear regression with depend variable "study satisfaction" (method: inclusion); *p<.05 ***p<.001

In order to analyse the development of interest facets of the whole sample during the first term, we used dependent t-tests. In contrast to H2, neither interest in school mathematics (M(T1)=3.70; M(T2)=3.76; p>.40) nor interest in university mathematics (M(T1)=3.22; M(T2)=3.23; p>.80) differs between T1 and T2.

In the whole sample, the means of the interest facets do not change during the term, although it is possible that some students develop a higher interest whereas other students' interest decreases. To identify different profiles of interest development amongst the students, we conduct a cluster-analysis with interest in school mathematics and in university mathematics at T1 and T2. One outlier was eliminated using the single-linkage method. The Ward-dendrogram indicated that a four cluster-solution is most appropriate to describe the data. This solution was improved using the k-means method. A MANOVA revealed that the four clusters differ significantly concerning their interest in school and university mathematics at both measurement points. The interest development of the clusters is described in the following (see also figure 1):

Cluster 1 (n=24; *highly interested students*): These students start with the highest interest in school mathematics as well as university mathematics and undergo no significant development during the term.

Cluster 2 (n=19; *mainly in school mathematics interested students*): These students already start with a high interest in school mathematics, which increases significantly during the term. In contrast, they show less interest in university mathematics and even lose interest during the term. It's the only cluster with increasing interest in school mathematics and decreasing interest in university mathematics.

Cluster 3 (n=24; *average interested students*): These students start with an average interest in school and in university mathematics. Their interest remains nearly constant during the term.

Cluster 4 (n=19; *low interested students*): These students start with the lowest interest in school and university mathematics. While their interest in school mathematics remains stable, their interest in university mathematics moderately increases.

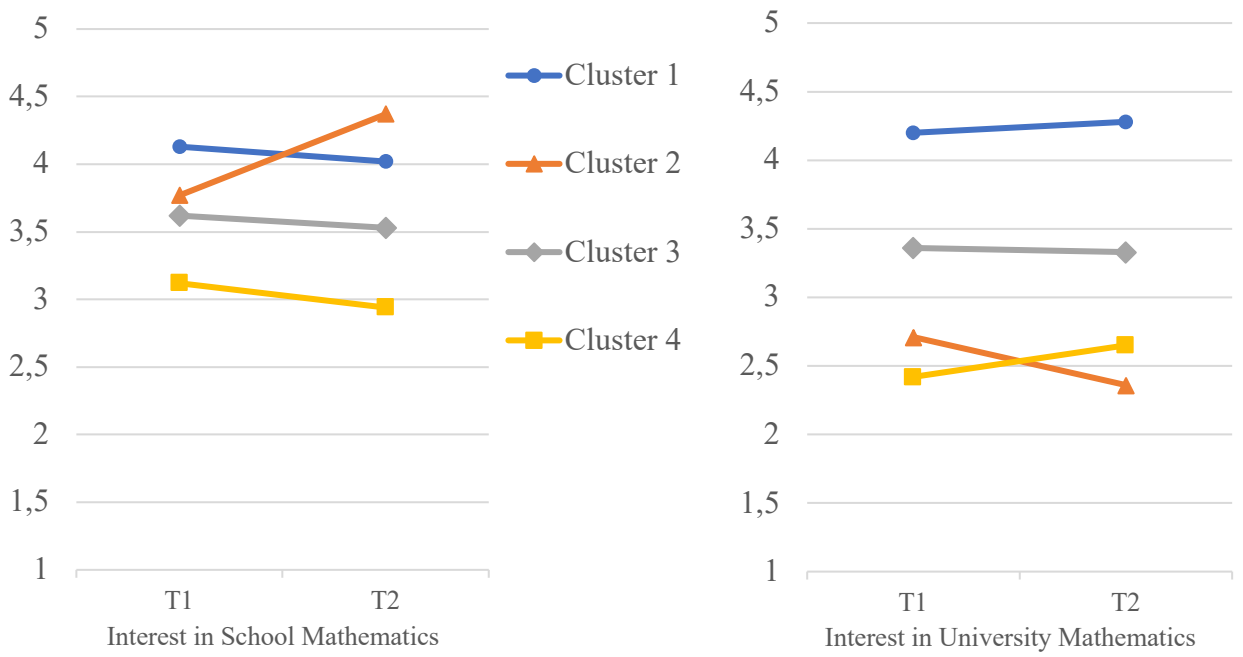

Figure 1: Interest development of the clusters; 1=totally disagree, 5=totally agree

To check whether the clusters differ concerning their study satisfaction and their school qualification grade, a MANOVA was conducted (table 3). The results indicate that the four clusters do not differ concerning their school qualification grade. However, significant differences in the study satisfaction can be observed (η^2=.46; p<.001). The *highly interested students* (Cluster 1) report the highest satisfaction while the *mainly in school mathematics interested students* (Cluster 2) are least satisfied with their studies. The *medium* as well as the *low interested students* (Cluster 3 and 4) report medium satisfaction.

Variable	Cluster 1 (n=24)	Cluster 2 (n=19)	Cluster 3 (n=24)	Cluster 4 (n=19)	η^2
School Qualification Grade	2.20	2.22	2.03	2.22	.02
Study Satisfaction	4.03	2.47	3.32	3.08	**.46*** **

Table 3: Means and results of the MANOVA; School Qualification Grade: 1=very good, 4=sufficient; Study Satisfaction: 1=totally disagree, 5=totally agree; ***p<.001

DISCUSSION AND OUTLOOK

Our results indicate that students' interest in university mathematics (at the beginning and in the middle of the first term) is positive associated with study satisfaction which is in line with recent research (Koisol et al., 2018). However, a negative relation between interest in school mathematics and study satisfaction was only observed at the second measurement point. This result might be explained by taking a closer look on the interest development during the first term.

Even though most studies report decreasing interest during the first term at university (Liebendörfer & Hochmuth, 2013; Sonnert & Sadler, 2015), in our sample the overall

interest in school as well as in university mathematics remained rather stable. This might be due to the fact, that our results rely on self-reports, which can be biased. Furthermore, the questionnaires were filled out during the lectures and students who do not regularly attend the lectures or have dropped out early – and therefore didn't participate in the second questionnaire – have not been part of our analysis.

However, a cluster-analysis enabled a more differential perspective: We identified a group of students with growing interest in school mathematics and decreasing interest in university mathematics. This group *(the mainly in school mathematics interested students)* turned out to be least satisfied with their studies of mathematics although they start with acceptable learning prerequisites. One might assume that these students probably began their studies with unfavourable expectations and get disappointed. In order to support students, it might be helpful to inform them before semester starts about main differences between school and university mathematics. Because low study satisfaction often goes together with study dropout, it is necessary to gain more information about the relationship between students' expectations, interest and study satisfaction. Our ongoing research will now focus on the relations between different facets of interest in mathematics and early dropout from university.

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EXPLORING STUDENTS' REASONING ABOUT FRACTION MAGNITUDE

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Numerous quantitative studies have shown that learners struggle to understand the magnitude of fractions. This is often explained by the natural number bias or the use of gap thinking. However, qualitative evidence that supports the interpretations of this quantitative research is scarce. We have carried out a qualitative study interviewing 52 seventh grade students from different profiles obtained in a previous quantitative study according to the way they reason about fraction and decimal comparison items. Results show a high consistency between students' reasoning and verbalizations given during the interview and their profile, supporting the existence of the hypothesized different ways of thinking about the magnitude of fractions, and indicating that these ways of thinking are somewhat stable over time.

THEORETICAL AND EMPIRICAL BACKGROUND

The importance of understanding rational number is well-documented in the literature during the last decades. It is a step prior to the learning of more advanced calculus and algebra concepts (Kieren, 1993). However, their learning is a complex task for primary and secondary school students (Behr, Lesh, Post, & Silver, 1983): most students show difficulties in understanding different aspects of rational numbers, especially fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Merenluoto & Lehtinen, 2002).

Current research focuses on examining whether these difficulties are due to the interference of natural number knowledge, a phenomenon known as natural number bias (Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). This phenomenon refers to the idea that difficulties with rational numbers may arise from an inappropriate application of natural number properties (Van Dooren et al., 2015), since students develop an erroneous concept of rational numbers as a simple extension of natural numbers (Kieren, 1993).

Research has documented how students from primary and secondary school make systematic errors in determining the fraction magnitude, due to their incorrect assumption that a fraction's numerical value increases when its numerator, denominator, or both increase (Behr et al., 1984; Resnick et al., 1989). Students incorrectly consider that 5/9 is larger than 2/3 because 5 is larger than 2 and 9 is larger than 3. When examining that possible bias, studies use fraction comparison items that are congruent with the natural number knowledge and items that are incongruent with this knowledge. Congruent items are items where the largest fraction has the largest

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numerator and denominator (e.g. 3/5 vs. 8/9), and incongruent items are items where the largest fraction has the smallest numerator and denominator (e.g. 2/3 vs. 4/9). Studies have shown that primary and secondary school students are more accurate and need less time to correctly answer congruent items than incongruent ones (Meert, Grégoire, & Noël, 2010; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). However, there are studies finding the opposite too: higher accuracy in incongruent items than in congruent ones (DeWolf & Vosniadou, 2015; Obersteiner & Alibali, 2018; Rinne, Ye, & Jordan, 2017). This gives rise to hypotheses about the existence of other incorrect ways of thinking, such as gap thinking or reverse bias thinking.

Gap thinking consists of comparing the difference (instead of the ratio) between numerator and denominator in both fractions, considering that a fraction is larger if the difference between the numerator and the denominator is smaller (e.g., 2/3 is larger than 7/9 "because from 2 to 3 there is a gap of one and from 7 to 9 there is a gap of two") (Pearn & Stephens, 2004). These students think additively, considering the difference between numerator and denominator, rather than multiplicatively, considering the ratios (Clarke & Roche, 2009; Moss, 2005). In the reverse bias thinking, students consider that the largest fraction is the fraction with the smallest denominator (DeWolf & Vosniadou, 2015; Gómez & Dartnell, 2018; Rinne et al., 2017), focusing on the belief that the smaller value of the denominator corresponds with the bigger size of the parts (e.g. "2/3 is larger than 3/5 because a whole divided in three pieces is larger than a whole divided in five pieces") (Pearn & Stephens, 2004). Therefore, students who use this reasoning seem to recognize that larger numbers in the denominator can lead to smaller fractions, but they do not fully understand the relationship between the numerator and the denominator (Rinne et al., 2017).

Both kinds of reasoning – gap and reverse bias– could explain previous contradictory results about the effect of congruency (González-Forte, Fernández, & Van Dooren, 2018). However, these types of thinking have been *indirectly* deduced, i.e. derived from quantitative studies using items that are congruent and incongruent for these types of thinking, assuming that when students perform better on the congruent vs. incongruent items, they also reason in that way. This is indirect evidence. In the current paper we aim for a validation by providing more direct qualitative evidence.

To the best of our knowledge, research that supports quantitative results with qualitative data about the above-mentioned incorrect ways of reasoning is scarce. Clarke and Roche (2009) carried out interviews with 6th graders after answering eight fraction comparison items. In this research, incorrect reasoning was identified: "4/5 is larger than 4/7 because 4 is closer to 5 than 4 is to 7" (gap thinking), "5/6 is equal than 7/8 because each has one left" (gap thinking) and "5/8 is larger than 3/7 because both numbers are larger" (natural number bias). Evidence for the use of such reasoning was also obtained in the study of González-Forte et al. (2018) with primary and secondary school students, who had to justify their answers.

Our research and objective

The current research is part of a larger quantitative study with primary and secondary school students. In this study 1262 primary and secondary school students answered a test with 31 fraction and decimal comparison items, specifically designed to detect errors that could be based on natural number bias thinking, gap thinking or reverse bias thinking. A cluster analysis revealed six different profiles: students who answered all items (or almost) correctly (*All Correct*), students who incorrectly solved all the incongruent items, both in fractions and in decimal numbers (*Full Natural Number Bias)*, students who had difficulties in the incongruent fraction items, but correctly solved both congruent and incongruent decimal items (*Fraction Natural Number Bias),* students who had difficulties only in the items in which a reasoning based on gap thinking leads to an incorrect answer (*Gap Thinker)*, students who had difficulties in congruent fraction items and correctly solved the incongruent items, and correctly solved both congruent and incongruent decimal items (*Fraction Reverse Bias)*, and students who solved the items without any clear pattern (*Remainder group)*.

The aim of this study is to validate previous results obtained in the quantitative study, by analysing the consistency between the students' reasoning on the test (profiles identified) and the reasoning and verbalizations given in an interview.

METHOD

Participants

Participants were 52 secondary school students (7th grade) from four Spanish secondary schools who had participated in our previous quantitative study. We interviewed 15 *All Correct* students, 14 *Gap Thinker* students, 15 *NNB* students (from *Full NNB* and *Fraction NNB*) and 8 *Fraction Reverse Bias* students. We chose 7th grade students since it was the grade with sufficient numbers of participants in each profile. Schools belonged to different cities and students were from mixed socio-economic backgrounds.

Instrument and analysis

The interviews were carried out individually and were videotaped. They consisted of two phases. In Phase 1, students had to explain how they had found the largest fraction. Items used in the interview were: 2/3 vs. 7/9; 4/5 vs. 5/8; 4/7 vs. 1/3 and 2/3 vs. 3/7. In Phase 2, students had to read three fictional students' answers given in plasticized cards. Some were based on Natural number bias, others on Reverse bias thinking and still others on Gap thinking (Table 2). Students had to indicate and justify with which of the students' answers they agreed and explain why they disagree with the others.

Item	Answer
1/3 vs. 5/8	Pere (Gap thinking): It's 1/3 because from 1 to 3 there is 2 and from 5 to 8 there is 3, so in 1/3 the difference between the numerator and the denominator is smaller. Marta (Natural number bias): It's 5/8 because 5 is larger than 1 and 8 is larger than 3, so the numbers are larger and 5/8 is larger. Andrés (Reverse bias thinking): It's 1/3 because 3 is smaller than 8, and if the denominator is smaller the fraction is larger.
4/7 vs. 3/4	Maria (Gap thinking): It's 3/4 because from 3 to 4 there is 1 and from 4 to 7 there is 3, so in 3/4 the difference between the numerator and the denominator is smaller. Roberto (Natural number bias): It's 4/7 because 4 is larger than 3 and 7 is larger than 4, so the numbers are larger. Alicia (Reverse bias thinking): It's 3/4 because 4 is smaller than 7, and if the denominator is smaller the fraction is larger.

Table 2: Fictional students' answers used in Phase 2

We analysed students' reasoning used in the two interview phases, to examine the consistency between the reasoning and the profile identified in the quantitative study.

RESULTS

Qualitative analysis supports our quantitative results since the majority of the students interviewed of each profile reasoned in the same way during the interviews, indicating a strong consistency in thinking. Following, we show some examples of students' answers in each profile interviewed.

86.7% of the students of *All Correct* profile remained consistent with their reasoning in the quantitative study. This means that these students used a correct reasoning in Phase 1 and did not choose any of the options offered to them in Phase 2. Tables 3 and 4 exemplarily show the verbalizations given by one of these students and indicate how the students indeed follow a correct type of reasoning.

Item	Answer	Reasoning
2/3 vs. 7/9	7/9	I turned the thirds into ninths. Then I compared 6/9 and 7/9.
4/5 vs. 5/8	4/5	5/8 is closer to a half, and 4/5 is much more than a half, so 4/5 is larger.
4/7 vs. 1/3	4/7	4/7 is more than a half, and 1/3 is less than a half.
2/3 vs. 3/7	2/3	2/3 is more than a half and 3/7 is less than half.

Table 3: Answers of an *All Correct* student in Phase 1

Item	Answer	Reasoning	
1/3 vs. 5/8	None	Andrés	We can use only this strategy when the numerators are the same.
		Pere	He says that from 1 to 3 there are 2... but it should be 3/3 minus 1/3 is 2/3.
		Marta	That both numbers are larger does not mean anything because, for example, 2/3 is larger than 5/8.
4/7 vs. 3/4	None	Maria	This strategy of comparing the difference between the numerator and the denominator is not correct. We have to subtract 4/4 and 3/4 and 7/7 and 3/7, and compare what is left.
		Roberto	It does not matter which number is larger. The denominator should be the same to say this.
		Alicia	We can use this strategy when the numerators are the same. As they are not the same, this rule does not work.

Table 4: Answers of an *All Correct* student in Phase 2

64.3% of the students of the *Gap Thinker* profile remained consistent in their reasoning. These students used a reasoning based on gap thinking in Phase 1 and chose Pere and Maria's answer in Phase 2. Thus they always considered as the largest fraction the fraction with a smaller difference between numerator and denominator. Tables 5 and 6 show the verbalizations given by one of these students and indicate how the students indeed followed a reasoning based on gap thinking.

Item	Answer	Reasoning
4/7 vs. 1/3	1/3	Because in 4/7 you have 3 left over, and in 1/3 you have 2 left over. As in 1/3 I have 2 left over, this is the biggest one.
2/3 vs. 7/9	2/3	In 2/3 I would have one without eating and in 7/9 I would have two without eating.

Table 5: Answers of a *Gap thinker* student in Phase 1

Item	Answer	Reasoning	
1/3 vs. 5/8	Pere	Andrés	This reasoning does not work for anyone.
		Pere	I agree with this. The less you left over, the larger the fraction is.
		Marta	This reasoning is not correct.

Table 6: Answers of a *Gap thinker* student in Phase 2

73.3% of the students of *NNB* profile were consistent with their reasoning. Also in the interview these students used natural number ordering for determining the magnitude of fractions in Phase 1 and chose Marta and Roberto's answer in Phase 2. They always considered as the largest fraction the fraction with a larger numerator and denominator.

Tables 7 and 8 show the verbalizations given by one of these students and indicate how the students indeed followed a reasoning based on natural number bias.

Item	Answer	Reasoning
4/5 vs. 5/8	5/8	I compared the numerators and denominators. In this case 5/8 is larger because the numbers are larger, 5 is larger than 4 and 8 is larger than 5.
2/3 vs. 3/7	3/7	Because 7 is larger than 3 and 3 is larger than 2.

Table 7: Answers of a *NNB* student in Phase 1

Item	Answer	Reasoning	
4/7 vs. 3/4	Roberto	Roberto*	I agree. It's the same. You have to compare the 4 with the 3 and the 7 with the 4. The larger the numerator and the denominator is, the larger the fraction is.

*This student did not say anything about Maria and Alicia's answers.

Table 8: Answers of a *NNB* student in Phase 2

75.0% of the students of the *Fraction Reverse Bias* were consistent with this kind of reasoning in the interview too. These students showed reverse bias thinking in Phase 1 and chose Andrés and Alicia's answers in Phase 2. They always considered as the largest fraction the fraction with the smallest denominator. Tables 9 and 10 show the verbalizations given by one of these students and indicate how the students indeed followed a reasoning based on reverse bias.

Item	Answer	Reasoning
4/7 vs. 1/3	1/3	Firstly, I look at denominators to know the size of the pieces. If the denominator is smaller than the other denominator, although I have fewer pieces, the pieces are larger.
2/3 vs. 7/9	2/3	Because the 3 pieces are larger than the 9 pieces.

Table 9: Answers of a *Fraction Reverse bias* student in Phase 1

Item	Answer	Reasoning	
1/3 vs. 5/8	Andrés	Andrés*	I agree with this, I think it's the best way to reason, because it's how I do it.

*This student did not say anything about Pere and Marta's answers.

Table 10: Answers of a *Fraction Reverse bias* student in Phase 2

Students who were inconsistent answered one of the two phases in the same way as their profile, but in the other phase they changed the way of reasoning or answered both phases of the interview with reasoning different of their profile.

DISCUSSION AND CONCLUSIONS

The high consistency between the students' reasoning and verbalizations given during the interview and their profile as it was determined in the prior quantitative study

clearly indicates the existence of different ways of thinking about the magnitude of fractions. These are somewhat stable over time, indicating that students' errors are not random. Furthermore, in the last part of the interview students were shown various types of incorrect reasoning and even the correct way of reasoning. The participants were largely persistent in the choice for their own way of reasoning.

These qualitative data support the quantitative data by showing firstly, that there are indeed several other wrong ways of thinking next to the natural number bias, such as gap thinking and reverse bias thinking. Secondly, these wrong ways of thinking are persistent over time. Students are even persistent in their way of reasoning when being shown the correct way of reasoning. Finally, the verbalizations students provided confirm the definitions of gap thinking and reverse bias thinking as they were proposed in previous quantitative research. As such, our results can provide an explanation for the apparently contrary results regarding to the congruency effect as obtained in previous studies (DeWolf & Vosniadou, 2015; Obersteiner & Alibali, 2018; Rinne et al., 2017): different students more or less persistently follow different incorrect ways of reasoning.

The existence of other wrong ways of thinking raises questions about when and why these errors appear in students. It is widely assumed that natural number bias is due to students' interference with their knowledge about natural numbers when they are working with rational numbers and thus mainly occurs in the beginning of the learning process. However, gap thinking and reverse bias thinking seem to be learned during the middle school. Future studies could focus on the emergence and development of these kinds of reasoning and their mutual interaction during development.

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HOW ENGINEERS USE INTEGRALS: THE CASES OF MECHANICS OF MATERIALS AND ELECTROMAGNETISM

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It is often said that engineers need to study differential and integral calculus to pursue their training. However, there is not much research examining how this content is used in professional courses, and how much of it is necessary to pass these courses. In this paper, we present our results concerning the use of integrals in two engineering courses: Strength of materials for Civil Engineering and General physics for Electrical Engineering. Using tools from the Anthropological theory of the didactic, our analyses show that conceptual aspects of integrals are used to define notions proper to engineering, and that the technical skills needed to calculate antiderivatives are mostly reduced to simple functions.

INTRODUCTION

Calculus (and, in particular, the use of integrals) plays a key role in solving problems in scientific and technical fields, especially within the various engineering disciplines. Nonetheless, research in engineering and mathematics education has shown that, for many undergraduate students, calculus is one of the most challenging courses of their programme, with very high failure rates. This can often lead students to abandon their career aspirations (Ellis, Kelton, & Rasmussen, 2014; Rooch, Junker, Härterich, & Hackl, 2016). With the considerable increase in studies examining university mathematics education in recent years, new research topics have emerged, such as the consideration of the different contexts and programs in which mathematics is taught at the tertiary level (Nardi, Biza, González-Martín, Gueudet & Winsløw, 2014).

Regarding these different contexts, researchers and educators alike seem to agree that traditional calculus content and teaching methods are not meeting current professional needs and do not allow students to adequately develop mathematical skills useful for the workplace (Sevimli, 2016). Loch and Lamborn (2016, p. 30) stated that "mathematics is often taught in a 'mathematical' way with a focus on mathematical concepts and understanding rather than applications. The applications are covered in later engineering studies." The disconnection between the content of calculus courses and how (and when) it is later used in a professional context has been identified as problematic. For instance, Jones (2015) points out that due to the disconnection felt by students in attempting to apply their mathematical knowledge to science and engineering, mathematics educators and teachers of STEM courses, including calculus, should be aware of and concerned about how the content of integral calculus is used and applied in the professional world.

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Regarding this disconnection, research has highlighted the importance of understanding the needs of other disciplines and their use of mathematical notions (González-Martín et al., 2017). Specifically, more research is needed "studying the use of mathematics (and the professional needs) of several categories of professionals, as well as how mathematics can be taught by targeting these professional needs." (p. 1958). Interested in better understanding engineers' needs regarding mathematics, we launched a research program to investigate how calculus notions are used in engineering courses and to identify possible ruptures between how notions are first introduced and used in calculus courses and how they are subsequently used in professional courses. We note the lack of research on how mathematics notions are used in professional courses.

In this first stage of our research, we are interested in better understanding how single-valued integrals are used in engineering courses, and whether there is a disconnection with the content of calculus courses. To do so, we analysed engineering textbooks, working under the principle that most tertiary instructors use textbooks as an important resource in planning their curriculum (e.g., Mesa & Griffiths, 2012). At previous conferences, we presented our partial results regarding the use of integrals to define bending moments for beams and first moments of an area in a Strength of Materials for Civil Engineering course (González-Martín & Hernandes-Gomes, 2017, 2018). Now having analysed the whole course, in this paper we present a summary of our complete results, as well as results from our ongoing analyses of a General and Experimental Physics course.

THEORETICAL FRAMEWORK

We use tools from the anthropological theory of the didactic (ATD – Chevallard, 1999), which considers human activities to be institutionally situated. A central element in ATD is the notion of praxeology, which is formed by a quadruplet [$T/\tau/\theta/\Theta$] consisting of a type of task T to perform, a technique τ which allows the task to be completed, a discourse (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. Human activity can therefore be described using these elements: the first two elements [T/τ] are the practical block (or know-how), whereas the second two [θ/Θ] form the knowledge block that describes, explains, and justifies what is done. These two blocks are important elements of the ATD model of mathematical activity which can be used to describe mathematical knowledge. Although ATD distinguishes between different types of praxeology, due to space limitations we only present our analyses in terms of tasks.

Knowledge—and praxeologies—may move from the institution where they emerge to other institutions that find them useful (Castela & Romo Vázquez, 2011). Such is the case, for instance, of mathematical notions that are used to solve engineering problems. In this process, one can speak of *transposition* effects on the praxeologies in question (Castela & Romo Vázquez, 2011; Chevallard, 1999). That is, in moving from one institution to another, some (or all) elements of the original praxeology may evolve. Therefore, it is important to analyse the types of tasks and techniques as well as the

discourses and theories employed in both institutions. In this paper, we present our results about the tasks in two engineering courses that call for integrals, and then discuss how their use of integrals relates to uses in calculus courses.

METHODOLOGY

To better understand how calculus concepts are applied in engineering courses, we have had exchanges with two teachers in an engineering programme: one teaches Civil engineering, the other Electrical engineering. Thanks to their input, we selected the courses we have analysed so far. We examined content from two courses: Strength of Materials for Civil Engineering and General and Experimental Physics for Electrical Engineering. Both courses are taught in the third semester of the engineering programme (second year) at the Brazilian university of the first teacher, and both use internationally recognized reference books (Beer, Johnston, DeWolf, & Mazurek, 2012, and Halliday, Resnick, & Walker, 2014, respectively).

Regarding the Strength of Materials course, we have analysed all appearances of the following notions (defined as integrals): first moment of an area (Q), second moment or moment of inertia (I), polar moment of inertia (J), bending moment (M), and centroid (C). It should be noted that the reference book (Beer et al. 2012) has 831 pages and five appendixes. Regarding the General and Experimental Physics book, we started with the content related to electromagnetism (chapters 21 to 24, for a total of 108 pages, from p.609 to p.716), covering all appearances of integrals.

Our methodology was applied in two stages:

- First, we analysed the general structure of the content related to integrals in the first-year calculus courses. The reference book is Stewart (2012). We identified the main tasks concerning integrals proposed to students, the techniques used to solve them and the explanations (technology).
- Second, we started our analyses of both reference books, using electronic versions. For the first book, we identified all appearances of the notions listed above, while for the second, we identified all appearances of integrals in the electromagnetism section, searching by keywords. We also identified all appearances of the symbol " $\int$ ". For all content defined using a single-valued integral, as well as for every instance of the " $\int$ " symbol, we identified the tasks where they are involved, as well as the techniques and explanations present, which allowed us to pinpoint the different praxeologies where integrals are used in both courses.

For instance, Figure 1 shows an example of a task from Beer et al. (2012). This task (from chapter 4) asks students to calculate the value of the bending moment on a bar. The technique uses the notion of centroid, which can be defined using integrals (see González-Martín & Hernandes-Gomes, 2018). In bars with symmetric sections, the centroid is on the axis of symmetry, and its coordinates can be found by using a given formula (which can be deduced by calculating an integral). Note that the formula is given to the students, and there is no need to calculate any integral. Also, students are told that centroids are placed on the axis of symmetry (this can also be deduced using

integrals). Once the value is obtained, it can be replaced in another formula. We see, therefore, that the justifications for using the centroid are based on a professional discourse, and that the content from calculus is not explicitly used.

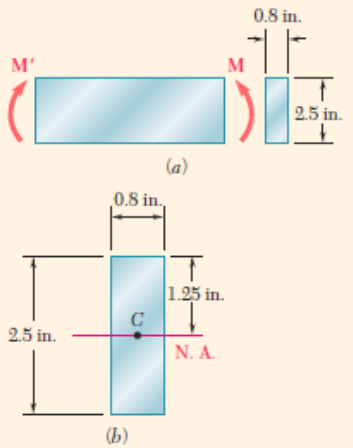

Fig. 4.14 (*a*) Bar of rectangular cross-section in pure bending. (*b*) Centroid and dimensions of cross section.

Concept Application 4.1

A steel bar of 0.8 × 2.5-in. rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar (Fig. 4.14*a*). Determine the value of the bending moment M that causes the bar to yield. Assume $\sigma_Y = 36$ ksi.

Since the neutral axis must pass through the centroid C of the cross section, $c = 1.25$ in. (Fig. 4.14*b*). On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(0.8\text{ in.})(2.5\text{ in.})^3 = 1.042\text{ in}^4$$

Solving Eq. (4.15) for M, and substituting the above data,

$$M = \frac{I}{c}\sigma_m = \frac{1.042\text{ in}^4}{1.25\text{ in.}}(36\text{ ksi})$$

$$M = 30\text{ kip·in.}$$

Figure 1: Task to calculate the bending moment of a bar, using the notion of centroid (Beer et al., 2012, p. 247).

DATA ANALYSIS

Figure 2 summarises our findings of all appearances of the notions *Q, I, J, M*, and *C* in Beer et al. (2012) and Figure 3 summarises our findings regarding chapters 21 to 24 in Halliday et al. (2014). The last column indicates the functions *f*(*x*) students need to integrate, and FT indicates the use of the Fundamental Theorem to calculate a definite integral ($\int_a^b f(x)dx = F(b) - F(a)$, Stewart, 2012, p. 386).

Regarding the book Mechanics of Materials (Beer et al., 2012) used in the Strength of Materials course, our results (Figure 2) show that, with regard to first moments of an area (*Q*), moments of inertia (*I*), polar moments of inertia (*J*), bending moments (*M*), and centroids (*C*), integrals are mostly used in the theoretical sections to introduce and define notions proper to engineering, as well as to deduce some properties. However, students can use given tables and formulae to find values to solve most of the tasks (as in the example shown in Figure 1 above). The actual technique does not rely on using integrals, and it is only in the explanation of the technique (technology) that integrals appear. However, in many cases the explicit justifications rely on a professional discourse which is not (at least for the student) explicitly related to explanations and properties that would appear in a calculus course.

Description of use	Terms used	Chapter – Sections	Rules of Integration Functions to integrate
The term appears in a theoretical explanation. It appears without an expression using the integral sign or the word *integral.*	*Q*	4.4(p.262); 6.1(p.422); 6.1C(p.424, 425); 6.4(p.440); 6.6(p.454, 459-460); 8.1(p.559); 9.5A(p.651); 9.5C(p.654); 9.6B(p.666)	
	I	3.6(p.196); 4.2(p.245); 4.7(p.292); 4.8(p.304-305); 5(p.347); 6.1C(p.424); 8.1(p.559); 9(p.600); 10.1A(p.696)	
	J	3.1C(p.155, 156); 8.2(p.563); A4(p.A11)	
	M	4.1A(p.240); 4.1B(p.242-244); 4.2(p.247); 4.4(p.260, 262, 276); 4.6(p.279); 4.10(p.323); 5(p.347-348); 5.1(p.349); 5.2(p.360); 5.3(p.371, 373); 5.4(p.383, 385, 386); 5.5(p.396); 6.1(p.419); 6.5(441); 6.6(p.454, 459-460); 8.1(p.559); 8.2(p.563); 9(p.600, 602, 603); 9.5C(p.655); 10.1A(p.695); 10.1B(p.698-699); 10.2(p.709-711)	
	C	1.2A(p.8); 4.2(p.245); 4.4(p.259-260, 262); 4.6(p. 273, 276, 279); 6.1C(p.424); 6.4(p.440-441); 6.6(p.454, 459-460); 8.3(p.575); 9.5A(p.651); 9.5C(p.654); A3(p.A10); A4(p.A11)	
The term appears in a theoretical explanation. It appears with an expression using the integral sign or the word *integral*, but no calculation is required.	*Q*	4.2(p.245); 4.6(p.274); 6.1(p.420-421); 6.3(p.437); A1(p.A2, A3); A2(p.A5); A4(p.A11)	
	I	3.6(p.195); 6.1(p.420-421); 11.2A(p.766); A3(p.A8)	
	J	3.1C(p.154); 11.2B(p.767, 768)	
	M	4.1B(p.241); 4.2(p.245); 4.6(p.273, 274); 5.2(p.361-362); 6.1(p.418, 420-421); 6.3(p.437); 9(p.601); 9.1(p.603-604); 9.5A(p.649-650); 11.2A(p.766)	
	C	4.8(p.303); 9.1A(p.607); 9.2(p.611); A1(p.A2, A3); A2(p.A5)	
The term appears in a theoretical explanation. It appears with an expression using the integral sign or the word *integral* and calculation is required.	*I*	3.1C(p.154); 11.5A(p.788-789)	$f(x) = k$; FT
	J	3.2(p.169)	$f(x) = k$; FT
	M	4.6(p.275); 4.10(p.321-322); 5.4(p.384); 9.1B(p.609); 9.3(p.623-624)	$f(x) = x^n$; FT; $f(x) = k$
	C	4.10(p.321-322, 324)	$f(x) = k$; $f(x) = 1/x$; FT
Concept application: It is involved in some calculations, but no calculation of integrals is required.	*Q*	6.3(p.438); 6.6(p.456, 457); 8.3(p.577-578); 9.5B(p.652-653); 9.5C(p.655); 9.6B(p.667); 9.6C(p.669); A1(p.A4); A2(p.A7)	
	I	4.2(p.247); 4.4(p.261, 264); 4.8(p.306); 6.1(p.422)	
	J	3.1C(p.155); 11.5B(p.792)	
	M	4.2(p.247, 248); 4.4(p.261, 264); 4.6(p.278); 4.10(p.325); 5.1(p.350, 351); 5.2(p.362); 5.3(p.372); 5.4(p.388); 6.3(p.438); 6.6(p.456); 9.5B(p.652-653); 9.5C(p.655, 656); 9.6B(p.667); 11.5A(p.789-790)	
	C	4.2(p.247, 248); 6.1(p.422); 6.6(p.458); 9.5B(p.652-653); 9.5C(p.656); A1(p.A4); A2(p.A6, A7); A5(p.A12)	
Concept application: It is involved in some calculations and the calculation of integrals is required.	*I*	A3(p.A9)	$f(x) = x^n$; FT
	J	A3(p.A10)	$f(x) = x^n$; FT
	M	5.4(p.387-388); 5.5(p.397); 9.1A(p.605; 606; 607-608); 9.1B(p.610); 9.2(p.611-612); 9.3(p.624-625); 11.2A(p.767); 11.8(p.806-808, 810, 813-814)	$f(x) = k$; $f(x)= x^n$; FT; Substitution rule for $f(x) = (x - a)^n$
Sample problem: It is involved in some calculations, but no calculation of integrals is required.	*Q*	6.2(p.428-429); 6.5(p.443-444); 6.6(p.461-462); 8.2(p.564-565); 8.3(p.580-581, 582-583)	
	I	4.3(p.250-252); 4.4(p.265-267); 4.6(p.281-282); 5.5(p.400-401); 6.2(p.428-429)	
	J	3.3(p.173)	
	M	4.3(p.250); 4.4(p.265-267); 4.6(p.280, 281-282); 5.1(p.352-354); 5.2(p.363-367); 5.3(p.373-376); 5.5(p.398-401); 6.2(p.429-430); 6.5(p.443-444); 8.2(p.564-565, 566, 568-569); 9.4B(p.639-640); 9.5C(p.657-659); 9.6C(p.670)	
	C	4.3(p.251-252); 4.4(p.265-267); 4.6(p.283); 4.7(p.291, 295); 4.9(p.310); 4.10(p.326-327); 6.2(p.428-429); 6.5(p.443-444); 6.6(p.461-462)	
Sample problem: It is involved in some calculations and the calculation of integrals is required	*M*	5.4(p.389, 390-391); 9.2(p.613, 615, 616-617); 9.3(p.626, 627-628, 629-630); 11.3(p.774-775)	$f(x) = x^n$; $f(x) = \sin(ax)$; $f(x) = \cos(bx)$; Substitution rule for $f(x) = (x - a)^n$; FT

Figure 2: Summary of the use of integrals in Beer et al. (2012)

It is also worth noting that students are asked to recall the calculation of an antiderivative primarily in cases involving constants, a polynomial, a sine or a cosine. The most complex functions to integrate in the analysed parts are a sum of functions of the form $(x - a)^n$, and this can be done either directly, or by substitution. We see that integration techniques for very complex functions (as taught in the calculus course) are not required. Instead the focus is on the meaning of what an integral represents, for the theoretical introduction of notions. However, note that once these notions are introduced, students usually do not need to recall that they are defined as an integral, since most techniques allow for the use of tables and given formulae.

Our results are very similar for the second book (Figure 3), although we note the use of substitution techniques for three more challenging functions: $f(r) = 2r.(z^2 + r^2)^{-3/2}$ (*), $\int \frac{RdR}{\sqrt{z^2+R^2}}$ (**), and $\int \frac{dx}{\sqrt{x^2+d^2}}$ (***, for which the antiderivative is directly given and can also be found in a table at the end of the book). In this textbook, integrals also are used mostly to introduce and define notions proper to electrical engineering, and it is rather the interpretation of an integral that allows for a proper analysis of the phenomena under study. Regarding the tasks, as with the previous book, many techniques call for the use of given properties or tables, which means they can be solved without the students being aware of the use of integrals. In most cases where students need to calculate an integral, immediate integration techniques are sufficient.

Description of use	**Terms used**	**Chapter – Sections**	**Rules of Integration Functions to integrate**
The term appears in a theoretical explanation. It appears with an expression using the integral sign or the word *integral*, but no calculation is required.	Electric Field E	22.4(p.638-639)	
	Electric Flux Φ	23.1(p.659-662)	
	Gauss' Law	23.2(p.662, 664-666)	
	Potential V	24.8(p.706-707)	
The term appears in a theoretical explanation. It appears with an expression using the integral sign or the word *integral* and calculation is required.	Electric Field E	22.4(p.640); 22.5(p.643-644)	$f(x) = k$; FT; Substitution rule (*); $f(x) = x^n$
	Potential Energy U	22.7(p.648-649)	$f(x) = \cos(x)$; FT
	Gauss' Law	23.4(p.671); 23.5(p.673)	$f(x) = k$
	Potential V	24.2(p.690-692); 24.3(p.694-695); 24.5(p.698-700)	$f(x) = k$; FT; $f(x) = x^n$; Improper Integral (a to ∞) of $1/r^2$; Substitution rule (**); Trigonometric Substitution (***)
Sample problem: It is involved in some calculations, but no calculation of integrals is required.	Potential V	24.6(p.702)	
Sample problem: It is involved in some calculations and the calculation of integrals is required.	Electric Field E	22.4(p.641-642)	$f(x) = \cos(x)$; FT
	Electric Flux Φ	23.1(p.662, 663-664)	$f(x) = k$; FT
	Gauss' Law	23.2(p.666-667)	$f(x) = k$
	Potential V	24.2(p.693)	$f(x) = k$; FT

Figure 3: Partial results of the use of integrals in Halliday et al. (2014)

FINAL REMARKS

In this paper, we present the results of our ongoing analyses of the use of integrals in two engineering courses (in the civil and electrical disciplines). We identify three different cases involving tasks where integrals are needed:

Integrals are used in theoretical sections, to define notions proper to engineering or to deduce properties.

Integrals are implicitly used in tasks proper to engineering, although the actual technique does not require students to calculate integrals or recall content used in calculus courses.

The calculation of an integral is required as part of the technique used to solve a task. In this case, most of the functions to integrate require the use of immediate integration techniques.

Our results indicate that, although integrals are used in both courses, students could succeed without using most of the techniques taught in calculus courses. Indeed, in both textbooks, there are few tasks in which the technique calls for an explicit use of integration or theoretical results as taught in calculus courses. It appears that, when integrals move from calculus courses to these two professional courses, the *transposition* effects make a large part of the mathematical discourse disappear, and justifications are mostly based on professional facts. Given these results, it is necessary to question the amount of time spent in calculus courses on integration techniques of very complex functions, as well as their relevance to the training of engineers.

Moreover, our results show that integrals appear mostly in the definition of notions proper to engineering and in deducing properties. This may indicate that more time should be spent in calculus courses developing conceptual knowledge and interpreting phenomena using the notion of integral. While still preliminary, our results seem to call for a rethinking of the content of calculus courses for engineers.

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DESIGNING EFFECTIVE PROFESSIONAL LEARNING PROGRAMS FOR OUT-OF-FIELD MATHEMATICS TEACHERS

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"Out-of-field teaching" refers to the practice of assigning teachers to teach subjects that do not match their training or education. The practice is an international phenomenon that seems particularly prevalent in the teaching of mathematics. The aim of this paper is to analyse the design features of a national professional learning program for out-of-field secondary mathematics teachers. Using a hybrid theoretical framework that conceptualises out-of-field teaching as boundary crossing, our analysis identifies structural and core features of the program that could contribute to teacher learning and identity development.

INTRODUCTION

In many countries, the need to advance education in STEM (science, technology, engineering and mathematics) is recognised as vital for meeting social and economic challenges and developing a scientifically, mathematically, and technologically literate citizenry. In these circumstances, education and teacher quality become political issues linked to a country's economic success, with teacher recruitment, preparation and retention assuming a high priority in national education policy agendas (Cochran-Smith, 2013). However, despite the existence in most countries of requirements or regulations specifying the qualifications needed for teaching specific school subjects, "out-of-field" teaching is an international phenomenon that results in teachers being assigned to teach subjects that do not match their training or education (Ingersoll, 2002). This practice seems particularly prevalent in the teaching of mathematics. Out-of-field mathematics teaching may be considered undesirable because teachers without subject-specific qualifications lack the content knowledge and pedagogical content knowledge needed to develop students' mathematical understanding (Baumert et al., 2010). In addition, out-of-field teachers tend to suffer from lack of confidence, stress and feelings of inadequacy, which hampers development of a professional identity in their out-of-field subject (du Plessis, 2016).

Research is beginning to emerge on the incidence, causes, and consequences of out-of-field teaching. For example, researchers representing the Teaching Across Specialisations (TAS) Collective have developed international comparative studies of out-of-field teaching in Australia, Ireland, Germany, Indonesia, the United Kingdom, and the USA highlighting the complexities involved in understanding this issue (Hobbs & Törner, 2019). In some countries, professional learning programs have been developed to upskill out-of-field teachers of mathematics; yet there has been little

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research to date on the effectiveness of such programs. It seems likely that programs supporting out-of-field teachers to build their professional knowledge and identities will have a different emphasis from professional development for in-field teachers. The aim of this paper is therefore to retrospectively analyse the design features of a national program for out-of-field mathematics teachers in Ireland, the Professional Diploma in Mathematics for Teaching (PDMT), in order to address the following question: What constitutes effective professional learning for out-of-field teachers of mathematics?

BACKGROUND AND CONTEXT

Underperformance of secondary school students in mathematics and the low uptake of Higher Level mathematics in the senior secondary years were a concern in Ireland in the early 2000s (NCCA, 2005), leading to a complete overhaul of the secondary school mathematics curriculum that was initiated in 2008. At around the same time, a national survey of mathematics teachers in Irish secondary schools (Ní Riordáin & Hannigan, 2009) established that 48% of respondents were teaching out-of-field. Causes of such a high incidence of out-of-field teaching are historically and culturally embedded, and arise from factors such as the small size of schools and the autonomy granted to school principals to hire and deploy teachers of any discipline in order to deliver a comprehensive curriculum (junior secondary students typically take up to ten subjects). Nevertheless, the findings of Ní Riordáin and Hannigan's survey were interpreted by education policy makers as a threat to the successful implementation of the new mathematics curriculum. As a result, the Department of Education and Skills (DES) agreed to fund a national program to develop out-of-field teachers' content and pedagogical content knowledge. The winning bid for delivering the PDMT was submitted by the University of Limerick and the National University of Ireland, Galway, leading a national consortium of 13 higher education institutions. The PDMT is a two-year, part-time postgraduate program, with teachers' tuition fees fully funded by the DES. Six cohorts will have undertaken the program between 2012 and 2020, the contract end date, resulting in the upskilling of more than 1000 teachers to meet the mathematics teaching requirements of the Irish Teaching Council.

Evidence emerging from international studies suggests that programs responding to out-of-field teaching vary in their goals and structures while displaying some common design features (Faulkner, Kenny, Campbell, & Crisan, 2019). We consider that the PDMT's unique characteristics make it a useful example for analysis. Unlike in other countries, where multiple providers offer different programs in different regions, often with short-term funding, in Ireland the PDMT represents a large-scale, sustained initiative that delivers a single university-accredited program across the whole country. While the program was designed to meet the pragmatic constraints of the government tender, we are now interested in developing a theoretically informed analysis of its key design features to make a contribution to knowledge about effective professional learning for out-of-field teachers of mathematics.

THEORETICAL FRAMEWORK

Although much is known about the design of teacher professional learning programs, a new framework is needed to capture teachers' learning when they move from a familiar in-field subject to an out-of-field subject where they have limited knowledge and experience of what and how to teach. Hobbs (2013) frames this process as boundary crossing between domains of knowledge and practice. A boundary is viewed as "a sociocultural difference leading to discontinuity in action or interaction" (Akkerman & Bakker, 2011, p. 133), where discontinuities can present opportunities for learning and identity development. Hobbs formulated a Boundary Between Fields (BBF) model to account for factors that influence out-of-field teachers' identity construction and thus counter deficit views of those who teach out-of-field. The model has three groups of factors: context, support mechanisms, and personal resources. Contextual factors include a school's geographical location and state governance structures, practices and policies. Support mechanisms can be provided by a school or system or sought out by teachers. Personal resources that teachers bring to the out-of-field experience include adaptive expertise; knowledge of content, pedagogy, curriculum, and learners; and dispositions such as confidence and commitment.

The interplay between context, support mechanisms, and personal resources influences out-of-field teachers' experiences of boundary crossing, which has implications for designing appropriate professional learning responses that are yet to be fully explored. In this paper we apply the approach proposed by Kenny and Hobbs (2015), who mapped the three sets of BBF factors onto the model for effective professional development created by Garet, Porter, Desimone, Birman, and Yoon (2001) from a survey of more than 1000 US mathematics and science teachers. Garet et al. distinguished between structural and core features of professional development. Structural features included the *form* of activity (e.g., workshops and courses vs activities such as coaching and mentoring embedded in the teacher's regular work day), *duration* of the activity, and the extent of *collective participation* by groups of teachers from the same school, department or year level. Core features were defined as the degree of focus on improving teachers' *content* knowledge, opportunities to engage in *active learning* involving meaningful analysis of teaching, and *coherence* of the professional development activities with broader educational agendas and opportunities for professional communication with colleagues. Garet et al. found that all core features – content focus, active learning, coherence – had a positive influence on teacher knowledge, skills and classroom practice. Amongst the structural features, sustained and intensive professional development was more effective than shorter duration activities. The integrated framework proposed by Kenny and Hobbs links the specific needs and motivations of out-of-field teachers highlighted by the Boundaries Between Fields model to the structural and core features of the Garet et al. model.

ANALYTICAL APPROACH

Table 1 presents a condensed version of the Kenny and Hobbs (2015) framework for designing effective professional learning programs for out-of-field teachers, with illustrative questions that guided our analysis of the key design features of the PDMT.

Structural features	Core features
Form *(Context)*	Content *(Personal resources)*
What organisational resources are available to coordinate and facilitate the program?	What disciplinary and pedagogical content is to be covered?
How will participants be selected?	How can the material be tailored to meet the needs of this teacher group?
Where and how will the program be conducted?	Active learning *(Personal resources)*
To what extent will blended learning approaches be used in delivery?	How can learning activities be designed to enable meaningful engagement in analysis of teaching and learning?
To what extent, and with what consequences, will teachers be released from regular duties?	How can learning activities be linked to teachers' daily work?
Duration *(Context; Support mechanisms)*	How committed are teachers to learning and practising in this discipline?
How many sessions are offered, how often, and with what duration?	Coherence *(Support mechanisms)*
What demands are made on teacher time and workload?	What are the expectations of the school, educational system, and teachers?
Collective participation *(Context; Personal resources)*	What supports are offered by the school in the school or system?
What are the desired characteristics and individual needs of participants?	What are the prospects and expectations for teachers' future career development?
How can participation be facilitated?	

Table 1: Framework for designing effective professional learning programs for out-of-field teachers (adapted from Kenny & Hobbs, 2015)

The underlined terms correspond to the framework of Garet et al. (2001), and the italicised terms come from Hobbs' (2013) Boundaries Between Fields model. As the authors of this paper are respectively the recently appointed and founding Directors of the PDMT, our analysis combines intimate "insider" knowledge of the program's historical development with a more "arm's length" external perspective on its current operations. To address the questions listed in Table 1 we drew on the PDMT tender documentation and course information provided to students, minutes of meetings of the group comprising DES officials and university personnel that monitors the

program, annual student evaluations of the program, and a published case study of the PDMT (Faulkner et al., 2019).

INDICATIVE FINDINGS

Selected design features of the PDMT are identified to highlight relevant structural features and core features, with attention to factors that influence teachers' experience of boundary crossing between fields and development of a professional identity as a teacher of mathematics.

Structural Features

The PDMT is managed and coordinated by EPI*STEM, the National Centre for STEM Education at the University of Limerick. Two full-time staff – the National Program Coordinator and Teaching Coordinator – support a teaching team dispersed across participating institutions. Requirements for the *form* of the program were stipulated by Ireland's Minister for Education and Skills such that graduates would meet the mathematics teacher accreditation requirements of the Irish Teaching Council, which regulates the teaching profession and teacher education programs. PDMT participants are selected according to the following criteria: they must be currently teaching mathematics in a secondary school in Ireland, qualified as a secondary teacher in a discipline other than mathematics, and registered with the Teaching Council as a secondary school teacher.

The *duration* of the PDMT is two years of part-time study with teachers accessing lectures, workshops, and tutorials in the evenings, week-ends, and summer vacation. The program comprises 10 mathematics modules presented in 30-hour blocks over six-week sessions, and 2 pedagogy modules presented via five Saturday workshops and a week-long summer school. Teachers are not released from regular duties in order to undertake the program, which makes significant demands on their time.

Participation of teachers across the country is facilitated by a blended learning format involving face-to-face and online interaction, delivered by a national consortium of higher education institutions. In the current version of the program, lectures are pre-recorded and made available to participants at a designated time for viewing at home. Video lectures include interactive tasks, and they may be accessed online after being posted. Additional supports include lecture notes, face-to-face tutorials and workshops at sites around the country, moderated online discussion forums, individual and group online tutorials, screencasts and applets for specific topics.

Implications of Structural Features for Teachers' Boundary Crossing Experience

Contextual factors such as working in a rural school with limited access to collegial support and professional learning opportunities have been found to exacerbate the challenges of teaching out-of-field (Hobbs, 2013). The form and duration of the PDMT, involving intensive online and face-to-face participation over a sustained period of time, may help to ameliorate these challenges. However, the demanding workload represents an additional hurdle that may adversely affect how teachers

perceive the discontinuity of the boundary crossing experience, especially when they are not given release time from their teaching duties to meet study commitments. While many blended learning tools are offered to support teacher participation and interaction with lecturers, tutors, and program management staff, one could argue that this does not necessarily support collective participation by teachers from the same school or year level who can share information on planning, resources, and students' learning needs. Collective participation of this kind requires support from colleagues and school leaders if out-of-field teachers are to develop new professional identities.

Core Features

The *content* of the PDMT is shaped by the non-negotiable requirements of the DES tender, especially the insistence by the Teaching Council that graduates would have to meet the same or equivalent requirements for mathematics teaching as fully qualified in-field secondary mathematics teachers. Thus the ten mathematics modules – each worth 6 ECTS credits and delivering mainly university level rather than school level content – focus on calculus, algebra (including linear algebra and number theory), probability, statistics, geometry, history of mathematics, problem solving and modelling. Two mathematics pedagogy modules, worth 9 and 6 ECTS credits respectively, are each spread over a full academic year and require attendance at five 3-hour week-end workshops and a week-long summer school. The pedagogy workshops are linked to the corresponding mathematics content module and are designed to develop topic-specific pedagogical content knowledge as well as address contemporary issues in mathematics education such as mathematical modelling, numeracy across the curriculum, and teaching mathematics with digital technologies. One of the mathematics pedagogy modules requires participants to undertake an action research project in their own mathematics classroom, with supervision for the project provided from a national network of experienced teacher-researchers.

When the PDMT was designed it was anticipated that participants would vary in their mathematical backgrounds, with most having taken only Ordinary Level mathematics at senior secondary school and a low to moderate amount of mathematics in their teaching degree. The program is tailored to these varying mathematical needs in a number of ways: for example, through recognition of prior learning via other professional development courses, a suite of online mathematics learning supports, and provision of flexible pathways through the program for those experiencing difficulties.

Active learning is encouraged by the blended learning format (for mathematics content) and the action research projects (for mathematics pedagogy). Otherwise, little systematic attention is given to meaningful analysis of teaching and learning in the participants' own classrooms in the ways suggested by Garet et al. (2001) (e.g., reciprocal peer observation of teaching, discussion and practising of classroom implementation of new teaching approaches, analysis of student work samples).

The *coherence* of the PDMT is strongly enhanced by its alignment with the national mathematics curriculum reform and the Teaching Council's standards for accrediting

teacher education programs and registering teachers. These national standards and frameworks communicate clear expectations to schools and teachers about requirements for curriculum delivery and teacher quality, and completion of the PDMT has enhanced graduates' prospects for gaining employment contracts as fully qualified mathematics teachers. Participants may experience another dimension of coherence identified by Garet et al. (2001), involving opportunities for professional communication with colleagues engaged in similar initiatives, but this is not an explicit goal of the program.

Implications of Core Features for Teachers' Boundary Crossing Experience

The personal resources and support mechanisms available to out-of-field teachers are likely to influence how they experience the program's core features. Teacher knowledge and confidence are major factors impacting on their experience of crossing the boundary from out-of-field to in-field teaching (Hobbs, 2013), and the PDMT is specifically designed to bolster these personal resources. But Kenny and Hobbs (2015) suggest that it is also important for school leaders to provide in-school support and time for teachers to implement new practices in their classrooms in order to help out-of-field teachers identify with their role as teachers of mathematics.

CONCLUSION

This paper presented an initial exploration of principles underpinning effective professional learning programs for out-of-field teachers of mathematics, based on both theoretical ideas and an empirical case. To capture these teachers' distinctive motivations, needs, and experiences we applied a hybrid framework in an attempt to understand what might be needed for boundary crossing between fields to be experienced as an opportunity for learning and identity development rather than an insurmountable challenge. Our analysis focused on a sustained, large-scale program designed to address national educational priorities in a specific context. This analysis suggested that the strengths of the program include its emphasis on mathematics content and how children learn it, the intensive and sustained engagement it fosters among participants, and its alignment with national education goals and standards. However, viewing the program through a boundary crossing lens brought into focus additional requirements that could be worth considering in future developments. These largely involve building professional learning communities involving collective participation and collaboration between teachers in the same schools or districts, learning activities that engage teachers in systematically analysing their own practice, and a focus on school leadership that provides time, mentoring, and other in-school support for teachers who are crossing boundaries between disciplinary fields. It remains to be seen whether such school-based participatory structures can be successfully incorporated into a large-scale national program such as Ireland's PDMT.

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INTERROGATING EQUITY AND PEDAGOGY: ACCESS TO MATHEMATICS IN AN INFORMAL LEARNING SPACE

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Prepare2Nspire is a near-peer mathematics tutoring program that serves marginalized middle and high school students through small learning communities. This informal learning space provides mathematics support that is in contrast to their formal classroom environments. This paper examines the combination of tutoring and mentoring (mentutoring) and the effect on students' participation in mathematics thinking and learning. Findings from this critical ethnography suggest that students increased their participation in mathematics and altered their mathematical identities.

INTRODUCTION

The double doors of the room are propped open, inviting you in. From the outside, you can see tables filled with young people; some with their heads pointed toward the table and others having conversations with people near them.

> As I entered the room, I could hear a constant murmur of conversation. The intonations of the conversations fluctuated between loud bursts and a consistent buzz of voices. The general feeling was one that projected energy into the room. I could see tables filled with students of various ages who appeared to be focused on the papers and books in front of them. I could see notebooks, pencils, textbooks and calculators in various positions on each of the 14 tables for easy access for students to use. I saw students talking and laughing while others had their heads bent over pieces of paper, pencils in hand, with their eyes intent on what they were doing. I could smell the warmth and lingering odor of many bodies placed together in one room (field notes, October 11, 2017).

Then, step in a bit further and you will experience mathematics. The description above is one of the author's introduction to an informal mathematics tutoring program, Prepare2Nspire (P2N). The mission of this afterschool program is to build mathematics confidence, connections and content for urban underrepresented youth through cascading tutoring and mentoring and to develop a Science, Technology, Engineering and Mathematics (STEM) pipeline to higher education.

Prepare2Nspire is situated in a working-class neighbourhood where the majority of the residents identify as African American. This urban neighbourhood is cited as having the highest crime rate in the region and a higher than average unemployment rate (Eligon, 2016). Many of the program participants attend neighbourhood schools that post some of the lowest academic performances in the state (Minnesota Department of Education, 2018). Program participants choose to belong to disrupt deficit narratives

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(Larnell, 2016) and to become stronger mathematics students. Disruption is inherent in the program's model through the use of learning communities.

A P2N learning community typically consists of a university undergraduate, four eleventh-grade and three eighth-grade participants. An intention of the cascading tutoring model is for the undergraduate to mentor and tutor (mentutor) younger participants. Eleventh graders provide the same experience for eighth graders. A typical P2N tutoring session includes focused time for mathematics and a community meal. The meal is a crucial factor in cultivating mentoring relationships amongst the participants as it provides an authentic time and space for conversation in a relaxed atmosphere.

There is fluidity in how the participants in the tutoring program take on roles from mentutor to mentutee all while making sense of using and learning mathematics in an informal space. Taking on each of these roles is empowering. Each participant is expected to engage with their community in authentic ways thus acting as natural mentors for one another. The mathematical implications become obvious while the richer cultural and social implications are important for continuing to work with diverse groups of students in both formal and informal settings.

THEORETICAL FRAMEWORK

To conduct research in a space such as P2N means confronting the existing inequities in mathematics education; especially for students of color. These inequities can be interrogated using theories related to the development of mathematical identity (Bishop, 2011), situated learning theory (Lave & Wenger, 1991) and cultivating positive peer interactions (Walker & McCoy, 1997; Walker 2012).

There is a myth that culture does not matter in mathematics teaching and learning (Nasir, 2013). This is troubling because many states continue to have large academic and opportunity gaps between students of color and white students (Barton & Coley, 2010; Ladson-Billings, 2006). Paying attention to "issues of race and culture in the way we teach mathematics has incredible power to disrupt the troubling opportunity gap," (Nasir, 2013, p. 13). We must acknowledge that such issues exist and address them. P2N inherently addresses issues of mathematics inequity while empowering students of color.

Participation in formal mathematics classes has a profound effect on the development of one's sense of self. The shaping of mathematical identity is a social construction (Boaler, 1997, 2002; Bishop, 2012) that is constantly evolving based upon narrative factors. These factors are often racialized narratives (Larnell, 2016; Nasir, 2013). Formal classroom settings perpetuate such narratives by placing students from underrepresented populations into remedial courses that make it nearly impossible to

advance. Students, in turn, begin to identify as being incapable and unable to perform academically. They are implicitly convinced that they are not capable of doing better. Students internalize these narratives and begin to have low self-worth even if they have the skill set or prior experience to show they can be successful. As Larnell (2016) describes, these "[deficit] identities intersect with already existing stories about other kinds of social identities" (p. 262).

One approach to influencing the formation of mathematical identity is to cultivate positive peer relationships (Tate, 1994; Walker, 2012) through the use of learning communities. Lave and Wenger (1991) suggest using a situated learning model where learning occurs within social relationships. Positive encouragement from peers is an effective and motivating factor for student success (Walker & McCoy, 1997). Therefore, community participation is imperative when cultivating mathematical identity.

Social factors and communication play a role in the shaping of mathematical identity (Bishop, 2012). Upon joining a community, students learn "in the margins." As they become more comfortable and competent they begin to play a more centralized role within their group (Lave & Wenger, 1991). An additional factor in the formation of identity is the feeling of empowerment. Struggling students should be put in empowering roles and learning will increase (Lieberman, 2013).

This research aims to observe how an informal mathematics space, comprised mostly of participants from underrepresented groups, gives participants an opportunity to engage academically and socially in a community centered around mathematics. Through program design, participants are placed into a situation that cultivates positive peer relationships with fluid roles. Our research questions are:

> How does participation in an informal mathematics learning space increase access to mathematics? How does an informal mathematics learning space combine tutoring and mentoring to alter underrepresented students' mathematical identity?

METHODOLOGY

The methodology for this research is critical ethnography. Barton (2001) asserts that

> politicizing ethnography is a defining characteristic of critical ethnography because it is rooted in the belief that exposing, critiquing, and transforming inequalities associated with social structures and labeling devices (i.e., gender, race, and class) are consequential and fundamental dimensions of research and analysis (p. 906).

P2N is an informal social structure that competes with the formal structure of schooling; mathematics classrooms, in particular. Formal mathematics classrooms are often seen as inequitable spaces (Gutstein, 2003; Larnell, 2016). The purpose of this research is to guide thinking toward equity and access for marginalized students.

