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Mellony Graven, Hamsa Venkat, Anthony A Essien and Pamela Vale

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RESEARCH REPORTS (L – Z)

REASONING-AND-PROVING TASKS IN HONG KONG SENIOR SECONDARY LEVEL PUBLIC MATHEMATICS EXAMINATION

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The reasoning-and-proving (RP) tasks in Hong Kong senior secondary level public mathematics examination are investigated. The results show that a predominant proportion of the RP tasks require the students to produce reasoning that merely involves calculation and memorisation of facts and procedures. The findings are compared and discussed with the curriculum aims. This study indicates some of the reasons for the marginal role that proof plays in school mathematics in Hong Kong.

INTRODUCTION

Proof can not only verify the truth of a mathematical statement, but also promote sense making and understanding in mathematics through explanation (Knuth, 2002). Consequently, mathematics educators from many countries suggest that proof and proof-related reasoning (RP) permeate school mathematics at different levels and across different content areas (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2003). In response to this international call, in recent years many studies have been conducted to investigate how RP is being treated in the curriculum (e.g., Stylianides, 2009; Thompson, Senk, & Johnson, 2012; Davis, Smith, Roy, & Bilgic, 2014; Fujita & Jones, 2014; Wong & Sutherland, 2018). The focus of these studies, however, has been exclusively on examining students' learning opportunities of RP from textbooks. Few studies have been conducted to examine the RP in examination papers and the influence of the latter on the former. It is a common perception that examination papers have an influence on the students' learning experience in general and learning opportunities of RP in particular, particularly in East-Asian countries (e.g., Hong Kong, Singapore, South Korea), just because these countries have a highly examination- driven culture. For instance, both Leung (2000) and Wong (2002) claimed that the learning experience of students in East-Asian classes is heavily aligned with the tasks in the public examinations. However, no direct evidence has been provided to support such a claim. This paper is intended to fill these gaps. Specifically, it studies the influence of examination papers on the students' learning opportunities of RP. This is done by *directly* examining the RP tasks of the senior secondary level public mathematics examination papers in Hong Kong.

THE CONTEXT: HONG KONG

Hong Kong launched its new senior secondary curriculum, Hong Kong Diploma of Secondary Education (HKDSE), in 2009 (CDC & HKEAA, 2015). This new curriculum consists of a Compulsory Part and an elective Extended Part, and is aimed at developing in students: "(a) the ability to think critically and creatively, to conceptualise, inquire and reason mathematically, […]; (b) the ability to communicate

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with others and express their views clearly and logically in mathematical language; […]" (p. 2). The indication of the need to teach RP should be evidently clear from these aims. Since 2012, HKDSE examination of mathematics has been conducted yearly to assess students' learning outcomes of the new curriculum, and the mathematical tasks presented in these examination papers have been set in a consistent format throughout these years. Like other East-Asian countries, Hong Kong has a well-established examination- oriented culture. Hitherto, the results of various studies have indicated, indirectly, that the HKDSE examination of mathematics has influenced teacher's curriculum decision making and lesson preparation, student's learning experience, and their perceptions of teaching and learning mathematics (e.g., Brown, Kennedy, Fok, Chan, & Yu, 2009; Choi, Lam, & Wong, 2012; Lee, 2019). In particular, according to these studies, teachers, in order to aid their students to excel at examinations, tend to design mathematical tasks which are aligned with those tasks presented in the public examinations. We believe, therefore, that a direct investigation of RP tasks in public examinations may shed light on how RP is being treated in school mathematics in Hong Kong, one of those high-achieving regions in school mathematics (Mullis, Martin, Foy, & Hooper, 2016; OECD, 2016).

ANALYTIC FRAMEWORK AND METHODS

Reasoning-and-Proving			
Making mathematical generalisations		Providing support to mathematical claims	
(a) Identifying a pattern	(b) Making a conjecture	(c) Providing a proof	(d) Providing a non-proof argument
Plausible pattern Definite pattern	Conjecture	Generic example Demonstration	Empirical argument Rationale

Table 1: Stylianides's Analytic Framework (Stylianides, 2009)

In this study, we adapted Stylianides's (2009) Analytic Framework to examine the mathematical tasks in HKDSE examinations of mathematics. Based on a particular conceptualization of reasoning-and-proving (RP) for doing authentic mathematics, this framework was developed and applied first in investigating the RP opportunities in junior secondary mathematics textbooks in the US. This framework defines four proof-related mathematical activities: (a) identifying patterns, (b) making conjectures, (c) providing proofs, and (d) providing non-proof arguments, and each is further subdivided as in Table 1 (for exact definitions of these terms, see Stylianides, 2009). In particular, there are two types of proof: generic example and demonstration. Wong (forthcoming) adapts this framework to investigate RP opportunities in senior secondary mathematics textbooks in Hong Kong and extends the classification of demonstration into (i) visual proof, (ii) paragraph proof, and (iii) two-column proof. The paragraph proof is further subdivided into (1) proof by definition, (2) proof by calculation, (3) proof by calculation and definition, (4) general direct paragraph proof, (5) proof by contradiction, and (6) disproof by counterexample (see Table 3). This

extended classification of demonstration, together with the marking schemes in the examination reports, allow us to investigate the different types of demonstration required in the tasks and the intended mathematical abilities that the task designers expect candidates to acquire at the senior secondary level.

There were several reasons for our adapting Stylianides's (2009) Analytic Framework with Wong's extension (Wong, 2017, forthcoming; Wong & Sutherland, 2018): (a) it is designed for studies in the domain of RP, (b) it has been adopted and/or adapted by different studies (e.g., Davis et al., 2014; Fujita & Jones, 2014), and (c) Wong's extension of this framework can enable us to compare his results of RP opportunities in textbooks with those in examination papers in Hong Kong.

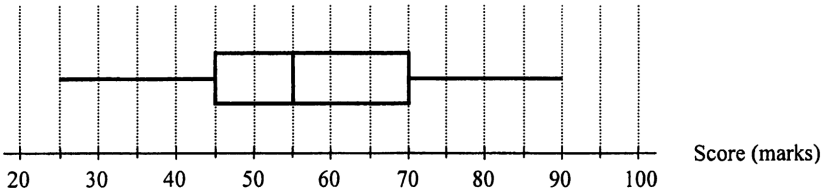
In this study, we focused on the Compulsory Part of the curriculum and examined all the examination papers and reports so far (HKEAA, 2012–2018). All mathematical tasks were examined – a task here refers to "any exercise, problem, activity, or parts thereof that have a separate marker" (Stylianides, 2009, p. 270). In total of 293 mathematical tasks in the examination papers, 103 of them were identified as RP-related tasks. These tasks were then analyzed and categorized into subcategories of RP-related activities, and those categorized as demonstration were further subcategorized into the eight classes of demonstration set out in Table 3.

PROCEDURES AND EXAMPLES OF ANALYSIS

The mathematical tasks in the examination papers were analyzed in three steps. First, we determined whether the tasks were RP-related. Tasks phrased with "prove that" or "explain your answer" were coded as RP tasks because these tasks explicitly ask the candidate to provide a support to her answer (see Example 1). Those tasks phrased with "find the equation of …", "express … (a variable) in terms of … (another variable)" and "describe the geometric relationship…" were also coded as RP tasks (see Example 3) because these tasks ask the candidate to make a mathematical generalization of a pattern that "describe[s] covariation between structures, properties, or variables" (Stylianides, 2009, p. 263). Then, the RP tasks were coded as "identifying a pattern", "making a conjecture", "providing a proof" or "providing a non-proof argument". Note that some tasks were dually coded as both "identifying a pattern" and "providing a proof" because (a) they explicitly state "describe the geometric relationship… explain your answer" (see Example 3), or (b) both the stated marks of the task and the marking scheme hint the candidate to not only identify a pattern but also provide a justification. Finally, tasks coded as "demonstration" were subcategorized according to Wong's extended classification of demonstration (for details, see Wong, 2017; Wong & Sutherland, 2018) (see Examples 1 and 2). Note again that according to the marking scheme, some demonstrations were categorized into multiple classes due to multiple solutions in the marking scheme (see Examples 2 and 3). Following Thompson et al. (2012), we adopted the consensus model for coding: each author coded all tasks independently and differences were then resolved by discussion among themselves.

Example 1

15. The box-and-whisker diagram below shows the distribution of the scores (in marks) of the students of a class in a test. Susan gets the highest score while Tom gets 65 marks in the test. The standard scores of Susan and Tom in the test are 3 and 0.5 respectively.



(a) Find the mean of the distribution. (2 marks)

(b) Susan claims that the standard scores of at least half of the students in the test are negative. Do you agree? Explain your answer. (2 marks)

Solution

(b) Note that if the test score of a student is lower than the mean, then the standard score of the student is negative. 1M
Also note that the median is 55 marks and the mean is 60 marks.
So, the median is less than the mean. either one
Therefore, the test scores of at least half of the students are less than the mean.
Thus, the claim is agreed. 1A f.t.
----------(2)

Figure 1: Task 15(b) of HKDSE examination 2013

In Figure 1, Task 15(a) is not coded as any RP-related activity. Task 15(b) is coded as "demonstration–general direct paragraph proof" as it does not emphasize calculation or manipulation (and interpretation) of algebraic expressions (i.e., proof by calculation, or proof by calculation and definition), nor involves one-step deductive reasoning derived directly by some definition, property or theorem (i.e., proof by definition). This task requires, instead, the candidate to have conceptual understanding of the definitions of mean, median and standard score in order to formulate an argument as given in the marking scheme.

Example 2

18. The equation of the parabola Γ is $y = 2x^2 - 2kx + 2x - 3k + 8$, where k is a real constant. Denote the straight line $y = 19$ by L.

(b) The points of intersection of L and Γ are A and B.

(i) Let a and b be the x-coordinates of A and B respectively. Prove that $(a-b)^2 = k^2 + 4k + 23$.

(ii) Is it possible that the distance between A and B is less than 4? Explain your answer. (5 marks)

Solution

(ii) $AB<4$ $\sqrt{(a-b)^2}<4$ $(a-b)^2<16$ $k^2+4k+23<16$ (by (b)(i)) $k^2+4k+7<0$ Note that $4^2-4(1)(7)=-12<0$. Also note that the coefficient of k^2 is positive. Therefore, $k^2+4k+7<0$ has no solutions. Thus, it is not possible that the distance between A and B is less than 4 .	1M 1A	for using (b)(i) f.t.
AB $=\sqrt{(a-b)^2}$ $=\sqrt{k^2+4k+23}$ (by (b)(i)) $=\sqrt{(k+2)^2+19}$ $\geq\sqrt{19}$ $>\sqrt{16}$ $=4$ Thus, it is not possible that the distance between A and B is less than 4 .	1M 1A	for using (b)(i) f.t.

Figure 2: Task 18(b)(ii) of HKDSE examination 2017

As shown in Figure 2, the marking scheme gives two different solutions to Task 18(b)(ii): The first one is an indirect "proof by contradiction" and the second one is a direct "proof by calculation". So, this task is dually coded.

Example 3

18. It is given that $f(x)$ partly varies as x^2 and partly varies as x . Suppose that $f(2)=60$ and $f(3)=99$.

(b) Let Q be the vertex of the graph of $y=f(x)$ and R be the vertex of the graph of $y=27-f(x)$.

(iii) The coordinates of the point S are $(56,0)$. Let P be the circumcentre of ΔQRS

Describe the geometric relationship between P , Q and R . Explain your answer.

Solution

(iii) The slope of QS $=\frac{-48-0}{-4-56}$ $=\frac{4}{5}$ The slope of RS $=\frac{75-0}{-4-56}$ $=\frac{-5}{4}$		
Hence, the product of the slope of QS and the slope of RS is -1 . So, $\angle QSR$ is a right angle. Therefore, QR is a diameter of the circumcircle of ΔQRS . Note that P is the circumcentre of ΔQRS . Thus, P is the mid-point of the line segment joining Q and R .	1M 1A	 f.t.
QS^2+RS^2 $=((-4-56)^2+(-48-0)^2)+((-4-56)^2+(75-0)^2)$ $=15\,129$		

QR^2 $=(-48-75)^2$ $=15\,129$		
Hence, we have $QS^2+RS^2=QR^2$. So, $\angle QSR$ is a right angle. Therefore, QR is a diameter of the circumcircle of ΔQRS . Note that P is the circumcentre of ΔQRS . Thus, P is the mid-point of the line segment joining Q and R .	1M 1A	 f.t.

Figure 3: Task 18(b)(iii) of HKDSE examination of 2018

In Figure 3, Task 18(b)(iii) is dually coded as "identifying a definite pattern" and "demonstration". The marking scheme gives two different solutions, both of which focus on (a) calculation (to show either the product of two slopes is -1 or $QS^2 + RS^2 = QR^2$) and (b) interpretation of the calculation results (∠QSR is a right angle and hence QR is a diameter of a circle) to draw a conclusion. So, this task is further categorized into "demonstration–proof by calculation and definition".

RESULTS AND DISCUSSION

	Frequency*	Percentage*
Identifying a plausible pattern	2	1.9%
Identifying a definite pattern	39	37.9%
Making a conjecture	0	0.0%
Providing a generic example	0	0.0%
Providing a demonstration	92	89.3%
Providing an empirical argument	0	0.0%
Providing a rationale	0	0.0%

*Due to dual coding, the sum exceeds 103 (or 100%).

Table 2: Frequency and distribution of RP subcategories

As shown in Table 2, there are relatively greater proportion of RP tasks in the examination papers (103 out of 293 tasks, or 35.2%) than RP opportunities in textbooks (Cf. Geometry: 15.2% (Wong, 2017) and Algebra: 13% (Wong & Sutherland, 2018)). Furthermore, a dominant number of these RP tasks require the candidate to provide a proof (demonstration: 92 out of 103, or 89.3%) but none of them accepts any non-proof argument as a support to the solution of the task. However, looked closely, Table 3 reveals also that among those demonstrations, proof by calculation (40.2%) and proof by calculation and definition (47.8%), both of which emphasize students' accurate calculation over mathematical reasoning, are the majority (Cf. Palm, Boesen, & Lithner, 2011; Wong, 2017; Wong & Sutherland, 2018). These results suggest that the pubic examination of mathematics in Hong Kong is calculation-focused and this in turn indicates that the task designers intend to assess the candidate's accurate and precise calculation and procedures of problem solving but not RP (general direct paragraph proof: 3.3%; proof by contradiction: 5.4%; two-column proof: 7.6%). This is in contrast to the curriculum aims. These results provide additional support to the claim

that proof plays a marginal role in secondary mathematics in Hong Kong, echoing the conclusion of Wong (2017) and Wong and Sutherland (2018).

	Frequency*	Percentage*
Visual proof	0	0.0%
Two-column proof	7	7.6%
Proof by definition	0	0.0%
Proof by calculation	37	40.2%
Proof by calculation and definition	44	47.8%
General direct paragraph proof	3	3.3%
Proof by contradiction	5	5.4%
Disproof by counterexample	0	0.0%

*Due to *dual* coding, the sum exceeds 92 (or 100%).

Table 3: Frequency and distribution of the classification of demonstration

A number of reflections on these results are in order. Hong Kong teachers' perceptions of the characteristics of the mathematical tasks presented in the examination papers very probably influence their perceptions of teaching (in particular, RP) and classroom practices (Lee, 2019). For example, teachers tend to adopt the mathematical tasks in the examination papers in their in-class activities for examination preparation (Brown et al., 2009; Choi et al., 2012). This explains why student's learning opportunities are mainly calculation-focused. However, these calculation-focused tasks require students only to recall and apply procedural knowledge (i.e., *imitative* reasoning) rather than *creative* mathematical reasoning (see Palm et al., 2011). As a bad consequence, such an overwhelming drilling of this practice would inevitably hinder students from developing logical reasoning, conceptual knowledge and, crucially, understanding of mathematics.

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TWO TYPES OF FRACTION KNOWLEDGE? – AN EMPIRICAL STUDY FOCUSING ON THE SEPARABILITY OF CONCEPTUAL AND PROCEDURAL KNOWLEDGE

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Many explanatory approaches for students' difficulties in dealing with fractions are based on the distinction between conceptual and procedural knowledge. In order to empirically confirm such assumptions and for finding corresponding solutions, valid assessment of conceptual and procedural knowledge is essential. However, previous studies show high correlations of these constructs. Hence, it is not clear, yet, if the theoretically assumed 2-dimensional structure of fraction knowledge is appropriate. Based on theoretically grounded conceptualizations and operationalizations this study thus investigates data from N = 235 students regarding the empirical separability of the two knowledge types. Confirmatory factor analyses show that the theoretically assumed model fits the data indeed better than other possible models.

INTRODUCTION

In spite of extensive efforts in research and practice of mathematics education, fractions continue to challenge mathematics teachers and students throughout the world (Lortie-Forgues, Tian, & Siegler, 2015). There is consensus among researchers that the distinction between conceptual and procedural knowledge is important in understanding and remedying these problems (e.g., Bempeni, Poulopoulou, Tsiplaki, & Vamvaloussi, 2018; Hiebert & LeFevre, 1986; Pantziara & Philippou, 2012). Studies focusing on textbook analyses indicate for instance that learning opportunities concerning fractions often focus on procedural knowledge (Alajmi, 2012; Son & Senk, 2010), which might be a reason for students often lacking conceptual knowledge of fractions. To empirically validate such assumptions and for finding corresponding solutions, theory based conceptualizations as well as operationalizations of conceptual and procedural knowledge are needed. In particular, identifying individual differences in students' knowledge structures and evaluating the effectiveness of different instructional methods with respect to conceptual and procedural knowledge requires a test instrument that provides valid measurement for this purpose. Therefore, it is essential to investigate whether conceptual and procedural fraction knowledge can be measured independently of each other and with a sufficient degree of validity (e.g., Crooks & Alibali, 2014; Rittle-Johnson & Schneider, 2015).

THEORETICAL BACKGROUND

The term *procedural knowledge* is used to describe knowledge of a sequence of steps or partial actions which are performed to achieve a specific goal (Hiebert & Lefevre, 1986; Rittle-Johnson & Schneider, 2015). In particular, procedural knowledge

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regarding fractions involves knowledge of how fraction arithmetic procedures such as addition of fractions work. Thus, procedural knowledge is used to solve computation tasks that are presented symbolically. *Conceptual knowledge* is commonly defined as knowledge of concepts and relations that are fundamental in a certain domain (Crooks & Alibali, 2014; Hiebert & LeFevre, 1986; Rittle-Johnson & Schneider, 2015). In particular, conceptual knowledge of fractions involves knowing different aspects of fractions, for example the part-whole aspect and the measurement aspect (e.g., Charalambous & Pitta-Pantazi, 2007).

Positive correlations between conceptual and procedural fraction knowledge were found in different studies (e.g., Hallett, Nunes, & Bryant, 2010; Jordan, et al. 2013). For example, Hallett and colleagues (2010) examined individual differences in the conceptual and procedural knowledge of 4th and 5th graders in the United Kingdom (N = 318), who had not received full instruction on fractions, yet. The authors formed a conceptual and a procedural scale by assigning each item to one of these scales depending on their judgement of whether the item's solution relies more on conceptual or on procedural knowledge. The resulting scales correlated highly on a manifest level ($r = .675$, $p < .001$). A similar correlation between the two knowledge types on a manifest level was found by Jordan and colleagues (2013): $r = .622$, $p < .001$. The authors examined general predictors (attentive behavior, language, nonverbal reasoning) and number-related predictors (number line estimation, calculation fluency) of 6th grade students´ conceptual and procedural fraction knowledge in the United States (N = 357). In this study the separability of conceptual and procedural knowledge was as well merely assumed, but not validated empirically in any explicit way. The results showed, however, that predictors contributed diversely to conceptual and procedural knowledge, which can be interpreted as an indicator of separability.

In addition to these similar results concerning the relationship between conceptual and procedural knowledge, there are also completely contradictory findings: For example, Lin and colleagues (2013) found no significant correlation between procedural and conceptual fraction knowledge of prospective teachers in Taiwan (N = 47) and in the United States (N = 49). Taking a closer look at the operationalizations used in this study suggests that using on the one hand purely nonverbal calculation tasks for procedural knowledge and on the other hand only verbal explanations for conceptual knowledge may lead to an artificially created separation of the constructs. More generally, findings of the Multitrait-Multimethod Analysis show that variance in test results does not only account for differences in the constructs to be measured, but can also be caused by their specific operationalizations.

This indicates that the results of previous research regarding the relationship between conceptual and procedural fraction knowledge must be analyzed in view of the conceptualizations and operationalizations of the two constructs in these studies. Accordingly, a closer look at the studies mentioned above shows that the conceptualizations of conceptual and procedural knowledge differ in their level of detail. Furthermore, the comparison of the different operationalizations shows that in

each case different approaches were used to assess the two knowledge types and thus systematic and comprehensive operationalization is missing so far (e.g., Crooks & Alibali, 2014). Therefore, it can be concluded that more research based on explicit domain specific considerations and a comprehensive operationalization is needed. This is a necessary first step for investigating how conceptual and procedural knowledge can be measured independently of each other with a sufficient degree of validity. Accordingly, Rittle-Johnson and Schneider (2015) criticized the lack of attention that is paid to validity concerning the measurement of conceptual and procedural knowledge.

RESEARCH INTEREST

The aim of this study is therefore to develop and validate a test instrument with respect to its objective to detect differences in students' conceptual and procedural fraction knowledge. This includes in particular evidence for the empirical separability of the two knowledge types.

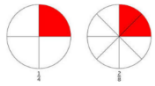
In this report we examine in particular the following research question:

Does the 2-dimensional model distinguishing between conceptual and procedural fraction knowledge show a better model fit than the 1-dimensional model and than a 2-dimensional model distinguishing between a verbal and a nonverbal factor?

TEST DESIGN, SAMPLE AND METHODS

Our conceptualizations of conceptual and procedural knowledge build on previous approaches since commonly occurring core elements of conceptual and procedural knowledge were included. In order to develop a systematic conceptualization, however, we also took into account different cognitive processes regarding both knowledge types, following the two-dimensional taxonomy for learning, teaching, and assessing by Anderson and Krathwohl (2001). The cognitive processes were adapted according to a mathematics specific taxonomy developed by Barzel, Leuders, Prediger, and Hußmann (2013), which was designed in order to systematically create tasks that support learners in knowledge organization.

Hence, the resulting taxonomy structures both knowledge types by different cognitive processes which are decisive for the acquisition of mathematical knowledge: *verbalization*, *application*, and *visual representation*. Table 1 shows an excerpt of this taxonomy regarding the concept 'equivalence of fractions' which is important for instance for the addition of fractions with uncommon denominators. In terms of procedural knowledge it refers to expanding and simplifying fractions.

	verbalization	application	visual representation
<u>conceptual knowledge</u> 'equivalence'	"Fractions that describe the same proportion are equivalent to each other."	know examples/ counterexamples $\frac{1}{2}=\frac{2}{4}$ $\frac{1}{2}\neq\frac{2}{8}$	roughening or refinement of models 

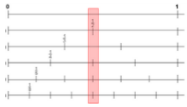
			recognize that equivalent fractions have the same position on the number line 
procedural knowledge ‘Expand fractions/ Simplify fractions’	“Multiply/divide numerator and denominator of a fraction by the same number.”	apply the procedure $\frac{1 \cdot 2}{4 \cdot 2} = \frac{2}{8}$ $\frac{2:2}{8:2} = \frac{1}{4}$	link between procedural and conceptual knowledge:

Table 1: Excerpt of the taxonomy to describe knowledge elements

Conceptual knowledge is explicitly formulated in definitions and theorems, while procedural knowledge is explicitly formulated in instructions. The application specifies the explicitly formulated knowledge. In terms of conceptual knowledge this means that examples and counterexamples are known. In terms of procedural knowledge, partial steps are internalized by the execution of the procedure and supplemented by knowledge about conditions of applicability. Finally, visual representation of conceptual knowledge is central for generating mental models. Visual representation of procedures represents a connection between conceptual and procedural knowledge. Therefore, to assess conceptual and procedural knowledge separately, knowledge elements from this part of the taxonomy are not operationalized.

The conceptualizations of the knowledge types according to the taxonomy were implemented in test items. In order to take into account existing research, operationalizations of conceptual and procedural knowledge from previous studies were reviewed and systematized according to the taxonomy with a focus on tasks that are suitable for paper-pencil testing. This analysis revealed three most commonly used approaches to assess conceptual fraction knowledge: 1) visualizing the part-whole aspect in translation tasks 2) applying the measurement aspect in size comparison tasks or number line tasks 3) verbalizing general principles of fractions in explanation tasks (for an overview of task types used to assess conceptual knowledge see Rittle-Johnson & Schneider, 2015). All three approaches were taken into account for a comprehensive operationalization.

Commonly, procedural fraction knowledge is captured by calculation tasks. In line with de Jong and Ferguson-Hessler (1996), we assume, however, that procedural knowledge also exists in a declarative manner. Consequently, procedural knowledge encompasses not only the application of procedures, but also the verbalization of (action-)sequences.

Following the approach by Hallet and colleagues (2010), all test items were assigned to either the conceptual or the procedural scale. In line with Faulkenberry (2013), the classification was based on the assumed knowledge type that participants use when solving the tasks. To confirm this classification of the items the scales were discussed with experts in mathematics education (N = 14). Controversially discussed items were subsequently revised or discarded. The final paper-pencil test instrument assessing conceptual and procedural fraction knowledge consisted of 71 items.

The data was collected by means of a paper-pencil test in a whole-class setting during two consecutive periods of 45-minute lessons. Participants were randomly assigned the test versions A - D. Test versions A - D contained the same items in different order. First, participants were told to complete the fraction test individually at their own pace. Students' responses were scored dichotomously (0 = incorrect or fragmentary; 1 = correct). About 20% of the data were coded by a second person. The degree of agreement was determined using Cohen's kappa for each item, which represents the degree of concordance in coding, while controlling for chance agreement. Adequate intercoder reliability was obtained with $k > .741$ for each item.

A total of N = 235 German students in grade 8 and 9 participated in the study. These students were from 10 different classes of different "Realschulen" (a type of school for students with an average level of academic performance). In Germany fractions are usually taught in 6th grade. Therefore, we assume that 8th and 9th graders had enough time to practice and consolidate their earlier instruction in the domain of fractions.

We used confirmatory factor analyses to investigate whether the data supported the theoretically assumed structure of fraction knowledge distinguishing between conceptual and procedural knowledge. Model comparisons took into account the 1-dimensional model and in addition a 2-dimensional model based on a language factor (verbal – nonverbal). Considering this second 2-dimensional model responds to the findings of Multitrait-Multimethod Analysis mentioned above and showing that variance in test results does not only account for differences in the constructs to be measured but can also be caused by their specific operationalizations.