Moreover, this research demonstrates how students learn to persevere which can alter their mathematical identities in both formal and informal settings.

Communities at P2N meet as table groups named after underrepresented scholars in science or mathematics. (For example, one community is named "Katherine Johnson," an African American mathematician known for her computation work for NASA.) Annually P2N serves approximately 30 eighth-grade participants, 60 eleventh-grade participants, with the help of 13 undergraduate tutors. Notably, most of the undergraduate tutors are STEM majors from underrepresented backgrounds. In order for students to see themselves as people who can excel in science and mathematics, they must see and know previous scholars who have succeeded in this way (Egalite & Kisida, 2018).

Over the course of one academic year, four of thirteen learning communities were observed for this research. Field notes were recorded and then coded for emergent themes. Two interactions highlight how the program supports learning and shows the simultaneity of tutoring and mentoring. Additionally, they demonstrate the complexities of the role of mentutor in the building of relationship, accountability, and mathematical content. This research focuses on findings related to mathematics learning in informal spaces and the relationship between tutoring and mentoring.

FINDINGS

This research is important for several reasons: changes in student confidence, mathematical identity, building of trust, and social and economic implications. For this paper, however, we focus on the benefits of learning communities in informal learning spaces for supporting marginalized students. Noteworthy, both researchers have taught mathematics in formal classroom spaces. As such the discussion will compare and contrast the relationships and behaviors that form mathematical identities.

"Hiding," a formal classroom tactic, is not an option in P2N's learning communities. Participants are held accountable by the members of their community. For example, four African American, eighth-grade students are members of the same community. Two of them, Marcus and Derrick, were sitting across from one another and together they were working on the same math problem. Marcus read aloud, "two x plus y is greater than four x minus three." He grabbed his pencil and ruler and immediately started to write on his paper. Derrick looked at what Marcus was doing and then looked back at his own paper. He gently placed his pencil on the table. Marcus began to demand affirmation that he was solving the question correctly by pushing his page toward me while asking, "Like this? Right? Is this how I do it?" While Marcus was speaking, Derrick quietly sat there seemingly awaiting for the time to pass and for his mentutor, Vanessa, to walk away. Instead, Vanessa encouraged him to get started by asking Derrick some questions about the math problem on his paper.

Vanessa: How do you think we should begin? [Derrick shrugged his shoulders.] What does this say?

Derrick: two x plus y…. [his speaking trailed off]

Vanessa: Do you know what you would do if there were an equal sign there rather than an inequality?

Derrick: I would move the two x.

Vanessa: How about if we go ahead and try that same thing?

Derrick: Okay.

Derrick quietly set to work. He correctly manipulated the inequality to be able to graph (field notes, October 25, 2017).

During this interaction, Marcus exhibited attention seeking behaviors. He was persistent and made sure to use his voice to get attention. He wanted to confirm that what he was doing was correct despite seemingly appearing to already know what he was doing. Derrick, however, is likely accustomed to being overlooked by his teacher in a formal space when other students begin to exhibit such behaviors. Classroom teachers, in contrast, are often presented with 30 students who all require time and attention. In an informal space and within a smaller community the ability to hide is much more difficult. Participants like Derrick are thus asked to confront problems that appear intimidating or that they would normally avoid.

There are several examples of this type of behavior. At P2N, students who struggle are asked to persevere, sometimes gently as shown above and sometimes more directly as evidenced by an undergraduate tutor's interaction with an eighth-grade African American student, Samuel, who is her mentutee. Roxanne, tutor, talks to her community members both as a mentor and as a tutor. She recognizes students who try to avoid their homework and encourages perseverance and hard work.

Roxanne: Excuse me sir, number nine right there [points to Samuel's paper].

Samuel: I need a calculator.

Roxanne: You don't need a calculator. I'm sitting here thinking this boy is smart but you used a calculator. Use your brain… let me see what your brain can do. [Samuel pretends to work, but two minutes later has not done anything.] Are you stalling on purpose?

Samuel: No.

Roxanne: Use this pencil. Now you have no excuse.

As this interaction comes to an end, the meal time begins. Community building is heightened during this time. Sharing a meal is an authentic way to allow for organic conversation and reflection. Participants connect with one another socially and are able to see their mentutors more as mentors and less as tutors.

The connection is evidenced through questions posed by two eleventh-grade female members of Roxanne's community, April and Lola. For example, they ask her "Have you ever cheated on a test? and "Have you ever fought?" April and Lola likely ask these questions because they are related to situations they are facing in their lives. The questions are personal and show how Roxanne can maneuver between mathematical and mentoring discourse. Roxanne answers the questions authentically while remaining a role model. She then turns back to Samuel to check his progress and asks him to complete the problems before he eats dinner. She encourages him through the process, brings him his dinner and then praises his efforts after he is able to correctly finish the problem.

Roxanne: You're not eating until you finish that problem. I'm serious. [Samuel gives her look of surprise. And looks quickly down at his paper as if to begin.]

Roxanne: If you had done the rest of these problems with your brain instead of with your calculator then you would know. [Samuel is quiet. He looks to the ceiling then back at his paper then toward Roxanne repeatedly as he finishes the problem.]

Roxanne: Okay. That's great work. I knew you could do it. [Samuel smiles.] Now, you're going to pick two more of these, no you're going to pick three, and do them without a calculator. [Roxanne and Samuel bargain over which questions he'll do.]

Samuel: I can't do them.

Roxanne: I am here to help you and show you. [Samuel shows hesitation with slow movement and avoiding eye contact with Roxanne.] Pick two.

Eventually Samuel is able to solve two more problems with Roxanne's guidance and encouragement. She holds him to a high standard, shows him that she believes in him and sits with him as he completes his work. Again, this is an example of an interaction that does not or cannot always happen in formal classrooms.

DISCUSSION

In this research report, a cascading, near-peer tutoring program is used to describe various ways in which community and identity can be formed around mathematics. An important component is the role of community for student growth and perseverance. The finding exists due to the profound use of community building and the ability for participants to engage culturally, emotionally and academically within this setting.

There is a need for educators to be more reflexive in the field about critical pedagogies. Critical pedagogy is an education process that integrates issues of self-identity, history, power, and the possibility for collective agency and struggle (Johnson, 1995). Student confidence is strengthened at P2N. We suggest some of the reasons are use of student voice, tutor advocacy, and diversity. It is important to build partnerships between

schools and community programs that are based around academics for students to build positive identities around mathematics.

Many of the students who participate in P2N are those who are not usually successful in formal classrooms. In an informal mathematics space like P2N, participants are empowered through the use of mentoring and tutoring. The strength of these identities and skills will hopefully transfer to their formal classrooms. As one participant describes bringing her ability to slow down and read through problems, acquired at P2N, into her formal mathematics classroom (field notes, December 12, 2017). There are powerful pedagogical suggestions for engaging students who are historically underrepresented. Students should be given many opportunities in both formal and informal learning spaces for peer (and better, near-peer) collaboration and communication. When participants end each tutoring session with "Math is hard...so is life… We accept the challenge!", we believe them and the hope is that they carry the acceptance of that challenge into their formal learning spaces.

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DYNAMIC GEOMETRY CONSTRUCTION: EXPLORATION OF REFLECTIONAL SYMMETRY THROUGH SPATIAL PROGRAMMING IN ELEMENTARY SCHOOL

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The goal of this paper is to explore the dynamic geometry environments (DGE) as a type of spatial computer programming language to learn geometry. Using classroom projects created by upper elementary students in Web Sketchpad (more specifically, closely following the construction process of a Leonardo da Vinci or mirror-writing machine by a team of two boys), I describe and analyse the computational thinking (CT) practices, which prominently came to the fore during the construction process, and show how these practices supported evolution of spatial reasoning and the learning of reflectional symmetry.

With the new era of rapidly growing virtual environments in every area of our daily lives, it is only natural to attempt and harness this unexplored force with seemingly unlimited potential for the goodness of education. At the Math and Coding Symposium in 2015, Richard Noss suggested, "It is impossible to be a citizen of the 21st century and not have some idea of what it means to write a computer program, of what is means to build a mathematical model". Kotsopoulos et al. (2016) underscored the benefits of integrating computational thinking (CT) and mathematics:

> When children write code, they come to (1) understand in a tangible way the abstractions that lie at the heart of mathematics, (2) dynamically model mathematics concepts and relationships, and (3) gain confidence in their own ability and agency as mathematics learners. (p. 1)

COMPUTATIONAL THINKING AND MATHEMATICS

Wing (2006) defined CT as "an approach to solving problems, designing systems, and understanding human behaviour that draws on concepts fundamental to computing" (p. 34). Among the plethora of CT definitions available today (e.g. see Weintrop et al., 2016; CSTA & ISTE, 2011), the frameworks of Hoyles and Noss (2015) and Brennan and Resnick (2012) were found helpful in analysing DGE-based programming. Hoyles and Noss defined CT as entailing abstraction, algorithmic thinking, decomposition, and pattern recognition. Similarly, but with a slightly different focus, Brennan and Resnick defined CT as involving three key dimensions: concepts, practices and perspectives. Within practices, Brennan and Resnick identified four main sets: being incremental and iterative, testing and debugging, reusing and remixing, and abstracting and modularizing. It is specifically the group of practices that is of main interest to me - due to its potential to help explain DGE programming live.

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SPATIAL PROGRAMMING

Jackiw and Finzer (1993) studied the potential of a DGE as a problem-solving domain, one which involved exploration of the process of expressing geometric relationships visually and by demonstration. They developed a notion of 'spatial programming', which they defined as "visual identity between a program and its output" (p. 295). They found that, in DGEs, there was no distinction between the geometric content domain and the spatial programming domain; students using it encountered programming as the central activity. "The distinction between programmer and user disappears; the two coalesce into one – the student" (p. 294). They also argue that, "Constructing a sketch in GSP [*The Geometer's Sketchpad*] is programming, in the straightforward sense of building a functional system which maps input to output. The unconstrained elements of the sketch […] constitute the program's inputs or parameters. The relationships between parts of the sketch […] correspond to a program's production statements. In GSP's case, the semantics of the production language are governed by traditional Euclidean constructions" (p. 295). They further state that a program's structure and its output are isomorphic, and that by manipulating the program's inputs, the student generated further output, meaning that manipulating is performed in the same domain as constructing the initial sketch.

Sinclair and Patterson (2018) argued that *Sketchpad* (Jackiw, 2012) can be an effective geometric computer programming language in the context of complex high school tasks: "In such a 'language', the forms of expression would be less linguistic or numerical and more geometric or spatial", the authors explained. After analysing finished DGE sketches created by high-school students in Belgium, they concluded that many CT practices associated with the use of propositional programming languages were also featured in the more spatial and temporal register of the geometric language of DGEs. While their analysis was focused on already-made sketches, I am interested in observing the construction process live in hope that it will provide additional insight into the phenomenon of using DGEs as CT tools that support learning of geometry. My research questions are: What kind of programming might be involved in an elementary school construction task? How does this programming support learning of geometry?

METHOD

The project described in this paper took place in a Grade 6/7 classroom in the spring in a high-density, affluent neighbourhood elementary school in Canada. Students participating in the project had been exploring geometry with a DGE using iPads for approximately one hour per week since the beginning of the school year. Two researchers led the project. We employed a team-teaching model, with one of us always being engaged with the whole class, or a small group, or individual students. Isometric transformations were selected as the content backdrop for these explorations, as they are part of the provincial curriculum for this grade level (though are usually presented through co-ordinates). From the topic of transformations, a sub-topic of reflection was

chosen as a starting point, with the hope that the concept of reflectional symmetry could offer a low-floor entry into the world of transformations.

The first lesson was introductory and 'unplugged': Students verbally created definitions of transformations based on knowledge from previous years. The second lesson involved an introduction of the concepts of symmetry and reflection, while the third and fourth lessons revolved around students working with reflectional symmetry. Following this, the students were told Leonardo da Vinci's story, shown a picture of the mirror-writing machine that he designed to encode his writings, and then were invited to create such a machine using a Basic Geometric Tools websketch (BGTWSP:http://www.sfu.ca/content/dam/sfu/geometry4yl/sketchpadfiles/BasicGeometryTools/index.html.). The researchers suggested that they start by placing a point (for the 'pen') and a segment (for the 'mirror') as a possible first step. The Continuous symmetry sketch was the 'black box', whose code the students had to uncover.

In this paper, I examine the work of one pair of students, Danny and Dexter, as they created their construction of a mirror-writing machine. They were good friends and, since the beginning of the year, both were very keen on working with Web Sketchpad. The boys worked together for approximately fifteen minutes during each of two sessions that were one week apart. At the end of the first session, they reached a partial solution and, by the end of the second session, they were finally able to carry out a workable procedure to their own and the researchers' satisfaction, creating a machine "that can write stuff backwards".

This teamwork was documented via audio recording and a screen capture tool, and later transcribed and analysed for the presence of prominent computational thinking practices identified by Brennan and Resnick (2012), and Hoyles and Noss (2015). Along with myself (the classroom teacher), the boys worked in a small office adjacent to the main classroom, so as to make clear audio-recording possible. I was mainly observing, but occasionally offered suggestions and scaffolds, especially during the 'being stuck' phases.

MIRROR MACHINES

During the construction process, the team had multiple restarts: the boys tried to start with intersecting two circles, using one circle with a semi-diameter, using two perpendicular lines, or adding triangles, but they had not been yet successful in creating a mirror-writing machine, getting either a translation as a result or no transformation at all. Only twelve restarts later, the boys were able to construct a mirror-writing machine and write the word "Hi" to their sheer delight (see some transitional sketches in Figure 1).

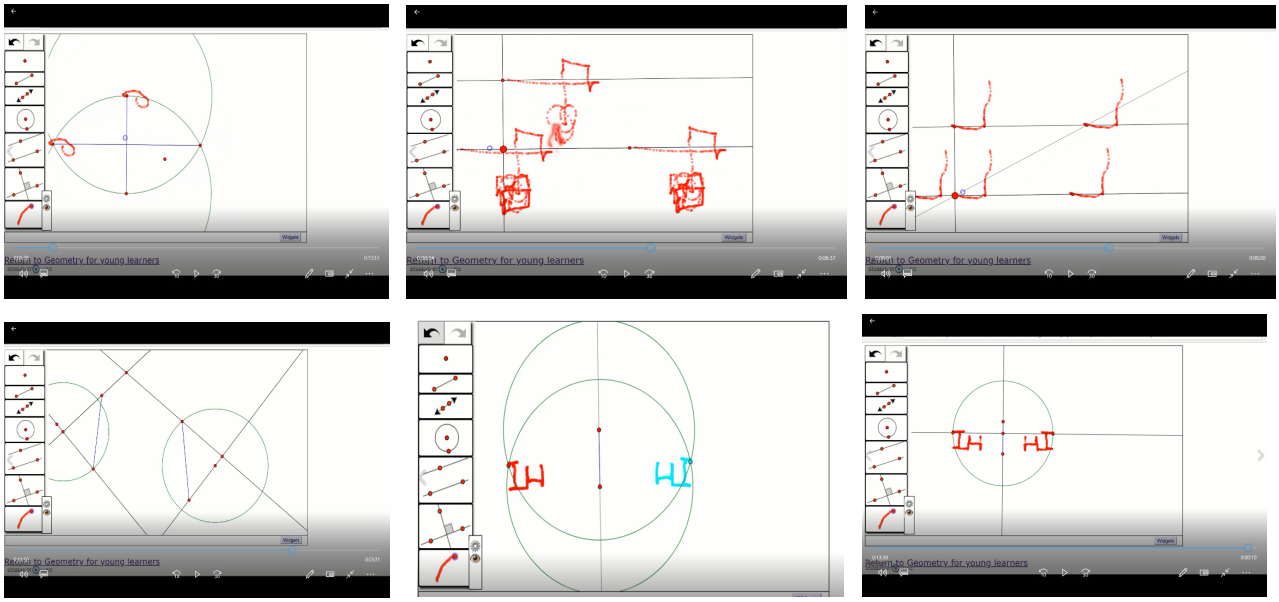

Figure 1: Various sketches

There are only two possible ways to build a mirror-writing machine: using two circles or using a circle and a perpendicular line. Neither of these starters suggested by the researchers appealed to the pair, and they followed their own course. There were two major pivots during the construction process, which seemed to indicate a breakthrough in the boys' spatial understanding of reflectional symmetry, and I will now highlight both of them.

Pivot 1

After the fourth restart, some five minutes into the session, Dexter constructed two parallel lines, each running through the two centres of the circles, one small and one large. He added tracers to the centres of two smaller circles and attempted writing with the right tracer, but no transformation happened. Danny took over and added tracers to the intersections of the larger circles. He dragged the pen around, but his letter L now had the angle of rotation at 60˚. The boys silently watched the action. Then the first disagreement happened:

> Dexter: Try making an L.
>
> Danny: No, it does the same thing.
>
> Dexter: No, look at this one. [Dexter took over and tried to draw an L.] See, this one.
>
> Danny: Yeah, but it's not backwards, it's just a different angle.
>
> Dexter: Oh yeah, but it's close. It's closer.
>
> Danny: No, it's not.

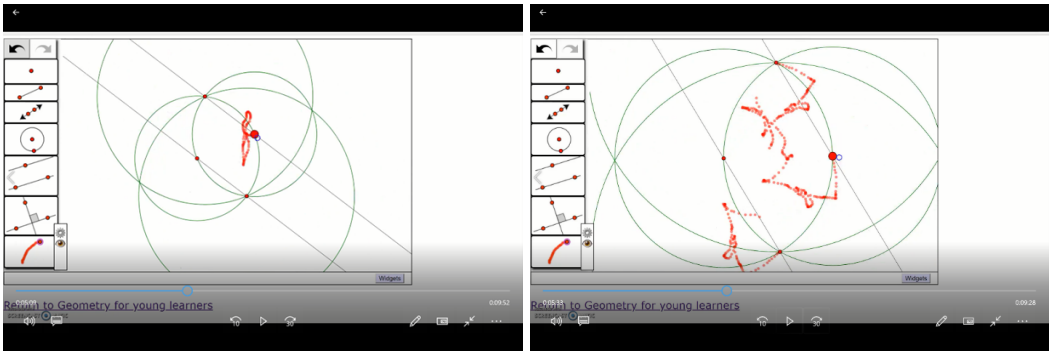

Figure 2: Rotations of the letter ‘L’

The boys could not yet see that if they added another 60-degree rotation increment, their ‘L’ would be a 180˚ rotation, which would make it upside down, and they would have an upside-down, mirror-writing machine (Figure 2). Danny saw the same shape even after a 120˚ rotation, but Dexter thought it was different enough to be considered ‘closer’ to mirror writing. It is not too early to mention that this was the first significant step towards reflectional symmetry: The boys began using their spatial reasoning skills and contemplating what degree of an angle constitutes a reflection.

Pivot 2

After several unsuccessful attempts that began with the segment–point starter, the team abandoned the suggested starter yet again and constructed a circle with a segment for the near-diameter, adding tracers to the end-points of that segment. When the circle dilated, the tracers drew a mirrored line (Figure 3).

> Danny: Yeah, see, if you spread the circle, it goes different ways. That’s it.
>
> Dexter: But then we need another circle.
>
> Danny: Why? It works.

Danny’s emphatic “That’s it” did not convince Dexter, who kept testing the construction. He was able to drag the entire circle without dilating it, so the tracers were translating rather than reflecting; Dexter then shrank the circle and the line was reflected again. He saw the limitations of Danny’s construction, but Danny was so excited seemingly to have found a solution, that he did not notice that the reflection turned back into the translation until later, when he had a chance to watch the video of it. This was the second major turning point, when the boys were seemingly beginning to realise that simply moving the tracers in the opposite direction was not enough and the relational movement of the two circles was needed. They demonstrated further development of their spatial reasoning skills.

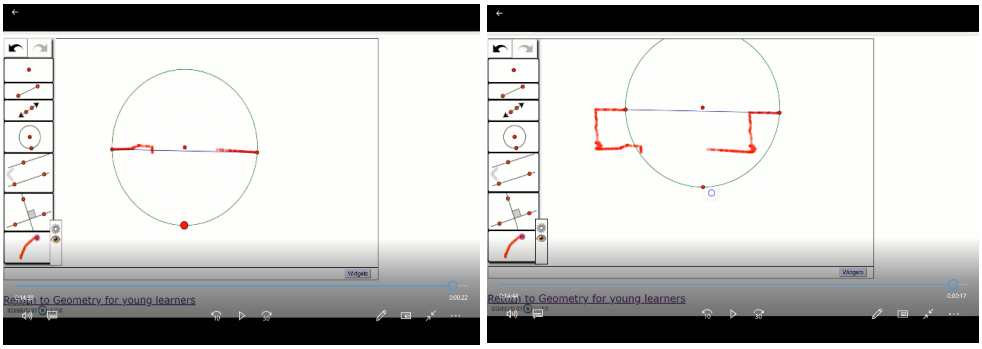

Figure 3: “That’s it!” … or is it?

ANALYSIS AND DISCUSSION

Upcoming analysis of the data described above will demonstrate that CT practices such as testing and debugging, and abstraction were prominent in the construction process.

Testing and debugging

According to Brennan and Resnick (2012), testing and debugging practices emerge when there is a breakdown in code. Most practices they described were developed through trial and error, transferred from other tasks or initiated by knowledgeable others. Sinclair and Patterson (2018) also found that breakdowns in the behaviour of the constructions were the primary source of debugging. Students had to determine why the relationships among objects were compromised under motion: “Such breakdowns can be seen as bugs if we consider them to be wrongly expressed relationships between the different objects in the sketch” (p. 69).

In the case of Danny and Dexter, testing and debugging was the most frequent practice observed. Both boys were relatively new to using GSP, being in their first year of its usage. However, by now, they had become comfortable interlocking two circles – circles intersecting each other’s centres – for various purposes (e.g. to construct perpendicular lines, congruent segments and regular polygons). Interestingly, debugging did not happen within this context. However, using less frequently explored widgets, like adding a perpendicular line or constructing a three-circles design, often resulted in soft constructions with variable angles, causing spatial relationships to be affected in undesirable ways and forcing the boys to debug.

Jackiw and Finzer (1993) commented: “In the programming of a sketch, a bug may be considered as an inaccurately or insufficiently expressed relationship between two or more objects. Bugs occur most frequently with novice GSP students, who readily arrive at a drawing of what they want, only to have the desired relationships between objects disappear when they drag a free node” (p. 303). One of the benefits of constructing in a DGE is access to immediate visual feedback: breakdown in code could be seen immediately as the boys dragged the design which would not behave in the desired way. They would take a couple of seconds to ponder and then start over. Instead of having to fix the alphanumeric code first, they would make adjustments right there in

the sketch. That saved a lot of time and after twelve debugging attempts they eventually reached the correct design.

Abstraction

Hoyles and Noss (2015) defined abstraction as "seeing a problem and solution at different levels of detail". When Dexter and Danny were designing a procedure for the mirror-writing machine, there was abstraction involved, in that they had to see the relationship between the writing pen and the movement and position of the circle as well as how that circle related to the other circle or the line perpendicular to the line of symmetry. When the boys first encountered the finished model of the mirror-writing machine, it looked like two symmetric points of different colours, and there were no geometric objects visible, so it was difficult – if not, at that point, impossible – for them to see the relationship between the objects that the inner mechanism of the machine was comprised of and that the writing the pen was producing. However, a couple of pieces of the puzzle were in place from the very beginning by mere presentation of the finished product of the mirror-writing machine, which did not reveal the mechanisms, but vividly demonstrated what the machine was capable of. This 'black box' experience was very important in helping the boys understand the behaviour they had to model: They already knew that movement had to be symmetric, but it was not at all obvious to them what the relation between the two points was.

This relationship was becoming more palpable, as the boys observed the mirrored pen in action: It either remained static or it would follow its own trajectory, or it would translate or rotate the shape produced by the main pen. At first, they heavily relied on intuition. At one point, for example, Dexter referred to the perpendicular widget as, "maybe we could use this somehow, 'cause it has the right angle tool", not being quite sure why he would want the right angle. Another time, after having written another translation, Dexter remarked, "It doesn't work, because there are no circles moving up and down". Even though the up-and-down movement is not enough to produce a string of letters – the relational dilation of the two circles was required – it was nevertheless a solid step towards being able to abstract. From all this, the boys painstakingly extrapolated the intricate and complex relationships among the objects involved: That mirrored writing meant to have a point on the opposite side, that circles had to be connected but not locked, that the perpendicular line needed to move up and down, or that there had to be a line of symmetry that would anchor the entire machine.

Also, their reasoning was slowly becoming more grounded (e.g., "dot will have to go to the other side, since it will be backwards", or "if you spread the circle, writing goes in different ways" or "the lines would have to be in different ways"). Even though the boys did not rely on the use of proper CT or mathematical terminology, their ability to abstract was evident from their gestures and their on-screen behaviour. This ability did not necessarily transfer into full understanding of the geometric relationships, but the above utterances indicated that many pieces of the puzzle were already in place.

CONCLUSION

In this paper, I focused on the process of construction in a DGE, hoping that it would provide additional insight into the phenomenon of using DGEs as CT tools that support spatial reasoning. I also carried out analysis of the CT live and demonstrated what kind of programming might be involved in an elementary school construction task. The process of debugging was very salient in this example, yielding twelve attempts with various tools and various configurations. The boys were able to test out directly whether the construction worked, and they could get visual feedback of what they constructed. The available tools of the Web sketchpad were providing hints for the boys of the kinds of things they could try, and they were responding to those tools more than to the researchers. In order to crack the 'black box' code and get access to the hidden geometric relations of reflectional symmetry, the boys needed to develop the ability to abstract. This they managed to do through extensive testing and debugging.

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LEARNERS' MATHEMATICAL MINDSETS AND ACHIEVEMENT

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This study seeks to contribute to understandings of high performers' mathematical mindsets. Two high performers, with a growth and a fixed mindset were purposively selected for interviews and their responses analysed using thematic analysis. The findings show some relationship between mathematical mindsets and achievement. A relationship between agency and mathematical mindsets is suggested. Learners' experiences within communities that practice mathematics, and what family members, peers and teachers said about mathematics and about the learners, influenced learner agency.

INTRODUCTION

Beliefs about personal attributes, such as intelligence, define an individual's mindset (Dweck, 2016). Mathematical mindsets are defined as beliefs about mathematical intelligence (Boaler, 2013). Two main groups of learners can be distinguished, those with a growth and those with a fixed mindset (Dweck, 2016). Learners with a growth mindset believe that mathematical intelligence can be developed through effort, and therefore is malleable (Dweck, 2016; Boaler, 2013). Learners with a fixed mindset believe that they were born with a level of mathematical intelligence which cannot be changed by hard work (Dweck, 2016; Boaler, 2013). A growth mindset is characterised by viewing mistakes and persevering on challenging problems as opportunities to learn and develop intelligence (Boaler, 2013; Dweck, 2010b). Avoiding mistakes and challenges are aspects of a fixed mindset (Yeager & Dweck, 2012; Matthews & Folsom, 2008).

Memnun, Hart and Akkaya (2012) define beliefs as personal views constructed by past experience. Given that mathematical mindsets are beliefs, we postulate that negative mathematical experiences may influence the construction of a fixed mindset. Graven (2015) is concerned that many South African learners have negative mathematical experiences contributing to low performance in mathematics. Another concern in South Africa is that more learners are choosing not to do mathematics instead opting for mathematical literacy, which is believed to be less challenging (Spaull, 2013). Matthews and Folsom (2008) argue that having a fixed mindset may be seen by choosing courses that a person believes are less challenging and therefore can do well on.

Some researchers argue that learners who receive growth mindset messages are likely to significantly outperform those who do not (Dweck, 2010a). Appreciating that

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challenges are part of learning and can be overcome with effort i.e. holding a growth mindset, is viewed as important for success in school and in life (Yeager & Dweck, 2012). High performers can also improve their achievement by engaging in tasks that are challenging to them (Dweck, 2010b). There is however some controversy surrounding the relationship between mathematical mindsets and achievement, with a weak relationship found by some researchers (e.g. Sisk, Burgoyne, Sun, Butler & Macnamara, 2018). This paper seeks to address the following questions:

1) Which factors are related to high performers' mathematical mindsets?
2) Are high performers' mathematical mindsets and achievement related to each other?

LITERATURE REVIEW

It is known that some high performers hold a fixed mindset. Dweck (2010a) found that high performers holding a fixed mindset were likely not to perform well in the future while those with a growth mindset were likely to improve. Dweck (2011; 2010b) argues that "even geniuses have to work hard" and be given opportunities to persevere on challenging problems for them to develop their abilities, citing Einstein as an example. Dweck adds that developing the intelligence of high performers to become better than they already are is important. Hwang, Reyes and Eccles (2016) however argue that holding "a fixed mindset does not necessarily predict low performance for relatively high achieving students" (p.16).

Claro and Loeb (2017) compared about 125 000 learners, with the same prior achievement, some with a growth and others a fixed mindset, and found that after one year learners with a growth mindset had mathematics test scores about 0.04 standard deviations higher whereas the test scores of learners with a fixed mindset were about 2 standard deviations lower. Dweck (2007) followed learners who were doing challenging mathematics courses and experiencing the challenge of moving from a lower grade to a higher grade and found that despite having similar skills and knowledge, learners with a growth mindset outperformed those with a fixed mindset. Anderson, Boaler and Dieckmann (2018) found that changing learners' mathematical mindsets influenced increases in achievement of about 0.1 standard deviations.

There is general consensus that teachers have the ability to influence how learners construct their mathematical mindsets. Changing teachers' mathematical mindsets and teaching practices, and encouraging learners to value mistakes and persevere on challenging problems have been shown to influence learners' mathematical mindset shifts from fixed to growth mindsets (Anderson et al., 2018). Yeager and Dweck (2012) argue that what teachers and parents say can influence how learners construct their mindsets. Studies focusing on peers and family members contributing to learners' mathematical mindset construction are however limited.

THEORETICAL FRAMEWORK

The current study is guided by Dweck and Boaler's theories of mindset and mathematical mindset respectively, Wenger's (1998) theory of communities of practice, and a theory of human agency (Mercer, 2011). A sense of agency is defined as understanding the choices that one has (Mercer, 2011). Exercising agency is expressing chosen choices in action, some of which may be imposed (Mercer, 2011). Agency and mathematical mindset are related to each other in that depending on how a learner exercises his/her agency, an existing mindset may be transformed. Mediated by agency, a learner's mathematical mindset may influence achievement, and achievement may influence mindset.

Wenger (1998) argues that social participation shapes our interpretation of the world and our actions. As learners interact with members of communities that practice mathematics they go through experiences that influence their views of mathematics, which may be related to their mathematical mindsets and achievement. A learner's mathematical mindset and the mindsets of teachers, peers and family members may influence each other. The academic performance of peers may influence how a learner constructs his/her mathematical mindset, including whether or not to persevere with mathematics. A learner's performance may influence how others choose to see the learner, in turn influencing how the learner constructs his/her mathematical mindset.

METHODOLOGY

A conveniently located school serving a community with children of low socio-economic status was selected. Fifty seven grade 10 learners volunteered to take part in the study. The learners responded to a mathematical mindset questionnaire and allowed us to access their grade 9 average mathematics test scores, which ranged from 77% to 15%. We purposively selected eight learners with a range of scores and mindsets to respond to an interview on mathematical mindset. In this paper we report on the two highest performers' interview responses, one with a growth and the other a fixed mindset.

Some of the interview questions were adapted from previous studies (e.g. Darragh, 2015). We had asked the same questions when we carried out a pilot study and made some changes e.g. replacing words and phrases that learners seemed not to understand. Learners were requested to be honest as they responded to the questions. Deductive and inductive approaches to thematic analysis were carried out. Themes already classified as relevant in relation to our study topic, theoretical framework and the literature, such as agency, teacher pedagogy, what parents and peers say and perseverance were coded. We also coded for responses that were not apparent in the literature but came up in the data, for example: views of classwork. The categories that will be discussed in this paper are: views of mathematics; what others say; and

achievement. Views of mathematics are opinions about mathematics, some of which are related to experiences within communities that practice mathematics. What others say incorporates what teachers, peers and family members say about mathematics and the learner. Achievement is the learner's academic performance based on test scores, and we also show how the learners view achievement as part of their mindsets.

FINDINGS

The interviews showed that views of mathematics, acquired through interaction with the teacher, peers and family members, and what others say about the learner and mathematics are related to agency, which is related to mathematical mindset. When learners explained how they felt about what others said about them and about mathematics, a relationship between agency and mathematical mindset was suggested. Besides mathematical mindset, getting good grades and a good paying job appeared to be related to achievement. The influence of achievement on mathematical mindset could not be clearly established, probably because the interview questions were not specific enough. Views of classwork and not understanding what mathematics is, appeared to be related to mathematical mindset.

Ben

Most of Ben's (pseudonym) responses suggest aspects of a growth mindset, with a few aspects of a fixed mindset evident. The interview shows that Ben has a largely growth mindset, which is evident through his views of mathematics, what others say and how he views his achievement.

Views of mathematics:

Ben understands that mathematics involves creativity, illustrating an aspect of a growth mindset (Anderson et al., 2018). However, the way in which he is taught mathematics in class influences a belief that mathematics is not useful:

> It's [math is] useful except for doing classwork. Outside you see many things that need mathematical formulas. Math makes me notice new and easy ways to solve matters, most of the time I come up with my own methods.

Ben understands the importance of making mistakes (Boaler, 2013). He believes that memorising rules is not a sign of being good at mathematics, suggesting a belief in depth in learning mathematics (Anderson et al., 2018), and that people are not born good at mathematics (Darragh, 2015), showing additional aspects of a growth mindset. A relationship between experience and Ben's views of mathematics is suggested:

> When you have not done a mistake you will not learn how to do things perfectly. Mistakes help you to see the wrong things that you were doing.
>
> You can memorise the [math] rules but if you can't use them in the correct manner you may not get the correct answer.

> No [people are not born good at math], because I started being great in math in grade 8.

What others say:

What Ben's family says seems related to his sense of agency and some aspects of a fixed mindset. The importance of agency in mathematical mindset construction is shown when Ben describes how he feels about what his family said but still chose to continue with mathematics. Ben's performance seems related to what his family says about him, which is related to his agency and mathematical mindset:

> They [family] don't have enough support. That makes me think that all I'm doing is useless. They are not used to encouraging me, last year they said I must do math lit [mathematical literacy] because math was difficult for me.

When his peers say that mathematics is difficult, Ben chooses not to be influenced by them, illustrating how he exercises his agency. Ben believes that there is a relationship between not understanding what mathematics is, and aspects of a fixed mindset:

> They [peers] say it's [math is] difficult. I see that they don't understand what they are saying and what math is.

Achievement:

When Ben talks about the time when he liked mathematics, a relationship between teacher pedagogy, agency, some aspects of a growth mindset and achievement is suggested. The importance of past experience and depth in learning mathematics is evident:

> I started liking math in grade 8. There was a teacher who showed us small things that we didn't notice in math, I was amazed…it was fun. It [performance] rose.

Ben sometimes reveals aspects of a fixed mindset which seem related to teacher pedagogy. However Ben perseveres in order to get good grades, illustrating a relationship between getting good grades, his agency and perseverance, despite having some aspects of a fixed mindset. A relationship between experience and mathematical mindset is suggested:

> The teacher that was teaching me math was very strict, I disliked the teacher and that made me to dislike the subject…I just wanted to pass so I worked hard and passed.

Cindy

Cindy's (pseudonym) interview responses suggest more aspects of a fixed mindset than a growth mindset. Cindy constantly talks about mathematics as being "difficult" but useful for "making a living", and she feels as if she has no choice but to do mathematics if she wants to be successful.

Views of mathematics:

Cindy does not talk about mathematics having other uses besides for getting a good paying job. Her sense of agency and perseverance appear related to getting a good

paying job despite her showing some aspects of a fixed mindset. A relationship between experience and Cindy's views of mathematics is suggested:

Interviewer: If math was not needed in a job that you wanted would you still do it?

Cindy: Not really, if it's not needed I can do it but I won't focus on it, I will focus on the job that I want to do.

I think that math is a very difficult subject but if you…try by all means to practice you can go somewhere. It [math] is useful because you can make a living out of it, you can't go anywhere without math.

Cindy believes that memorising rules is a sign of being good at mathematics but does not believe that people are born good at mathematics suggesting that she incorporates some aspects of a growth mindset and some of a fixed mindset. She seems to value mistakes and doing challenging problems, suggesting some aspects of a growth mindset (Boaler, 2013):

I can say so [memorising rules is a sign of being good at math], because you need to apply such rules when you want to pass math.

[Making mistakes]…may be because I followed the wrong method or I jumped some steps when doing a sum, it's also all about not focusing.

It does help [doing hard problems] because you can be used to such problems.

What others say:

When Cindy talks about how her peers make her feel, a relationship between what peers say, agency and mathematical mindset is suggested. What the teacher says, and a need to get a good paying job seem to be interrelated with Cindy's sense of agency and how she constructs some aspects of a fixed mindset. Cindy seems to feel as if identifying with mathematics is an imposed choice (Mercer, 2011):

They [peers] say math is difficult and it's just a waste of time. It affects me to listen to them and have that belief that math can really be difficult.

He [teacher] says I need to do my best and try…he assists me. It makes me think I can make a living out of math and if I don't love math I won't go anywhere.

Achievement:

When Cindy talks about the times when she liked and did not like mathematics a relationship between her experiences and mathematical mindset, and that of mindset and achievement are suggested. Cindy seems not to like doing new and challenging problems suggesting an aspect of a fixed mindset, and there seems to be a change in achievement, over time, related to mathematical mindset (Claro & Loeb, 2017):

In grade 9 [she liked math], because we were doing interesting topics. I was really getting high marks.

This year [she did not like math], they are introducing us to many topics that we have not done before and it's really challenging. It's [performance] average, it's not like last year.

DISCUSSION AND CONCLUSION

As learners interact with their teacher, peers and family members while talking about and doing mathematics they construct mathematical mindsets which to some extent appear related to achievement. A learner's sense of agency and how the learner exercises it is related to mathematical mindset and achievement. Experience and what others say about the learner and mathematics, including their mathematical mindsets, are related to how a learner exercises his/her agency. Teacher pedagogy is related to agency which is related to mathematical mindsets, which in turn appear related to achievement.

In this study, both high performers have some aspects of a growth and fixed mindset. Ben has a largely growth mindset while Cindy has a somewhat fixed mindset suggesting that some high performers can construct a fixed mindset. The finding that Cindy has a somewhat fixed mindset and feels that her academic performance is decreasing when she compares her grade 9 and 10 performances, suggests that high performers with a fixed mindset are likely not to perform well in the future (Dweck, 2010a). The time when Cindy liked mathematics her performance was good but when she did not like mathematics her performance was not good, supporting Anderson et al.'s (2018) argument that mathematical mindsets are related to achievement. However when Ben liked or did not like mathematics his performance was still good.

Ben believes that not understanding what mathematics is, is related to aspects of a fixed mindset. Anderson et al. (2018) argue that a strategy for changing learners' mathematical mindsets may be to assist them in redefining mathematics as "a product of student thinking and purposeful struggle" (p.15). Even though Ben believes that mathematics is useful in life, he appears not to believe that the way mathematics is taught in class shows its use in real life, influencing how learners construct their mathematical mindsets.

As suggested in the current study, some learners with a fixed mindset may be high performers likely to avoid new and challenging topics they encounter in higher grades, leading to a decrease in achievement (Claro & Loeb, 2017; Dweck, 2007). Some researchers argue that high performers can also improve their achievement in school and in life through persevering on challenging problems (Yeager & Dweck, 2012). Studies focusing on the mathematical mindsets of high performers, such as the current study, should therefore be considered important (Dweck, 2011).

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STUDENTS' GENDERED EXPERIENCES IN UNDERGRADUATE PROGRAMS IN UNIVERSITY MATHEMATICS DEPARTMENTS

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Examining undergraduate student experiences through a gendered lens provides insight into the gender gap in participation in university-level mathematics. In this paper, we report on findings from a comparative case study involving first- and third-year undergraduate students at two large Australian universities. From individual interviews with 23 undergraduate students, we found that students' experience increasingly gendered interactions as they progress through their tertiary studies. In particular, the women participants recounted times where their gender had affected their treatment in university and described how perceptions of gender stereotypes changed the ways in which they interacted with others in their undergraduate mathematics studies.

INTRODUCTION

In many countries, concerns have been raised regarding the lack of participation of students in STEM (science, technology, engineering, and mathematics) fields at the university level due a shortage of skilled professionals to meet the needs of a world that is "becoming increasingly technological and significantly more mathematical" (Australian Academy of Science [AAS], 2016, p. 37). In Australia, a very small proportion (0.4%) of students enrolling in tertiary education plan to pursue degrees in the mathematical sciences (AAS, 2016). Consequently, several government and scientific organizations have stressed the need for increased participation in the mathematical sciences at the tertiary level (AAS, 2016; Australian Mathematical Sciences Institute [AMSI], 2017). Additionally, women remain a minority of students in university programs in the mathematical sciences, and women's proportion of the enrolments has declined in recent years (AAS, 2016; AMSI, 2017; Johnston, 2015).

Tertiary mathematics education is an expanding field of research, and experts in the field (Coupland, Dunn, Galligan, Oates, & Trenholm, 2016) suggest that research is needed about students' experiences, since much of the existing research has focused on the teaching and learning of specific mathematical topics. In particular, there is a paucity of research conducted in Australia on the lack of participation, particularly by women and gender minorities, in undergraduate mathematics degree programs.

Gender is a social construct, and the ways that students experience mathematics vary due to the gendered constructs placed on mathematical knowledge (i.e., a "hard" discipline associated with men's knowledge and abilities (see Ernest, 1998). Therefore, understanding how students of all genders experience tertiary mathematics is crucial in deconstructing this kind of gendered knowledge. The gendering of mathematics has

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long had impacts on the participation of women in the field (Correll, 2001), and there is evidence that gender bias continues to be a factor that discourages women from staying in the field (Ganley, George, Cimpian, & Makowski, 2018). However, it should be noted that participation alone does not constitute "successful" retention, as women may be at risk of significant anxiety due to internal and external pressures to remain in the field of mathematics (Hall & Suurtamm, 2018).

Here, we report on an ongoing study in which we address issues of gendered student experience in undergraduate mathematics degree programs, via a multimodal methodology, photovoice, combined with individual interviews.

THEORETICAL FRAMEWORK

This study is framed by a feminist and social constructivist epistemological stance (e.g., Butler, 1999; Fosnot, 2005). We view knowledge as a human construction that is gendered and culturally, socially, and historically situated. Furthermore, we view disciplinary knowledge of mathematics, as well as views of mathematics and mathematicians, as socially constructed, gendered, and linked to the specificities of time and place. With regard to the context of the study, we apply this lens to the students' experiences in mathematics degree programs by viewing their learning as "both a process of active individual construction and a process of enculturation into the mathematical practices of the wider society" (Cobb, 1994, p. 13).

In interpreting this study's contributions to the broader understanding of student participation in mathematics, we have adopted the socio-cultural perspective outlined by Piatek-Jimenez (2015), whereby we interpret students' participation (or lack thereof) in mathematics as a choice. Hence, we consider the students in our study to have agency in their educational and vocational journeys and to have made an active choice to study mathematics. Our findings provide insight into how students' university experiences have influenced their decisions to remain in the field and increase understanding of what is happening at the undergraduate level that may be contributing to the attrition of potential mathematics students in the Australian context.

METHODOLOGY

In the following sections, we provide an overview of the study's methodology, namely comparative case study and photovoice. Then, we discuss the data collection, participants, and analysis methods.

Comparative Case Study

As a case study, our research involves "the study of an issue explored through one or more cases within a bounded system (i.e., a setting, a context)" (Creswell, 2007, p. 73). According to Stake's conception (1995, 2005), our research project is an instrumental case study, as we are focusing on a broader issue, of which the case is representative, and a collective case study, as it is an instrumental case study extended to several cases, also known as multiple case study design. The broader issue is the differential experiences and participation by gender and year level in studying university

mathematics, as illustrated by the cases of the first- and third-year students at each institution. To investigate this issue, we are collecting data using a modified version of photovoice methodology (Wang & Burris, 1997).

Photovoice

Photovoice involves participants taking photographs that are relevant to their lives in order to "promote critical dialogue and knowledge about important community issues through large and small group discussion of photographs" (Wang & Burris, 1997, p. 370), with the goal of reaching policymakers. Photovoice is highly accessible to marginalised groups, such as women in mathematics, and promotes agency and action within these marginalised groups (Wang & Burris, 1997). Hence, this method served as a conduit to empower the local communities at the participating institutions to address inequities in their experiences and highlight the strengths of their respective mathematics programs. As a consequence, our findings provide rich insight into the gendered experiences of students in university-level mathematics programs and identify elements of student experience that support or hinder access to mathematics.

We are using a modified version of photovoice, beginning with individual semi-structured interviews about each participant's educational experiences. Then, per the photovoice process (Wang & Burris, 1997), the participant takes photographs to represent the supportive and challenging aspects of the mathematics department. In focus group interviews, participants discuss the photographs. Photographs can focus and encourage discussion, as well as provide a different mode in which participants can express themselves (Whitfield & Meyer, 2005). Supported by the interview facilitator, participants discuss themes that they see across the photographs.

Data Collection and Participants

We have data from three sources – individual interviews, focus group interviews, and photographs – with the latter two intertwined. The individual interviews were audio-recorded, the focus group interviews were video-recorded, and the photographs were provided electronically for further analysis. These multiple data sources allow for triangulation, wherein "various strands of data are braided together to promote a greater understanding of the case" (Baxter & Jack, 2008, p. 554).

Data were collected from two comparable, prestigious Australian universities (herein referred to as University X and University Y), both with large mathematics departments. Data collection will be completed by February of 2019 (i.e., before the new academic year commences). In Table 1, we provide information about the participants to date. These participants have all completed individual interviews and are in the midst of undertaking the photovoice process.

We view gender as a non-binary social construct, so we asked each participant to identify their own gender at the beginning of each individual interview as part of a set of demographic questions. In so doing, our study was reflective of all genders. However, participants only gave responses within binary categories, so we have organised our findings to be split as "women" and "men" to reflect these responses.

	Number of first-year participants			Number of third-year participants		
Institution	Men	Women	Total	Men	Women	Total
University X	3	5	8	2	3	5
University Y	4	2	6	1	3	4
Total	7	7	14	3	6	9

Table 1: Participant information by year level and gender.

Three focus groups have been completed, with one consisting of first-year students and two consisting of third-year students. We divided students into focus groups with three participants each, as this group size provided space for all participants to share their photographs and contribute fully to the discussion. Since all of the individual interviews have been completed, we report on those findings here, and plan to share findings from the other data sources at the conference.

Data Analysis

The multiple data sources and participant groups necessitated a complex and multi-stage approach to data analysis. The individual interviews were analyzed through a process of emergent coding (Bogdan & Biklen, 2007; Creswell, 2014). That is, all of the participants' responses were read several times to get an overall sense of the data. Then, coding categories were created based on the reading of the data and then applied to the responses to each question.

Due to the importance placed on the participants' explanations of the photographs, they are currently being analyzed within the context of the focus group interviews, in which the photographs were discussed. We will further analyze the photographs' content, using content analysis methods (Riffe, Lacy, & Fico, 2014), after the aforementioned initial analysis, to provide additional detail and description that may not be evident in the focus group interview videos.

FINDINGS

The individual interviews provided a wealth of information about the participants, spanning from memories and beliefs to educational experiences. For the purposes of this paper, we will discuss findings from two of the questions that most related to gender and mathematics as they have yielded interesting information about the undergraduate student experience in Australia. Additionally, we will address concerns raised by women participants about their university experiences. When discussing our findings, we will focus on gender differences as well as year-level differences, if the latter is relevant.

Have you ever felt that you were treated differently because of your gender?

There was a division between year levels in the way that participants at both institutions answered this question. Most of the first-year participants (85.7%) articulated that they

did not believe that they had experienced any differential treatment due to their gender while at university. Two of the first-year women participants reported negative interactions with educational staff (7.1%) or members of the public (7.1%), but these occurred outside of university (i.e., during high school or in conversation outside of university). For instance, Participant 8 from University X shared the following example:

X8: In terms of maths, I did have a teacher that said boys are better. Girls should go do English.

Interviewer: Was that at university?

X8: No, that was in high school. But at uni, I haven't encountered anything like that.

In contrast to the first-year students' positive experiences at university with regard to gender-related treatment, 22.2% of the third-year participants (both women) shared instances from their undergraduate studies where they had felt that their gender had impacted their experience, as illustrated by the following quotation:

> I don't know, even that I'm blonde and female, it's like very – last picked sort of scenarios happen quite a bit or just assuming – even if I would go to write on the board and I'd halfway get through something and then someone else would swoop in and be like, "Actually, I know better" but they're writing the same thing as me. (Participant 9, University X)

Additionally, two third-year participants who did not feel like they had been treated differentially described interactions similar to those recounted by Participant 9 from University X, but felt that this kind of behaviour did not relate to gender:

> I don't think I've ever had anyone who's targeted me specifically because I'm female. I've had some people do some crappy things, but I don't think it's because I'm female and there was a perceived gender difference; I think they were just twats. So real analysis, this particular proof I worked quite hard on, I explained it to someone, and then they explained my proof back to me like I hadn't just come up with the entire proof. And they asked me, "Do you understand this?" And I was like, "Yes, I wrote this." But again, I don't think that was a gender thing. (Participant 2, University X)

These interactions, regardless of whether the students interpreted them as gendered or not, highlight a challenge for the women in our study – navigating situations where they feel that their work is disparaged by fellow students.

Have you ever seen any differential treatment by gender?

Extending beyond our participants' personal experiences, we also examined any interactions that they had observed. Across both institutions, none of the first-year participants believed that they had observed differential treatment due to gender. In contrast, five (55.6%; two men and three women) third-year students shared instances where gender had influenced treatment of fellow students. The majority were interactions during class where a man or men would take over a problem that a woman

was working through or where assistance was targeted specifically at women students, as in the following example:

> X2: I do know friends who have experienced this, stories of – two of my friends, Ashley, one girl Ash, one boy Ash, working at the board, and people going up to girl Ash to ask if she needed help when both of them were struggling at the board in the same physical location, like a metre apart from each other.
>
> Interviewer: Who asked if they needed help, was it a student, staff?
>
> X2: It was a tutor, I believe. I think it was a tutor.

While this kind of behaviour may have been an innocent interaction, it is clear that many of our women participants, and a small portion of the men participants, interpreted these behaviours to be targeted due to the gender of the students that they observed.

Additionally, Participant 7 from University X recounted a story that he heard from a student from another university (University Y) expressing transphobic views, refusing to sit in the same row as another student who identified as transgender. While only one participant mentioned hearing about transphobic views, these stories warrant further consideration to better understand the impact of these experiences on women's and gender minorities' desire to persist in the field of mathematics.

Internal pressure for third-year women students

The third-year women participants also described internal challenges that they faced as part of their university experience as women in mathematics. Participant 2 from University X explained:

> In the early years when it's compulsory, there's a pretty good gender balance, and as you go further and further up, there are less girls, I've noticed in the tutes. And failing at a concept often feels like failing as a girl, or as a female in mathematics. I feel very representative of that.

Participant 5 from University Y also reflected on how her fears had impeded her experience and engagement in mathematics:

> I find myself not speaking up in classes because I've also internalised like, "Be quiet." And also I fear that if I ask a question I'm going to look really stupid, which is definitely something that I took from high school. Like, that's something that I haven't shed yet. Looking stupid is something that I struggle with and that's because growing up, everyone was like, "Oh, you're really clever" and so that became a very key tenet of my personality, well not my personality – my identity.

These experiences may reflect a significant challenge common to women progressing through undergraduate mathematics. Contextualised in the historical gendering of mathematics and the impacts of societal beliefs on participation in mathematics, the way that women internalise behaviours and feel pressure to excel may be limiting participation beyond the undergraduate level.

CONCLUDING REMARKS

Analysis of the current dataset of individual interviews with 14 first-year students and nine third-year students has revealed gendered experiences present in the undergraduate mathematics programs at two large Australian universities. While not universal to all participants, many of the stories shared illustrate a need for further investigation into the breadth and depth of the challenges described by women studying undergraduate mathematics. Similar to Hall and Suurtamm's (2018) findings, our findings highlight significant considerations around the experiences of women in mathematics and what is considered "success" in reducing gender inequities in mathematics. While the women in our study had persisted to the undergraduate level, barriers remain in the ways that women are treated that become increasingly frequent as they progress to higher levels of education.

Many of the negative interactions that the women participants described were acts by men in mathematics, purposefully or not, disparaging the work of women who were participating in mathematics. This kind of gender bias is a strong predictor of the gender gap in various college majors (Ganley et al., 2018), and the implied societal values imparted to current and future undergraduate students can have very tangible impacts on the performance of women in mathematics (Correll, 2001). Hence, our study provides a novel addition to the understanding of factors influencing the participation of women in mathematics and a glimpse into the lived experiences of undergraduate students in Australia.

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MAKING AND OBSERVING VISUAL REPRESENTATIONS DURING PROBLEM SOLVING: AN EYE TRACKING STUDY

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This paper presents a case study of following visual attention during collaborative geometry problem solving. We first analyse the emergence and spread of the incorrect idea of curved lines as the optimal shortest solution. Then, we examine the different visual representations made for solution ideas and how these are observed by the collaborating peers.

INTRODUCTION

This paper will present a 20 minute episode of collaborative problem solving in a Finnish grade nine mathematics classroom. The episode is rich in students using different representations for the ideas they are creating, making this an interesting case study on multimodality. We have been using mobile eye-tracking devices, which provides us an insight not only of what representations students are using when they work with ideas on different media, but also to what visual presentations the students are attending to during collaboration.

Eye-tracking is a method to study student cognition as it occurs. Hartmann and Fischer (2016) compare it to mind-reading: the target of a fixation (maintaining of the visual gaze on a single location) tells what we think about, and the fixation duration corresponds to processing time. Glöckner and Herbold (2011) summarize research evidence to suggest that gazes related to more automatic processes would have shorter fixations (below 250 ms) and more elaborate information processing generally requires long fixations of more than 500 milliseconds. However, there is also much evidence to the effect that sometimes the gaze-mind connection breaks and the fixation and thoughts are not aligned (e.g. Schindler & Lilienthal, 2017).

Our research will focus on examining student eye movements in the context of collaborative problem solving in geometry. This study will address three under-examined research areas. First, while eye-tracking research is strong in the context of language processing (Rayner, 1998), the method has been far less used in the area of mathematics (Hartmann & Fischer, 2016). Second, eye-tracking research has mainly been conducted in laboratory situations and studies conducted in real classrooms have only recently started to appear (e.g. McIntyre, Mainhard, & Klassen, 2017). And third, research on multiple persons interacting (e.g. Rogers, Speelman, Guidetti, & Longmuir, 2018) is so far extremely limited. Although methodologically challenging, moving into these poorly explored areas is important for mathematics education.

Information in a learning context is often visual. Gestures and diagrams are part of multimodal communication (Radford, 2008; Arzarello, Paola, Robutti & Sabena 2009).

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The student cannot look at all possible sources of visual information at the same time, but must choose what to attend to. We need to study visual attention in the context of mathematical behaviour in ecologically valid ways. In this paper we will present results from the MathTrack project that uses mobile gaze-trackers to study students' collaborative problem solving.

METHODS

Participants

The data was collected during a grade ninth mathematics lessons in Finnish lower secondary school. The teacher and a collaborative group of four target students volunteered for wearing gaze-tracking glasses throughout the lesson.

Apparatus

We recorded the actions and conversations of the problem-solving session using audio recording and three stationary video cameras in the classroom. The gaze-tracking device consisted of two eye cameras, a scene camera, and simple electronics attached to 3D-printed frames (see, Toivanen, Lukander, & Puolamäki, 2017). The accuracy of the device was approximately 1.5 degrees of visual angle. We used software that computed the gaze target coordinates on the scene video. The camera frame rate depends on lightning conditions, and maximum rate in optimal conditions is 30 frames/second. Data was recorded on laptop computers that were carried in backpacks allowing subjects to move freely in the classroom.

Procedure for data collection and analysis

The research group visited two mathematics lessons. The purpose of the first lessons was to help students to get used to the equipment and to test the equipment in the data collection environment. The students solved a non-routine mathematics problem collaboratively in groups of four students. The actual data collection and the collaborative geometry problem solving took place during the second lesson. The goal of the problem task was to find the optimal way to connect four imaginary cities located at the vertices of a square with cable.

The problem solving session lasted almost an hour. The students worked both with paper and pencil as well as with GeoGebra software. In addition, three video cameras recorded the teacher and target students' activity and we collected students' written work. We also recorded screen capture videos form student computers and conducted post lesson interviews with the teacher and focus students, but these have not been used in the current analysis.

The students were seated in pairs sitting opposite to each other. The students in the same pair could easily see each other's computer screens while seated, but not the screen of the other pair. However, all students could see at least two other students' notebooks.

We had scripted most of the problem solving session for the teacher. During the lesson, the teacher first gave general instructions for the lesson and students got their GeoGebra software started (3 min 45 s), then posed the problem on the board and students worked on the task individually (4 min 33 s). Next, students worked with a partner (4 min 10 s), and then the four students worked together (26 min 42 s; fast solvers were provided an extension task). Finally, the solutions were collected on the board and discussed (4 min 54 s). Furthermore, the students downloaded a GeoGebra app to further examine the correct shape (5 min 57 s). In this article, we focus on the phase during which the four students were working together.

When students were working individually, in pairs, or in groups of four, the teacher's activity consisted of roaming in the classroom and stopping to scaffold one group at a time. The teacher was instructed to provide encouragement and to ask questions that required students to explicate their thinking but to not provide hints on how to solve the problem.

The episode was transcribed using eye-tracking video to provide rich information about visual attention to gestures and images. Where necessary and possible, multiple videos were examined to better discern details.

RESULTS AND DISCUSSION

During individual and pair work, the students as a group had produced all the common easy solutions (Figure 1) and each student had found the solution "X", which is best of these solutions.

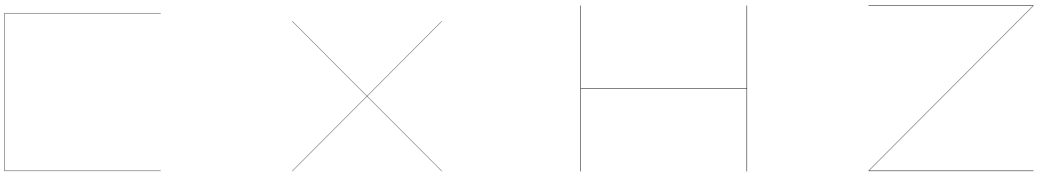

Figure 1. The common easy solutions, which we will call "*C*", "*X*", "*H*", and "*Z*".

I begin the description where the students are challenged to move beyond these basic solutions. The teacher comes to check how the students are doing, and while she discusses with students, they start interacting as a group. In what follows, T is for teacher, S1 to S4 the four students. S1 and S3 are seated as a pair, and S2 and S4 as another pair.

Teacher	How are you doing here?
S3	We thought that this [pointing at *X* on her screen]. We were thinking, if one can get through here [tracing the diagonal from one vertex to the centre and then along the other diagonal to the other vertex].
Teacher	Yes, it's possible.
S3	Then this is [our favourite]
S4	I feel there is something, just "Ha!" [Expressing surprise, smiling, gestures a rapid short hand movement.]

	[Teacher and S2 smile.]
Teacher	You can still continue thinking. Some good elements from different solutions. Brainstorm still a little further. [Moves away.]
S3	[To S4.] So that means that there is still some better solution.
S4	So there is.
S1	Circle.
S2	Some triangle.

For this first segment, I want to point out four things. First, the eye-tracking data allowed us to see what the student was actually pointing at and what her question was about. The same level of detail will be informative in some of the events that will come later. Second, we see that the students know that it is common that a problem has an unexpected solution and they joke about it. As the teacher encourages them to explore further, they know that there must be another, better solution to be found. S1 suggests a circle. This idea catches fire in the group and very much influences the process that follows. I will next report how this idea develops and after that I will discuss later why this might have been such a tempting idea.

Teacher	[Comes to S2 and S4.] What do you have here? … Which, in your mind, would be the most convincing?
S4	That. [Points an *X* on her screen.] Would it be circle? [Makes a small unclear gesture in the air.]
Teacher	You can test it. You can get the length of the circle, when you take the circle tool. [Points at the tool on GeoGebra menu bar.]
S3	But how…?
S1	I said circle before (----). [Pointing at S4, smiling.]
Teacher	[Smiles to S1.] Yes. You try it, too.
S1	I don't want to (---).
S2	It does not have to be a whole circle, it can be only. [Pointing at *X* on own screen, tracing an incomplete circle to connect all four vertices.]
S1	[Begins to draw.]
S3	How can the circle even (--) [Draws a circle in her notebook.] …
S2	(To S3, smiling.) Do you know the thing, circle and inside it a square, and then circle and square and circle and square [gesturing with hands, making first a square with both hands, and then 'collapsing', them into a smaller circle and repeating this a few times.]
S3	Yes, but. Aha! [Surprised.] But, but, it becomes longer, still. Or does it?

The teacher makes the decision not to call the circle either a good or a poor suggestion. Instead, she invites S4 to test it. As S1 claims ownership for the idea, she suggests him

to test it, too. S2 realizes immediately, that the connection between two vertices could be left out. Only S3 opposes the idea, but also joins drawing it.

The teacher then announces to the class that it is time to move on to work in groups of four. She soon returns to challenge this group to reconsider the circle as a solution. Meanwhile S4 used GeoGebra to compute the length of the circle as the solution connecting the four vertices. She got a solution that it would be shorter than *X*, possibly due to not having the figures in the same scale. However, the solution is not questioned, even though S1 is puzzled by the apparent fact that the solution he suggested is so good.

S4	No, it IS the fastest! [Cheering.] S1, you are the best! [Gestures V for victory with both hands.] Eight point four. Okay, it is the fastest. S1 you are a god.
S1	Weird. [Shakes his head a little.]
S2	And, if you remove that part, because that is not needed. [Pointing at S4 screen, circle between two vertices.] …
Teacher	I would ask about this circular solution, what are the benefits in this in comparison to, for example this, there they are straight. [Points at the solution *C* on S2 screen.]

I would suggest that this phase of events is influenced by reading incorrectly the social cues given by the teacher. For S3, the social pressure might be at play. However, S2 points out to an image in her mind, probably something like Figure 2.

A circle has beauty and power in mathematics. Although the discussion ends with a conclusion that the straight lines are shorter, the students present some quite interesting arguments.

S3	Don't have to stop, just goes on wheeeee. [Rolls head in small circles.]
S4	[The verbal explanation was incomprehensible and could not be translated. However, her gestures focus on turning around the vertex.]
S3	…If you change the lines of the square into a circle, the circle will be bigger. [S4 agrees.]

Here, we see two kinds of arguments. The first kind refers to physical movement, and there, the curved path feels better. The other argument refers to the circle being the optimal solution for the ratio between area and perimeter.

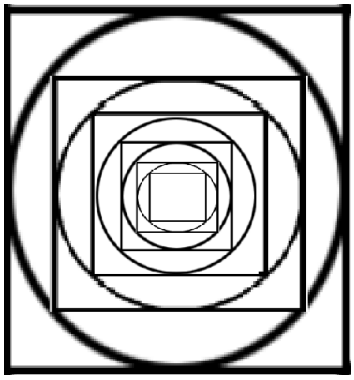

Figure 2. Squares inscribed in circles inscribed in squares.

The students finally agreed that the curved lines are just "extra length" and continued generating new ideas. I will present those that remained in the discussion and developed further. However, we see the curved lines to follow in their ideation, see *X'* and *H'* in Figure 3.

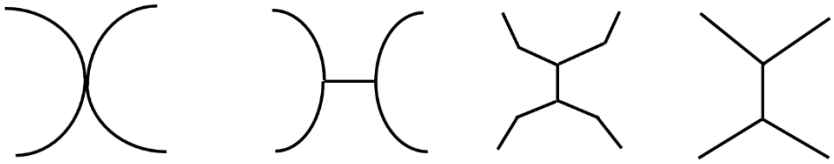

Figure 3. The evolving solutions, which we will call *X', H', Y'*, and *Y*.

The following stage in student progress is best described as a diagram (Figure 4). The students generated several ideas that were visually presented either as gestures, or images, either on paper or on computer. These ideas were effectively communicated among the students in the group and the arrows present the flow of information from one student to another, when a student was looking at another student's gestures or drawings.

In this diagram, you can see, for example, how *H'* was drawn by S4, and then observed by S2 and S3, who both produced their own versions of the solution somewhat later However, the students struggled to produce the solution with GeoGebra. In fact, this led to S2 and S4 making little progress, as they struggled with the tools to produce a solution of the right shape. On the other hand, S3 decided to approximate *H'* using lines, leading first, accidentally, to small *Y*, which she deleted, and then to the solution *Y'*, which was developed further into the optimal solution *Y*.

	Student 1			S3			S4			S2		
Time	HG	PP	GG	HG	PP	GG	HG	PP	GG	HG	PP	GG
22								*H'*				
23											*H'*	
24					*H'*						*X'*	
25				*X'*								

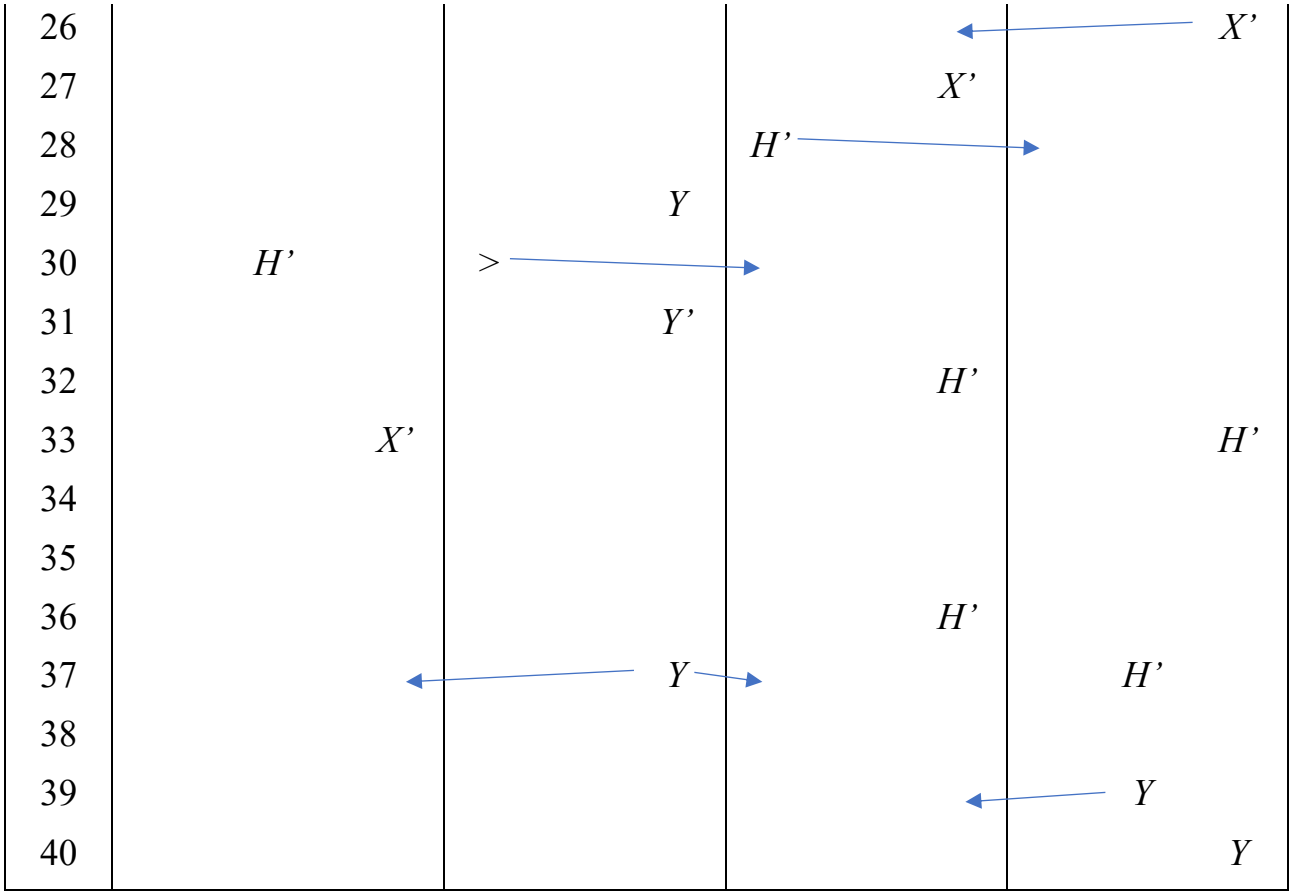

Figure 4. Diagram of visual representations and their observations. The arrow presents the direction of information flowing from visual representation to observer. HG = Hand Gesture, PP = Pen on Paper, GG = GeoGebra. The symbol ">" in the graph refers to a student gesturing a shape like that.

CONCLUSIONS

This case study illustrates how an erroneous idea may emerge and survive in collaborative problem solving. The original the idea of circle as the solution was not grounded on anything more than a false intuition. However, it got rooted in the group's discourse, probably because students were reading the social cues falsely. Although the students discussed and logically refuted the idea, new solution attempts with curved lines appeared and spread in the group.

However, the erroneous solutions (*X'* and *H'*) both have some resemblance to the optimal shortest path (*Y*), providing opportunities to find the optimal solution. Moreover, the use of technology both assisted and hindered the finding of optimal solution. Student S2 and S4 had their focus on how to draw curved lines, which effectively prevented them from generating new ideas. The student S3, on the other hand, used line segments to make and approximation of the curved line, which opened her the opportunity to find the optimal solution.

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STUDENTS' USES OF ONLINE PERSONAL ELABORATED FEEDBACK

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Technological developments made it possible to collect, analyze, and assess automatically students' mathematics work, offering important tools for the assessment of students while they are working in rich digital environments. Automated online formative assessment has the potential to promote reasoning processes by improving the learners' mathematics thinking or behavior (Shute, 2008). A commonly used type of online formative assessment is elaborated feedback. We explore whether students use elaborated feedback in the course of their reasoning process, and if so, how. We focused on the process of conjecturing, which is part of students' work on an online example eliciting task (EET) using the Seeing the Entire Picture (STEP) platform. We present two case studies of the students' uses of the elaborated feedback.

INTRODUCTION

Online Mathematics Formative Assessment

We refer to formative assessment activities as "all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (Black and Wiliam, 1998, pp. 7-8). Researchers have differentiated between two main types of online formative assessment: verification feedback and elaborated feedback. Verification feedback gives simple information about the correctness of the students' answer. Elaborated feedback provides an explanation of why a response is correct or incorrect. Shute (2008) identified six different types of online elaborated feedback. For the current research, we used: "attribute isolation elaborated feedback" (AIEF) which consists of observations on the requirements of the task and on its mathematical characteristics, including the nature of mathematical objects and actions involved, and the mathematical reasoning processes entailed.

Shute and Rahimi (2017) argued that complex feedback might not be useful to learners. The context in which the feedback is given, however, has great influence on the students' learning process. Therefore, a relevant question is whether AIEF given to students in a context of mathematical reasoning has the potential to develop students' mathematical reasoning.

Conjectures and example eliciting tasks

We focused on conjectures, which are a main component of mathematical reasoning and learning. When conjecturing, students reconstruct their knowledge and enhance

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their ability to prove (Boero, Garuti, and Lemut, 2007). For studying the potential of elaborated feedback to support reasoning, we needed an environment that supports mathematical inquiry and reasoning as well as automatic online elaborated feedback. To this end, we used the Seeing the Entire Picture (STEP) platform. STEP is an interactive diagram in a multiple linked representation (MLR) environment that supports example eliciting tasks. The STEP platform also provides various formats of personal feedbacks (Yerushalmy et al., 2017).

In example eliciting tasks students are asked to construct examples in an MLR environment to support their answer. Example eliciting is a vital element in the reasoning processes and in conjecture posing. Example eliciting may also be indicative of the students' mathematical reasoning (Zaslavsky and Zodik, 2014). Yerushalmy, Nagari-Haddif, and Olsher (2017) stressed the importance of example eliciting as an e-task design principle. According to Yerushalmy et al., asking students to submit as different as possible examples encourages them to develop a rich and varied example space. Example spaces are collections of learner-generated examples that fulfill a specific function. The example space can be automatically analyzed to provide feedback to students. The STEP platform, therefore, constitutes a novel pedagogical tool that supports reasoning processes as well as rich feedback. Yerushalmy et al. showed how design principles can affect the personal example space. In the present study, we assumed that giving elaborated feedback to the students, which reflects their reasoning, can affect their conjecturing process. We examined whether and how students use the AIEF while working in rich digital environments that support example eliciting tasks and promote reasoning processes.

METHODOLOGICAL CONSIDERATIONS

The current research is part of a wider study in which we explore how students use the personal AIEF. Here we present a case study of two pairs of students, chosen from the advanced mathematics stream from high schools in Israel. Each pair carried out a different activity consisting of a task-based interview.

First, the students performed an online example eliciting task (EET). The students were asked to submit three representative examples, as different as possible, for their conjectures. To be able to examine whether the students used the feedback to produce new conjectures or to expand their personal example space, the students had the option to go back to the task and to resubmit new examples.

Asking the students to create three examples as different as possible supported the construction of a personal example space that could be assessed and analyzed with STEP, to produce an elaborate feedback for each example submitted.

Each AIEF consists of a list of mathematical characteristics of the task (e.g., the figure is rectangle, the figure has equal sides). The list of characteristics was prepared in advance, as part of the task design. STEP can analyze the submitted work, and mark the identified characteristic of the submitted example in blue and the non-identified ones in gray. By that STEP automatically produced the AIEF (See Figures 2 and 3 for students' submissions and AIEF). We hypothesize that giving the students elaborated feedback that consist of three lists of characteristics (one list for each submission) that differ by the indication whether the characteristics of each example were or were not identified, enables students to analyze the differences and similarities between the submitted examples and thereby support the conjecturing process.

The data were collected through the STEP platform, and the students' work was also recorded using the Camtasia screen recorder and video editing software, to triangulate with elements collected automatically. We analyzed the following aspects of the students' use of the feedback: raising conjectures, changes in the characterization of the personal example space, using a variety of characteristics from the feedback, and usage of language.

FINDINGS

Pair 1: Ella and Anna

Ella and Anna were students in the 9th grade. The task they were given to elicit examples was formulated as follows: "A, B, C, and D form the quadrilateral ABCD. They are all dynamic and can be dragged. If possible, create 3 examples that are as different as possible from each other, in which the perpendicular bisectors to the sides of ABCD meet in a single point." Figure 1 shows a screenshot of the task applet.

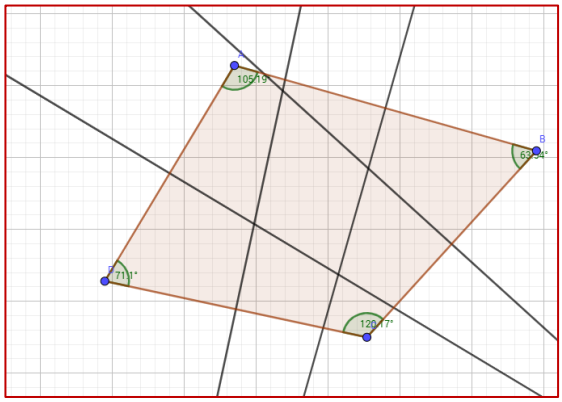

Figure 1: Task applet

The task has been found effective for raising conjectures (Olsher, forthcoming). We used the GeoGebra software to design the task applet. To enable students to focus on building quadrilaterals, the various features of GeoGebra were not available to them. To support eliciting examples of various types of quadrilaterals, the applet provided measurements of squares and angles. As explained in the methodology section, the three examples were analyzed by STEP and produced AIEF. Figure 2 shows a screenshot of the student submissions, and the online AIEF they received:

Figure 2: Submissions of Pair 1 and feedback to students

The students' first submission indicated a narrow example space consisting of alterations of a single shape: a rectangle. Below are some of the students' reactions to the feedback:

Anna: All the feedbacks are the same.

Ella: Yes. When you submit a rectangle.

Anna: No. But it doesn't have to be like that. It can also be a parallelogram [pointing at the attribute: "parallelogram" written in the feedbacks].

Ella: So it happens every time that there is a pair of equal sides.

Anna: No. Whenever there is a pair of equal and congruent sides in the quadrilateral.

STEP appears to have considered the exact same five characteristics for each example: "All angles are equal", "Rectangle", "Parallelogram", "Sum of adjacent angles is 180º", and "sum of opposite angles is 180º". At this point, the students could use the characteristic: "rectangle" to verify the conjecture they formulated (rectangles are shapes in which the perpendicular bisectors to the sides meet in a single point). Instead, they used the elaborate feedback as a list from which to choose another conjecture.

To formulate a new conjecture, the students used the following characteristics: equal sides, and congruent sides. Therefore, in addition to using the elaborated feedback as a list from which to choose conjectures, the students also used the feedback to formulate a new conjecture. Note that the students mentioned gray (shape-not-identified characteristic) as well as blue (shape-identified characteristic) attributes from the feedback.

The students had the option to go back to the task and resubmit new examples, and they returned to the applet. First, they refuted the conjecture that parallelogram bisectors to the sides of ABCD meet in a single point. Next, they found several examples in which the perpendicular bisectors to the sides of ABCD meet in a single point. Confused by the variety, they decided to submit three examples and to get ideas from the feedback. In other words, they deliberately asked for the feedback help in formulating conjectures. Figure 3 shows a screenshot of the students' submissions and the elaborated feedback they received.

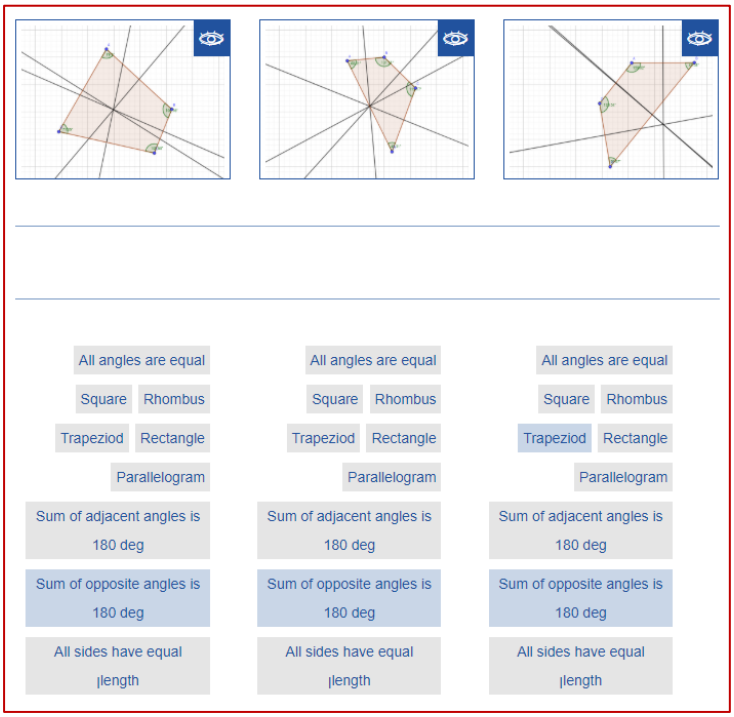

Figure 3: Second submission of Pair 1 and feedbacks

Unlike the students' first submission, the second submission indicated a rich example space that contains a variety of shapes. The feedback on the first submission encouraged students to go back to the applet and continue the inquiry. We assume that the first round (Figure 2) drove the students to expand their example space. The students' reaction to the feedback was as follows:

Ella: Well, that explains it.

Anna: Opposite angles are supplementary.

Ella: This is our assumption. Opposite angles are supplementary.

As we can see, the students used the feedback to formulate a new conjecture.

Pair 2: Guy and Shai

Guy and Shai were students in the 12th grade. The example eliciting task they received was formulated as follows:

"f(x) (blue) and g(x) (green) are linear functions. h(x) is the product function of the two linear functions. Drag the blue points to create new functions. What kind of product functions appears when multiplying two linear functions? Formulate your answer in the dialogue box, and create three examples that are as different as possible from each other." Figure 4 shows a screen of the task applet:

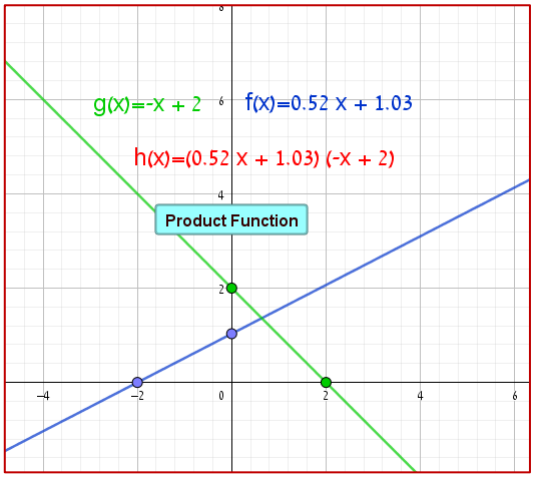

Figure 4: Task applet

The task was chosen because the product function of two linear functions has a potential of a varied and rich examples space that may cause students to inquire and conjecture. Similar to the previous task, the GeoGebra features were limited to dragging the two linear functions and to the "Product Function" button that supported the immediate construction of the product function. And similar to the previous task, the students were asked to submit three examples as different as possible, and received the AIEF that contained information about the attributes of the submitted functions.

After exploring with the applet, Guy and Shai formulated the following conjecture and submitted three examples: "When two linear functions have the same slope tendency, the quadratic function is positive. Otherwise, the quadratic function is negative."

Similar to Pair 1, the submission of Pair 2 indicates a narrow example space that contained variations of the same quadratic function. Knowing that the blue characteristics represent mathematically identified characteristics of the submissions, Guy and Shai focused on the blue characteristics only. They found that the identified characteristics of the submissions corresponded to the conjecture they had formulated.

We analyzed the following aspects of the students' use of feedback:

Raising conjectures: Pair 1 used the AIEF to formulate new conjecture; Pair 2 used it to verify their conjecture.

Example space: the example spaces of both pairs of students were rather narrow, but Pair 1 used the AIEF to expand their example space.

Language usage: Pair 1 was helped by the feedback to formulate a new conjecture; the conjectures of Pair 2 resembled the formulation of the AIEF.

Using a variety of characteristics from the feedback: Pair 1 mentioned a variety of characteristics of their feedback that marked in blue and in gray, whereas Pair 2 mentioned only characteristics of their feedback that were marked in blue.

The differences between the two pairs are reflected in each aspect of the students' use of feedback. Pair 1 used the elaborated feedback to formulate a new conjecture; Pair 2 used the feedback to verify their conjecture. The main similarity between the two pairs was that they both used the elaborated feedback as a supporting tool in the conjecturing process.

DISCUSSION

The aim of the research was to examine whether and how students use the AIEF when working in a rich digital environment that supports example-eliciting tasks. We focused on four aspects of the students' use of feedback, and presented evidence of the uses of AIEF for each aspect. Students used the language of the elaborated feedback to formulate conjectures, and Pair 1 even used the elaborated feedback to formulate a new conjecture. Both pairs used several characteristics from the feedback throughout the conjecturing process. Pair 1 used the feedback to formulate a new conjecture. Pair 2 used the feedback to verify their conjectures and the examples. Pair 1 used the feedback to expand their example space.

There may be several explanations for the differences between the two pairs: different subject meters, different classroom norms, etc. An additional explanation may be the fact that Pair 1 was surprised by the feedback they received. The students did not consider parallelograms as an option. The feedback may have motivated them to resume the inquiry. Pair 2 was not surprised by the feedback at all. Perhaps because of that, the students had no reason to continue with the conjecturing process. Both pairs used the elaborated feedback to support their reasoning process.

Previous studies have raised the question whether elaborated feedback is useful (Shute and Rahimi, 2017). We presented here two case studies in which the AIEF produced by automatically analyzing students' examples provides information about mathematical attributes of the students' examples and reasoning. It was found to be useful when working in a rich digital environment conjecturing about example eliciting tasks. The students' use of the AIEF exposed an important relationship between

example eliciting tasks and elaborated feedback, marking another step in exploration of personal elaborated feedback.

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THE CASE FOR SELF-BASED METHODOLOGY IN MATHEMATICS TEACHER EDUCATION

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How might it be possible to research the growth of a mathematics teacher educator in a way that is meaningful to self and others? In this theoretical/methodological report, I explore issues pertinent to research where self is both the observer and the observed. By starting from a research problem, this paper presents the case for self-based methodologies in mathematics teacher education through an examination of self-observation from research in the domain of psychology and existing self-based methodologies in teacher education literature in pursuit of a legitimate approach to enquiry. This paper is a positive move towards one such methodology.

A (NON-)IDEAL-TYPICAL RESEARCH REPORT

In his opening plenary lecture at PME-42, Mogens Niss characterised the 'ideal-typical' mathematics education journal paper and, in contrast to this, appealed to the mathematics education community about the importance of publishing "non-empirical papers that focus on an issue…on analysing, comparing or linking theoretical frameworks, or on presenting and analysing methods" (Niss, 2018, p. 47)

In the same plenary lecture, Niss reminded us of Arcavi's (2000, p. 145) distinction between "Problem-driven research" and "Theory-driven research" in mathematics education. In the examples of research that Arcavi proposed in his paper, he placed himself closer to a problem-driven orientation and emphasised how:

> (a) a broad theoretical predilection underlies all what we do (but does not blind us) and (b) theory (or, in some cases, theories) is to help us find ways and insights to conduct the research. (Arcavi, 2000, p. 163)

This paper is one such (non-)ideal-typical research report that was partly motivated by a comment made to me during my own PME-42 presentation (Helliwell, 2018) which was roughly "the problem is you are researching yourself, you need to research somebody else". This problem-driven research report presents a response to that comment and my search for a self-based methodology (SBM) that is underpinned by the theoretical foundations of self-observation. My intention in this paper is to examine some of the issues that such SBMs present (such as those I can only assume my audience member was talking about) by exploring existing self-observational approaches and in doing so move towards the formulation of a methodology that enables me to make sense of my own lived experiences as a mathematics teacher educator (MTE). As Arcavi proposed as a legitimate approach to research, I begin with the problem from which point I then "pursue it" and "shop around for frames which may help me to make sense of what I find." (Arcavi, 2000, p. 145).

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BEGINNING WITH THE PROBLEM(S)

My research problem

Behind every piece of research lies at least one human being. Sometimes there is a personal motivation for research activity, other times it is a professional one. For the last three years I have been collecting data in different forms and from different settings. Before this time, I was a secondary school teacher of mathematics for 13 years, a role in which I had developed expertise. In my move to a new role, as a university MTE, working primarily with prospective teachers of mathematics but also with groups of more experienced teachers of mathematics with a common interest (e.g., developing the use of mathematical reasoning in their classrooms), I found that the expertise I had developed as a teacher, was necessary but not sufficient for me working with these teachers. I needed to develop a new expertise and became interested in understanding both what this expertise looked like and how the development might happen. I have a desire to better understand myself in my role as an MTE so that I can develop my practice and hopefully the practice of others with similar interests. The problem I have set myself is to make sense of *what* and *how* I am learning as an MTE by placing myself as the subject of research. By researching my own growth and understanding more about the process of becoming an MTE, I hope to contribute to the growing area of research and knowledge on MTE learning. One such contribution is the development of an SBM, which this paper is a positive move towards.

A problem for the mathematics education community?

At a symposium last December that was held to mark the retirement of a dear friend and colleague, Laurinda Brown, Olive Chapman gave a seminar on SBMs in MTE learning. She talked of the recent meeting of the North-American chapter of the IGPME group (PME-NA 2018) where she had been involved in a working group whose interest was in developing the use of studies within mathematics education that privilege the self. Both in the working group's paper (Suazo-Flores, Kastberg, Ward, Cox, & Chapman, 2018) and during the symposium, a tension was expressed between the importance of developing research methodologies that aim to understand and improve the practice of MTEs and the pressure as research academics of publishing research in established and 'prestigious' *mathematics* education journals where articles documenting SBM are the exception (e.g., Hjalmarson, 2017).

In contrast to this shortage of papers within mathematics education journals, self-based study is a well-established genre of research within the teacher education community more broadly. Much activity originated from a group of teacher educators that, in 1994, began a special interest group (SIG) of AERA (American Educational Research Association) known as the 'Self-study of teacher education practices' (S-STEP) SIG. The S-STEP community initiated an international biennial conference (the Castle Conference) that began in 1996 and from this point onwards, there has been an ever growing collection of books published, including in 2004, the *International handbook of self-study of teaching and teacher education practices* (Loughran, Hamilton,

LaBoskey, & Russell, 2004) and (to date) the 19-volume S-STEP book series, volume 1 and 2 of which were published in 2005. Around the same time, the journal *Studying teacher education: A journal of self-study of teacher education practices* was launched and since then more than 35 issues have been published which include a modest number of articles with a mathematics education focus. Is it time for the mathematics education community to more fully embrace this new genre of research?

The remainder of this paper is one attempt at bringing to light this possibility through firstly exploring the notion of self-observation from research within the field of psychology to examine existing SBMs from (mathematics) teacher education literature towards the articulation of an SBM for my own research, as an MTE.

SELF-OBSERVATION

The observation of 'objects' (including other human beings) is viewed as a fundamentally different activity (both ontologically and epistemically) to the observation of the 'self'. One familiar distinction between 'objective' knowledge and self-observation is that the former is based on "public, verifiable, external, and transparent entities" while the latter concerns only "private, idiosyncratic, and internal experience" (Clegg, 2013, p. 5). With these characteristics in mind comes the necessary consequence of any enquiry based on self-observation as epistemological solipsism where the firm assumption is that an observer can only ever know their *own* mind. From this position, it becomes meaningless to suggest that self-observation as a method has anything to contribute to the production of general knowledge. At the level of experience, however, the distinction between external/internal objects of perception can of course be abandoned, since, as Clegg puts it, there is "no meaningful way to conceptualise an "external" perception" (p. 6), such terms are in fact contradictory. In the words of Maturana (1987, p. 65) "everything is said by an observer", that is, no knowing is independent of the observer, no experience is independent of the experiencer, there is no true subjective/objective divide:

> Rather than pretending to create objective observer-independent knowledge or retreating into an inner subjectivity, we can use critical methods together with inner subjectivity to bring about a maximum of intersubjectivity, that is, understanding the Self to understand the Other (Roth, 2005, p. 15)

In a self-based enquiry, the self both observes and is observed. One prominent method of self-observation from the field of psychology is that of Introspection (where, having been trained, it is considered possible to carefully and objectively analyse the content of your own consciousness). The practice of self-observation within the phenomenological traditions, where the focus is on uncovering the essential features of lived-experience, is also prevalent. Over time, phenomenology's conceptual approach to self-observation was transformed from "an *observation of the self* to the *self's detached observations of experience…* and finally to the self*'s engagement with the meaning of lived experience"* (Gantt & Thayne, 2013, p. 166, emphasis in original).

> The observing self cannot be separated from the world of lived experience that constitutes it as an observing self and, thus, the self cannot observe itself in an "objective" or detached manner. This does not, however, make self-observation an unreliable or unscientific form of psychological inquiry. Rather, it simply means that we must acknowledge and take account of the relational and experiential context in which all self-observation necessarily takes place. (Gantt & Thayne, 2013, p. 167)

According to Clegg (2013) a good theory of self-observation needs to account for and make sense of "the differences in inter and intrasubjective agreement across both experience-near and experience-distant self-observations" (p. 14). He goes further to make clear the importance of accepting the difference between the observer's and interpreter's frames of reference and to place emphasis on "the negotiations that bring these multiple and nonstandard frames of reference into relation" (p. 14). This shift towards seeing the value not only from working on one's own experiences in-the-moment, but also with retrospective accounts of one's own experiences with others, leads me to consider carefully the practice of self-observation as described within the narrative traditions in such forms as narrative inquiry and autoethnographic writings. Such approaches are becoming more commonly used within educational research and teacher education specifically. I will later be exploring these approaches in more detail in the context of mathematics teacher education.

The self of self-observation

The status and nature of the self are issues that receive a great deal of attention within fields such as sociology, psychiatry, developmental psychology, philosophy of mind, social theory, cultural studies, and cognitive neuroscience. Whether self is a social construct, an experiential entity or indeed exists (in reality or virtually) is a topic of much debate amongst interested parties. Some traditions, such as Buddhism, question the legitimacy of the notion of self entirely, as do several philosophers, both modern and contemporary. Furthermore, rather than self taking some form of experiential reality, some authors claim that self only exists in linguistic form, as a narrative fiction, as self that is constructed. For others the self is understood as pure ego-pole, independent of experience, that remains unchanged throughout the life of the individual. Since self as pure ego-pole is viewed as pure subject of experience, it cannot be objectified, that is, the self is not something, which in itself can be experienced.

According to philosopher and psychologist, Harré (2001), "the word "self"... appears in person-centered discourses in at least three psychologically diverse contexts: perception, reflection, and social interaction" (p. 60). From each context there exists an associated form of self that Harré calls Self 1, Self 2, and Self 3 respectively. It follows that any SBM underpinned by an ontological position that involves multiple selves, requires multiple methods of data collection and analysis that can take account of this plurality.

EXISTING SELF-BASED METHODOLOGIES

Attempts have been made, including by members of the PME-NA 2018 working group (Suazo-Flores et al., 2018), to draw out distinctions between such SBMs as "narrative (a look at a story of self), autoethnography (a look at self within a larger context), and self-study (a look at self in action, usually within educational contexts)" (Hamilton, Smith & Worthington, 2008, p. 17) all of which are used in studies that privilege the self in research design. It is acknowledged, however, that the boundaries between such methodologies are blurred:

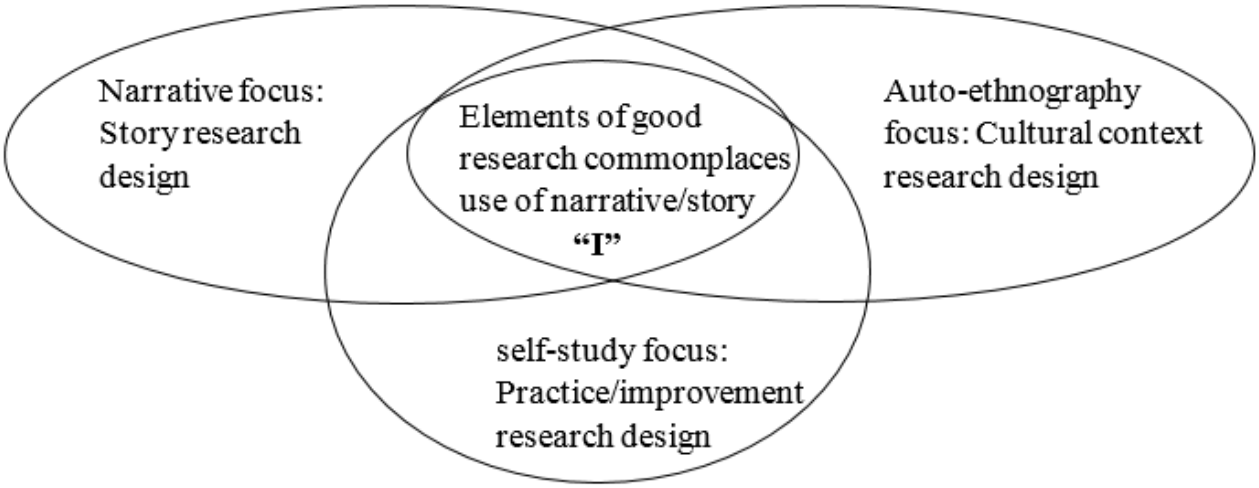

Figure 1: Venn analysis of the three methodologies (Hamilton *et al.,* 2008, p. 24)

Following is an articulation of each SBM from above, and Mason's (2002) Discipline of Noticing as a methodology, which I have utilised within my own research.

Self-study

The term self-study points to the *focus* of the study rather than any particular approach or set of methods. Across existing self-study literature, a range of research methods are utilised and consequently a range of reporting styles are evident. The aim of self-study is to support practitioner researchers in seeing differently what they are seeing from the inside. This insider's perspective is essential to handling the complexity of teaching about teaching and this complexity "requires a familiarity with practice in concert with maintaining a distance from practice in order to see what is happening while it is happening" (Loughran, 2006, p. 35). The impact of self-study is immediate change in practice since those engaged in self-study refer to the experience of "I" as a "living contradiction" (Whitehead, 2000, p. 93) when it is recognised that a value is held (such as social justice) yet it is denied in practice. It is the study of these living contradictions that is so powerful in creating the conditions for change.

Narrative inquiry

Narrative refers to the stories that people tell. Within narratives, lived experiences are organised into meaningful episodes that allow the narrator to interpret their worlds. The construction of narratives can be seen as the construction of what we call a Self, and, in this construction, one gradually transforms perceptions of direct experience into

that of knowledge. Self-based narrative research focuses not only on the experiences of the researcher but also on the meaning given to those experiences. In narrative inquiry, records of experiences are usually taken but then layered with reflection on those experiences (Suazo-Flores et al., 2018).

Autoethnography

According to Roth (2005) "auto/ethnography and auto/biography are genres that blend ethnographic interests with life writing and tell about a culture at the same time it tells about a life" (p. 4). Autoethnography is an autobiographical genre of research and writing that displays multiple levels of self and consciousness. An evocative image of the autoethnographer is triggered by Ellis (2004) in her autoethnographic novel where she refers to the gaze of the autoethnographer "first... focusing outward on social and cultural aspects of their personal experience; then... inward, exposing a vulnerable self" (p. 37). Through this movement back and forth, the "distinctions between the personal and the cultural become blurred, sometimes beyond distinct recognition" (p. 38). Autoethnographers use self-reflection and personal writing (of many forms) to explore personal experience in order to understand it in relation to the wider cultural context.

The discipline of noticing

As part of my own study I looked initially to Mason's (2002) Discipline of Noticing (DN) as a methodology for my research as an MTE (see Helliwell, 2018). Mason acknowledges that DN stems from a phenomenological underpinning and he makes a strong case for working with lived experience as an alternative to much research that has become a task of refining and using existing theoretical frameworks. For Mason, the central focus of noticing is one's "own inner experience" (p. 183) that is "awakening…. an inner witness who watches and comments but does not interfere" (p. 184). This inner witness refers to being awake in the moment so that an increasing set of possibilities for action become available. The awakening of this inner witness comes about through engaging with DN, which incorporates re-entering moments by reflecting on accounts-of incidents. Through this post-spective process of reflecting on these incidents it is possible to consider alternative ways of acting in the moment. Through labelling particular phenomena that have occurred, it opens up the possibility of recognising that phenomena again in the future and triggering a different response and by communicating accounts of this process, it can become useful for others to try out new ways of acting for themselves. In my own research, I have been reflecting on the experience of working as an MTE by paying attention to what I notice, in the process of transcribing conversations with mathematics teachers. By returning to Harré's self-model, in light of these existing methodologies, I offer a final examination.

TOWARDS A SELF-METHODOLOGY FOR STUDYING MTE LEARNING

Self 1, the self in the context of perception, "is manifested in the structure of perceptual fields, each of which is centered on the location in space and time of the embodied perceiver" (Harré, 2001, p. 60) and is close to the phenomenologists' understanding of the self in the first-person perspective. Self 1 plays the major role in telling stories of

our encounters with the material environment. A self-based study that pays attention to Self 1, looks to methodologies linked to frameworks of noticing and phenomenology. Self 2, the self in the context of reflection on oneself as a person, is the totality of a person's attributes, including their self-beliefs. The capacity to develop a self-concept in this way only comes about through looking at oneself from the outside, through objectifying the self, which is usually then articulated using a narrative form. A self-based study that pays attention to Self 2, looks to methodologies linked to narrative traditions such as narrative enquiry and autoethnography. Self 3, the self in the context of social interaction, refers to the person we are taken to be by others and points to the contrast or consistency between how we perceive ourselves and how we are perceived by others. This self, as a social process, can be described as dialogical, in that impressions of self get interpreted by others which in turn get re-interpreted into the self. A self-based study that pays attention to Self 3, looks to methodologies where the focus is on exploring living contradictions between self-concept and others. In adopting a multiple self-model, ideas, principles and methods from each of these four methodologies (self-study, narrative, autoethnography, DN) can be brought together a legitimate methodology for an MTE developing her expertise.

In embracing SBMs in researching MTE growth and the development of expertise and recognising research based on SBMs as valid contributions to knowledge, much fruitful work can be done in the area of MTE learning. Questions around what, if anything, makes self-based research distinctive for a *mathematics* teacher educator in contrast to teacher educators more broadly could be explored.

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VALIDATION OF A DEVELOPMENTAL MODEL OF PLACE VALUE CONCEPTS

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A profound understanding of the decimal place value system challenges primary school learners worldwide. Besides procedural place value understanding – which is transcoding and mapping of number symbols, number words and quantity representations – children have to develop place value concepts, which refer in particular to the relation of the bundling units (e.g. 1 ten = 10 units). In this paper we propose a five-level developmental model of place value concepts and present two empirical validation studies. In a cross-sectional study (N=749) we show in a Rasch-analysis that the items operationalizing the model form a one-dimensional hierarchy as proposed. A longitudinal study (N=195) reveals learners' progress within the model sequence.

INTRODUCTION

There is a growing body of empirical research highlighting difficulties children have while developing a sound understanding of the decimal place value system (e.g. Cobb & Wheatley, 1988; Fuson, Wearne, Hiebert et al., 1997; Hart, 2009; Herzog, Ehlert & Fritz, 2019; Kamii, 1986; Ross, 1989). In this paper we present a developmental model of place value concepts and its empirical validation.

From a historical perspective, children's difficulties with the place value system are no surprise. Compared with analogue representations of quantities and additive number systems, the fully developed place value system including a symbol for zero is rather new (Ifrah, 1998). In particular the history of the zero sign, whose development took more than 2000 years, illustrates the difficulties humans had to develop a complete place value system and might explain, why many children struggle understanding its principles (Herzog et al., 2019; Ifrah, 1998; Kamii, 1986). This raises the questions, which properties define the decimal place value system and what knowledge is necessary for its understanding.

THEORETICAL BACKGROUND

There can be four properties determined that structure the decimal place value system (Herzog et al., 2019; Ross, 1989):

Positional property: Each digit in a number contains information about which bundle size it represents.

Base-ten property: All bundles are powers of ten.

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Multiplicative property: The digits' face values have to be multiplied with the corresponding face values.

Additive property: The resulting components of a number (decimal decomposition) are summed up to construct the number.

Another main property is the continued bundling principle as it facilitates the iterated re- and unbundling in powers of ten. This leads to a specific relationship between the bundling units (ones, tens, hundreds etc.): 10 units equal one ten, 10 tens equal one hundred etc. (Herzog et al., 2019; van de Walle, Karp & Bay-Williams, 2004). These properties make the place value system to an astonishingly efficient tool for enumeration, writing and reading numbers and calculation (Fritz & Ricken, 2008; van de Walle et al., 2004).

Place value understanding

Van de Walle et al. (2004) distinguish between procedural and conceptual place value understanding. *Procedural place value understanding* refers to the understanding and mapping of number words, number symbols and quantity representations in a way that allows "translating" number representations between these codes. This model can be seen as an application of the triple code model on the decimal place value system (Dehaene, 1992; Herzog et al., 2019; van de Walle et al., 2004).

Transcoding – the translation process between the number codes – has been focused by researchers for a longer time. Effects of number size and distance, several effects regarding the digits as well as processing routes of multi-digit numbers have been discussed (Nuerk, Moeller & Willmes, 2015). Linguistic influences were investigated, in particular how the structure of number words in different world regions affect transcoding processes (Miller, Smith, Zhu & Zhang, 1995; Miura, Kim, Chang & Okamoto, 1988; Pixner, Zuber, Hermanova, Kaufmann, Nuerk & Moeller, 2011; Zaslavsky, 1999). Typical errors children do in preschool age (70053 for "sevenhundredfiftythree") indicate that the integration of the abovementioned properties is a main learning step for children (Byrge, Smith & Mix, 2014). However, only a small minority of learners seems to have persistent difficulties with transcoding (Herzog et al., 2019).

Conceptual place value understanding focusses the meaning as well as the relation of the bundling units (Cobb & Wheatley, 1988; Herzog et al., 2019; van de Walle et al., 2004). For example the meaning of ten as bundling unit is hardly to be separated from its relation to units: A ten is a bundle of ten units. Therefore, place value concepts are not about how to read, write or manipulate numbers, but on the reason why we can do it like we do (Herzog et al., 2019).

Without a conceptual understanding of the place value system, learners cannot understand the iterated bundling of the bundling units. In this case, the bundling units remain unconnected beside each other (Cobb & Wheatley, 1988; Fuson et al., 1997). An understanding of the relation of the decimal bundling units allows representing numbers in different ways: For example, 43 can be represented as 3 tens and 4 units

(canonical), but also as 2 tens and 14 units or 1 ten and 24 units (non-canonical). This knowledge has to be considered as main part of conceptual place value understanding (Herzog et al., 2019; van de Walle et al., 2004).

Certainly, procedural and conceptual place value understanding are not separate domains, but interact. In particular procedural place value understanding bases on place value concepts and is unlikely to develop without any conceptual basis (Herzog et al., 2019; van de Walle et al., 2004).

Developmental model of place value concepts

In order to structure the concepts that form a sustainable place value understanding, we developed a model of place value concepts (Herzog, Ehlert & Fritz, 2017; 2019). The model is based on existing theories and models from the last 30 years as well as an analysis of the place value system on the one hand (Cobb & Wheatley, 1988; Fuson et al., 1997; Resnick, 1983; Ross, 1989). On the other hand, the model was operationalized and empirically tested throughout the process of development in several pilot studies in Germany. The empirical results led to a revision and adaption of the model, so that theoretical considerations and empirical findings were integrated (Herzog et al., 2019). In an empirical study, the model hierarchy could be underpinned in South Africa, too (Herzog et al., 2017). The model specifies four central conceptual levels and an additional starting level:

Pre-decadic level: On this level, children perceive numbers as unitary entities that are not structured into decimal bundling units (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989). Thus, to children a number like 25 is not structured into tens and units. That does not necessarily mean that children do not understand decompositions of smaller numbers or even more complex decompositions (e.g. 5x5) at all. But they do not see the decimal structure of numbers initially.

Level I (Place values): The first concept, learners develop is the knowledge that there are decimal bundling units (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; van de Walle et al., 2004). They can name and detect digits' positions within numbers. This knowledge allows them to (de-)construct numbers canonically, while non-canonical representations cannot be handled yet. The number range in which children know the place values relies on instruction; however, tasks get not significantly more difficult with growing number range (Herzog et al., 2019).

Level II (Tens-units relation with visual support): The first step towards an understanding of the relation between tens and units as decimal bundling units relies on visual support in form of structured manipulatives (e.g. base-ten blocks) (Cobb & Wheatley, 1988). With help of these manipulatives children can verify the identity of ten units and one ten, for example through counting processes. Therefore, non-canonical representations are understood if visual aids are given. However, abstract representations are not yet understood. Moreover, this concept is limited to tens and units while bigger bundling units are not involved.

Level III (Tens-units relation without visual support): On this level, learners understand the relation of tens and units in an abstract way. The bundling unit "ten" now has a cardinal meaning that is independent from visual aids or counting processes. This knowledge is limited to tens and units. On the same level, children start to integrate hundreds and tens when they have visual representations. Analogue to level II, they can verify and use the relation between hundreds and tens if they have visual support.

Level IV (Decimal-bundling-units relation in general): On this last level, children internalize their knowledge about the bundling units so they do not rely on visual aids at all (Fuson et al., 1997). They can now process non-canonical representations abstractly. Although the number range naturally depends on instruction as new digits' names have to be learned, the general principle of the continued bundling is generally understood. With this concept the development of place value concepts is considered to be completed (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989).

EMPIRICAL VALIDATION

While the model hierarchy is in line with existing research and the content's structure, the proposed model needs an empirical validation. We tackled the model's validity with two studies.

Study 1: Model hierarchy

In the first study we operationalized the levels of the model and tested the resulting items with a total of N=749 children (340 girls) from grades 3, 4 and 5. The sample is distributed equally across the grades (grade 3: 270 (36%), grade 4: 236 (31.5%), grade 5: 243 (32.4%). Data were collected at the beginning of the school year in all grades.

The final test contained 36 items based on the model proposed above (Herzog et al., 2019). Items were designed according to one specific conceptual level each (12 items on level I, 8 on levels II to IV each). The item collection shows a good internal consistency (Cronbach's α=.894).

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           ########   |  III_4A
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Fig. 1: Person-Item-Map of the Rasch analysis with level clusters.

To test the hierarchy of the concepts as proposed by the model, we used a one-dimensional Rasch analysis. This probabilistic test model allows determining the items' specific difficulties (Dunne et al., 2012). The model is supposed being affirmed, if items operationalizing the same conceptual level form consistent clusters according to the model hierarchy. In addition, we investigated how the learners' performance is distributed across the grades 3 to 5.

The Rasch model shows acceptable to good MNSQ-infit measures (.68-1.3 for all items and .8-1.2 for 30 of that) (Wright & Linacre, 1994). In general, the items used to operationalize the model form consistent clusters that order as suggested by the model. 6 items show difficulty measures that belong to neighboured levels (see fig. 1). All

"misplaced" items can be explained by instructional and curricular influences (3 items), use of specific strategies that change the items' conceptual level (2 items) or too complex task wording (1 item).

We compared the mean raw scores of learners separated by grade. Table 1 shows the mean scores and standard deviations. An ANOVA revealed significant differences between the grades ($F(2,748)=76.827$, $p<.001$). Post-Hoc-tests following Scheffé showed significant differences between grade 3 and 4 as well as grade 3 and 5 (both $p<.001$); grades 4 and 5 did not differ significantly ($p=.414$).

	total	3	4	5
N	749	270	236	243
M	22.77	18.64	24.68	25.51
(SD)	(7.55)	(6.54)	(6.49)	(7.6)

Table 1: Distribution of raw scores across the grades

These results underpin the model hierarchy of place value concepts. Obviously we can distinguish specific concepts that children develop in primary school. However, the results also reveal that instruction affects the model validity. A transfer to other grades needs a specific validation and maybe a revision of the operationalization. The distribution of performance across the grades indicates that learners in higher grades have more elaborated place value concepts than learners in lower grades, pointing at a real development during primary school. However, it seems as learners do mostly progress during grade 3 while there is only little development in grade 4.

Study 2: Individual learners' development

To give evidence that the model describes a real development of concepts, it is necessary to investigate if learners really rise through the levels and gain conceptual knowledge. For this purpose the learners from grade 3 and 4 were followed in a longitudinal study for one school year.

Measurement times were beginning (T_1), middle (T_2) and end (T_3) of grade 3 and 4 respectively. Data from a total of N=195 (87 girls) could be collected at all measurement times. For this study the same task collection in altered order was used as in study 1.

The learners' raw scores in all three tests were investigated in a variance analysis with repeated measurements. Table 2 shows means and standard deviations from all measurements for all learners and separately for grades 3 and 4. For all learners significant differences were found ($F(2,388)=70.350$, $p<.001$, part. $\eta^2=.266$). Learners in grade 3 showed bigger learning effects ($F(2,170)=60.571$, $p<.001$, part. $\eta^2=.416$) than the fourth-graders ($F(2,216)=21.225$, $p<.001$, part. $\eta^2=.164$). In post-hoc-tests (Bonferroni) growths were significant in all groups and between all measurements except grade 4 from T_2 to T_3 ($p=.459$).

Group		T_1	T_2	T_3
all	M	24.37	27.34	28.29
(N=195)	(SD)	(6.41)	(6.09)	(6.1)
Grade 3	M	21.43	25.24	26.63
(N=86)	(SD)	(6.38)	(6.68)	(5.94)
Grade 4	M	26.67	28.99	29.61
(N=109)	(SD)	(5.43)	(5.03)	(5.92)

Table 2: Development of raw scores over one year

The longitudinal study shows that learners individually progress significantly in their conceptual place value knowledge. We therefore argue that the proposed model validly describes and assesses learners' individual learning trajectories. The model of place value concepts thus is appropriate to structure teaching and assessment in class. However, the study reveals that learners in the first develop place value concepts during grade 3and the beginning of grade 4. This is in line with the results from study 1. This might root from the curriculum and teaching.

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CONTRIBUTION OF ACADEMIC MATHEMATICS TO TEACHER LEARNING ABOUT THE ESSENCE OF MATHEMATICS

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This study investigates the contribution of academic mathematics courses to teacher learning about the essence of mathematics. Analysis of interviews with eleven secondary school mathematics teachers, who graduated from a master's program that included a strong emphasis on academic mathematics studies, identified references to four topics regarding the essence of mathematics: (1) Wide and varied, (2) Lively and developing, (3) Rich in connections, and (4) Structured deductively. The findings show that whereas all teachers (but one) reported that they expanded their knowledge about the essence of mathematics, which, in turn, contributed to their classroom teaching, substantial differences existed among the teachers in the extent to which they referred to each of these four topics.

INTRODUCTION

In many countries, the education of secondary school mathematics teachers traditionally includes a strong emphasis on advanced mathematics courses at the college or university level. This tradition has been reconsidered in recent years, and the relevance of academic studies of mathematics to secondary school mathematics teaching is being debated (e.g., Dreher et al., 2018; Murray et al., 2017). As part of a comprehensive research program that addresses this issue, the current study examines what might be the contribution of academic mathematics courses to teacher learning about the *essence of mathematics*, i.e., about what is this discipline called mathematics.

THEORTICAL BACKGROUND

The literature on the relevance and contribution of academic mathematics studies to mathematics teaching in secondary school is rather limited. Theoretical contemplations suggest a potential contribution in two dimensions of subject-matter knowledge that appear to be critical for teaching. One dimension is knowledge of specific topics, concepts, and procedures, and the other is a more general epistemological knowledge about what mathematics is and what doing mathematics entails (Ziegler & Loos, 2014).

The existing empirical research literature, which is mainly based on interviews with teachers, mostly reports on the contribution of academic mathematical studies to teaching at the level of knowledge about what mathematics is and what doing mathematics entails (e.g., Even, 2011; Wasserman et al., 2018; Zazkis & Leikin, 2010). Teachers referred to such aspects as doing mathematics as problem solving, the role of intuition in doing mathematics, and the use of mathematics in other disciplines. Yet, the literature is lacking conceptual frameworks for analysing this contribution.

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In a recent study (Hoffmann & Even, 2017, 2018) we addressed this shortcoming of suitable conceptual frameworks. In that study we found that expanding teachers' knowledge about what mathematics is was one of the main objectives of mathematicians who taught advanced academic mathematics courses to practicing secondary school teachers. Based on interviews with these mathematicians we developed a conceptual framework for analysing teachers' knowledge of what mathematics is. The framework comprises of three main aspects: (1) the *essence* of *mathematics*, (2) *doing* mathematics, (3) and the *worth* of mathematics. Each of these aspects includes between two to four topics; a total of nine topics (see **Error! Reference source not found.**).

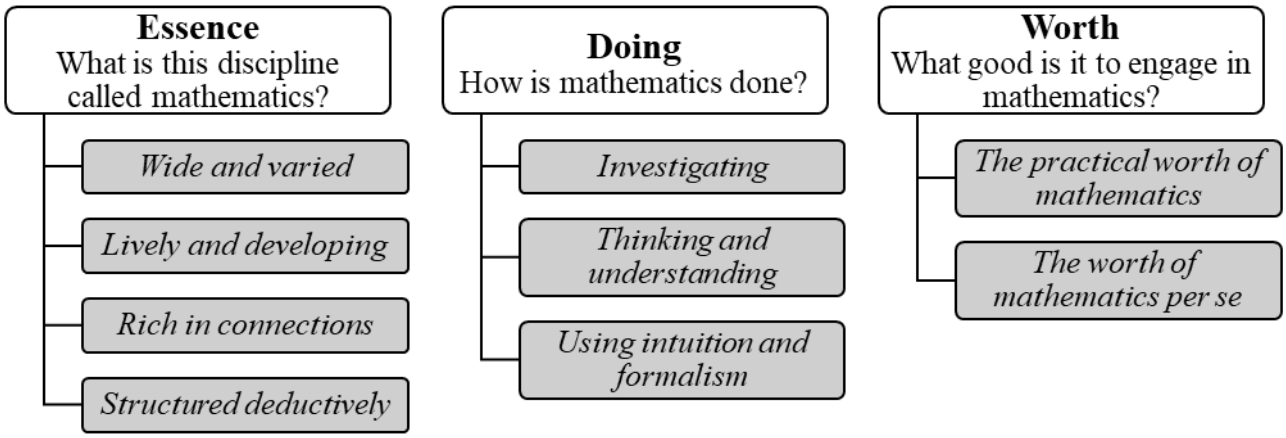

Figure 9: A conceptual framework for teacher knowledge about what mathematics is

This framework serves us now in studying the relevance and contribution of academic mathematics courses to teaching mathematics in secondary schools. This paper focuses on one aspect, namely, the *essence of mathematics*, which deals with the question: What is this discipline called mathematics? The research question is: *What do teachers learn about the essence of mathematics during academic mathematical studies and how does this knowledge contribute to their work?*

METHODS

Setting and Participants

The study is situated in a two-year master's program for practicing secondary school mathematics teachers. A major component of the program comprises eight academic mathematics courses tailored for teachers, designed and taught by research mathematicians. Four of these courses deal with topics in the school curriculum at an advanced level: algebra, analysis, geometry, and probability and statistics. Three courses are devoted to the use and application of mathematics in other domains: computer science, natural sciences (applied mathematics), social sciences and everyday technologies. One course deals with the history and philosophy of mathematics. In addition, a final project that involves an independent study of an unfamiliar mathematical topic is carried out under the guidance of a mathematician.

Eleven program graduates participated in the study; eight women and three men. All had experience in teaching mathematics in upper secondary school. Their teaching experience varied considerably from 3 to 23 years.

Data Collection

The main data source included individual semi-structured in-depth interviews with the teachers. The interviews took place between 0.5-2.5 years after graduation, and lasted between 45-90 minutes. The aim was to learn how the mathematics courses contributed to the teachers' knowledge about the nature of mathematics, and how that knowledge contributed to their teaching. The interview consisted of eight open-ended questions. The main questions were:

1. Has there been any change in the teacher you were before the program and the teacher you are today?
2. The program has two main components: courses in mathematics education and courses in mathematics. Have the *mathematics courses* contributed to you as a teacher?
3. Have you learned anything new about what mathematics is from the *mathematics studies* in the program?

Following each question, the interviewees were asked to explain their responses and to give examples from their experiences in the program and their teaching. Additional data sources were participant observations in most courses and informal conversations with the teachers. The aim was to strengthen the internal validity of the study.

Data Analysis

We used the conceptual framework in **Error! Reference source not found.** for a nalyzing the interviews, employing the method of directed content analysis (Hsieh & Shannon, 2005). This is a qualitative data analysis method, which uses the categories of an existing theoretical framework for coding scheme. During analysis additional categories could be created and definitions of existing categories could be modified as arising from the data.

After full transcripts of the interviews were made, everything said by the teachers that concerned the nature of mathematics was coded in an iterative and comparative manner. For every category found in a transcript, if the teacher did not link it to contributions of the academic mathematics courses, the coded text was omitted. As the interview questions were neutral by means of the conceptual framework, that is, none of the nine topics were mentioned or implied by the interviewer, we regarded the extent to which a teacher referred in the interview to a certain topic to indicate that the topic was significant for that teacher (Morgan, 1993). Hence, in each transcript, we counted the number of words related to each category and calculated the percentage of the text associated with each category out of the total coded text.

FINDINGS

Analysis revealed that all three aspects and all nine topics of the conceptual framework for teacher knowledge about what mathematics is (**Error! Reference source not found.**) were mentioned in the interviews. Also, no additional aspects or topics of knowledge about the nature of mathematics were identified. Thus, the resulting categories were identical to the theoretical framework.

In the case of the essence of mathematics, which is the focus of this paper, 10 (out of 11) teachers reported that the academic mathematics courses contributed to their knowledge about this aspect, and also that this new knowledge contributed to their teaching. Yet, not all ten teachers mentioned all four topics included in *the essence of mathematics*: (1) *wide and varied*, (2) *lively and developing*, (3) *rich in connections,* and (4) *structured deductively*. Table 1 presents for each teacher and topic the number and percentage of interview words related to contribution of the mathematics courses to the teacher's knowledge and practice.

Teacher	Wide and varied	Lively and developing	Rich in connections	Structured deductively	Total coded text
A	-	567 (48%)	-	31 (3%)	1175
B	325 (16%)	368 (18%)	58 (3%)	93 (4%)	2073
C	-	203 (7%)	700 (25%)	-	2761
D	-	-	-	-	336
E	-	116 (3%)	575 (14%)	-	4041
F	110 (14%)	-	-	338 (42%)	800
G	312 (17%)	-	-	-	1794
H	135 (14%)	-	546 (55%)	-	991
I	-	229 (9%)	523 (21%)	648 (26%)	2529
J	484 (20%)	214 (9%)	186 (8%)	-	2395
K	1342 (35%)	317 (8%)	443 (11%)	294 (8%)	3881
Median	319	229	483	194	

Note. Percentages are out of the total number of words in the coded text. Shaded cells indicate that the teacher reported also on contribution to classroom teaching.

Table 3: Numbers and percentages of words in teachers' interviews associated with contribution to the four topics of the *essence of mathematics*.

As shown in Table 1, the teachers varied considerably in the number of topics to which they referred (0-4). Still, a similar number of teachers (5-7) referred to each topic. Moreover, topics that received considerable focus in a teacher's report were topics on which that teacher reported on contribution to practice as well. In the following, for each topic, we describe characteristics of the contribution of the mathematics courses to the teachers' knowledge and practice.

Wide and varied

As shown in Table 1, six teachers reported on contribution of the mathematics courses related to the topic mathematics as *wide and varied*. These teachers reported that they became aware of how wide mathematics is, and what a huge amount of mathematics exists, which they still and never would know. While one teacher (K) related to this category exceptionally lengthy, throughout 1342 words, the median length of word count per teacher in this category was 319 words. Teachers reported that they learned how wide and varied mathematics is, in terms of the fields of mathematics that exist, the kinds of questions one may ask, the assumptions one can make, the ways one may try to solve or model problems, and the many levels of complexity a mathematical concept could have. The following excerpt from J's interview exemplifies this.

> I: From the mathematical studies, was there anything you learned about what mathematics is? Did you learn something new? What is this discipline?
>
> J: Mostly perhaps how many areas it covers. Like when I'm studying mathematics, I learn that I don't know... Actually, you see that mathematics is like a whole world. It is impossible to know it. ...You learn whatever you learn, but it is more than that. It gives you, like, the feeling of the size of the [pause], like, basically this.

Five of the six teachers reported that their new knowledge about mathematics as a wide and varied discipline contributed to their teaching. They said that their new knowledge enabled them to widen students' perceptions about mathematics and revealed to them new and interesting frontiers. The following excerpt from K's interview manifests this.