RESULTS

Confirmatory factor analysis

According to the global fit measures the 2-dimensional model based on the knowledge types showed good agreement with the empirical data (x^2 = 20.568; *df* = 13; $P(x^2)$ = .082; *TLI* = 0.975; *CFI* = 0.984; *RMSEA* = 0.050; *SRMR* = 0.030) and was superior compared to the other models regarding the global fit indices. The Bayesian Information Criterion (BIC) which allows for a direct comparison of the models showed the lowest BIC value for the 2-dimensional model distinguishing between conceptual and procedural knowledge. Differences in BIC values were higher than 10, indicating a strong effect (Raftery, 1995). Therefore, the comparison of the fit statistics suggested that the 2-dimensional model based on knowledge types was the most likely model underlying students' performances. Since conceptual and procedural knowledge

were highly correlated *(r = .79,* $p < .001$*)*, a chi-square difference test was conducted to get further evidence for discriminant validity. The null hypothesis assumes that there is no significant difference between the 2-dimensional model distinguishing between knowledge types and the 1-dimensional model. With $\Delta\chi^2$ (1, N = 235) = 29,162, $p < .01$ the difference was significantly greater than zero. This means that there is a significant difference in the goodness of fit between the 2-dimensional model based on knowledge types and the 1-dimensional model.

DISCUSSION AND DIRECTIONS FOR FURTHER RESEARCH

In this study, we developed a test instrument that enables to detect differences in students' conceptual and procedural fraction knowledge. This is necessary for in-depth research into reasons and remedies for students' persistent problems in dealing with fractions, since it is assumed that the distinction between these two knowledge types is key in this context (e.g., Hiebert & LeFevre, 1986). More generally, this responds to the need for research regarding the validity of measures of conceptual and procedural knowledge, in particular with respect to the question as to how conceptual and procedural knowledge can be measured independently of each other and with a sufficient degree of validity (Rittle-Johnson & Schneider, 2015). As Rittle-Johnson and Schneider (2015) pointed out, answering this question is necessary before more progress can be made in understanding the relations between conceptual and procedural knowledge. In line with results of previous research, the latent variable analyses showed high correlations between conceptual and procedural knowledge (e.g., Hallett et al., 2010). However, the findings of this study provide multiple pieces of evidence in terms of discriminant validity. Before data collection, the allocation of items to the scales for conceptual and procedural fraction knowledge was confirmed by experts of the field (researchers in mathematics education). The question of whether the data has the theoretically assumed 2-dimensional structure was investigated by means of comparison of different models. The results confirmed that this 2-dimensional model was indeed superior to a 2-dimensional model distinguishing between a verbal and a nonverbal factor as well as to the 1-dimenstional model. Beyond these results presented in this report, a further piece of evidence in terms of discriminant validity refers to the analysis of the correlations between the two knowledge types and general cognitive abilities. Our findings hence indicate that conceptual and procedural fraction knowledge can indeed be empirically separated. Nevertheless, it should be noted that strictly speaking the validity of the model can only be interpreted with respect to the sample that was investigated and thus future studies should explore whether it holds true for other samples as well. It should however be noted that compared to the typical sample of 5th or 6th graders who have just received fraction instruction, considering a sample of 8th and 9th graders may be in fact more meaningful regarding the question of empirical separability, since several scholars assume that the differences in knowledge types decrease over time or with increasing expertise (e.g., Hallett, 2012; Baroody et al., 2007).

Future research can also benefit from this study as it provides essential information on how conceptual and procedural fraction knowledge can be conceptualized and operationalized systematically by taking into account different cognitive processes. Moreover, the application of the conceptual framework to other (mathematical) domains is promising for further research.

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ENHANCING MATHS CURRICULUM THROUGH TEAM-BASED LEARNING

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Team-Based Learning (TBL) is an instructional method developed by Michaelsen et al. (1982), where students participate in pre-class preparatory work and then apply the knowledge learned during class. TBL has been shown to produce a variety of learning gains over traditional lecture-based pedagogies. We conducted a pre-post analysis of student performance using TBL in mathematics class to examine students' engagement and success using TBL. Students' feedback about the TBL lessons was also gathered through focus group discussions. This paper reports the preliminary findings of the comparison of a TBL group with a non-TBL group of Grade 7 students. Based on our findings, we hope to further develop resources to enhance a TBL curriculum for secondary-level mathematics for low achieving students.

INTRODUCTION

Team-Based Learning (TBL) is an instructional approach designed to combine the principles of Problem-Based Learning, Participant-Centered Instruction, and Constructivism. Popularized by Larry Michaelsen in the late 1970s, TBL first gained prominence in medical education as a framework to develop intern and resident doctors (McMahon, 2010). TBL has since been adopted primarily in tertiary education throughout health sciences and business curricula, and more recently, in teacher training (Samad et al., 2015). This framework provides participants with opportunities to discover inconsistencies between their current understanding and new information in order to build new knowledge (Samad et al., 2015; Hrynchak & Betty, 2012). One of the values of TBL is that it can be used as a complete course framework strategy but is versatile enough to be effective when delivered as part of a hybrid design (Michaelsen & Sweet, 2008).

TBL requires students to participate in pre-class preparatory work and then apply the knowledge they learnt to in-class lessons. Upon arrival to class, participants complete the Individual Readiness Assurance Test (iRAT). The iRAT is a multiple-choice test assessing knowledge gained from the learning resources provided by the instructor. Following the iRAT, participants proceed with the Team Readiness Assurance Test (tRAT), which takes places as soon as the time limit is up for the iRAT. The tRAT is comprised of the same multiple-choice questions but participants complete the tRAT with teammates that have been pre-assigned by the instructor. The iRAT and tRAT are designed to assess participant readiness before advancing to the higher-level problem solving required in the application stage. In addition, the tRAT is designed to

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provide immediate feedback to the teams to confirm whether their understanding of the concepts are correct, based on their agreed upon answers, or if they need to recalibrate their understanding. After participants have completed the iRAT and tRAT, the teacher offers a mini-lecture elaborating on the content and answering questions students may have. The participants in the course then undertake the Application Exercise (AE), which are designed to allow participants to think critically about complex concepts. This stage involves intra-team discussion and larger class discussions, with the emphasis on the application of knowledge as opposed to simple rote learning

In Singapore, many schools and faculty have relied on traditional pedagogies that are teacher-led and lecture-based. While lecture-based teaching has been widely criticized due to its lack of participant engagement (Mennenga, 2013; Di Leonardi, 2007), TBL has demonstrated in a number of studies to be a transformative instructional approach with team processes at its core (Sisk, 2011). Advantages of TBL include better academic performance, greater participant-to-instructor engagement, and greater participant satisfaction (Frame et al., 2015; Zgheib et al. 2016; Thomas & Bowen 2011; Vasan et al., 2011; Sisk, 2011). The key to TBL's effectiveness is cultivating students as active learners based on four essential elements: 1) teams that are strategically formed, 2) students being accountable to themselves and their team, 3) immediate feedback from instructors to enable correction of misconceptions and promotion of content retention, and 4) application exercises that require student interaction and decision-making with real-life problems. Since many students struggle with mathematics, it is important to find innovative pedagogies that improve learning for all students.

METHODOLOGY

Participants

A total of 79 Grade 7 students (56 Males and 23 Females) from School A in Singapore participated in this study. School A is a specialised school for students in the Normal Technical stream. Students of School A are predominantly weak in mathematics and have a very low intrinsic interest in solving maths problems. For this preliminary study, two other two mainstream schools (School B and School C) were also recruited to serve as a baseline for comparison with School A. Refer to Table 1 for a breakdown of students in each group. The schools were contacted via email correspondence, and permission was obtained from the respective principals before proceeding with the research. Informed consent was obtained from students and teachers involved in the study. Ethics approval was obtained from the Institutional Review Board of Nanyang Technological University (IRB-2018-07-008).

School	Number of students in TBL group	Number of students in Non-TBL group
A	39 (2 classes)	40 (2 classes)
B	36 (1 class)	.
C	.	26 (1 class)
Total	75	66

Table 1: Number of students in each group separated by school.

Design

The study uses a qualitative and quantitative approach to examine the effectiveness and feasibility of incorporating TBL into mathematics lessons for secondary school students. The study design consisted of a pre-post analysis of test results to obtain insight about students' performance, as well as a focus group discussion to gather feedback about the TBL lessons.

Quantitative Research Questions

1. Does TBL improve student motivation to learn math?
2. Does TBL improve student achievement in math?

Qualitative Research Questions

3. What components of TBL have the greatest impact on the motivation of students to learn math?
4. What impact do the Application Exercises, especially breakout activities and escape room games have on the attitudes of students to learn math?

Students were divided into two groups, a TBL group and non-TBL group. Refer to Table 1 for a breakdown of the groups by schools. For the pilot study, students from School B went through TBL lessons while students from School C went through non-TBL lessons. Those in the TBL group attended lessons that were delivered using the TBL framework, while those in the non-TBL group attended lessons that were as per the teacher's usual teaching style. Student performance between the two groups were compared to find out whether the overall results of students in the TBL group were significantly better than the students in the non-TBL group. In addition, a pre- and post-test was conducted to find out whether student performance in the TBL group improved more than student performance in the non-TBL group.

The qualitative component of the study included focus group discussions to gather feedback from students and teachers about the TBL lessons. This allowed for greater understanding of what students enjoyed about the TBL lessons and provided insight into how future TBL curriculum could be structured in order to maximise the benefit for teachers and students. 18 students from School A were invited to participate in the focus group discussion. These students were all from the TBL group and shared their

experiences with the TBL lessons. Audio recordings were taken and then transcribed. Common key words were identified, coded into categories, and grouped together to form themes (Kolb, 2012).

Materials and Procedure

Prior to the research lessons, the teachers were given lesson plans as a guide to conduct the lesson. Lessons were observed by two research staff, and a procedure checklist was recorded and compared to account for inter-rater reliability. All students completed the pre-test before starting the research lessons. Each class spent approximately eight hours on the topic of simplification of algebraic expressions. Students completed their pre-lesson learning through Deck.Toys, an interactive online platform for engaged learning. After which, the students attempted the iRAT, followed by the tRAT. The iRAT and tRAT are administered digitally through InteDashboard™, because it saves time by allowing for immediate feedback for the teachers to track the progress of the students' scores. After students completed the tRAT, the teachers addressed any questions and consolidated the lesson. Worksheets and homework were given as application exercises for the students, and the teacher went through textbook and workbook examples to explain mathematical concepts. Upon finishing the research lessons, students completed a post-test.

RESULTS

Pre- and Post-Test

The mean scores for the pre-test and post-test are shown in Table 2 and 3 respectively.

School	Group	*N*	Mean	*SD*
A	TBL	39	2.00	2.86
	Non-TBL	40	0.90	1.53

Table 2: Pre-test mean score and standard deviation.

School	Group	*N*	Mean	*SD*
A	TBL	39	6.28	2.70
	Non-TBL	40	4.63	2.64

Table 3: Post-test mean score and standard deviation.

Assumptions of parametric tests were not met because assumptions of normality were violated ($p < .05$) as assessed by Shapiro-Wilk's test (Shapiro & Wilk, 1965). A Mann-Whitney U test (Mann & Whitney, 1947) revealed that there was no statistically significant difference in pre-test scores ($U = 602$, $p > .05$) between the TBL and non-TBL groups, while it revealed otherwise for the post-test scores ($U = 511$, $p < .01$). While both groups scored similarly for the pre-test, the TBL group ($M = 6.28$) scored significantly higher than the non-TBL group ($M = 4.63$) for the post-test.

The difference score was tabulated by subtracting the pre-test score from the post-test score to find out the improvement within-group. Assumptions of normality were met but not for homogeneity of variance ($p > .05$). An independent-sample t-test that is corrected for unequal variances showed that there was no significant difference in the difference score between the TBL and non-TBL group (t (76) = .934, $p > .05$). The non-significance meant that both groups improved similarly regardless of how the lessons were delivered.

In order to examine whether School A's TBL and non-TBL groups performed similarly to their other school counterparts, the post-test results of all schools were compared. The assumptions of parametric tests for the post-score were met. A one-way ANOVA showed that there was a statistically significant difference between all four groups for the post-test score, (f (3,140) = 3.40, $p < .05$). Tukey's post hoc test revealed that School A's TBL group performed similarly to School B's TBL group ($p > .05$). Likewise, School A's non-TBL group performed similarly to School C's non-TBL group ($p > .05$). The results suggested that the post-test scores were consistent across the pilot study and current study; TBL groups performed better than the non-TBL groups post research lessons.

Focus Group Discussion

One common theme of the discussion was that students found TBL lessons to be helpful in terms of learning and teamwork. When asked whether TBL has helped the students to enjoy the learning experience, 15 out of 18 students agreed. The following are selected student comments about their learning experience with TBL.

Student 1: When we're asked a question on the spot… some of our brains can't really think that fast… maybe this programme can help us to help one another and then maybe can try and think faster by ourselves… it's something like that for me.

Student 2: So for Deck.Toys we can actually compete with each other. And we can also earn points.

This implied that collaborative learning was prominent in TBL and by brain-storming for the answers together; it helped the students to think more effectively as individuals. The students also enjoyed the self-learning component of TBL because of its interactive nature. When asked to name what teamwork skills the students have gained, some of the student's response were as follows.

Student 3: Teach[ing] somebody.

Student 4: When something is wrong, there is discussion about the question. Then the person that knows will teach us. Then we will know that it's wrong.

The students' feedback shows how TBL helped the students learn from one another through peer-to-peer teaching, building up their collaborative learning and teamwork skills.

Another common theme was leadership. Students enjoyed the opportunity to be group leader and facilitate the team discussion with TBL. When asked which component of TBL did the students learn the most from, one student responded the team test, and his elaboration is shown below.

> Student 5: Because I'm the leader… each question we take turn and we get to explain why it's the answer.

A good leader is needed to manage the team discussion and ensure that everyone has an opportunity to voice out their opinion. TBL provides a platform for leadership. Being the group leader is an important role, and the students enjoyed taking up the responsibility.

Lastly, positive attitude was a common theme brought up. Having observed some of the research lessons, the following is the interviewer's comments about the students from School A.

> Interviewer: I'm very impressed by [School A] students, is that your willingness to help one another to learn… I always hear you say 'never mind', 'try', and 'try on your own first'… I hope that you will continue with this wonderful attitude.

The comment suggested that TBL fosters a positive learning environment that can motivate students to perform better and gain more self-confidence.

DISCUSSION

This preliminary study was conducted as part of a larger study at the National Institute of Education to examine how TBL can enhance the mathematics curriculum in secondary schools in Singapore. The preliminary findings suggested that TBL is effective in engaging low achieving students in the learning of mathematics. The group of students taught using the TBL approach performed significantly better than the group of students taught through a more traditional approach. One of the key success factors of TBL was that the collaborative nature of TBL encourages peer learning. The component of the pre-class preparatory work using lessons from the online platform Deck.Toys encouraged independent learning and allowed students to learn at their own pace as well as to be responsible for their own learning. In the case of normal technical students who are less motivated to learn mathematics, the pre-class preparatory work was conducted in the classroom with teachers' monitoring. The findings of this study suggest that the independent pre-learning component of the TBL approach is effective in supporting students to learn regardless of whether the pre-learning takes place outside of class or in class.

Limitations and Future Direction

There are certain limitations to take note of for future research. One point to note is that there may be teacher bias as different teachers have different teaching styles, so the learning experience may be different across classrooms depending on the teacher. In order to minimise this bias, we plan to conduct a workshop for educators, teaching them how to conduct and facilitate the research lessons. In addition, the sample size of 79 is too small and not representative of the entire secondary school population in Singapore. As such, we hope to minimise this limitation by recruiting more students from different levels in secondary school for the larger part of the study. Finally, due to time constraints, we were unable to carry out the full application activities that were intended for TBL. The only reinforcements that the students received were in the form of worksheets and homework, which may not be the most engaging for long-term learning. For the larger study, we plan to incorporate puzzles and games for the application activities so students can apply the knowledge that they have learnt by solving the challenges as a team.

There is promise for TBL to be translated to the education setting in secondary schools in Singapore. Traditionally, TBL has been implemented most effectively with highly motivated learners in tertiary education (McMahon, 2010). Through this study, we hope to extend the benefits of TBL to low achieving learners as well.

This study provides insight into how students feel about TBL lessons and what they learn from this unique pedagogy. Given the positive feedback from students that experienced the TBL lessons, it is encouraging to see how students enjoyed the lessons and learnt soft skills such as teamwork and leadership in addition to achieving higher scores in mathematics than the pilot groups. The students also encouraged one another to try harder, and this helped to foster a positive learning environment. While more work needs to be done before the full benefits of TBL can be reaped, it is promising to see how students who completed the TBL curriculum scored significantly higher than students who went through regular lessons for the post-test.

Conclusion

This study examined the feasibility and effectiveness of implementing TBL in mathematics class in secondary schools through experimental manipulation and focus group discussions. Pre-test scores were found to be similar, which meant that students' baseline knowledge levels on simplification of algebraic expressions were similar. The TBL group scored significantly better than the non-TBL group for the post-test, suggesting that the TBL group improved performance compared to the non-TBL group. Moreover, the TBL group consistently performed better than the non-TBL group for the post-tests when comparing School A's results with the baseline results established by the pilot study. However, when within-group differences were analysed, no significant results were found. Focus group discussions revealed that students enjoyed the TBL lessons, and that the TBL lessons were useful in reinforcing leadership, teamwork and positive learning attitudes among the students.

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A FRAMEWORK FOR DETERMINING THE EMBEDDED INSTRUCTIONAL APPROACH IN MATHEMATICS TEXTBOOKS

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In this article, I introduce a framework developed for determining the embedded instructional approach in mathematics textbooks, and demonstrate the application of the framework on two consecutive editions of the same textbook series. I show how the analysis of the sequencing of presentation formats using the framework illuminates major differences and similarities in the approach to teaching and learning, advanced in each textbook. The underlying implications for teacher textbook use are explored, especially in the context where the textbook remains a major resource for teaching.

INTRODUCTION

The research reported here is drawn from a larger study (Leshota, 2015) investigating the relationship between the affordances of the textbook and teachers' capacity to design instruction using the existing curricular resources, that is, teachers' pedagogical design capacity (Brown, 2009). All the participating teachers used the same prescribed textbook. However, as data collection commenced, a new edition of the textbook was released as South Africa was undergoing curriculum revision of the National Curriculum Statement (NCS), the new version being called the Curriculum and Assessment Policy Statement (CAPS). Teachers were provided with the copy of the new edition. This meant that at the start of data collection, each teacher had both textbook versions to use as they saw fit. For ease of reference, the old textbook edition shall be referred to as the Pre-CAPS textbook, and the new one, as the CAPS textbook.

The research focused on the topic of Functions at grade 10 level whose content did not change from the old curriculum to the new one. However, an initial inspection of the textbooks showed some differences in the layout of the content. A closer look also pointed to differences in how the properties of the functions were investigated. For example, in the Pre-CAPS, the activity involved all three function classes being drawn at the same time and then investigating which of these classes exhibited a certain property, such as symmetry. In the CAPS textbook, each function class was drawn and the different properties featuring in the function investigated.

From a socio-cultural perspective, textbooks are tools used by humans to achieve goal-oriented activity (Vygotsky, 1978). Furthermore, textbooks have a potential to enable or constrain human activity and as a result, they are cultural artifacts that mediate human activity (Wertsch, 1998). Textbooks mediate between authors and teachers, between teachers and learners, and between the mathematics itself and the learners. In the context of this study, textbooks are seen as a powerful tool for 're-sourcing' (Adler, 2000) teachers who have been perceived as insufficiently prepared for teaching

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mathematics, and therefore have a social as well as cultural influence on teaching mathematics.

It was thus important for the study to understand what opportunities for mediation of function, its properties, processes and applications, the different approaches opened up in the classroom, and prompted a more systematic approach to investigating the embedded instructional approach in textbooks. Hence, the search through the literature and then development of a framework that would enable this investigation.

STRUCTURE OF TEXTBOOKS – A LITERATURE REVIEW

The field of textbook research is a new emerging field (Trouche & Fan, 2018). In an earlier synthesis of research on textbooks and curricular resources, Remillard (2005) noted that that there was less work done on conceptualising curricular resources than on writings about teaching, despite the fact that these resources are critical to teachers' work of teaching. To date it appears there is one study that actually analysed the approach of the textbook (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002); pointing to a gap in literature. The framework introduced here contributes to this gap in the field, providing a tool for analysis of the embedded instructional approach in textbooks.

Research widely concurs that curricular resources "represent much more than static collections of tasks and lesson plans" (Remillard, 2005, p. 234). Otte (1986) conceptualises text and textbooks as "objectively given structures" and "subjective schemes". In other words, there is what can be seen in terms of the content that is supposed to be learned, the sequencing of topics, and exercises to be worked on by learners, when looking at a textbook. On the other hand, there is the individual interpretation and meaning given by each teacher when encountering the textbook. At the same time, teachers operate within a social system and so set of constraints. The analysis of the structure therefore provides building blocks for the current article.

Remillard (2012) identifies the *structure* of curriculum materials as one of five 'forms of address' by which materials interact with teachers. Typical analysis of the structure entails determining "how the various components are organized, the mathematics content included or excluded through the representations, and the valence or emphasis of the content, including how the content is represented" (p. 111). Such analyses communicate the embedded views of authors of these materials, a notion asserted by Brown (2009) in his definition of the structure. For Brown, the content and structure of curriculum resources are representational of the physical objects, tasks and concepts. Teachers bring these to life in the classroom, meaning that teachers should be able to perceive the pedagogical intentions communicated through the textbook.

Valverde et al. (2002), assert that the structural and pedagogical features of textbooks are characterised through the form and style of textbooks. According to the authors, the form and style are more important than the substance itself, as they provide the context for learning and as such "can either help or hinder the conveyance of the content themes or substance" (p. 17). The authors hypothesise that how the textbook

sequences its pedagogical situations and content marks its unique 'signature'. They suggest that analysing the sequencing of presentation formats (or building blocks) in each textbook, is an important aspect in the textbook analysis.

Pedagogical situations and content may also be sequenced according to performance expectations of learners. Performance expectations presuppose an approach to learning mathematics. In the case of Functions in this study, the sequencing of performance expectations would illuminate the conceptions of Function presupposed by the textbook: be either a *pointwise* and *correspondence* view of functions, or a *global* and *covariation* view (Ayalon, Watson, & Lerman, 2017; Even, 1998; Niss, 2014). To this effect, Ensor et al. (2002) note the existence of two distinct pedagogic approaches of textbooks (and teachers), the *deductive approach* and the *inductive approach*.

> The first approach is deductive. Here, the teacher (or textbook) initially states appropriate definitions or concepts, which are then exemplified and followed by exercises for students to practice. …The second pedagogic approach by contrast is largely inductive. Rather than starting with definitions, the teacher (or textbook) introduces a topic by engaging students in a range of activities that can be regarded as instances of the concepts, which students are to master. Activities lead to definitions and from this point opportunities may be provided for students to practice. (pp. 22-23)

They found that inserting a textbook that was inductive in approach in settings that were largely deductive, did not change teaching practices by the teachers. The authors noted that teachers who preferred the deductive approach that emphasised on procedures, did not use the textbook with a different approach, or if they did, would be quite selectively, and not coherent with the intentions of the textbook. They recommended the incorporation of textbook use in in-service teacher education as a means to facilitate changing one approach to another in textbooks and in classrooms.

Developing the Framework

The framework illustrated in figure 1 has been developed from the literature on the conceptualisation of curricular resources presented. Figure 1 depicts the *mathematics content* and embedded instructional *approach* as the main components of the structure of the textbook. The *content* illuminates the *nature of content* with respect to content areas to be covered and their ordering, in each chapter. The *approach* is defined in terms of its organisation and sequencing of the content. The organisation of content reflects two major components, the *presentation formats* (building blocks of content) and the *performance expectations* of learners. Sequencing the presentation formats indicates whether the approach of the textbook is largely inductive, referred to as *quasi-inductive* in the framework, or largely deductive, called *quasi-deductive* in the framework (Leshota, 2015). Adopting the definitions from Ensor et al.(2002), a quasi-inductive approach of the textbook would be one that builds from investigative activities for learners, versus a 'didactic' approach that emphasises procedures through worked examples, the quasi-deductive approach.

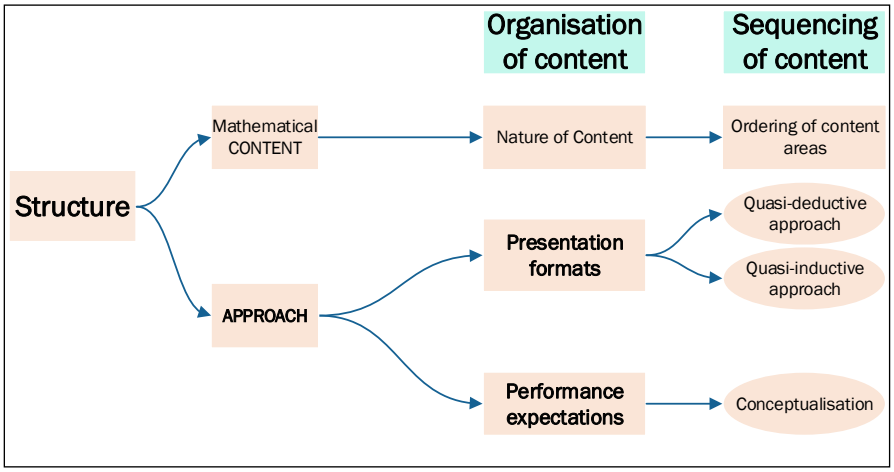

Figure 1: A Framework for Analysing Affordances of a Mathematics Textbook

The sequencing of performance expectations of learners (not part of the article) on the other hand illuminates the conception of a particular topic reflected by the textbook.

The question for this article is what could be learned about the embedded instructional approach advanced in a mathematics textbook from the sequencing of presentation formats in the textbook.

METHODOLOGY

The first chapters on Functions in the Pre-CAPS and CAPS textbooks served as data for the analysis in this article. These were chapter 11 in the Pre-CAPS textbooks and chapter 7 from the CAPS textbook. According to the curriculum, learners have done straight-line graphs of the form $y = mx + c$ in the previous grade. In grade 10, the curriculum deals with determining properties of function classes and their vertical transformations. The functions involved include: the quadratic function, $g(x) = ax^2$; the rectangular hyperbola, $h(x) = \frac{a}{x}$; the exponential function, $k(x) = ab^x$; and the three basic trigonometric functions, $\cos x$, $\sin x$, and $\tan x$.

The first step in the analysis was to determine and list all presentation formats in each chapter in the order that they appeared. For example, in the Pre-CAPS textbook, the first presentation format was a short *narrative* outlining what the chapter dealt with. As the first presentation format, it was assigned a letter 1. The *narrative* was followed by an investigative *activity* for learners, which was assigned the letter 2, and so forth. A similar process was performed on chapter 7 of the CAPS textbook. Table 1 shows the different presentation formats and their sequencing in the two textbooks.