> I: Have there been any changes in you as a teacher following the program?
>
> K: …It made me understand that mathematics is not something rigid and closed but something very very wide… and it enabled me as a teacher… to convey to the students that mathematics is very very interesting, and very deep, and it covers all domains, and I didn't have that before.
>
> I: How do you convey it? ...
>
> K: …When we study vectors and we learn norms, then every plane has one norm. Today, after studying differential geometry, I know that surfaces have norms in different directions. I mean, you can talk now about that a norm is not one norm to a plane. If the surface is a bit curved, then you have infinite number of norms. You can convey this to a certain extent, just open it, and curious girls fly with it further.

Lively and developing

Six teachers reported on contribution of the mathematics courses related to the topic of mathematics as a *lively and developing* discipline. The median length of coded text in this category was 229 words. The teachers reported that they learned that mathematics did not appear suddenly from nowhere but was rather developed slowly. They also learned how and why today's mathematics has evolved. B illustrates this:

> …it is very important how mathematics has evolved... Let's say Euclid, his book of the elements of geometry, and then over the years how things evolved, and so on, and projective geometry, and so on, and then like all the developments of recent years. There is something in this conception that straightens out my head. I can understand much more deeply. How it was created.

Five of the six teachers said that their new understandings about mathematics as a lively and developing discipline contributed to their teaching. They reported that they started to present students with the background of how things have developed in order to deepen students' understanding and raise their interest. For example, A described:

> I: Is there anything else you teach differently that you can point out?
>
> A: …if I talk to them about the development of mathematics then it is to teach differently. Most definitely. … anything new from the curriculum that I want to bring to class… I try not to dump it on them, but to prepare them for it… I may not always be able to do that... But whenever I can explain where it came from and what people did in order to reach this discovery, I do that.

Rich in connections

Seven teachers said that their participation in the academic mathematics courses contributed to their understanding about mathematics as *rich in connections*. The median of coded text in this category was the highest of all categories in the aspect of the *essence of mathematics*, with 483 words. The teachers talked about their understanding of the connections between different topics and the importance of making connections while dealing with mathematics. For example, B said: "One of the most powerful things that I had as an insight, it is the connection of all the topics to each other and the integration between them." And E said: "I saw the connection between the different topics much more, how one domain uses another in order to prove, to make progress, to illustrate."

Teacher C demonstrated her development of knowledge on the connectivity of mathematics by referring to the topic of conic sections. She explained how differently she knew this topic before participating in the program:

> I taught circles, taught ellipses, taught parabolas, taught hyperbolas in 12th grade in Cartesian geometry, and I knew how to lead my students so that they succeed in the matriculation exams on any question on these subjects. Still for me it was four equations of four shapes. I never saw it as one family of quadratic functions that are part of a general thing.

And she continues later in the interview:

> What I told you before about the connections between things. This phenomenon of compartmentalization, of putting things in separate drawers. Maybe it's convenient for us to put things in drawers, but when things suddenly connect and everything can be here and there, then suddenly you feel that you understand.

Five of the seven teachers elaborated on the ways their new knowledge about the connectivity of mathematics contributed to their teaching, by solving problems in multiple ways, and by using concepts and representations from various fields. For example, C said: "Mainly I think about yesterday and tomorrow, and not just about the present. When I teach something, I try to connect it to what was before and what will be later and to other domains." For instance, she stresses for students that the straight line they sketch in geometry, the linear equation $f(x) = mx + n$ they see while

learning functions, and the equation $Ax + By + C = 0$ that they meet in Cartesian geometry, are all the same mathematical concept in different representations.

Deductively structured

The contribution of mathematics courses to the topic mathematics as *deductively structured* was mentioned by five teachers, relatively shortly, with a median of 194 words, the lowest from all categories in the aspect of the *essence of mathematics*. The teachers did not mention the logical structure of mathematics but rather referred solely to the elements of the deductive structure.

Three teachers mentioned contribution to teaching in this category. They said that the mathematics courses contributed to their knowledge regarding the essence of mathematical axioms, definitions and proofs, and how mathematics is established by axioms and definitions. For example, K said:

> In any mathematical course. It suddenly pops up. You didn't look for it until you became aware of it. When you are aware that you look for definition of everything, I just look for it... I suddenly explain to the kids: That is a definition, you cannot argue about it, let us define it properly. I also explain what 'well defined' is... I think it made me a much more organized teacher, and it makes order in the students' heads...

CONCLUSION

As shown in this paper, the conceptual framework we developed, based on interviews with mathematicians about what they wished to teach secondary school teachers about mathematics (Hoffmann & Even, 2017, 2018), was useful for examining the contribution of academic mathematical studies to secondary school teachers' knowledge and practice, in relation to the *essence of mathematics*. Analysis of interviews with teachers, who graduated from a program comprised of eight academic mathematics courses, generated the same four topics associated with the *essence of mathematics*, as the analysis of the interviews with the mathematicians who taught in that program: mathematics as (1) *wide and varied*, (2) *lively and developing*, (3) *rich in connections,* and (4) *structured deductively*.

Our study provided tangible examples of how academic mathematics courses that focus on specific fields (geometry, analysis, etc.) contributed to teachers' learning about the *essence of mathematics*, which in turn contributed to their practice. For instance, teacher K's use of the concept of norm she studied in the course on differential geometry for demonstrating her developed knowledge about mathematics as a *wide and varied* discipline, and her use of this new knowledge in teaching as a means to raise her students' interest in, and widen their perspectives about, mathematics.

Our findings suggest that all participating teachers, but one, learned something new about the *essence of mathematics* in their academic mathematics studies. Yet, even though all the teachers studied the same academic mathematics courses with the same instructors, different teachers attended to different topics of the *essence of mathematics*. More research is needed to better understand these variances among teachers, and how they relate to teacher characteristics and to course instruction.

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PREPARING IN-SERVICE TEACHERS FOR THE DIFFERENTIATED CLASSROOM

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Teaching in heterogeneous classes is currently one of the biggest challenges for teachers in Germany. In order to teach mathematics adaptively, teachers must be able to select and use adequate tasks. This study addresses teachers' abilities to evaluate tasks with respect to their potential for differentiation. 78 teachers took part in a paper-pencil-test that assessed their perspectives on the potentially differentiating features of tasks, such as openness or accessibility. It was found that their arguments were predominantly based on the categories adaptivity, context/language, type of knowledge and layout. By cluster analysis four argumentation profiles could be identified.

INTRODUCTION AND THEORETICAL FRAMEWORK

One of the central challenges for teachers is to handle the students' heterogeneity (e.g. with respect to their achievement) in the classroom. There are various teaching strategies to deal with heterogeneity: for example, it is possible to group students. Research shows on the one hand that learning outcomes can increase in homogeneous groups, on the other hand other studies show a benefit for students in heterogeneous groups (Slavin, 1993; Lou et al., 1996; critical discussion at PME39: Forgasz, 2015). Unfortunately, most of the research on grouping does not reveal how the teaching was done in the different groups (e.g., diagnostic instruction, Brown, 1984; adaptive teaching, Beck et al., 2008). Another possibility is to focus on suitable tasks. Tasks are significant in teaching mathematics (Hiebert et al., 2003) and the central steering instrument for learning mathematics in schools. It is known from previous research that one key element of effective teaching is adaptivity and this plays an important role with regard to differentiation (e.g., external differentiation Boaler, 2016; internal differentiation, adaptivity Klieme & Warwas, 2011). "Adaptivity" is the ability to adapt teaching to heterogeneous learning groups. "Differentiation" refers to a variety of strategies of a teacher that aims at improving adaptivity. Among those strategies, task quality plays a crucial role for differentiated teaching (Brändström, 2005). Therefore, using adaptive mathematical tasks with a high differentiation potential is considered as one possible strategy to cope with students' heterogeneity. In this study we look at the teachers' abilities with regard to their competences selecting adequate tasks for differentiated teaching. In particular we investigate which task characteristics they focus on.

To judge the quality of tasks, one can refer to a number of characteristics (e.g., Stein & Smith, 1989; Blömeke et al., 2006; Leuders, 2015). In this study we focus on characteristics of tasks that are suitable to address learners with different abilities and different prior knowledge. The following two examples (taken from the test) show

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 368-375). Pretoria, South Africa: PME.

different characteristics of tasks with respect to adaptivity. In Example 1 (Fig. 1) students have rarely a possibility to work on different levels or in different ways. There is almost no variety in task types and students would have to work on more or less the same level in all items. In Example 2 students can begin on an easy level in the first item (1); the next items are more challenging because they require working backwards (subtraction). In (3) it may not be obvious for all students how to find a solution. Moreover, the question whether there are multiple solutions, requires argumentation and problem solving. Part (4) is open in several ways: Students can figure out easy or complex solutions, they can try to find the same or different numbers on the left and right side (symmetric or non-symmetric solution); they can try to use visualizations to find further solutions etc. – depending on their individual level of achievement. The characteristics of this task (across all four parts (1) to (4)) can be described as problem solving, openness and creativity. Another important characteristic is accessibility – all students should be able to begin working on item (1); high-performers, however, may begin with item (3) or (4). Finally, this task can be described as 'adaptive' because it can be solved in multiple ways.

Task-Example 1 (almost no differentiation potential, less adaptivity):

Compute.

a) $\frac{1}{4}+\frac{2}{3}$ b) $\frac{1}{3}+\frac{2}{5}$ c) $\frac{1}{2}+\frac{4}{9}$ d) $\frac{5}{6}-\frac{2}{5}$ e) $\frac{3}{4}-\frac{1}{3}$ f) $\frac{4}{5}-\frac{1}{2}$

Task-Example 2 (high potential for differentiation, adaptive for low- and high performing students):

Add the numbers. The result is always in the middle of two numbers. Search for several solutions in (4).

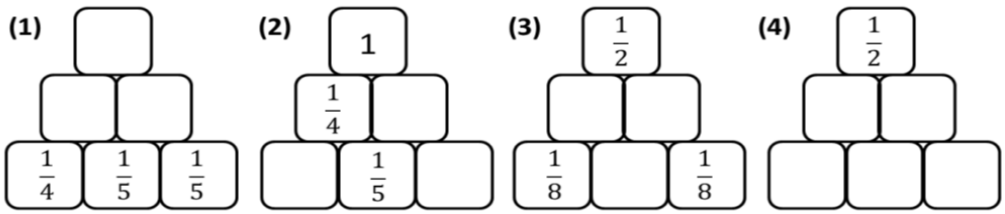

Figure 1: Task examples.

When teachers reflect on the tasks and their characteristics, they analyze the affordances of the task and the tasks' adequacy for learners with different abilities. Therefore, we aim at assessing teachers' arguments when choosing or evaluating tasks with respect to their potential for differentiation. We address the following research questions:

(1) To what extent do teachers succeed in identifying the differentiating potential of tasks compared to an expert view?

(2) What criteria do teachers have for assessing tasks with regard to their potential for differentiation?

(3) Is it possible to identify groups of teachers with different kinds of criteria?

METHODS

The participants of the study were N = 78 in-service teachers at secondary schools in Germany with several years of teaching experience. These participants completed a paper-pencil test instrument including 8 tasks, which had to be evaluated with respect to the question whether they are appropriate for differentiation (in figure 1 two of the eight tasks are listed). The teachers were asked to give reasons for their evaluations; they had to describe how adaptivity is realized in the task (or not). Three experts first assessed the tasks in terms of their differentiation potential. The expert rating was carried out by three independent persons who have been studying the topic of differentiation for several years and also have teaching experience. The agreement of these three persons was fair to good (with a Fleiss kappa value of .48; Table 2). The test instrument was then designed such that tasks with varying potential for differentiation (weak, medium and strong) from an expert's point of view were mixed but were not explicitly marked as such. First, the teachers were asked to indicate on a four-point-Likert-scale for each task how suitable they considered the task for use in heterogeneous learning groups (unsuitable (1), rather unsuitable (2), rather suitable (3), suitable (4)). In a next step, the teachers were instructed to describe two aspects of each task that were relevant for their evaluation of the task. First, the teachers' evaluations of the tasks were compared with the expert judgement (Table 2). Afterwards, teachers' comments were analysed using a specially developed rating manual (Table 1).

Development of the rating manual

A rating manual was used to evaluate the teachers' open answers. In a first step, this manual was developed based on theoretical considerations. At this point, the rating manual included 7 categories. During the evaluation it became necessary to add categories and therefore the manual was further developed inductively. In particular, it was found that teachers often evaluated the tasks by focusing not on aspects that characterize a task in terms of its differentiation potential but on other aspects of tasks such as "layout" or "task" context. Moreover, the teachers formulated their reasons often from the perspective of their students. Finally, the complete manual included 23 categories (see figure 3). We evaluated each category according to whether it was mentioned regarding lower or higher achieving students. The procedure (Table 1) was followed for all categories C1 to C23. The rating scheme was used to evaluate the teacher comments in the test. The procedure was as follows (Table 1): If a category matched a teacher's comment, the value "1" was assigned; we assigned "–" if the comment was negative and "+" if the comment was positive. If the commentary was formulated with respect to higher or lower achieving students, first a "1" was assigned in "general". For average, higher, and lower achieving students, the values "1", "+", and "–"were assigned, respectively.

category	category description	category rating
Goal differen-tiation [C2]	Teacher describes that task pursues different goals in terms of content for those with higher and lower achievement.	In general: ☐ no: 0, yes: 1 + – For low achieving: ☐ no: 0, yes: 1 + – For high achieving: ☐ no: 0, yes: 1 + –

Table 1: Example for the rating system.

Some examples of the rating can be found in figure 2.

Teacher comment	increasing the difficulty	Rating of the category
Level differen-tiation [B1]	The term "step task" is specified or described by the teacher (for example, the task is divided into several processing steps (usually a), b), ...); if necessary, a flexible start can also be provided).	☐ not described 0 ☒ circumscribed 1 ☐ explicitly named 2
Difficulty level of the task [C7]	The teacher describes the level of difficulty / level of the task or whether the learners are overwhelmed.	In general: ☐ no: 0, yes: ☒ + - For low achieving: ☐ no: 0, yes: 1 + - For high achieving: ☐ no: 0, yes: 1 + -
Teacher comment	too hard for weak students	
Fit [C1]	The teacher describes that the task (in the form in which the processing is taken) is only suitable for lower performers / only for stronger performers / for both groups.	In general: ☐ no: 0, yes: ☒ + - For low achieving: ☐ no: 0, yes: 1 ☒ - For high achieving: ☐ no: 0, yes: 1 + -
Difficulty of the task [C7]	The teacher describes the level of difficulty / level of the task or whether the learners are overwhelmed.	In general: ☐ no: 0, yes: ☒ + - For low achieving: ☐ no: 0, yes: 1 ☒ - For high achieving: ☐ no: 0, yes: 1 + -

Figure 2: Rating examples.

RESULTS

For answering the first research question, the teachers' evaluations were compared to the expert judgements.

Task number	0	1	2	3	4	5	6	7
Expert 1	4	2	3	4	3	1	1	4
Expert 2	4	1	4	4	3	1	2	4
Expert 3	4	2	2	4	3	1	1	3
Share Teacher Value 1	6%	23%	14%	8%	5%	16%	18%	11%
Share Teacher Value 2	43%	44%	31%	25%	46%	33%	38%	25%
Share Teacher Value 3	44%	26%	40%	29%	28%	31%	29%	30%
Share Teacher Value 4	8%	8%	14%	38%	19%	15%	15%	30%
Average Experts	**4,0**	**1,7**	**3,0**	**4,0**	**3,0**	**1,0**	**1,3**	**3,7**
Average Teachers	**2,5**	**2,2**	**2,5**	**3,0**	**2,6**	**2,5**	**2,4**	**2,8**

Table 2: Teacher ratings compared to expert ratings.

Overall the expert judgment on all 8 tasks was almost met only with 2 tasks, namely tasks 2 and 4 (see Table 2). Assessments of teachers were rarely extreme (i.e., they rarely chose 1 and 4 on the rating scale). It is interesting to see what justifications

teachers give when selecting or not selecting tasks in relation to a differentiated assignment of tasks in a practice lesson at the end of a unit (still without intervention). The decisions were mainly based on the aspects of "technical work", "difficulty level" and "layout".

Code	total number of answers	Condensed codes
C4 Technical work	256	Task Content
C7 Difficulty	200	Specific Adaptivity
C10 Layout	191	Task Structure
C3 Openness	134	Specific Adaptivity

Table 3: Most frequently mentioned aspects in the test.

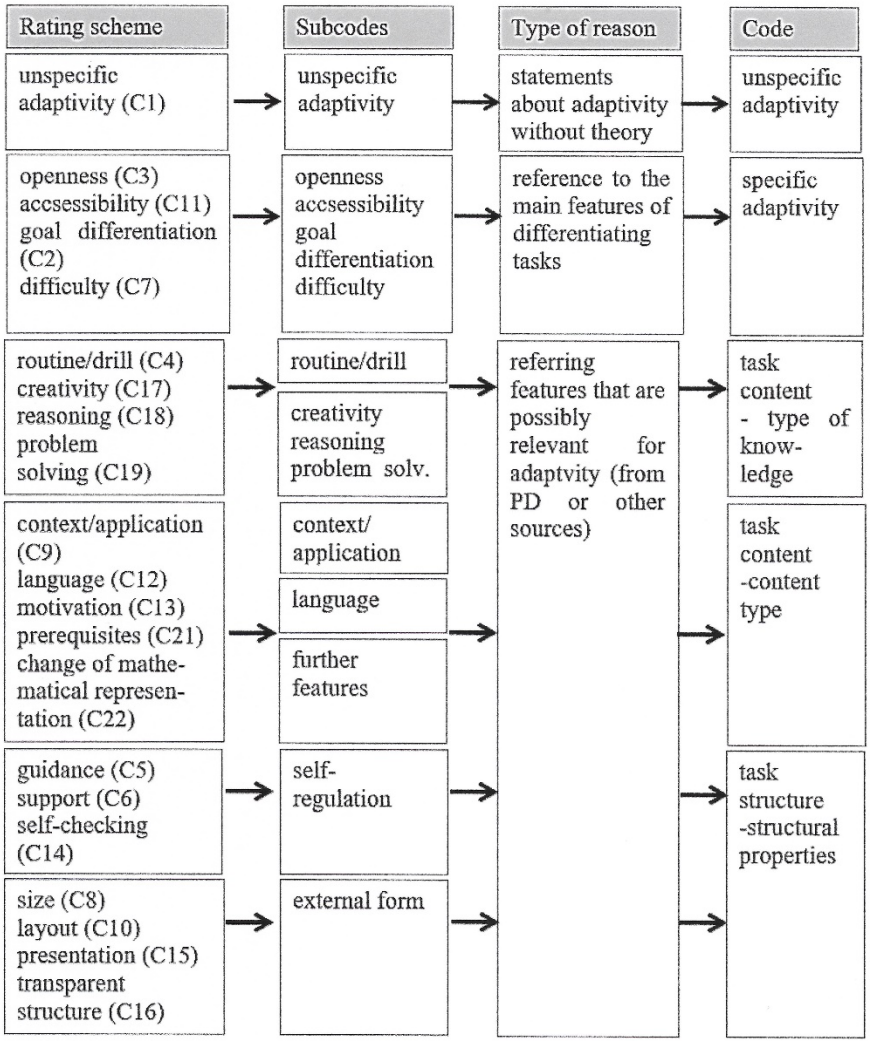

Figure 3: Condensed category system.

To cluster the teachers' answers, the 23 existing codes were condensed. In particular this was done in view of the fact that not all codes could be mentioned for task reasons

in each of the eight tasks. Figure 3 displays the results. For example, a teacher may mention a task feature eight times over all eight tasks, although this aspect is actually characteristic of only two tasks. Thus, a categorization had to be developed, which at the end considers the actual occurrence of the features over all tasks. The result is the (standing above) grid in figure 3.

For answering the third research question a dimensional analysis was conducted. Since the codes "unspecific adaptivity" and "self-regulation" were mentioned very infrequently, these codes were excluded for cluster analysis. Thus, the cluster analysis was based on the four codes "specific adaptivity", "task content (knowledge type)", "task content (content type)" and "structural properties (external features)". The frequencies of these four types of arguments per teacher were normalized, so that for all four codes a value between 0 and 1 could be ascribed to each teacher. Thus every teacher was characterized by his or her individual argumentation profile.

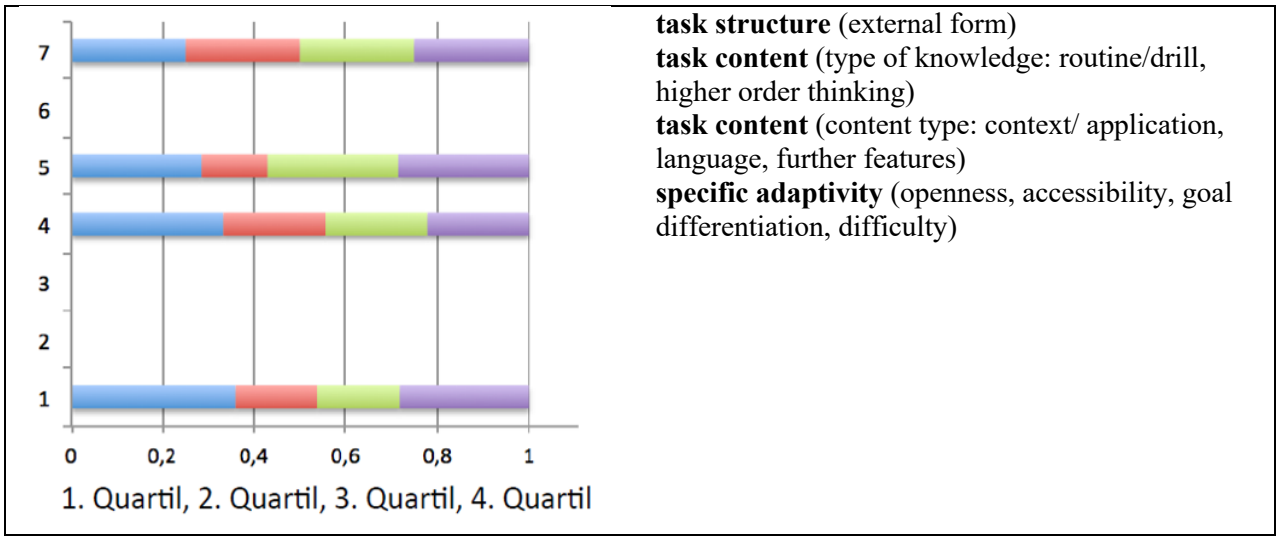

Figure 4: Distribution of the arguments mentioned by the teachers among the quartiles.

The distribution of the arguments within the whole group of teachers is shown in fig. 4. Since every quartile contains the same number of teachers, one can say that the all four argument types are mentioned seldom (1. quartile) to often (4. quartile) by a relevant part of the teachers: the arguments are well distributed over the group. But are the argument profiles of each teacher different or are there certain types within the group? This was investigated by a latent cluster analysis (LCA), in which the data was described best by a four-cluster model (see fig.5).

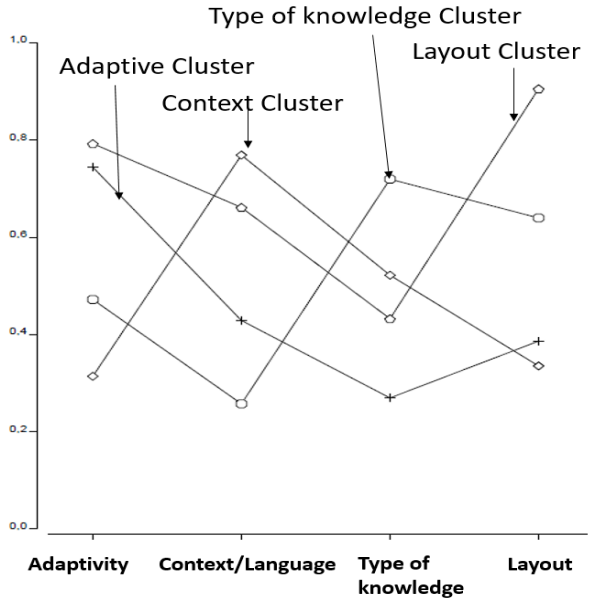

Figure 5: Profiles of the cluster analysis (cluster sizes between 20 and 35)

In the group of 78 teachers in particular, two clusters show high values for adaptivity, while one of these clusters has a high layout-value, the other is low in this aspect. The other two clusters have low adaptivity values but differ significantly in the context category. This shows, that one group of teachers values real-life tasks.

CONCLUSIONS

The results show that teachers are not always aware of different types of task with respect to their potential for differentiating. Overall, there was only little agreement between teachers and expert ratings of the differentiation potential of tasks (research question 1). Thus, we conclude that teachers often do not recognize the differentiation potential of a task and also assign tasks as a potential for differentiation that does not exist at all.

An unexpected feature of a task often referred to by teachers as justification for existing or non-existent differentiation is the “layout” feature. Teachers focus in their justifications on task features that are easy and obvious to recognize, such as “layout”, “level of difficulty” or “technical work” (research question 2). In addition, teachers with different foci on task characteristics can be identified. Here four groups of teachers can be distinguished (research question 3): A group of teachers called increasingly aspects of the code “specific adaptivity” (e.g. degree of difficulty, openness), another group from the code “task content (knowledge)” (e.g. technical work), another group from the code “task content (content type)” (e.g. language) and the fourth group from the code “task structure” (e.g. layout). The cluster with a high value for “context” (figure 5) is to be viewed critically, since learning-relevant characteristics of tasks with regard to the potential for differentiation are obviously overlooked by the teachers.

This study has several limitations: The findings may depend on the specific items. Moreover, teachers in this test evaluate the tasks with regard to their practicability in the classroom and this leads to a certain degree of confusion. For example, they focus

on the length of the text of a task or on the fact that the task (in general) is not suitable for their class. There is another limitation of the study: Of course, the 78 participating teachers were a selection of motivated colleagues, as they had registered for a series of further training courses. In this respect, the results are to be interpreted against this background.

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MENTAL COMPUTATION FLUENCY: ASSESSING FLEXIBILITY, EFFICIENCY AND ACCURACY

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In this paper, we elucidate the steps taken to develop an assessment of mental-computational fluency, with the aim to capture the key features of the construct, namely procedural flexibility, efficiency, and accuracy. Over 200 children in third and fourth grade completed the assessment. Using Rasch analyses, we investigated which items contributed to producing a measure and which did not. The findings allowed us to modify the assessment and produce a unidimensional scale for measuring mental-computational fluency with addition. We discuss the value of this assessment in terms of advancing learning theory and educational practice.

PROCEDURAL FLUENCY

As in many countries, reform in mathematics education in Australia has required greater emphasis be placed on children developing proficiencies relating to understanding, fluency, problem solving, and reasoning (Australian Curriculum Assessment and Reporting Authority, 2010). This study is focused on children's development of fluency and in particular procedural fluency, defined by the National Council of Teachers of Mathematics (NCTM) as:

> the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another (NCTM, 2014, p.1).

Embedded in this definition of procedural fluency is the expectation that children will have more than one procedure for solving mathematical problems and can switch between procedures depending on problem features, and so demonstrate procedural flexibility. Children should also be able to apply an increasing understanding of mathematical ideas to modify procedures and remove redundant steps, and so increasingly rely on efficient procedures, and make few errors when doing so (displaying accuracy).

Rittle-Johnson (2017) explained how procedural fluency develops as procedural knowledge and conceptual knowledge interact and influence each other in an iterative process through experience in solving problems. The procedures and concepts exhibited by children as they build procedural fluency with mental computation are well documented in the literature, particularly for the operations of addition and subtraction, and are outlined in the next section.

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Mental computational fluency

As children develop procedural fluency with mental addition and subtraction, they demonstrate a good understanding of the structure of numbers (place value), the positioning of numbers (number magnitude), and the meaning of the operations (Verschaffel, Greer, & De Corte, 2007). They apply a range of procedures to solve problems, which are often characterised in the literature as being one of three types: split strategies, jump strategies, and compensation strategies (Blöte, Klein, & Beishuizen, 2000; Torbeyns, Verschaffel, & Ghesquière, 2006). Both split and jump strategies make use of *standard* partitioning (i.e., with reference to place-value) and rearranging to make adding easier. Split strategies involve partitioning two or more addends [e.g., 34 + 48= (30 + 40) + (4 + 8)] and jump strategies involving partitioning one addend [e.g., 76 + 13 = (76 + 10) + 3]. Compensation strategies involve changing (adding to or subtracting from) one addend, to make use of a known fact, and then compensating for this change: [e.g., 17 + 9 = (17 - 1) + (9 + 1) or 17 + 9 = (17 + 10) - 1].

There is another type of partitioning strategy, often described in the literature relating to single-digit addition, referred to as decomposition strategies (Siegler, 1987). Decomposition strategies can be thought of as non-standard partitioning strategies, where at least one addend is partitioned to make use of a known fact: for example, a doubles fact [e.g., 25 + 26 = (25 + 25) + 1] or a tens fact [e.g., 47 + 6 = (47 + 3) + 3].

While there is considerable research in the field identifying the different types of strategies children use for mental computation with addition and subtraction, and the understandings children require in order to apply these strategies meaningfully, there are few suitable assessments for evaluating mental computation fluency. Some assessments depend on self-reports of strategy use to capture indicators of efficiency and flexibility (e.g., Beishuizen, 1993). These involve labour-intensive data collection methods and so are not suited to large-scale research or everyday classroom practice. Other assessments produce problematic results, including data with a bimodal distribution (e.g., Brown & Alibali, 2018), making them unsuitable for measurement purposes. More broadly, assessments designed to measure computational fluency have focused on speed and accuracy, and ignored key features of the construct, namely flexibility and efficiency (e.g., Calhoon, Emerson, Flores, & Houchins, 2007; Foster, 2018).

Baird, Andrich, Hopfenbeck, and Stobart (2017) point out the huge impact educational assessments can have on teaching and learning when they function as a communicative device, making explicit to educators what represents quality learning. These authors explicate that the notion of quality (including how proficiency of a construct is operationalised) will depend on four spheres of influence, which are theory based, empirically driven, expert devised, and policy driven. They argue that the washback effects of assessments can serve the goals of education particularly well, when learning theory and assessments (aligned with assessment theory) are strongly associated and have reciprocal effects upon each other. Relating specifically to procedural fluency,

Rittle-Johnston (2017) also expressed the need for researchers to develop and validate measurement tools to build theories of mathematical development, which are more comprehensive than those that currently exist.

In the next section, we explain the steps we took to construct an assessment designed to measure mental computational fluency with addition (MCF-A). We outline findings from an analysis of responses from over 200 children to identify items that contribute to producing a scaled measure and an investigation of items that did not. It is worth mentioning that previously we pilot-tested an assessment that included both addition and subtraction items. We found children struggled with the format of the assessment and switching between operations, and so included only addition items in this version of the assessment.

ASSESSMENT DESIGN

To construct an assessment for measuring mental computational fluency with addition (MCF-A), we used a novel format for items that required students to explain a strategy used by Emma (a fictitious student). Instructions to children were:

> Emma is good at adding numbers. She uses clever strategies to make adding easier. Your job is to try and think like Emma. Explain what Emma did to get the number in the box. For example, to solve 3 + 4, Emma thinks, that is the same as $\boxed{6}$ + 1. What numbers did Emma add together to get 6? (Answer: 3 + 3).

According to Rasch Measurement Theory (Andrich, 2016), a critical component in designing assessments to measure a construct is to generate items that range from being very easy to get right (or endorse), to items that are very difficult. Furthermore, item difficulty must be invariant regardless of who attempts these items, provided they are drawn from the population for which the assessment was designed. Items that adhere to this principle produce scores that display a Guttman structure, which is ideally needed before scores from items can be added together to form a measure. The Rasch model represents a probabilistic Guttman structure and can be compared with data from an assessment to evaluate the extent to which items on the assessment provide scores that adhere to this structure.

While items for assessing a construct on either end of a continuum are often easy to write, it is harder to write items that operate at regular intervals along the continuum. To create items for assessing mental computation fluency that function along a continuum of difficulty, we included items with single-, double- and triple- digit addends, and items with two or three addends. Since we used items with three addends, it was appropriate to include a new type of procedure, one not previously mentioned in the literature, which we labelled *noticing strategies*. Noticing strategies do not involve partitioning or changing an addend, only changing the order of addends to make adding easier. These involve combining addends that have a particular association or relationship, such as tens facts or doubles [e.g., 3 + 5 +7 = (3 + 7) + 5].

We hypothesised three factors would influence item difficulty: (i) the size of the addends, (ii) the number of addends, and (iii) the type of strategy represented.

Predicting item difficulty based of addend features was straightforward; however, previous studies have produced mixed findings in relation to which strategy is more difficult, possibly due to differences in national curricula (Verschaffel et al., 2006). We reasoned that strategies that can be used to solve both single-digit and multi-digit problems (i.e., noticing and non-standard partitioning strategies), would be easier than strategies used to solve only multi-digit problems (e.g., standard partitioning strategies), as children would have opportunities to practice these at a younger age. We used a grid (see Table 1) to help construct items that differed in difficulty. Table 1 is organised such that items differ in terms of addend features (organised in rows) and strategy type (organised in columns). The item numbers represent the predicted order of difficulty and the order of presentation in the assessment.

Noticing	Non-standard partitioning	Standard partitioning	Compensating
	1. 4+5=$\boxed{8}$+1		3. 9+3=$\boxed{13}$-1
	2. 4+7=$\boxed{10}$+1		
4. 5+8+5=$\boxed{10}$+8			
5. 6+3+6=$\boxed{12}$+3			
6. 8+3+2=$\boxed{10}$+3			
			7. 8+19=$\boxed{28}$-1
			8. 4+28=$\boxed{30}$+2
	9. 10+11=$\boxed{20}$+1	10. 28+13=$\boxed{38}$+3	
		11. 21+26=$\boxed{40}$+7	
12. 11+3+9=$\boxed{20}$+3			
13. 6+18+34=$\boxed{40}$+18			
	14. 48+45=$\boxed{90}$+3	17. 76+21=$\boxed{96}$+1	16. 44+49=$\boxed{94}$-1
	15. 36+37=$\boxed{72}$+1	18. 56+33=$\boxed{80}$+9	19. 22+49=$\boxed{21}$+50
		20. 23+44=$\boxed{64}$+3	
21. 67+45+43=$\boxed{110}$+45		22. 45+67+82=$\boxed{180}$+14	
	23. 235+238=$\boxed{470}$+3	25. 955+445=$\boxed{1000}$+400	24. 456+356=$\boxed{800}$+12

Table 1: The assessment of MCF-A comprising 25 items.

METHODS

To investigate how the assessment of MCF-A functioned, we asked 203 children in Years 3 and 4 to complete it. Using convenience sampling, participants were selected from three metropolitan public primary schools in Melbourne, Australia. These schools

served a range of demographics with one school community relatively advantaged, one school relatively disadvantaged, and one school community similar to the national average, based on indicators of socioeconomic status. Participation rates in each school were 53%, 33% and 71% respectively. The researchers (authors) and a research assistant administered the assessment, which children individually completed during a 60-minute mathematics class. Most children finished the assessment after 30 minutes. Data were analysed using RUMM software (Andrich, Sheridan, Lyne, & Luo, 2000; version 2030) to evaluate how well the observed data fitted expectations of the Rasch model.

RESULTS

An initial analysis of the data indicated the assessment had targeted the population well, with the exception that 12 participants did not score on the assessment. Person-ability scores ranged from 4.6 to 3.0 logits (M = 0.8, SD = 1.9) and the item-difficulty scores ranged from -2.9 to 0.4 logits (M = 0.0, SD = 1.4). The total item-trait statistic (chi-squared = 321.93, df = 50, p = 0.000) indicated some modifications to the scale were needed. Ideally, the probability statistic should be greater than 0.05 (without Bonferroni adjustment).

Noticing	Non-standard partitioning	Standard partitioning	Compensating
	1. 4+5=$\boxed{8}$+1		3. 9+3=$\boxed{13}$-1
5. 6+3+6=$\boxed{12}$+3			
			7. 8+19=$\boxed{28}$-1
			8. 4+28=$\boxed{30}$+2
		11. 21+26=$\boxed{40}$+7	
		17. 76+21=$\boxed{96}$+1	
21. 67+45+43=$\boxed{110}$+45	23. 235+238=$\boxed{470}$+3		

Table 2: Misfitting items on the MCF-A

The poor functioning of item 3 and item 7, which were predicted to be the easiest for use with a compensating strategy, are noteworthy given these items were considerably more difficult than expected. Instead of being ranked 3rd and 7th in difficulty as predicted, they were ranked 15th and 18th respectively (based on the initial analysis of data, before items were removed). While it might be that recognising use of a compensation strategy was more difficult than expected, this point is not clear given these items did not fit the Rasch model. It could be that items solved using a compensating strategy are more sensitive in picking up differences in teaching approaches. In other words, a compensatory strategy may be emphasised in some

classes or schools and not in others. Alternatively, it could be that these items were not well constructed.

Item response curves for Item 3 and Item 7 indicating that at least some children were guessing the answer, suggesting a problem with how the items were constructed. We reasoned that Item 3 (9 + 3 = $\boxed{13}$ - 1) could actually be efficiently solved using a counting-on strategy (where children count on three). A better item might be 9 + 7 = $\boxed{17}$ – 1, as a compensatory strategy is clearly more efficient for this problem than a counting-on strategy. We also noticed that Item 7 (8 + 19 = $\boxed{28}$ – 1) might have been made more difficult (unintentionally) due to the ordering of the addends. A better item would be 19 + 8 = $\boxed{28}$ – 1.

The inconsistent ordering of the addends might also help explain why responses to Item 8 (4 + 28 = $\boxed{30}$ + 2) did not fit the Rasch model. The item's characteristic curve indicated it was over discriminating. This item might have distracted students of lower fluency ability and be less confusing for some students if it was written as 28 + 4 = $\boxed{30}$ + 2 or 4 + 28 = 2 + $\boxed{30}$.

Item characteristic curves for Items 1 and 11 indicated both items were discriminating. We reasoned that Item 1 (4 + 5 = $\boxed{8}$ + 1) did not work well for students of higher fluency ability because some students might just know (i.e., can retrieve) the answer of nine; hence, the item was not a good example of a problem efficiently solved using a non-standard partitioning strategy. Similarly, Item 11 (21 + 26 = $\boxed{40}$+7) did not work well because some students might just know the answer. We examined the item character curves for the other ill-fitting items but possible explanations for misfit were less forthcoming.

After removing the nine ill-fitting items, the test-of-fit summary statistics for the modified assessment (with 16 items) produced scores that did not deviate from perfection imposed by a Guttman structure (chi-squared = 321.93, $df = 50$, $p = 0.031$). (Note, the probability statistic needed to be greater than 0.003 with Bonferroni adjustment.) All items produced scores fitting the Rach model and there was no evidence of differential item functioning according to school. As all items were dichotomous (marked either correct or incorrect), the ordering of thresholds did not have to be checked. Furthermore, a principal components analysis of residuals indicated the scale to be unidimensional. While the results are promising (illustrated in Figure 1), further work to the assessment is needed as 27 children did not score. The power of analysis of fit was good but not excellent.

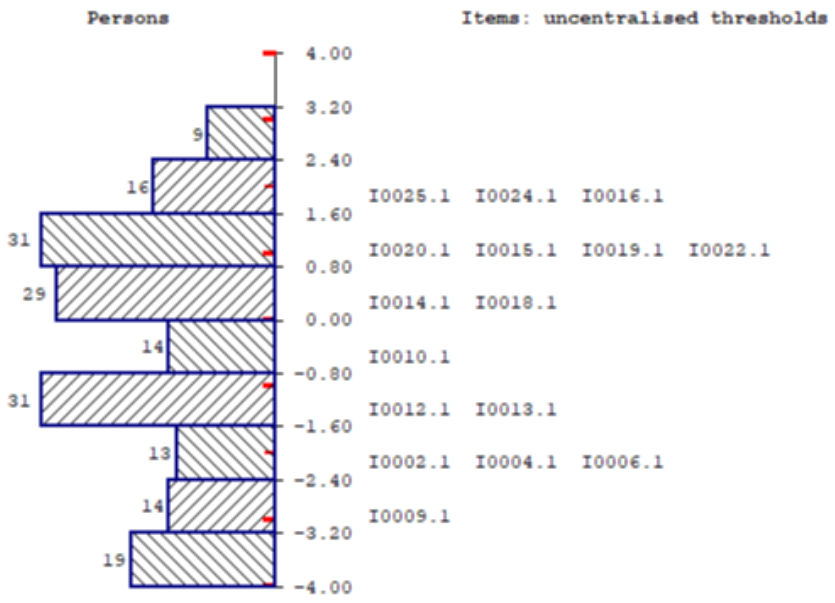

Figure 1: Item map illustrating person scores and item scores (representing item difficulty) in logits, using the modified assessment of MCF-A.

Children who found it difficult to score would have benefitted from more items that were easier and possibly from clearer instructions.

DISCUSSION

Applying Rasch measurement theory, we designed an assessment to measure computational fluency with mental addition for children in Years 3 and 4. Testing the assessment to see if it produced scores that were consistent with the Rasch model provided much useful information. The analyses identified problematic items and illuminated possible reasons for why misfit occurred. After removing these items, we found the assessment targeted the population fairly well and produced a scaled and unidimensional measure of a mental computational fluency.

We contend that this assessment of MCF-A has great potential for communicating what students need to know and be able to do. The novel format of items operationalises important aspects of procedural fluency as defined by the NCTM (2014). The MCF-A requires flexibility, as children need to generate a range of potential solution procedures when considering the strategy Emma had used. It requires knowledge of efficient procedures because the intermediate step used by Emma represents an efficient means of solving the problem. The child also has to accurately recall or calculate the appropriate number fact corresponding to the intermediate step arrived at by Emma. In this sense, the assessment addresses discipline-expert devised criteria of quality learning. It also builds on a substantial body of empirical findings revealing the different types of mental strategies children use to add together multi-digit numbers. Furthermore, we believe the assessment has much potential for improving educational practice and contributing to learning theory. It meets the criteria articulated by Baird et al. (2017) and the need expressed by Rittle-Johnston (2017) for better tools to build

comprehensive, integrative theory of how fluency develops. As the removed items represented key strategy types, they were theoretically important and so need to be rewritten. The next version of the assessment is currently undergoing testing.

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CULTURALLY DIVERSE STUDENTS' PERCEPTIONS OF MATHEMATICS IN A CHANGING CLASSROOM CONTEXT

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In classroom contexts where professional learning and development focused on ambitious teaching is undertaken, there are new expectations and roles for students. The aim of this study was to explore culturally diverse students' perceptions of mathematics within a changing classroom context. Data are drawn from four classes of Year 5 - 8 students (9-12 year-olds), including an open-ended questionnaire item and small group interviews. Results indicate that as changes were introduced into the classroom, there were shifts in the student perceptions related to their vision of mathematics, what it meant to be good at maths, and how to improve in maths.

INTRODUCTION

In many Western countries, an ongoing challenge for researchers and educators over the past decade has been persistent inequities in mathematics teaching and learning for particular groups of learners. Growing awareness of the gatekeeping role of mathematics on students' future education, employment and life choices (Hunter & Hunter, 2018; Tate, 2013) has resulted in a need to investigate and address issues related to equity within mathematics education. Within the New Zealand context, there is a changing student population that is increasingly culturally diverse. This includes indigenous Māori students, as well as the largest group of Pāsifika students in the Western world. Students of a Pāsifika background are not from a single ethnicity, nationality, language or culture but are a diverse group including those born in New Zealand who identify themselves with the Pacific Islands culture and language and those who have migrated to New Zealand from the Pacific Island nations (Coxon, Anae, Mara, Wendt-Samu, & Finau, 2002). Frequently, the cultural background of both Māori and Pāsifika students has been perceived as a deficit within the schooling system (Hunter & Hunter, 2018; Turner, Rubie-Davis, & Webber, 2015). At the same time, mathematics teaching and learning practices in classrooms have perpetuated racial disparities with frequent use of ability grouping, in which Māori and Pāsifika students are over-represented in "low-ability" groups and related repetitive and procedural teaching (Anthony & Hunter, 2017; Turner et al., 2015).

Both the widespread use of ability grouping and associated procedural teaching in New Zealand mathematics classrooms has been challenged by *Developing Mathematical Inquiry Communities* (DMIC), a whole-school formative professional learning and development (PLD) research project. The DMIC project is a transformative re-invention of pedagogical practices designed to support teachers' development of ambitious mathematics pedagogy (Kazemi, Franke, & Lampert, 2009) and culturally responsive teaching (Gay, 2010). Implemented in schools that serve disadvantaged

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communities, key components include the use of teacher designed group-worthy, culturally appropriate tasks; instructional practices that support respectful social interactions; and the development of a range of mathematical practices including providing mathematical explanations, mathematical argumentation, and justification. A number of previous studies (e.g., Kazemi et al., 2009; Hunter & Hunter, 2018) have focused on the role of the teacher in shifting pedagogy towards ambitious practice, fewer studies focus on student perceptions in changing classroom contexts. For students inducted into reformed classroom communities, there are shifts in their role as a learner. Students are required to engage in ways of learning that privilege different forms of knowledge and participation (Hodge, 2008; Hunter, 2016; Pratt, 2006).

While the teacher takes an important role in reconstructing their pedagogical practices to align with ambitious mathematics pedagogy, it is also important to acknowledge the student role within the classroom. A number of studies (e.g., Clarke & Roche, 2018; Esmonde, Brodie, Dookie, & Takeuchi, 2009; Hodge, 2008; Larkin & Jorgensen, 2015; McDonough & Sullivan, 2014; Moyer, Robison, & Cai, 2018) have recognised and advocated for the need to consider student perspectives when researching mathematics classrooms. Some of these studies have used video diaries or photo-elicitation (Hunter, 2016; Larkin & Jorgensen, 2015) or creative interviewing techniques (McDonough & Sullivan, 2014) to investigate student views and beliefs related to mathematics teaching and learning. Other studies (e.g., Hodge, 2008; Moyer et al., 2018) report on perceptions and beliefs of students who have experienced different forms of mathematics teaching. For example, Hodge (2008) examined younger students' perceptions in regards to their role in the mathematics classroom. The researcher interviewed students after they had spent the first year in a classroom with reform based instruction methods and then again after they had moved to more traditional orientated instruction in the second year. The students gave markedly different responses in regards to what it meant to be a good mathematics student in each class. Similarly, Moyer and colleagues interviewed high school students who had been taught using different types of mathematics curricula in middle school, reform or traditional. The researchers found no difference in emotional disposition or perceived competence between the groups of students. However, there was a significant different in the students' vision of mathematics and beliefs related to how they thought they best learnt mathematics. Students who had experienced a traditional curriculum in middle school showed a stronger preference for individual work and instrumental instruction.

A further set of studies have investigated student perceptions of specific aspects of mathematics lessons such as the use of contextual tasks (Clarke & Roche, 2018) or cooperative groupwork (Esmonde et al., 2009). For example, Clarke and Roche (2018) asked students to provide feedback on three different types of tasks in regards to enjoyment, learning and difficulty. Esmonde et al. (2009) interviewed students about their beliefs related to the efficacy of group work and positive and negative group interactions. An over-arching theme of these studies is that students are able to both reflect upon and analyse their experiences of classroom practices and discuss their

perceptions of mathematics and teaching and learning. The focus in this paper is an exploration of diverse students' perspectives over a school year related to learning mathematics in a changing classroom context. Specifically, we investigate students' initial views of mathematics, what it means to be successful at mathematics, and how mathematical performance can be improved. Within this paper, the initial data is contrasted with student perspectives at the end of a school year after their teachers had participated in the DMIC professional learning and development project.

The theoretical framing of this paper is based within a socio-cultural perspective. In this view, individuals participate in the everyday activities within a classroom community of practice (Lave & Wenger, 1991) and thereby learn the ways of thinking and acting valued by the community. Through participation, a sense of what it means to be a member of the community and a sense of self in relation to the community is developed. For different students there can be a greater or lesser sense of belonging to the community. This is related to how students come to understand what it means to do mathematics in the classroom and to what extent they identify with this (Boaler, Wiliam, & Zevenbergen, 2000).

RESEARCH DESIGN

This research reports on a case study of four classrooms at one school and is part of a larger study focusing on the implementation of DMIC and student and teacher perspectives. It was conducted with Year Five to Eight students (aged 9 – 12 years old) from four classrooms at a low socio-economic, high poverty, urban school in New Zealand. Ninety-one students completed the written questionnaire for the study. Forty-four students also agreed to participate in a small group interview including 22 male and 22 female students. The students were predominantly of Pāsifika descent with other students from an indigenous New Zealand Māori background and a small number from South East Asia and India.

To explore the students' perspectives both a written questionnaire and small group interviews were undertaken. The questionnaire consisted of two open-ended questions and 19 likert scale statements and was undertaken in February and November. The interview was conducted in small groups of two or three students with 14 questions. Due to the ethics procedure, the initial interviews were undertaken later in the year in May after the DMIC professional development had begun. The final interviews were undertaken in November. These included questions focused on their classroom experiences of learning mathematics both in the past and present, their disposition towards mathematics, perceptions of how they learnt mathematics or succeeded in mathematics, the teachers' role within mathematics lessons, and their perceptions of participating in mathematical practices in the classroom. Interviews were audio-recorded and wholly transcribed. The results reported in this paper draw on one of the open-ended written responses (*What is mathematics?*) from the questionnaire and the interview data for the following questions: 1) What makes someone good at maths? 2) What would you need to do to get better at maths?

Data analysis used a grounded approach in which codes, categories, patterns, and themes were developed. A coding system was utilised to determine how to examine, cluster, and integrate the emerging themes (Creswell, 2008). Researchers had regular meetings to identify and discuss the emerging themes. The analytic tool, NVivo was utilised to code the data and student responses to the interview questions could be coded in multiple nodes. To ensure reliability of the coding, two members of the research coded each of the interview data independently and then crosschecked the analysis. For cases where there were contradictions, the researchers discussed these differences until a consensus was reached. Insights gained from the students' surveys and interviews are presented in the following sections.

RESULTS

Each section in the results will provide an overview of the themes identified from the initial and final student responses to the specific questions.

What is mathematics?

The open-ended survey question asked students to write a description of their perceptions of mathematics. In the initial phase, student responses mirrored a traditional mathematics classroom with a strong focus on number.

Most students gave responses that indicated their belief that mathematics was synonymous with number and calculations. Although the open-ended nature of the question allowed students to draw on a wider definition of mathematics, 51% of the responses solely referenced number, timestables, or operations (addition, subtraction, multiplication, division). A smaller group of student responses (28%) included problem-solving, typically amongst a list of words associated with number. For example, one student wrote: "timestables, dividing, plus, take-away, problem-solving, equals". A small group of students (4%) included other mathematical topics (e.g., measurement or geometry). Other responses included thinking (8%) and brainy (7%).

Responses to the open-ended survey question at the final phase of data collection were noticeably different. While number was still a prevalent response, with 28% of student responses only referencing number or calculations, the large majority of students offered additional descriptions of mathematics.

Most commonly, students mentioned the words learning and problem-solving. References to learning were made within two different contexts. The most frequent response (33%) was to refer to learning mathematics as participatory practices of working together, sharing ideas, and asking questions. For example, one student wrote:

> solving of problems, friendly arguing, sharing lots of ideas and your smart thinking, working as a team/family together, participating and contributing, learning ratios, fractions, division, adding, multiplication, subtraction, decimal numbers.

Other student responses (12%) viewed mathematics as a way of learning and developing: "maths is another way to learn, it's like a language with numbers that I think everyone should know". Alternatively, another student responded: "learning tool,

powers up your brain". Student responses (15%) also commonly referred to problem-solving as well as topic aspects of mathematics. Interestingly, a small group of students (9%) described mathematics with reference to current and future utility: "an activity that can led you to your future education and job".

What makes someone good at maths?

At the beginning of the study, the most common response (20%) that students attributed to making someone good at maths was listening. This included both listening to the teacher and to their peers: "listen to my teacher, listen to other people if they're sharing". These responses generally indicated a passive view of listening, with only one student providing a response that indicated they viewed listening for understanding as important: "listening and asking questions". A smaller group of student responses (11%) identified sharing mathematical ideas and explanations as what made someone good at maths. Generally, these were short responses that did not detail why or how this made someone good at maths: "telling people, helping, sharing, explaining". Other student responses (11%) noted that working productively with others made someone good at maths: "working together being as a family". Finally, a small number of student responses (9%) noted the importance of taking a risk.

In contrast, at the end of the year, a shift was evident with student responses (23%) putting greater emphasis on sharing mathematical ideas and explaining their thinking. Interestingly, the responses indicated a responsibility on the behalf of the person explaining to ensure others understood: "making sure everyone understands what you are saying and doing". Similarly, another student stated: "sharing ideas and ask if they have got any questions". Students continued to privilege listening as making someone good at maths (20%) and again this largely indicated a passive view, however at this point, three students provided responses that indicated listening for understanding. A larger number of student responses (18%) emphasised being able to work collaboratively. In the final interviews, three new themes were noted with two of these related to mathematical disposition. A number of student responses (14%) referred to effort: "if you put in this much effort (indicates wide with hands), you get this much in return (indicates wider)". Also related to disposition, student responses (9%) noted that someone good at mathematics was willing to take a risk, make mistakes and keep persevering with confidence: "someone who is focused, has made mistakes, confident, shy a little bit, a person who can stand up to what they think. Anyone can be good at maths, you just have to believe". Finally, a group of student responses (7%) noted mathematical practices as components of being good at maths: "challenging them [peers] and asking questions" or: "we need to respect each other even if we disagree with an idea, we can't say it in a bad way".

What would you need to do to get better at maths?

In the initial interviews, students most commonly stated that to get better at maths you needed to ask questions (18%) and similar to the earlier section to listen to both peers and the teacher (18%). Mirroring the previous responses, most of the statements related

to listening implied a passive listening role with only one response indicating listening for understanding. A number of students (14%) also noted their belief that to get better in maths, it was important to study and practice at home: "practice at home my timestables". A small group of student responses noted aspects related to disposition including effort and willingness to take a risk.

A notable change in the final interviews were the increasing range of descriptions and attributes in the student responses related to how an individual could get better at maths. The number of student responses that valued listening to improve in maths increased slightly (23%), with this increase related to an increased number of students indicating listening for understanding: "listen to other people doing their strategy so I can use their strategy to help with my learning". The number of student responses that indicated passive listening remained the same. Other student responses (20%) identified providing explanations and revoicing their peers' explanations as a means of getting better at maths. For example, one student response was: "explaining my ideas to my group, each and every detail so they can all understand". Another common student response (14%) noted the importance of effort: "try really hard and give it what I've got". In the final interview, a much smaller number of student responses identified asking questions (7%) or practice at home (7%).

DISCUSSION AND CONCLUSION

Clearly, the classroom environment has a key influence on students' views of mathematics and their beliefs about success in relationship to mathematics. Similar to the findings of Moyer and colleagues (2018), the findings of the current study show shifts in these students' vision of mathematics. Initially, many of the student responses indicated that they viewed mathematics as solely related to number and calculation. Notably, towards the end of the first year of change within their classrooms, student responses indicated a wider perception of mathematics with an emphasis on participatory practices and also a view of mathematics as a way of learning and developing. A key difference in the current study from previous studies is that these students had the same teachers within a school where PLD was being undertaken rather than experiencing different curricula (e.g., Moyer et al., 2018) or different teaching styles from two different teachers (Hodge, 2008).

Similarly, shifts were noted in student perspectives of what it means to be good at mathematics and how to improve in mathematics. The perception of the importance of listening to others was maintained throughout the year, although some small shifts were noted in relation to the type of listening, passive or listening for understanding. The initial privileging of listening over explaining as a form of meaning-making is similar to the findings of Pratt (2008) with students in the United Kingdom. In the current study, although an emphasis on listening was maintained, key shifts included widening definitions of what it meant to be good at mathematics and ways in which to improve mathematical performance. The shift in student responses paralleled ambitious teaching practices advocated in the DMIC PLD such as facilitating student questioning, providing challenging mathematical tasks and the introduction of key mathematical

practices such as explanations, justification, and mathematical argumentation. Also of interest in the student responses were links to their cultural background and key values, for example, respect (e.g. behaving in a respectful manner when disagreeing) and reciprocity and collaboration (e.g. ensuring others understood their explanations and working as a family).

Similar to previous research studies (e.g. Hodge, 2008; Larkin & Jorgensen, 2015; Moyer et al., 2018), the current study highlights that students are able to reflect on their visions of mathematics and what makes someone successful in this area. Future research may investigate more closely the relationship between these diverse students' cultural backgrounds and their perspectives. It will also be beneficial to analyse the student responses in conjunction to the teacher actions within the classroom.

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WHAT IT MEANS TO DO MATHEMATICS: THE DISCURSIVE CONSTRUCTION OF IDENTITIES IN THE MATHEMATICS CLASSROOM

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This paper explores how teachers and students discursively construct mathematical activity as they interact in mathematics classrooms, and the consequences these have on the identities of mathematics teachers and mathematics students. Using a conversation analytic approach we examine how these identities are discursively constituted through the social actions that are performed in the interaction. We offer two contrasting examples that both involve students calculating values using known procedures, but where the identities of mathematics teacher or student are constructed differently.

INTRODUCTION

Language is constitutive; our choice of words that we use to describe concepts or tasks and activities creates the meaning that these have for our students. The ways that teachers and students describe the activities of mathematics therefore constitute the nature of the mathematical activity, which in turn influences what it means to be a teacher or a student of mathematics. In this paper we examine how the way that teachers introduce and begin to work on tasks affects the construction of these identities. Whilst there has been a great deal of research into identity in the mathematics classroom (e.g. Lerman, 2011) this has largely focused on researcher's categorisations of identity types. In this paper we take an ethnomethodological approach to examining identity that takes identity as something that is discursively constructed in interaction (similar to the approaches taken by Wood (2013) and Cobb & Hodge (2011)).

THEORETICAL BACKGROUND

An ethnomethodologically based analysis of identity, such as the conversation analytic (CA) one used in this paper, is based on a turn-by-turn analysis of the identities that the participants in an interaction orient to. It is the identities that the participants themselves draw upon, both directly and indirectly, that are of interest rather than researcher identified identities such as teacher or student (Schegloff, 1997). Having said this, that the participants are orienting to these identities of teacher and student is often apparent through the structure of turn-taking and the actions performed within these turns (Ingram, 2012), as they are in the extracts below. The teacher usually controls who can speak when, generally controls the topic and asks the questions (that they already know the answer to), whereas students answer these questions (Mehan, 1979).

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Ethnomethodological approaches do not enter into the debate about the ontology of identity, or any analysis about the relationship between identity and what participants think or feel. Instead the analysis focuses on how participants display and orient to specific identities and the consequences this has on the interactional activities (Benwell & Stokoe, 2006). This conceptualisation of identity means that identities are indexical to the interaction in which they are oriented to, and are also dynamic and fluid; adapting, altering and developing in interaction.

Using a conversation analytic approach, Zimmerman (1998) distinguishes between three different types of identity that are oriented to in interactions: discourse identities, situated identities and transportable identities. It is the discourse identities that we focus on in this paper, and these are the identities oriented to by participants in a turn-by-turn basis such as speaker, listener, questioner or answerer. These discourse identities are often reflexive and have consequences for the identities of the other participants. For example, by assuming the discourse identity of speaker, the other participants are required to assume the discourse identity of listener (though they may not actually do this). Situated identities relate to the context in which the interaction is occurring, such as teacher and student where the interaction is occurring in a classroom. Transportable identities are carried by participants across different interactions in different contexts, such as female, mother, but are not necessarily oriented to within an interaction.

An analysis using CA focuses on the social actions being performed by turns within the sequential context in which they occur. The design of these turns, including the choice of words, constitutes the nature of mathematical activity that occurs in the interaction. In this paper we examine two transcripts of whole class interactions to show how the participants' identities of mathematics teacher and mathematics student are discursively constituted in the interaction, and how these identities constrain what it means to do mathematics.

RESEARCH METHOD

This paper uses data from a larger study of interactions in secondary mathematics classrooms (students aged 11-18) in the UK. The extracts are taken from two different lessons, one taught by Tim and the other taught by Todd (all names are pseudonyms). The lessons were video recorded and then all whole-class interactions were transcribed using Jefferson transcription notation (Sidnell, 2010) but there was no intervention or direction given to the teachers about how or what to teach, so the lessons are to some extent *naturally occurring* (Ingram & Elliott, 2019). The transcriptions have been simplified in this paper for ease of reading.

A conversation analytic approach has then been used to develop a detailed description of how the identities of mathematics teacher and mathematics student are constituted in interaction and the nature of these identities in terms of the mathematical activities they involve.

THE DISCURSIVE CONSTRUCTION OF THE IDENTITIES OF MATHEMATICS TEACHER AND STUDENT

In this section we compare and contrast the introduction of the first task of a lesson, showing how each teacher discursively constructs the nature of activity and as the interaction develops how the identities of mathematics teacher and student are reflexively constituted. The first three extracts are from Tim's lesson on limits of sequences, whilst the last three are from Todd's lesson on frequency tables.

18	Tim:	okay (0.6) your fir:st thing today I've put a problem on the board, I will
19		have a problem on the board in about (0.3) 30 seconds, okay I want you to
20		look at that (.) first question is quite an easy one, the second question we
21		have to need to think about in terms of (.) what it actually means, (1.3) okay
22		and I want you to try your best and try and understand (.) how far you can
23		get it done, okay. here is your problem. have a go at this (.) I've just inherited
24		twelve thousand pounds, (0.4) okay and being the generous man that I am
25		I want to donate (.) some of that to charity (.) but because I'm not totally
26		generous, (1.2) okay I'm going to donate one quarter of the twelve thousand
27		pounds, then the following week I want to donate a quarter of that amount,
28		following week a quarter of that amount okay how much will I donate in
29		each of the first four weeks, the first few are obviously easy. how much will
30		you donate in total okay

Extract 1: Tim introducing the first task.