Category	Presentation Formats	Pre-CAPS	CAPS
Explanatory Text	Narrative	1,6	16
	Definition/explanation		1,3,4,8,19,20
	General discussion	3,5,7,11,15	
	Note box		6,11,14,18,27
	Summary tables		30,31,32,33
Exemplification	Activity	2,4,10,16	26,28,29
	Worked example	8,12,13	2,5,9,10,12,13,17,21,23,24
	Practice exercise	9,14	7,15,22,25
	Check your skills	17	34
	Apply your skills	18	
	Problem solving exercise	19	

Table 1: Presentation Formats in the Pre-CAPS and CAPS Textbooks

Table 1 shows two main categories for the presentation formats. The *explanatory text* category catered for all presentation formats involving some form of explanation, and included five different types; *narrative, definition or explanation* of a mathematical concept*, general discussion* providing a more elaborated set of notes*, note box,* usually attached to specific examples, and *summary tables.* The *exemplification* category included presentation formats involving all forms of exercises and examples. Investigative *activities*, *worked examples*, which detailed solutions to specific questions, and *practice exercises* appeared within the chapters, while three types of end-of-chapter exercises were determined, *check your skills, apply your skills,* and *problem solving* exercises. The numerals under the last two rows indicate when a particular presentation format occurred. For example, *narratives* occurred twice in the Pre-CAPS textbook, as the first (1st) item and at the sixth (6th) place; and once at the sixteenth (16th) place in the CAPS textbook.

The second step was to determine the type of approach advanced by each textbook. For this purpose, a matrix table in Table 2 was drawn with the numerals represented on the top row, and the different presentation formats on the first column. A symbol,Θ, was used to mark where each presentation format occurred (red for the Pre-CAPS and black for the CAPS textbook). For example, the caption in figure 2 taken from Table 2 indicates that a *narrative* was the first (1st) and sixth (6th) presentation format in the Pre-CAPS textbook, while a *definition/explanation* was the first (1st), third (3rd), fourth (4th) and eighth (8th) presentation format in the CAPS textbook.

Presentation formats	1	2	3	4	5	6	7	8
narrative	Θ					Θ		
explanation/ definition	Θ		Θ	Θ				Θ

Figure 2: A caption from Table 2

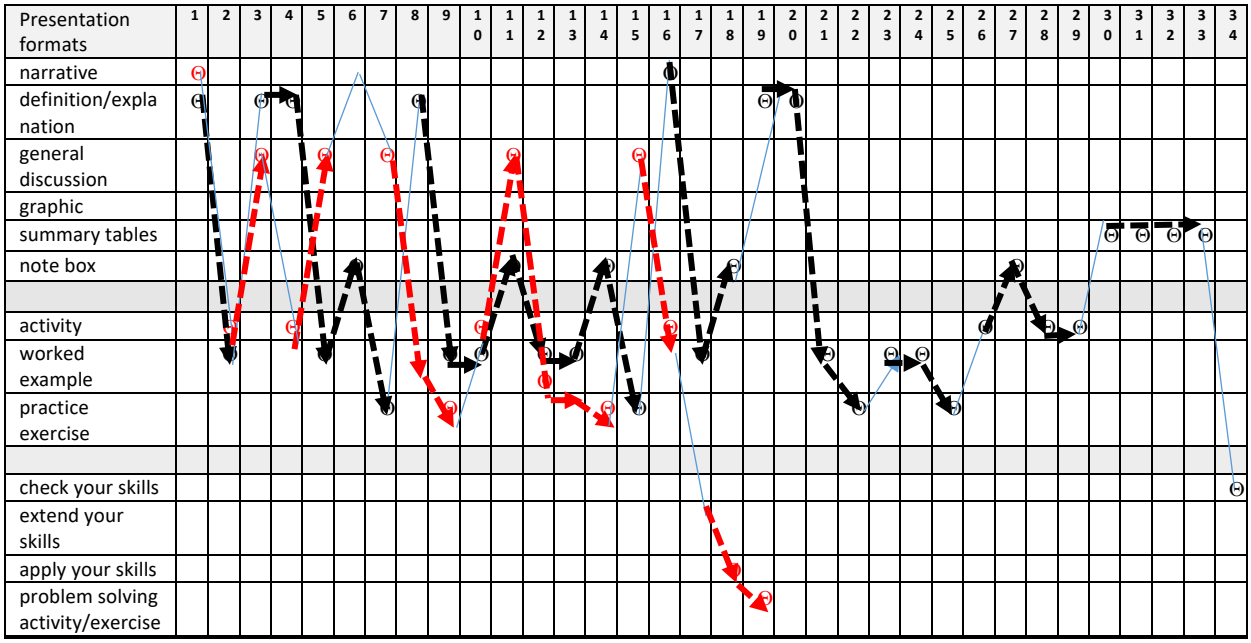

Table 2: Sequencing of presentation formats in the Pre-CAPS and CAPS textbooks

By joining consecutive symbols for each textbook with a straight line, some sequencing patterns emerged, from which interpretations about the approach in each textbook could be made. The red graph on Table 2 represents the sequencing of presentation formats in the Pre-CAPS textbook, while the black graph represents those in the CAPS one.

RESULTS

The Pre-CAPS textbook

The red graph on Table 2 shows a dominance of '*activity - general discussion'* pattern. This occurs at three positions: 2-3; 4-5; and 10-11. This indicates that in this textbook, a mathematical 'concept' (in this case, properties of functions), is established through an investigative set of exercises (*activities*), after which a lengthy discussion and elaboration of the mathematical concept featuring illustrations and definitions is presented. A closer look at the textbook shows that this pattern occurs when establishing the features of the different function classes.

A second type of pattern occurs at positions 7-9 and 11-13. This is a '*general discussion - worked example - practice exercise'* pattern. Here a general discussion on concepts is provided, and these concepts are then exemplified, followed by practice exercises. This section deals with notation and terminology. In the last pattern, 15-16 before end-of-chapter exercises, a *general discussion* on how functions are applied in real life starts, followed by an investigative activity featuring real life applications.

The analysis shows that the Pre-CAPS textbook advances an approach that is largely investigative, a *quasi-inductive* approach. It is only when notation and terminology issues are dealt with, that the textbook adopts a more 'didactic' *quasi-deductive* approach.

The CAPS textbook

The black graph on Table 2 depicting patterns in the CAPS textbook starts with a '*definition – worked example'* pattern at position 1-2. From there, the graph depicts a dominant pattern of '*definition – worked example (note box) – practice exercise'* at various positions: 3-7; 8-15 (where some four examples have been worked out); 16-18; and 19-25. Several *note boxes* provide additional clarifications or explanation to the worked examples. From position 26-29, there is a pattern of '*activities - summary tables*', followed by end-of-chapter exercises. In this regard hence, this textbook depicts a highly *quasi-deductive* approach to teaching.

A further comparison of the two textbooks shows some similarities with respect to the approach for particular content areas. For example, in both textbooks, the sequence patterns dealing with determining properties of functions are similar: '*activities – general discussion*' in the Pre-CAPS textbook, and, '*activities – summary tables*' in the CAPS textbook. This shows that for determining the features of function classes, both textbooks advance an investigative, quasi-inductive approach. However, for introducing notation and terminology, both textbooks advance a 'didactic' quasi-deductive approach punctuated by '*general discussion – worked example – practice*' pattern in the Pre-CAPS and '*discussion/explanation – worked example – practice*' in the CAPS textbook.

DISCUSSION

The question for this article was what could be learned about the embedded approach(es) advanced in the textbook(s) from the sequencing of presentation formats. The analysis of the sequence patterns succeeded in differentiating between a dominantly *quasi-inductive* approach in the Pre-CAPS textbook versus a largely *quasi-deductive* CAPS textbook. Thus, the process has illuminated the uniqueness of each textbook in how it approaches the content, as suggested by Valverde et al. (2002). The two chapters analysed have the same author; and share the same mathematics content, but they exhibit quite different approaches to teaching this content. This is a powerful information for teachers to know as it helps teachers in their decisions about how they should interact with the textbook. Furthermore, the analysis has illuminated the similarity in approach that the two textbooks advance for specific content areas, for example, notation and terminology. This shows the potential of the framework as a tool for studying and understanding better, the embedded instructional approaches in textbooks.

In conclusion, I concur with Ensor et al. (2002) that textbook use should be incorporated in in-service teacher education and professional development activities for mathematics teachers. My findings of tacit textbook use by teachers from the larger study (Leshota, 2015) could be as a result of clashing approaches between the teachers and their textbooks.

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MATHEMATICAL PREPARATION AND CONFIDENCE OF PROSPECTIVE ELEMENTARY TEACHERS IN CHINA

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We focused on both prospective teachers' (PTs) confidence about their knowledge preparation and the extent of their knowledge of the specific topic of fraction division. The results revealed how these PTs' confidence may or may not be supported by their knowledge preparation for teaching fraction division, a concept they would be expected to teach as part of the country's curriculum standards. The results also illustrated the importance of specifying knowledge components in mathematics in order to help build or support PTs' confidence for classroom instruction.

Accumulated research findings in past decades have led to the understanding that teachers' knowing mathematics for teaching is essential to effective classroom instruction (e.g., Li & Kaiser, 2011; RAND Mathematics Study Panel, 2003). Corresponding efforts have also been reflected in teacher preparation programs that call for more emphasis on prospective teachers' learning of mathematics for teaching (CBMS, 2012; Li, Ma, & Pang, 2008). Such efforts can presumably increase the quality of teacher preparation and prospective teachers' confidence and ultimate success in their future teaching careers. However, previous studies (Li & Kulm, 2008; Li & Smith, 2007) revealed a wide gap between sampled prospective middle school teachers' high confidence and their limited mathematics knowledge needed for teaching fraction division in the U.S. Much remains to be learned about the extent of knowledge in mathematics and pedagogy that prospective teachers have and what else they may need to know for building or supporting their confidence. As part of a large research study of elementary school teachers' mathematical preparation, this paper focused on a group of PTs' confidence and knowledge of mathematics and pedagogy on the topic of fraction division in China.

The topic of fraction division is difficult in school mathematics not only for students (Li, 2008), but also for prospective teachers (Li & Kulm, 2008; Simon, 1993). Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as "invert and multiply." However, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, or real world contexts (Greer, 1992; Li, 2008). The selection of the topic of fraction division, as a special case, can provide a rich context for exploring possible depth and limitations in prospective teachers' knowledge in mathematics and pedagogy. Specifically, this study focused on the following two research questions:

(1) What is the confidence of prospective elementary school teachers regarding their knowledge preparation for teaching?

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(2) What is the extent of prospective elementary school teachers' knowledge in mathematics and pedagogy for teaching fraction division?

CONCEPTUAL FRAMEWORK

To be able to help students learn mathematics with understanding, teachers need to have mathematics conceptual knowledge for teaching (MCKT; Li & Howe, 2017). By MCKT we mean *topic-based conceptual knowledge packages that are needed for understanding, explaining, as well as teaching specific mathematics content topics with connections*. It can be specified as containing the following three topic-based knowledge components that can and should be acquired by mathematics teachers:

(a) Having knowledge and skills directly associated with a specific content topic;
(b) Being able to connect and justify the main points of a content topic, and to place it in wider contexts;
(c) Knowing and being able to use various representations for teaching the content topic, and being able to teach the relations between them.

Clearly, specific MCKT varies from one content topic to another. The task of specifying MCKT is needed but enormous for different content topics. Nevertheless, teachers' acquisition of MCKT would enable them to develop a profound understanding of mathematics content topics they teach as termed by Ma (1999). Given the dramatic variations across mathematical content topics, we focus on the MCKT that teachers would need to have for teaching fraction division.

The conceptual complexity of the topic of fraction division is evidenced in a number of studies that documented relevant difficulties prospective and practicing teachers have experienced (e.g., Borko et al., 1992; Simon, 1993; Tirosh, 2000). Although both prospective and practicing teachers can perform the computation for fraction division, it is difficult for teachers, at least in the United States, to explain the computation of fraction division conceptually with appropriate representations or connections with other mathematical knowledge (Ma, 1999; Simon, 1993). Teachers' knowledge of fraction division is often limited to the invert-and-multiply procedure, which restricts teachers' ability to provide a conceptual explanation of the procedure in classrooms (e.g., Borko et al., 1992). Because the meaning of division alone is not easy for prospective teachers (e.g., Simon, 1993), fraction division is even more difficult (Li & Kulm, 2008; Ma, 1999). The findings from previous studies help provide specifics of these three components of MCKT as follows:

(a) Having knowledge and skills about fraction division, including conceptual and procedural knowledge (e.g., Borko et al., 1992), and solving problems involving fraction division (e.g., Greer, 1992)
(b) Mathematical connections and justifications of main points related to fraction division, including fraction concept; addition, subtraction, and multiplication of fractions (e.g., Ma, 1999; Tirosh, 2000)

(c) Representational variations and connections for teaching fraction division such as explaining the computational procedure for "division of fraction" with different representations (e.g., Li & Huang, 2008; Li & Kulm, 2008)

The specifications of these three components of knowledge provided a framework for the current study and served as a guideline for selecting items to examine the extent of PTs' knowledge and specific difficulties with fraction division.

METHODOLOGY

Subjects

The participants were prospective elementary school teachers sampled from five institutions located in three provinces in China. These prospective teachers were in their last stage of study in the program. They had already taken all of the required mathematics courses and were completing the mathematics methods course at the time of their participation in this study. A total of 319 responses were collected and used for analyses and reporting.

Instruments and data collection

A survey was developed for this study, containing two main parts with three items for Part 1 and seven items for Part 2. Part 1 contains items on elementary teachers' knowledge of mathematics curriculum and their confidence in their readiness for teaching. Part 2 has seven main items that assess elementary teachers' three knowledge components of MCKT on the topic of fraction division. Most items were taken from previous studies (Li, Ma, & Pang, 2008; Li & Smith, 2007), with some items adapted from school mathematics textbooks and others' studies (e.g., Tirosh, 2000). Given the limited page space, only three items (note: each item containing two questions) from Part 2 and PTs' responses to these items are included for analyses to provide a glimpse of sampled PTs' confidence and MCKT.

The survey was administrated at regular class time by instructors in five institutions. Participants were notified that the survey was for research purposes only and should be completed anonymously.

Data analysis

Both quantitative and qualitative methods were used in analysing and reporting the participants' responses. Specifically, responses to the items in Part 1 were directly recorded and summarized to calculate the frequencies and percentages of participants' choices for each category. To analyse participants' solutions to the items in Part 2, specific rubrics were first developed for coding each item, and subsequently, the participants' responses were coded and analysed to examine their solutions/answers.

RESULTS AND DISCUSSION

In general, the results showed interesting relationships between PTs' confidence and their mathematical preparation for teaching fraction division, which illustrates the

importance of specifying knowledge components in mathematical preparation in order to help build or support PTs' confidence for classroom instruction.

For PT's confidence, the results from the survey indicated that (1) participating PTs in China did not know well about their national curriculum standards in general; (2) the majority of these PTs were confident in the knowledge preparation they received for future teaching careers; but (3) they knew very well about the topic placement of "multiplication and division of fractions" in mathematics curriculum. The results suggested that these PTs tend not to feel over confident.

For specific knowledge components of MCKT, these PTs' performance revealed that their mathematical preparation was sound in the content topic itself, especially in the procedural aspect, and relatively weak conceptually in connecting the content topic with other topics both mathematically and pedagogically. The seemingly mixed results in their responses actually suggest that these PTs' confidence was built upon or supported by what they know that can and should be specified in concrete terms or knowledge components. The following sections are organized to present more detailed findings corresponding to the two research questions.

Prospective teachers' confidence in elementary school mathematics

The following items are from Part 1 of the survey to illustrate PTs' confidence of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the National Mathematics Standards? On a scale of four choices (High; Proficient; Limited; Low), 78% and 10% of the participants chose "Limited" and "Low", respectively. Small percentages of the prospective teachers felt to have high (1%) or proficient (11%) understanding of their national mathematics standards.

For item 2-(2): Choose the response that best describes whether elementary school students have been taught the topic – *Multiplication and division of fractions.* On a scale of five choices (Mostly taught before grade 5; Mostly taught during grades 5-6; Not yet taught or just introduced during grades 5-6; Not included in the National Mathematics Standards; Not sure), 86% participants indicated that the topic is "mostly taught during grades 5-6" (a correct choice), and most of the remaining (9%) chose the first response ("Mostly taught before grade 5", a partially correct choice if only fraction multiplication is considered). The results, in contrast to the participants' response to item 1, suggested that these PTs know very well about the content topic placement in mathematics curriculum, although they did not feel confident in knowing about their national mathematics standards.

For item 3-(2): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of "Number – Representing and explaining computations with fractions using words, numbers, or models?" On a scale of three (Very ready; Ready; Not ready), 52% of the participants thought they were "ready", while 14% chose "very ready," and 33% "not ready." The

results indicated that the majority of these PTs were confident in their preparation for teaching fraction computations, including fraction division. There was also a large percentage of prospective teachers who are not confident. The diversity in responses suggested the need of learning more about their confidence and possible connections with their knowledge preparation.

Taking together, prospective elementary school teachers' responses to the Part 1 suggested that these PTs in China tend not to feel over confident, although they actually knew very well about some specifics. In fact, the results are consistent with what has been reported about in-service mathematics teachers in East Asian countries (Mullis, Martin, Gonzalez, & Chrostowski, 2004) and China in specific (Li & Huang, 2008). The consistency in the general response pattern between PTs in the current study and in-service teachers in other studies suggested that culture likely plays an important role in expressing confidence by teachers in East Asia including China.

The extent of prospective elementary school teachers' preparation in MCKT for teaching fraction division

These PTs' responses to Part 2 allowed a closer look at the participants' three knowledge components of MCKT, especially on the topic of fraction division. Results indicated that these PTs do very well on items related to fraction division computation and problem solving (MCKT knowledge component 1). For example, for the problem "Say whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without solving. Explain your reasoning." 96% of these PTs answered the problem correctly (i.e., the first numerical expression is greater than the second one). Among those who provided the correct answer, 79% did not use fraction division computations. The common explanations include (a) "If the dividend is the same, the smaller the divisor, the larger the quotient." and (b) "2/3 (8/12) is smaller than ¾ (9/12)". And many respondents provided both reasons. The other 17% used the computation rule for fraction division (i.e., converting division into multiplication, then followed by comparing 3/2 and 4/3) to reach the correct answer. A very small percent of sampled PTs (3 respondents, 1%) used both methods.

Moreover, these PTs also had great performance in solving multi-step word problems that involve fraction division. For example, 94% participants solved the following problem correctly.

> Johnny's Pizza Express sells several different flavour large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was 3/4 of the number of pepperoni pizzas sold, and 2/3 of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?

Specifically, 64% used a multi-step computation method to get the answer (e.g., 24 x ¾ =18, 18÷2/3 =27), about 27% used a combined computation method (e.g., 24 x ¾ ÷ 2/3 =27), 2% adopted an algebraic approach to set up and solve an equation for solution, and a few (about 1%) provided more than one solution approach. About 6%

of these respondents did incorrectly, resulting from either computation errors (e.g., providing a computation as 24 x ¾ ÷ 2/3 = 36) or misunderstanding of the problem (e.g., providing a computation as 24 ÷ 2/3 = 36).

For the knowledge component 2 of MCKT, PTs were asked to explain "the meaning of fraction division, and how fraction division relates to other content topics" that aims to assess their knowledge of fraction division and ability of connecting and justifying possible association between fraction division and other content topics. The results suggested that 73% provided correct explanations to the first sub-question. Among those correct answers, 9% responded as "the meaning of fraction division is the same as the division of whole numbers", 28% provided their answers as "if knowing the product of two factors and the value of one factor, it is an operation to find the value of the other factor", 18% responded with an answer that combines the above two, 12% explained the meaning of fraction division as "partitioning a number into several parts, taking one part or several parts. The rest of answers include comparing fraction division with ratio, or operations involving whole number or fraction. For the second sub-question, only 18% of these PTs provided correct explanations, with 13% indicating "fraction division is an inverse operation of fraction multiplication" and 4% mentioning that "fraction division relates to inverse number, for example, if divided by a number equals to multiplying its inverse number". The vast majority (82%) either stated what content topics may relate to fraction division but failed to explain how (38%), provided some other wrong explanation (8%), or simply did not answer this sub-question (36%).

There were several items used to assess PTs' knowledge component 3 of MCKT. As an example, PTs were asked to explain how to explain/teach given computations of fraction division. In particular, the problem of "How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$?; Why $\frac{2}{3} \div \frac{1}{6} = 4$?" (adapted from Tirosh, 2000) was included in the survey. For the first fraction division (i.e., explaining why 2/3 ÷ 2 = 1/3?), 95% provided valid explanations for dividing a fraction by a natural number (i.e., 2/3 ÷ 2 = 1/3). The dominant explanation (50%) used the meaning of fraction such as, "dividing a whole into three equal parts, each part should be 1/3, so 2/3 mean to have two such parts. Dividing 2/3 into two equal pieces, so each piece should be 1/3." The other 18% were dominated by explanations that were based on the algorithm, "dividing a number equals to multiplying its reciprocal", 14% explained with a drawing or number line, and about 5% provided correct explanations with two or more different approaches. For the second fraction division (i.e., explaining why 2/3 ÷ 1/6 =4?), 84% provided valid explanations but the dominant explanation was based on the fraction division algorithm (38%). 27% provided their explanations mainly as "changing fractions so that they have the same denominator first, and then using the meaning of fraction for solution, for example, changing 2/3 into its equivalent fraction 4/6, and 4/6 has four 1/6". Some (9%) used drawings or a number line to help explain. Taking together, these PTs did very well in explaining these two fraction divisions (95% and 84%,

respectively). Their explanations were dominant with an approach that relies on either the meaning of fraction or the fraction division algorithm. Moreover, they performed better in explaining a fraction divided by a whole than explaining a fraction divided by a fraction.

The results from these PTs' responses on MCKT items revealed their strengths in many aspects of MCKT, as specified in the framework. However, PTs' strengths across these aspects varied to a certain degree. It appeared that these PTs have solid performance on items related to fraction division computation itself, especially in the procedural aspect and problem solving, but relatively weak conceptually in connecting the content topic with others both mathematically and pedagogically.

CONCLUSION

The findings from this study helped shed a light on the relationships between these PTs' confidence and their mathematical preparation for teaching fraction division. Specifically, these PTs didn't feel over-confident about their understanding of national mathematics standards, but they knew very well about the curriculum placement of selected content topics. They also had better confidence in terms of their readiness to teach elementary school mathematics. Such confidence was likely supported by their solid knowledge and skill directly associated with fraction division, a knowledge component that is also typically required for school students. At the same time, their relatively weak performance on items that are conceptually demanding in mathematics or pedagogy likely failed to support their confidence in readiness for teaching. Such knowledge differentiations, as specified in the MCKT framework, help provide an important and feasible lens for us to know the strength and weakness of teachers' knowledge. For the case of China in this paper, the results suggested that PTs likely gain much more on mathematics, somehow less on mathematical pedagogy, and very limited on connections of mathematical ideas through their program studies. In turn, such results helped illustrate what teacher preparation programs need to do more in mathematical preparation in order to help build or support PTs' confidence for classroom instruction.

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AFFORDANCES OF SOLVING COUNTING PROBLEMS IN A COMPUTATIONAL ENVIRONMENT

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In this paper, we present results from a study in which undergraduate students solved counting problems in a computational environment using simple Python programming. We present affordances of this experience, both in terms of the students' combinatorial reasoning and in terms of their computational activity. We argue that there are compelling reasons to have students leverage computing to solve counting problems. We briefly discuss limitations and conclude by offering directions for future research.

INTRODUCTION AND MOTIVATION

Researchers and policy makers in Science, Technology, Engineering, and Mathematics (STEM) education have made the case that computing is an increasingly integral aspect of scientific work, and it should be a fundamental component of our preparation of mathematics and science students (e.g., Weintrop et al., 2016; Wing, 2006). Given the vibrant state of computing in STEM, we are motivated to examine ways in which engaging in computing interacts with students' learning of mathematical content.

In this paper, we present results from a study that investigates the effects of Python coding on students' reasoning about the domain of combinatorics. Specifically, we focus on undergraduate students' solving of "counting problems," which are problems that answer how many elements in a set satisfy certain conditions. We address the following research question: *What are affordances of having students solve counting problems in a computational setting, and how do these affordances benefit students?* By affordances, we mean unique ways in which the computational setting seems to enhance students' combinatorial reasoning and activity.

RELEVANT LITERATURE AND THEORETICAL PERSPECTIVES

In this section, we frame our work within existing literature on combinatorics education, and we present theoretical perspectives that guide our work by clarifying key terms and ideas we use in our study. There is plentiful evidence that counting problems are difficult for students to solve (e.g., Batanero et al., 1997; Lockwood & Gibson, 2016). Some reasons for these difficulties are that counting involves non-routine problems, many of which seem disconnected (Kapur, 1970), and counting problems often involve very large sets of outcomes, making them difficult to verify (Eizenberg & Zaslavsky, 2004). There is further evidence that students struggle to understand, justify, and appropriately apply counting formulas (e.g., Batanero, et al., 1997; Lockwood, 2014). Researchers have attempted to address such difficulties in a number of ways – including having students reinvent fundamental combinatorial

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concepts (e.g., Lockwood & Schaub, in press) and by examining ways of thinking and strategies (e.g., English, 1991, Lockwood, 2014) that might be beneficial for students.

As a conceptual framework, we draw on Lockwood's (2013) model of combinatorial thinking (Figure 1). In this model, Lockwood defines counting processes as the enumeration process that one engages as they solve a problem – the steps or procedure the individual does. Sets of outcomes are the collection of objects being counted, and the cardinality of the set of outcomes is typically the answer to a counting problem. Lockwood (2013, 2014) has argued for the importance of making meaningful connections between the counting process and the set of outcomes of that process. In particular, she argued for students to develop a set-oriented way of thinking toward counting, which is a "way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" Lockwood, 2014, p. 31). In this report, we frame our perspective on counting through the lens of this model. We designed our study accordingly, with the intention of helping students reinforce the relationship between counting processes and sets of outcomes.

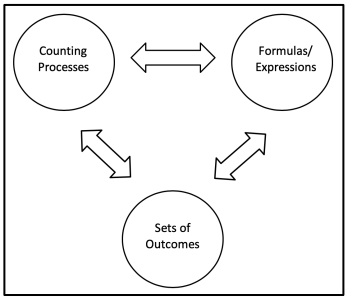

Figure 1: Lockwood's (2013) model of students' combinatorial thinking

One practical way to reinforce this relationship is through listing outcomes. Lockwood and Gibson (2016) found that even partially systematically listing outcomes was positively correlated with students answering counting problems correctly. We thus infer that listing is potentially a productive combinatorial activity. However, while there is merit in partially listing sets of outcomes, complete listing is often infeasible to do by hand (as problems often have enormous sets of outcomes). Thus, our interest in computing is grounded in the notion that computing can help students connect counting processes and sets of outcomes (through the instantaneous generation of entire sets of outcomes through programming). This perspective affected our design of the study, as we sought to put students in situations where they could use computing to reinforce the relationship between counting processes and sets of outcomes.