Tim constructs the activity in his lesson as solving problems that involve 'thinking' (line 21), 'understanding' (line 22) and 'having a go' (line 22). He begins by asking his students to 'look' (line 19) at the problem. The problem is given using words, but Tim has used the word 'look' rather than 'read'. Consequently, Tim is asking his students to go beyond reading the question, and to also think about the problem. Furthermore, Tim describes the second part of the question as something that "we have to need to think about in terms of what it actually means" (lines 20-1). The choice of the pronoun 'we' here aligns Tim with his students in the role of problem solvers, referring to the generic processes that a problem-solver goes through when they encounter a problem; the processes of thinking about the problem, working out what it means as well as what to do as the first stages of solving a problem (Mason et al., 2010). Thus the identity of mathematics teacher includes problem posing, and the identity of mathematics student includes problem solving, thinking, trying, understanding, and having a go.

33	Tim:	wee:k two:, how much am I donating if I'm donating a quarter of that.
34		(.) Sam?
35	Sam:	seven point s-, seven (.) point five, no seven hundred and fifty
36		(1.5)

37 Tim: ok a quarter of that, seven hundred and fifty points. ok. I want you to try
38 and work out (0.8) the next (0.5) two weeks, and then I want you to think
39 about (0.3) how much are you going to end up donating in total.

Extract 2: Tim asking how much is donated in week 2.

In Extract 2, Tim repeats Sam's answer of seven hundred and fifty (line 37), consequently showing that this answer is appropriate and correct. Tim also adds the units 'pounds' to the end of this answer. This revoicing is usually discussed in the literature as a device that teachers use to encourage students to give more complete or mathematical answers, but here it also focuses attention on the specific problem of donating money, rather than a generic calculation of a quarter of three thousand. Here the situational identities of teacher and student are being oriented to by who is asking the question, evaluating the answers, and setting the tasks of what to do next, and who is answering the question.

58 Tim: some of you used calculators, some of you didn't. ok that's good. I don't
59 mind either way. (1.7) I want you thinking about it. ok. the values you got
60 for the first three weeks were three thousand, (1.0) seven hundred and fifty,
61 (.) one eighty seven fifty and forty six eighty eight?
62 [yep,] if you round it.
63 Simon: [forty seven]
64 Tim: ok (0.7) what I was wanting to think about is what (0.3) is actually
65 happening. some of us talked about when do you s:top, do you stop.
66 Steve: nope
67 Tim: why not. hands up. (1.7) why not. Seb?
68 Seb: because the number: 9(.) keeps getting smaller, cus of ((inaudible))
69 Tim: so it keeps getting smaller
70 Seb: yep
71 Tim: but will there be a point where we actually s:top?
72 SS: yes
73 Tim: why

Extract 3: Tim asking do you stop.

In Extract 3, line 58 Tim refers to his students using calculators. By mentioning this he is making it relevant (in a CA sense) to the interaction. In line 59 he then emphasises that he wants the students "thinking about it" before listing the values for the earlier calculations, reinforcing the earlier constitution of mathematics students as thinkers. Tim contrasts the performance of and the answers to the calculations with what he wants his students to do, thinking. There is no discussion of the how the values were calculated and he stated that he did not mind whether students used calculators. The answers to the calculations are listed by Tim without comment and without inviting the students to offer them. Calculations are part of the activity, but attention is on their use

in solving the problem rather than the calculations themselves, contrasting 'working it out' (line 38) and 'thinking' (line 59). Thus both the identities of teacher and student involve the action of calculating, but it is the activity of thinking that is oriented to as something that students should do.

The emphasis so far on thinking about (to use Tim's words) what is going on leads to Tim's question in line 65 of 'do you stop'. The introduction of the question of whether you stop references earlier conversations ("some of us talked about" in line 65). By mentioning these discussions Tim is making them relevant to the current interaction and indicates that the question is something that is worthwhile discussing; the answer needs to be thought about. Discussing is something teachers and students do together. Consequently, Tim is suggesting that the answer of whether you stop or not is not immediately obvious. This is developed further in line 67 where Tim asks "why not" twice, before re-asking the question in line 71 and asking why in line 73, thus asking his students to offer explanations for each alternate answer. In summary, in the introduction of this problem, Tim has discursively constructed his students as people who think and find meaning, as well as people who participate in mathematical activities such as specialising, discussing, explaining, arguing and justifying.

The next three extracts come from one of Todd's lessons and illustrate the differences in what it means to be a mathematics student.

276 Todd: sometimes in the exam they won't give you that extra column they'll just
277 give you these two, and they'll expect you to know (.) that it might be useful
278 (.) to put this extra column on, do you know what I mean. And in a minute,
279 when you do some practice from the text book it's the same thing. They
280 just give you this bit of the table and they expect you to use your initiative
281 (.) to draw in the extra column to do it. ok. well let's go through these then,
282 the mode, the median, the mean and the range. I think we'll leave the mean
283 till last because it's a bit like the mean one. um Peter and (.) Paul, paying
284 attention now specially, right any offers anyone for telling me what, m-why
285 of course we always want to know why (.) what the mode, the median the
286 mean and the range are. (1.7) … ((transcript omitted)) Phillip?
287 Phillip: er um days absent three is the mode because it's the most common one

Extract 4: Todd introducing the task of finding mean, mode, median and range.

In Extract 4, Todd's introduction includes references to doing "some practice" (line 279), needing "to know" (line 277) and using "your initiative" (line 280). The question asked in lines 284-286 begins and ends as a what question, but with a request for why sandwiched between. Phillip's answer in line 287 includes the remembering of a definition and procedure for finding the mode and similarly in line 358 in Extract 5 Pierre is remembering a definition and procedure for finding the median. The interaction in lines 313-319 in Extract 5 is all about remembering and applying the procedure for calculating the range. In this introduction Todd constructs his students

as people who do practice, use their initiative, pay attention, and tellers and Phillip adds to this remembering definitions, applying them to questions and using them to explain answers.

313 Peter: is the range a hundred and seven- seventeen
314 Todd: range a hundred and seventeen. the range is the biggest number take away.
315 the smallest number. the biggest number is a hundred and twenty-five, the
316 smallest number is eight, a hundred and twenty-five take away eight. Paul
317 Paul: no because the (.) the range is going to be in days absent so it'll be eight
318 Todd: ah. remember Peter. this table does not have any numbers a hundred and
319 twenty-five in there.
((transcript omitted))
358 Pierre: is the median um ta-add up all the (.) frequencies so [that] adds up to five
359 hundred

Extract 5: Calculating the range from a frequency table.

In Extract 5 Peter phrases his answer in line 313 as a question, and later in line 358 Pierre does the same, thus orienting to the teacher's role of evaluating. Yet it is Paul who makes the evaluation in line 317. Todd's response to Peter constructs the teacher also as someone who applies definitions in his calculation of the range. The role of student as someone who remembers continues in line 318 where Todd explicitly asks Peter to remember, but also in Pierre's turn in line 358-9 where he remembers the algorithm for calculating the median.

There is a brief change of focus in line 499 in Extract 6:

499 Todd: Peter in the corner. What does that number there represent, this five hundred
500 Peter: er::m how many (.) times, (0.9) um how many people there was
501 Todd: good how many people were surveyed. Pip. what does that one thousand
502 seven hundred and sixty represent.
503 Pip: um the total um number of days off
504 Todd: if you add up everyone's days of absence it will add up to one thousand
505 seven hundred and sixty, so as Phillip said, what we're going to do now is
506 one thousand seven hundred and sixty, divided by five hundred it's going
507 to give you (0.7) what is it Phillip
508 Phillip: three point five two?
509 Todd: three point five I'm going to call that. Three (.) point five. So the average, the mean average number of days absent (.) is about three and a half…

Extract 6: Calculating the mean from a frequency table.

In lines 499 and lines 501-2 Todd asks the students what these two numbers mean. Todd accepts Peter's and Pip's answers but his acceptance of Pip's answer focuses back on the procedure for calculating the mean and the role of these numbers in the

calculation. Again here the students are remembering features of a frequency table. Todd also constructs the identity of teacher as someone who demonstrates procedures, and identifies which calculations to perform, but the students as the ones who perform the calculations. So in this introduction the identity of teacher has been constructed as one that involves posing questions, demonstrating, and evaluating and the identity of student as someone who practices, pays attention, remembers, applies and uses definitions.

CONCLUSION

In this paper we have shown how the identities of mathematics teacher and mathematics student are discursively constructed in interaction, and how these interactions constitute the nature of these identities. Teachers and students orient to and perform different mathematical activities in interaction, making particular activities relevant to the identities of mathematics teacher and student. In the extract from Tim's lesson mathematics students are discursively constructed as people who think and find meaning, as well as people who participate in mathematical activities such as specialising, discussing, explaining, arguing and justifying. In the extract from Todd's lesson, students are discursively constructed as people who practice, pay attention, remember, apply and use definitions. Whilst all of these activities are part of what it means to do mathematics, in each case students are only asked to do or perform a small set of these actions. We have only shared brief extracts from the lessons for each of these teachers, but for each teacher these same actions are requested of or performed by students consistently across all four lessons that were video recorded. In the study, this consistency is apparent across the majority of teachers who videoed more than one lesson, with only one exception. This suggests that what it means to be a student of mathematics as constructed through classroom interactions is specific to the classroom that these interactions are situated in, and that it is not only the nature of tasks that influence the nature of this identity.

Transcript conventions (Sidnell, 2010)

Convention	Name	Use
[text]	Brackets	Indicates the start and end points of overlapping speech.
(0.5)	Timed silence	Indicates the length, in seconds, of a silence.
(.)	Micropause	A hearable pause, usually less than 0.2 seconds.
.	Period	Indicates falling pitch or intonation.
?	Question Mark	Indicates rising pitch or intonation.
,	Comma	Indicates a temporary rise or fall in intonation.
-	Hyphen	Indicates an abrupt halt or interruption in utterance.
underline	Underlined text	Indicates the speaker is emphasizing or stressing the speech.
:::	Colon(s)	Indicates prolongation of a sound.

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ON THE LEARNING OF GROUP ISOMORPHISMS

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Undergraduate mathematics students' encounter with the notion of group isomorphism is, more often than not, one of the major challenges that these students face in their first Group Theory course. This study focuses on students' responses to the notions of kernel, image and group isomorphism. Analysis suggests the great majority of students face difficulties that are mostly related to unclear grasp of the definitions of the three aforementioned notions. This has an unfavourable impact on the proof process as well as the use of notation. For the purposes of this qualitative research study there has been used the Commognitive Theoretical Framework.

INTRODUCTION

Research in the learning of Group Theory proves significant, since novice students consider this module as one of the most demanding in their syllabus. It "is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Group Theory module requires deep understanding of the abstract notions involved. An important element that causes students' difficulty with Group Theory is its very abstract nature (Hazzan, 2001). The deductive way of teaching Group Theory is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p140). The aim of this study is to investigate undergraduate mathematics students' responses to the notions of kernel, image and group isomorphisms by using the Commognitive Theoretical Framework (CTF) (Sfard, 2008), due to its great potential to investigate mathematical learning in both object level and meta-discursive level (Presmeg, 2016).

THEORETICAL FRAMEWORK

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different notions such as metaphor, thinking, communication, and cognition (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an **autopoietic system** of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). Moreover, CTF defines discursive characteristics of mathematics as the **word use**, **visual mediators**, **narratives**, and **routines** with their associated metarules, namely the **how** and the **when** of the routine. In addition, it involves the various objects of mathematical discourse such as the

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signifiers, **realisation trees**, **realisations**, **primary objects** and **discursive objects**. It also involves the constructs of **object-level** and **metalevel rules**.

Mathematical discourse involves certain objects of different categories and characteristics. **Primary object** (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). **Simple discursive objects** (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. **Compound discursive objects** (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, group is an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166)

Sfard (2008) describes two distinct categories of learning, namely the **object-level** and the **metalevel learning**. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p. 253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p. 254).

LITERATURE REVIEW

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields. The first reports on the learning of Group Theory appeared in the early 1990's. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic notions. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al. 2014). In the context of this research strand, Ioannou (2012) has, among other issues, focused on the intertwined nature of object-level and meta-level learning in Group Theory.

A significant milestone in the learning of Group Theory is the introduction of the rich and multifaceted notion of group isomorphism. Literature suggests that for novice students, the notion of isomorphism is a "complex and compound concept, composed of and connected to many other concepts which in themselves may be only partially understood" (Leron et al., 1995, p. 153). Leron et al (1995) also suggest that

understanding of group isomorphism requires understanding of the notions of group, function and quantifier, and conversely, learning about group isomorphism may substantiate the understanding of the aforementioned notions. Moreover, there is a connection between the isomorphism tasks to the notions of the order of group, the order of the elements of the group, commutativity and others.

The high level of abstraction and the conceptual difficulties are interconnected with the students' cognitive perplexity in Group Theory, which, climaxes at the introduction and proof of the First Isomorphism Theorem (Nardi, 2000). Indeed in a typical introductory module in Group Theory, First Isomorphism Theorem is the last and most crucial result introduced and is probably a "container of compressed conceptual difficulties" (Nardi, 2000, p179), since it involves all the preceding concepts.

METHODOLOGY

This study is part of a larger research project, which conducted a close examination of Year 2 undergraduate mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Abstract Algebra (see Ioannou, 2012). The course was taught in a research-intensive mathematics department in the UK. It was mandatory for Year 2 undergraduate mathematics students, and a total of 78 students attended it. It was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. The course assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of semester. The gathered data includes the following: Lecture observation field notes, lecture notes, audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 staff interviews (5 members of staff who gave 3 interviews each), student coursework, and student examination scripts. For the purposes of this study, the data of the 13 volunteers has been analysed, following the principles of Grounded Theory (Glaser and Strauss, 1967).

DATA ANALYSIS

The aim of this study is to analyse undergraduate mathematics students' first encounter with the notions of kernel, image and group isomorphisms. It is a ramification of a larger study, which focused on the analysis of these students' first holistic experience with Group Theory (Ioannou, 2012). The analysis of the 13 students' data suggests that all 13 have shown signs of incomplete object-level learning of the notions of kernel, image and group isomorphism. The relevant mathematical task that they had to solve in the coursework is the following:

5. Prove that the following are homomorphisms:
(i) G is any group, $h \in G$ and $\phi : G \to G$ is given by $\phi(g) = hgh^{-1}$.
(ii) $G = \mathrm{GL}(n, \mathbb{R})$ and $\phi : G \to G$ is given by $\phi(g) = (g^{-1})^T$.
(Here $\mathrm{GL}(n, \mathbb{R})$ is the group of invertible $n \times n$-matrices over $\mathbb{R}$ and the ^{T} denotes transpose.)
(iii) G is any abelian group and $\phi : G \to G$ is given by $\phi(g) = g^{-1}$.
(iv) $\phi : (\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ given by $\phi(x) = \cos(x) + \mathrm{i}\sin(x)$.
(v) G is any group, N is a normal subgroup of G and $\phi : G \to G/N$ is defined by $\phi(g) = gN$.
In each case say what is the kernel and the image of ϕ. In which cases is ϕ an isomorphism?

Figure 1: Mathematical task on group isomorphisms

The analysis of the thirteen students' solution suggests that ten of the thirteen students produced flawless answers regarding the first part about the definition of homomorphism. In what follows, I analyse some typical examples of the errors involving incomplete object-level learning of kernel and image, in the process of proving that a given map is in fact an isomorphism.

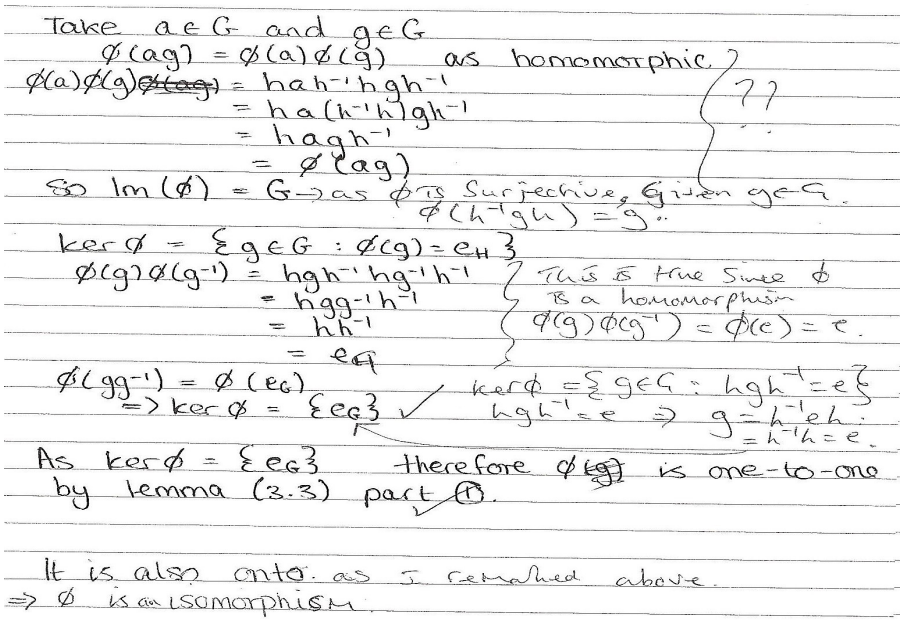
Take $a \in G$ and $g \in G$
$\phi(ag) = \phi(a)\phi(g)$ as homomorphic
$\phi(a)\phi(g) = hah^{-1}hgh^{-1}$
$= ha(h^{-1}h)gh^{-1}$
$= hagh^{-1}$
$= \phi(ag)$
??
So $\mathrm{Im}(\phi) = G \to$ as ϕ is surjective, given $g \in G$.
$\phi(h^{-1}gh) = g$.
$\ker\phi = \{g \in G : \phi(g) = e_H\}$
$\phi(g)\phi(g^{-1}) = hgh^{-1}hg^{-1}h^{-1}$
$= hgg^{-1}h^{-1}$
$= hh^{-1}$
$= e_G$
This is true since ϕ is a homomorphism
$\phi(g)\phi(g^{-1}) = \phi(e) = e$.
$\phi(gg^{-1}) = \phi(e_G)$
$\Rightarrow \ker\phi = \{e_G\}$ ✓
$\ker\phi = \{g \in G : hgh^{-1} = e\}$
$hgh^{-1} = e \Rightarrow g = h^{-1}eh$
$= h^{-1}h = e$.
As $\ker\phi = \{e_G\}$ therefore ϕ is one-to-one by lemma (3.3) part (1).
It is also onto as I remarked above.
$\Rightarrow \phi$ is an isomorphism.

Figure 2: Part of Student A's solution of 5i

Student A's reasoning is occasionally not explicit. She does not produce full narratives for proving what is the kernel and image of the homomorphism, suggesting probably some difficulty with the definitions of these two notions, as well as understanding of the connection between image and subjectivity, and kernel and injectivity. As the markers' comments suggest, her narratives need to be more detailed, predominantly by referring to the definitions of image and kernel. Since her object-level learning is incomplete, the application of metarules is problematic. The word use becomes rather awkward. Instead of stating that φ is an isomorphism, she mentions that φ is *isomorphic*. The use of the word 'isomorphic' instead of 'isomorphism' suggests that Student A has not yet objectified the notion of isomorphism. This notion may not have been reified yet, since Student A refers to the notion of isomorphism as being a

procedural activity and not an object. In general, the above analysis is in agreement with Student A's initial perception about this task and the d-objects of homomorphism, kernel and image.

> I found it easy enough to show that they're homomorphisms. The kernel's… it's just the image… that I'm having problems with at the moment, yeah. So I know someone in my course that – he did it and he got help in his seminars, so he said he'd try and explain it to me? So hopefully, I'll understand it then, if not I'll have to go ask the lecturer. Student A

Other typical inaccuracies regarding the notions of kernel and image were apparent in Student B's solution. In particular, there were occasions of problematic use of notation, for instance instead of writing e_G he writes e_H. This inaccuracy is rather important, because it is probably a result of incomplete object-level learning of the d-object of group, and the identity element in particular. Furthermore, it possibly suggests that he is not yet aware of the fact that the identity element of Gis the same for every subgroup. In addition, there are problems with his metalevel learning resulting inaccuracies in the process of proving. For instance, he does not justify why $g = e$ when he concludes that $\ker(\varphi) = \{e_G\}$. There is no explicit explanation saying that since $\ker(\varphi) = \{e_G\}$, therefore φ is injective. Student B has probably not yet objectified the idea of φ being injective or surjective, therefore an isomorphism, in relation to the kernel and image.

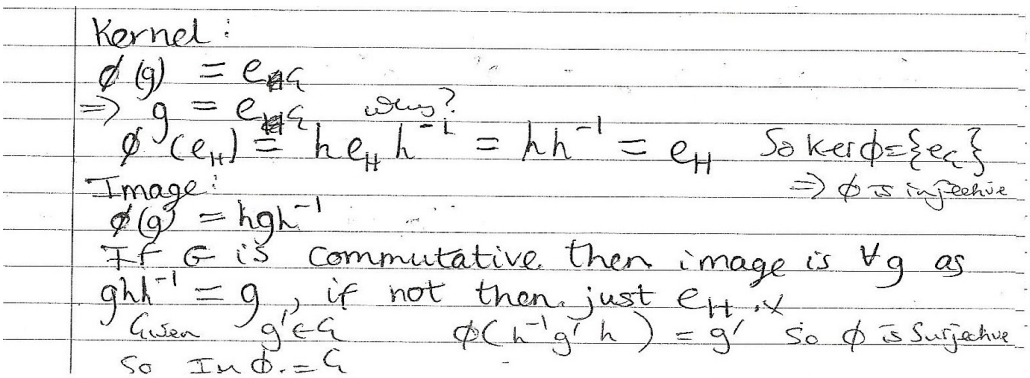

Figure 3: Part of Student B's solution of 5i

His performance is relatively mirrored in his somewhat optimist impression regarding the task, as discussed in the second interview.

> Yeah, 5, I mean, proving the homomorphisms, wasn't too much of a problem, sometimes saying what the kernel, the image was, was a bit – harder, and – which were isomorphisms, basically though, that was fine, once I'd kind of – went over the definition of an isomorphism, I mean, it just kind of pretty much relies on what you've – get in your kernel image, but um... cos I mean it's probably the whole visualizing, I was just a bit kind of – it's kind of – this is abstract, it's the whole abstract concepts of kernels and images and – not all – you know, not kind of meeting them everyday, things... but – yeah, I think I'm getting there... Student B

Student B expresses his need to have a visual image of the d-objects of kernel and image and links his difficulty to cope with them with the level of abstraction. He emphasises that the new d-objects are different from the 'usual' mathematical d-objects

('not kind of meeting them everyday'), and indirectly suggests that his approach to objectify them should be different.

Another representative example of incomplete object-level learning of the notion of image was apparent in Student C's solution. In particular, in this mathematical task, $\mathrm{im}(\varphi) = G$, since φ is surjective. Instead, Student C wrote that $\mathrm{im}(\varphi) = hgh^{-1}$, without explicitly stating that the image of the homomorphism in this case is the group G itself indicating also problematic application of the governing metarules, showing lack of precision, clarity and rigor. In addition, her attempt to prove that the homomorphism is bijective, and therefore an isomorphism, is problematic because of her incomplete object-level learning of the kernel. In particular, Student C was not yet able to conclude that when $\ker(\varphi) = \{e_G\}$ then φ is one-to-one and therefore isomorphism (since it is already proven that it is surjective as well).

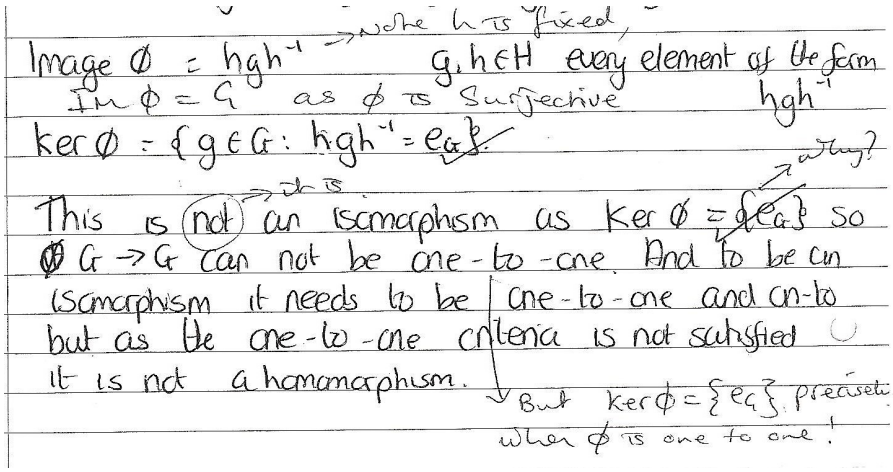
note h is fixed,
Image ϕ = hgh^{-1} $g, h \in H$ every element of the form hgh^{-1}
$\mathrm{Im}\,\phi = G$ as ϕ is surjective
$\ker\phi = \{g \in G : hgh^{-1} = e_G\}$
why?
This is not an isomorphism as $\ker\phi = \{e_G\}$ so $\phi: G \to G$ can not be one-to-one. And to be an isomorphism it needs to be one-to-one and on-to but as the one-to-one criteria is not satisfied it is not a homomorphism.
But $\ker\phi = \{e_G\}$ precisely when ϕ is one to one!

Figure 4: Part of Student C's solution of 5i

Student C' impression, as this has been expressed in her interviews is in agreement with her performance, expressing her initial difficulty with the d-objects of kernel, image and isomorphism.

> Um... like the first couple of parts, but obviously didn't get in – again, they were getting a bit harder towards the end because they were getting more difficult examples… so yeah... and again, it's just trying to get your head round all like the concepts, I think, you just need to keep going over them and then – obviously the questions will become easier to do. And then... because obviously we've only just done this one as well, so I think I need to go through it a couple more times, just to see whether I can um, do it. Student C

The above analysis possibly suggests that incomplete object-level learning affects unfavourably metalevel learning and application of metarules in the given context. Moreover, application of metarules is possibly context sensitive, and even if metalevel learning is complete in a certain mathematical discourse, the application of the same metarules might be problematic in a different one.

Another typical error is the incomplete object-level learning (or occasionally the imprecise use of notation) regarding the sets and their elements. For instance, Student D seems to have incomplete object-level learning of the d-objects of kernel and image.

These problems were revealed when he considered kernel to be the identity element of G, instead of the set containing the identity, as the excerpt below suggests. Instead of writing $\varphi(h^{-1}gh) = g$, Student D stated that φ is surjective whenever any element $g \in G$ can be written as $h^{-1}gh = g$, without considering the role of homomorphism.

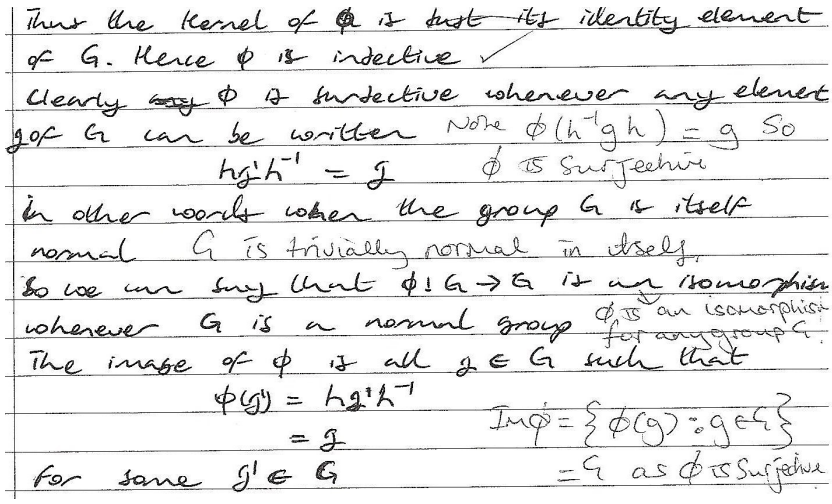
Thus the kernel of ϕ is just its identity element of G. Hence ϕ is injective ✓
Clearly ~~any~~ ϕ is surjective whenever any element g of G can be written Note $\phi(h^{-1}gh) = g$ So
$hgh^{-1} = g$ ϕ is surjective
In other words when the group G is itself normal G is trivially normal in itself,
So we can say that $\phi: G \to G$ is an isomorphism whenever G is a normal group ϕ is an isomorphism for any group G.
The image of ϕ is all $g \in G$ such that
$\phi(g') = hg'h^{-1}$
$= g$ $\quad Im\phi = \{\phi(g) : g \in G\}$
for some $g' \in G$ $\quad = G$ as ϕ is surjective

Figure 5: Part of Student D's solution of 5i

CONCLUSION

The first major signs of incomplete object-level learning in Group Theory, for the majority of students occurred when the notions of kernel and image were introduced (see also Ioannou, 2012). The analysis above suggests that there are clear indications of incomplete metalevel learning, which were revealed through the lack of explicitness in the thirteen students' mathematical narratives, and by the absence of important steps in the application of the routines involving these d-objects. Another indication of a problematic encounter with these d-objects was the increasingly problematic use of vocabulary and notation. The d-objects of kernel and image were problematic to the majority of students, mainly because these novice students were not able yet to objectify them properly. Kernel and image have not been fully objectified as algebraic structures linked directly with the notion of homomorphism and giving valuable information about the injectivity and subjectivity, respectively, of this homomorphism. Moreover, students have possibly not realised that $ker(\varphi)$ is an element of G and $im(\varphi)$ is an element of H. In a future study, I aim to investigate further students' encounter with the aforementioned notions and how these are related to the learning of the First Isomorphism Theorem.

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PRINCIPLES IN THE DESIGN OF TASKS TO SUPPORT PRE-SERVICE TEACHERS' NOTICING ENHANCEMENT

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Professional noticing is understood as a knowledge-based reasoning process. Particularly, noticing students' mathematical understanding involves the use of different domains of knowledge to attend to students' strategies, to interpret students' understanding and to decide how to respond considering students' understanding. In this paper, we discuss the principles in the design of tasks to support pre-service teachers' enhancement of noticing students' mathematical understanding using learning trajectories as a tool.

INTRODUCTION

Professional noticing has been conceptualized from different perspectives. Mason (2011) stated that noticing implies a "movement or shift of attention" (p.45). Based on Mason's work, Sherin (2007) characterized professional noticing as two main sub-processes: selective attention (noticing) and knowledge-based reasoning. Noticing is linked with the ability of teachers to focus their attention on a relevant classroom situation while reasoning involves using what teachers know about the subject matter to make sense of an idea, making connections between specifics of the classroom and broader principles of teaching and learning (van Es & Sherin, 2008). In this sense, professional noticing is the ability to use specific knowledge in classroom situations and bridges the gap regarding the relationship between professional knowledge and professional practice (Seidel, Stürmer, Prenzel, Jahn, & Schäfer, 2017).

In noticing children's mathematical understanding (Jacobs, Lamb, & Philipp, 2010), attending to children's strategies implies that teachers identify mathematically significant details. Therefore, attending to requires "not only the ability to focus on important features in a complex environment but also knowledge of what is mathematically significant" (Jacobs et al., 2010, p.195). Interpreting children's understanding involves that teachers' reasoning is consistent with both the details of the specific children's strategies and what is known about children's mathematical understanding. In other words, teachers must attend to children's strategies and "also have sufficient understanding of the mathematical landscape to connect how those strategies reflect understanding of mathematical concepts" (p.195). Finally, deciding how to respond implies to select an activity that can help students progress in their understanding. Therefore, it involves the use of "knowledge about children's mathematical development to identify a reasonable next step" (p.195). This perspective suggests that noticing students' mathematical understanding is a knowledge-based reasoning process (Mason, 2002; Sherin, 2007).

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NOTICING CHILDREN'S MATHEMATICAL UNDERSTANDING AS A KNOWLEDGE-BASED REASONING PROCESS

When pre-service teachers notice children's mathematical understanding (attending to, interpreting and deciding) they have to use their knowledge (subject matter knowledge and pedagogical content knowledge). Considering the MKT framework (Ball, Thames, & Phelps, 2008), attending to children's strategies implies that teachers identify important mathematical details in students' procedures (common and non-common) and the roots of their mistakes (Specialized Content Knowledge, SCK). To interpret students' understanding, teachers must coordinate what has been attended with what is known about children's mathematical understanding. Therefore, in interpreting, knowledge for explaining procedures, understanding common and uncommon strategies and explaining the origin of their errors (SCK) is required. Furthermore, knowledge about which aspects of the concept are the easiest or the most difficult ones for students, which are the most common errors related to a concept and how a mathematical content develops over time (Knowledge of Content and Students, KCS) is also needed to interpret different levels of understanding. Finally, deciding how to respond involves to take into account which aspects of the concept are the easiest or the most difficult ones for students; which are the most common errors related to the concept and how a concept develops over time (KCS); and which are the strategies or representations more adequate for introducing the concept (Knowledge of Content and Teaching, KCT). Furthermore, teachers should use their knowledge about which are the best sources and materials to help students progress in their understanding (Knowledge of Content and Curriculum, KCC).

As in the Spanish context teachers' educators cannot visit pre-service teachers at schools, the design of learning environments focused on developing professional noticing gives pre-service teachers the opportunity to learn of and about the practice. However, its development is a challenge since pre-service teachers not only must acquire the necessary knowledge but also learn to use it to attend to, interpret and decide (knowledge-based reasoning process). Previous research has shown that providing pre-service teachers with students' learning trajectories (information about how children learning develops over time) can help them to notice students' mathematical understanding (Callejo, Pérez, Moreno, Sánchez-Matamoros, & Valls, 2017; Edgington, Wilson, Webb, & Sztajn, 2015; Ivars, Fernandez, Llinares, & Choy, 2018; Sztajn, Confrey, Wilson, & Edgington, 2012). The objective of this paper is to present the principles in the design of tasks to support pre-service teachers' enhancement of noticing students' mathematical understanding using learning trajectories as a tool.

PRINCIPLES IN THE DESIGN OF TASKS TO SUPPORT PRE-SERVICE TEACHERS' NOTICING: LEARNING TRAJECTORIES AS A TOOL

We take into account three notions in the design of tasks addressed to enhance pre-service teachers' noticing: Learning Trajectory Based Instruction (LTBI, Sztajn et al.,

2012), semiotic tools from sociocultural perspectives of learning (Wells, 1999) and authentic tasks (Brown, Collins, & Duguid, 1989).

A learning trajectory (LT) is a construct that involves hypotheses about "the order and nature of the steps in the growth of students' mathematical understanding, and about the nature of the instructional experiences that might support them in moving step by step toward the goals of school mathematics" (Daro, Mosher, & Corcoran, 2011, p. 12). Therefore, a LT provides pre-service teachers with information regarding different students' strategies (common and uncommon), the common students' errors and their origin (SCK), and levels of understanding (growth of students' mathematical understanding) including which aspects of the concept are the easiest or the most difficult ones (KCS). Furthermore, it provides pre-service teachers with a set of different instructional activities to reach the learning objective including the most adequate strategies and representations for introducing the concept (KCT) and the use of the best materials and resources to help students progress in their understanding of the concept (KCC) (instructional experiences).

Sztajn et al. (2012) define Learning Trajectory Based Instruction as "teaching that uses students' LTs as the basis for instructional decisions" (p. 152). We use this notion in the design of tasks in teacher training programs to enhance noticing, since LTs create "a more integrated understanding of instruction based on how the logic of the learner becomes more sophisticated over time" (p. 152). Therefore, using learning trajectories as a tool provides pre-service teachers with (i) a frame to focus their attention on interpreting students' mathematical understanding and on responding with appropriate instruction (Sztajn et al., 2012); and (ii) a mathematical language to describe students' thinking (Ivars et al., 2018; Wickstrom, Baek, Barrett, Cullen, & Tobias, 2012).

From a sociocultural perspective of learning, Wells (2011) describes the learners as "agents-acting-with mediational –means". Tasks in our teacher education programs are designed with the objective that pre-service teachers use the information provided in the learning trajectories to notice students' mathematical understanding. Therefore, we use learning trajectories as mediational-means. In other words, learning trajectories are meaning-making tools that mediate pre-service teachers' communicative and reflective actions (semiotic tools).

Finally, the tasks designed can be considered as authentic activities defined by Brown et al. (1989) as the "ordinary practices of the culture" (p. 34). We use real classroom vignettes that consist of written students' answers to different activities, videos of students' solving activities or videos showing interactions student-teacher.

Following, we illustrate the design of a task in the context of noticing students' fractional understanding (Ivars et al., 2018).

THE DESIGN OF A TASK: AN EXAMPLE

The task has three elements: (i) two primary school fraction activities (Figure 1), (ii) the answers of three primary school students to these activities with different level of understanding (Table 1), and (iii) three questions as a guideline: **Q1**) Describe how

each pair of students has solved the activity identifying how they have used the mathematical elements involved and the difficulties they have had with them. **Q2**) What are the characteristics of students' understanding that can be inferred from their responses? Explain your answer. **Q3**) How could you respond to these students? Propose a learning objective and a new activity to help students progress in their thinking.

To answer the task, pre-service teachers have to use the information provided in a learning trajectory of the part-whole meaning of fraction. This LT was designed considering previous research on how students' thinking about the part-whole meaning of fraction develops over time (Battista, 2012) (the LT as a semiotic tool). This LT contains information about the mathematical elements of the part-whole meaning of fraction, and its development over time (students' levels of understanding). Characteristics of these levels of understanding are exemplified by different students' answers and by examples of instructional activities that could help students progress between levels. For instance, activities of identifying and representing a fraction given a whole, and activities aimed at identifying and representing a whole when a part is given, using different representations: continuous contexts, discrete contexts and the number line. In some of these activities, the use of materials and resources are exemplified (such as the use of Cuisenaire Rods).

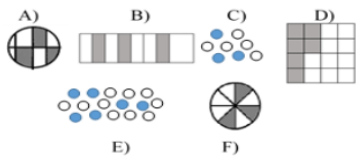
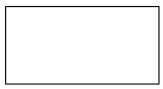
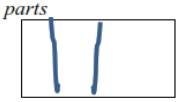
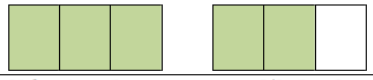
	Activity 1	**Activity 2**
	1. Which figures represent 3/8? 	2. This figure represents 5/3 of the whole. Represent the whole 
Student 1	*The figures that represent 3/8 are A), B) and F) because there are three parts of 8 shaded*	*There are 3 parts* 
Student 2	*F) represents 3/8. A) and B) do not represent 3/8 because the parts are not equal sized. In C) there are 3 dots shaded and in E) there are 6 dots shaded. D) represents 6/16*	*I split the whole in 3 equal sized parts and then I take five parts like that.* 
Student 3	*A) and B) do not have equal sized parts and do not represent 3/8. C), D), E) and F) represent 3/8.*	*If the given figure represents 5/3, first I must split the figure into five equal sized parts that represent the five thirds. Then I must shade 3 parts representing 3/3, which is the whole.*

Figure 1: Activities and primary school students' answers of the task

	Student 1 (Level 1)		Student 2 (Level 2)		Student 3 (Level 3)	
Mathematical Elements \ Activity	1	2	1	2	1	2
The parts of a whole must be equal sized	No	No	Yes	Yes	Yes	Yes
A part could be divided into other parts	No		No		Yes	
A part (unit fraction) as an iterative unit to reconstruct the whole		No		No		Yes

Table 1. A summary of the characteristics of students' answers of the task

EXAMINING ROLE OF TASK TO SUPPORT PRE-SERVICE TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL UNDERSTANDING

In this section, we illustrate how pre-service teachers use the knowledge from the LT to notice students' mathematical understanding. Data come from a study where 29 pre-service teachers (PTs) answered this exemplified task. All of them were able to attend to students' strategies and 28 of them used the LT to interpret students' mathematical understanding. However, only 17 out of the 29 PTs provided suitable instructional activities that help students progress in their understanding. Following, we show how PTs used the knowledge provided by the LT to attend to, interpret and decide through a pre-service teacher's answer (PT05). The discourse provided by PTs is considered as evidence of the use of the LT as meditational-mean to endow meaning to the situation.

Describing (attending to) students' answers

The following excerpt shows how PT05 uses Specialized Content Knowledge (SCK) to identify the mathematical details in the student 2's answer and to explain the origin of the errors (this information was provided in the LT. Emphasis added):

> **Student 2 (Activity 1).** He recognises that <u>the parts of the whole must be of equal size since he states that figure F represents 3/8 but A and B don't represent 3/8</u> because their parts are not of equal size. He has still difficulties in recognising that a part could be divided into other parts <u>since he doesn't realise that figure D represents 3/8</u> (although he says that D represents 6/16, he doesn't realise that this is equivalent to 3/8). Additionally, regarding figures C and E, he only notices the shaded square points but he doesn't take into account that they also represent 3/8 in a discrete context.
>
> **Student 2 (Activity 2).** He has taken into account that the parts of the whole must be of equal size since he divides the figure given into parts of equal size. However, he considers that the rectangle given is the whole, so he draws another rectangle to represent 5/3.

This PT identifies mathematical details in the student 2's answers when she identifies that the student recognises that the parts of the whole must be of equal size or when she identifies the student's difficulties in recognising that a part could be divided into

other parts. Furthermore, this PT explains why this student has difficulties with this mathematical element "although he says that D is 6/16, he doesn't realise that this is equivalent to 3/8", and the origin of the student 2's error in the activity 2 "However, he considers that the rectangle given is the whole, so he draws another rectangle to represent 5/3".

Interpreting students' understanding

To interpret students' understanding, PTs have to coordinate the mathematical details they have attended with their knowledge about students' understanding of the fraction concept. The type of task designed provides the context in which PTs can use Specialized Content Knowledge (SCK) and Knowledge of Content and Student (KCS) to relate the mathematical elements with the proficiency levels of students' understanding. An example of this type of discourse is the excerpt of PT05:

> **Student 2**. Student 2 shows features of the level 2 of understanding considering the students' understanding levels given in the LT. Although he recognises that the parts of the whole must be of equal size, he has still difficulties in recognising that a part could be divided into other parts. Furthermore, he has difficulties in recognising the whole in an activity of reconstructing the whole.

This PT interprets student 2's understanding recognising that this student is at level 2. She explains that the student "recognises that the parts of the whole must be of equal size […]" but this student has difficulties with the mathematical elements "a part could be divided into other parts [and] recognising the whole…" In this sense, this PT relates what she has attended to with the information from the LT about the students' most common errors and the levels of students' understanding.

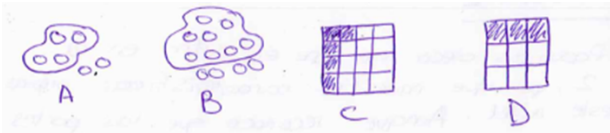
Responding to students: Proposing a learning objective and a new activity

Providing a learning objective and an instructional activity to help students progress in their understanding of the fraction concept create a context where pre-service teachers can connect learning and teaching. This requires that PTs use the knowledge about how the understanding of a concept develops over time, which aspects of the concept are the easiest or the most difficult ones for students and which are the most common errors related to the concept (KCS) to formulate the learning objective. Furthermore, to design a new activity, PTs need to use the knowledge about the instruction, for instance, which are the most adequate representations for introducing the concept (KCT), and the best sources or materials to help students progress in their understanding (KCC).

Following, an excerpt of PT05 is shown. Considering her interpretation of the student 2's understanding, she proposes an objective focused on *recognising that a part could be divided into other parts*. This objective helps student 2 progress in his understanding. Furthermore, this PT proposes an instructional activity of recognising

2/3 in different representations (discrete and continuous) in which the mathematical element identified as learning objective is considered.

> **Student 2**: Objective: Recognising that a part could be divided into other parts.
> Activity: Which of these figures represent 2/3?



This instructional activity includes two modes of representation (discrete and continuous) and involves the necessity of identifying a whole to recognise the representation of 2/3. The fact that students have to identify that 8/12 is equivalent to 2/3 in figure B or 6/9 is equivalent to 2/3 in figure D focuses students' attention on recognising that a part could be divided into other parts.

FINAL COMMENTS

We have shown the principles that we consider in the design of tasks addressed to enhance pre-service teachers' noticing: The notion of Learning Trajectory Based Instruction (Sztajn et al., 2012), the notion of a semiotic tool (Wells, 1999) and the use of authentic tasks (Brown et al., 1989). Through the PTs discourse to an exemplified task we can see how they use the knowledge provided in the LT to notice students' mathematical understanding. Therefore, the principles used in the design of tasks can be considered useful to support pre-service teachers' enhancement of noticing students' mathematical understanding (noticing as a knowledge-based reasoning process). Furthermore, the authentic tasks designed bridge the gap between professional knowledge and professional practice (Seidel et al., 2017) since PTs have to use their knowledge in practical situations such as interpreting students' understanding.

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PRE-SERVICE TEACHERS' NARRATIVES IN KINDERGARTEN TEACHER EDUCATION

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Developing teachers' knowledge, awareness and noticing ability in and for attending and developing students' mathematical understanding needs to be a core element in teacher education – mainly taking as a starting point the students' reasoning. Pursuing such gold one possible perspective is to consider (pre-service) teacher's narratives on observed practices focusing on what they notice, and the knowledge required in and for such noticing and to provide a fruitful feedback – Interpretative Knowledge. In the scope of this paper we discuss pre-service teachers' noticing ability and interpretative knowledge when observing a teachers' practice in the scope of number sense.

THEORETICAL BACKGROUND

Kindergarten is an educational stage which has gained a relevant attention into the mathematical research community in the last decades (Lin, Tsamir, Tirosh, & Levenson, 2013). Such focus considers that children, at this stage, can understand mathematical processes (concrete and abstract ones), and the way they acquire expertise in mathematics can define the acquisition of knowledge, in mathematics and other subjects, in the future educational stages (e.g., Baroody, 2000; Jordan, Kaplan, Ramineni, & Locuniak, 2009).

There are a wide range of studies conducted with kindergarten children focusing on their ability to understand different mathematical concepts such as patterns (Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015), parallelism (Sinclair, Moss, & Jones, 2010), or symmetry (Sinclair & Kaur, 2011). Nevertheless, such focus of attention is leaving aside the teachers' role and the teachers' knowledge for teaching such concepts.

To teach mathematics, particularly at kindergarten, teachers are required to be in possession of specialized professional knowledge. This knowledge needs to be developed during teacher education programs since such specialization does not develop over time in the daily work in classrooms (e.g., Ribeiro, Mellone, & Jakobsen, 2013). In fact, previous research has shown that often kindergarten teachers seem not to be (still) in possession of such adequate knowledge to teach (e.g., Hassidov & Ilany, 2015). Therefore, it is crucial to focus on this issue. In this context, as teacher educators, we are interested in how to enhance pre-service kindergarten teachers' (PTs) knowledge and skills to teach mathematics.

Professional noticing has been identified as a competence that teachers must acquire since it allows teachers to be aware of aspects, in a teaching learning situation, that other people could not identify (Roller, 2016). Jacobs, Lamb, and Philipp (2010)

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 416-423). Pretoria, South Africa: PME.

characterized professional noticing of children's mathematical understanding as a set of three interrelated skills: attending to children's strategies; interpreting children's understanding; and deciding how to respond on the basis of children's understanding. This competence is understood as a knowledge-based reasoning since teachers must attend to a classroom situation and then interpret the situation, considering their available knowledge (Sherin, 2007) to decide what to do next. Therefore, this competence highlights the need of specialized knowledge.

Such knowledge is well captured by the notion of Interpretative Knowledge – IK (Mellone, Jakobsen, & Ribeiro, 2015; Ribeiro, Mellone, & Jakobsen, 2016) which refers to the teachers' knowledge required and involved in and for interpreting. Therefore, it corresponds to the knowledge for giving meaning to students' productions and for providing a fruitful feedback that can help students develop their mathematical knowledge. Such IK is grounded in the teachers' content knowledge and when we consider the specificities of such knowledge, the Mathematics Teachers Specialized Knowledge – MTSK (Carrillo et al., 2018) conceptualization is assumed. Therefore, kindergarten teachers are required to be in possession of a knowledge that allows them to, amongst others, understand the hows and whys related to the mathematical topics they are dealing with (Knowledge of Topics-KoT), in order to leave the door open to future learning; the connections that can be performed, and its implications, within and between topics (Knowledge of the Structure of Mathematics-KSM) and what comprises to "do mathematics" in terms of the mathematical activity to be developed (Knowledge of Practices in Mathematics-KPM).

Narratives can be used as a means for capturing and studying practice and a form of expressing teachers' practical understanding of mathematics teaching (Chapman, 2008). We perceive narratives as a story in which the author relates to a sequence of significant events that has an internal logic and makes sense to him/her. In this sense, PTs' narratives describing what they notice on a teaching-learning situation are tools that can help them focus their attention on important aspects of the situation (Ivars & Fernández, 2018). In fact, Ivars and Fernández (2018) underline narratives as a tool to enhance primary school PTs' noticing during their practices at school.

In our study, we use narratives as a tool to enhance pre-service kindergarten teachers' noticing during their period of school practice. In these narratives, PTs had to identify a significative (for them) mathematical teaching-learning situation (associated to the acquisition of students' mathematical competence), interpret it and propose next teaching decisions. Our aim is to analyse how PTs notice children's mathematical understanding understood as a knowledge-based reasoning. Therefore, we analyse PTs written narratives to identify the knowledge PTs use when they notice children's mathematical understanding. We extend previous studies in two ways. Firstly, we focus on kindergarten teacher education and, secondly, we carry out a deeply analysis of how pre-service kindergarten teachers use their knowledge to notice children's mathematical understanding using the IK framework.

The present study

The study presented here is part of a larger study where 21 PTs had to write narratives during their period of practice at schools. In the Spanish context, the degree to become kindergarten teacher consists of four years. During the last year PTs are enrolled in a period of eight weeks of school practice. The first two weeks they focus on observing the school tutor while on the remaining six weeks PTs must develop and implement a didactic unit.

During the observation period, PTs were asked to write a narrative identifying and describing a noteworthy teaching-learning situation related to children's mathematical learning. PTs are provided with a guide to focus their attention on attending to, interpreting and deciding (Ivars & Fernández, 2016) and they should mobilize the knowledge acquired at least in two previous compulsory mathematics education courses (one related to numerical sense the other to geometrical sense).

Due to space constrains, and from the richness of data, from the 21 PTs narratives we will focus on one of the PTs' narrative (Alicia's narrative) as an example of how PTs use their knowledge to notice children's mathematical thinking through a written narrative. Alicia was observing a four years kindergarten classroom where children were working with the number-word sequence and the principles of counting.

In the Spanish context kindergarten school comprises children from 3 to 6 years old and its attendance is not compulsory. Nevertheless, there is a curriculum (Decree 38/2008) indicating that a kindergarten teacher should teach, amongst others, the following contents: the cardinal and ordinal number, the construction of number sequences by adding a unit, the representation of collections of objects through the cardinal number and the use of the number-word sequence to count elements in everyday situations.

Considering the focus of Alicia's narrative, during the mathematics education courses, PTs have participated in discussions where they had to interpret children's mathematical understanding related to the acquisition of the number-word sequence and the principles of counting (analysing videos or written answers) and to propose activities that help children progress in their understanding, using a theoretical document with information about how children learn this mathematical content. This theoretical document includes information about how students acquire the principles of counting (Gelman & Gallistel, 1978): stable order principle, the one-to-one correspondence principle, the cardinal principle, the abstraction principle, the irrelevance-order principle and the most common errors. The one-to-one correspondence principle implies coordinating two processes: partitioning, which implies the one-to-one mental transference of the objects in the group to be counted to a counted group and tagging which implies to give a tag to each of the elements counted. In the acquisition of this principle children can make some errors: (i) errors in the partitioning process such as ticking off an item more than once or skipping an item; (ii) errors in the tagging process such as using the same tag twice; and errors in

the coordination of the two processes such as continuing tagging elements that had been counted.

The research question posed is: Which knowledge does a PT mobilize to notice students' mathematical understanding when writing a narrative about children's acquisition of the principles of counting during her period of practices at school?

METHOD: ANALYSIS

The narratives were analyzed independently by three researchers with the focus on the knowledge used by this PT when she attends to children's strategies, interprets children's understanding and proposes following next steps. Afterwards, similarities and differences were discussed until reaching a consensus.

Therefore, we focus our attention on what mathematically important details PTs identify when describing children's answers; on the interpretations of PTs to children's answers, and how such interpretation addresses children's mathematical understanding; on how PTs provide specific instructional steps to implement forward based on children's mathematical understanding; and on the knowledge involved in and for doing such identification, interpretation, and proposition for the following instructional steps.

RESULTS AND DISCUSSION

Alicia, in her narrative, describes the context and the activity in which children are engaged in the classroom "an activity of making a correspondence between an amount of fruits and a card with its cardinal number":

> The students and the teacher are seated in a U-shape where the teacher puts different autumn' fruits in a row and asks different children to count how many fruits there are and to choose the corresponding card with the cardinal number.

Next, she describes the objective pursued in the activity:

> The objective, in this activity, is to acquire some principles of counting: stable order principle, the one-to-one correspondence principle and the cardinal principle.

Alicia is able to identify the important mathematical elements involved in the activity– principles of counting (KoT). Counting this set of elements implies having acquired the stable order principle, the one-to-one principle and the cardinal principle. Alicia does not mention the other two principles (the abstraction and the order-irrelevance principles) probably because within the proposed activity, these two principles are not brought into play (its acquisition cannot be observed). By doing so, she focuses on the immediate observable evidence (what can be seen directly) and thus, she leaves aside a more global view on what children need to learn and the aspects of teachers' knowledge required for the development of such learning. This does not mean that she does not have such knowledge for herself (KoT), but highlights the need to (us, as educators) rethink also some of the focus of attention and guidelines for observation. This is relevant since if we, as teacher educators, make more explicit the focus on the noticing process as well as on the content of the IK, pre-service teachers could provide

better informed decisions afterwards. Thinking on the IK, and its intertwined nature with the noticing skill, we can consider, as a core element in and for providing "true" meaning to children's comments, the knowledge grounding noticing and also dimensions/aspects non-observable to the eyes (e.g., what is beneath the actual discussion and the implications of such mathematical dimensions, having children's understanding and productions as a starting point).

Following, Alicia describes the next classroom interaction (the teacher [school tutor] puts six chestnuts on the floor on a row):

Teacher:	How many chestnuts are there?
Sara:	One, three, four, six, eight, nine.
Teacher:	Are you sure there are nine?
Sara:	Yes.
Teacher:	Please, try it again.
Sara:	One, two, four, eight, five, nine.
Teacher:	Can we help Sara to count the chestnuts?
Students:	One, two, three, four, five, six.
Teacher:	Now you, Sara.
Sara:	One, two, three, four, five, six.

When Alicia opts by including this interaction into her narrative, she reveals an awareness of the teacher awareness on the student' difficulties in counting, and points out the importance of the teacher's particular actions for helping the student (Sara) in developing her knowledge on the principles of counting, having as a starting point her own difficulties. This interaction highlights the importance of Alicia's knowledge attributed to the one-to-one and the cardinal principles as shaping the teacher's goals.

In fact, Alicia interprets Sara's understanding of counting principles as we can observe in the following excerpts of her narrative:

> […] Sara doesn't understand the notion of cardinality since, the last number recited does not only represent the last element counted, but also the total amount of elements.
>
> […] Sara has not acquired the stable order principle because, if we count the collection with the number-word sequence, we would say: one, two, three, four, five and six.
>
> […] in the case of Sara […] the only mathematical concept she has assimilated is the one-to-one correspondence. She shows difficulties with the order when she is counting each object; those difficulties may be because the order of the number-word sequence has not been acquired yet.

Alicia attributes Sara's difficulties in obtaining the cardinal of a set of elements to her problems with the order of the number-word sequence. Nevertheless, Alicia points out that Sara is able to assign one counting word to each of the chestnuts in the set. This identification allows Alicia to interpret that Sara understands the one-to-one correspondence principle as she states that "the only mathematical concept she has assimilated is the one-to-one correspondence". In that sense, although Alicia identifies the Sara's difficulty (which is linked with giving meaning to the coordination of

partitioning and tagging -KoT), it seems that she is assuming that the one-to-one correspondence does not necessarily link quantities to its corresponding word (KoT related to the ways definitions are perceived). Such interpretation seems to show an absence of connections between the elements of the principle of counting (KSM) – understanding them as isolated components–as if each one could coexist on its own and even so, at any random moment.

When thinking on how to improve the quality of the discussion and of the students mathematical learning and understanding, Alicia proposes a modification for the initial activity–to start with a smaller number of objects and then adding more:

> In the case of Sara, who, from the students, is the one who reveals more difficulties in this activity, the teacher should have started with only two chestnuts and then add more, maybe in this way, she would have counted each element correctly without the help of the other students.

Alicia (pseudo)mobilizes a hermeneutical listening (Davis 1997) as she is able to give "some sort" of meaning to Sara's difficulties concerning the counting process but considering the feedback provided, she does not focus on helping Sara to reach a broader (complete) understanding of the counting principle (stable order principle). This principle is linked to the acquisition of the different levels of the number-word sequence (Fuson, 1991). Therefore, the focus was on the action and not on the mathematical ideas to be discussed (fruitful feedback) and her new proposed action is grounded on her own understanding of the mathematical concepts itself which she perceives as being grounding Sara's difficulties. Such focus on the actions (deriving from a mathematical discussion into a pedagogical one) is related with Alicia's own knowledge of the topic to be addressed – counting principle (KoT); the connections between its elements – e.g. correspondence (KSM) and the role of generalization and problem solving (KPM).

FINAL COMMENTS

Using as a framework the IK can provide a deeper insight into what knowledge pre-service teachers mobilize when they notice students' mathematical understanding. From our results, we can see how Alicia uses specialized knowledge to attend to the mathematical elements of the activity, to interpret children's understanding (IK), being able to relate the important mathematical elements of the activity (principles of counting involved). Furthermore, she uses some dimensions of specialized knowledge when interpreting Sara's production since she is able to relate Sara's difficulties with some of the counting principles previously identified. These results suggest that writing narratives allows this PT to focus her attention on specific aspects of the teaching-learning situation, particularly, on children's mathematical understanding. Therefore, writing narratives can be seen as a powerful tool to enhance pre-service kindergarten teachers' noticing (understand as a knowledge-based reasoning) during their practices at schools.

Furthermore, the analysis of Alicia's narrative reveals PTs difficulties in and for interpreting children's reasoning as a starting point to provide next instructional steps focused on children's understanding. This information is important since teachers' educators need to be aware on the nature and content of the knowledge used by PTs in order to notice important aspects of teaching-learning situations for the design of teacher training programs.

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PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING MATHEMATICS: WHAT MATTERS FOR PRESERVICE PRIMARY TEACHERS IN MALAWI?

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This paper examines how Malawian preservice teachers perceive the mathematical knowledge needed for teaching in primary schools. Drawing on the practice-based theory of mathematical knowledge for teaching, the study highlights the main elements constituting the pedagogical knowledge required for teaching mathematics in remote areas of scholarly education. Data collection instruments included a questionnaire survey and individual interviews with three entrant preservice teachers attending a teacher-training college in Malawi. The results yielded by thematic data analysis show that preservice teachers conceived components of pedagogical content knowledge in different but complementary ways to those noted in pertinent literature.

INTRODUCTION

For decades, education researchers have been exploring practices and teaching techniques to learn what can improve teachers' knowledge. They have argued that, by possessing a specific type of knowledge, a teacher is able to organize potential activities that can transform subject matter content into forms more comprehensible to students (Abell, 2007; Grossman, 1990; Hurrell, 2013; Marks, 1990; Shulman, 1987). Thus, the knowledge of how teachers think, construe, and evaluate their own teaching provides valuable evidence on what comprises and enhances the nature of such knowledge.

A particularly significant contribution to this line of research was the idea of *Pedagogical Content Knowledge* (PCK), developed by Lee Shulman (1986). In his view, PCK intersperses pedagogical and subject matter knowledge, and allows teachers to make scientific subjects meaningful and useful to learners (Shulman, 1986). This idea was further elaborated by a research team led by Debora Ball. The group made observations of preservice teachers' teaching practices and identified six domains pertinent to effective teaching of mathematical subjects (Ball, Thames, & Phelps, 2008). Those domains led to the development of the *practice-based theory of mathematical knowledge for teaching* (MKT).

Drawing on a larger study about teacher education in Malawi, this paper explores how preservice teachers perceive the elements that compose the teacher's mathematical knowledge for teaching, in particular, the PCK for teaching mathematics. Consequently, the general goal is to contribute to a new understanding of how those domains take place in remote areas of teacher education.

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TEACHERS' KNOWLEDGE DOMAINS

The particular forms of content that embody teachers' professional knowledge have become a central issue of the 21st century educational practice, raising questions whose answers can affect many educational institutions around the world. In this context, the practice-based theory of MKT comprises of the following six fundamental domains of teacher' knowledge in mathematics: *Common Content Knowledge* (CCK), *Horizon Content Knowledge* (HCK), *Specialized Content Knowledge* (SCK), *Knowledge of Content and Students* (KCS), *Knowledge of Content and Curriculum* (KCC), and *Knowledge of Content and Teaching* (KCT). Together, those domains constitute a promising conceptual framework for research on teacher knowledge (Ball, Hill, & Bass, 2005).

CCK is the first domain that incorporates the subject matter knowledge. Ball et al. (2008) described it as the type of knowledge "used in settings other than teaching" and related to circumstances typically common by others who know mathematics (p. 399). In contrast, SCK refers to a specialized knowledge unique to the work of teaching. This domain entails the mathematics knowledge and skills that a teacher needs to possess in order to create necessary conditions for students to learn the subject. SCK reflects the teachers' capacity for dealing with the teaching tasks, such as "looking for patterns in student errors" and "understanding different interpretations of the operations in ways that students need not explicitly distinguish" (Ball et al., 2008, p. 400). In addition to these domains, HCK concerns "a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires" (Ball & Bass, 2009, p. 1). HCK helps preservice teachers be aware of "how the content being taught is situated in and connected to the broader disciplinary territory" (Jakobsen, Thames, Ribeiro, & Delaney, 2012, p. 4642) and "how mathematics topics are related over the mathematics span included in the curriculum" (Ball et al., 2008, p. 403).