The Joint Task Force for Computing Curricula note that "In a general way, we can define computing to mean any goal-oriented activity requiring, benefiting from, or creating computers" (2005, p. 9). For mathematics in particular, we follow Lockwood, DeJarnette, & Thomas (2018) and define computing as *the practice of using tools to perform (mathematical)* ***calculations*** *or to develop or implement* ***algorithms*** *in order to accomplish a (mathematical) goal*. From a calculational perspective, computing might involve using a tool to complete numerical or symbolic calculations, and from

an algorithmic point of view, computing might involve developing or implementing a logical sequence of steps telling how to produce output information from given input.

Weintrop et al. (2016) developed a "taxonomy of practices focusing on the application of computational thinking to mathematics and science" (p. 128). We use this taxonomy of practices, especially the computational activities associated with Computational Problem Solving Practices, to characterize computing activity. Practically, for the results described in this paper, computing activity meant the students engaged in basic programming tasks in Python, and this primarily included (in Weintrop et al.'s terms) *Preparing Problems for Computational Solutions* (which they define as "reframing problems so that existing computational tools can be utilized" (p. 138)) and *Programming* (which "consists of understanding and modifying programs written by others, as well as composing new programs or scripts from scratch" (p. 139)).

METHODS

Participants and Data Collection

We interviewed three pairs of undergraduate students in paired teaching experiments (Steffe & Thompson, 2000) – two pairs for ~15 hours, one pair for ~10 hours. Generally, the teaching experiment methodology involves repeated task-based interviews that allow for a researcher to explore students' reasoning over a period of time and to observe how they think about and learn particular mathematical concepts. The aim is not necessarily explicitly to teach directly (that is, the end goal is not only that students will learn certain material), but rather the goal is "for researchers to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). In this report, we present results from one pair recruited from vector calculus. Charlotte and Diana (pseudonyms) were sophomore and freshman chemistry majors, respectively. They were novice counters (had not taken counting in college), and they had no programming experience in high school or in college.

The students sat together and worked at a computer, coding in the language Python in the environment PyCharm (which has a window in which they could write code and a window that displays the output). They worked as a pair on tasks that covered a variety of counting problems and situations. In some tasks, we gave them code to interpret, which served to familiarize them with the syntax. More often we gave the students a counting problem and asked them to solve and code it. Generally, their lack of programming experience did not interfere with their ability to explore problems and write and edit code (the programming the tasks required was mostly limited to for loops and if statements). We videotaped the interviews and also recorded the screen. The students worked well together and took turns programming, and both contributed ideas. The interviewer (the first author of this paper) regularly asked the students to explain their work. Also, the interviewer regularly prompted students to predict the output of their code, and this was motivated by the desire to help students connect their counting processes with the set of outcomes (Lockwood, 2013).

Data Analysis

All interviews were transcribed, and we created enhanced transcripts that included relevant screenshots from the video. The research team reviewed the transcripts and identified episodes in which affordances or limitations were clearly evident, and they then reviewed those episodes so as to characterize the nature of the affordances or limitations they observed. The research team is using the qualitative analysis software MaxQDA to systematically code these affordances and limitations. For the purposes of this report we report on one key episode that highlights multiple affordances.

RESULTS

To present results from this study, due to space, we only discuss four affordances in detail. Thus, the affordances we mention do not comprise a comprehensive list from our data set, but rather they are examples of the kinds of affordances students may gain from counting within a computational setting. In the final presentation we will discuss additional affordances that we do not have the space to elaborate here. We describe a situation in which the students were attempting to solve the following License Plates Problem: *A license plate consists of six characters. How many license plates consist of three numbers (from the digits 0 through 9), followed by 3 lower case letters (from the first 5 letters in the alphabet), where repetition of characters is allowed? Write some code to solve this problem. What is a mathematical expression that represents the number of outputs of your code?* This occurred in the second of nine 60-90 minute episodes with the students. We share some highlights from their work on this problem, and in doing so we characterize the affordances that emerged in their work.

In working on the problem, the students began by recognizing that they could visualize the license plates by writing six dashes representing six positions, and that the first three positions could be 0 to 9, and the last three *a* to *e*. Diana wrote the dashes and options in Figure 2, which is a common way to represent such outcomes.

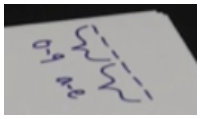

Figure 2: Diana writes dashes and options for positions in the license plate

They then turned their attention to the code and the program. They had copied some code that had two nested loops and were editing it. Diana suggested that they consider multiple for loops for letters and numbers, and Charlotte suggested perhaps they simply repeat the variables with the two loops they had in front of them.

Diana: Maybe we should figure out a way to have it recognized that there's three letters – or three numbers in a row, followed by two letters in a row. So, we could do like a sorting system and like repeat the sets. So, like for *i* in numbers, for *j* in numbers, for *k* and numbers, and then keep going with the letters – for *l* in letters, for *m* in letters, for *n* in letters.

Charlotte: Do you think it would be easier if we kept like these [referring to two the nested loops] … But then when it comes down to print, do you like, *i* comma, *i* comma, *i* comma, *j*, *j*, *j*.

The students had thus suggested two possibilities for how to proceed. We will argue that the computational setting fostered efficient experimentation as they could get immediate feedback on their two ideas. They first proceeded with Charlotte's suggestion, and they wrote the following code (Figure 3).

```
Numbers = ['0','1','2','3','4','5','6','7','8','9']
Letters = ['a','b','c','d','e']

Licenseplate = 0
for i in Numbers:
    for j in Letters:
        print(i,i,i,j,j,j)
        Licenseplate = Licenseplate+1
print(Licenseplate)
```

Figure 3: Code representing Charlotte's idea on the License Plate problem

The interviewer asked them what they thought the output of the code would be, and they had the following exchange:

Diana: I think it will definitely give us strings of three numbers, followed by three letters, but I'm a little worried it might do only the identical numbers, followed by identical letters […] But I don't know. I don't know how it will read it for sure.

Charlotte: Yeah. Kind of same thinking. I'm not exactly sure what it'll do.

The students ran the program, and Pycharm yielded a list of 50 outcomes. The list was generated as Diana had predicted, printing only the outcomes with repeated elements. The first few elements were 111aaa, 111bbb, 111ccc, and the last few were 999ccc, 999ddd, 999eee. By surveying the set of outcomes, the students saw that this was what Diana had guessed but not what they wanted for the problem. The interviewer reminded them of Diana's different idea, and Diana then wrote the code in Figure 4.

Before they ran this program, we asked them to predict what the code would do and how the outcome would be organized. Diana gave the following response.

Diana: Yeah. It'll probably do – it'll keep these – because these are the starting points. So, like the triple 0 and then it'll probably go like 001, 002, 003, and include all those different combinations. And in terms of like a mathematical expression, I was thinking, maybe it might multiply the options by each other. So, like with our paper here [points to the dashes written in Figure 1] – since this is 0-9, then there are 10 options. And there's 10 options here, and there's 10 options here. It might do 10 times 10 times 10 times 5 times 5 times 5. Because there's 5 options here.

```
Numbers = ['0','1','2','3','4','5','6','7','8','9']
Letters = ['a','b','c','d','e']

Licenseplate = 0
for i in Numbers:
    for j in Numbers:
        for k in Numbers:
            for l in Letters:
                for m in Letters:
                    for n in Letters:
                        print(i,j,k,l,m,n)
                        Licenseplate = Licenseplate+1
print(Licenseplate)
```

Figure 4: Code representing Diana's idea on the License Plate problem

Diana explicitly pointed to the dashes they had written in Figure 1 as she spoke. This suggests to us that she was making an explicit connection between the representation of the code and the positions she had written. Based on her guess about the mathematical expression, the students anticipated a total of 125,000. They ran the program and found that, indeed, the total was 125,000, and the outcomes had been organized as Diana had discussed in the previous excerpt.

Summary of Results: Characterizing Affordances

This brief episode highlights a handful of affordances that we want to discuss. We do not claim that the programming was necessary for the students (indeed, they could have solved the problem using non-computational methods). However, we do contend that the programming offered some unique benefits that would not have been so easily realized by hand. The computing offered the students an opportunity to generate **immediate feedback**, in the sense that they could instantly see the output of their code. Importantly, in this combinatorial context, the output of the code exactly represents the set of outcomes of a counting process (in the sense of Lockwood, 2013). From a combinatorial perspective, then, such feedback is especially valuable because it gives students immediate access to the *entire* set of outcomes, which is often impossible without computational aid. We highlight two additional affordances that this immediate feedback enabled. The availability of such feedback facilitated **experimentation** for the students. That is, they were able to explore two different ideas efficiently. This is evidenced by the fact that both Charlotte and Diana could offer and implement solution strategies. Again, while multiple solution strategies are not an inherently computational feature, we suggest that the computational context did seem to facilitate the efficient and thorough exploration of multiple strategies in ways that would be restricted by by-hand exploration. For example, they simply could not have generated complete sets of outcomes that reflected two different processes without the use of the computer. The availability of immediate feedback also provided numerical and structural **verification**. In Charlotte's approach, the immediate feedback of seeing the list of outcomes revealed that the nature of the outcomes was not what they intended to count, and this confirmed that answer was incorrect. In Diana's approach, when they ran the problem they were able to gain numerical verification that her code generated the expected number of outcomes. Researchers have indicated that one reason counting problems are so difficult is that they are difficult to verify (Eizenberg & Zaslavsky,

2004). Given this established issue, we argue that the verification the computational setting provides is a considerable affordance for approaching counting in this way.

We briefly mention one additional affordance, which is that the computational setting seemed to facilitate **connections between mathematical representations.** This is a valuable practice in mathematics education (Pape & Tchoshanov, 2001), and we argue that the computational setting enhanced it by offering additional representations of the code itself and the entire list of outcomes, which complement other combinatorial representations such as tree diagrams. Further, because we often asked students to predict what the output of their code would be, we regularly observed students making explicit connections between the code, the list of outcomes, and written expressions. We suggest that the availability of the entire list of outcomes made it relatively easy for them to recognize explicit connections among multiple representations.

DISCUSSION AND CONCLUSION

Although we do not have space to discuss them at length, there are also potential limitations of using a computational environment. In one situation the students had first incorrectly solved a problem by hand, and then they correctly coded that incorrect answer. The act of computing did not help them notice the error or ensure that they would fix it, nor it did preclude them from arriving at an incorrect answer. An additional limitation is that focusing on computing too heavily may prevent students from engaging in mathematical approaches to a problem.

The affordances we highlighted in this brief example demonstrate potential ways in which computational activity may help students reason about counting problems, which has implications for helping students successfully solve counting problems. While ours is a narrow focus on combinatorics, these results serve as an instantiation of a broader phenomenon that warrants further investigation – namely, the role that computational activity can play in students' mathematical thinking and activity.

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YOUNG CHILDREN EXPLAIN REPEATING PATTERNS

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Patterning instruction needs a shared language for talking about the structure of patterns. In order to develop a basis for such discussions, this paper focuses on young children's explanations of an ABC pattern. Categorization of verbal explanations shows that young children make different structural aspects relevant. Although most of the explanations are not complete from a mathematical point of view, they offer qualitatively rich starting points to further develop children's structural understanding of repeating patterns.

INTRODUCTION

There is increasing evidence that competencies regarding repeating patterns are especially important for mathematics learning if children focus on the underlying structure of the pattern: the unit of repeat and its repetition (Papic, Mulligan, & Mitchelmore, 2011; Threlfall, 1999). However, Radford (2012) highlights that the ability to analyze and generalize the structure of a pattern does not develop spontaneously, but rather depends on cultural influence and on some kind of education. Recent studies indeed confirm: Young children's understanding of repeating patterns is scarcely developed in most cases but can be supported by 'good' instruction (Björklund & Pramling, 2014; Lüken, 2018; Papic et al., 2011). Such instructions require from children and their educators to find a shared language for talking about patterns: Which structure is there to be seen? In order to develop a good basis for such discussions with young learners, we investigate children's spontaneous *explanations* of a given repeating pattern. In doing so, we do not primarily focus on the correctness of their answers, but rather on different structural aspects that they make relevant in their interpretations of the pattern.

THEORETICAL BACKGROUND

Repeating patterns

Patterns are an abstract and generalizing phenomenon which is not easy to define (McGarvey, 2012). In a mathematics education context, we consider the term 'pattern' either as a synonym for any regularity ("What is the pattern?") or as a name for a mathematical object that is structured in a regular way ("This is a pattern."). As an example, we can think of six wooden cubes in the following colors: *green, purple, orange, green, purple, orange* (see Fig. 1). A pattern in line with the first description is any regularity concerning the color. For example, correct answers to the question "What is the pattern?" might be: "Every third element is orange", "There is an equality between every element in the pattern and one of the first elements.", or "A bit orange, a bit purple, a bit green." (see section "Results") However, these answers do not refer

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to the whole structure of the example-pattern. For a complete explanation, the structure of the pattern has to be taken into account: the unit of repeat and its repetition, e.g. "Green, purple, and orange, and on and on." (see section "Results"). Thus, patterns in line with the second description can be characterized by their underlying structure. In this paper, we use the *green, purple, orange, green, purple, orange* pattern which is a *repeating pattern* (see Fig. 1). A repeating pattern is a periodic sequence of elements which can be reduced to a smallest subset, the so-called *unit of repeat* (Threlfall, 1999). This unit of repeat is repeated in the form of a geometric translation. In our example, the unit of repeat is either *green, purple, orange* or *purple, orange, green* or *orange, green, purple*. In more general terms, this repeating pattern is called an ABC pattern which indicates that the unit of repeat has the length 3 and consists of three different elements (Liljedahl, 2004).

In order to grasp the structure of a pattern, children have to learn to interpret a pattern in a certain way. They have to grasp the *relevant* structural aspects: "The recognition and use of a small sequence as an element to be repeated does, however, bring an awareness of a relationship between a part of the pattern and the whole as a repeat of it." (Threlfall, 1999, p. 25) Activities like *copying*, *extending, repairing, creating,* and *translating* repeating patterns, can be correctly solved without referring to the structure: by matching elements or by a recursive strategy (McGarvey, 2012; Rittle-Johnson, Zipper, & Boice, 2019; Wijns, Torbeyns, De Smedt, & Verschaffel, 2018). Thus, it is important to talk with children about their perception of the pattern. An activity that allows children to do just that and has hardly been investigated up to now is the activity of *explaining*: "Tell me about the pattern!", "Look at this pattern and explain." If children are invited to explain a pattern, they can use *language* and other available resources, e.g. gestures or the wooden cubes, to make their individual interpretations explicit – with all the shortcomings and potentials of *learning* processes (Moschkovich, 2011; Schleppegrell, 2004).

Children's explanations of repeating patterns

There is little research on children's explanations of repeating patterns and, building on such results, on the question of how we might help children in everyday conversations to grasp the structure of a pattern. Below, three selected studies are shortly presented.

Rittle-Johnson, Fyfe, McLean, and McEldoon (2013) explored 4-year-old children's verbal explanations during a translation task (n = 65). After producing a correct translation of a model pattern, children were asked to explain the correct solution. They were asked both, "What is my pattern?" and "How is your pattern the same as mine?" Children's explanations were coded with the help of seven categories. The categories comprised non-pattern explanations, in which children named the colors or shapes without reference to their position in the pattern, gave other vague explanations, or refused the answer. A lot of explanations were given by labeling the items in order, i.e. naming either the shape or color of consecutive elements or pointing to each element

in order. Children also referred to the pattern's elements as same or different, or explicitly linked individual elements in the new pattern to the model pattern. Rittle-Johnson et al. (2013) consider these explanations as more sophisticated because children abstracted beyond naming characteristics of individual pattern elements. The sophistication of children's explanations increased after the experimenter gave a high-quality explanation that included the shared, general labels of 'same' and 'different', e.g. "For my pattern, I put two that were the same and then one that was different. Then I started over again." (Rittle-Johnson et al., 2013, p. 383).

McGarvey (2013) in her research on assessing young children's attention to pattern and structure categorized four types of children's responses. They are based on her working experience with children from age 4 to 7 on various patterning tasks including identify, describe, copy, extend, compare, and create patterns. The first type of responses, which McGarvey called *inattention to pattern and structure*, show that children have no verbal definition of the term pattern and do not attend to the model pattern when describing it. This category relates to Rittle-Johnson et al. (2013) nonpattern categories. The second category (*direct comparison*) in which the elements of the pattern are labelled in order, is similar to one of Rittle-Johnson et al. (2013) categories, too. Although children describe patterns in the third category (*recursion strategy*) also by labelling the elements in order, it becomes clear in their verbal responses to other tasks that they focus on the relationship between consecutive elements in the pattern. Based on the last element, they are able to predict the next. For the *core unit strategy* children explicitly name the unit of repeat.

Fyfe, McNeil, and Rittle-Johnson (2015) hypothesized that labels used to describe patterns and relations might influence children's relational reasoning. In their study, 62 pre-schoolers (M_{age} = 4.4 years) translated and described eight repeating patterns. During an intervention, half of the children were exposed to concrete labels, i.e. the experimenter described the patterns by referring to the changing perceptual dimension – the color or shape of the elements (e.g., blue-red-blue-red). The other half was exposed to abstract labels, i.e. the experimenter described the patterns using the letters of the alphabet (e.g., A-B-A-B). Children exposed to abstract labels solved more tasks correctly than children exposed to concrete labels. Fyfe et al. (2015) conclude that using abstract language helps children to discover relations between the elements of two structurally identical patterns and, thus, facilitates children's pattern translation.

These three studies clearly show that children talk about patterns in very different ways. But, as far as we know, there is still no study of children's explanations that systematically compares explanations of *one* pattern and that is, at the same time, open to children's idiosyncratic ways of explaining. From our point of view, just such insights can be helpful in order to develop appropriate activities, tasks, and helpful impulses for supporting children in their work on repeating patterns. For that reason, in this paper we focus on the following question: Which structural aspects do children make relevant in their explanations of a repeating pattern?

METHOD & ANALYSIS

Context of this paper is a study investigating patterning strategies in early childhood. 254 children from 15 kindergartens and 4 primary schools in a metropolitan area in the north-western part of Germany participated in the study. Children were ranging in age from 2 years 11 month to 7 years 8 month (M_{age} = 5 years 3 month; SD = 17.19 month). None of the participating kindergartens were using a specialized curriculum focused on patterning, and teachers reported doing no patterning activities at all. The first-grade teachers reported having spent one to two lessons on copying and extending repeating patterns. Both practices are representative for the German educational system.

In a one-on-one interview, each child completed 24 patterning tasks; first eight with an AB pattern, then eight with an ABC pattern and last eight with an ABCC pattern. Children were always invited to explain their thinking and their ways to find solutions. For this paper, we exclusively look at the first of the eight ABC tasks *explain ABC pattern*. Children were presented an ABC pattern made out of two units of repeats from green, purple, and orange wooden cubes (see Fig. 1). The presentation was accompanied by the prompt, "Please, look at this pattern and explain what you see." All interviews were video recorded and children's answers were transcribed.

The transcribed explanations to the task *explain ABC pattern* were analyzed by means of "qualitative content analysis" (Mayring, 2008). In a first deductive step, we defined two initial categories from the above summarized literature: "figurative interpretations" and "structuring interpretations". The first type of interpretation is related to the pattern, but not in an obvious structuring way. The second type is explicitly structuring, however. Whereas we did not differentiate the figurative interpretations any further, we did so with the structuring interpretations. For these interpretations, we deduced in a second deductive step two subcategories from the structure of the pattern (see above): *unit of repeat* and *repetition.* Following an iterative process, the children's explanations were coded by the help of these categories and subcategories. In this process, particularly the existing subcategories were inductively modified and revised. This way, we now differentiate four categories for the structuring interpretations which focus on different structural aspects. In the following section, the categories are described with regards to content but not quantified yet.

RESULTS

Figurative interpretations. Some children interpret the representation of the pattern as a representation of a concrete object.

1 "This is a snake." [German original: "Das ist eine Schlange."]

2 "This is a wall." ["Das ist eine Mauer."]

Interpretations of this type obviously refer to the visible appearance of the six wooden blocks and compare it with something that looks in some way similar. Thus, a snake has also an elongated shape (1st example), and a wall is also made up of single cuboids (2nd example). The interesting point about those figurative interpretations is that some

of them might be indeed good starting points for children's thinking and talking about patterns and others not – at least not in an obvious way. For example, the exterior shape of the representation of a pattern is not that important at all: From a mathematical perspective, it is not relevant that the six blocks form together an elongated shape and look, as a consequence, a bit like a snake. But a pattern is, like a wall, composed of single elements. Thus, the latter interpretation might be a good cause to talk about the structure of the pattern and, for that reason, it can be regarded as a pre-form of a structuring interpretation.

Structuring interpretations. Some children interpret the representation of the pattern with a more or less explicit reference to structural aspects of the pattern. This does not necessarily mean that the child is talking about the structure of the pattern, but it does mean that the child shows some kind of structuring in his or her explanation that can be regarded as useful for that topic. The analysis of children's explanations indeed indicates that there is more about explaining a repeating pattern than exclusively talking about a unit of repeat and its repetition. Figure 1 illustrates our four categories of structuring interpretations which reflect four different aspects of a repeating pattern.

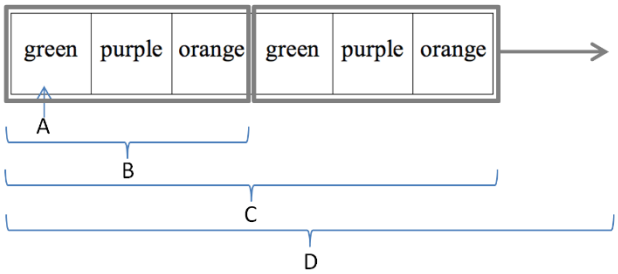

Figure 1: Categories for structuring interpretations

A) **Relevant characteristic**. In the given task, the relevant characteristic of the cubes is their color. Thus, in order to interpret the pattern as such, children have to identify the relevant dimension and, consequently, focus on the color of the wooden cubes, but not on their size, their material, or their exact shape.

1 "A bit orange, a bit purple, a bit green." ["Ein bisschen Orange, ein bisschen Lila, ein bisschen Grün."]

2 "Two orange ones, two purple ones, and two green ones." ["Zwei orange, zwei lilane und zwei grüne."]

3 "Green, purple, orange, green, purple, orange." ["Grün, lila, orange, grün, lila, orange."]

Some children classify the cubes by color (1st & 2nd example), and some of them also determine the number of cubes that belong to each of the classes (2nd example). These explanations refer to some regularity concerning the color: the cubes can be sorted according to color, and each color appears twice. Other children assign the color to each of the six cubes (3rd example). All these explanations are similar in identifying the color as the relevant dimension. Thus, they can be regarded as appropriate starting points for thinking and talking about the structure of patterns.

B) **Unit of repeat.** In the given task, the unit of repeat consists of three single cubes in three different colors: green, purple, and orange. Thus, in order to talk about the structure of the pattern, children have to express in some way that three successive cubes constitute a unit.

1 "Green- purple-orange!" ["Grün-lila-orange!"]

2 "(*pointing at a set of three cubes:*) Orange, purple, green." ["Orange, lila, grün."]

Some children name the unit of repeat without talking about its repetition yet. They just name the unit by naming the three different colors (both examples). Some of them also support their verbal explanation by a gesture which makes clear that the three cubes in three colors form a unit (2nd example). Those explanations do not make the repetition relevant yet, but it should be noted that the unit of repeat can actually not be identified without recognizing which part of the pattern repeats. Thus, we can conclude that those children who only name the unit have already structured the pattern in a useful way, but they only explain part of their analysis.

C) **Repetition.** In the given task, the unit of repeat is "green, purple, orange", which is represented twice by the six wooden cubes.

1 "Green, purple, orange, again green, purple, orange." ["Grün, lila, orange, wieder grün, lila, orange."]

2 "Orange, purple, green. Orange, purple, green." ["Orange, lila, grün. Orange, lila, grün."]

Some children name the unit of repeat twice and, thereby, explicitly highlight the repetition: "again" (1st example). Other children indicate the repetition of the unit of repeat just by their prosody (2nd example). Their explanations sound as if they consist of two complete sentences. Both examples show that the explanation of the repetition requires that children explicitly highlight the distinction between the units of repeat: Where does the first representation of the unit end and where does the second begin?

D) **Continuability**. If children have identified the unit of repeat and its repetition, they *can* start continuing the pattern by repeating the unit of repeat. It has to be noted that this aspect of continuability goes beyond the given task. The children were asked to explain what they see – and not what they do not see, but what they can imagine. Nevertheless, some children already touch this aspect of continuability.

1 "Always green, purple, orange." ["Immer grün, lila, orange."]

2 "Green, purple, and orange, and on and on." ["Grün, lila, orange und immer so weiter."]

Those explanations generalize to a certain extent the aspect of repetition: "Always" (1st example), "and on and on" (2nd example). If we have identified the unit of repeat, we can repeat this unit any number of times. There is no need to think about an end. At least by means of our language, we can even think and talk about an infinite pattern that we cannot show with concrete wooden cubes at all.

DISCUSSION

The four categories of structuring interpretations that we differentiate are not mutually exclusive, but rather build on one another. For example, a child who explicitly refers to one of the 'correct' units of repeat has obviously recognized the relevant dimension of the six wooden cubes. Nevertheless, there are utterances that can be assigned to one category, but not to the one(s) before. The following explanation is a case in point: "They alternate." ["Sie wechseln sich ab."] This utterance indicates that something repeats, but without pointing out what actually repeats. Thus, this explanation can be assigned to the category of "repetition", but not to the categories of "relevant characteristic" and "unit of repeat".

Anyhow, this reflection shows that the categories can be used as an analytical framework. Researchers and educators can analyze a child's explanation of a pattern quite quickly and can take into account both sides of the coin: We can see what a child makes already relevant in his or her explanation and which aspects are not explicit parts of his or her interpretation yet. Concerning the language, there are obviously different ways of explaining the same aspect. Concerning the content, some children do not focus on the (structure of the) pattern but count the cubes or give a figural interpretation. Nevertheless, some of these non-structural explanations can be regarded as pattern-related like the wall-example. Other children indeed develop a structuring interpretation of the pattern. As a first step, they identify the relevant dimension of the elements. By classifying the colors, children refer to a pattern as a regularity (a pattern in the pattern). Furthermore, some explanations explicitly take the structure of the pattern into account and either describe part of it ("unit of repeat") or even give a complete explanation (subcategories C & D). Interestingly, the children's explanations show us that there might be an important difference between an *actual* repetition of the unit of repeat, which is represented by the given cubes (subcategory C), and a *possible* repetition that can be imagined even as infinite, but that is not represented in the concrete task at all (subcategory D).

In our opinion, such scientific or pedagogical work with the categories really needs a clear formulation of goals. We have to be aware of the structure of a pattern in order to be able to identify which structural aspects children make already relevant and which not. From a scientific perspective, we need further research on children's explanations of repeating patterns (and of other pattern types, too), but with other concrete tasks. In this way, we will see how far the categories and subcategories presented in this paper are appropriate for other tasks as well and, consequently, how far they could serve as an appropriate basis for developing 'good' activities, tasks, and helpful impulses for children's learning about patterns.