The second set of domains (KCT, KCS, and KCC) resides in the pedagogical content for teaching—a singular type of knowledge that allows teachers to create and be oriented in a conducive learning environment. For Freire (1996), this is the most fundamental skill, indispensable from the beginning of teaching, that helps educators, teachers, and preservice teachers to understand that "teaching is not about transferring knowledge but creating possibilities for its production or its construction" (p. 47) (Translation by the authors).

The domain of KCT addresses methodological aspects that allow teachers to understand and choose the appropriate ways to meet students' learning needs. KCT concerns lesson design and use of appropriate activities in the classroom (Herbst & Kosko, 2014). Such knowledge helps teachers to develop effective teaching practice and conceptualize their own teaching. Additionally, KCT can contribute to the future teachers' learning by addressing specific ways to organize lessons that help to them deal not only with the ongoing adaptations in the classroom environment (Goodnough, 2006) but also with problems that permeate much of our current society (Freire, 1996).

The second domain, KCS, pertains to the development of teachers' specialized content knowledge (Philipp et al., 2007). KCS mainly focuses on the students' mathematics learning and understanding, i.e., how they think about mathematics, their limits, and difficulties they experience when learning the subject. From this context, Ball et al. (2008) explained that:

> When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking. (p. 9)

While KCS supports teachers' ability to listen to their students, KCC helps them to identify, select, and decide how a particular set of curriculum materials can benefit students' learning. This last domain, in the context of initial teacher education, appears as a potent tool for preservice teachers to explore the curriculum and instructional materials relevant for teaching (Ball et al., 2008). Possessing an effective KCC, however, implies not only knowing the curricular content that need to be followed, but also how those content areas can be better introduced in the lessons.

The introduction of these last three components by Shulman (1986) and their reconceptualization by Ball et al. (2008) has been an important step toward a better comprehension of teachers' knowledge. Moreover, as Depaepe et al. (2013) pointed out, although a significant body of research on PCK has been conducted in the US and European countries, there is evident paucity of studies focusing on the African countries. Thus, further research is necessary with the aim of providing not only new insights about the idea of PCK, but also discussing the possibilities and limits in applying those concepts in African contexts.

PRESERVICE PRIMARY TEACHER EDUCATION IN MALAWI

There are eight teacher-training colleges in Malawi. They operate with local primary schools in neighboring regions by offering a two-year program for candidates aspiring to become primary school teachers. As a part of this program, candidates review the basic topics taught in primary schools, analyze lesson models given by in-service teachers, and develop teaching activities for lower and upper grades. At the core of their training, a modular structural curriculum combines studies with a special focus on the pedagogical content knowledge for teaching and competencies for ensuring learning with understanding (Malawian Institute of Education, 2017).

Although this new curriculum appears adequate for preparing preservice teachers to teach in Malawi, a recent study conducted at the eight Malawian teacher colleges indicated that most preservice teachers begin their college program with a poor understanding of basic mathematics. Consequently, their essential knowledge and skills for teaching mathematics typically do not improve as a consequence of the teacher education (Kasoka, Jakobsen, & Kazima, 2017). These results highlight the need to better understand what beginner teachers do in their teacher-training programs

in Malawi, how they understand and evaluate their own lessons, and on what bases they choose to act in particular ways instead of others.

THE RESEARCH DESIGN

The larger research study that this paper draws upon was based on a qualitative multiple-case study (Stake, 2009) with 23 primary preservice teachers selected due to the diversity in their backgrounds and teaching subject preferences. The broader study was divided into three research moments following the teacher-training calendar. The first moment consisted of asking all 23 primary preservice teachers to complete a questionnaire inquiring into their previous teaching experience and subject preferences for teaching, and individual interviews about characteristics of effective mathematics teaching in Malawi. The current investigation focuses solely on the information obtained within the first moment from three preservice teachers presenting similar traits on those criteria.

The data was organized into three units of analysis bounded by the domains of teachers' PCK (KCT, KCC, and KCS). Based on these domains, three distinct categories emerged from the data: *Decision-making in teachers' KCT*; *Relations between KCT and KCC*; and *Adaptations of the classroom activities from students' contributions and level.* These categories provided insight into the particularities of Malawian preservice teachers' perceptions of PCK for teaching mathematics.

FINDINGS

Decision-making in teachers' KCT

The passage presented in this category occurred during the interview with Martin, a preservice teacher with previous teaching experience, and a preference for teaching sciences and mathematics in primary schools. The starting point was based on the type of "knowledge needed to decide on the best examples and representations to use for given instructional objectives" (Herbst & Kosko, 2014, p. 24). This idea is perceived as a significant element in KCT, as it is vital for the work of teaching mathematics (Ball et al., 2008). The evidence that frames this notion is given below:

Researcher: Martin, is it important for teachers to think about the design of the lesson or any strategy before starting the classes?

Martin: Yes, it is very important!

Researcher: Why? Don't you have a [teacher] manual? You have to follow it, right?

Martin: Yes, we have to follow the manual! **But a good teacher also has to think about what is good or bad for the learners.** He needs to think about the lesson before, so if something happens, he or she can manage what to do. The big problem is that, in Malawi, there are so many students, that teachers cannot pay attention to everyone all the time. If they do that, they will never finish the content. **So, the teacher needs to decide when it is time to give attention to students and when it is time to move on**.

Researcher: How does he know it is the time to move on?

Martin: **If the majority understand the topic, it is time to move on**. Learners can take revision lessons at the end of the term, so there is no problem if one or two students have problems. They will have an opportunity to review it. So this I something teachers need to consider when they are teaching.

For Martin, the knowledge about instructional practice design combines the knowledge of the organization and conduction of mathematical tasks in classrooms. Decision-making was also considered as an integral part of the primary school teacher's skill management for teaching, an idea that resembles KCT as the "knowledge of strategies and representations for teaching particular topics," as proposed by Borko and Putman (1996, p. 677). However, the context in which such skill is applied was a distinctive characteristic in Martin's view: "*The number of pupils in Malawian primary classrooms usually ranges from seventy-five to more than one hundred.*" Martin also implied that teaching under such circumstances should be based on more than following manuals; it demands organization, decision-making, and a sense of time and place for learning.

Relations between KCT and KCC

This section contains a segment of the interview by Carlos, a preservice teacher with a similar background and preferences as Martin. Carlos's responses not only describe a singular characteristic for an effective teaching in Malawi, but also show how KCC might interact with the domain of KCT from a broad curricular perspective.

Researcher: Carlos, how do you describe an effective teacher?

Carlos: Ok…, **I think an effective teacher needs to be resourcive** [resourceful]**.** He needs to know how to use the right [teaching] resources to engage the learners in learning a topic.

Researcher: How does he know if the [teaching] resource is good or bad for the learners?

Carlos: If the resource is too complicated, it is, of course, not a good resource.

Researcher: Can you give an example of a good resource?

Carlos: **It is hard to say because in Malawi we don't have enough material**, you know… but… if you are teaching in standard four, you can use small stones to teach multiplication by nine, but just multiplication of small numbers. Multiplication by nine is part of the curriculum in standard four, you know… But, in standard five, if you use stones to teach multiplication of big numbers, it won't be a good idea because **it will take too much time for the students to count the stones**. So, for teaching multiplication involving big numbers, **you need to think about using other sources**.

In this passage, Carlos used the word *"resourcive"* to describe the way a teacher considers effective teaching resources in the classroom. This characteristic reflects the teacher's capacity for selecting, revising, and using appropriate didactical materials to facilitate student learning. Carlos cogitated that if a teacher is capable of discerning the advantages and disadvantages of using didactical resources, he/she could create conditions that are more favorable for the learners, so that they can learn by doing. If

the material is incompatible with students' learning pace, it would be for the teacher to search for proper teaching resources, explains Carlos.

Although Shulman (1987) described curriculum knowledge as a major category that includes "the grasp of the materials and programs that serve as tools of the trade" (p. 8), the knowledge and skills related to teaching materials were only recently introduced into the KCC domain (Sleep, 2009). Thus, the example provided by Carlos—*t*he use of small stones to teach multiplication of small numbers by nine—depicted KCC not only as the teacher's ability to familiarize with the tools for delivering mathematical lessons, but also his/her capacity for understanding how simple objects can help a concept become a learning need for students.

Adaptations of the classroom activities from students' contributions and level

The passage below is an excerpt of the interview conducted with Clara, a preservice teacher who also acquired some teaching experience before starting her training program. She too has expressed affinity for teaching mathematics and science foundations in primary schools. The passage portrays her perceptions of focal features in KCS.

Researcher: If a student uses a different method to solve a problem, what do you think the teacher needs to do?

Clara: I think the teacher needs to consider it because, in Malawi, you know… students have many different methods, so the teacher must consider it.

Researcher: But if you are teaching multiplication, for example, you know… there are many ways to teach it. How do you choose it?

Clara: **Well… first, the learners have to learn the method of the teacher. Then, you can discuss the other methods**. If the teacher stops the class to explain every single method, it will confuse those learners who are trying to learn.

Researcher: But how do you know which method is better?

Clara: **Usually, the method of the teacher is easier. A good teacher will always choose the easy way for the learners.**

Researcher: So, would you use the same method in all classrooms?

Clara: No, **it would be different because you have to use the method according to the level of students.** In standard four, for example, you have to use methods that show why this number is like this, why this is this… **so the method also has to show the details**. In standard five, you don't need to explain the details, you can just apply it.

The transcription above reveals a significant aspect of teachers' KCS. Although students' ideas should be considered in the classroom, methods proposed by the teacher are still dominant. For Clara, the knowledge about the students' capacities can be beneficial for teachers to better understand and react to the students' ideas while teaching, but the use of multiple approaches might not be helpful for those who are still learning the content.

According to Ball et al. (2008), anticipating what students are likely to think and what they can find confusing is an essential skill that helps teachers balance and adapt their work according to students' contributions and level of learning. Moura (2010) reminds us that every idea exchanged is valuable for improving the quality of teacher's work. By knowing and interacting with their students, teachers can change their conceptions and learn new ways of acting that would incite development of learners' reasoning ability (Moura, 2010).

CONCLUSION

All three segments presented in this paper focused on PCK; more specifically, they reflected what preservice teachers in Malawi perceived as crucial for the domains of KCT, KCC, and KCS. Although each of these components assumed different forms of conceptualization, they provide a new perspective on how PCK takes place in remote areas of scholar education. Moreover, over the last decades, the concept of PCK has received different conceptualizations by the academic community, posing a challenge that many educational institutions find difficult to put in practice. These insights are relevant not only for teacher educators in Malawi, who can use them to consider their activities based on preservice teachers' specificities and needs for teaching mathematics, but also for deepening and developing the theory of MKT.

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UNDERSTANDING OF WRITTEN SUBTRACTION ALGORITHMS: WHAT DOES THAT MEAN AND HOW CAN WE ANALYSE IT?

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In mathematics education, supporting children in developing strategic thinking and therefore conceptual understanding is an important goal. One of the topics that are questioned in the light of understanding is written algorithms. To teach an algorithm in a way that provides understanding requires having a grasp on what it means to have conceptual understanding of this algorithm. To approach this question, we conducted an exploratory study with 222 children from 12 fourth-grade-classes. We raised children's conceptual understanding of crossing the tens boundary and developed a detailed coding-scheme. In this way, we could work out some key aspects describing conceptual understanding of written subtraction algorithms.

INTRODUCTION: ALGORITHMS AND UNDERSTANDING

In mathematics education one goal is "to prepare students for a technological society" (Anghileri 2006, p. 364) by supporting them to develop strategic thinking and therefore conceptual understanding of mathematical contents (see also e.g. Verschaffel et al. 2007).

But there are many examples like the following given by Wertheimer that show that this goal is not yet reached satisfactorily and too many pupils work mechanically instead: Wertheimer gave students problems like

$$\frac{375 + 375 + 375}{3} = ?$$

and had to realize that only a few of them saw the structure of the problem and did not solve it in a mechanical way (Schoenfeld 2008, p. 470).

The requirement of understanding is discussed especially in those mathematical topics, where working mechanically seems to be the main goal as standard written algorithms. Kamii and Dominick (1998) examined children's performance in calculating comparing children who had learned the standard algorithms and children who had not. Kamii and Dominick (1998) conclude that algorithms

> "are harmful to children's development of numerical reasoning for two reasons: (a) They 'unteach' place value and discourage children from developing number sense, and (b) they force children to give up their own thinking" (p. 58)

or in other words: algorithms prevent children from understanding. Some findings support Kamii and Dominick's statement: Selter (2001) observed children that learned different strategies including the written algorithm for addition and subtraction up to

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1000. His results show that children use a steady procedure regardless of the special problem so that some of them (try to) solve 701 – 698 by written subtraction.

Interpreting these results, one should consider that we know from Brownell and Moser (1949) that teaching the algorithms mechanically vs. meaningfully makes a difference in the conceptual understanding children gain. With that in mind, it is important to recognize that the children that participated in Selter's study got lessons that emphasized the written algorithms instead of strategic thinking (Selter, 2001). Kamii and Dominick (1998) do not report the content of the lessons taught in the studied classrooms. Thus, the statement of Kamii and Dominick (1998) that teaching algorithms "force children to give up their own thinking" (p. 58) should be considered with caution.

Usiskin (1995) expresses another point of view. For him deeper insights in mathematical structures like the place value ideas can be supported by encouraging conceptual understanding of the way of functioning of the algorithm. All these findings, and reasons, call for the emphasis of understanding the algorithm instead of only producing fast and correct results. A consequence for research and teaching practice is therefore, to find a way to soundly analyse children's understanding of algorithms.

THEORETICAL BACKGROUND

Teaching an algorithm in a meaningful way requires having a grasp on what it means to have conceptual understanding of this algorithm. In our study, we chose to examine the written subtraction because written subtraction can be carried out in different ways and the different ways have different requirements concerning understanding. The procedures differ by the way the difference is calculated and the way the tens boundary is crossed. Calculating the difference x between a and b can be done by subtracting ($a - b = x$) or by an indirect addition ($b + x = a$). The tens boundary can be crossed by regrouping ("I change one hundred in 10 tens") or by equal addition based on the equivalence relation between differences ($a - b = (a + 10) - (b + 10)$: "I add ten tens at the tens place of the minuend and one hundred at the hundreds place of the subtrahend"). There is international discussion on which procedure should be chosen, and some comparative studies tried to get empirical evidence (e.g. USA: Brownell and Moser, 1949, Germany: Mosel-Göbel, 1988; Jensen and Gasteiger, in press, Italy: Fiori and Zuccheri, 2005). In Germany it was prescribed to teach indirect addition from 1958 until 2001 (Kultusministerkonferenz, 1958), now in almost all Federal States of Germany the choice for an algorithm is on the part of the teachers. For this choice the possibility of understanding the way of functioning by children is an important argument (Brownell and Moser, 1949; Mosel-Göbel, 1988; Gerster, 1982/2012).

There are few studies examining and comparing the conceptual understanding of written subtraction procedures resp. crossing the tens boundary. Brownell and Moser (1949) and Mosel-Göbel (1988) show better understanding of the "regrouping" than "equal addition" but there are several limitations concerning these studies. They were conducted more or less long time ago. It can be claimed that compared with today,

aims and methods of mathematics education differ in some extend. Furthermore, these studies do not hold a replicable coding scheme to comprehend their idea of conceptual understanding. Therefore, these studies do not offer a method to measure this understanding.

A possibility to measure understanding could be to ask children to explain their approach and to evaluate the explanations. Klein (2009) differentiates an explanation-HOW as answer to a question "how…" – in this case the answer is a description – and an explanation-WHY as answer to the corresponding question "why …" – in this case the answer can be characterized as reasoning. Our assumption is that children who understand the algorithm will not only be able to describe their approach – and hereby demonstrate their ability of mechanical performance – but rather give reason for it. But reasoning is not enough to show conceptual understanding, because there are several kinds of explanation-WHY: Klein (2009) characterizes different contexts which allow different reasonings. For example, in an explanation in everyday life it is legitimate to use an intention or an initial situation as reason, whereas this would not be acceptable in an academic context. Brunner (2014) describes different kinds of reasoning and contrasts this everyday life reasoning as one side of a range with mathematical proof on the other side and emphasizes that in mathematics education children should reason mathematically and not only in everyday life contexts.

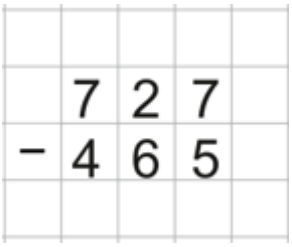
To approach measuring conceptual understanding in the context of written subtraction algorithms we analysed children's reasons and descriptions of crossing the tens boundary to explore whether it is possible to differentiate between the explanation-HOW and the explanation-WHY and how these explanations can be further characterized.

EXPLORATORY STUDY

Methodology/Research Design

To get a better grasp on the conceptual understanding of the procedures "regrouping" and "equal addition", we conducted an exploratory study with $N = 222$ children from five primary schools in Germany – six classes and teachers for each procedure ("subtraction with regrouping" $n_r = 113$, "indirect addition with equal addition" $n_e = 109$). All children learnt the algorithm about nine months prior to our study, so it can be assumed that they developed routine in using written subtraction.

We developed two tasks to raise children's conceptual understanding of the learned way of crossing the tens boundary. The first task was identical for both groups (regrouping and indirect addition) and the second one was adjusted on the taught algorithm. The first task was to solve a written subtraction and to explain the approach:



Calculate the task.

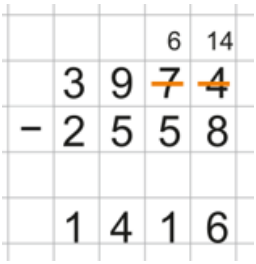
Now explain your approach.

*First I*_____

_____

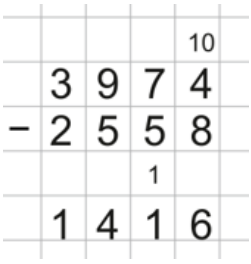
The second task resented a calculation and the children had to reason the notation:

task for regrouping:



Why was the seven crossed out and a six was noted above?

task for equal addition:



Why a little 'one' was noted?

Analysis of children's conceptual understanding

For an evaluation of the explanations given by the children in the two tasks the answers were analysed with a detailed qualitative content analysis. Because some children provided only in one of the two tasks a reasoning through referring on the mathematical relation and both explanations complemented each other, the deliberations of both tasks of each child were considered in the analysis as one explanation.

We built categories and characterized them in a way that fits both procedures (see table 1).

To represent the differences in children's ability of reasoning with a focus on substantial mathematical relations, we evaluated two aspects in each category:

a) On the one hand, we coded how the child described the conducted steps in crossing the tens boundary in each of the two involved columns. Therefore, it was evaluated how the numbers of bundles and the place value were named in each case (from code 1 to 4 described in a).

b) On the other hand, we evaluated how the child described the relation between the conducted steps in the two columns where crossing the tens boundary is necessary (described in b).

Children's explanations were analysed according to these two aspects and the results were combined to an overall evaluation of the whole explanation of crossing the tens boundary.

Code	Approach of the child
0	**Erroneous description:** The descriptions of the approach in the columns contain errors (e.g. numbers/place values are named incorrectly; approach is totally wrong)

1	**Description of the notation resp. calculation: a)** The child uses numbers without naming the place value or names incorrect place value or gives a vague description (e.g. "take some away"). **b)** The child does not comment the relation between the two columns.
2	**Description is very vague regarding the naming of the place value and without relating the bundles: a)** Description remains very vague ("some"), place value is used wrong or the name of the bundle is missing in at least one of the two columns. **b)** The child relates the two columns but without showing awareness of the bundling of ten.
3	**Description is vague with naming the right place value but without relating the bundles: a)** The child describes the approach in both columns with naming the right place value. **b)** see code 2
4	**Description of the full procedure crossing the tens boundary with naming the right place values and their numbers: a)** The child describes the approach in both columns with a correct naming of place value and numbers of bundles. **b)** The child relates the two columns and refers to the different size of bundles (e.g. 'ones and tens' or 'tens and hundreds').

Table 1: categories for the evaluation of the explanations given by the children

Findings

Our first finding is that it is not trivial to draw the line between an explanation-HOW and an explanation-WHY. In the first task all children felt required to describe their approach (how). Moreover, some gave elements of an explanation-WHY. In the second task, many children gave reasons for their approaches with explicit verbal pointers ("because", "so", "therefore").

For many children it seems to be satisfactory to reason only through the lack of possibility to calculate in the present column without crossing the tens boundary:

> "First I *calculated from 5 to 7. Then I thought: 'From 6 to 2 is not possible', so I calculated from 6 to 12. Afterwards I wrote a little 1 as carryover under the 4. And 4+1 equals 5, so I calculated from 5 to 7.*" (All translations from German by the authors.)

This child reasons a motivation for writing a one as carryover and so this explanation cannot be characterized only as explanation-HOW. But this reasoning seems more an everyday life explanation than a mathematical reason for the legitimacy of the approach, because an initial situation (not possible to calculate from 6 to 2) is used as reason. Therefore, an explanation in such a way was not evaluated as a clear explanation-WHY (and hence classified as Code 1), because the child does not give a mathematical legitimation for using the "12" and for the "little 1 as carryover".

From Code 2 on, explanations can be characterized as explanation-WHY and then differentiated in their quality. In the following examples (Code 2) the explanations stay

vague and it is unclear whether the child has a conceptual understanding of the mathematical relations:

> "Why a little 'one' was noted? *When you calculate from 8 to 4 this is not possible. Because of this you have to calculate from 8 to 14. And because of the 10 there gets a 1 in the next column.*"

"Why was the seven crossed out and a six was noted above? *Because you cannot calculate 4-8, therefore you cross out the 7 and you write a 6 above und 10 are carried to the 4.*"

In these explanations, at least basic approaches of explanation-WHY can be found (in the second explanation in a more or less implicit manner) which can be seen as related to the mathematical relations.

The third category is different from the second one accounted for by giving a description of a relation between the two columns with right naming of bundles but without describing the precise relation between the bundles:

"*…one ten has gone over to the ones.*"

An example for a complete explanation-WHY (Code 4) is the following explanation:

> "First I *calculated 7-5, this equals 2. Then the task was 2-6, but this is not possible. So I have to take one of the hundreds and make 10 tens out of it. 12-6=6. There are only 6 hundreds remaining and 6-4=2.*"

This child names the appropriate bundles "hundreds" and "tens" and in addition describes the relation between these two kinds of bundles (that one can make 10 tens out of one hundred (unbundling)). This explanation can be evaluated as a generalizable description of the rules. This aspect of generalizability is an important step towards mathematical reasoning and is stressed out in the context of evaluating reasoning skills (Brunner, 2014).

By using the developed coding schemes we found that children could explain regrouping much better than equal addition (Jensen & Gasteiger, in press): Most of the children that learned the equal addition described it in a mechanical way and no child could be evaluated with code 3 or 4. In the "regrouping"-classes, half of the children could give explanations in category 2 or higher and five children could give a complete reasoning. Altogether three quarters of the children provide no explanation-WHY.

Discussion and implications

Our analysis showed a possible way of grasping conceptual understanding of written subtraction algorithms and in this way we could show some key aspects in conceptual understanding: Referring to the place value and therefore the kind of bundles that are worked with are an important indicator for conceptual understanding. However, beyond that the relation between the actions in both columns is crucial as reason for the way of functioning of the algorithm. This relation is unclear for most of the children, so that many of them stay in a mechanical state (explanation-HOW and sometimes explanation - WHY but with everyday life reasoning) and do not refer to

place value and the meaning of noted numbers. It is already common in teaching an algorithm to work with manipulatives like tens blocks that can emphasize these ideas. Moreover, our findings give some clues how these actions could be verbalized to get a better understanding of the important ideas.

Developing own strategies and using informal written arithmetic instead of just learning the algorithm (e.g. Verschaffel et al., 2007, Treffers, 1983) could help as well to get a better understanding. One informal strategy uses the idea of number decomposition into its place values. That can lead to calculating column-wise and at the same time being aware of the meaning of these columns and in further consequence support the aim to lead children to choose the use of written algorithms purposely versus mental calculation or informal strategies.

These approaches get even more importance when regarding Clarke (2005) findings that essential basics in calculating that are needed for building the conceptual understanding described above are not available.

Thus, in the discussion about teaching algorithms anyway there should be added the idea of delaying the moment for teaching so that there is more room to lay the groundwork for conceptual understanding. How this teaching can be designed must be content of further research.

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NOTIONS, DEFINITIONS, AND COMPONENTS OF MATHEMATICAL CREATIVITY: AN OVERVIEW

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In this paper, we present an overview of notions of mathematical creativity in contemporary empirical educational research. We conducted a review using two databases and coding articles with respect to definitions and components of creativity that were used and/or referred to. By doing so, we found six different notions of creativity as reflected in the articles. Statements, which composed creativity, could either be assigned to these notions or represent further specifications.

INTRODUCTION

The concept of mathematical creativity is difficult to grasp. Are well-known mathematicians – as persons – creative? Or is it just their proofs – as products – that are creative? Or is it rather the process leading to a solution that is creative? These questions refer to Rhodes' (1961) *four Ps of creativity*, in which Rhodes distinguishes between creativity of a person, a process, a product, or press (the environment). On the other side, there are conceptualizations of creativity focusing, e.g., on the moment of illumination after some time of conscious and unconscious confrontation with a problem (see Hadamard, 1945). Creativity can be examined from different perspectives. Research, including research from mathematics education, makes use of this diversity and investigates creativity in a multifaceted way. These different approaches as well as the increasing interest in this field lead to a growth of research on mathematical creativity (Leikin & Sriraman, 2017). The aim of this study is to systematically capture and qualitatively illustrate different notions of creativity in contemporary mathematics education research.

THEORETICAL BACKGROUND

In the following, different perspectives on creativity are described to give a short introduction into creativity. Both psychological and mathematics education perspectives are taken into account, as both are strongly interwoven in the field of creativity.

One of the pioneers in this field, coming from psychology, was Guilford. In his work "the structure of intellect", Guilford (1956) deals with intelligence, naming convergent and divergent thinking as essential components of creativity. In particular, divergent thinking is assumed to be composed of fluency, flexibility, originality, and elaboration. Torrance (1974) refers to this seminal work and developed the Torrance Test of Creative Thinking (TTCT) to measure creativity. Some researchers in mathematics education use this framework by assessing creativity through examining students' fluency, flexibility, and originality (e.g., Leikin & Lev, 2013).

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 440-447). Pretoria, South Africa: PME.

Hadamard (1945), in the field of mathematics, provides another approach to creativity. He describes the path to an invention in mathematics through a sequence of stages of initiation, incubation, illumination, and verification, and builds on the fundamental example of Poincaré (1948). This approach is also used in mathematical education as a framework. Liljedahl (2013), for example, emphasizes the affective moment of illumination as an essential part when working creatively.

In research on creativity, Rhodes (1961) identified the 4Ps of creativity, namely person, product, process, and press. Rhodes points out that different aspects can be examined and described as creative. For example, people can be considered creative, or only a single product or process, such as a famous work of art or a new process. With press, Rhodes refers primarily to environmental influences that are in constant interaction with the other strands.

These frameworks are different from each other and address various scopes. It is therefore a particular challenge to outline the construct of mathematical creativity as clearly as possible in theory, so that the research based on it is coherent and can be discussed within the respective framework. Any vagueness in the theoretical orientation can lead to difficulties in a discussion, which should be carefully prevented (Joklitschke, Rott, & Schindler, 2017). Of course, there have been literature reviews before (e.g., Sriraman, 2009); our focus lies on the notions of creativity in contemporary research. In order to present this systematically and completely, we conducted a systematic review, which is intended to give an overview and facilitate a well-founded discussion. Therefore, the following research question arises: *Which notions of mathematical creativity are referred to in current empirical articles in mathematics education research?*

METHODOLOGICAL CONSIDERATIONS

Searching procedure

To answer the research question, a systematic review was conducted. Gough, Oliver, and Thomas (2013) have described how a systematic review can be realized in the field of politics; Nilsson, Schindler, and Bakker (2018) used this approach in order to elaborate on theories in statistics education and thus successfully transferred it to mathematics education research. In preliminary work, Joklitschke, Rott, and Schindler (2018) used this to create a foundation for this article. This article can therefore be seen as a continuation of the former article. On the one hand, the paper at hand expands our previous by adding a second database (namely MathEduc), which is specific for mathematic education research and thus complements the psychological side of the database PsycINFO®. On the other hand, this article deals with the qualitative findings arising from a systematization of the prior study's findings: We use the following eight keywords that were identified in our previous works, which are used as synonyms for creativity and therefore suitable for our literature search: creativ*, invent*, innovate*, illuminat*, divergent think*, aha*, overcome* fixation, and bisociat*

Thus, two databases were searched for articles: PsycINFO® and MathEduc, with only articles that were published in *Web of Science* listed journals or in A* or A journals (Törner & Arzarello, 2012) being included. By using two databases, the psychological as well as the mathematical educational perspective was taken into account.

The search interval for publications is 10 years, ranging from 2007 to 2016 (at the time of the search, 2017 and 2018 were not yet fully included in MathEduc).

Coding of the articles

The articles found were then analyzed. For this, all parts of the articles up to the research question or the methods, respectively, were read and so-called *statements* were coded. Statements are sentences with a reference to the literature that contribute to the theory construct of creativity – this means that sentences without reference or about empirical findings were not coded. This strict procedure makes it possible to work out the theories which the authors of the articles refer to.

The statements were marked and then inductively merged into categories. In order to give a first qualitative insight into the notions of creativity, statements of the category *definitions* and *components* were considered in this article. *Definitions* are statements, which explicitly define creativity; *components* are statements, which name a closed list of properties. The *definitions* were then clustered for coherence of content. They therefore reflect different notions of mathematical creativity. The same was done with the *components* afterwards.

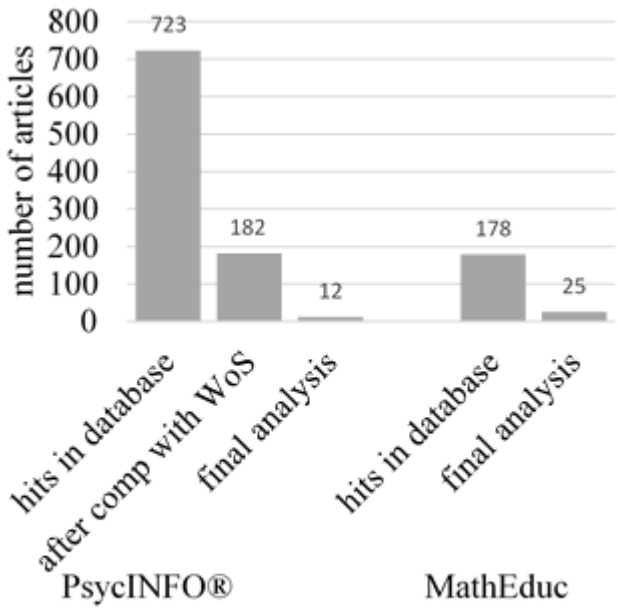

Figure 1: overview about the number of found and included articles in two databases

RESULTS

Overview of included articles

In total, 37 articles from two databases were included in our analysis (Fig. 1).

Notions of creativity

In those 37 articles, the definitions were grouped into coherent groups, thus reflecting different notions of creativity. In the following, these notions (which can be recognized by the italic headings) are presented and illustrated with examples.

Person-, process- and/or behavior-based notion of creativity. One of the most prominent notions of creativity found in our review focusses on who or what can be creative, or on the interplay between these facets, similar to Rhodes (1961). Shriki (2010, p. 160), for example, gives the following definition: “Sriraman (2009) […]

argues that […] 'it is sufficient to define creativity as the ability to produce novel or original work (pp. 14–15).'" With this statement, Shriki refers to products which can be regarded as creative or not.

Flexibility, originality and/or other aspects as notions of creativity. This notion of creativity originates in the works of Guilford (1967) and Torrance (1974). Levav-Waynberg and Leikin (2012, p. 76) write in their study: "The tool is based on Torrance's (1974) definition of creativity, according to which creativity is based on four mutually related components: fluency, flexibility, novelty, and elaboration", referring to the set of four aspects as definition of creativity. The definition by Chen, Chang, and Kuo (2016, p. 428), highlighting mental flexibility as a characteristic of creativity, is rather focused on one of those facets. Referring to Krutetskii (1976), they "define […] mathematical creativity as mental flexibility that enables students to think outside the box and apply novel approaches to solve problems (Haylock, 1985, 1987, 1997; Sriraman, 2004)". This group of notions also includes many statements of the category *components*. Common are constellations such as *fluency*, *flexibility*, and *originality*, but also other combinations such as the one by Leikin and Lev (2013, p. 185), which mention the "creation of new knowledge and flexible problem-solving abilities" as characteristic.

Creative mathematical reasoning (CMR) as notion of creativity. This notion of creativity differs from other notions because it is inherently linked to reasoning. Lithner contrasts creative mathematical reasoning with algorithmic reasoning (see Lithner, 2008). In this context, Jonsson, Norqvist, Liljekvist, and Lithner (2014, p. 22) refer to this definition as follows: "Lithner (2008) defined CMR as fulfilling all of the following criteria: (i) Creativity; a new reasoning sequence (new to the reasoner) is created, or a forgotten one is re-created, in a way that is sufficiently fluent and flexible enough to avoid restraining fixations; (ii) Plausibility […]; and (iii) Anchoring […]."

Notion of creativity as sequence of stages. This kind of notion of creativity comprises statements that could be summarized representing a sequence of phases in a certain way. The descriptions of most of the statements go back to the model of Hadamard (1945) with his stages of *initiation*, *incubation*, *illumination*, and *verification*. Liljedahl (2013, p. 254) regards Poincaré's (1948) remarks as so fundamental that he defines the illustrations of the passing through the stages and the experience of the illumination of these stages as creativity: "[…] so deep were his [Poincarés] insights into his acts of invention and discovery that it could be said that he not so much described the characteristics of mathematical creativity, as defined them."

Notion of creativity as divergent thinking. In our review, we also found statements that deal with Guilford's (1967) and Torrance's (1974) remarks of divergent thinking in the conceptualization of creativity. "[C]reativity is defined in a number of ways by scholars, including the capacity to produce ideas related to divergent thinking rather than convergent thinking (Guilford, 1959)" (Kim, Roh, & Cho, 2016, p. 39). Other authors do not only emphasize the divergent aspect, but in particular argue that the interplay between convergent and divergent thinking is characteristic for creativity.

Notion of creativity as a learning activity. We found one statement that, in contrast to the others, was very wide-ranging. Liu, He, and Li (2015, p. 142) cite this definition here: "Similarly, Kember and Leung (2009) defined 'creative thinking' as a learning activity of students in Hong Kong.". However, this type of definition has only been used once.

Distinctions from a clear definition. Some statements refer to the fact that there is no clear definition or that the authors deliberately distance themselves from a definition, e.g., "Mann (2006) argues that there are more than 100 contemporary definitions of creativity" (Leikin & Lev, 2013, p. 184). Although this is not regarded as a notion of creativity, those statements contribute to the research interest here in the sense that the authors explicitly point out how difficult it is to grasp the concept of creativity.

Additional components of creativity

Hence, we see that the statements of the category *definition* can be clustered into six different notions of creativity (see Fig. 2). If we take a look at the statements that have been categorized as *components*, we see that many of those could be assigned to these notions (most could be assigned to the notions "*flexibility, originality and/or other properties as notions of creativity*" and "*notion of creativity as divergent thinking*"). However, there are also some statements that extend the spectrum of notions. In the following, two main groups will be presented.

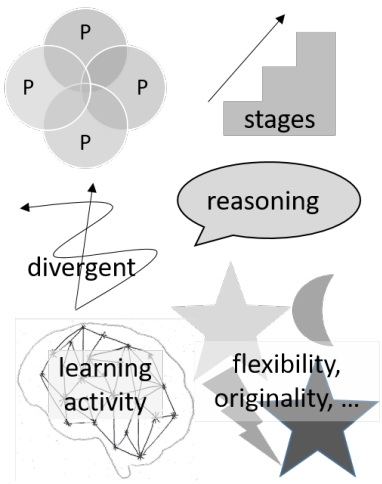

Figure 2: Pictograms of the notions of creativity

The statements presented here relate more to the scope of creativity. Leikin and Lev (2013, p. 185) formulate: "Leikin (2009) suggested that […] creativity […] in school-children requires a distinction between relative and absolute creativity." With this assertion, the authors refer to the discussion of *who can be creative?* Historically, mathematical creativity was only assigned to outstanding professional mathematicians. Meanwhile, however, research assumes that students can also be creative. In our review, some *components* address this contrast. Other terms than *absolute* and *relative* in this context are *big-C* and *little-c*.

Another group of *components* focuses on the range of creativity. Many researchers make a distinction between domain-general and domain-specific creativity, such as Peng, Cherng, Chen, and Lin (2013, p. 53): "[d]omain-general means that a creative individual can extract his/her creativity in any domains, whereas domain-specific means that an individual can only extract his/her creativity in a certain domain […] (Hong & Milgram, 2010; Silvia, Kaufman, & Pretz, 2009)." Other authors specify this further by speaking of subdomain-specific or task-specific creativity.

In our analysis, we could also find further statements within the category *components*. These could either be assigned to the above notions or to the scopes presented here, or did not represent a significant part and are therefore not further discussed here.

DISCUSSION AND OUTLOOK

The literature review presented here constitutes a continuation and extension to Joklitschke et al. (2018). In the previous study, preparations were made such as the search for synonymous terminology, which extend and complete the hits in the search. The extracted statements were also grouped into categories. In the present study, a further database was added, namely MathEduc, which is specifically directed towards mathematics education research. With these two databases, we were able to increase the number of journals even further (from seven to eleven different journals). A qualitative examination of the coded statements in the *definitions* category revealed six different notions of creativity (see Fig. 2 and subheadings in the results part). This list is, of course not irrefutable. In our analysis, we only looked for *definitions* and *components* and only in well-chosen journals. Some statements also deliberately distance themselves from *definitions*. In addition, the statements of the category *components* were also qualitatively examined and could largely be assigned to the perceptions of creativity. Further, *components* could be clustered into new groups and thus represent different areas of application. Here, the comparisons of absolute and relative creativity as well as the consideration of domain-related and domain-specific creativity are predominant. The additional scopes represented by the *components* show that essential properties often cannot simply be represented by a definition and that it is therefore reasonable to have a wider view regarding the notions of creativity.

There are some limitations to our approach: (1) The coded statements do not necessarily reflect the opinion of the authors of the respective articles, but merely represent which conceptualizations are mentioned. (2) The propositional content does not have to agree with the authors' opinion of the primary source; there may always be different formulations or slightly different interpretations. (3) There were many sentences, which conceptualize creativity but were not linked to a reference and, therefore, not coded. With this restriction we wanted to make sure that we capture those notions that are theoretically based. (4) We have only read those contributions that fit under the framework conditions mentioned above. Thus, conference papers or chapters in books, older and newer articles, as well as articles of other journals were not considered. In order to continue the study presented here, it is now interesting to see what other descriptions of creativity are given in the articles considered. *Descriptions* do also form a category within the extracted statements. In addition, statements were coded that reflect a *relation* between creativity and another construct. Here is of interest to see in which contexts creativity is addressed. The review reflects the notions of mathematical creativity in current research and thus further contributes to the theoretical field in this area.

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SECOND GRADERS' FIRST MEETING WITH VARIABLE NOTATION

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This paper reports on an empirical study of second-grade students' emerging understanding of variable notation in the context of functional thinking. I describe the mathematical practices that became taken-as-shared in the classroom and the levels of sophistication in the students' thinking about variable notation, as the teacher implemented a hypothetical learning trajectory inspired by early algebra research. The findings confirm the results of former research on students' understanding of variable notation and contribute with knowledge about the mechanisms that promote shifts in young students' thinking when they explore functional relationships.

INTRODUCTION

Generalizing, representing and reasoning with generalizations are widely viewed as core aspects of early algebraic thinking (Stephens, Ellis, Blanton, & Brizuela, 2017). However, there are different views on the role of variable notation in relation to these forms of algebraic thinking. Few would disagree that algebraic thinking ultimately involves conventional variable notation, but the question is when and how to introduce it? Radford (2011) argues that algebraic thinking in the early grades is not about using letters but about reasoning with indeterminate quantities in ways, that do not include conventional notation. Fujii and Stephens (2008) argues that number sentences and quasi-variable thinking should be used in the early grades as a bridge from arithmetical operations to the idea of variables. Finally, Brizuela, Blanton, Sawrey, Newman-Owens and Gardiner (2015) argue that children should be introduced to variable notation from the beginning of their schooling. I address this issue by contributing with an analysis of the development in second graders' (8-9 years) understanding of variable notation as it emerged in the classroom. My research questions were: Which conceptual understandings related to variable notation can second-grade students develop, as they explore functional relationships, and which mechanisms promote shifts in their thinking about variable notation?

Research on elementary students' understanding of variable notation

It is well documented that adolescents often have problems with the concept of variable and variable notation. The difficulties include understanding the use of letters as generalized numbers or as variables (Vergnaud, 1985) and the use of mathematical symbols to express relationships between quantities (Bednarz & Janvier, 1996). One could expect that such difficulties would be even more conspicuous for younger children, but recent research question this. In particular, research on students' understanding of variable and variable notation within the context of functional

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thinking seem promising. In one study, Carraher, Schliemann, and Schwartz (2008) found that students in grade 2-4 successfully could use variable notation to represent a (simple) linear functional relationship between two amounts of candy. Blanton, Stephens, Knuth, Gardiner, and Kim (2015) found that third-grade students participating in a 1-year early algebra intervention were more successful representing a linear relationship of the form $y = mx$, $m \in N$, with variable notation than with words. Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015) found that even first-graders could represent functional relationships using variable notation and develop understanding of variables as varying quantities that can be treated as an object in reasoning with general forms. These findings suggest that difficulties with variable notation are less about age and premature use of formalism, than about the learning opportunities offered to students and the conflicts the notation creates in relation to their experiences and understandings from non-mathematical contexts. In this line Brizuela et al. (2015) argue that variable notation can act as a mediating tool to facilitate children's reflections about indeterminate quantities and that conceptual understanding do not necessarily need to precede the introduction of variable notation, but rather that meanings and symbols can co-emerge. The different perspectives suggest that more research on how children understand and represent variable quantities is needed. The study presented in this paper is based on research that supports the belief that understanding of algebraic concepts and symbols can co-emerge, even in 2nd grade. It has much in common with a study performed by Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens (2017), which presents an empirically developed progression in first-grade children's thinking about variable and variable notation.

RESEARCH DESIGN

The study reported here forms part of a long-term early algebra project (2017-2020), framed as a classroom design study (Cobb, Jackson, & Sharpe, 2017), in which I collaborate with an expert mathematics teacher on ongoing analysis of 2-4 grade students' algebraic learning and how to support this learning. We centre our research on the use of a classroom teaching experiments (CLT) and on the design of a hypothetical learning trajectory (HLT), consisting of (a) learning goals, (b) an instructional sequence and (c) a developmental progression that specifies increasingly sophisticated levels of thinking in which students might engage (Clements & Sarama, 2004). In this study, the HLT was based on findings from early algebra research regarding children's functional thinking, especially findings which show that students already in first grade are able to explore correspondence relationships between two sets (Blanton, Brizuela, et al., 2015). Lessons in the instructional sequence were based on variations of tasks, which have been used successfully in previous research on children's functional thinking (Blanton, Stephens, et al., 2015; Carraher et al., 2008) These tasks focused on identifying and representing correspondence relationships between two sets and/or to justify and reason with such relationships. The functions in

the sequence progressed from the type $y=ax$, over $y=x+b$ and $y=x-b$ to $y=ax+b$ (See Table 1 for examples).

Week	Examples of tasks and function types
1	"Rectangles with the (fixed) side 4": $y = 4x$ or $y = x + x + x + x$. Find relationships between the height and the total numbers of unit squares in the rectangle.
2	"Saving": $y = x + 3$ or $y = x - 3$. Lucca and Albert save an equal amount of money each week. Before they started saving, Lucca had 3 d.kr. which she has kept. How can you express the relation between how much money Lucca has and how much money Albert has? In words? In numbers? In letters?
3	"Tables and chairs": $y=6x$ and $y=4x+4$. Find a relationship between the number of tables and chair (in different settings).
4	"Towers of centicubes": $y=4x + 2$. Find a relationship between the number of centicubes in a "tower" and the number of visible squares on the "tower".

Table 1: Examples of tasks in the planned instructional sequence

We designed the instructional sequence as a four-week course, each week including four 45 min lessons. Participants were one second-grade class (25 children) in a Danish elementary school.

Implementation of the instructional sequence

Each lesson typically consisted of three phases. In the first phase, the teacher set the scene for the mathematical problem of the day by telling a story aimed at engaging the students in the problem and giving them a context to support their thinking. In the second phase, the students worked in small groups with the problem, and the third phase consisted of a class discussion on the students' different strategies and solutions. In this discussion, the students should focus on explaining their thinking, describing their strategies, and justifying their solutions. The teacher was focusing on highlighting mathematical points and guiding the students' social and socio-mathematical norms (Cobb, 1999), e.g. by emphasizing that, they were expected to try to make sense of the explanations given by their peers, to indicate agreement or disagreement and by signaling what counted as an acceptable mathematical explanation or an insightful solution. As part of the discussion, the teacher introduced variable notation as a way (among others) to represent variable quantities. In particular, the teacher introduced variable notation in the context of representing student's rules for functional relationships.

I observed and videotaped the lessons. After each lesson, I met with the teacher to discuss observations regarding the students' thinking about concepts addressed in the tasks and to discuss possible revisions for the subsequent lesson. In our ongoing

interpretation, we focused not only on classroom tasks and tools, but also on the emerging mathematical practices in the classroom and on the nature of the social- and sociomathematical norms, and the quality of classroom discourse as potential supports for the students' learning. Thus, we took the position, that students' mathematical learning is situated with respect to classroom social norms and that the teacher and the students establish mathematical practices collectively (Cobb, 1999).
The teacher conducted individual, semi-structured 30-min pre- and post-interviews with five students, selected to represent the academic level of the class. Both interviews were based on the students' work on function tasks. The purpose of the pre-interview was to assess the students' current understanding of concepts related to functional thinking and to identify aspects of students' current reasoning on which instruction could capitalize. The purpose of the post-interview was to assess the students' individual mathematical practices as well as to identify levels of sophistication in their thinking about variable notation.

Data analysis

The empirical data consist of video recordings and the students' written work from the lessons and the interviews. I used a grounded theory approach (Strauss & Corbin, 1990) and my data analysis focused specifically on identifying

- different levels of sophistication in student's thinking about functional relations and representations of these relations
- mathematical practices that became taken-as-shared in the classroom, that is practices that did no longer need explanation or justification (Cobb, 1999).

I did the analysis in five phases. First, I sorted sequences of the video recordings into two groups: 1) the forms of functional thinking that student expressed in the sequence ('Identifying', 'Representing', 'Justifying' and 'Reasoning') and 2) the social norms, socio-mathematical norms and mathematical practices expressed in the sequence. Secondly, I focused on 'Representing' and 'Mathematical practices'. I revisited the sequences in these categories and constructed memos to characterize the mathematical practices the students engaged in, and the incidents in which students used variable notation or discussed their understanding of variable quantities. The memos consisted partly of descriptions of the mathematical practices and students' thinking, and partly of possible interpretations of this thinking.
In the third phase, I sorted my memos according to the type of practices and the type of thinking expressed in them. I gave each group of memos a preliminary code, which reflected the type of practice and thinking within each group. Then I organized the memos of the mathematical practices according to the temporal order in which they appeared in the classroom, and the memos of students' thinking according to the level of sophistication expressed in them. In the fourth phase of the analysis, I reviewed the video sequences, and coded them according to the mathematical practices and the levels of thinking exhibited. Throughout this process, I refined existing codes and created new codes to reflect findings from the analysis. I continued this refinement

until no new findings emerged in the data. In the fifth phase of the analysis, I focused on relationships between the mathematical practices, I had identified, and the sequences categorized as social and socio-mathematical norms. My intention with this part of the analysis was to describe the relation between the identified mathematical practices and the social and sociomathematical norms in the class (Cobb, 1999).

FINDINGS AND DISCUSSION

I identified five different mathematical practices that emerged in the class during the CTE (See Table 2).

Practice number	Classroom mathematical practices
1	Using a function table to look for patterns
2	Describing calculations related to functional relations to look for patterns in them
3	Representing (general) functional relationships in different ways
4	Making sense of expressions with letters and reason with these expressions
5	Representing functional relationships with variable notation

Table 2: Classroom mathematical practices

The transition from one of these mathematical practices to another mathematical practice involved a shift in the focus of attention and in the nature of the discussions in the class. For example, the transition from practice (1) to (2) involved a shift in the students' attention from recursive thinking to correspondence thinking. It also involved a shift in the nature of the discussions in the class, such that the focus was on the relation between the context and the numbers, rather than just on the numbers. One of the tasks was about the relationship between the height and the total numbers of unit squares in a rectangle with a fixed side length of 4. Most students relatively quickly filled out a function table and discovered that the numbers in “the left side” of the table grew by 1, and the numbers in “the right side” grew by 4. Their arguments were based on calculations of the differences between the numbers. A shift occurred, when the teacher asked the students how they calculated the total numbers of unit squares. Some students explained that they could see the same number of units four times on their drawings of a rectangle, and when they knew the number of units in the height of the rectangle, they could just add this number four times. The calculation related to the rectangle with the side length 4 was thus 4 + 4 + 4 + 4. In similar ways, other students argued that the calculation related to the side length 5 could be 5 + 5 + 5 + 5. In the new mathematical practice (2) that emerged attention was thus directed to the correspondence relationship, and the students' arguments were based on the quantities represented. The mechanisms that promoted the shift can be understood as a combination of the task involved, the students’ drawing of rectangles (they could *see*

the total numbers) and the students' mathematical explanations and justifications. The students in the class were used to explain and justify their mathematical thinking with reference to the quantities represented. Other students therefore had the opportunity to interpret the relationship between the independent and the dependent variable in terms of actions on e.g. a geometric representation. Along with representations of the functional relationships that the students explored, this sociomathematical norm supported the emergence of the new mathematical practice (2). It is likely, that the shift would not occur, or that the shift would have been different, if the sociomathematical norms in the class were different.

The transition from mathematical practice (4) to (5) provides another example of the mechanisms that promoted shifts in students' thinking. To support the students' understanding of variable notation, the teacher focused on tasks, which had algebraic notation as a starting point. For example, the teacher described a family's "rules for pocket money" with the expression $A + 15$ and explained that the rule showed how much money children in different ages would get. The students' task was to make sense of the expression. At the beginning of a class discussion on this issue, a student suggested that the term could mean that you would get 15 d.kr. in pocket money when you had reached a certain age. Another student suggested that the expression could mean that you would get 15 d.kr. more for every year you grew older. However, these students changed their perception as a consequence of others' arguments in the discussion that had the character of an investigation. In this context, it was crucial that it was a social norm in the class to try to make sense of explanations given by others and to indicate agreement or disagreement.

Table 3 summarizes the levels of sophistication that I found in students' thinking about variable notation and the following quotes exemplify the levels. The levels show a possible progression in the students' development, but students may skip levels or revert to lower levels when the settings change. However, the five students in the post-interview all showed thinking at level 4 or 5.

Level	Code	Characteristic
0	Pre-symbolic	No meaning is given to letters in a mathematical context
1	Pre-algebraic	Letters are given meaning associated with past experiences, but not in a way that is meaningful in an algebraic context
2	Letters as labels	Letters denote an object or a person, but it is unclear whether the letter at the same time represents a number value
3	Letters as specific values	Letters denote a specific numerical value

4	Letters as variables	Letters denote an unknown numerical value that can vary within a domain
5	Variable notation as object	Expressions with letters are perceived as objects that can be used to reason with.

Table 3: Levels of sophistication in children's thinking of variable notation

Student 1: You cannot use letters in mathematics". (Level 0)

Student 2: H must be 8, because H is the 8th letter of the alphabet". (Level 1)

Student 3: H represents Heidi´. (Level 2)

Student 4: H represents how many years Heidi is. So it must be 44". (Level 3)

Student 5: S can actually be a lot of numbers. It can be, for example, 4, 10 or 100. It can be numbers up to infinity" (Level 4)

Student 6: $4 \cdot n + 2$ cannot be an odd number, because all the numbers in the 4-table are equal and they remain equal when adding 2" (Level 5).

There are some differences between this part of the findings of the study reported here and the results of a similar study in an American grade-one classroom (Blanton et al., 2017). However, the most striking is the similarities between the analyses, which have been done independently. Findings in both studies suggests that young students can learn to think in sophisticated ways about variable notation, and indicates that conceptual understanding of functional relationships and symbols can co-emerge. This, of course, does not mean that the question of when and how to introduce variable notation is settled. Such a decision should be based on a broader basis. Among the questions, whose answers can further inform the role of variable notation in early algebra, is the question of how students understanding of variable notation and their understanding of functional relationships relate to each other. Answering this question requires more studies to shed light on *the ways* in which students' conceptual understandings and understandings of variables co-emerge.

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INDICATORS OF PROSPECTIVE MATHEMATICS TEACHERS' SUCCESS IN PROBLEM SOLVING: THE CASE OF CREATIVITY IN PROBLEM-POSING

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This study examines the effects of the fluency, flexibility, and originality components of creativity, as reflected in problem-posing, on the problem-solving abilities of prospective teachers. A problem-posing task regarding pattern subject and a problem-solving test were administered to prospective teachers, 12 of whom provided further data through semi-structured interviews. Prospective teachers with high scores in the flexibility and originality dimensions were found to make more in-depth analyses of the math concepts in the task, and were able to come up with more problems involving different math concepts. The results of the quantitative analysis showed that flexibility and originality scores were better indicators of problem-solving achievement than scores related to fluency.

INTRODUCTION

Problem-solving (PS) is one of the basic skills emphasized in studies of math education. In recent years, two other skills, problem-posing (PP) and creativity, have attracted researchers' attention and have started to be emphasized in education policy documents. The PP and PS skills affect each other, and people who are successful in one are usually also successful in the other (Xie & Masingila, 2017). Leung (1997) has defined creativity as "bringing into being," and in this context, posing problems can be seen as a sign of creativity. The skill of creativity, on the other hand, is usually measured using the dimensions of originality, flexibility, and fluency that were identified by Torrance (1988). Based on the fact that creativity underlies both the PP and PS processes (Silver, 1997), how would the scores achieved in different individual creativity dimensions, measured using PP, affect PS achievement? This study aims to answer this question through both qualitative and quantitative approaches, and thus contribute to a better understanding of the relationships between PP, PS, and creativity.

THEORETICAL BACKGROUND

PP is defined as the generation of new problems or the reformulation of existing problems. There is a strong relationship between PP and PS, with each contributing to the other (Cifarelli & Sevim, 2015; Kilpatrick, 1987). Kilpatrick (1987) defined PP as an important component of PS; Kapur (2018), on the other hand, indicated that PP with solution generation is a more beneficial preparatory activity than PP without solution generation. PP contributes to PS because it is an inquiry-based activity (Silver, 1994).

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Xie (2016) argued that successful PS depends on the development of an in-depth understanding regarding the structure of the problem. In the process of PP, on the other hand, the math concepts in the activity are analyzed in depth, and the validity of the problems is questioned (Xie & Masingila, 2017). Thus, PP contributes to PS success by facilitating the questioning process of the PP structure. In this context, the number, variety, and rarity of the problems created in PP tasks serve as predictors of PS achievement. Torrance (1988) referred to these phenomena as originality, flexibility, and fluency, and defined them as the main components of the creativity skill. While fluency deals with the total number of problems posed by each participant, flexibility refers to the number of problems with different structures posed by each participant. The dimension of originality, on the other hand, refers to the frequency of the posed problems that are created by less than 10% of participants (Harpen & Sriraman, 2013).

There are some studies in the literature combining PP, PS, and the components of creativity. Such previous studies measured the creativity skills of students using PP (e.g., Harpen & Sriraman, 2013; Yuan & Sriraman, 2011). In the study by Harpen and Sriraman (2013), the creativity skills of high school students were examined; the authors found that creativity scores were not as high as expected in each dimension, and that the problem types posed were concentrated within limited categories, such as length and area. Other studies made use of experimental methods and involved some of the components of creativity in the process. For example, Chen, Dooren, and Verschaffel (2015) found that an intervention based on PP and PS was more effective at improving students' PS skills. In their analysis of the development of PP skills, they looked into the dimensions of appropriateness and originality, among which the dimension of appropriateness corresponds to the fluency dimension of creativity. The authors identified a statistically significant improvement only in originality, in that originality requires the addition of new data to the situation described in a problem (Yuan & Sriraman, 2011). Accordingly, this component of creativity can be seen to make a greater contribution to the development of PS skills by facilitating in-depth inquiry. The studies mentioned provide no guidance on which components of creativity make a greater contribution to PS achievement, whereas the present study aims to answer this question through the combined use of qualitative and quantitative approaches.

METHODOLOGY

Participants. The study was conducted with the participation of 64 prospective mathematics teachers attending the final year of a primary mathematics education program at a university. Prospective teachers graduate after a four-year program and undergo theoretical and practical training on PP and PS in two mathematical instruction courses taken in their third year. All of the participants had PP-related experience, but none of them had attended any courses focusing exclusively on PP or PS. Each prospective teacher was assigned an identification code (e.g., PT_1, PT_2).

Data Collection and analysis. A PP task involving patterns and a corresponding problem-solving test (PST) were prepared for the study. The PP task involved a pattern

concerning the ordering of blocks. The test was administered to participants during a class hour, after which a six-item PST was prepared based on the variety (fluency, flexibility, and originality) of the problems posed by the participants. Figure 1 presents the PP task and two sample questions from the PST.

Problem-posing task: The following picture depicts the first three steps of a pattern of blocks. Please try to pose as many different and difficult problems as possible, and any problems that you think are somehow connected to this pattern.

1.Term 2. Term 3. Term

Sample Questions from the Problem Solving Test:

1.Term 2. Term 3. Term

1. In the pattern shown above, blocks are colored white, red and blue following an order. What is the color of the last block at the top in the 14th term of the pattern?

2. Given this pattern, what is the ratio of the number of blocks in the 18th step to the number of blocks in the 72nd step?

Figure 1: Problem-posing task and sample questions from the PST

Responses to the PP task were subjected to a two-step analysis based on the analysis schema used in previous studies (Chen et al., 2015; Harpen & Sriraman, 2013; Yuan & Sriraman, 2011). In the first step, the goal was to identify problems that were incompatible with the structure of the task or that contained mathematical errors. These responses were coded as non-viable problems. In the second step, responses coded as viable problems were analyzed on the basis of fluency, flexibility, and originality. For the statistical analyses, quantitative scores were calculated for each creativity dimension. The fluency score was calculated by assigning one point to every viable problem. The first two authors of the study created problem categories by separately coding the papers of 20 participants, after which common problem categories were created by comparing these analyses. The resulting categories were then used to analyze the responses of the other participants. The responses that did not fit into any of the categories were discussed again, and new problem categories were added. Consequently, a total of 25 problem types were identified in the participants' responses. The flexibility score was based on a calculation of the total number of viable problems with different types. To assign the originality score, problem types used by less than 10% of the participants were identified. Given the total of 64 participants, any problems duplicated by seven or more participants were not considered original; following the analyses, 13 problem types were coded as original. If participants posed two original problems in their responses to the PP task, these participants were assigned an originality score of 2.

To examine the process of PP in terms of creativity, interviews were conducted with a total of 12 participants, including both those who produced original answers and those who did not. In these interviews, participants were asked to answer certain questions such as *What was your thinking like when you posed the problem? Are there any problems that you consider to be different and interesting among your responses? If*

so, how was your thinking different when you posed these problems? These questions were aimed to clarify the reasons behind the participants' different fluency, flexibility, and originality scores when posing problems.

The solution to each problem on the PST was scored out of two points. A score of 0 was assigned if no solution was provided, or if the solution conflicted with mathematical rules and procedures; 1 point was assigned if the procedure in the solution was logically valid but there was a calculation error; and 2 points were assigned if both the approach and the final answer were correct. Thus, the maximum total score on the PST was 12 points. A multiple linear regression analysis was conducted to examine how creativity scores based on PP could predict PS achievement.

RESULTS

The prospective teachers posed 291 problems, averaging 4.55 problems each. Of these, 17.2% (50 problems) were classified as non-viable problems. A total of 18 of the problems in this category contained mathematical errors and 32 were not related to the pattern. One of the problems that contained a mathematical error was as follows: *Blocks with side lengths of 100 cm, 50 cm, and 30 cm are used to form staircase, as shown in the picture. Which term corresponds to the stairs with a volume of 1000 m^3?* (PT_2).

The individual participants posed a minimum of one and a maximum of nine viable problems. The mean fluency score was 3.77, and there were 25 problem types in the viable problem category (Table 1). The maximum flexibility score received by a participant was 8, and the minimum was 1. The mean flexibility score was 3.19. To calculate originality scores, problems posed by less than 10% of the participants were identified (Table 1: Q_{3b}, $Q_{4(a-b)}$, $Q_{5(a-b-d)}$, Q_7, Q_9, Q_{10}, Q_{11c}, and $Q_{12(a-b-c)}$). The maximum originality score received by a participant was 3, and the minimum was 0. The mean originality score was 0.34. One participant, PT_7, posed a problem that was very similar to the first sample from the PST reported in Figure 1. This problem asked the color of the last step of the eighth term in the pattern and was coded as original. Another example that was coded as original was as follows: *One step is left when Ali climbs the steps three at a time, and two steps are left when he climbs four at a time. Which term of the pattern is the first to meet these conditions?* (PT_8).