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MULTIPLE ARTIFACTS IN THE MATHEMATICS CLASS: A TENTATIVE DEFINITION OF SEMIOTIC INTERFERENCE

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This work aims at laying the base of a model for studying how the mathematical learning process triggered by the use of an artifact in the classroom can be affected by the use, even in different moments and for different purposes, of another artifact by the same students. Taking a Peircean framework, we define semiotic interference as an enchaining of signs emerging from the contexts of use of different artifacts, referring one to the other. We test our tentative definition on two examples; one is taken from the literature while the other one taken by original data. Finally, we discuss the potential operativeness of this definition in the data analysis and its generative potential in terms of new research problems and questions.

INTRODUCTION

Mathematics has been shaped in its history by the use of several tools; and for many years research in mathematics education investigated the contribution of tools to the process of teaching/learning mathematics (Monaghan, Trouche, & Borwein, 2016). Among the different research approaches, we place ourselves in the stream of research adopting a semiotic perspective in studying the role of artifacts in the mathematics classroom (e.g. Bartolini Bussi & Mariotti, 2008; Presmeg, 2010; Radford et al., 2017). We share the view that mathematics meaning-making develops through a complex interplay of signs. Following Wartofsky (1979), we call *artifact* any object produced by human beings for accomplishing a task: utensils, texts and books, scientific instruments, ICT.

Here, we focus on the issue of the presence of more than one artifact in the same class. The importance of addressing this issue is acknowledged in literature (e.g. Trouche, 2004). However, with few exceptions which we will examine in the next section, how the use of two or more artifacts triggers the process of meaning-making in mathematics remain a rather unexplored issue and it deserves further research.

SYNERGY BETWEEN ARTIFACTS: LITERATURE REVIEW

The theme of the relation between different artifacts within the same class has been addressed by few researchers. We can find an example in the work by Maschietto and Soury-Lavergne (2013): they attentively study the signs produced by students while using a manipulative (the Pascaline) with the aim of including such signs in the design of a software. They call *duo of artifacts* the couple made of a software and a manipulative by which it is inspired.

The possible linkage between an historical text recalling some features of another artifact has also been studied. Maracci and Mariotti (2010) noticed that, after the

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introduction to a Dynamic Geometry Environment (DGE), high school students were able to interpret a text by Euler in which dynamic features of the notion of function appears. Signs related to the software are used by to make sense of the historical text. Similarly, Maffia (2018) describes how a primary teacher proposes to translate into modern language a text by Tartaglia about the so-called "surface-numbers". The students already worked with rectangular slips of paper as representation of multiplications. Signs referring to the rectangular slips of paper are used to interpret the text and then to construct a definition of surface-numbers.

Faggiano, Montone and Mariotti (2018) design a complete teaching sequence aimed at introducing axial symmetry by using a manipulative and a DGE. The tasks are designed to allow student to reflect on the use of one artifact while using the other. This potentiality is actually expressed. Indeed, the authors observe that the functional dependence between the points involved in the geometric transformation is constructed via a semiotic chain involving signs related to both artifacts. This phenomenon is defined by Faggiano and colleagues as *synergy* between artifacts.

These studies share both a semiotic perspective on mathematics teaching/learning and the role of artifacts, and a commitment to design contexts of use of different artifacts with the aim of exploiting possible connections between them.

THE CONSTRUCT OF SEMIOTIC INTERFERENCE

As a first step in our research, we try to define a model for studying how the mathematical learning process triggered by the use of an artifact in the classroom can be affected by the use of another one, even in different moments and for different purposes. We refer to this phenomenon as *semiotic interference*. We adopt Pierce's framework as described by Presmeg (Presmeg, 2006; Presmeg et al, 2016) and by Sáenz-Ludlow (2006). We consider as *sign* a triplet made of: a *representamen*, the perceivable part of a sign (like a visible gesture, an inscription or an audible word); an *object*, that is something for which the sign stands for; an *interpretant*, that can be defined as "the sign's relationship to its semiotic object with respect to its interpreter or sign agent" (ibidem, p. 188). Through communication (with the self or with others), signs are continuously translated into new signs. In this continuous interpreting process, students construct shared meanings that converge (desirably) towards the mathematical community's interpretation (ibidem). This process is called *semiotic chain* (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008).

We say that we can observe semiotic interference between two artifacts when the interpretant of a sign whose object belong to the context of an artifact is translated by a student in a new sign whose object belong to the context of another artifact. In other words, semiotic interference is *an enchaining of signs emerging from the contexts of use of different artifacts and referring one to the other*.

The diagram in Figure 1 is an elaboration from Sáenz-Ludlow (2006, p. 188), and it depicts (a hyper-simplified version of) the semiotic interplay which may occur when semiotic chains originate from to the use of different artifacts. Signs referring to the

activity with an artifact evolve being translated into new signs referring to the activity with the other one (Ri, Oi and Ii stand for representamen, object and interpretant). The background colour fades as the semiotic enchaining progresses in order to represent that, through this process, signs are supposed to be more and more decontextualized and generalized, to evolve towards target mathematical signs. "From the continuous translation of signs into new signs a concomitant sequence of interpretants evolves giving rise to the emergence of meaning." (ibidem, p.188). The expression "semiotic interference" is used to refer to a semiotic phenomenon; it is not regarded as positive or negative phenomenon by itself.

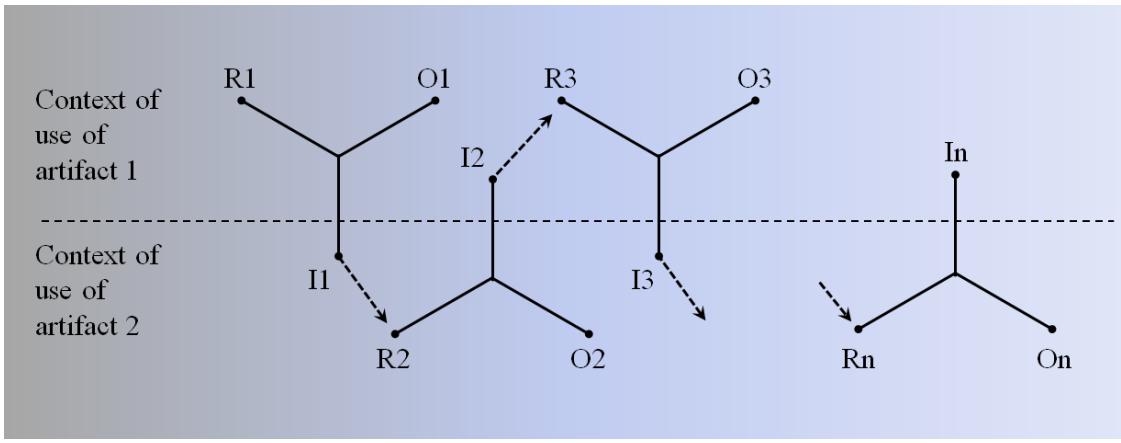

Figure 1: Semiotic interference in a semiotic chain.

All the above-mentioned studies can be interpreted in terms of semiotic interference. In the following sections we show the kind of analysis prompted by our tentative definition on two examples: the former consists in original data from a still in progress study, and the latter is drawn from the mentioned literature. We will eventually discuss the operative potential of this definition in the data analysis and its generative potential in terms of new research problems and questions.

FIRST EXAMPLE: STRAWS AND ABACUS

Emma is a third-grade student. During second grade, her teacher introduced the positional representation of numbers by using straws as proposed in the guidelines of the "Percontare-project" (Baccaglini-Frank, 2015). Through several tasks, Emma and her peers were invited to count straws by creating groups of ten and tying together ten straws of the same group. The tied groups were put in a box and the remaining straws were put in another one. The positional representation of the number was obtained counting the number of groups (tens) and the number of remaining straws (units).

In third grade, Emma have the same mathematics teacher. The teacher wants to introduce hundreds, she decides to abandon straws and to involve the abacus in her classes. The abacus was introduced asking students to use it to represent different two-digits numbers (as in Bartolini Bussi & Mariotti, 2008). After many similar tasks, the teacher asked to her students 'If we add one unit-bead to 99, what does it happen?'. Here the word "unit-bead" represents a sign aimed at connecting the role of beads in the abacus to the mathematical notion of unit. This analysis is in accordance to a case

reported by Bartolini Bussi and Mariotti: students use the words beads and tens-beads to distinguish between units and tens.

Figure 2 shows the text produced by Emma to explain what she obtained after using the abacus to complete the task. Emma writes "Now in the units there are 10 tens-beads and I tie them and I transform them in a ten-bead but now in the tens there are 10 tens, I tie them and I transform them in a hundred-bead".

In Emma's text we can notice a semiotic interference expressed by using the word "tie" ("lego" in Italian). This sign shares the representamen with the sign that was used to refer to the activity of (concretely) tying a group of straws. The object of the sign used in the past was a concrete activity, in this text the object has changed. The sign "tie" cannot refer to the activity of tying beads (never realized by Emma). If we look to this sign with the expert's eye, its object seems to be the operation of changing unit (from units to tens) as it can be performed with the abacus: taking off the beads from the unit stick and putting one bead in the tens stick. As Sáenz-Ludlow (2006) claims, the translation of the previous sign ("tie" as a concrete action) in a new one ("tie" virtually referring to beads) allows to get closer to the mathematical shared culture.

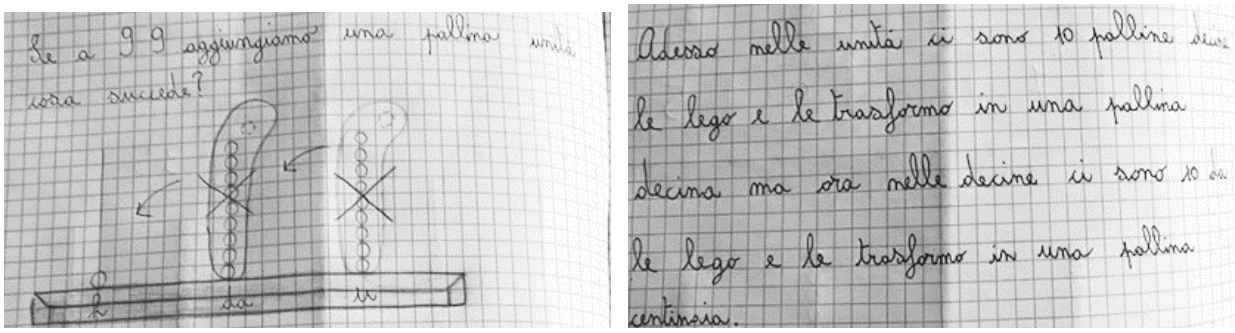

Figure 2: Emma's written text.

The two signs have many common features, but they are not the *same* sign (even if they have the same representamen) because the object (at least) is different. The fact that the two signs share the same representamen, suggests that Emma can see something similar (a pertinence) between the two contexts. The process of interpretation that is realized by the child, leads her to generate, for the new sign, an interpretant based on that activity that she performed with the straws. The new sign "tie" is based, in the choice of the representamen and in the attribution of the new interpretant, on a translation of the interpretant of the sign "tie" that was introduced in the context of straws.

The new sign has its roots in the interpretant of the sign born in the context of straws, but the two signs are different. This interpretation is strengthened by the fact that Emma uses the word "tie" to refer both to the change between units and tens (a change that was actually performed with straws) and to the change between tens and hundreds. In this protocol (fig. 2), the first time in which "tie" is used, it could share an interpretant that is very similar to that of the sign produced in the context of straws. But the second time, the interpretant is more decontextualized because, by using a representamen

related to the context of straws, it refers to the general operation of passing from tens to hundreds, an activity that was never performed with straws. So, we have an enchaining of three signs, all with the same representamen (the word "tie"): the first one has as object the activity of tying straws, the second refers to the activity of changing 10 unit-beads with 1 tens-bead and the third one can be interpreted as replacing 10 tens-beads with one hundred-bead.

We can also notice that the child invents the sign represented by the word "hundred-bead" ("pallina-centinaio" in Italian). It was never used by the teacher or the schoolmates before. Because of the similarity of the representamen, we can conjecture that this sign has an interpretant generated by those of the signs "unit-bead" and "tens-bead" that were shared in the class-group. There is an enchaining of signs that is not an interference: the context is always the same artifact (the abacus).

SECOND EXAMPLE: PAPER, PIN AND DGE

The second example is taken from Faggiano and colleagues (2018). Fourth grade students are involved in a teaching sequence about reflection in the Euclidean plane. First, children use paper to realize the reflection of polygons by folding the paper and using a pin to realize little holes on the vertexes of a given polygon. When the paper is opened, the holes are the vertexes of the image of the polygon through the reflection. Later, students are introduced to the use of the tool "symmetry" of a DGE. The authors' analysis shows a synergy between the artifacts: the cross-reference to them fosters the construction of the mathematical meaning of functional dependency. We agree with the authors, but we look more closely at how this cross-reference took place from a semiotic point of view, according to the lens that we have introduced.

We show an excerpt drawn from table 2 (p. 1177). Pupils must explore and explain the behaviours of different points in the DGE with respect to dragging: A is a free point, point C is obtained by symmetry. Students can observe that both point A and the symmetry axis can be dragged and C moves accordingly. The teacher asks: 'What moves? And what does not move?'. F meets "difficulties in understanding why the two points behave differently when she drags them" (p. 1177). She overcomes such difficulties referring to the previous folding activity.

F. when we drag point A, point C moved, but the line did not! … I cannot explain it … no, but why should it be normal …, but perhaps because C was created by us so … so in the same way as we did with the paper …

With her right thumb up, she gestures behind her.

F. … point A is our black figure, … thanks to the line … since the line moved … first the red and then the blue … so C moved. [...]

In the first part of the excerpt, the objects of the discourse refer to the activity with the DGE: "drag point A", "point C moved", etc. are representamens of signs whose objects are images on the screen and their behaviour. When F has difficulties to explain the behaviour of the points in the DGE, a new sing appears: A is "our black figure". We can notice several interesting features: first, it is unusual, even for older students, to

use the term "figure" to denote a point; then the adjective "our" is used, suggesting something very familiar for the utterer; finally, the figure is called "black" – that is the colour used to draw polygons to be reflected in the paper-folding activities. The phrase "our black figure" expresses the interpretant of the sign "point A" which is translated into the representamen of a new sign referring to the paper-folding activity. Similarly, in the second part of the utterance, the "line moved" is the representamen of a sign referring to the behaviour of a line in the DGE, and it prompts the expression of the interpretant "first the red and then the blue [line]" translated into the representamen of a sign referring to the paper-folding activity (where lines were red and blue). The situation is even more complex and intriguing: once these utterances are produced, the expression "line moved" can become an interpretant of the sign "first the red and then the blue [line]" in the follow-up of the activities, re-enforcing fruitful cross-reference to the use of the artifacts.

Again, we can draw on the work by Sáenz-Ludlow (2006) to say that the translation of the sign originated within the activity of paper-folding in a new sign in the context of DGE, helped in getting closer to the shared mathematical culture; e.g. from a figure that is actually black, to a sign for the independent variable in a reflection.

DISCUSSION

We have defined semiotic interference between two artifacts as the phenomenon that is observed when the interpretant of a sign whose object belong to the context of an artifact is translated in a sign belonging to the context of another one. It is a mean to study learning environments in which we can find several tools, new technologies with more traditional manipulatives (Faggiano et al, 2018; Maschietto, 2018; Maschietto & Sourey-Lavergne, 2013) and texts (Maffia, 2018; Maracci & Mariotti, 2010). Such environments have become more and more common in schools.

The proposed definition allowed us to interpret the relationships between the signs in the excerpts from the above-mentioned literature and other original data. The examples that we presented allow us to discuss our definition in terms of operativity, meaning the possibility of clearly detect and describe the phenomenon while analysing data. The adoption of Peircean framework revealed to be important for distinguishing the signs from their representamen that can appear as identical, even when the object and/or the interpretant are different. The performed analysis leads us to claim that, when the interpretant of the sign is based on a sign coming from the context of a different artifact, it is more than just a "cross-reference" (Faggiano et al., 2018) or an "evoking" (Maffia, 2018; Maracci & Mariotti, 2010) between the artifacts. The observed enchaining suggests that semiotic interference plays a role in the process of teaching/learning intended as a continuous interpreting process (Fig. 1) in which teachers and students approximate more and more the culturally established mathematical signs "by means of successive networks of interpretants individually generated with the collaboration of the classroom participants" (Sáenz-Ludlow, 2006, p.190). In both the examples, the

translation of a sign conducted to the generation of a new sign that we can describe as "closer" to culturally shared mathematical signs.

We do not claim that semiotic interference is always desirable, even if in literature this seems the case. Faggiano et al. (2018) define synergy as the expression of the semiotic potential of two artifacts linked to the same mathematical content. According to our definition of semiotic interference, synergy is the particular case in which semiotic interference leads to a semiotic chain linking the activity with the artifacts to the mathematical signs that are objective of a teaching intervention.

We conjecture that is not always the case. For instance, in Emma's protocol, we found a case of semiotic interference that was not expected by her teacher. The simultaneous or consecutive presence of more than one artifact is not a rare case and so semiotic interference could occur often in our classes. What does it happen when a teacher does not recognize a semiotic interference in the signs that are proposed by the students? What do teachers need to recognize and manage those situations? How can we deal with cases, if any, in which semiotic interference block the decontextualization of signs from the context of the artifact(s)? How can we design tasks in order to exploit the potential of semiotic interference between artifacts?

The fact that many research problems and questions arise after defining this construct suggests that this first step in constructing a theoretical model for studying this kind of learning environments will be productive in developing further research.

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MATHEMATICAL PROBLEM SOLVING TASKS IN THE EYES OF EGYPTIAN TEACHERS

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University of Cambridge

This work aims to explore the contextualisation of the Egyptian national mathematics curriculum across a variety of private schools catering to students of different sociocultural backgrounds. In light of the Goodson Change Model, the study maps out contextual relationships between an imposed teaching culture-represented by the national curriculum-that is rigid and teacher-centred and a schooling culture - represented by the private schools- that in contrast appreciates inquiry and student activities. In light of this dynamic the study adopts teacher focus groups to explore mathematics teachers' perceptions about problem solving tasks. Results show that across varying sociocultural levels and despite being governed by a rigid national curriculum, the schooling culture still dominantly influences teachers' perceptions.

INTRODUCTION

The teaching and learning of mathematical problem solving has received considerable attention in literature. It is commonly associated with equipping students with the capacity to become independent and critical thinkers (Schoenfeld, 1992). Nevertheless, there is no unanimous agreement on how the term 'problem solving' is globally perceived. Depending on the educational context there seem to be variations (Apple, 2004; Ernest, 1995). The educational context in Egypt is particularly interesting as both the political and cultural contexts are quite hierarchical and reserved by nature and would therefore not naturally align with an educational agenda that encourages independence and critical thinking (Naguib, 2006). This work seeks to explore how contextual factors play a role in shaping Egyptian teacher perceptions about mathematical problem solving tasks.

THEORETICAL FRAMEWORK

This work builds on the theoretical grounding of the Goodson Change Model (Goodson, 2000). Along with other researchers (Alexander, 2000; Apple, 2004; Ernest, 1995), Goodson (2000) argues for education to be governed and affected by the context where it is situated. He distinguishes between three contextual agents that influence the formation of an educational identity. First of all, there is the internal agency, represented by, for instance, the local stakeholders and the school culture. This internal context is in turn part of a wider -so called- external context which is often represented by external agents affecting the school such as standardised examinations, imposed curricula or school inspection mechanisms (Alexander, 2000). Tensions usually exist between the two aforementioned contextual spheres. These tensions are balanced,

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according to Goodson (2000) by the personal agency of teachers; i.e. their degree of involvement and investment in either of the sources of contextual influence.

This study focuses on exploring the aforementioned inner and outer spheres of contextual influence in relation to how they affect teacher perceptions about mathematical problem solving tasks. While I acknowledge that the definition of problem solving has received considerable attention in scholarship (Polya, 1945; Schoenfeld, 1992), this work builds specifically on how mathematical tasks are designed to incorporate a problem solving element. In order to study that, I make use of Stein, Smith, Henningsen, & Silver (2000)'s conceptualisation of mathematical tasks. According to the authors (Stein et al., 2000), mathematical tasks can be broadly classified into four categories, namely 'memorisation tasks', 'procedures without connections tasks', 'procedures with connections tasks' and 'doing mathematics tasks'. The latter type of tasks is described as being tasks that build on a student's conceptual understanding of mathematics. The tasks present a problem situation that can only be resolved when students go through an iterative process of knowledge construction, collaboration and progress monitoring. I choose to refer to the authors' characterisation of 'doing mathematics tasks' as problem solving tasks for the scope of my work.

LITERATURE REVIEW

There have been many studies exploring teacher perceptions, how they develop and what factors they are influenced by (Ernest, 1982; Martino & Zan, 2011). Scholarship has argued for a need to situate an exploration of teacher perceptions within the social, cultural and political contexts that control them (Alexander, 2000; Apple, 2004). In his work on identifying various purposes in teaching and learning mathematics, Ernest (1995) places significant emphasis on an existing co-relation between a school's sociocultural standpoint, the purpose for which mathematics education is directed towards and the pedagogical nature of teaching and learning mathematics within that particular context. In order to better comprehend the socio-cultural and political context of a given school, it is vital to consider the historical roots (Apple, 2004; Bourdieu, 1986).

A wider study from which this work has been derived has addressed in great detail the historical mapping of education in modern Egypt and how this has influenced different schooling types across the socio-cultural spectrum of Egyptian schools. While it is out of scope of this work to delve into the details of that historical mapping, I think it is still worthwhile to relate the schooling culture to the political context in which it is situated (Apple, 2004). In his comparative study of four different schooling cultures, Alexander (2000) places specific emphasis on the political decision to adopt a centralised or de-centralised schooling model and argues for that decision to strongly relate to the power tensions between the aforementioned inner and outer circles of contextual influence (Goodson, 2000). With that in mind, it is worth mentioning that since the formation of the republic of Egypt (1957), the national curriculum for mathematics has been standardised. It has also become a centralised system that is

solely governed by the Ministry of Education (MOE) and strictly imposes itself on all forms of national schooling in Egypt. Literature (Naguib, 2006; Sika, 2010) claims that in such a system mathematical knowledge is viewed as a fixed set of rules and the transmission of it is mostly focused on memorisation and on the mastery of procedures.

Foreign (non-governmental) entities that govern private schools have various historical roots which have strongly impacted the formation of the inner contextual circle of influence that, for the scope of this study, is reduced to the school's cultural context. The particular strand of private schooling that this study is focused on has historically been associated with more student centred, inquiry based forms of knowledge creation (Heyworth-Dunne, 1968). Despite being privately owned, they are still subject to the centralised control of the MOE and are still expected to adopt the national curriculum. This is why I refer to these schools as national private schools. Unlike the status of private schools in some other countries, in Egypt national private schools serve a wide range of socio-cultural groups and they constitute a major part of schooling serving around 2 million students annually (CAPMAS, 2017).

Alexander (2000) argues that schools are microcultures in themselves, adapting the imposed curriculum (external context) to the local schooling culture (internal context). The strength of those microcultures is derived from the personal agents that drive them, which according to Goodson (2000) are mainly the teachers. For the case of Egypt, being a collective culture (Al-Omari, 2003), teachers' choices and practices are governed and influenced by each other making it unlikely for a teacher to adopt a certain view if this view is not collectively perceived as best practise. Research conducted in various other educational settings in the Middle East also confirms this viewpoint (Thomas, 2008). This is why, for the scope of this work, I choose to associate the element of personal agency, which is the third pillar of Goodson's (2000) framework, with the collective view of teachers at a given school. Figure 1 maps out the inner and outer circles of contextual influence for the specific case of national private schooling.

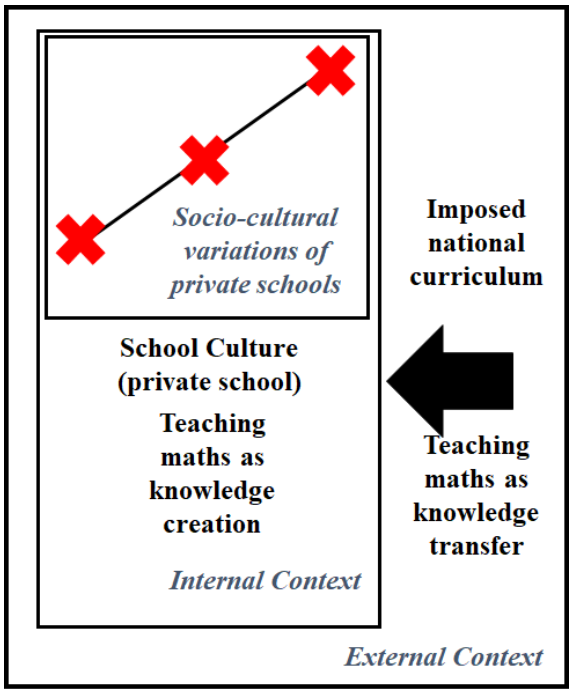

Figure 1. Contextual Mapping

As presented in Figure 1, the inner circle of private schooling seems to be dominated by an externally imposed national curriculum framework. This inner circle itself comprises schools of different socio-cultural standings. In this study, I first intend to explore relationships between inner and outer contextual circles and how those are affecting the element of personal agency, which for the scope of this work is referring to a group of mathematics teachers and their colleagues' perceptions about problem solving. Secondly, in light of Ernest's (1995) association of the nature of teaching and learning mathematics with a school's socio-cultural standpoint, I intend to explore the variation of teacher perceptions about mathematical problem solving across the socio-cultural spectrum of national private schooling. The research questions are as follows:

(1) How do Egyptian mathematics teachers working in the context of a private school that follows the national curriculum characterise problem solving tasks?

(2) How do these perceptions vary across the socio-cultural spectrum of national curriculum private schooling?

METHODOLOGY

This study adopts a multiple case study design (Yin, 2009) as it seeks to explore teacher perceptions about mathematical problem solving across a range of national private schools that vary in terms of socio-cultural standing (Figure 1). The categorisation of schools was based on their geographical location, the school fees and discussions with the leadership team of the partner institution that governs this whole strand of private national schooling. The wider study from which this work is derived has assessed six schools. For the scope of this work, I will report on the research conducted at only three of those schools. These three schools were selected as the teacher sample was quite comparable in terms of group size and group gender variation. Also, the schools have

been classified as high (H), middle (M) and low (L) with regards to their socio-cultural standing. Each case study focuses on one school.

The data collection took the form of teacher focus groups. As previously mentioned, being a collective culture, it was important to capture teacher perceptions in a group setting (Thomas, 2008). The sample at school H comprised a group of 4 female and 2 male teachers. The sample at school M comprised a group of 3 female and 3 male teachers. The sample of group L comprised a group of 4 female and 3 male teachers. All teachers interviewed are familiar with the school culture as they have been working at the school for several years and are themselves Egyptian. Also, all of the teachers are products of the national curriculum schooling system.