The interviews revealed that the participants took three aspects into account when posing problems: i) prior experience, ii) the visual characteristics of the pattern, and iii) context and associations. Participants with high fluency and flexibility scores emphasized the first and second items, and those with high originality scores emphasized the third item. The participants had prior experience with questions similar to Q_3, Q_8, and Q_{14} from textbooks, and 41.6% of the 291 responses (121 problems) featured these types of problems. In addition, 47, 30, and 20 participants posed problems regarding one of these categories in their first, second, and third answers, respectively. Participants who had difficulties coming up with different problems stated that starting out with a problem of this type allowed them to better analyze the pattern. Some of the participants said that they were more focused on avoiding any

mathematical errors in their problems, and therefore posed routine problems with less risk of making an error. One participant, PT_9 gave the following explanation: *I thought really hard about how to come up with a different problem, but this led me nowhere. I thought maybe I should start with a simple question and the rest will follow. This is why I posed this problem* [a problem of type Q_8]. *We see these types of problems very often when studying the patterns subject.*

Problem types	
Q_1. Volume	Q_7. Slope
Q_2. Viewing the stairs from different perspectives	Q_8. The number of blocks in a given term
Q_3. Pattern rule	Q_9. Climbing up/down the stairs corresponding to a term with three steps
a. Asking about the general term	Q_{10}. Asking about the number of rectangles on the side at a given term
b. Finding the block numbers that are perfect squares	Q_{11}. Asking questions given the lengths of edges
Q_4. Climbing up/down the stairs with a single rule	a. Asking about the number of blocks given the height of the stairs
a. Given the p-step movement and the total number of blocks, asking for the total number of steps to climb the stairs	b. Asking about the lengths of the side edges at a given term
b. Knowing the position on the stairs according to the starting point after moving the p-step backward k-times, asking for the total number of blocks	c. Asking about the distance from a given point on the stairs to another point outside
Q_5. Creating new patterns by adding data to the existing pattern	Q_{12}. Climbing up/down the steps with two rules
a. Posing problems by placing consecutive numbers on the side faces	a. Asking about the total number of steps needed to be taken to reach the kth step on the stairs by climbing up p_1 steps and climbing down p_2 steps in every round
b. Coloring the side faces white (W), red (R), and blue (B) following an order; asking about the color of a block	b. Asking the location reached after k steps are taken by climbing up p_1 steps and climbing down p_2 steps in each round
c. Asking questions about situations in which each block consists of a different number of other identical blocks	c. Asking which terms meet both of the criteria p-steps at a time + remaining d_1 and s-steps at a time + remaining d_2
d. Coloring the side faces W, R, and B following an order; asking about the rule for blocks that contain WRB faces	Q_{13}. Providing prices for coating/painting individual blocks and asking about the total cost of any stair corresponding to the given term
Q_6. Comparison	Q_{14}. Asking the order of the term by providing the number of blocks
a. Multiplication b. Addition	Q_{15}. Area

Table 1: Problem types identified in the responses of prospective teachers

Some of the participants said that the visual features of patterns, such as surface areas and volumes, attracted their attention, and indicated that they focused on these features when posing problems. Categories Q_1, Q_2, Q_6, and Q_{15} in Table 1 are of this type, and

together made up 22% of the responses (64 problems). The participants received higher fluency and flexibility scores by identifying a visual feature that attracted their attention, and then adapting this feature to problem structures with which they were familiar. One participant, PT_{12} gave the following explanation: *I first took note of the visual. I thought of volumes because there were blocks, and I posed a volume problem* (Consistent with these explanations, PT_{12} posed Q_1, Q_2, and Q_3 type problems, in this order.). The generation of original problems, on the other hand, was based on the creation of stories that fit the pattern and allowed for associations to be made. Participants who posed original problems were focused on creating contexts that fit the pattern and adding different mathematical concepts. Creating an interesting problem that fit the pattern was found to encourage the use of different mathematical concepts. Furthermore, some of the participants said that after creating a well-thought out and interesting scenario, a variety of problems followed naturally, contributing to variety, which was measured by flexibility. All three of the problems posed by PT_5 were coded as original, and the participant gave the following explanation:

> *I tried to come up with a different problem based on stair. With stairs, you first think of asking about the number of steps. I thought about how I could make it more interesting. Then, I thought about climbing up and down the steps. I wrote the problem. Then, I wrote new questions changing the rules about climbing up and down the steps. For example, I thought about moving by turning backward while climbing up/down following a certain rule.*

A total of 59 participants were included in the PST, with a mean test score of 8.22 and a standard deviation of 2.57. A multiple linear regression was employed to estimate the effects of fluency, flexibility, and originality scores on PST achievement. The results showed that creativity scores significantly predicted PST achievement (F (3,55) = 21,642, p <.000, R^2 = .516). Thus, 51.6% of the variance in the PST achievement was explained by the fluency, flexibility, and originality scores. The standardized regression coefficients (β) for flexibility (β = .358, p = .042<.05) and originality (β = .291, p = .003<.05) indicated that they make a statistically significant contribution to explaining the PST achievement. On the other hand, although the zero-order correlation was found to be moderate between fluency and PST, the effect was not statistically significant (β = .302, p = .083) at the .05 level.

DISCUSSION

The results of this study indicate that the flexibility and originality dimensions of creativity predicted PS success, whereas the fluency dimension did not have a predictive effect on it. As supporting evidence for this finding, Silver and Cai (2005) indicated that fluency scores can be seen as an insignificant way to evaluate students' creativity. According to findings gathered from interviews, we found that participants with high fluency scores posed simple problems based on the visuals of the pattern in order to avoid making mathematical errors by benefitting from the problem types they had previous experience with. Posing these types of problems increased their fluency scores, but did not allow an in-depth analysis of the mathematical structure within the

PP task. Similarly, Singer and Voica (2015) emphasized that participants began to pose problems of familiar models, and that these types of responses limited their creativity. On the other hand, participants with high flexibility and originality scores were found to examine the pattern conceptually, to involve different mathematical concepts in the process, and to care more about quality than quantity. Thus, a higher level of flexibility and originality scores allowed the participants to conduct a more in-depth analysis of the mathematical concepts in the activity, making a bigger contribution to their PS success. This is the main reason why the flexibility and originality dimensions are statistically better indicators of PS success. Therefore, we recommended that PP training focusing on improving PS skills should try to ensure that the problems posed are different and original, rather than emphasizing quantity.

Even though originality is the dimension that receives the most emphasis in all definitions of creativity (Kontorovich, Koichu, Leikin, & Berman, 2011), statistically, flexibility was found in the present study to be the dimension with the strongest effect on PS success. Arguably, the main reason behind this finding is that the sample size is a decisive factor in the determination of originality scores. Similarly, when the regression coefficient and the p-value for the fluency score are considered together, it can be argued that statistical significance was not achieved due to the small sample size. Thus, further studies need to be conducted with broader samples and with different grade levels to better understand how the dimensions of creativity predict PS.

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EXPLORING RELATIONSHIPS BETWEEN NUMBER OF HOURS OF PROFESSIONAL DEVELOPMENT, MATHEMATICS KNOWLEDGE FOR TEACHING, AND INSTRUCTOR'S ABILITY TO MAKE SENSE OF PROCEDURES

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This paper explores relationships between the amount of time community college mathematics faculty spend participating in professional development (PD) and their mathematical knowledge for teaching (MKT) with the one measurement of the quality of instruction-making sense of procedures. We present the results of an analysis of data collected during the Fall 2017 semester from a group of community college instructors in the U.S.

BACKGROUND

For several decades now, researchers have attempted to identify and describe the special effect teacher knowledge plays in student achievement (Hanushek, 1972; Hill, Rowan, & Ball, 2005; Rowan, Chiang, & Miller, 1997). Meanwhile other researchers have interrogated the impact professional development (PD) experiences have on teacher knowledge (National Academy of Education, 2009; Bell et al., 2010) and how teacher knowledge influences instructional practices (e.g., Fenema & Franke, 1992; Sowder, Phillip, Armstrong, & Shappelle, 1998). It is logical to hypothesize that robust teacher knowledge anchors classroom instruction and leads to student success. The number of upper-level undergraduate or graduate-level mathematics courses taken by community college instructors is a typical measure used in the United States to determine their mathematical proficiency. The minimum qualification for a community college mathematics-teaching job in the U.S. is a Master's degree and 18 upper division mathematics courses. Unfortunately, this measure (number of mathematics courses taken) does not provide an accurate picture of the specific mathematics that teachers hold or of how their mathematics knowledge manifests. For example, a study of prospective secondary mathematics teachers at three major institutions showed that, although they had completed the upper-division college mathematics courses required for the mathematics major, they had only a cursory understanding of the concepts underlying elementary mathematics (Ball, 1990). As incorporating making sense of procedures in instruction is a necessary component of assisting students in seeing mathematics as a study of relationships rather than steps to be memorized and without sense making mathematical knowledge is "fragile" (Hiebert, 1999, p. 12), thus, it will be used as a measure of instructional quality in this paper. We present results from an

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investigation of the relationship between an instructor's self-reported hours of professional development (PD), their score on a mathematical knowledge for teaching (MKT) questionnaire, and one of the measures of quality of instruction-instructors sense-making of procedures.

THEORETICAL FRAMEWORK

The impact of teachers' subject matter knowledge on their students' learning has been examined by looking at the quality of instruction the teachers offer their students or by investigating their students' gains. Hill et al. (2008) described research attempting to link teachers' mathematical knowledge with their students' learning as either deficit or affordance. In deficit studies, the instructor's low mathematical knowledge was accompanied by "the presence of mathematical errors and poor mathematical choices in the classroom" (Hill et al., 2008), while in the affordance studies teachers with deeper mathematical knowledge facilitated richer lessons for their students. Studies using proxies such as number of mathematics and teaching methods courses taken (Monk, 1994) have generally shown that there is no positive correlation between the number of courses taken by an instructor and student performance data (Hill et al., 2008).

Research on teachers' knowledge has flourished following Shulman's (1986) presidential address at the 1985 American Educational Research Association's annual meeting, where he introduced the idea of pedagogical content knowledge (PCK). This knowledge, which Shulman called the "missing paradigm" in research on teaching, linked knowledge of teaching pedagogy with knowledge of the specific content that was taught. As a result of Shulman's speech and the research that followed, Ball and her colleagues introduced the term mathematical knowledge for teaching (MKT) (e.g., Ball & Bass 2002), which describes the mathematical knowledge required for the work of teaching.

While much research has been done to understand the mathematical knowledge needed for teaching at the elementary school level (e.g., the MKT instrument for elementary and middle school teachers, Hill et al, 2008), significantly less research has looked at the mathematical knowledge needed to teach high school and college mathematics. As Speer et al. (2015) noted, there are reasons to believe these constructs apply to instructors of all levels. However, Speer and colleagues located only five instances of empirical investigations of post-secondary mathematics instructors and their practice with only two of these instances including an analysis of the instructor's knowledge. These instances were narrowly focused on implementation of inquiry-based instructional materials and practices.

Professional development (PD) of in-service teachers is a common strategy used to increase teachers' MKT and improve their practice. There is little debate among policymakers, researchers, educators, administrators and reformers about the importance of high-quality, ongoing PD for teachers as a necessary component for improving U.S. education (American Federation of Teachers, 2002; National Academy

of Education 2009). Further, there is general agreement that features of high-quality PD likely impact teacher knowledge and their practice including: (a) focusing on deepening subject matter knowledge specific to teaching, including understanding how students learn and the specific difficulties they may face in understanding the concepts, and (b) involving sufficient time for significant learning (e.g., a program or a course lasting 40 or more hours distributed over a year or more, Bell et al., 2010).

Different instruments and frameworks have been developed and used to observe and assess the quality of the interactions that occur between instructors and students with mathematical content. Hill et al. (2008) developed the mathematical quality of instruction (MQI) instrument which assesses the quality of enactment of elements such as richness of mathematics, "the use of multiple representations, linking among representations, mathematical explanation and justification, and explicitness around mathematical practices such as proof and reasoning" (p. 437). The lack of empirical clarity about the relationship between teacher knowledge, teaching practice, and student learning suggests that we do not yet have a sound understanding about the complex interactions that make professional learning successful (Bell, et al., 2010). In this paper we consider the question: What links exist between a community college instructor's hours of professional development (PD), mathematical knowledge for teaching (MKT), and their enactment of sense making in the classroom?

METHODS

Both quantitative and qualitative methods were used to help us understand the role of PD and MKT on the instructors' quality of instruction. Our data came from Intermediate and College Algebra courses taught at six community colleges located in three states in the U.S. These courses are often considered a mathematical gateway to a career in science, technology, mathematics, and engineering (STEM) as well as careers in other fields. Among the topics key to success in future mathematics and STEM courses are linear equations or functions, rational equations or functions, and exponential equations or functions–topics typically taught in Intermediate Algebra (IA) and College Algebra (CA) courses at community colleges. These courses are designed for U.S. students who desire to enter a STEM or Business-related field but have inadequate mathematical preparation. We collected instructor data via surveys (beliefs, MKT, number of PD hours in the last year--self-reported), video data of classroom practice, and student data (course grades, beliefs, and performance on a pre/post mathematics algebra and precalculus concept readiness (APCR) assessment). We hypothesized interesting relationships may be visible when juxtaposing the amount of PD hours an instructor reported receiving and their MKT; thus, creating the PD-MKT plane (Figure 1).

We used median MKT scores and median number of hours of PD, self-reported, to define four quadrants—Q1 (High PD, High MKT), Q2 (Low PD, High MKT), Q3 (Low PD, Low MKT), and Q4 (High PD, Low MKT). Further, we acknowledge that it is unlikely that the instructors' MKT fits perfectly into a dichotomy (High MKT vs. Low MKT); rather MKT fits on a continuum. Nevertheless, we found it helpful to organize the data in this manner to allow us to analyse the quality of instruction enacted by the instructors in different quadrants.

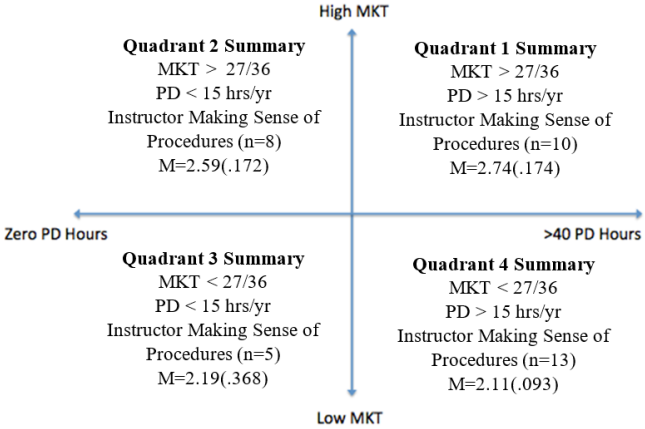

Figure 1. Distribution of Mathematical Knowledge for Teaching score, Professional Development hrs/yr and Instructor Sense Making of Procedures.

In addition to the quantitative data in Figure 1, we videotaped each instructor between 2 and 5 times, depending on the duration of a class period, in the Fall of 2017. The class sessions ranged in duration from 45 to 150 minutes and were taught by 37 different instructors (15 part-time and 26 full-time). Topics taught were linear, rational, and/or exponential equations and functions, chosen because they offer opportunities to analyse instruction on key algebraic concepts in IA and CA across multiple institutions. As part of this study, we developed a video analysis instrument called EQIPM: Evaluating Quality of Instruction in Post-secondary Mathematics (Cawley, et. al., 2018) (see Figure 2). EQIPM was based on the Mathematical Quality of Instruction (MQI) instrument by Hill et al. (2008) and Quality of Instructional Practices in Algebra (QIPA) instrument by Litke (2015). EQIPM describes and qualifies instructional practices (via a score from 1 to 5) from video-recorded class sessions that are deemed representative of the instructor's practice.

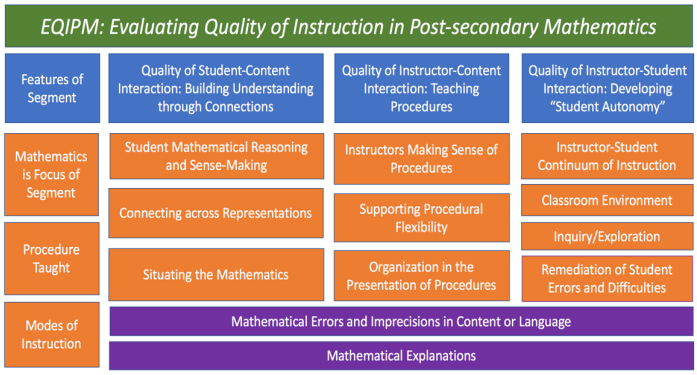

Figure 2. Structure of EQIPM

We used the instrument to addresses four dimensions of instruction: (a) Features of the Segment, (b) Quality of Instructor-Student Interaction, (c) Quality of Instructor-Content Interaction, and (d) Quality of Student-Content Interaction (see Figure 2). In this paper, we focus on the Instructors Making Sense of Procedures code as a way of describing the affordances the instructors offered to their students, see rating definitions in Figure 3. For this code we sought to identify ways in which instructors used mathematical relationships or properties to motivate a mathematical procedure by capturing all instances in which instructors make salient mathematical properties, relationships, and connections embedded in a

specific mathematical procedure. We hypothesize instructors' engaging in sense making with procedures elevates the richness of the mathematics afforded to their students.

RESULTS

Figure 1 shows the classification of the sample of instructors (n=36) on the PD-MKT plane. Since the axes (PD median and MKT score median) are arbitrary we chose to place instructors who landed on the axes in the higher quadrant. Ten (28%) teachers placed in the Q1 (High PD, High MKT), 8 (22%) teachers placed in Q2 (Low PD, High MKT), 5 (14%) teachers placed Q3 in (Low PD, Low MKT), and 13 (36%) teacher placed Q4 (High PD, Low MKT) quadrants. Instructors who received a high number of PD hours are expected to grow in their content knowledge (high(er) MKT scores) while instructors with few hours of PD are expected to have low(er) MKT scores. Interesting cases lie in quadrants II and IV low (Low PD, High MKT) and (High PD, Low MKT), respectively. The mean MKT of the instructors was 26.54 with a standard deviation of 4.8. The hours of PD data have a mean of 18.1 hours and a standard deviation of 14.5.

Statistics for Instructors Making Sense of Procedures Code.

Of the 804 segments in which a procedure was taught, we identified 228 (28%) in which instructor sense-making was not present; 223 segments (28%) had a rating of 2, 256 segments (32%) had a rating 3, while 97 segments (12%) had 4 or 5 (8% and 4%, respectively). The mean of the rating for the Making Sense of Instruction code for each instructor across all segments coded was highest in Q1 and lowest in Q4. The mean rating for this code of individual instructors higher than the population mean in Q1 only. Also of note, only 2 of the instructors in Q2 had 0 hours of PD in the past 12 months while the remaining 21 instructors had 6 or more hours of PD in the past 12 months. Instructors 11022 (Quadrant 1) and 10322 (Quadrant 3) had the highest and the lowest average EQIPM ratings for this code (3.417 and 1.33, respectively).

Discussion of Video Data.

Of the 808 segments, 353 (44%) were rated 3 or higher meaning that the instructors were making a genuine effort to make sense of procedures they were teaching. Nearly 70% of these segments with a high rating for this code were in Q2. As an example of an instructor in Q1, during a lesson on analysing the behaviour of rational functions Instructor 11523 used three examples for discussion. In the last segment of the lesson focused on horizontal asymptotes, she refers to these examples as Case 1, Case 2 and Case 3 because it is possible to "put your rational function into one of these three cases". The instructor asks students to compare the function equation with the graphical behaviour in these three cases and to make a generalized statement about this behaviour. She states that when the degree of the expression in the numerator is lower than the degree of the expression in the denominator (Case 1) the horizontal asymptote will always be y = 0 because the leading term in the denominator grows faster than the leading term in the numerator and therefore the function "will always approach 0." The

instructor moves to Case 2 in which the degree of the expression in the numerator equals the degree of the expression in the denominator. After students observed the horizontal asymptote was "y equals the ratio of the leading coefficients", she asks "Will that always happen?" In both cases, the instructor speaks about the behaviour "as x gets larger and larger" the leading terms of the numerator and the denominator will determine the behaviour of the function still occur to justify the generalization.

We follow the previous example from Q1 with an example from Q4. While teaching a lesson on exponential functions, Instructor 10322 provided students some "rules" for transformations of functions. He says, "you can really look at like just what number they are adding or subtracting and where they are adding and subtracting and it will follow those rules." The instructor led the students in confirming that the "rules" worked by applying them [rules] to three pairs of expressions (2^x and 2^{x+2}), (2^x and 2^{x-1}), and (2^x and $2^x + 2$) then identifying how the graph of 2^x would be transformed to get the second graph. It is important to note that this instructor was speaking about transformations of functions but wrote expressions instead. The instructor graphed the pairs of functions on his graphing calculator then he sketched them on the board. The instructor's approach for teaching transformations compared the given functions to a template that indicated the type of transformation. There was not an attempt to make sense of the procedures he was teaching. For example, he did not talk about how horizontal shifts of exponential functions impact the input of the function (x-values) or attempt to make sense of the impact of the shifts (other than to say the graph goes left 2). Moreover, he did not highlight differences in the graphs of the FUNCTIONS based on the transformations made to the original graph of 2^x: for example by noting how the coordinates of points change, or using algebraic operations such as $2^{(x+2)} = (2^x)(\,2^2) = 4(2^x)$ to explain the relative shapes of the graphs. For these reasons this segment was rated as a 1 for making sense of procedures.

CONCLUSIONS

In our analysis of the EQIPM ratings generated by the Instructors Making Sense of Procedures code we were searching for connections between that code and instructor knowledge as measured by MKT and hours of PD as reported in the past year. Based on our data, the relationships among the three quantities we measured (time of PD engagement, MKT and Instructor Making Sense of Procedures as a measure of quality of instruction) are somewhat elusive. There does not appear to be a discernible underlying relationship when focusing on segments of instructors making sense of procedures between the number of hours spent on PD and MKT, with some instructors who had few hours of PD scoring high on their MKT. This might be explained by some limitations regarding the instructor data. For example, instructors self-reported their PD work or professional learning completed during the previous 12-months. An additional limitation relates to the MKT scores. As previously noted, little attention has been paid to MKT of post-secondary instructors. To measure the MKT of the instructors we used an instrument designed for Middle School teachers because it focused on the algebra topics we were observing. We acknowledge that analyses of

one code, Instructor Making Sense of Procedures, is not enough to get a full picture of mathematical quality of instruction. Future direction of this work will involve analysing more data and expanding our analyses to encompass the whole Quality of Instructor-Content Interaction dimension. This is acknowledged to be a preliminary report of our findings. We are also refining our EQIPM instrument to help us better capture the quality of mathematics instruction. For example, following the analyses reported on this paper, we have modified the instructor Making Sense of Procedures to include instructor making sense of the mathematics because making sense is not limited to making sense of procedures. Our next step is to explore our data using the new code and an expanded analysis of all the EQIPM dimensions.

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BRIDGING THE COGNITIVE GAP – STUDENTS' APPROACHES TO UNDERSTANDING THE PROOF CONSTRUCTION TASK

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In this paper we examine how undergraduates bridge the cognitive gap between a theorem statement and its proof. Therefore, we analyse nine proof construction processes with special regard to the activities that undergraduates apply in service of understanding the given statement. The analysis reveals 12 different activities. Comparing successful and non-successful proof attempts, three activities seem to be crucial for developing an appropriate understanding of the theorem statement. These are inferring, enriching and questioning. Leaning on the concepts of cognitive unity and mental problem representations, we discuss to what extent each individual activity supports the understanding process and, therefore, the proving process in general.

INTRODUCTION

There is a broad consensus that analysing a theorem statement before attempting to prove it is crucial for a successful proof construction (Selden, 2012). Nevertheless, students often consider this first step in proving as obvious and, therefore, lay little emphasis on it (Schoenfeld, 1985). Previous research on the process of proof construction shows that the time undergraduates spend on understanding the given statement varies greatly (Kirsten, 2018). However, there seems to be no causal relationship between the duration of analysing and exploring the proving tasks and the quality of the written proof. In contrast, in some cases students put a lot of effort into understanding the given statement, but could not achieve any solution.

Hence, in this study we focus on qualitative differences in students' behaviour instead of quantitative time differences. Therefore, we examine the cognitive processes aimed at analysing and exploring the given statement on a microscopic level and try to identify effective and non-effective ways to develop understanding of the proving task.

THEORETICAL FRAMEWORK

The first approach to describe different activities in understanding the given problem was made by Pólya (1945). His first principle of problem solving *Understanding the problem* contains various techniques such as restating the given statement, making a drawing or clarifying the words used in the statement that are applicable to proving tasks. Beside the content of the given statement, Selden (2012) emphasises that in case of proving it is also crucial to analyse the statement's logical structure. This includes ascertaining the hypotheses and the conclusion in an informally stated theorem as well as differentiating between the if- and the only-if-part. While various manuals exist, in which mathematicians and mathematics educators suggest techniques that might support understanding, there is a lack of empirical research on the activities

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undergraduates really perform and deem helpful in constructing proofs. In order to investigate different kinds of activities in understanding a stated theorem, we base our research on the concepts of mental problem representation and cognitive unity.

Mental problem representation

Mathematical problems at school and at university level are usually presented as texts, which contain the relevant conditions as well as additional irrelevant information. Research on understanding and solving mathematical word problems has shown that successful problem solving requires taking the relevant information off the specific task-context and transforming the text into a more general problem representation (Reusser, 1990). This problem representation can serve as a basis for exploring relevant relations and concepts and applying mathematical operations. According to Kintsch and Greeno (1985) an appropriate problem representation consists of two components: the propositional text base and the situation model. The propositional text base refers to the different aspects mentioned in the text and their conceptual meaning. The main concepts and relations are summarised, but description remains based on the task-specific structure. The situation model, instead, concentrates on the concepts and relations themselves and is detached from the specific task formulation. Beyond that, it includes additional information that are associated with the concepts given in the text or derived directly from the text base (Kintsch & Greeno, 1985). Hence, the situation model can be regarded as an abstract and enhanced representation of the mathematical problem, which supports flexible and creative thinking. This allows the problem solver to anticipate possible actions and to select appropriate strategies.

Transferring the concept of mental representation on proving tasks reveals one possible account to explain how an insufficient understanding of the given problem might cause impasses in the proving processes. If the understanding of the problem remains limited to the text base and its structure, it becomes even harder to choose effective strategies and search for promising approaches. Thus, developing an appropriate mental representation, which includes the relevant information of the text as well as additional inferred information, seems to be crucial for progressing in proof construction.

Cognitive unity

The concept of cognitive unity describes the deep connection that under special circumstances exists between the production of a conjecture and the construction of its proof (Garuti, Boero & Lemut, 1998; Pedemonte, 2007). Students as well as mathematicians usually base their proofs on informal arguments developed in producing the conjecture. In these cases cognitive unity becomes visible in particular content, linguistic or even structural aspects that continue from argumentation to proof (Pedemonte, 2007). Empirical research indicates that cognitive unity can foster the proof construction process as conjecturing activities support insights into the problem situation (Garuti et al., 1998). Conversely, cognitive discontinuity can in some cases account for obstacles in students' proving processes. In common teaching practice, undergraduates are often faced with proving tasks that already provide a statement

estimated to be true. As there is no need to engage in conjecturing in this case, cognitive unity breaks and leaves a gap between the theorem statement and its proof. To re-establish continuity, it is necessary to reconstruct the conjecturing process by exploring the conditions of the given statement (Garuti et al., 1998). Combining the concepts of cognitive unity and mental problem representation, producing a conjecture or rather re-establishing continuity can be considered as one way to develop a situation model. By exploring the conditions and limitations of a conjecture, inferences are made and relevant relations are highlighted. Thus, a rich representation of the problem is built.

RESEARCH QUESTIONS

The aim of the study is to investigate how undergraduates approach to bridge the cognitive gap between a statement and its proof when they are asked to prove a specific theorem. Therefore, we analyse the different kinds of activities students perform in service of understanding the proving task and their potential to develop an appropriate problem representation. We suppose students who successfully write a proof having a richer situation model than those who do not progress substantially. Hence, comparing successful and non-successful proving processes might indicate effective and non-effective ways of developing a situation model and, therefore, of understanding a statement. Moreover, we question the connection between understanding and conjecturing by discussing how the observed activities might contribute to re-establishing continuity. In particular, our research is guided by the following questions: (1) What activities do undergraduates engage in when they are trying to develop understanding of a theorem statement? (2) To what extend do successful and non-successful proving processes differ in their approaches to understanding?

METHODS

The presented study is part of the project *Apropos*, in which 21 proving processes of undergraduate mathematics students have been analysed according to an adjusted version of Boero's proving model (Kirsten, 2018). The participants had recently begun their studies and were attending their first mathematics courses at university. In order to create a situation as authentic as possible, data collection was organised during the participants' weekly meeting and they were allowed to work in groups of two or three. Each group worked on two proof construction tasks that could be approached by applying a prominent theorem of real analysis. Working on these tasks, participants were told to write a proof as if they were submitting it like a regular exercise. The participants' proving processes were videotaped, transcribed and finally analysed leaning on Schoenfeld's (1985) method of protocol analysis. Thus, the proving protocols were partitioned into phases, each describing a relevant step in proof construction (see Kirsten, 2018). Besides analysing the processes, we additionally rated the students' performance in proving. Therefore, each of the written proofs was graded by two independent coders on a four point scale, where a score of a 4 corresponds to a complete and valid proof and a score of a 0 describes a proving attempt with no substantial progress. We consider a proving process successful, if the corresponding

proof was graded with a score of a 3 or higher. Interrater reliability in coding was satisfying for both types of analysis (partitioning: $\kappa \geq 0.81$, rating proofs: $\kappa \geq 0.74$).

To gain deeper insights into the understanding processes, we selected nine cases out of the whole sample for a fine-grained analysis. Following the sampling strategies of maximum variation and homogeneous sampling (Patton, 1990), we considered those proving processes as information-rich cases that are similar regarding the time spend on task analysis, but differ in proving performance. Accordingly, we chose four successful and four non-successful proving processes that match spending approximately a fifth of the proving process on understanding the given statement. Additionally, the proving process of Marcus and Lena was included to the sample, because they were the only participants who received a score of a 4. Concentrating on processes linked to the phase of understanding, analysis was guided by the principles of summarising content analysis according to Mayring (2014). By paraphrasing, generalising and reducing, a system of categories was developed that describes the different activities participants perform in service of understanding and exploring the statement of a theorem to be proved.

RESULTS

Analysing nine proving processes, we could identify 12 different kinds of activities that contribute to the students' understanding of the given statement (see Table 1).

Activity	**Description**
Extracting	Emphasising or copying out relevant information
Inferring	Deriving implicit preconditions, anticipating possible steps in proving, identifying informal arguments
Visualisation	Drawing a sketch or illustrating the situation with gestures
Enriching	Adding further information by recalling prior conceptual or strategical knowledge or looking up definitions
Questioning	Checking plausibility of the statement, analysing the exact formulation of the statement, validating prior considerations
Repeating	Rereading words, phrases or the whole task
Paraphrasing	Restating the statement in their own words
Transformation	Translating natural language into a symbolic notation
Focusing	Restricting one's consideration to a certain aspect or sector
Generating examples	Considering specific examples
Analysing special cases	Discussing extreme or special cases
Classification	Describing the task as an existential or universal proposition

Table 1: Activities occurring in the understanding process of undergraduates

Some of these activities such as extracting, visualisation, enriching or questioning occur in nearly all proving processes. Others like focusing, transformation or analysing special cases could only be observed in two cases each. As we have analysed proving processes of two different proof construction tasks, some of the mentioned activities might be task-specific. Activities like generating examples, transformation and focusing in our sample only occur in proving processes corresponding to the task *extreme point*, which is related to the sufficient condition of extrema. Beyond that, there is a task-related difference in the number and frequency of activities occurring in the understanding process (see Figure 1). While students working on the task *fixed point*, which is a corollary of the intermediate value theorem, only used three to six different activities, proving processes related to the task *extreme point* contain eight to ten different activities. One exception is the proving process of Marcus and Lena, who work successfully on the latter task, but engaged just in four different activities.

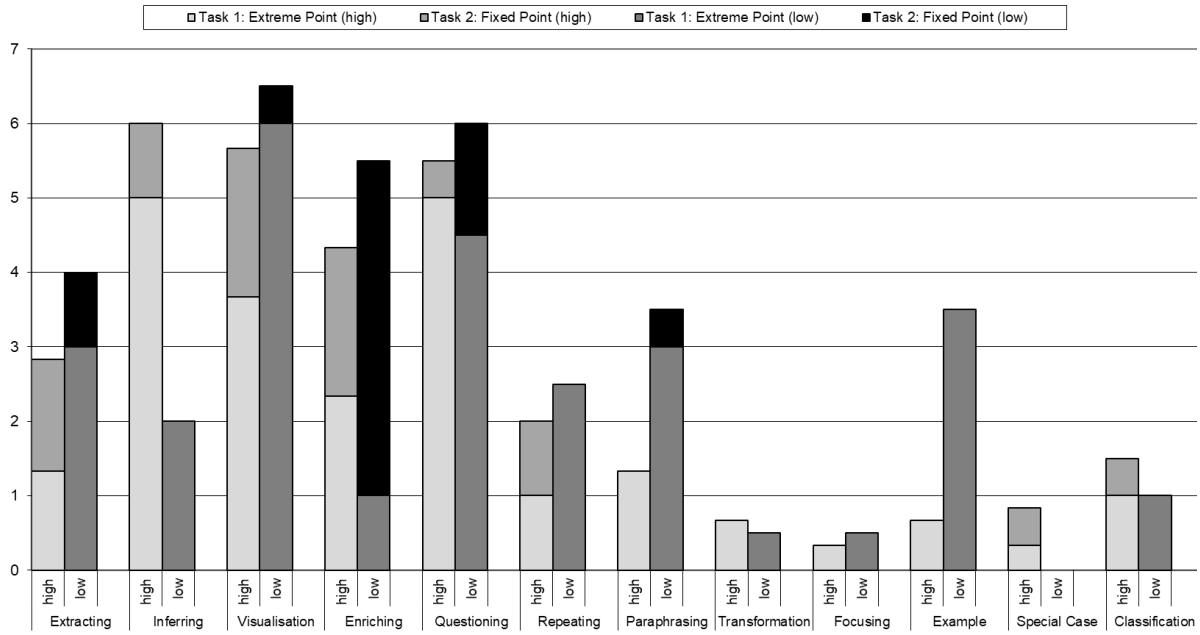

Figure 10: Average occurrence of each activity distinguishing students with highly and lowly rated proofs as well as different proving tasks (extreme point and fixed point)

Comparing successful and non-successful proving processes, we focus on the activities *Inferring*, *Enriching* and *Questioning*, which seem to be crucial for examining differences in how students engage in developing an appropriate problem representation of a theorem statement.

Inferring

While undergraduates with lowly rated proofs engage more frequently in restating the proving task and discussing example functions, students with highly rated proofs concentrate on deriving further information from the task context (see Figure 1). The following consideration occurred in the proving process of Fiona and Thomas after they had made a rough drawing on the fixed point task.

Thomas: this should be x … and this $f(x)$ [refers to his sketch of a function] and the function is continuous in this interval…than there is a fixed point … [first points at $(x|f(x))$ and then draws a diagonal]…well, the function and the diagonal intersect in any point…that is what we have to show

In the above excerpt Thomas first recalls the premise and combines it with his drawing. Considering the assertion, he recognises that all possible fixed points are located on a straight line. Hence, Thomas turns the problem of the existence of a fixed point into a similar one of an intersection, which is more familiar for them. Similar to Thomas and Fiona, other, mainly successful, students inferred an informal argument by exploring the conditions on an operative and semantic level. By doing so, they overcome the task-related structure and enable themselves to discuss the given concepts agile-minded.

Enriching

Enriching occurred, when participants add further information to the text base by recalling prior knowledge or consulting a textbook. Although this activity could be observed in all single proving processes, there are slight differences in how students put this activity into practice. Participants who wrote a lowly rated proof focused much on the textbook as they looked up the definitions of unfamiliar terms, worked through term-related theorems and examples or leafed through the textbook hoping to find clarifying explanations. In contrast, participants being successful in proving more frequently recall prior knowledge gained at university or school from memory, discuss it on a content-related level and exchange experiences from earlier proving approaches. The following consideration occurred in the proving process of Tom and Lukas. It illustrates how students use their prior knowledge to make sense of the given assertion.

Tom: if the second derivative is greater than zero … well, that means that there exists a minimum, doesn't it … yes, I'm sure … but it's like this, if it's less, then it's a maximum, if it's greater, it's a minimum… but the text says greater than or equal … that means it doesn't have to be a minimum, it can also be a point of inflection as it is in x^3

Questioning

Questioning includes checking the plausibility of a theorem statement, validating assumptions and drawings that have been made before as well as analysing the formulation of the premise and the assertion in detail. Especially the last action could be observed more frequently in successful proving processes. Participants discuss what the preconditions mean to a concrete function or search for reasons why the assertion was formulated exactly in this way. The example below shows Lukas and Tom questioning the formulation of the statement and discussing possible consequences for their proving approach.

Lukas: Can't we just show that a minimum exists and then we are finished?

Tom: Yes, that it is, yes, you are right … but then the task could say "there exists a y with a second derivate greater than zero", without the equality… but the task says

> greater than or equal…if we are able to show greater than, it's okay, but maybe it's not possible to show the statement excluding the equality

As this excerpt demonstrates, questioning the theorem statement can support students to become aware of the decisive aspects of the statement, explore possible as well as invalid variations and get a feeling for the statement in general.

DISCUSSION

The analysis of nine proving processes reveals particular activities that have the potential to support students' understanding of the proving task and, by doing so, their proving process in general. Comparing to students with lowly rated written proofs, successful students recall prior conceptual and strategical knowledge, analyse the formulation of the given statement in detail as well as derive further information. These activities are similar to processes that occur in producing a conjecture as they aim at exploring the decisive aspects of a problem situation and (re-)constructing a statement formulation. Hence, these activities are supposed to develop a situation model that allows agile-minded thinking and anticipating possible approaches to proving. Other activities like paraphrasing, visualisation or generating examples may support the students making sense of the theorem statement, but in many cases remain task-specific. Engaging solely in activities like this can account student's difficulties to proof construction as they are not able to overcome a task-related propositional text base and, therefore, restrict their thinking to the information given in the task. Moreover, we find the proof construction task to be relevant for students' engagement in understanding the theorem statement. While the concept of extreme values is usually known from school, the concept of continuous functions is often introduced first at university level. Due to an intuitive understanding of the concepts, it might be easier for students to apply understanding activities in the task considering extreme values. These findings highlight the relevance of conceptual knowledge and rich concept images for analysing and exploring the task environment.

Concluding, we hypothesise that some of the students' difficulties with proof construction is due to their inability to overcome a task-related propositional text base and develop an appropriate situation model. Activities such as inferring, enriching and questioning can contribute to this process as they reconstruct the conjecturing process and, by doing so, bridge the cognitive gap between a theorem statement and its proof. To this extent, the study supports the considerations made by Garuti et al. (1998) that successful proving requires re-establishing continuity. Further research should highlight the importance of understanding processes in proof construction and develop effective instructional designs that encourage students to engage in activities such as inferring, enriching and questioning in order to build an appropriate problem representation. For this purpose, it can be worth considering research into reading comprehension regarding mathematical textbooks and especially proofs (e.g. Sheperd & van de Sande, 2014; Weber, 2015).

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THE POWER OF THEIR IDEAS: HIGHLIGHTING TEACHERS' MATHEMATICAL IDEAS IN PROFESSIONAL DEVELOPMENT

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In the study discussed here, I examine the potential of noticing and making teachers' ideas the focal point of instruction to develop teachers' mathematical knowledge and contribute to building a community of learners. The theoretical framework used in the study is Whiteness. Sample research findings are provided that demonstrate why focusing on teachers' mathematical ideas supports their mathematical learning, while also modelling for them the value of being part of a learning community.

INTRODUCTION

In my work as a Mathematics Teacher Educator (MTE), I teach content and methods courses for prospective teachers as well as lead professional development (PD) sessions for practicing teachers. One of my primary goals for PD sessions for practicing teachers is to use their ideas as a means to support them developing deeper mathematical knowledge and stronger mathematical agency. This involves noticing teachers' mathematical ideas and having them share these ideas with their peers (Sherin & van Es, 2009; van Es & Sherin, 2008). More than just noticing their mathematical ideas, the focus of instruction in the PD sessions *is* the teachers' ideas and my instruction is essentially informed by the participating teachers' ideas. This entails working actively to both highlight and build on teachers' ideas during instruction as a means to support teachers reflecting on and potentially revising their mathematical thinking (Schoenfeld, 1985).

In this paper, I describe a research study in which teachers in a series of mathematics professional development sessions solved problems and experienced inquiry-based instruction as learners. During the 2016-17 and 2017-18 academic years, I led monthly full-day professional development sessions for more than 30 primary and secondary-level teachers of mathematics who teach in culturally and linguistically diverse, rural schools in northern New Mexico, USA. The vast majority of the participating teachers are culturally and linguistically diverse women, primarily of Hispanic and Native American descent. In each "Institute" session offered, teachers regularly engaged in problem solving activities, shared their solutions with one another and the whole group, and examined other teachers' solutions. In addition, teachers learned about practices that they could use in their instruction that support inquiry, such as establishing classroom norms in support of students regularly sharing their mathematical thinking with one another (Cobb, & Yackel, 1996). In 2016-17, the problems assigned in the sessions were aligned with the Common Core State Standard Mathematics (CCSSM) Standard, "Operations & Algebraic Thinking" (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

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The United States does not have national mathematics standards, however a majority of states have adopted a version of the CCSSM as their state standards. During 2017-18, problems assigned aligned with the CCSSM Standard, "Number & Operations—Fractions."

One of my primary goals for the Institute sessions was to use participating teachers' mathematical ideas as a means to help teachers develop deeper mathematical understandings. I also work to make teachers' mathematical ideas central in PD sessions I lead for the following three reasons: (1) To champion the notion that we all bring mathematical knowledge and ideas to bear to solve problems (Schifter, 2005); (2) To endorse teachers' knowledge and ideas as mathematically valid and important; and (3) To distribute the mathematical authority in our sessions (Cobb, & Yackel, 1996). An important consideration for me in designing the Institute sessions is the fact that many of the participants are elementary school teachers who are culturally and linguistically diverse women whose ideas have not historically been taken seriously in the mathematics classroom (Becker, 2003). In my work as a MTE, I work to create a community of learners in which every member has opportunities to contribute mathematical ideas. The research question for the study described here is: How can noticing and making teachers' ideas the focal point of instruction further develop teachers' mathematical knowledge and contribute to building a community of learners?

For this study, Whiteness is the theoretical framework. Whiteness in mathematics education has historically subjugated culturally and linguistically diverse students, while reproducing privilege for white students (Battey, 2013; Joseph, Haynes, & Cobb, 2016; Martin, 2013). As a PD provider, an important part of my work is considering white privilege and how it operates to provide differential educational opportunities based upon race and class (Battey, 2013; Martin, 2013). Since the vast majority of Institute participants are culturally and linguistically diverse people, it is imperative that as a white male I specifically highlight the work of these Institute participants as a means to work against white privilege and toward the building of an inclusive professional development community. Such noticing is purposeful in that it challenges instruction in which culturally and linguistically diverse students are constructed as mathematically deficit, while positioning students from the dominant culture as the most mathematically capable (Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2007).

RESEARCH METHODOLOGY

To illuminate the value placed on centering the work of the Institute sessions on participating teachers' mathematical ideas, sample tasks developed for use in the Institute will be shared, as well sample teacher solutions to those tasks. To study the potential of noticing and making teachers' ideas the focal point of instruction to further develop teachers' mathematical knowledge and build community, I did four things: (1) Collected copies of and took photos of teachers' solutions to tasks posed during Institute sessions; (2) Engaged in my own journaling about what transpired during sessions; (3) Reviewed video of sessions; and (4) Examined teacher evaluations of

sessions. All data subsets were analysed using interpretive methods (Creswell, 2014). Each data subset was read or viewed as a whole, followed by a period of open coding to allow for the emergence of themes. An iterative process of coding, memo writing, focused coding, and integrative memo writing followed (Creswell, 2014). Creation of the codes went through multiple revisions, as the data were repeatedly read and reviewed to check the consistency of themes. This process continued until either no new categories were developed or consistency was achieved. After a set of themes was obtained from the dataset, I searched for commonalities and differences in the data subsets. I also sought both confirming and disconfirming evidence by searching for supportive and non-supportive evidence (Miles, Huberman & Saldana, 2013).

USING TEACHERS' IDEAS TO ENRICH UNDERSTANDING

I offer three examples to address how noticing and making teachers' ideas the focal point of instruction enhanced their mathematical knowledge and contributed to building a community of learners. For each example, the teachers solved tasks that were designed to align with a CCSSM Standard (NGA & CCSSO, 2010). Culturally and linguistically diverse Institute participants created all the solution strategies included below.

In the first example, I built on a participating teacher's idea to develop notions of number sense through examination of a mathematical property. The following task was offered to participating teachers that approximates a fifth-grade CCSSM Algebra Standard: Judy says that to find 5 times 26, she can find 10 X 26 (260), and then divide this number by 2 to get 130. Write an equation that demonstrates her mathematical thinking.

As teachers worked on the task individually and then with group members, I walked around observing and listening to how teachers explained their solutions to one another. I then selected one of the participating teachers to share her ideas about how to solve the task on chart paper for the whole group to view because I believed that through examining her solution with the whole class, mathematical insights about the mathematical properties could be derived. After writing her solution strategy on the chart paper, she shared this strategy with the entire class:

(5 X 2) X 26 =

∨

10 X 26 = 260

260 ÷ 2 = 130

In her solution, the teacher noted that multiplying 5 by 2 was equivalent to multiplying by 10. She also argued that after multiplying 26 by 10, it was necessary to divide by 2 to arrive at the appropriate result since 10 ÷ 2 equals 5. The teacher who shared the work above, then proceeded to produce the following equation for the whole class that synthesized the solutions that she had previously presented:

5 X 26 = (10 X 26) ÷ 2

In the whole class discussion that ensued, I worked to validate and build on the teacher's initial solution to demonstrate how the teacher had used the transitive property of equality, namely that if a = b and b = c, then a = c. To do this, I circled the first expression (5 X 2) X 26 and labelled it as expression "a". I continued by circling expression b, 10 X 26 and expression c, 260, and then reviewed how a = b, b = c, and finally that a = c.

a c

(5 X 2) X 26 = 10 X 26 = 260

b

Next, I asked whether (10 X 26) ÷ 2 = 10/2 ÷ 26/2? The discussion that resulted included consideration of whether dividing the quantity (10 X 26) by 2 should be distributed by dividing each term of the quantity by 2. This led to consideration of whether 10 X (26 ÷ 2) = (10 X 26) ÷ 2? The realization that these expressions are equal led a participating teacher to share how she teaches GEMA (grouping symbols, exponential operations, multiplicative operations, and additive operations) rather than PEMDAS (parenthesis, exponents, multiplication and division, addition and subtraction) to support student learning.

In the second example, a teacher created a mathematical model that challenged teachers to examine their beliefs about what their students are capable of achieving. The following sample task was offered to participating teachers that approximates a fifth-grade CCSSM Fraction Standard: Use a visual fraction model to show (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)

Once again, I walked around observing and listening to how teachers explained their solutions to this problem to one another. I then selected a teacher to share his solution with the whole class based on my belief that this solution was unconventional and would challenge teachers to consider their beliefs about sharing such solutions.

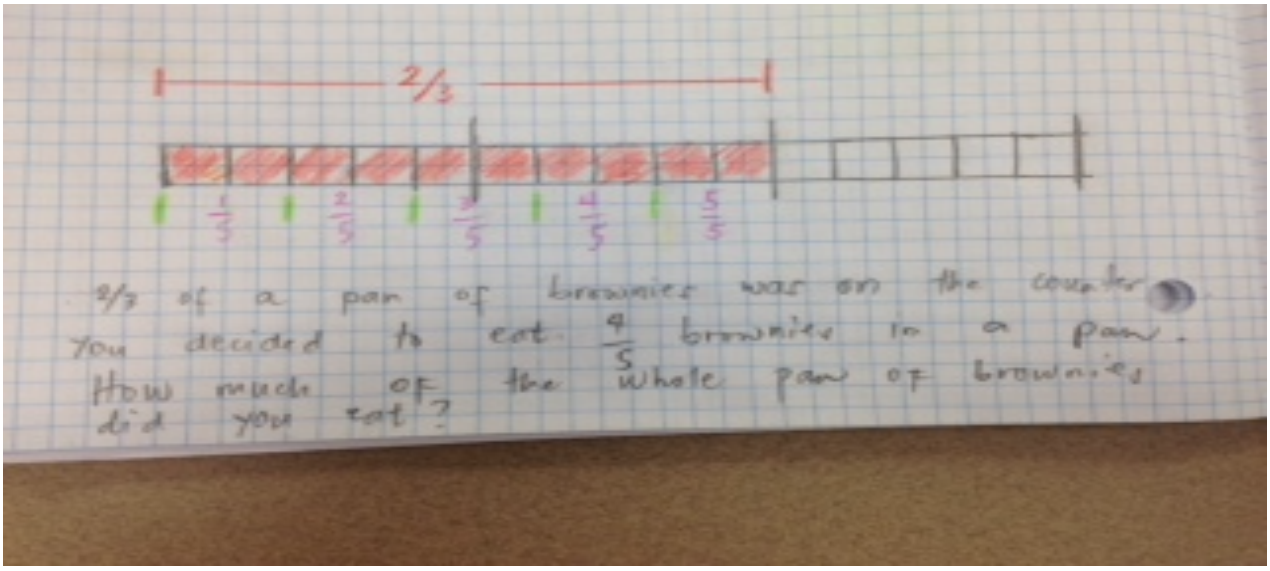

Figure 1: A teacher uses a fraction strip model to find 4/5 of 2/3.

When this teacher presented his solution to the class, there were questions about his strategy. First, using a fraction strip to represent a pan of brownies was not intuitive to many of the participants. So, the model did not align well with how many of the teachers viewed a brownie pan, a rectangle that may even be a square. However, the model works, even if it does not align well with a practical context. While I could have demonstrated a model similar to this participating teacher's, through soliciting a teacher's solution to the task, I was able to validate his ideas as mathematically worthy. This example also led teachers to reflect on their beliefs about what their students are capable of achieving. Some teachers doubted their students could create the mathematical model illustrated in Figure 1, while others challenged their colleagues to give students a chance to generate models to solve problems without first being shown these models.

In the third and final example, teachers were once again asked to create a mathematical model to solve a problem. The following sample task was offered to participating teachers that approximates a fifth-grade CCSSM Fraction Standard: There are 64 fifth graders at Blanco Elementary. Three-eighths of them bring their own lunch to school. Use a mathematical model to determine how many fifth graders bring their own lunch. As usual, I selected a teacher to share her model with the whole class after first observing and listening to how teachers explained their solutions to this problem to one another.

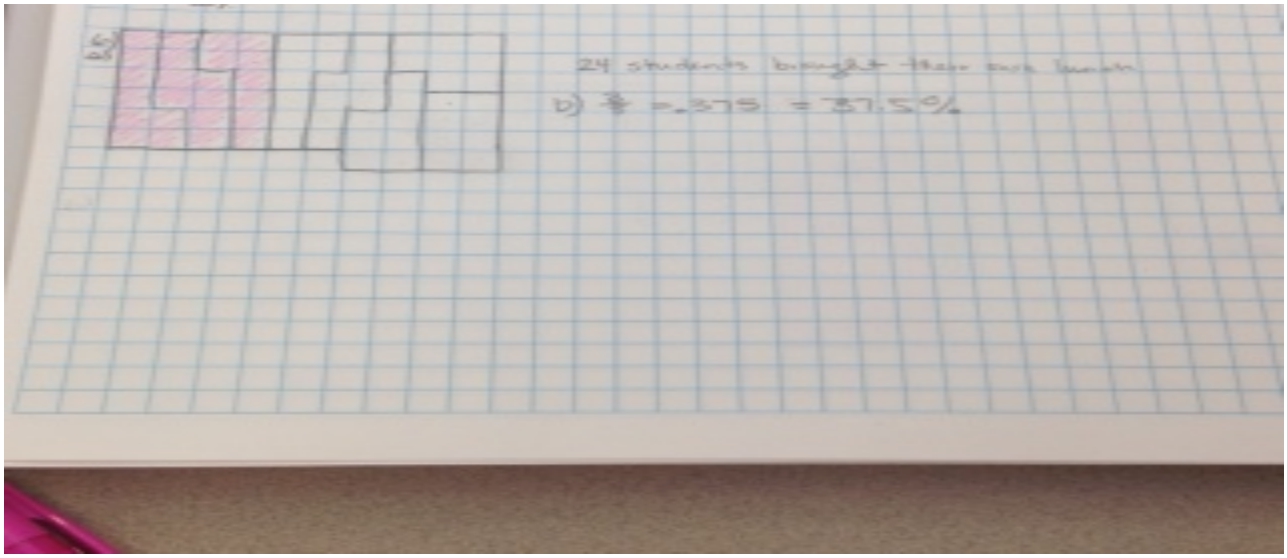

Figure 2: Teacher's representation showing 3/8 or 37.5% of 64 is 24.

In this case, a teacher solved this problem in a way that I did not anticipate. Moving from left to right, she constructed a rectangle with 6 rows and 10 columns, increasing the number of rows to 7 by the 7th column so that she could have a total of 64 squares to represent the 64 total students. Dividing the total number of squares (64) into 8 groups, she then shaded 3 of her 8 groups, demonstrating that 24 students (24 shaded squares) at Blanco Elementary brought their lunch to school.

When this teacher presented her solution to the whole class, I worked to support her ideas because they are mathematically valid, knowing some might criticize her less than conventional model. My goal here was two-fold: First, I wanted to validate a non-conventional solution strategy that was mathematically correct. This was done to pursue a second goal, which was to once again challenge teachers to consider their beliefs about what their students could achieve. While an 8 X 8 square would be the model that many would most likely choose to solve this problem, I wanted teachers to consider how students may actually have novel mathematical ideas that could serve to enhance other students' mathematical understandings.

DISCUSSION AND IMPLICATIONS

In this study, I examined how noticing and making teachers' ideas the focal point of instruction could further develop teachers' mathematical knowledge and contribute to building a community of learners. Using Whiteness as a guiding framework, culturally and linguistically diverse participants' mathematical ideas were continually on display throughout the Institute, and these ideas were intentionally used as a means to further develop teachers' mathematical knowledge. In their evaluations of the Institute, teachers wrote that they liked having their ideas highlighted throughout. Some teachers also promoted the idea that they needed to provide more space in their classrooms for greater inclusion of their students' mathematical ideas.

In the first example, I deliberately selected a teacher's solution that incorporated the transitive property of equality; a powerful property that I knew teachers had some

exposure to but lacked understanding of. The examination of order of operations was inspired by the teacher's final equation that I knew from experience was worthy of consideration since so many teachers teach PEMDAS in a manner that can lead to miscalculations (Dupree, 2016). An intentional strategy here was to take advantage of participating teachers' mathematical ideas to organically push their thinking to consider ideas that most certainly will emerge in their instruction. The final two examples supported the development of teachers' mathematical knowledge, while inspiring reflection among them about the capacity of their students to generate mathematical models to solve problems. From participating teachers' responses to their colleagues' mathematical solutions and from information gleaned from evaluations completed, teachers frequently offered how the Institutes helped them learn mathematics and become more confident in their mathematical abilities.

In the final example, I chose to validate a particular teacher's non-conventional solution, which also extended participating teachers' mathematical knowledge. This teacher is a Native American woman. By highlighting and validating her ideas, I demonstrated that I valued the contributions that every member of the group makes. In addition, intentionally having women and culturally and linguistically diverse teachers publicly share their mathematical solutions to problems granted legitimacy to their thinking and removed me as the sole mathematical authority in the room. As a white male, it is imperative in this work that I explicitly notice and then affirm the contributions of the Institute's diverse participants. Displacing myself as the white male who possesses all mathematical knowledge highlights that I am serious about building a community in which every participate is valued.

An implication of the research findings of this study is that noticing and making diverse teachers' ideas the focal point of instruction is a practical approach to incorporate equity and access to mathematics in one's teaching. Such instruction is equitable for the simple reason that it validates the ideas of members of underrepresented groups in the learning process, highlighting that everyone has ideas to contribute to the group. Furthermore, teachers have indicated that they feel a sense of ownership and belonging in the community we have developed and that they are willing to defend it. This came became clear recently when a male principal who is a former mathematics teacher joined our group and attempted to demonstrate his mathematical superiority on several occasions. During some heated discussions that took place, some participating teachers, culturally and linguistically diverse women asserted that this was their community and that he was welcome to join but needed to respect them and their ideas. Lastly, after experiencing instruction that supports them to develop deeper mathematical understandings through actively constructing mathematical ideas (Gravemeijer, Bruin-Muurling, Kraemer, & van Stiphout, 2016), teachers want the same for their students. Specifically, that their students will experience mathematics as a sense making activity (Kitchen et al., 2007).

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TEACHING GRAPHING FORMULAS BY HAND AS A MEANS TO PROMOTE STUDENTS' SYMBOL SENSE

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In this study, we investigated whether teaching graphing formulas by hand could promote students' symbol sense. During five lessons of 90 minutes, 21 grade-12 students worked on graphing tasks. To assess the effects of this intervention, nine months later, a symbol-sense test was administered to these students and to a control group of 93 students. Six students from the treatment group, who thought aloud during the test, showed their graphing-formulas symbol sense, but also some transfer, i.e., using symbol sense when solving algebraic problems, like taking a global view, switching of representation, and qualitative reasoning. The treatment group used more symbol-sense and scored significantly higher on the test than the control group. The results suggest that graphing formulas might be a means to teach symbol sense.

INTRODUCTION

Many students in secondary school have difficulties with algebra, in particular with developing symbol sense (Arcavi, Drijvers, & Stacey, 2017; Kieran, 2006). Regular education often focuses on basic skills, and not on symbol sense. A lack of symbol sense leads to defensive strategies, like learning procedures, and to an overreliance on basic skills (Arcavi et al., 2017). Symbol sense is difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2010). In a previous study, we found that graphing formulas by hand, based on recognition and heuristic search, improved students' insight into algebraic formulas, i.e., the ability to identify the structure of a formula and its components, to decompose formulas into sub-formulas, and to reason with and about formulas (Kop, Janssen, Drijvers, Van Driel, submitted). In this study, we investigated whether teaching this would enable students to use this insight and other aspects of symbol sense when solving algebraic problems.

THEORY

Symbol sense has many aspects, among which the ability to read through an algebraic expression, to make rough estimates of the pattern that would emerge in a graphical representation, and to have a feeling for the power of symbols and know when to abandon symbols (Arcavi, 1994). Drijvers (2010) described symbol sense as complementary to basic skills. Basic skills involve procedural work with a local focus and an emphasis on algebraic calculations, whereas symbol sense involves strategic work with a global focus and an emphasis on algebraic reasoning, and so it functions as a compass when using basic skills. It is needed in all phases of the problem-solving

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cycle, from a realistic world problem, via a mathematical problem and a mathematical solution, to a real-world solution. Pierce and Stacey (2001) used algebraic insight to capture the symbol sense involved in the transformation of a mathematical problem to a mathematical solution. This algebraic insight has to do with recognition of objects, key features, dominant terms, and the meaning of symbols, and with the ability to link representations (Pierce & Stacey, 2001). In this paper we use the term symbol sense to refer to these aspects. To promote students' symbol sense, we taught them how to graph formulas by hand.

In expertise research, it has been established that to graph formulas effectively and efficiently, one needs recognition and heuristic search (Kop, Janssen, Drijvers, & Van Driel, 2015). A repertoire of basic function families with features is needed for recognition (Slavit, 1997). Heuristics like decomposing a formula into sub-formulas and qualitative reasoning are needed when graphing more complex functions and recognition falls short. Qualitative reasoning is about the global features of a function, such as its global graph and infinity behavior and is characterized by estimation and approximation.

Graphing formulas based on recognition and heuristic search involves many aspects of symbol sense. One has to start with a global view to identify the structure of a formula and to recognize the building blocks (basic functions) of the formula and the key features of the graph. If needed, a formula might be decomposed into basic sub-formulas and/or qualitative reasoning can be used to transform basic function, to compose the sub-graphs, and/or to find the key features of the graph.

In short, graphing formulas requires a great deal of symbol sense. However, symbol sense involving the solution of algebraic problems is broader. One has to start the problem solving with a global view to consider different strategies: for instance, to abandon a symbolic representation in favour of a graphical representation or to use qualitative reasoning instead of starting (error-prone) calculations, and to predict and monitor results during the problem-solving process. Below is an example to illustrate these aspects of symbol sense:

> Task: Can the outcome of $y = -0.1(x-3)(x-10) + 40/(x-3)$ be larger than 70?
>
> Approach: First one can consider a strategy like translating the problem into an inequality ….>70, using a global view on the equation $y = -0.1(x-3)(x-10) + 40/(x-3) = 70$ and reason like: 2nd degree polynomial + 40/'something with x' = 70 gives a 3rd degree polynomial equation, that almost certainly cannot be solved exactly. Another strategy could be linking it to a graph; with a global view one sees a 'parabola with a maximum + a broken function'. Can this function be larger than 70? Looking at the tails of the graph or ….? The parabola might give a maximum larger than 70; therefore, calculations are needed. On the other hand, the broken sub-function has a vertical asymptote, so, in the neighborhood of x=3, when x is a little bit larger than 3, the function has to be larger than 70; and there are no problems with the outcomes of the parabola.