The focus group discussion was devised around a Task Sorting Activity. Each group of teachers were provided with a set of mathematical tasks, that have been pre-classified from the literature as fitting in one of the four broad categories of mathematical tasks referred to earlier (Stein et al., 2000). The tasks also vary in form. Some tasks are purely numeric. Others include a narrative or a visual element. Teachers were not made aware of the task pre-classification and were asked to single out the tasks that they would classify as problem solving tasks. Subsequently, the focus group discussion explored the reasons behind the teachers' choices and more generally discussed features that the teachers associate with problem solving tasks. Similarly, teachers were also asked to single out the tasks that would align with the requirements of the national curriculum. This was followed by a more general focus group discussion about national curriculum requirements for mathematical tasks.

The data analysis scheme follows the constant comparative method in coding and comparing all features that teachers associate to problem solving across the case studies (Glaser, 1965). The same method was also used for reporting task features that are associated with national curriculum requirements. The saturated code is represented by a set of features that collectively have been associated by the teachers to problem solving or to national curriculum requirements respectively. For each case, the popularities of task features that teachers associate to problem solving have been recorded with the help of a frequency count analysis. Finally, across the cases, popular codes have been contrasted.

FINDINGS

Table1 maps out the most popular codes that according to teachers characterise national curriculum requirements for mathematical tasks. Table 2 maps out the most popular task features that according to teachers relate to problem solving. Table 2 also contrasts findings across the three cases H, M and L.

A task where the solution procedure is already known
A task that is formulated in a similar way to another task students have worked on before
A task that is solvable only using one approach

A task that relates to one sole mathematical concept
A task that establishes a daily life connection
A task that requires 1-2 steps to solve
A task that addresses the level of the average student
A task where the solution is based on memorisation

Table 1: National Curriculum Task Characteristics

Task Features	H	M	L
A task that triggers students to think	x	x	x
A task that is formulated in a way that is unfamiliar to students	x	x	x
A task that can be solved using multiple approaches	x	x	x
A task that combines different mathematical concepts	x	x	x
A task that establishes a daily life connection	x	x	x
A task that requires multiple steps to solve	x	x	
A task that is used to measure the student's level	x		
A task that requires the student to devise a yet unknown formula in order to solve the problem			x

Table 2: Problem Solving Task Characteristics: Teachers' Perceptions

Tables 1 and 2 show the contrast between characteristic features of national curriculum mathematical tasks and features the teachers think are inherent in problem solving tasks. Table 2 also shows the distribution of teacher problem solving task classifications across the spectrum of socio-cultural levels of schooling. These findings will be further discussed in the next section.

DISCUSSION

When considering teacher responses with regards to their classification of mathematical tasks, in light of the understanding of the inner and outer spheres of influence, it becomes clear that teachers seem to be less affected by national curriculum requirements when describing their own perception of features specific to mathematical problem solving tasks. That being said, it is important to mention that, being guided by the national curriculum, most teachers have never been exposed to any other type of mathematical tasks other than the ones whose features are described in Table 1. One teacher refers to that fact by stating that: "We have no other resources for mathematical tasks. How would we know how and where to find problem solving tasks?". There seems to therefore be minimal teacher exposure to features that according to literature (Schoenfeld, 1992) relate to problem solving and yet Table 2 shows some traces of task features that are very different to the ones that relate to the national curriculum. It would therefore be sensible to conclude that the teachers, as

personal agents, are strongly affected by the school culture, or in other words the internal context, when it comes to forming their perceptions about mathematical tasks. Future work may seek to explore which of the contextual spheres has superiority in affecting curriculum enactment decisions.

Across the socio-cultural levels of schools there seems to be generally agreement about the perceived characteristics of problem solving tasks. One exception is the feature which relates to the knowledge creation process, referred to only in case H (A task that requires the student to devise a yet unknown formula in order to solve the problem). The extended study from which this work is derived shows how other national private schools that have been classified in the same socio-cultural cluster (H) seem not to make reference to that particular task feature. Their input seems to generally align with that of cases L and M. It seems therefore that the school culture is a dominant influence on teacher beliefs irrespective of their socio-cultural standpoint.

Returning to the research questions, it can be concluded that, firstly the inner contextual circle has indeed greater impact on teachers' personal agency as opposed to the outer circle of influence. Secondly, a similar private schooling microculture seems to extend across the three case studies, suggesting that most teachers have similar thoughts about mathematical problem solving tasks irrespective of where they stand on the socio-cultural ladder.

Findings of this study might be of interest to policy makers. Current research into reforming the national mathematics curriculum might want to consider de-centralising the national curriculum and contextualising it based on the cultural footprint of each schooling context.

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SOCIAL COGNITIVE THEORY AS AN INTEGRATED THEORY OF MATHEMATICS TEACHERS' PROFESSIONAL LEARNING

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Mathematics teachers' professional learning has frequently been treated as either predominantly participationist or acquisitionist – as social learning in becoming part of a community of practice or cognitive learning in the acquisition of knowledge and beliefs. Our experience within mathematics education suggests that professional learning is neither purely acquisitionist nor participatory and that a theoretically integrated treatment of both, together with affective processes of learning, helps shed more light on the complexities of professional learning. In this theoretical paper we firstly show how Social Cognitive Theory and its component theory of self-efficacy attend to this integration; and, secondly, illustrate this integration with an example from our research into preservice mathematics teachers' training.

INTRODUCTION

In the past, professional learning theory has often been characterised as either *acquisitionist* or *participationist* (Borko, 2004; Sfard, 1998). That is, theory tended to foreground the individual, cognitive, *acquisitionist* or the social, community, *participatory* aspects of professional learning. What we do in this paper is consider a particular learning theory, *Social Cognitive Theory* (SCT) (Bandura, 1986) and its component self-efficacy theory (Bandura, 1997), which helped us address the issue of theory dichotomy. This was prompted by our work and research in mathematics teachers professional learning (TPL) (e.g. Watson, 2013; Watson & Marschall, in press). Our interest in SCT resulted from our observations of and empirical work with professional learning rather than from a review of theory. Indeed, in this paper, we neither aim to critique other theories which, in the last two decades, have tried to attend to a comparison and an integration of the two dichotomised aspects of learning, nor do we suggest a grand theory of TPL. Our theoretical perspective comes from practice and is informed by social psychological theory.

Our interest in professional learning began from the perspective of practice and through our work in mathematics teacher education. In our work we noticed the importance of teachers' affective experiences and teachers' agency in their professional learning. We came to recognise the level of emotional demand on trainee teachers during their training (Watson & Marschall, In press) and we observed the extent to which practising teachers would implement and sustain ambitious teaching approaches based on their level of confidence and emotion (Watson, 2014). For this reason, we considered professional learning in terms of the development of teacher self-efficacy (Bandura, 1997), with its overarching theory, SCT (Bandura, 1977, 1986, 1997). As we interpreted SCT in the context of mathematics TPL, we found that it provided a useful

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theoretical integration of *acquisition* and *participatory* metaphors of learning. It is an account of this theoretical investigation that we present here.

In this paper, we explain how SCT (Bandura, 1986) and its component self-efficacy theory (Bandura, 1997) integrate the dichotomised cognitive or *acquisitionist* and social or *participationist* treatment of theory of mathematics TPL (see, for example, Borko, 2004; Sfard, 1998) within one theoretical approach. While we have limited space to offer a full account of SCT and self-efficacy theory, our contribution here is in explaining, at a theoretical level, how this integration works and illustrating this with an example from preservice secondary mathematics teacher training. Overall, we suggest that this theoretical integration, that not only accounts for *acquisition* and *participation*, but also attends to affective aspects and teacher agency, could improve understanding of TPL.

We begin with a brief description of the *acquisition* and *participation* learning metaphors (discussed by Sfard, 1998). We then introduce a triadic causality framework, which we use to explain the dichotomised nature of the two approaches. Using the triadic framework, we then describe SCT and self-efficacy theory and explain how they integrate the two learning metaphors. Finally, we illustrate this integration with an example from our research into mathematics initial teacher education (ITE).

THE ACQUISTION AND PARTICIPATION METAPHORS FOR LEARNING

The *acquisition* metaphor represents the accumulation of knowledge; where concepts, ideas, meaning, facts and representations are internalised, appropriated or transmitted through cognitive and psychological processes (Sfard, 1998). Here, knowledge acquisition is seen as transfer, "from a social to an individual plane and internalised by the student" (Sfard, 1998, p. 6), but is considered primarily as an individual activity (Sfard, 2006). The *acquisition* metaphor implies learning programmes that develop or change beliefs, or that develop teachers' knowledge.

The *participation* metaphor suggests social bonds and the development of identity in becoming part of a community of practice (Sfard, 1998). "*Participationists* conceptualise developmental transformations as changes in what and how people are doing and claim that patterned collective activities are developmentally prior to those of the individual" (Sfard, 2007, p. 568). Research within this perspective focuses on membership in communities of practice or discourse, which are governed by their particular rules and norms (e.g. Sfard, 2007). In professional learning this suggests a focus on professional community, where learning involves changing the discourse an individual participates in (Sfard, 2007).

In reality, however, learning theories are neither purely one or the other but rather "acquisition-oriented or participation-oriented" (Sfard, 1998, p. 7), and any approach to understanding professional learning should attend to both metaphors (Sfard, 1998). In line with this view, there have been many attempts at utilising both metaphors, although executed mainly by employing pluralistic methodological research

approaches (e.g. Borko, 2004; Kaiser, Blömeke, König, Busse et al., 2017). For example, in the context of teacher professional development, Borko (2004) proposed a *situative* approach, which makes use of "multiple conceptual frameworks and multiple units of analysis" (p. 4); using psychological frameworks when focusing on individuals and their knowledge evolution and development of understanding, and sociological frameworks when focusing on group of students and learning through participatory activities. What makes SCT different is that, instead of employing integration of the two metaphors on a methodological level, it presents a *theoretically integrated* approach.

CONCEPTUAL FRAMEWORK – THE BPE MODEL

In order to illustrate the *acquisitionist-participationist* dichotomy we use a triadic causality BPE model, as developed by Bandura (1977, 1997). In the model, behaviour and actions are denoted as 'B'; a person's individual thinking, affective states and cognition as 'P' and the environment, social context or culture as 'E'. Although we acknowledge that *acquisition* or *participation* are not pure phenomena (there is a degree of overlap between them), for the purpose of our argument we present the most idealised forms of each metaphor i.e. almost purely *acquisitionist* or *participatory*.

In the *acquisition* metaphor (illustrated in Figure 1a) learning process involves the acquisition of knowledge and beliefs (P) about teaching. These are developed in response to the social and cultural context (E): E →P. Actions and behaviours, practice and pedagogy (B) are a consequence of the teachers' knowledge and beliefs: P→B. In the *participation* metaphor (illustrated in Figure 1b) the main influence is of the social context (E) on behaviour (B): E→B.

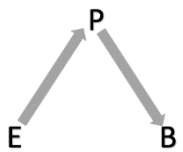

Figure 1a: Acquisition metaphor

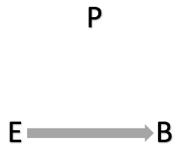

Figure 1b: Participation metaphor

This simplified visualisation illustrates the underlying dichotomy in the two metaphors of learning. We will use this model to show how SCT and self-efficacy not only integrate the two metaphors but also pay attention to affect and offer a sophisticated account of individual agency.

SOCIAL COGNITIVE THEORY

SCT integrates both cognitive and social aspects of learning presenting a reciprocal triadic relationship between behaviour (B), the individual (P) and the environment (E) (illustrated Figure 2), which Bandura (1977) explains as follows:

> Personal and environmental factors do not function as independent determinants, rather they determine each other. Nor can "persons" be considered causes independent of their behavior. It is largely through their actions that people produce the environmental conditions that affect their behavior in a reciprocal fashion. The experiences generated by

> behavior also partly determine what a person becomes and can do which in turn, affects subsequent behavior (p. 9).

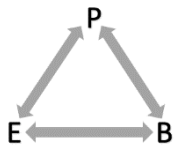

Figure 2: Reciprocal causation (Bandura, 1977, p. 10).

This presents a "transactional view of self and society, [where] internal personal factors in the form of cognitive, affective and biological events; behaviour; and environmental events all operate as interacting determinants that influence one another bidirectionally" (Bandura, 1977, p. 6). As well as integrating the social and cognitive aspects of learning, SCT incorporates affective aspects and provides a sophisticated conceptualisation of individual agency within a collective social context. Although *agency* is often treated as a property, capability or characteristic of the individual (Biesta, Priestley, & Robinson, 2015), SCT treats it as an emergent and evolving state, "achieved by individuals through the interplay of personal capacities and the resources, affordances and constraints of the environment by means of which individuals act" (Priestley, Biesta, & Robinson, 2015, p. 19). In other words, reciprocal causation provides opportunities for and simultaneously sets limits on individual self-direction for action (Bandura, 1997). This action, however, is guided by the individual's agency and governed by their self-efficacy.

Self-efficacy is the belief an individual has in their capability to be successful in a domain (Bandura, 1997). It was originally presented as a component of SCT, though its popularity as an operationalisation of SCT, it has latterly become a *guise* of SCT. It carries within it the triadic reciprocal causation of SCT, the transactional nature of agency, and reflects the affective aspects of confidence and motivation. It acts on behaviour,

> by impacting goals, outcome expectations, affective states, and perceptions of socio-structural impediments and opportunities (Bandura, 1997). It is a belief about one's capabilities, developed through a self-assessment and assessment of the context, based on which the individual mentally models a course of action. Individuals who feel that they will be successful on a given task are more likely to be so because they adopt challenging goals, try harder to achieve them, persist despite setbacks, and develop coping mechanisms for managing their emotional states (Bruce & Ross, 2008, p. 347).

In learning to teach mathematics, the trainee teacher has a degree of agency as they construct and implement mental models of elements of practice and pedagogy that they have learnt through observing experienced teachers. They become self-efficacious in implementing culturally embedded practices, i.e. traditional teacher-centred approaches. The more self-efficacious the teachers become, the more likely they are to deviate from culturally accepted norms or use innovative approaches.

How self-efficacy is acquired has important implications for understanding and theorising teachers' professional learning and development. Bandura proposes four sources of self-efficacy: direct experience, vicarious experience, verbal persuasion, and physiological and affective states (Bandura, 1997).

Direct experience is the strongest source of self-efficacy, which develops self-efficacy through relating outcomes to the approach employed by the individual (Bandura, 1997). This implies the need for both cognitive processing and individual reflection during the learning process. As the individual develops mastery, the adopted approaches become less conscious and increasingly automated, routinised and based on sophisticated heuristics (Bandura, 1997). This goes some way in explaining why traditional teacher-centred practices are so common in mathematics classrooms and why they can be difficult to change (Watson, 2014) as well as why, once established, self-efficacy tends to remain stable over time (Bandura, 1997).

A second major source of self-efficacy is through vicarious observational processes. Individuals save considerable time and energy observing self-efficacious teachers and accumulating models of practice that they can use, adapt and absorb into their own developing practice. Lortie (2002) characterises teaching as an apprenticeship of observation, in which cultural patterns within practice and pedagogy are transmitted through generations. A third source of self-efficacy is social support and verbal persuasion. As teachers develop, encouragement from others helps them overcome dips in confidence and become more expansive in their teaching. In ITE these are offered by experienced mentors and colleagues, both in school and educational institutions. The fourth factor, physiological and affective states, affect the individual's level of self-efficacy in two ways. On the one hand, stress and tiredness undermine a belief in one's capabilities. On the other hand, excitement and arousal can increase self-efficacy (Bandura, 1997).

The four sources of self-efficacy allow us to see the role of *acquisition, participation,* agency and affect in TPL. In England, university-based ITE involves the development of practice within the school context by collaborating and learning from experienced colleagues. These are complemented by faculty-based sessions, where trainee teachers are given theoretical tools to help them process, plan and reflect on their experiences. All of those create opportunities for preservice teachers to acquire knowledge, learn from others, participate in professional activities as well as reflect on practice and guide their own courses of action. The following example from our own research provides an account of this interplay.

ILLUSTRATION OF SCT IN INITIAL TEACHER EDUCATION

Alison is a trainee secondary mathematics teacher, whose development of self-efficacy is investigated over the course of a year-long ITE programme (see Watson & Marschall, in press). Alison's professional learning involves direct and vicarious experiences. As she begins to teach, Alison attends to planning, initially parts of and finally full, lessons. In terms of SCT, during this process, she cognitively constructs *a*

mental model of actions in the lesson; a script that is to guide her action. The source of the models comes from observation of practice, even though, as SCT suggests, Alison's actions are not mere imitations of what she has observed. During the process, Alison expresses a degree of agency; she constantly assesses her capabilities and devises plans of action she believes she can execute effectively.

Affective aspects also have a role in Alison's mental modelling as well as in her performance. She herself often describes how big a part they play in her lesson planning and teaching, particularly in the early stages of her training. This is reflected in her account of her overall experience, which she summarises by saying: "…there were times when I felt like I wasn't really progressing, to go from being terrified to do a 10-minute starter, to actually enjoying teaching them for two hours is quite a big improvement!" (Watson & Marschall, in press, p. 17). Throughout the year, emotion and affect play important part in Alison's training. In our experience, this level of emotional demand is not uncommon for the majority of trainees.

While Alison *acquires* knowledge through *participation* in a community of practice, there is, as SCT suggests, a reciprocity, or transaction, between the *cognitive* acquisition and the *social* participation. The development of Alison's abstract knowledge does not represent a mere *acquisition* of concepts but it is also practical and it relates to Alison's sense of self: affective and agentic. Alison's pedagogic knowledge and her actions in the classroom are deeply connected to her affective experiences, as illustrated by the following example:

> [The lesson was] about using rectangles and trapezia to estimate the areas under curves, and I think it was the first ever thing I've taught where the students were more (rather than less) happy with it than I thought they'd be … and it really boosted my confidence.

In short, Alison's self-efficacy reflects knowledge in action, affective aspects and agency, acquired within a community of practice.

IMPLICATIONS AND CONCLUDING REMARKS

SCT and self-efficacy integrate the *acquisitionist* and *participationist* metaphors of professional learning. The nature of the integration in SCT is transactional, the teacher negotiates their institutional and social context, with consideration of their personal resources: their knowledge, capabilities and dispositions. SCT takes into account affective states of an individual, where environmental and contextual effects together with physiological and emotional conditions can undermine teachers' level of self-efficacy. This is especially important for trainee teachers, who do not yet have enough experience to fall back on; in such situations mentoring and social support become critical in supporting them (Bandura, 1997). The transactional nature of self-efficacy allows for individual teacher agency: existing culturally embedded practices are sustained, but teachers can adapt the approach to reflect their own personality.

SCT provides a *theoretical* integration of dichotomised learning metaphors (as opposed to a methodological integration employed by other approaches). The key theoretical link involves affect and agency. SCT does not consider a teacher learning

as acquiring knowledge and beliefs that are largely abstract or that *just* reflecting the behaviours developed through participation. It simultaneously considers the role of affect and agency in guiding and governing this learning.

Our current account has gone so far as to show *how* the theoretical integration takes place within SCT. We tentatively suggest, however, that, due to this integration, the application of SCT and self-efficacy to mathematics teachers' professional learning could contribute to improving the effectiveness of teacher training programmes.

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USING A COMMUNITY OF PRACTICE TO SUPPORT MATHEMATICS TEACHER EDUCATORS' GROWTH AND DEVELOPMENT

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Mathematics teacher educators (MTEs) play a significant role in helping prospective teachers to develop the mathematical knowledge they need for teaching. However, research has shown that the majority of MTEs receive little to no training or support to do this important work (Masingila, Olanoff & Kwaka, 2012). To address this need, we formed a Community of Practice (CoP) where two novice MTEs worked with an experienced MTE to develop their mathematical knowledge for teaching teachers (MKTT) and improve their teaching of mathematics content courses for prospective elementary teachers. In this paper, we discuss how we formed our CoP, the MKTT that we gained from working with the CoP, and what we learned from participating in the study, and how we mutually benefited from the experience in different ways.

BACKGROUND

Mathematics teacher educators (MTEs) play a significant role in helping prospective teachers (PTs) to develop the mathematical knowledge they need for teaching. However, research has shown that the majority of MTEs in the United States have little experience teaching students at the level of mathematics that they are preparing PTs to teach (e.g., primary school), and that they receive little to no training or support either in their preparation programs or in their jobs (Masingila, Olanoff, & Kwaka, 2012). In order to attempt to improve our teaching of mathematics content courses for prospective elementary teachers, two novice MTEs (Nov1 and Nov2) worked with an experienced MTE (Expt) as part of a mentored teaching experience at Syracuse University, where Nov1 and Nov2 were graduate students and Expt was a faculty member. We chose to study our teaching critically and form a Community of Practice (CoP) to support one another in improving our teaching and developing Mathematical Knowledge for Teaching Teachers (MKTT).

Wenger, McDermott, and Snyder (2002) defined CoPs as "groups of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (p. 7). Through research and participating in a CoP, members develop and articulate new knowledge in response to questions and problems they have about their practice. A CoP offers a platform for its members to engage in negotiating shared understandings, learning, meaning making, and identity. Wenger (1998) identified three dimensions of the community which has practice as the source of coherence: (a) CoP members interact with one another, and determine norms and relationships through *mutual engagement*, (b) CoP members are held together by their understanding of a sense of

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joint enterprise, and (c) CoP members seek to produce, over time, a *shared repertoire* of communal resources (e.g., language, routines, artifacts, and stories). Forming a CoP allowed us to work together to gain knowledge about and improve our teaching of mathematics content courses for PTs.

The basis for our mutual engagement came in the form of Nov1 and Nov2 needing to have a mentored teaching experience by Expt. We were each invested in the project, and our joint enterprise of improving our MKTT, as we all were interested in improving our teaching of mathematics content courses for prospective teachers. Through our work, we developed a shared repertoire of lesson plans, reflective memos, meetings, and ways of interacting with one another.

THEORETICAL FRAMING

Our CoP revolved around reflecting on the process of teaching mathematics content courses for PTs, part of our shared repertoire. We guided our reflections around research on reflection and inquiry, as well as research on MKT. Below; we briefly review the literature that guided our work.

Reflection and Inquiry

Chapman (2008) states that reflection on teaching practice is an inherent part of the work of teacher educators, with this reflection involving: "examining, framing, and attempting to solve the dilemmas of classroom practice; and being aware of and questioning the assumptions and values [they bring] to teaching" (p. 121). We chose to situate our reflection using a position of *inquiry as stance* (Cochran-Smith & Lytle 1999). Teacher educators in an inquiry community "generate local knowledge, envision and theorize their practice, and interpret and interrogate the theory and research of others" (p. 289); they produce the knowledge they need "to teach well … when they treat their work as a site for intentional investigation at the same time they treat the knowledge and theory produced by others as generative material for interrogation and interpretation" (Cochran-Smith, 2003, p. 16). Inquiry as stance is critical in examining both one's own work and the work of others. We used an inquiry as stance framing for examining our teaching practice and the practice of the members of our CoP; inquiry as stance provided a framing for our joint enterprise (Wenger, 1998) of developing MKTT.

In studying our practice through the CoP, we took on the role of "reflective practitioners" (Schön, 1983). Schön defines two ways to reflect on one's practice: *reflection on action* and *reflection in action*. The former refers to ways in which members of a community of practice (in our case, mathematics teacher educators) reflect on past experiences with the intention of refining their work to achieve their instructional goals, while the latter refers to "thinking on your feet" (p. 54), the reflection that occurs while one is in the process of teaching. Because part of our CoP focused on peer observation, we were able to add another type of reflection to our process: *reflection on the actions of others*. We observed each other's teaching both directly and through reading their memos, and we were able to think deeply about

choices that they and we made and reflect on these decisions. We were bound together in mutual engagement (Wenger, 1998) through our reflective practitioner roles, and our reflections became part of our shared repertoire.

MATHEMATICAL KNOWLEDGE FOR TEACHING

Research on teachers' knowledge has flourished following Shulman's (1986) presidential address at the 1985 American Educational Research Association's annual meeting, where he introduced the idea of pedagogical content knowledge (PCK). As a result of Shulman's speech and the research that followed, Ball and her colleagues introduced the term *mathematical knowledge for teaching* (MKT) (e.g., Ball & Bass, 2002), which describes the *mathematical* knowledge required by the work of teaching.

Ball and her colleagues (Ball, Thames & Phelps, 2008) developed a framework for MKT in which they sought to expand Shulman's descriptions of content knowledge and pedagogical content knowledge to include sub-categories of the mathematical knowledge that teachers need to know. They broke mathematical content knowledge into three sub-categories: *common content knowledge (CCK), specialized content knowledge (SCK),* and *horizon content knowledge*. The first of these categories, CCK, refers to the mathematical knowledge that everyone needs to know, whereas SCK refers to the mathematical knowledge that is unique to the work of teachers. The researchers similarly broke pedagogical content knowledge into *knowledge of content and students (KCS), knowledge of content and teaching (KCT),* and *knowledge of the curriculum*.

Significantly less research has looked at the mathematical knowledge needed by MTEs to help PTs develop MKT (Castro Superfine & Li, 2014). A number of researchers have determined that there is a category of knowledge needed by teacher educators that goes beyond the knowledge needed by their students (PTs) (e.g., Jaworski, 2008). Additionally, many frameworks for teacher educator knowledge provide a structure similar to a framework for teacher knowledge (Olanoff, Welder, Prisad & Castro Superfine, 2018). In looking at the parallels in the ways that MTEs learn and develop knowledge, versus the ways that mathematics teachers learn and develop knowledge, we hypothesized that Mathematical Knowledge for Teaching Teachers (MKTT) would have similar knowledge categories to the MKT framework discussed above (Ball, Thames, & Phelps 2008).

The initial joint enterprise of our CoP involved developing our MKTT. However, as we progressed through the semester, we added the additional enterprise of studying portions of our shared repertoire in order to better understand some of the aspects of MKTT we developed and how our CoP helped us in this enterprise. Thus, our CoP became dual purpose: to develop our MKTT and also to research what we had learned.

RESEARCH METHODS

We formed our CoP when we each taught a section of the same two mathematics content courses for PTs during two consecutive semesters. During the first semester,

the course content focused on whole number and operations, number theory, probability and statistics, and functions. The content for the second semester course focused on rational numbers, geometry, and measurement. These courses were taught using the same textbook and general lesson plans, and with an emphasis on PTs learning mathematics via cooperative problem solving. PTs worked together in small groups to solve problems with the goal of developing deeper understandings of the K-8 level mathematics and their own MKT. The role of the instructors of the courses was to help facilitate the PTs' problem solving and knowledge development. For two semesters, Nov1 and Nov2 observed Expt teach her class before teaching their own classes later in the day. Expt also observed Nov1 and Nov2's classes several times each semester. In order for us to reflect on our teaching and observations, and to have a record of these reflections, each of us wrote a memo after each lesson session. We met briefly before each lesson and weekly after we had taught all of the lessons for the week to discuss our observations, what was going well, what was not going well, and where we thought we needed to go next. We audio-recorded the weekly meetings and transcribed them to add to our data set.

We used both ongoing and retrospective analyses of the data (daily memos and meeting transcripts). Ongoing analysis occurred during the two semesters and was the basis for continued reflection on our teaching and learning about our teaching, the testing of emerging hypotheses, and the strategies for promoting further development of the prospective teachers' mathematical understandings. Retrospective analysis involved examining the larger corpus of data through a carefully structured review of all the relevant data.

RESULTS

We found evidence that working in our CoP had helped us to develop our MKTT (the goal of our joint enterprise) in a number of ways, through our meetings and observations, and our reflections, memos, tasks, and revised lesson plans. In this paper, we share one finding, which was that as teacher educators we developed an enhanced understanding of the mathematics content of elementary mathematics (MTE CCK and SCK).

Mathematics Content Knowledge

As former mathematics majors, we entered into teaching these courses feeling confident with the elementary mathematics material. However, we soon realized that much of this material was more complex than we had originally thought. Ma (1999) described elementary mathematics as fundamental. She wrote:

> Fundamental mathematics is elementary, foundational, and primary. It is elementary because it is at the beginning of mathematics learning. It is primary because it contains the rudiments of more advanced mathematical concepts. It is foundational because it provides a foundation for students' further mathematics learning. (p. 124)

As teacher educators, we had not looked at some of the mathematics since we were in elementary school ourselves, and at that time, we were not studying it with the depth, breadth, and thoroughness that we would need in order to teach it, and to help prospective teachers develop the deep understandings that they would also need in their teaching. The following example on counting methods illustrates one instance where our mutual engagement in our CoP helped us develop mathematics content knowledge.

Thinking differently about counting methods

One activity in our curriculum involved using combinations and permutations to determine the number of possible outcomes in a given situation. Having taught this activity before, we discussed that our students did not seem to develop a deep understanding of permutations and combinations. Instead, they learned the words and where the keys for permutations and combinations were on their calculators, and then they punched the appropriate keys and got an answer. Alternatively, if we did not allow them to use their calculators, they plugged numbers into a formula, but did not understand where the formula came from and why it was appropriate to use in a given situation. In discussing our plan for this lesson in our weekly meeting, we decided that rather than define permutations and combinations during class that we would try to get our students to think about the situations in the problems and use the fundamental counting principle on every problem. If the order did not matter in a certain situation, they would need to divide their answer by the total number of orders. In the memo that follows, Nov1 describes how she introduced the problem: *At State University, a group of seven students wishes to select a committee of four to negotiate student activity fees with the Dean of Students. How many committees can be selected from the group of seven?*

In the past, we have given notes for this activity, but ... this time, I didn't even really define permutation and combination in terms of their formulas. I talked about the first question [7C4] as we have 4 slots, so we have $7 \times 6 \times 5 \times 4$ ways of picking people to fill them, but then we have to divide by the number of ways of arranging 4 people. I'm not sure how it will work, but I liked it better than them just punching in 7C4 on their calculators. (Nov1, Memo 10/31/07)

At first, this method of talking about combinations seemed to be much better than what we had done in the past. The students really seemed to be thinking about whether or not the order mattered in a problem, and when they needed to divide by the number of orders to figure out the answer. However, when Nov1 was asked a question about a homework problem, *An experiment consists of tossing a coin 8 times and observing the sequence of heads and tails. How many different outcomes have exactly 3 heads?* she found it difficult to explain the problem without using the words choose or combination. Here is an excerpt from her memo following this discussion:

In my head, I knew the answer was just 8 choose 3. You have 8 coins and you want to choose 3 of them to be heads. This makes total sense to me and I usually teach it that way. However, since I am not doing "choose", I had a lot of trouble explaining it...I

still have trouble working out why you would draw three slots instead of 8. It sort of makes sense, but I can't really explain it. I just don't know what the best way is. (Nov1, Memo 11/05/07).

Through talking with members of the CoP and reflecting through her memo (part of our shared repertoire), Nov1 was able to eventually make sense of how to explain the problem both to herself and to her students. Had she not had the benefit of the CoP, it would have been easy for her to go back to encouraging the PTs to use formulas. Nov2 summarized this idea in a memo:

> The temptation here is very high for both the instructor and the students to fall into using the formulae for factorial, combination and permutation in which case the students may not get to critically think about what they are doing. The instructor needs to consciously work on avoiding getting trapped into that so that he/she can lead the students into actually thinking about why they are doing what they are doing. (Nov2, Memo, 10/31/07).

Expt later mentioned the same thing in a CoP meeting where we were discussing how the CoP had been beneficial. "I think I hadn't thought about before, about how to teach, some of these ideas without referring to combinations and permutations" (Expt, Meeting Notes, 1/31/08).

It can be difficult to teach in a way that supports students in developing deep, connected understandings. We found that elementary mathematics has complexities of which we were unaware before we began teaching mathematics content courses for PTs and reflecting deeply on our teaching through our CoP. Like the majority of US teacher educators teaching these classes, we did not have experience teaching elementary school ourselves (Masingila, Olanoff, & Kwaka, 2012). Through our mutual engagement and shared repertoire (e.g., reflections, memos, tasks, lesson plans), we came to realize the importance of looking deeply at the underlying mathematics behind the representations, algorithms, and definitions that we use, and we believe that understanding elementary mathematics with strong levels of depth, breadth, and thoroughness is an essential component of MKTT. Being able to discuss and work through the challenging content with the CoP allowed us to expand our own MKTT by learning new mathematics ourselves and discussing ways to encourage PTs to work with the content with high levels of depth, breadth, and thoroughness.

CONCLUSIONS

Participating in our CoP was extremely worthwhile for our development as MTEs. While the work we did took a lot of time and effort, we believe that the benefits and opportunities for growth, both individually and as a group, outweigh the challenges. While it is possible for MTEs to reflect on their teaching alone by writing memos and examining student work, they would miss out on the benefits of the CoP. Without forming a CoP, there is no opportunity to reflect on the actions of others, to receive feedback on your teaching, or to see other ways of doing things. Additionally, participation in a group provides accountability for its members.

We call on the Mathematics Education Community to encourage its members to form CoPs by providing a Special Interest Group or forum for interested people to meet with others who share this interest. Improving the teaching of mathematics for future teachers will benefit the community and the population at large.

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SPEAKING ABOUT THE MATHEMATICS CLASSROOM: A COMPARISON OF THE PROFESSIONAL LEXICONS OF TEACHERS IN AUSTRALIA AND THE CZECH REPUBLIC

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The International Classroom Lexicon Project documented the professional, pedagogical vocabulary employed by teachers worldwide to describe the phenomena of the middle school mathematics classroom. This paper reports aspects of the professional lexicons in use by middle school mathematics teachers in Australia and the Czech Republic. A comparison of both content and structure revealed differences in: phenomena named (and absent); local systems employed for the organisation of terms by each teaching community; and, the distribution of teacher and student agency across the set of terms. Access to lexicons of other countries offers possibilities for practice that may expand the professional repertoire of mathematics teachers around the world and usefully inform teacher reflection on their practice.

INTRODUCTION

Researchers have been documenting the professional language of middle school mathematics teachers as part of The International Classroom Lexicon Project. This project set out to identify the vocabulary (lexical terms accompanied by descriptions and examples) that teachers use to name the phenomena in the classroom in 10 communities worldwide: Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan, the USA, and more recently, Korea (Mesiti & Clarke, 2017). In this paper we compare and contrast the lexicons of the Australian and Czech Republic middle school mathematics teachers to indicate the variation possible within two Western teaching communities with different pedagogical traditions and different lexico-grammatical forms and features. The documented differences can be attributed to these origins.

THE TEACHER'S PROFESSIONAL LEXICON

Researchers have previously argued that the [English] language of practice in teaching seems particularly underdeveloped (Lortie, 1975; Lampert, 2000; Grossman, Compton, Igra, Ronfeldt, Shahan & Williamson, 2009) and have advocated for a "framework for teaching with well-defined common terms for describing and analyzing teaching" (Grossman & McDonald, 2008 p. 187). More recently, McDonald and her colleagues have indicated that "the lack of a common language limits our ability to engage in research aimed at understanding the impact of core practices" (McDonald, Kazemi, & Kavanagh, 2013, p. 381). Lampert (2000) has concluded that

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the lack of opportunities to work collaboratively with peers on the problems of practice result in "a language of practice [that] remains flat or nonexistent" (p. 90) whilst Connell (2009) has similarly observed that the organisational culture of the teaching profession does not appear to support the informal "processes by which practical know-how is passed to new teachers in on-the-job learning" (p. 223). A documented lexicon would help support the initiation and purposeful fostering of professional discussions, similar to the practice of Lesson Study in Japan (Lewis & Tsuchida, 1998), that lack such a formal tradition.

The process of creation and validation of the Czech lexicon confirmed the difficulty for Czech teachers to speak about lessons in general terms. Czech teachers tended to pass many evaluating judgements and discuss the conception of the lesson and assess its quality rather than describe the classroom interactions in general terms. They also tended to use everyday language rather than pedagogical terminology. It corresponds with the traditions of Czech approach to education influenced by its rich history (among others the ideas of Comenius, 1907). Entries in the Czech Lexicon focus mainly on pupils' and teachers' observable activities, mathematical content as such is less important. This gives the Czech lexicon a generic character as many of the terms are applicable to other classroom settings aside from the mathematics classroom.

However, teachers do engage in discussion with their colleagues about their practice, and, in this paper, we compare the documented lexicon in current use by teachers in Australia and the Czech Republic. These lexicons encrypt the pedagogical history and values of its community and as such, provide an insight into the evolution of mathematics classroom practice and the implicit theories of learning and instruction that have become locally institutionalised within the professional language of mathematics teachers. The teachers in one language have access to terms, and constructs, that may not be available to others. The actions, interactions and events of the mathematics classrooms are described and categorised in different ways by different communities.

THE RESEARCH DESIGN FOR DOCUMENTING THE LEXICONS

Research teams included junior and senior researchers and at least two experienced teachers of middle school mathematics and accorded authority to teacher voice in the generation of the lexicon. The Australian team included three academics and three high school teachers of mathematics, two of whom had taught extensively at years seven and eight, whilst the third was a more recent member of the profession. The Czech team included three academics and two very experienced teachers that now hold leadership positions in schools.

Each team contributed video material, time-stamped transcripts as well as supporting material related to one lesson of year eight mathematics. These lessons were re-packaged as "three-ups", that is, three camera angles with a time-code and subtitles all visible in one viewing window. A stimulus package of nine lessons, one from each team, was constructed and distributed to each local research team for project-wide use.

These nine lessons presented a variety of classroom settings and instructional approaches both familiar and unfamiliar to the research team members.

Each team began with the prompt "What do you see that you can name?" and recorded their responses on a standardized template. The nine videos were intended to stimulate thinking about the possible terms of the lexicon, however, it was not necessary that each term recorded had to be present in the video stimulus material. The essential point was to record single words or short phrases that are familiar to at least two-thirds of middle school mathematics teachers with an agreed meaning.

Operational definitions were developed for each of the terms. National surveys were subsequently developed to collect information about teachers' level of familiarity with each of the terms, the extent to which they endorsed the descriptions, examples and non-examples, and the frequency with which they used the terms (or phrases) in conversation with their colleagues. Opportunities for commenting on the clarity and appropriateness of the descriptions and the examples and non-examples for each term were also provided. The goal was to establish that the constituent terms of the lexicon were not only familiar to the teaching community whose classroom phenomena were encoded in the lexicon, but also, that their meaning was represented in a way that teachers were happy to endorse.

COMPARING LEXICONS: NEGOTIATING OUR UNDERSTANDING OF CLASSROOM PHENOMENA

The essential set of elements for the specification of a lexical term included: the named term in the original language (the closest English approximation is in brackets if appropriate); a description; examples (and non-examples if appropriate) (in the original language as well as English). A selection from the Australian and Czech lexicons is given in Figure 1.

The Australian researchers (speakers of English not Czech) and Czech researchers (speakers of English and Czech) independently reviewed the two entire lexicons and began by matching lexical terms by name and then by pedagogical intention: whether identical, similar or absent. This structured comparison of the two lexicons was followed by a number of face-to-face meetings, where the preliminary pairings were scrutinised. This approach allowed for critical distinctions in meaning to emerge. This involved long explanations that identified which particular elements of the phenomena were critical for the terms to be named and subsequently paired. The challenges faced by the researchers were overcome with much discussion, detailed explanation and reference to examples from the classroom.

The Australian lexicon consists of 61 terms organised into five categories whilst the Czech lexicon consists of 49 terms organised into 10 categories. A selection of terms is included in Figure 1 whilst the organisational structures are given in Figure 2.

	Term	Description	Examples and Non-examples
Examples from the Australian lexicon	**checking**	The process by which a teacher or student • checks answers, by determining the exactness or correctness of solutions; or • checks progress, by determining whether the expected amount of work has been completed.	Example: • The teacher makes notes in her chronicle to indicate the timely completion of homework. • The teacher makes a mental note or observation. *Non-example:* • *The student annotates their workbook solutions.*
	explaining (explanation)	Make an idea or situation clear to someone by describing it in more detail or by revealing certain facts.	Example: • The teacher gives reasons why a particular mathematical process works in all cases. *Non-example:* • *The teacher merely provides a solution.*
Examples from the Czech Lexicon	**shrnutí** (summarization)	Učitel/žák shrnuje slovně. Teacher/pupils recapitulate and sum up verbally.	Examples: • Učitel vysvětluje řešení/části řešení úlohy. Teacher explains the solution/part of the solution of a problem. • Žáci vysvětlují na pokyn učitele. Pupils explain in response to a teacher's request.
	vysvětlování (clarification)	Učitel vysvětluje žákům učivo, které již znají, jiným způsobem. Teacher explains a topic familiar to their pupils in a new way.	Example: • Učitel reaguje na nepororumění učivu a vysvětluje žákům učivo jiným způsobem. Teacher responds to pupils' failure to understand a topic and explains it to pupils in an alternative way.

Figure 1: A selection of lexical terms from the Australian and Czech Lexicons

Australian Lexicon	
Category Name	**Number of Terms**
Administration	8
Assessment	10
Classroom Management	5
Learning Strategies	27
Teaching Strategies	49

Czech Lexicon			
Category Name	**Number of Terms**	**Category Name**	**Number of Terms**
Domácí úkol (Homework)	2	Organizační formy vyučování (Organisational Forms of Instruction)	4
Druhy úloh (Types of Problems)	4	Procesy podporující učení žáků (Processes Supporting Student Learning)	5
Fáze vyučovací hodiny (Lesson Stages)	2	Samostatná práce žáků (Pupils' Individual Work)	3
Hodnocení (Assessment)	7	Vyučovací metody (Teaching Methods)	10
Organizace vyučování (Lesson Organisation)	8	Využití didaktických prostředků (Use of Didactical Means)	4

Figure 2: Organisational structure of the Australian and Czech Lexicons

Identically-named terms (identical in name and intention) and similarly-named terms (identical pedagogical intention) are given in Figure 3. Czech terms that are absent from the Australian Lexicon and Australian terms that are absent from the Czech Lexicon are given in Figure 4.

Identically-Named Terms					
	Australia	**Czech Republic**		**Australia**	**Czech Republic**
1	Assigning Homework	Zadání domácího úkolu (Assignment of homework)	9	Monitoring	Monitorování (práce žáků) (Monitoring (of pupils' work))
2	Board Work	Zápis informací na tabuli (Written record on the board)	10	Motivating	Vnější motivace žáka při výuce (External motivation in a lesson)
3	Clarifying	Vysvětlování (Clarification)	11	Practising	Opakování a procvičování vědomostí a dovednosti (Revision and practice of knowledge and skills)
4	Disciplining	Udržování kázně (Maintaining discipline)	12	Reciting	Sborová odpověď (Group response)
5	Formative assessment	Hodnocení formativní (Formative assessment)	13	Scaffolding	Scaffolding
6	Giving Praise	Pochvala žáka (Commendation)	14	Summative assessment	Hodnocení sumativní (Summative assessment)
7	Group work	Skupinové vyučování (Group teaching)	15	Whole class discussion	Hromadné vyučování (Collective teaching)
8	Individual work	Individualizované vyučování (Individual teaching)			

Very Similar Terms			
Australia	**Czech Republic**	**Australia**	**Czech Republic**
Assigning Homework	Zadání úlohy (Assigning a task)	Worked Example	Řešení úlohy řízené učitelem (Teacher-controlled solving of a task)
Demonstrating	Výklad spojený s demonstrací (Explanation with demonstration)		

Figure 3: Terms that appear in both the Australian and Czech Lexicons

Czech Terms (absent from Australian Lexicon)	
Hromadné vyučování (Collective teaching)	Organizační pokyn (Organizational instruction)
Individualizované vyučování (Individualized teaching)	Úloha důkazová (Proof problem)
Individuální konzultace se žáky (Individual consultation with pupils)	Úloha určovací (Determining problem)
Organizační otázky učitele (Teacher's organizational questions (non-mathematical))	Vizualizace (Visualization)

Australian Terms (absent from Czech Lexicon)	
Feedback	Positive Reinforcement
(use of a) Hook	Reasoning
Justifying	Reflecting
Modelling	Wait Time

Figure 4: Distinctive terms from the Australian and Czech Lexicons

DISCUSSION

In comparison, the Australian and Czech Republic lexicons offer interesting similarities and differences:

The organisational structure

The Australian organisational categories were inspired by a university class of practising teachers who were invited to group the lexical items, as well as provide category names for their groupings. Some terms appear in more than one category; 23

terms belong to both the Learning Strategies and Teaching Strategies categories. The ten Czech categories were inspired by the components and familiar structures of Czech middle school lessons of mathematics. The 49 terms in the Czech Lexicons are organized across twice as many categories as the Australian one which also indicates differences in degree of specificity of categories (Novotná, Moraová, Hošpesová, Žlábková & Bureš, 2016).

In comparing the organisational structures of the lexicons both the category names, as well as terms grouped within the category, were examined. The following findings are noticeable: a) only one category appears in both lexicons (*Assessment; Hodnocení*); b) the Australian category *Learning Strategies* appears similar in intention to the Czech category *Procesy podporující učení žáků (Processes Supporting Student Learning)* c) four of the Czech categories *(Domácí úkol (Homework*), *Druhy úloh (Types of Problems*), *Fáze vyučovací hodiny (Lesson Stages)* and *Využití didaktických prostředků (Use of Didactical Means*) are overlooked in the Australian lexicon organisational schema (and terms within these categories are noticeably absent from the Australian lexicon); and, d) the Australian category, *Teaching Strategies*, is comparable with the Czech categories, *Organizační formy vyučování (Organisational Forms of Instruction), Vyučovací metody (Teaching Methods)* and *Organizace vyučování (Lesson Organisation*) as many of the terms within these categories shared pedagogical intention.

The named phenomena

There are 20% fewer lexical items in the Czech lexicon and 15 terms (almost a third of the entire terms in the Czech lexicon) are present in the Australian lexicon. An additional set of 6 terms are very similarly named and have identical pedagogical intention (see Figure 3).

The lexicons differ with level of specification with regard to a particular classroom phenomenon. For example, whilst both lexicons identify the phenomena of both *Formative* and *Summative Assessment* the Czech lexicon has indicated an additional four assessment-related terms: *Hodnocení neadresné (Assessment-non-directed)*; *Hodnocení povšechné (Assessment – general)*; *Sebehodnocení (Self-assessment)*; and *Vrstevnické hodnocení (Peer-assessment)*. Another example is that of the classroom phenomena of *Shrnutí (Summarization)*. There are four terms in the Australian Lexicon related to the recording or reporting of the main points of a discussion (or text material), these include: *Summarising, Recapping, Re-Teaching* and *Reviewing*.

Of interest is the term *Scaffolding* which appears in both the Australian and Czech lexicons. This is an example of a term that has been appropriated into the Czech lexicon from the English language and may indicate the consequences not only of the internationalisation of English, perhaps too the function of the English language as the preferred communicative protocol for the international community.

The distribution of agency

If we adopt the more commonplace definition of agency, that is, "agency can be described as the ability or potential to act," (Priestley, Biesta & Robinson, 2015, p. 22) the prevalence of certain terms within the lexicons reflect the importance attached to teacher and student actions. Terms were analysed according to who performs the action: teacher only; student only; teacher or student; teacher and student. Of the 61 lexical terms in the Australian lexicon, students' voice and participation in the classroom is a significant characteristic, as 36 of the 61 terms relate to student action (*Raising Hands, On Task, Practising*). However, the teacher's role was more prominent with 50 of the 61 terms related to the teacher action (*Differentiating, Scaffolding, Wait Time*).

A distinctive feature of the Czech lexicon is the prevalence of terms that reflect the importance attached within Czech education to the teacher-student relationship. For example, the term *Řešení úlohy řízené učitelem (Pupils solve the problem with help from their teacher)*, is interesting in that it characterises some aspect of the interaction between teacher and student. However, it should be noted that the Czech term anticipates an action by the student at the invitation of the teacher (Clarke, Mesiti, Cao & Novotna, 2017). So, while foregrounding a particular student activity, the term nonetheless accords the teacher the authority to initiate the designated phenomenon, since it is the teacher who invites.

A particularly interesting category of terms is those in which teacher and student/s are wholly complicit and in which the agency or authority to initiate resides legitimately with both or either. An obvious Australian example is *Whole Class Discussion*. In such activities, teachers and students are both complicit and co-agentic. The question of agency offers insight into the institutionalised power differentials maintained in our classrooms.

CONCLUSION

In this paper we have shared not only the processes by which each lexicon was documented and organised but also by which the two were compared. The contrasting of the Australian lexicon with the Czech lexicon was involved and fruitful and revealed differences and similarities in context and structure. The comparison was able to highlight different emphasis regarding mathematics classroom phenomena as indicated by the number of related terms, the number and type of organisational categories and the encrypted agency of the terms.

Comparison of lexicons of teachers offers us the opportunity to reflect critically on our professional vocabulary and the pedagogical practices named, invisible or absent from our mathematics classrooms. Access to the lexicons of other countries offers possibilities for practice that may expand the professional repertoire of mathematics teachers around the world and usefully inform teacher reflection on their practice.

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CHANGES IN THIRD-GRADE STUDENTS' UNDERSTANDING OF POLYGONS

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The present study focuses on investigating the effects of a teaching module on the achievement of the understanding of polygons and their attributes with nine years old students. Data come from the answers of 59 pupils in a Spanish primary school to a pre- and post- test, designed ad hoc, about geometrical figures. A hierarchical similarity and implicative similarity analysis was conducted in order to examine how the changes on students' answers in both tests are from the connections among variables. Results indicate that (i) there is prevalence in the use of the perceptual register opposite to the verbal register, and (ii) the improvement of the understanding of polygons and the use of the attributes depends on the common attribute that characterizes a set of polygons.

INTRODUCTION AND THEORETICAL BACKGROUND

Students' ideas and conceptions about geometrical figures are developed in a progressive way. Initially, pupils recognise the figures by perceptual similarity, then they recognize the attributes and finally they based on formal definition (Satlow & Newcombe, 1998; Battista, 2007). The main emphasis in the domain of mathematics education is to develop teaching processes based on the expected cognitive development which facilitate the concept understanding. This emphasis is on the role of experience in geometry learning. According to Marchis (2012), geometrical notions and properties are in real world problems and pupils are expected to learn geometry by exploring their environment formally through the teaching or informally through everyday activities. The many and various activities construct the students' geometric ideas and those affect their future learning experiences in higher grades (Browning, Edson, Kimani, & Aslan-Tutak, 2014).

The process of understanding of the geometrical figures has been developed by Duval (1999) who defined the figure apprehension model based on three different ways to organize the visual aspects in geometric figures. The first one is related with the recognition of the figures (perceptual apprehension), the second one related to the definition, the statements and the descriptions of the figures constitutes the discursive apprehension, and the last one, the modification of a figure constitutes the operative apprehension. For working with geometrical figures, children had to identify the attributes of these through of the dimensional deconstruction (Duval, 2017). The dimensional deconstruction of geometric figures is the central process of geometric visualization, which allows the perceptual recognition of a figure, in a configuration of figurative units of smaller dimensions (from 2D to 1D).

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In geometry is necessary to combine the use of at least two representation systems, one for discursive expression (written or verbal) and another for visualization, since a geometrical figure is associated both discursive and visual registers. However, this association is cognitively complex (Duval, 2006). For this reason, it is recommended that teachers encourage students to describe and explain why a figure belong or does not belong to a shape category, adapting the language to the geometric thinking level of children (Clements et al., 1999; Elia & Gagatsis, 2003).

The development of the understanding of geometric figures is a transition that evolves from the general domain of geometric figures to the characteristics of the figure. That is, the construction of geometric concepts is a progressive development from the perceptual to the conceptual (Clements & Battista, 1992). The understanding of geometric figures is linked to the use of minimum conditions, therefore, recognizing the critical attributes of a geometric figure to consider when a figure belongs or not to a category (Bernabeu, Moreno & Llinares, 2018).

Previous researches have argued that the teaching of geometry in the first years is of poor quality, because this instruction is repetitive (Sarama & Clements, 2009; Petridou, Ilia & Gagatsis, 2015). However, it has been shown (Petridou, Ilia & Gagatsis, 2015) that if the instruction is modified and based on the use of visual examples of geometric figures with different critical attributes (e.g., number of sides) or non-critical attributes (e.g., orientation, size, colour), is more possible for the child to develop a more consistence geometrical thinking and discourse. Through the use of discourse and verbalization, describing figures or belonging to a certain class (Sarama and Clements, 2009) and also, by mentally or physically modifying the figures (Duval, 1995), children can develop thinking and achieve a more coherent geometric discourse. Therefore, the aim of this study was to examine the effects of a teaching module on the achievement of the understanding of polygons and their attributes with nine years old students. For this reason, we proposed the following research questions:

- What changes could we observe in the use of the verbal and perceptual registers by the 3rd-grade students after the teaching module?
- What changes do we identify in the use of the attributes after the teaching module by the 3rd-grade students?

METHODOLOGY

The present paper is part of a research with a pre- and post-test and a teaching module.

Context

The participants of this study were 59 Spanish-speaking students (37 boys and 22 girls) of third-grade (9 years old). They studied at a public elementary school in Spain and they had mixed communication and math skills, according to their teachers.