In regular education, attention is paid to basic function families, but symbol sense (Arcavi et al., 2017) and graphing formulas by hand are largely ignored. We expected that explicit teaching of graphing formulas by hand would teach students to take a global view, to develop a repertoire of basic function families, and to learn qualitative reasoning when graphing formulas. We also expected some transfer, i.e., using this symbol sense when solving algebraic problems. This led to the research question: How can explicit teaching of graphing formulas be a means to promote students' symbol sense?

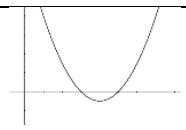
METHOD

In this study a written symbol sense test of 45 minutes, consisting of eight explicit graphing tasks and 12 algebraic problems, was administered to 114 grade-12 students from six different schools. We report in this article on the algebraic problems, all of which could be solved by using graphs and/or qualitative reasoning. The treatment group consisted of 21 students (from the author's school). The intervention, a series of five lessons of 90 minutes about graphing formulas, took place nine months before the test. During these lessons, the students were taught, using whole tasks and reflection questions, to graph formulas by hand, using basic function families and qualitative reasoning. The whole tasks were related to different levels of recognition, which were found in expert research: instant recognition of the graph or recognition of a function family, decomposition of a formula into sub-formulas, recognition of key graph features, and no recognition of the graph and using strategical search for graph features (Kop et al., 2015).

Two examples to illustrate these whole tasks are given below:

Task 4 about the recognition of key graph features (inspired by Swan (2005)):

Given two equivalent formulas and a graph, what features of the graph can be read from the formulas?

$y=(x-4)^2-1$	$y=(x-5)(x-3)$	

Reflection question: Give three formulas of functions from which you can simply read the zeroes of the functions.

Task 5: What happens to the y-values of the functions: $= 0,6^x \cdot x^{60}$; $y = (4x^3-6)/(x+3)^2$; $y = 52.7/(1+62{,}9 \cdot 0{,}692^x)$, when $x \to +\infty$? Choose between: $y \to +\infty$; $y \to a \neq 0$; $y \to 0$; $y \to -\infty$

The test questions were based on problems that have previously been used in assessing students' algebraic competencies. Students were asked to explain their answers. The internal consistency of the questions was acceptable: Cronbach's Alpha was 0.72. Below some examples of the algebraic problems:

2) Give the number of solutions of the equation: $5\ln(x) = \frac{1}{2}x - 10$

3) Give the number of solutions of the equation $2^x = 2^{-x} + 3$?

4) Can the outcome of $y = -0.1(x-3)(x-10) + 40/(x-3)$ be larger than 70?

7) What outcome(s) can y have when $y = 24 - 0,01(x+5)^4$?

8) When x is very large, the function $f(x) = (3e^{-x^2} + x^2)^3 + 70/x^4$ can be approximated by: Pick the best alternative out of : A. $y = x^6$; B. $y = x^5$; C. $y = 70x^{-4}$; D. $y = 27e^{-3x^2}$; E. none of these

14) Pick the correct alternative: A maximum of the function $y = x(14-2x)(8-2x)$ is situated in A. [-4;0]; B: [0;4]; C: [4;7]; D: [7;14]

The written material was analysed to determine correctness and strategies: using a graph, using qualitative reasoning, using calculations and/or algebraic manipulations.

Written tests give only limited information about students' thinking processes. Therefore, six students from the treatment group were asked to think aloud during the test. This was videotaped and transcribed. The thinking-aloud protocols of the six students were analysed in greater detail. It was registered whether the students used qualitative reasoning (Q), recognition of function families and/or key features (R), or strategic work, e.g., changing representation or monitoring (S), and whether they showed a lack of symbol sense (starting calculations/manipulations (C), e.g., expanding brackets, when not needed). In one problem, more than one strategy could be used.

RESULTS

We first present the results of the six students who thought aloud during the test. We give examples of their symbol sense and of their lack of symbol sense. We then compare the results of the written tests of the treatment group with those of the control group.

In problem 2, student S instantly switched to a graphical representation and used his/her knowledge of the logarithmic function family:

> "$5\ln(x)$ goes like this (draws a graph); $1/2x - 10$ goes not that steep; this one ($5\ln(x)$) goes flatter here; here, there is an intersection, and here too; so, two solutions" (encoding: S,R).

In problem 4, student A started to make calculations but monitored his/her progress and reconsidered his/her strategy; then s/he recognized a vertical asymptote and used qualitative reasoning (what the graph looks like near the vertical asymptote):

> "First expand the brackets $x^2 - 13x + 30 + 40/(x-3)$; can this be larger than 70?; I'm going to try to find the turning point; then see whether it is a parabola with a max or something like that, but there is also a broken function; let's see whether it is a parabola and see whether the turning point is beneath or above 70 and then 40…..? (tries to calculate) No, this will not work; I think I will calculate some points; There is a vertical asymptote at x=3; so, when x-3 is very small then this part becomes very large and dominate the rest of the

function; …x-3 can be infinity small and then the fraction will be very large and easily above 70" (encoding: R,Q)

In problem 7, student K used qualitative reasoning:

"To the power of 4 means that the outcome will be positive, regardless of the value of x; so, the whole ($-0.01(x+5)^4$) will be negative and the maximum value 24; and further, the larger x the larger the whole will be and so, the more negative it will be" (encoding: Q)

In problem 8, student Y used decomposing the function in sub-functions and used her/his repertoire of function families:

"When x is very large then $70/x^4$ fades because it becomes very small, approaches 0; therefore alternative C is not correct; when x is very large then x is very large then $y=e^{-x}$; it becomes $y=x^6$; $e^{-l\arg e}$ becomes almost 0, so, when substituting something very large in e^{-x^2} it approaches 0; only $(x^2)^3$ is left; I doubt 5 or 6, but it is multiplying, so answer A" (encoding: R,Q)

In problem 14, student S considered his/her strategy before switching to a graph, recognized key features of the graph (the zeroes), and used qualitative reasoning about infinity behaviour of the function:

"Hmm, not nice to expand the brackets and to differentiate the function; but is there a better way to do it? We can say that there will be a zero at 0, and when $14-2x=0$, so, at 7 and at 4; what shape do we get? For large x it is positive multiply negative multiply negative, so positive; for a very negative number we get a negative outcome (followed by a graph and the correct answer)" (encoding: S, R, Q, S)

Examples of students' lack of symbol sense were also found.

In problem 2, student I started with manipulations that did not lead to a solution:

"Long time since I did one like this. How do I do it? I transform this equation: $\log_e(x^5)=\frac{1}{2}x-10$ (thinks). Can I solve this equation? I can transform it into $x^5=e^{\frac{1}{2}x-10}$, but do you make any progress with this? x^5 can only be positive or negative. No, I do not know." (encoding: C)

Student A solved problem 2 correctly, but used a lot of calculations instead of graphs:

"How do I find the zeroes? I just solve it; I think because there is no quadrate in $5\ln(x)-1/2x+10$ that …; first make a derivative: $5/x-1/2$ and looking for a turning point; so, make it equal to 0; we see that there is one turning point at $x=10$ and when we substitute 10 we get two zeroes." (encoding: C)

Student S solved problem 4 incorrectly, because s/he only paid attention to the infinity behaviour of the function

"Let's see, when x is very large then this is very small ($40/(x-3)$); but it is only positive; the zeroes are at 3 and 10 and for large values of x this will be very large, this very large, but then negative, so, it is negative (looks at $-0.1(x-3)(x-10)$); then +40; turning point is in between, that is at 6.5; there it will not be very large, definitely not 70; for small values, much will be subtracted; no, it cannot be larger than 70"

Table 1 shows the numbers of correct answers and the number of problems in which the strategies qualitative reasoning and recognition (Q and/or R), switching of representation (S) and calculation/manipulation (C) were used. For example, student Y had a score of 3.8 (on 12 problems) and used recognition and/or qualitative reasoning (R, Q) in 11 problems; s/he used the strategy changing the representation (S) six times, and calculation/manipulation (C) five times. See Table 1.

Student	Score	Q and/or R	S	C
I	4.5	10	3	5
Y	3.8	11	6	5
K	9	11	6	1
S	8	12	7	1
M	4	11	5	4
A	7	9	6	6

Table 1: Scores and strategies from the thinking-aloud protocols

All students' written materials were analysed to determine their correctness and strategies (using a graph, qualitative reasoning, calculation/manipulation). The differences in strategy use between the treatment group (T) and control group (G) are shown in Table 2.

Percentage	Qs 2		Qs 3		Qs 4		Qs 7		Qs 8		Qs 14	
	T	G	T	G	T	G	T	G	T	G	T	G
Blank	0	31	24	44	14	23	29	52	10	26	19	52
Calculation	38	31	19	27	5	42	0	15	0	1	19	38
Graph	52	23	52	20	81	33	24	2	91	72	33	2
Reasoning	10	14	5	9	0	1	43	28	0	1	29	9

Table 2: Strategy use of treatment group (T) and control group (G)

The mean score of the treatment group was 5.29 with a standard deviation of 2.20, whereas the mean score of the control group was 2.53 with a standard deviation of 1.59. This difference was significant ($t(24,95)=5.46$; $p<0.001$) and the effect size was huge (Cohen's $d = 1.74$).

CONCLUSIONS AND DISCUSSION

The aim of the current study was to investigate whether students in the treatment group would use symbol sense, which they had learned during the teaching of graphing formulas, to solve algebraic problems. The thinking-aloud protocols showed that the students used their repertoire of basic function families for recognition and qualitative reasoning if they used the graphing of formulas to solve the algebraic problems. But they also used symbol sense in a broader sense: for instance, abandoning a symbolic

representation in favour of a graphical representation, using qualitative reasoning instead of starting calculations, and working strategically during the problem-solving process. Although we also found examples of a lack of symbol sense, the results seem to suggest that there was some transfer: the students used symbol sense learned through graphing formulas to solve algebraic problems.

The analysis of the written tests indicated that students in the treatment group used a graph and/or qualitative reasoning more often and that they were more successful in solving these algebraic problems. The explicit teaching of graphing formulas, based on recognition and qualitative reasoning, might explain the differences we found in scores and strategies between the treatment group and the control group. The findings of this study suggest that explicit teaching of graphing formulas might be a means to promote students' symbol sense. As there was only one treatment group and there was no randomized experiment, more research is needed. The algebraic problems used in this study were limited to those that could be solved using graphs and/or qualitative reasoning, so without much calculation and manipulation. For future research we suggest continuing on this path. Often, when symbol sense is taught, manipulations of algebraic formulas play a large role. We expect that technology in the future will do a lot of the algebraic manipulations, and that qualitative reasoning will become more important. Qualitative reasoning, with its global approach and estimation, was an important factor in our intervention. Unlike in regular textbooks, the focus from the start was on global shape, rough sketches, and approximations. Experts use this kind of reasoning often, but it is hardly taught in secondary school and it was very new to these students.

Before the intervention, students in the treatment group had problems with the sketching of basic functions. This suggests that, without explicit teaching of graphing formulas, Dutch students in grade 12 were not able to graph formulas using basic function families and qualitative reasoning. As a consequence, they would not be able to use these aspects of symbol sense to solve algebraic problems. This seems to be supported by the results of the control group in this study. A limitation of this study was that only Dutch students were involved. However, we expect that grade-12 students in other countries are in a comparable position; we have confirmed this through personal discussion with teachers from other countries.

Symbol sense is a broad concept with many aspects. A lack of symbol sense results in problems with solving algebraic problems. Symbol sense is difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2010). Suggestions for the teaching of symbol sense have been given in the literature, but these often require use of small tasks and/or snapshots (Friedlander & Arcavi, 2012; Pierce & Stacey, 2007). These seem to be small and local interventions. Graphing formulas might be a more structural and integrated way (using symbol sense in combination with basic skills) to pay attention to the development of symbol sense.

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TOWARDS COGNITIVE FUNCTIONS OF GESTURES – A CASE OF MATHEMATICS

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Despite the growing body of research on gestures in mathematics education, little is known about the self-oriented functions gestures can play in mathematical cognition. This paper adapts a framework from psychology suggesting four cognitive functions of gestures and explores its potential for the context of mathematics. We present an example from an interview on fractions to illustrate the manifestation of the cognitive functions in a mathematical context. Furthermore, we discuss the potential of the framework in the context of mathematics.

INTRODUCTION

Over the last 20 years, an increasing interest in understanding better how gestures influence mathematical thinking and learning becomes apparent in the literature. A lot of research, especially in mathematics education, focuses on gestures as semiotic resource (e.g. Arzarello, et al., 2009), shaping the social interaction and the learning process through fulfilling representational and epistemic functions (Dreyfus et al., 2014; Krause, 2016). Research based on embodiment theories also suggests *that* gestures fulfil cognitive functions (Clark, 2013) but little is known about *how* gestures contribute concretely to cognitive processes in mathematics.

Studies from the field of psychology, carried out in clinical settings and dealing with arithmetic equivalence problems, suggest that gesturing can help solving such problems through reproducing a grouping strategy (e.g. Goldin-Meadow, Cook, & Mitchell, 2009), which can lighten the cognitive load (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001), and can lead to a more general conceptual understanding of arithmetic equivalence (Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014). However, although this research suggests potential cognitive functions, we claim that these results cannot be generalized for mathematics in general.

But what do cognitive functions of gestures in the context of mathematics look like? In our paper, we adapt a framework for cognitive functions of gestures from psychology (Kita, Alibali, & Chu, 2017)—a framework that also includes the abovementioned Cognitive Load Reduction Hypothesis (Goldin-Meadow et al., 2001)—to test its applicability for the context of mathematics. In particular, we present the first part of a larger study on gestures' self-oriented functions in mathematical cognitive processes. In this first part we re-analyze existing data to explore how the four cognitive functions of gestures proposed in the framework manifest themselves in the context of mathematics. We will provide an illustrative example of the on-going analyses and discuss preliminary results.

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THEORETICAL FRAMEWORK

Theories of embodied cognition assume that all thinking, in our case, mathematical thinking, is deeply influenced by our interaction as humans in the physical and cultural world (Barsalou, 2008), assigning perception and action an integral role in (mathematical) cognitive processes. Gestures can now be understood as some kind of physical interaction with the world, grounded in the same generating system as concrete actions. In particular, *representational gestures* denote those "gestures that depict action, motion or shape, or that indicate location or trajectory", (Kita et al., 2017, p. 245) an understanding of gestures that encompasses the often applied taxonomy of *deictic*, *iconic* and *metaphorical* dimensions of gestures as introduced by McNeill (2005). However, representational gestures embrace gestures in a wider sense, not only considering those movements performed in presence of speech (co-speech gestures), but also those performed in thinking processes that lack simultaneous verbal expression (co-thought gestures). Both of these potentially have self-oriented functions in that their "production also affects gesturers' own cognitive processes and representations" (Kita et al., 2017, p. 245).

The Gesture-for-Conceptualization Hypothesis

Based on an extensive corpus of empirical research in psychology, Kita, Alibali and Chu (2017) organize findings and theories concerning self-oriented gestures and how they can influence and enhance problem solving performance and synthesize them in their *Gesture-for-Conceptualization Hypothesis* (GFCH):

> The key theses of the gesture-for-conceptualization hypothesis are (a) gesture activates, manipulates, packages, and explores spatio-motoric representations for the purposes of speaking and thinking and (b) gesture schematizes information, and this schematization process shapes these four functions. (p. 246)

Following Kita et al. (2017), gestures *schematize information* in that they represent information in a condensed manner, focusing on a subset of aspects potentially relevant for the gesturer for solving the task at hand while leaving out other aspects.

Activating spatio-motoric representations through gesture means to call up inactive representations or to inhibit active representations from vanishing, e.g. by ephemerally reproducing them, also in a condensed manner. *Manipulating* such representations through gesture can facilitate the enactment of demanding mental operations, e.g. by virtually enacting the actual or imagined action, like in mental rotation tasks. Through *packaging* information, gestures can support the building of information units to organize spatio-motoric information relevant for processing or expressing complex ideas. *Exploring* spatio-motoric information through gesture can help to examine complex situations to find and focus on the information most useful and helpful in order to solve the task (Kita et al., 2017).

Investigating how far this framework is useful for understanding better students' cognitive processes when dealing with mathematical problems we ask the following question, focusing on the four cognitive functions:

How do the cognitive functions of *activating*, *manipulating*, *packaging* and *exploring* manifest themselves in mathematical situations?

However, the analytical approach is a deductive one, aiming at testing the hypothesis in a new context. Developing further functions—perhaps specific to mathematics—is not within the scope of this paper but will be addressed in the ongoing study.

METHODS

To integrate a wide pool of settings and mathematical topics, we considered and re-analyze data from some of our different past and ongoing studies in which we focused on gestures' role in mathematical thinking and learning (e.g. Krause & Salle, 2018). These settings include open interviews, task-driven interviews, and working with worked-examples while thinking aloud, dealing with functions, basic arithmetic, fractions and complex numbers.

In all cases, video data was collected and transcribed, where the transcripts considered the co-timing of speech and gesture by using squared brackets to indicate the beginning and the end of a movement of the gesture and provided a description of the movement. For gestures performed in absence of speech, this description is given within the brackets, in for co-speech gestures the description follows the brackets.

The analysis consisted of three steps: First, we identified episodes and scenes that include gestures. Second, the gestures were interpreted in the context of the situation and, in the case of co-speech gestures, related to the spoken utterance. Third, the gestures were examined with respect to their potential to fulfil one of the cognitive functions stated in the GFCH. For this, evidence for a gesture to fulfil a cognitive function is drawn from gestures' potential contribution towards solving the task.

THE CASE OF THEO

The following case presents the analysis of a part of an interview with Theo, a 6-year student, against the background of the cognitive functions of gestures. It illustrates how the gestures help Theo to find an explanation for why not adding the denominators when adding two fractions.

1 Well ,if one now this … *(puts both his hands on the table, Fig. 1a)* [two thirds] [plus *(moves the right hand slightly up and down on the table)* ..] plus [seven ninth] *(moves both hands equally on the table to a location slightly below)* for example *(moves the left hand with open palm towards the right hand on the table)* [calculates directly] *(s. Fig. 1b)*

2 Eh one gets an entirely different <u>number</u>. Ehm .. that ,well one is just not allowed to [add' the denominators *(locates both hands with slightly opened palms such that finger tips touch the table) … (smiles)* stoooop] ,is complicated

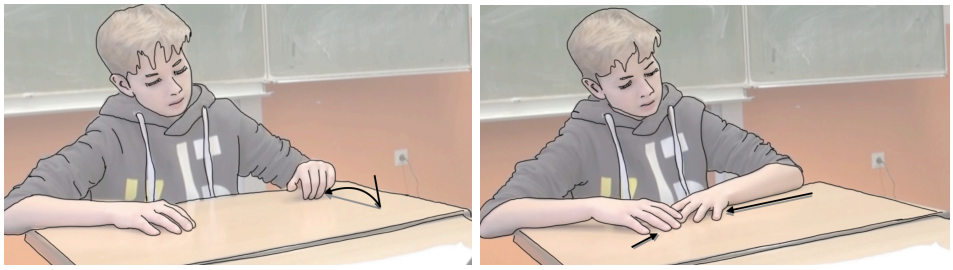

Figure 1a and 1b

3 ,ehm seven twelfth. …I

could you maybe again with pizzas, don't know if that is easier for you if you for example-

4 eeeh (puts both hands on the table again) (incomprehensible)

5 Yes ,for pizzas. for [one pizza] (*s. Fig. 2a)*

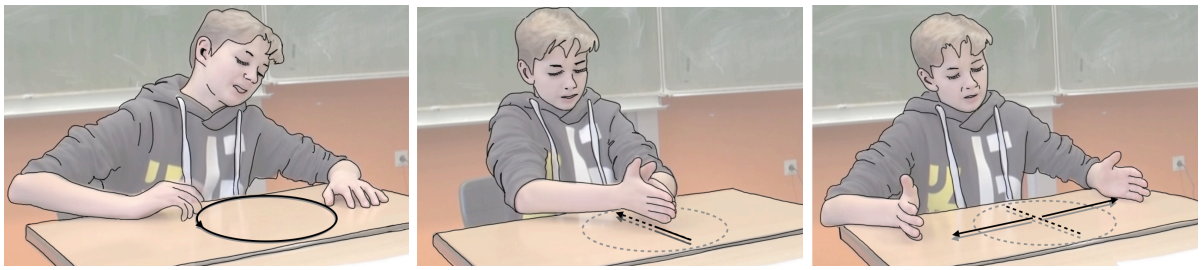

Figure 2a, 2b and 2c

6 If one by [..] (*moves the right hand on the table where he drew the circle before*) ehm ,nine I think .. is not that good- by six. If one divides by six- [(*draws straight lines with his right hand approximately on the right side of the circle he drew before in gesture; first one vertical line, then several straight lines in different angles to the right*) (.)]

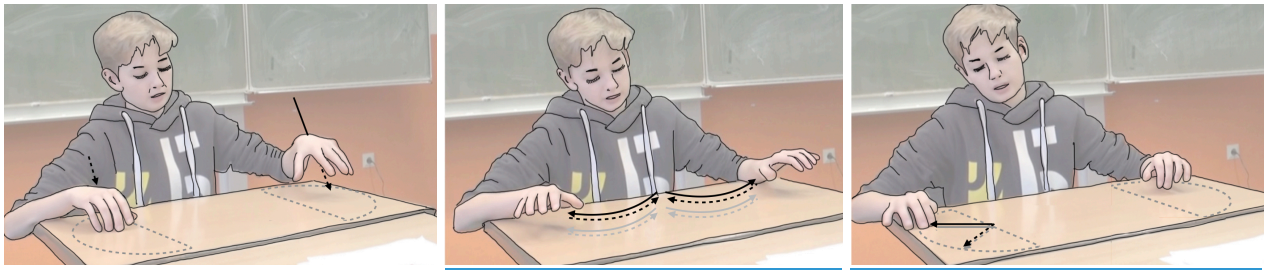

Figure 3a, 3b and 3c

7 ,so *(puts both hands on the table, s. Fig. 2b)* [by-] first *(s. Fig. 2b and 2c)* [[divides] [by the half']] then *(puts the right hand on the right side of the table, palms open and directed down)* [one have once] *(lifts the hand slightly and puts it back down)* [one half'] plus *(s. Fig. 3a)* [one half.] makes *(moves both hands together on the table and back apart, Fig. 3c)* [one whole.]

8 if one [divides this again] *(slightly moves the open right hand above the table such that the tip of the middle finger touches the table)* in- *(moves the hand over the table in a straight line)* [the-] *(briefly holds the right hand above the table)* [the one half'] in [th-r-ee'] *(s. Fig 3c)*

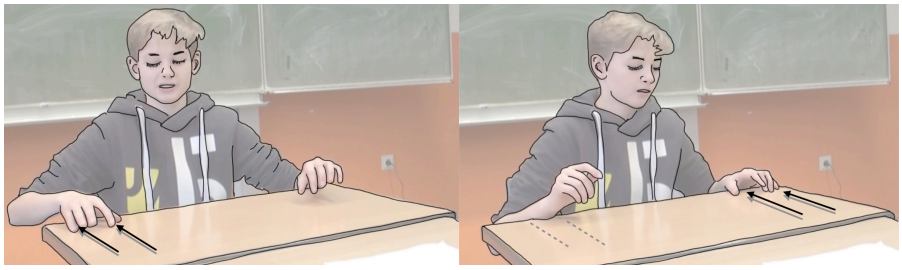

Figure 4a and 4b

9 then one has *(places the open right hand on the table)* [ ] *(s. Fig. 4a)* [three-] thirds' *(moves the hand back up)* [ehm no.] not correct. But yes *(places the hand on the table again and moves it down on the table)* [three thir-] no.

10 [[three sixths] *(moves right hand approx. 30cm above the table and then moves it towards him, similar to before on the table; then holds the right hand above the table in the air)* of the whole pizza']

11 plus *(Fig. 4b)* [one half.]

12 and *(moves the left hand down on the table similar to the right hand before but in a smaller movement)* [three sixths] plus one half would then be four- sevenths. And four sevenths [would then be] *(moves the hands, slightly open, palm directed down, together above the table)*

13 would have to be [a whole pizza] *(moves both hands in two half circles symmetrically and through this draws a circle on the table)* but that cannot be.

Theo provides a first, innermathematical and example-based approach to his explanation (lines 1-3, "you get an entirely different number"), that however seems to leave him unsatisfied (2: "is complicated"). Taking up a suggestion by the interviewer, he then frames a second approach in the concrete, real-life context of dividing pizzas and constructs a counter example for his explanation.

These two approaches are reflected in Theo's gestures:

First, he refers to a top-bottom structure of the symbolic notation of fractions as top-bottom-structure while referring to the two fractions in speech. In this, the first fraction seems to be located on the left side of the table, affiliated to the left hand (1: "two thirds", Fig. 1a). While recalling the second fraction (1: "plus …"), his right hand wanders above the table in a downward movement, perhaps anticipating the reference to the fraction as performed with the left hand before. However, the delay in verbalization might lead to both hands moving simultaneously co-timed to "seven ninth" (1) before moving the left hand towards the right hand while saying "calculates directly" (1, Fig. 1b). By this 'calculating directly' he might refer to the separate addition of numerator and denominator, since "one gets an entirely different number" (2). Moving the left hand to the right hand at the same time, he might indicate the addition of the one fraction to the other by joining them metaphorically.

When starting his second approach to the explanation, Theo establishes the simplified representation of "one pizza" (6) by drawing a circle in front of him on the table (Fig.

2a). He then starts visually dividing the imagined circle into parts by drawing straight lines with his finger (6). Seemingly not satisfied with this approach, he starts over, clarifying to "first divide by the half" (7), dividing the imagined circle first with his hands through 'cutting' a straight line through the middle (Fig. 2b), before separating the two halves by moving his hands away from each other to the left and right on the table (Fig. 2c). Following this, he locates "one half" and "one half" with each hand on one side of the table (Fig. 3a), moves the hands together to join them to "one whole" (7) and then immediately back apart to where the two imagined halves are located on the table (Fig. 3b). He divides the right half in "th-r-ee" (8, Fig 3c) by drawing two straight lines and while naming the fraction represented by this, his gesture refers to the top-bottom-structure of the symbolic notation again (9, 10, Fig. 4a & 4b). In his gesture, he established a representation of "three sixths of the <u>whole</u> pizza" (10) on the right side and one half on the left side (11). Theo then argues that adding these two fractions by adding the numerators and denominators "would then be four- sevenths" (7) but, at the same time, a whole pizza, reminding that the two sides joint make one whole by moving both hands—thus both halves—towards each other, a claim he established earlier (see line 7, Fig. 3a).

The cognitive functions of Theo's gestures

Within both approaches, gestures **activate** spatio-motoric representations that potentially help Theo structure his explanation. This concerns first the symbolic representations of the two considered fractions as top-bottom-structure on the two sides of the table. The symbolic top-bottom structure reduces the concrete fractions to *schematic representations.* Locating them on two opposite sides of the table, associated with two different hands allows Theo to treat them metaphorically as objects that can be 'put together'.

In the pizza-context, Theo **activates** the spatio-motoric representation of a pizza, simplified to a circle in front of him on the table. Using this *schematic representation,* he can now **explore** this representation and with that the situation used for his explanation, as can be seen in line 6, where he first divides the right half of the imagined circle in parts by drawing straight lines. In this instance, the **manipulation** of the circle through his gestures seems to show him that this subdivision does not work for him, though it is hard to say whether he does not consider it appropriate for constructing his explanation or for communicating his train of thoughts. However, in both cases it fulfils a self-directed function in the sense of the GFCH.

He furthermore **manipulates** the spatio-motoric representation through gestures when he develops it further by dividing it in two halves. As before, the two halves—and the respective associated fractions—are associated with one side of the table. Since this helps him to organize his complex thought into smaller meaning units, this can be seen as a case of **packaging** spatio-motoric information. Having broken down the complexity and reduced the amount of aspects to consider for one stream of his argumentation, he can combine both schematic representation—the symbolic top-bottom-structure and the circle representing the pizza—for his explanation.

In short: He activates spatio-motoric representations that help him capture the situations in two different ways that are eventually combined. Through manipulating these representations, he can explore the situation to construct an example—and its representation—to carry out his explanation. Packaging the information by metaphorically locating different components in space and reducing the focus on one of them furthermore supports Theo's cognitive processing towards constructing his explanation.

FURTHER COMMENTS AND DISCUSSION

This paper aimed at presenting a potential framework on cognitive functions of gestures in processes of mathematical thinking and learning. The case study we presented reflects mainly two phenomena that have been found so far across the data we consider in our ongoing study:

First, the vast majority of scenes in which we identified a cognitive benefit of gestures did not restrict themselves to one gesture with one cognitive functions but rather to multiple functions that synergized. The necessity to verbalize thoughts–through thinking aloud or through responding to an interviewer–might lead to more gestures, considering that gesture and speech production are strongly related (McNeill, 1992). In particular, we noticed that the activation of spatio-motoric representations played an important role, especially when no further representations were provided. Perhaps not surprisingly, the students first need to establish a representation in order to be able to manipulate or explore it in the cases and gestures provide a powerful tool to provide idiosyncratic representations as 'tool for embodied thinking'.

Second, we faced methodological issues across the data, probably related to them all requesting to verbalize their thoughts and train of actions. Different to the studies on which Kita et al. (2017) base the GFCH, our methodology does not use experimental settings but detects potential cognitive functions of gestures within an interpretative approach. This made it rather difficult to identify cognitive functions in the existing data since the component of communication ideas to a third person was a strong one. While gestures certainly can fulfil cognitive and communicational functions at the same time, it often seems hard to get evidence for a self-oriented function. While as one source of evidence for a self-directed function of the Theo's gestures we considered his gaze—he constantly looked at the gestures he produced but stared in the air to the upper left while gesturing when he seems to struggle with an idea (Zurina & Williams, 2011)—future research needs methodological revisions. For example, data could be gathered by capturing the students process of engaging with and solving a task but might request the verbalization of a solution just subsequently. While this leads to different difficulties with respect to the interpretation of gestures, it might facilitate a focus on co-thought gestures.

It has to be kept in mind that it is not within the scope of the preliminary study to identify all gestures fulfilling cognitive functions and to describe the data comprehensively but rather to test the waters for adapting the framework and to see

how the functions can manifest themselves within the mathematical contexts. Further research should extend this scope to get more insights about the *how* of the cognitive benefit of gestures.

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EXAMINING KNOWLEDGE DEMANDS FOR TEACHING CLASS INCLUSION OF QUADRILATERALS

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Though teacher knowledge has been assessed using paper-pencil tools, little is known about how teachers use their mathematical knowledge for teaching in in-the-moment challenging instructional situations while attempting to change their practice to encourage student reasoning. This paper illustrates using the case of teachers' discussion on class inclusion, how analysis of these challenging situations serves to not only identify the knowledge demands for teaching geometry but also illustrates the dynamic nature of mathematical knowledge for teaching as it may support use and development of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) to structure the classroom discourse. The need for safe space for teachers to explore their teaching and learn from their mistakes is needed by reflecting on and discussing their teaching with other educators.

INTRODUCTION

Teachers' knowledge of geometry has been studied mostly in the context of measuring prospective or in-service teachers' knowledge through the development of paper-pencil tools and have been found to be dismal (Jones, 2000). These tools have been largely based on two major frameworks prevalent in literature. One of these frameworks is van Hiele's theory (1959) which has been used for over 50 years since it was proposed and has been used to assess both students as well as teachers' knowledge of geometry. Another framework is by Ball, Thames and Phelps (2008) about Mathematical Knowledge for Teaching (MKT). However, these paper-pencil tools have the limitation of not being able to capture the situations in which this knowledge is used or developed "in-situ" in the process of classroom interaction. Further, the use of tests to show deficits in teacher knowledge may lead to teachers' distrust towards researchers. In the Indian context, where we are faced with a situation of lack of highly qualified mathematics teachers, we need to characterise the challenging situations that teachers face in addressing students' learning and the topic-specific knowledge of mathematics for teaching that is required to overcome that challenge while working with the teacher.

Attempts have been made to identify topic specific knowledge of geometry used by teachers in the moment of teaching. A study by Chinnappan, White and Trenholm (2018) identified a symbiotic relationship between the SMK and PCK. In this paper, we illustrate and use a topic specific framework to identify knowledge demands and relation between SMK and PCK, while analysing classroom interactions of teachers making an attempt to change their instructions to encourage student reasoning. The main research question addressed in this paper is

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How can analysis of challenging instructional situations contribute to our understanding of knowledge demands for teaching geometry, particularly class inclusion?

RELATION BETWEEN TEACHERS' CONTENT KNOWLEDGE AND TEACHING OF GEOMETRY

It has been well recognised that both SMK and PCK play a crucial role in determining the quality of instruction (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). However, the relationship between SMK and PCK has not been explored at depth especially in in-situ situations that illustrate the dynamicity of this knowledge (Takker & Subramaniam, 2017; Chinnappan, White & Trenholm, 2018).

SMK for teaching class inclusion can be identified using van Hiele theory (1959). According to van Hiele theory, the concept of class inclusion develops at level 2 when the learner is able to establish interrelationship between properties within and among figures, create and identify figures as examples and non examples of a class, deduce properties of figure, identify similarities and differences in figures and is able to classify correctly. It also includes giving and evaluating the alternative definitions for their correctness and knowledge of necessary and sufficient properties to arrive at the minimal definition, give figures as examples, non examples and also be able to evaluate non-routine figures for recognising a shape. A common theme that runs across the development of knowledge of geometry is understanding that the process of generalisation is a deductive process, often mistakenly assumed to be an inductive process.

A teachers' knowledge of the teaching of geometry or PCK is built over these foundation blocks, wherein a teacher uses this knowledge to generate and sequence the tasks for teaching or assessment. A teacher uses the knowledge of students' thinking combined with SMK to come up with examples, challenging examples and even counterexamples to make students think and focus on a particular or connected property of a shape. The understanding of correct and minimal definition guides the task implementation as well as evaluation of alternative and partially correct definitions given by students. A teacher may use the knowledge of different representations to depict and elicit statements of class inclusion from students and work on their alternative conceptions of shapes and hierarchical classification. The knowledge of deductive process of generalisation guides the norms of classroom interaction and how the generalisations are stated and validated. Teacher's response to visual and perceptual approaches by students are guided by this knowledge of the nature of mathematics.

METHODOLOGY

The instructional episodes reported in this paper are selected as cases from two teachers (Sunil and Shruti – Pseudonym) from a cohort of 10 teachers associated with Connected Learning Initiative (CLIx)[1] since 2016 to support the implementation of Geometric reasoning module. The selection of the instructional episode case was done based on similar content used in lesson and then within case and cross case patterns

were analysed to come up with the theory of role of challenging situations in teacher learning (Eisenhardt, 1989).

All teachers belonged to and worked in a rural setting in the state of Chattisgarh, India. The teachers were associated with the initiative since 2016 and had undergone around 9 days of face-to-face time in workshops prior to study. Based on reports of challenges faced in managing classroom discussion, follow up classroom support of around 4 weeks was provide to teachers by research team after a two-day workshop discussing the module in detail. During the follow up, researchers did classroom observations and follow-up discussions to discuss the activities, lesson plan, content related doubts and analysis of students' understanding. The data from the professional development workshop and classroom observations was collected by writing detailed notes by the researchers through participant observation and in some cases, audio was recorded after teachers' permission.

Sunil was 39 years old high school teacher (Male), had 8 years of experience of teaching high school maths and has the qualification of M.Sc in Physics and diploma in IT, though no formal qualification in the education field. Shruti was a 42 years old high school teacher (Female) with 11 years of experience and had M.Sc maths and B. Ed as a formal qualification. Both the teachers were motivated to implement the modules and engaged heartily with the researcher team on the discussions of content, pedagogy and student learning. However, Sunil had a very relaxed and conversational approach to teaching mathematics making students comfortable in expressing their thoughts and trying to engage all learners in classroom discussion. Shruti was a bit authoritative in classroom transaction. She felt that mathematics is too difficult for some students though she understood the importance of engaging students in discussion and supporting reasoning and expressing their thoughts.

FINDINGS

In this section, we report challenging episodes that made gaps in teachers' knowledge explicit and provided opportunities to us as teacher educators or teachers' themselves to reflect on their knowledge. These challenging events arose in situations when teachers attempted to support students' engagement in reasoning about shapes and came across contingent situations in practice.

The normal instruction for teaching geometry in these rural schools was that of a teacher explaining concept of shapes through examples of stereotypical figures from the textbook, listing its properties and definition and expecting students to memorise them. The teaching involves explanations of concept or definition followed by questions, most of which are replied in chorus or in few words. The attempt in the CLIx initiative was to encourage teachers to focus on developing property based reasoning through challenging and non-routine shapes for recognition and classification of shapes thereby developing the meaning of the definitions which may have been learned by rote by the students in earlier grades. The pedagogical pillars to provide safe space to

students to learn through their mistakes and learning through collaborative work was emphasised in the workshops.

The transcripts presented from both the lessons are one week after the intervention started. Teachers were discussing tasks of class inclusion using several challenging figures, for example, if the particular figure is a square or not, identify the shape if it was not a square and give reasons based on properties of shape.

Sunil's discussion on class inclusion

The lesson from which the episode is discussed, Sunil was discussing several figures from the task described above and one of them was a rectangle. He started the discussion on rectangle as indicated in the transcript below. In the transcript, T indicates the teacher, Sn indicates the different students speaking and Sch indicates chorus response of students.

36 T: What is a rectangle?

37 S1: One which has opposite sides equal.

38 Teacher draws a parallelogram in figure 1 and labels the opposite sides as 2 and 7 cm respectively.

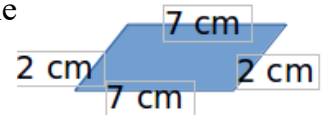

Figure 1: Parallelogram

40 T: Would you call this a rectangle?

41 S *chorus*: Yes

42 T: *Arre? (what!!)*

45 S *chorus* : No

46 S2: Because all its angles are not 90 degree

In the above episode, one needs to note that instead of responding to the student's incomplete definition of rectangle with an evaluative statement or explanation of the correct definition, the teacher responded by making a figure which satisfies the property given by the student but is not a rectangle. It is thus a counterexample given by the teacher which is contingent on the response given by the student. Knowledge of geometry is involved in both the pedagogical tasks, but giving a counterexample based on student response requires in-the-moment use of both SMK and PCK to respond in mathematically appropriate way to students' definition of rectangle. The teachers made the figure having the property of opposite sides but varied the angle using the knowledge of similar and different properties of parallelogram and rectangle. However, the decision to respond by drawing a counterexample figure rather than the correct definition involves knowledge of PCK of how to draw students' attention towards the missing necessary property of the ninety degrees angles of the rectangle. He expected that students would experience conflict through the figure, identifying that the property is same but the figure looks different from rectangle, but is surprised when student agrees. It made student realise the mistake and change the answer. However, in continuation of the pattern of giving reason and justification for their response in previous task, the student S2 justifies their response by identifying that angles are not ninety degrees which is the necessary property of the rectangle. Thus, by using the

counterexample, the teacher was able to highlight the need of the necessary property of angles being ninety degrees and help students take the step towards arriving at the minimal definition. One can still doubt, however, that whether the students have understood the concept of opposite sides being parallel too since this property can be deduced from opposite sides being equal. Identifying that this point needs to be discussed, again involves both SMK of properties and PCK of selecting the points to be discussed an assessed for understanding. The discussion returns to this point in the following transcript when the teacher asked the students to recognise a rotated parallelogram (not in horizontal or vertical orientation).

67 T: What is this?

68 S *chorus*: Rectangle

69 S1: In rectangle opposite sides should be equal and ninety degree angle should be there. (In response teacher points to the figure drawn earlier of the rectangle and the parallelogram.)

70 S2: Opposite sides are equal but …

71 S3: First one is a rectangle because opposite sides are equal. In parallelogram, it is not necessary to have a ninety degrees angle.

73 S4: Rectangle can be called as parallelogram because… but we cannot call a parallelogram as a rectangle.

75 Teacher repeats the assertion by S4 followed class repeating it in chorus.

In the above episode, all the students were able to successfully identify a rotated rectangle as a rectangle, thus were able to understand that orientation of the figure does not constitutes as the property of the figure itself. The students then gave their observations which represents the modified definition including the necessary property and statements of generalisations about parallelogram by S3 and relation between rectangle and parallelogram by S4. The generalisation by S3 is correct while that of S4 is partially incorrect. Teacher not evaluating and repeating the S4's assertion in line 73 indicates gap in SMK of not being able to correctly evaluate a generalisation which is behind the gap in PCK of not identifying students' misconception and ways to address students' misconception. In line 73, student is arguing visually considering the static figure of a parallelogram always having two acute angles and two obtuse angles and thus excluding the rectangle from the category of the parallelogram. It is confirmed that the teacher agrees with this line of thinking, since in the next task of analysis of square and rectangle, the teacher concludes that square can be a rectangle since it fulfils all the necessary properties of the rectangle (correct) but incorrectly concludes that rectangle cannot be made into a square and thus is not a square. This is also reasoning based on empirical nature of drawing where the argument is based on physical properties of drawing rather than mathematical properties. There is also a gap in the use of mathematical language as the teachers is not careful in use of qualifiers or terms that indicate generality like "All squares are rectangle" is a different statement from "A rectangle *can become* a square" constraining thinking in generality of shapes. Thus although the norms for giving reasons and articulating thinking have been established,

the knowledge demands during teaching indicate gaps in teachers' knowledge about the process and nature of class inclusion especially the process of generalisation and therefore students' understanding fall through the cracks.

Shruti's discussion on class Inclusion

We now discuss an episode from Shruti's class on 30 July 2018 wherein a similar discussion took place about the relationship between square and the parallelogram. Here the teacher was discussing a challenging example from the module (rhombus) and was asking students to evaluate if it was a square or not. Students identified that it is not a square and some identified it as a rhombus while others as a parallelogram. When students were not able to articulate what a parallelogram is, the teacher asked the students to make different figures of parallelogram and herself identified the property of a parallelogram as "one who has opposite sides equal and parallel". What happened afterward is given in transcript below.

96 S4: 2 acute angles and 2 obtuse angles

97 T: Very good! What is the main property?

98 S *chorus*: All sides equal

99 T: It is not necessary. (Draws a parallelogram with one pair of sides longer in horizontal orientation) Is this not a parallelogram?

100 S *chorus*: Opposite sides are parallel…

101 T: (Draws a rectangular candy and explains) Opposite sides will not ever meet, the angles of the parallelogram cannot be ninety degrees…. It is always 2 acute angle and 2 obtuse angle…. Or in some case, it may be possible…. In some cases, rectangle can also be a parallelogram.

In the above episode, in line 96 the student gave the property of 2 acute angles and 2 obtuse angles for a parallelogram based on a visual stereotypical image of it. Though wrong, teachers praised the student indicating it to be correct but tried to make student think about the necessary property as the "main property" and also gave a correct counterexample when students responded as "all sides equal" based on the visual figure of rhombus in front of them. Just like Sunil, she was also using her knowledge of SMK of necessary properties to generate PCK through the use of counterexample in the moment of teaching. However, the most interesting part in this episode is when teacher realised her own mistake of giving the property on the basis of visual diagram of the parallelogram rather than based on its necessary properties in line 101. While discussing the properties of the parallelogram, she at first said that angles of a parallelogram cannot be ninety degrees but in the next instance after a long pause and looking at diagram, she realised that a rectangle is a parallelogram and in that case the angles would be ninety degrees. Here we see an instance of teacher deepening her own SMK while responding to student's incorrect assertion. It is possible that this reflection may not have occurred during the classroom, would have occurred later or not at all. However, the potential of development of SMK as well as PCK while responding to and evaluating students' assertions cannot be denied.

THE WAY FORWARD

The analysis of challenging episodes related to class inclusion from the teaching of two teachers struggling to change their practice to encourage students' reasoning indicates how this type of analysis can shed light on the knowledge demands for teaching class inclusion in terms of both SMK and PCK and their interdepenencies. The main knowledge demand is knowing and supporting the process of generalisation among students and to be able to consistently operate at the level of analysing properties of figures mathematically rather than perceptually in contingent situations. The episodes also illustrate the dynamic nature of mathematical knowledge for teaching. The classroom interaction not only gave opportunity for teachers to develop their PCK contingent to student responses to develop their thinking through use of counterexamples and to identify necessary properties of the shape but also allows opportunities to deepen their own SMK through reflection and student engagement. However, to support teachers' exploration and learning from practice one need to create safe space for teachers to explore their teaching and learn from their mistakes perhaps by reflecting on their teaching with other educators.

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CLASSROOM SOCIAL CLIMATE IN THE CONTEXT OF MIDDLE SCHOOL GEOMETRY

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Research on psychosocial classroom learning environments has a strong tradition due to the early discovery of a relationship between positive classroom climate and academic performance and motivation, engagement, participation, and attitude towards school and teaching. But only recently has attention turned to mathematics classroom climate, and more research is needed in this context. In this paper, we propose a possible further development of existing classroom climate models specifically designed to assess students' perceptions of psychosocial classroom learning milieu in the context of school geometry by using drawings. The results are discussed not only with regard to the further development of the adapted classroom climate model but also with regard to their theoretical and practical implications.

INTRODUCTION

Classroom is a social context for learning, which with time develops a distinct social climate having certain demand characteristics (e.g., Evans, Harvey, Buckley, & Yan, 2009; Moos & Moos, 1978; Trickett & Moos, 1973). As such, classroom climate influences students' growth, and their academic, social and emotional development (Evans et al., 2009). The nature of climate in every classroom is unique, as it is a function of its different factors, such as norms and rules, student task-related interaction, styles of leadership, composition of the group members (e.g., Moos & Moos, 1978; Trickett & Moos, 1973). According to Bülter and Meyer (2015) and Gruehn (2000), the climate-creating determinants help the teacher to create a working alliance with the students, and thus, to achieve positive effects with regard to each student's self-confidence, social behavior, performance, and attitude towards school. In contrast, a negative classroom climate may lead to social and emotional behavioral disturbances, and thus, have a negative impact on students' performance (Evans et al., 2009). Mainly questionnaires have been employed to research this complex construct (e.g., Bülter & Meyer, 2015; Fraser, 1989, 1998). However, the use of creative methods, such as drawings provides a multi-dimensional, and holistic view of young students' latent experiences in the mathematics classroom (e.g., Ahtee et al., 2016; Kuzle, Glasnović Gracin, & Klunter, 2018; Pehkonen, Ahtee, & Laine, 2016).

Despite several decades of research (e.g., Bülter & Meyer, 2015; Evans et al., 2009; Fraser, 1989, 1998; Moos & Moos, 1978) concerning students' perceptions of the classroom learning milieu (e.g., cohesiveness, satisfaction, goal direction), the research area on social aspects of classroom environment (e.g., communication, interaction, social working structures, norms) is still considered a fruitful and interesting research direction (e.g., Hannula, 2012). Furthermore, studies in this area (e.g., Ahtee et al.,

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 511-518). Pretoria, South Africa: PME.

2016; Pehkonen et al., 2016) do not holistically assess the classroom social environment, but rather focus on its few aspects, and thus, do not give a comprehensive picture of what is happening in the classroom. In this report, we focus on the classroom social climate in the context of school geometry. The goals of this report are twofold: (1) to present a modification and further development of existing classroom climate models in the context of middle school geometry, and (2) to empirically evaluate its utility when using participant-produced drawings.

THEORETICAL PERSPECTIVE

The classroom climate refers to shared subjective representation of important characteristics of the classroom as a learning environment involving the physical environment of the classroom, the social relations between teachers and students or students among each other, expectations with regard to performance and behavior, the way in which teaching and learning processes take place, and the specific norms and values in the classroom (Eder, 2002). Within different disciplines various approaches to the conceptualization and assessment of environments have been used (e.g., Trickett & Moos, 1973). One often-pursued of these approaches is based on the concept of "*perceived environment*" (e.g., Eder, 2002; Moos & Moos, 1978; Trickett & Moos, 1973). This approach is characterized on the contention, that the environment of a particular setting is defined by the shared perceptions of its members along a number of environmental "*dimensions*" over a longer period of time (Moos & Moos, 1978). According to Trickett and Moos (1973), nine dimensions of classroom climate can be used in conceptualizing the individual dimensions characterizing diverse psychosocial environments. These fall under three general conceptual categories: (1) relationship, the degree to which individuals in the environment help and support each other, and to which they are involved in the class and its activities (i.e., involvement, affiliation, teacher support); (2) personal development, the degree to which self-enhancement can occur (i.e., task orientation, competition); and (3) system maintenance and system change, the degree to which the environment is orderly, clear in expectations, maintains control, and is able to change (i.e., order and organization, rule clarity, teacher control, innovation). On the other hand, Evans et al. (2009) conceptualized classroom climate as a function of three different components: academic, referring to the pedagogical and curricular elements of the learning environment; management, referring to discipline styles for maintaining order; and emotional, referring to the affective interactions within the classroom. Thus, the classroom climate is a multi-faceted construct "made up of a large number of components, which can be reduced to factors in a variety of ways" (Evans et al., 2009, p. 141) depending on the type of the environment (Fraser, 1998).

We propose a possible further development of existing classroom climate models (e.g., Eder, 2002; Evans et al., 2009; Fraser, 1989, 1998; Trickett & Moos, 1973) to better understand structure, functions, and processes in a mathematics classroom. This was guided by three principles. First, consistency with literature describing characteristics of the mathematics classroom. Secondly, individual insider characterization of the

classroom through students' eyes by using drawings. Thirdly, the age-appropriateness of the model without sacrificing its depth. We conceptualize classroom social climate as a function of three conceptual categories, namely *Interpersonal Relationship*, *Personal Growth*, and *Order*.

Interpersonal Relationship refers to nature, the intensity of personal relationships, and the mutual influences of the teacher and the students within the classroom, including social, pedagogical and mathematical aspects. The Verbal and non-verbal communication of the teacher, Verbal and non-verbal communication of the students and Organization are conceptualized as relationship dimensions (see Table 1).

Dimension	Subdimensions	Scales
Verbal and non-verbal communication of the teacher	Teacher's position in the classroom	In front of the blackboard, Among students, At the desk, Somewhere in the classroom
	Support by the teacher	Assistance, Positive feedback, Negative feedback, Mathematics related question, Mathematical statement, Observation, Non-mathematical comment, Passive
Verbal and non-verbal communication of the students	Students' position in the classroom	At the blackboard, At the table, Next to the teacher, In front of the blackboard, Amongst other students, Somewhere in the classroom
	Participation	Working on assignments at the table/on the blackboard, Listening, Responding, Questioning, Asking for assistance, Review, Discussion, Positive expression, Negative expression, Non-mathematical comment, Passive
	Affiliation	No communication with other students, Student-student communication, Student-student encouragement, Student-student help request, Student-student support, Negative comments towards other students
Organization	Working method	Teacher-centered instruction (frontal), Individual work, Group work, Working with a partner, Work/discussion while sitting in a (half-) circle

Classroom seating arrangement	Traditional classroom arrangement, U-shaped arrangement, Mixed arrangement, (Half-)circle arrangements, Group tables

Table 1: Description of Interpersonal Relationship category.

Personal Growth refers to the goal orientation and clarity of the lesson objective. On the one side, a lesson goal can be clearly represented by mathematical content or an assignment on the backboard, the teacher identifying the goal of the lesson or students working on their assignment. On the other side, the lesson objective can be pursued by using different teaching materials specific to geometry (e.g., geometric forms, models, tools), which can be utilized by class protagonists (teacher, students).

Order refers to the social norms and maintenance of order in the classroom. Since social norms are shared principles of behavior that are considered acceptable in a group, not only the teacher, but also the students are responsible for proper conduct, keeping order, and behaving properly.

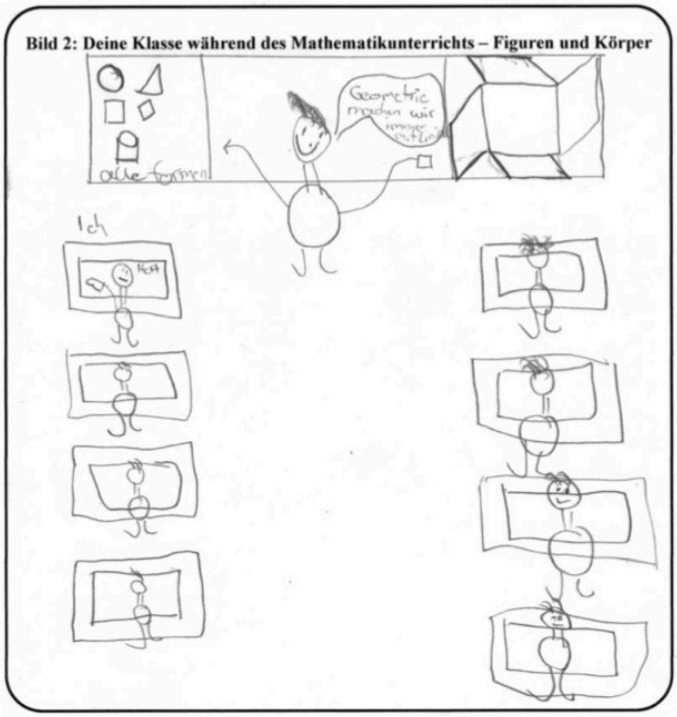
RESEARCH PROCESS

For this study, an explorative qualitative research design was chosen. The study **participants** were grade 3-6 students. For the purposes of this paper, we report on drawings of 30 grade 5 students from several urban schools in the federal state of Brandenburg (Germany). This age group was optimal for the purposes of this paper as the quality of drawings was high, and allowed for rich insights into classroom social climate. Typical case sampling as a type of purposive sampling was utilized as a way of collecting rich and in-depth data.

Three **main sources of data** were audio data, student work, and a semi-structured interview. Student work was based on an adaptation of the instrument from the work of Ahtee et al. (2016) and Pehkonen et al. (2016). The research data were **collected** in a one-to-one setting between a student and the first author of the paper. Each student was given a blank piece of A4-paper with an assignment given by a fictional 12-years old bright girl by the name of Anna: "Dear _____, I am Anna and new to your class. I would like to get to know your class better. Draw two pictures of your mathematics lessons. The first drawing should show what your arithmetic lessons are like and how you view them. The second drawing should show what your geometry lessons are like and how you view them. Include in each drawing your teaching group, the teacher, and the pupils. Use speech bubbles and thought bubbles to describe conversation and thinking. Mark the pupil that represents you in the drawing by writing "ME". Thank you and see you soon! Your Anna." After the students had finished drawing, the drawings were used as a catalyst for a semi-structured interview. Multiple data sources were used to assess the consistency of the results, and to increase the validity of the instruments.

The analysis of drawings involved the first author and another coder coding the data independently. The analysis contained the following steps: (1) analysis of drawings

with respect to our conceptualization of classroom social climate, (2) confirmation of the interpretation by content analysis of the data from semi-structured interview, and (3) coding of dimensions and respective subdimensions included in the student oral data. Different components of the classroom social climate were first assigned one of the categories (1 and 2), before assigning specific subdimensions (3). If the descriptor was not given, then both coders discussed together the nature of the classroom social climate element before developing a new scale and extended the coding manual. These were then validated through an iterative process, and constant comparison in order to obtain full agreement. Lastly, the descriptive statistics were calculated. Figure 1 illustrates the coding with accompanying codes (e.g., D = dimension, letters A to C = subdimensions, ordinal numbers = subscales, T = teacher, S = student). The number in brackets gives the number of drawn persons who fall into this category.



Coding of 1. category "Interpersonal Relationship":
D1A.1.T: position in the classroom; in front of the blackboard
D1A.11.T: support by the teacher; mathematical statement ("We do geometry always with a ruler.")
D1B.2.S (8): position in the classroom; at the table
D1B.11.S (8): participation; listening
D1B.29.S: affiliation; unidentifiable
D1C.1: working method; frontal
D1C.7: seating arrangement; traditional
Coding of 2. category "Personal Growth":
D2.A1: orientation; goal of the lesson
D2.B1; teaching material; 2D shapes
D2.B2; teaching material; 3D shapes
D2.B3; teaching material; 2D models
Coding of 3. category "Order":
D3A.3: keeping order; unavailable

Figure 1: An example of coding.

FINDINGS

The analysis of the students' perceptions of geometry classroom learning milieu showed that all conceptual categories and respective dimensions of our classroom social climate model were present in the students' drawings. Category *Interpersonal Relationship: Verbal and non-verbal communication of the teacher* was represented through the teacher's position in the classroom and support during the geometry lessons. In 70% of the drawings the teacher stood in front of the blackboard, otherwise the position of the teacher was at the desk (13.3%), among students (10%) or was not drawn (6.7%). In 50% of the drawings, *Support by the teacher* was not shown or not possible to identify. Otherwise, mathematical statement (26.7%), assistance (16.7%), and mathematics related question (16.7%) scales dominated. Similarly, *Verbal and*

non-verbal communication of the students was represented through its respective subdimensions, namely *Students' position in the classroom*, *Participation*, *Affiliation*. Slightly more than half of the students (56.7%) sat during their geometry lessons at their tables and worked on their assignments (26.7%). However, 30% of the drawings did not reflect any mathematical thoughts, but included statements such as "I'm tired", "What's for lunch?". The drawings did not reveal any elements pertaining to asking questions, asking for assistance or making negative statements about geometry. In 33.3% of the drawings, *Participation* of the students was either not shown or unidentifiable. *Affiliation* of the students (i.e., cooperation and mutual acquaintance of the students among themselves in the classroom) was in 60% of the drawings not shown or not possible to identify. However, when this aspect was identified in the drawings, the students did not discuss the assignment with other students but rather worked quietly on the assignment. The students' drawings also revealed the overall *Organization* of the assignments and the classroom. With respect to *Working method*, the teacher standing in front of the classroom and teaching, with students making notes was present in 50% of the drawings, followed by student individual work (13.3%). Group work was represented in one drawing only. In addition, in 26.7% of the drawings, the working method was not shown. The chosen working method was associated with *Classroom seating arrangement*. Nearly half of the drawings (43.3%) reflected a traditional classroom arrangement with tables in rows. Only a few drawings portrayed tables being arranged in U-shape or in groups. However, in 30% of the drawings either one table or no tables were drawn so that the classroom seating arrangement could not be identified.

With respect to the category *Personal Growth* both dimensions, namely *Orientation* and *Teaching material* were represented in the students' drawings. The goal of the lesson was transparent in 93.3% of the drawings (e.g., mathematical content or assignments were illustrated on the blackboard). In addition, in one-third of the drawings the mathematical content was indicated by the teacher (e.g., the teacher explained how to solve a mathematical problem, the teacher informed the students about the lesson plan). To reach the lesson goals, different *Teaching materials* specific to geometry were illustrated, such as 2D shapes (66.7%), 3D solids (26.6%), and 3D models (16.7%). Lastly, with respect to the category *Order* in the majority of the students' drawings (86.7%), behavioral demands on the part of the teacher or the students were not drawn. Altogether there were only five instances in three drawings where either the students or the teacher instructed students how to behave.

DISCUSSION AND CONCLUSIONS

Characteristics of learning environments are powerful predictors of students' academic success (e.g., Evans et al., 2009; Gruehn, 2000). The study results showed that our model of classroom social climate was theoretically coherent, and consistent with the data generated in the geometry context, with all categories and dimensions being present in the students' drawings. Most students' drawings represented episodic-

autobiographical images (De Beni & Pazzaglia, 1995) illustrating the occurrence of a single episode (i.e., geometry lesson) at a particular time and place.

The study results show that positive indicators of classroom social climate were present in grade 5 students' drawings of their geometry lesson. The high proportion of not drawn behavioral demands on the part of the students and teachers (86.7%) in the category *Order* is an indicator that the geometry lessons are orderly regulated. If rule clarity was not clear, and the teaching discipline explicitly lacked, the share of behavioral cues would have been higher. A small percentage (16.7%) of behavioral demands on the part of both teachers and students can be an indication that social behavior or discipline is insufficient, and thus, would represent a negative teaching climate (Evans et al., 2009). Nonetheless, the lack of behavioral demands clearly prevails, which is an indication of well-designed lessons with good teaching discipline and social behavior of the students (Bülter & Meyer, 2015).

According to Evans et al. (2009) and Gruehn (2000), improved school performance is a favoring factor of a positive classroom climate. This indicator was not explicitly found in the data. However, in 93.3% of the drawings the goal of the lessons was clearly identified from the students' perspective. This indicates that in geometry lessons a specific teaching goal is being pursued, which may guide the students in the direction of increased performance, and the formation of interest in the subject. Bülter and Meyer (2015) and Evans et al. (2009) also argued that in classes with positive classroom climate increased engagement, and participation can be observed. The drawings revealed a broad spectrum of *Participation* in geometry lessons; while 43.4% of the students worked on the assignment at the table or blackboard and 6.7% of the students participated in the discussion of the tasks, almost a third of the students (30%) did not follow any mathematical thoughts. These divergent results do not allow any conclusive statements about the classroom social climate since none of the scales from the *Participation* subdimension clearly dominated.

This study was an exploratory study with a small sample, so not all elements our classroom social climate model were present in the students' drawings, nor can we generalize the results. Furthermore, the absolute frequencies of "unavailable" and "unidentifiable" codes of some scales were high. This may indicate that some students had difficulties drawing, some aspects can be expressed by drawing in a limited way, and the necessity to improve the semi-structured interview guide.

Despite these drawbacks, drawings have opened up a new way of gaining insight into students' perceptions of psychosocial classroom learning milieu in middle school geometry. In practice, the drawings may be used as a classroom tool to promote a dialogue between students and their teachers (Anning, 1997). Additionally, they may provide teachers with feedback about how their lessons are perceived by students, and to plan and implement changes for the future lessons (Anning, 1997).

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