Test and teaching module

We designed a test consisted of six tasks about geometrical figures, which was used for the pre- and post-phase of the study. Each task included some items, altogether

were twelve items which were grouped in four focuses: (a) Recognizing polygons and non-example of polygons (Task 1 – Item 1a, 1b, 1c, 1d and 1e); (b) Identifying a common attribute in a polygons' set (Task 2, 3 4 - items 2, 3 and 4a); (c) Drawing polygons with conditions (Task 6 - item 6a); and (d) Classification of figures (relationships among figures) (Task 4, 5 and 6: item 4b, 5 and 6b). The last five tasks of the test were adapted from the metaphor of Drawing machine (DM) developed by Battista (2012), which let make figures with some conditions. In this research, we selected only the items in which students should use verbal and perceptual registers. In the items 1b, 1c, 1d, 1e, students had to identify the missing attribute in the non-example of polygon to turn it into a polygon, explain this transformation and draw the final polygon. The figures considered were: a figure with open, crossed and curves sides (1b); a figure with open and curved sides (1c); a figure with curved and crossed sides (1d); and a figure with open and crossed sides (1e). By other hand, in the items 2, 3 and 4a, students had to identify the common attribute of polygons' set, explain what the attribute was and draw an example of polygon complies the attribute. The attributes were: concavity and convexity (task 2); six sides polygons and non-six sides polygons (task 3); and polygons with a symmetrical axis and without symmetrical axes (task 4).

The teaching module consisted of 10 sessions of 50' each, two per week, during five weeks, from May to June of 2018. In this module, we worked critical attributes of polygons (closed, straight and non-crossed sides); and types of polygons and their attributes (e.g., concave and convex polygons; polygons according to the number of sides). The teaching module was implemented by a member of the research group who adopted the role of the teacher in both classes. First of all, teacher introduced the content of the session and students discussed tasks with the whole group. At the end of the session, students realised several tasks individually similar to the ones performed during the session. Some tasks during the sessions consisted on: classifying geometrical figures; constructing polygons according to their attributes with materials (e.g., Meccano); drawing figures on the digital board.

ANALYSIS

We generated 20 variables from the seven items of the test. Firstly, we named the variables with the initials "Pr" for the pre-test' tasks and "Po" for the post-test' tasks. We also named each task with a capital letter, for example the task 1 is the letter A, the task 4 is the letter D. In task 1 (A), the items 1b (Figure 1), 1c, 1d and 1e have two variables each one. One variable to identify the students' explanation of the turn of the non-example of polygon into a polygon (for example PoAd1), and other variable about the drawing from this turn (for example PrAb2). In tasks 2 (B), 3 (C) and the item 4a (D) we considered four variables. We assigned a lowercase letter *a* for the common attribute that DM can make and *b* for the attribute that cannot make it. We put number *1* for the explanation of the common attribute of polygons' set and *2* for the drawing of an example of polygon that fulfils the attribute. For example, we named PoDa1 the

answer in which students explain the common attribute and PoDa2 the drawing of an example that fulfils the common attribute identified in the polygons' set.

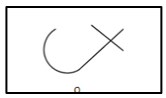

Figure 1: Non-example of polygon of the item 1b

For the analysis of the data, we used a computer software called C.H.I.C. (Classification Hierarchique, Implicative et Cohesitive) (Bodin, Coutourier, & Gras, 2000), by obtaining the hierarchical similarity connections and the implicative diagram of the variables' relations. In the hierarchical similarity, the clusters are established in particular levels of the diagram and can be compared with others. In the hierarchical similarity is formed some thick links in red which represent the most significant relationships among variables. The hierarchical similarity diagram shows how the variables are grouped according to the homogeneity with which the students operated them. The implicative graph is executed to know the implicative relations among the variables, and certainly, the students' answers of the same task in the pre and post-test.

RESULTS

We divided the results into two sections: comparison of variable's successful level, and comparison of the similarity diagram and implicative diagram from pre- and post-test.

Comparison of variable's successful level

Regarding the tasks, we observed two groups (Figures 2 and 3) group 1 includes the variables which belong the items about turning the non-examples of polygons into polygons and, group 2 includes the variables about identifying a common attribute of a polygons' set, explaining it and drawing an example of polygon with this attribute.

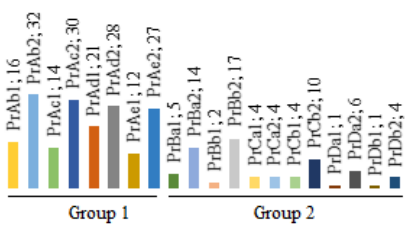

Figure 2: Bar chart of the frequencies of success in the pre-test

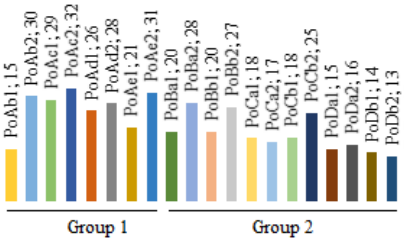

Figure 3: Bar chart of the frequencies of success in the post-test

If we observe level of success of group 1 on the pre-test (Figure 2) and the corresponding on the post-test (Figure 3), the variables of explanation how to turn the non-example of polygon into a polygon (Ab1, Ac1, Ad1 and Ae1) have lower success level than the variables of drawing the final polygon from the turning (Ab2, Ac2, Ad2 and Ae2), although on the post-test this difference is not very important. Attending to group 2, we observe differences among the variables of the pre-test and post-test. Firstly, we identify that the variables of explanation the turning on the post-test (Figure

3) have higher success level than the same variables on the pre-test (Figure 2). Secondly, the variables of drawings the final polygon from the turning on the post-test (PoAb2, PoAc2, PoAd2 and PoAe2) have higher success level than the same variables on the pre-test but this difference is slight.

In the Group 2, identifying a common attribute of a polygons' set, both on pre-test and on post-test, the variables about the drawing an example of polygon from the identification of the common attribute of the polygons' set (Ba2, Bb2, Ca2, Cb2, Da2, Db2) were higher than the variables about the explanation of this attribute (Ba1, Bb1, Ca1, Cb1, Db1, Db1). However, on the post-test the explanation and drawing variables had not a great difference.

Moreover, we can observe on the post-test' variables a higher success levels than on the pre-test' variables. The success level about the explanation of concave polygon (Ba1) raised from 5, on the pre-test, to 20, on the post-test, and the explanation of convex polygon (Bb1) from 4, on the pre-test, to 20 on the post-test. By other hand, the success level about the drawing of an example of concave polygon (Ba2) raised from 14, on the pre-test, to 28 on the post-test, and the corresponding of convex polygon (Bb2) raised from 17, on the pre-test, to 27 on the post-test.

We show an example of this difference in Figures 4 and 5. Ana, a girl of third-grade, drew a convex polygon instead of a concave polygon and she explained that the DM could make these polygons *because the line is not crossed, there is not any curved line and because the lines are straight*, which is a wrong answer*;* in the place of drawing and explaining the convex polygons, the student drew another convex polygon and did not explain anything. On the post-test, Ana drew an example of concave polygon and explained that the DM could make this figure *because has diagonals outside the figure* (the concave' definition that we gave during the teaching module); and she drew a convex polygon and explained that the DM could not make this figure because *it has not diagonals by outside (the figure)*.

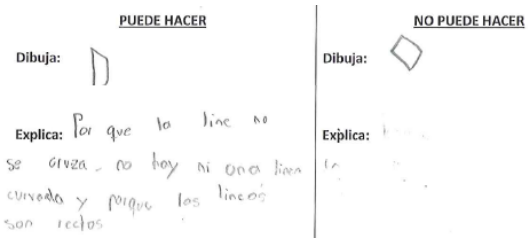

Figure 4: Answer on pre-test to explain and draw concave and convex polygon

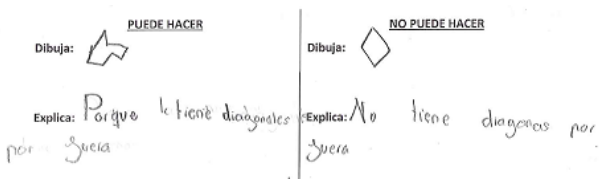

Figure 5: Answer on post-test to explain and draw concave and convex polygon

Comparison of the similarity diagram and implicative diagram

We made a different analysis (similarity and implicative diagram) with each type of task. In the similarity and implicative diagram of the both kind of tasks, is formed two groups, one belongs to the variables of the pre-test and another belongs to the variables

of the post-test. Additionally, in both cases there were relationships of invariants between the variables of explanation and drawing on the pre- and post-test. We payed attention to the tasks of identifying the missing attribute in the non-example of polygon to turn it into a polygon.

The similarity diagram shows two groups (Figure 6). In the group 1 all the variables correspond to the Pre-test (from PrAb1 to PrAe2) and almost all the variables of the group 2 correspond to the Post-test (from PoAe1 to PoAe2). We observe, in both groups, the same relationship among three variables relating to the explanation of turning the non-examples of polygons into polygons liked: a figure with open, crossed and curved sides (PrAb1 and PoAb1); a figure with curved and crossed sides (PrAd1 and PoAd1); and a figure with open and curved sides (PrAc1 and PoAc1). Another relationship is produced with the same variables of the drawing the final polygon after the turning: a figure with open, crossed and curved sides (PrAb2 and PoAb2); and a figure with crossed and curved sides (PrAd2 and PoAd2). That means there was an invariant among the same variables on the pre- and post-test. In both groups, the variables of the explanation about turning the non-example of polygon into a polygon have a similarity relationship and, on the other hand, the variables of the drawings are also linked among them. This meant that the variables of the explanations and drawings from the turn the same non-example of polygon into a polygon are not linked.

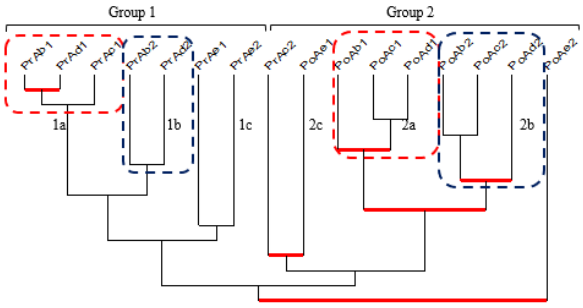

Figure 6: Similarity diagram of the pre- and post-test variables.

Furthermore, we can observe the same invariants in the similarity diagram of the statistical levels of 99% (Figure 7) and 95% (Figure 8) with the same variables. In both statistical level, we can observe in the chains A and B that from the variable about the explanation of turning the figure with open, crossed and curved sides into a polygon (PrAb1 and PoAb1) continues with the variable about the explanation of turning the figure with curved and crossed sides into a polygon (PrAd1 and PoAd1). As of the variables of the explanations emerge the variables about the drawings. Firstly, is the drawing of turning a figure with curved and crossed sides into polygon (PrAd2 and PoAd2), and from this emerges the drawing of the transformation of the figure with open and curved sides (PrAb2 and PoAb2).

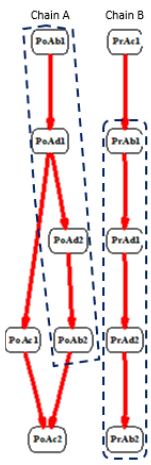

Figure 7: Implicative diagram of the statistical level of 99%

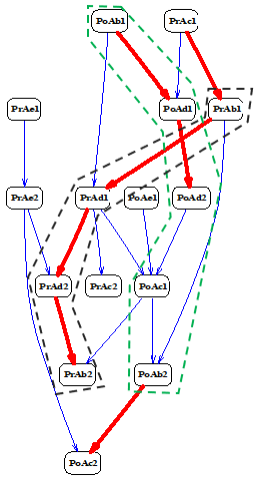

Figure 8: Implicative diagram of the statistical level of 95%

DISCUSSION

This research aims to examine the effects of a teaching module on the achievement of understanding of polygons and their attributes in third-grade school. Results indicate that (i) there is prevalence in the use of the perceptual register opposite to the verbal register, and (ii) the improvement of the understanding of polygons and the use of the attributes depends on the attribute that characterizes the geometric figure.

The difference in the use between the verbal and perceptual register in the students' solutions of the task on the pre- and post-test, in spite of the teaching module performed, is showed in the both diagrams (invariants). This phenomenon would be explained because the students are not yet cognitively prepared to use the visual and verbal register in their answers equitably (Duval, 2006).

In fact, producing invariants in the diagrams between the pre- and post' variables, it evidences of enhancing the quality of students' answers. Thereby, the students' understanding of polygons is improved, because they acquired a set of attributes which let them understand polygons and the relationships between the attributes to classify the geometrical figures shown in the post-test' answers (Clements & Battista, 1992). Moreover, we can intuit students become to develop the dimensional deconstruction of figures (Duval, 2017). They became to decompose the figures in smaller dimensions to stablish relationships between the geometrical concepts and the parts of the figures, to explain the attributes of the figures or drawing an example of polygon fulfils the attributes. For this reason, it is recommended encourage students to use the verbal and visual register to explain the figure's belongings, adapting the mathematical language to their levels (Clements et al., 1999; Elia & Gagatsis, 2003).

It would be recommended doing the same research (pre-, post-test and teaching module) with older students to verify if the invariant between the pre- and post-test' variables would be altered. Furthermore, it would be advisable to increase the number of participants to carry out the statistical analysis.

Acknowledgements

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TEACHING-PRACTICES THAT PROMOTE META-LEVEL LEARNING: THE CASE OF COMPLEX NUMBERS

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This paper focuses on teaching-practices in secondary school mathematics from the commognitive perspective. Specifically, the paper focuses on identifying accomplished teacher's practices visible in her teaching towards the meta-level learning of complex-numbers. For this purpose we analysed a video-taped 90-minute lesson on complex-numbers in 12th grade. Our findings include specific teaching-practices that were detected for dealing with the challenging task of teaching towards meta-level learning. We categorized the practices according to their role in promoting meta-level learning: those that relate to students' former discourse on numbers, those relating to the new discourse and those related to justifying transition between discourses.

INTRODUCTION

Teaching complex-numbers has long been identified in the research literature as a hurdle not easy to overcome (Tirosh & Almog, 1989). In addition, research under the constructivist paradigm showed that two familiar ways to teach complex-numbers demand the accommodation of mental schemata at least in a minor form. Furthermore, it is not reasonable to learn complex-numbers leaning on assimilation alone (Vinner, 1988). The commognitive paradigm as our over-arching theoretical framework affords a different look at teaching and suggests an answer to the question of why teaching of complex-numbers is so challenging. According to former commognitive studies, there are transitions in the mathematics discourse that include changing the meta-rules that govern the discourse. One example for such transition regards learning that "minus times minus is plus". Until the introduction of negative numbers, students considered mathematics operations on numbers as a part of the laws of nature (3 + 2 is 5 because if you take 3 pebbles and add 2 pebbles you get 5 pebbles), and therefore, believed that to justify the definition of such operations it suffices to find a suitable everyday model. Now, when learning multiplication of negative numbers, they are expected to accept a new type of justification, one that requires showing that the operation on the new set of numbers is a logical implication of some primary assumptions (for further discussion, see Sfard, 2007). This kind of change, the change in meta-rules of discourse, will be called here "meta-level learning".

In this paper, we clarify why we consider expanding the discourse on numbers from real numbers to complex numbers as a case of meta-level learning, and we observe one accomplished teacher in an attempt to identify the teaching practices she employed while trying to promote this special type of meta-level learning.

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TEACHING ACCORDING TO THE COMMOGNITIVE PERSPECTIVE

The commognitive perspective views mathematics as a unique type of communication with special ways of saying and doings, and considers learning mathematics as changing one's mathematics discourse, that is, modifying one's ways of communicating about mathematics. There are various settings in which learning may occur. Settings designed for learning include tacit agreements about the *what*, *how* and *who* that characterize the agreed-upon change in discourse. Sfard (2008) has defined the teaching-learning agreement, as "a situation that arises when the discursants are unanimous, if only tacitly, about at least three basic aspects of the communicational process: about which is the leading discourse, about the discursants' own respective roles as those who learn or those who teach, and about the nature of the expected change" (Sfard, 2008, p. 299). From this Tabach and Nachlieli (2016) define teaching as *the communicational activity which motive is to bring a change in the learners' discourse towards a leading discourse*. Such a definition of teaching includes all communicational actions (verbal, non-verbal, emotional, etc.).

PRAQTAL: PRACTICES FOR QUALITY TEACHING

Project PRAQTAL (PRActices for Quality Teaching) focuses on identifying and conceptualizing practices for quality teaching of secondary school mathematics and physics. The study reported in this paper is a part of this project. Our team arrived at the need to explicate practice as an attempt to improve a teacher-education program. The suggested change was from more "reflective" and "principles" based programs towards "practice" based. We noticed that the task of shifting the preservice teachers' traditional-teacher-centered approach towards more reform-student-centred approach calls for explicit discussion and explication of practices. Explicating teaching practices is at the heart of various other teacher education programs (Grossman, 2018). Some projects are content specific, some focus on specific grade levels, and others are general, but most grew from the needs of training programs for preparing teachers to practice. A prime example is the University of Michigan *Teaching Works* project that identifies and teaches high-leverage instructional practices aiming at preparing beginning teachers to be "skillful teachers who are committed to and able to support their (students') growth." (http://www.teachingworks.org/). Additional projects are discussed in Grossman's book *Teaching core practices in teacher education* (2018).

Adopting the commognitive conceptual framework, we define teaching practice as a routine. Routine is a repetitive pattern of behaviour, prompted by situations that the actor had identified as similar. Following Lavie, Steiner and Sfard (2018), we define a teaching-practice similar to their definition of routines: *the task the teacher saw herself performing together with the procedure she executed to perform the task* (p. 9). Two examples of such tasks could be: having to address a student's idea that the teacher had not thought of earlier or eliciting students' thinking. During a lesson a teacher needs to perform many different tasks. Our goals in PRAQTAL are thus to identify typical, interesting or challenging tasks that teachers set to themselves before, during and after

a lesson, and for each task we identify the various procedures that teachers execute to perform the task. The plural form is used here to denote that our point of departure is that for each task, different teachers may consider different procedures to perform.

IDENTIFYING TEACHING PRACTICES

For the purpose of project PRAQTAL, we videotape lessons taught by competent teachers. For each video-taped lesson, we focus on the teacher's teaching-actions that we have a reason to assume would be repeated when a similar task is pursued. The task is our interpretative answer to the question: *What may the teacher be trying to achieve by those actions?* The tasks are recursive in nature – to achieve a certain task, some sub-tasks should be achieved in advance. For example, to achieve the task of "promoting purposeful discussions in mathematics lessons", Smith and Stein (2011) identified five sub-tasks that a teacher should pursue, e.g. anticipate students' responses, or monitor students' work during group-work.

In this paper we focus on a specific type of task, the task of promoting students' meta-level learning. We wish to unpack this large-size task to sub-tasks by analysing teachers' actions in a 12th grade lesson on complex-numbers. Our research question is: *What practices are used by an experienced teacher to promote meta-level learning of complex-numbers?* The idea of meta-level learning is discussed in the literature (e.g. Sfard, 2007). Therefore, we first describe object-level and meta-level learning, and then detail the methods of our study and our findings. Let us mention that although the project is directed toward *quality*-teaching-practices in this paper we do not focus on the *quality* dimension of the practices.

OBJECT-LEVEL AND META-LEVEL LEARNING

As stated earlier, commognition (Sfard, 2008) views learning mathematics as changing the learner's ways of communicating about mathematics (in other words changes in her mathematical discourse). Those changes could be at the *object-level* and include expanding one's discourse about a familiar mathematical object. For example, learning how to find a y-intercept of a familiar function. Object-level learning may also involve endorsing new narratives about familiar objects. An example could be learning that the base angles of an isosceles triangle are congruent, after the learner is already familiar with triangle congruency theorems. This is considered object-level learning as all that is needed for this kind of learning is to apply the familiar congruency theorems.

In contrast, meta-level learning involves changing the meta-rules of the discourse. These changes can be expressed in various manners. Thus, for instance, the change may occur in executive rules of discourse, as is the case when, for the first time, students are required to justify their claims in the mathematics classroom. Another type of meta-level change may occur when a new mathematical object is introduced. For example, when students learn about fractions, they need to give up many of the previously endorsed claims about operations on numbers. Thus, for instance, multiplication is no longer an action that "makes bigger". Therefore, this kind of

learning is a transition to a discourse in which familiar words must be used differently (Sfard, 2007).

Based on previous studies (Sfard & Ben Zvi, 2007; Nachlieli & Tabach, 2012) teaching for meta-level learning requires designated teaching-practices which are basically different from teaching-practices that promote object-level learning. In the next section we explain why we consider the introduction of complex numbers as meta-level learning.

EXPANDING STUDENTS' MATHEMATICAL DISCOURSE ON NUMBERS: COMPLEX-NUMBERS

According to the Israeli mathematics curriculum, complex-numbers are introduced to advanced-track students in the twelfth grade. By that time, students have already developed a rich discourse on numbers. Throughout the years, they have expanded their mathematical discourse from that on natural numbers to unsigned rational, to rational, and finally to real numbers. Initially, the defining property of numbers was that they answered questions about quantity, such as “How much?” or “How many?” The introduction of negative numbers extended the use of the term “quantity” so that it could apply not only to what there is, but also to what is lacking. By the time they entered the twelfth grade, the students learned to work with real numbers and got convinced that the equation $x^2+1=0$ has no solution and the expression $\sqrt{-1}$ is meaningless.

Mastering the chapter on complex-numbers, students must acknowledge that the word *number* would signify from now on also objects of the type of a+bi, where “a” and “b” represent real numbers, and “i” is the square root of (-1). Within this new discourse, the narrative that "the expression $\sqrt{-1}$ is not a number" does not hold any longer. From the very first moment the students must realize that the new “numbers” cannot be presented as points on the number-line, do not answer to the question "how many?" and cannot be ordered according to size (there is no such thing as the size of a complex number!). In addition, numbers are now signified with the help of the letter “i”, a symbol that, so far, signaled only variables and parameters. All this implies a far-reaching change in the use of the word *number*, and thus in the meta-rules of numerical discourse. This change in discourse implies the need for meta-level learning.

As stated previously, it is reasonable to assume that promoting students’ meta-level learning is profoundly different from teaching for object-level learning. The adoption of new meta-rules that cannot be derived from an existing discourse is a challenging teaching goal. In the next section we look at a teacher grappling with this challenge.

METHOD

To identify the practices of the teacher trying to promote student’s meta-level learning, which is required in the transition to the discourse of complex numbers, we analysed the introductory lesson that took place in an advanced 12th grade mathematics

classroom in an Israeli high-school. The lesson was 90-minute-long, and it was video-taped and transcribed in its entirety. The lesson was taught by the 2nd author.

Our analysis of the lesson included segmenting the transcript according to teacher's actions. For each segment we asked: *What is the task that the teacher may be trying to perform in those actions? Could this task promote meta-level learning?*

Before focusing on the specific teaching-practices we describe the lesson's thematic structure: the teacher's choice, inspired by the assumption that ontogeny of discourses can be helped by explaining their historical roots, was to walk her students through a version of the historical development of the discourse on complex numbers. This was to be done in five steps: (1) making students realize that as long as our discourse on numbers includes only real numbers, no number, when squared, would yield a negative number (thus, there is no such thing as a square root of a negative number). (2) concluding that a 3rd order polynomial has at least one real root, (3) introducing the students to the suggestion, made in the 15th century by the Italian mathematician Girolamo Cardano, that the *real* roots of some cubic polynomial function could be calculated with the help of a well-defined formula. (4) Introducing students to Rafael Bombeli's suggestion that square roots from negative numbers should be recognized as full-fledged mathematical objects – as numbers, and could therefore be allowed for use in Cardano's formula; (5) justify the claim that these new objects can be called numbers (more precisely, "complex-number") by showing that it is possible to define operations on these entities that preserve the basic properties of the operations on real numbers; and finally, (6) show that real roots of cubic functions can, indeed, be found with the help of these new, “complex” numbers.

FINDINGS: TEACHING-PRACTICES THAT PROMOTE META-LEVEL LEARNING

We found that all the teacher’s identified actions fell into one of the following three categories: (1) those related to the students' current discourse on numbers and aimed at preparing the ground for the changes to-come; (2) those related to the new discourse on numbers, and (3) those related to transition between the discourses; and we present the teaching-practices accordingly. For each teaching-practice, we identify the task (in italics), followed by detailing some of the procedures the teacher execute in the apparent attempt to perform this task.

Teaching-practices performed in the context of students' familiar discourse on numbers

Explicating some of the narratives and rules in the students' former discourse on numbers that are about to change. The procedure performed by the teacher towards this task included initiating a discussion about roots of quadratic functions at the beginning of the lesson. When students solved the given quadratic equation and stated that: "there is no solution (as the discriminant was negative)", the teacher focused the discussion on the following questions: What is it that makes us say there is no solution? More specifically, the teacher asked: “What property of the graph of the function tells

us there is no solution?" and "why can't we find a number that is the square root of a negative number?". In addition, the teacher emphasized precise use of mathematics language by stating that "there is no **real** solution".

Helping students identify incompleteness in the existing discourse on numbers (that in the future could be resolved by the acceptance of complex-numbers as new mathematical objects). The teacher constructed a scenario that positions two narratives that if are endorsed, leads our discourse to a dead-end: (1) any cubic function has at least one root (its graph intersects the x-axis in at least one point) (2) a familiar and productive formula for finding the roots of a cubic function, fails to do so in some cases as it requires to calculate a square root of a negative number.

Teaching-practices performed in the context of the new discourse on numbers

Providing students opportunities to operate, if only ritually, on new mathematical objects or, more specifically, engaging students in applying familiar numerical operations to the new mathematical objects. After introducing $\sqrt{-1}$ as a number, the teacher asked students to calculate expressions in which the formerly known operations +, - and × were applied to complex-numbers.

Making efforts to convince students that accepting the new objects as numbers indeed makes sense, or in other words, showing students that it is reasonable to endorse this new meta-level narrative. One must notice that the "reason" we are pointing at here is not necessarily mathematical reason. For example, the advantage of being able to solve the cubic equation after executing ritual work on complex-numbers, provides the students with a good reason to, at least tentatively, accept this new "creature" ($\sqrt{-1}$) as a number: "So, I think that this exercise is here to give us the feeling that Bombeli was not talking nonsense, and that if we adopt those things that till now we wouldn't have looked at them at all, but would have thrown them away instantaneously, if we are looking at them just a little deeper, we may benefit from them" (the benefit: finding the solution to the given cubic equation) "And indeed, the mathematical world did benefit, and we are going to look deeper."

Teaching-practices performed in the context of transition between discourses

Support the students in tentatively accepting rules that may overcome the above-mentioned incompleteness. To resolve the incompleteness identified in the former discourse on numbers, the teacher suggests suspending tentatively the disbelief in $\sqrt{-1}$ as a number "let's see what happens if we do not throw this (the square root of a negative number that is to be calculated using Cardano's formula) away". Using those new "numbers" is only a tool for finding out whether this would be productive or not in resolving the incompleteness.

Marking explicitly boundaries between students' former discourse on numbers and the yet-to-be discourse on numbers. Throughout the lesson the teacher explicated what students know about numbers and what is still to be learned by pointing at situations in which the former discourse is no longer applicable.

DISCUSSION

In this paper we presented six teaching-practices that aim at teaching towards meta-level learning. What can we learn from identifying those teaching-practices? As we asserted earlier, learning complex-numbers is considered meta-level learning, that is, the expected change includes changing meta-level rules that govern the discourse on numbers (such as – what is a number? And - how do we accept a mathematical object as a number?). Therefore, students cannot "make sense" of the new objects within the familiar discourse on numbers as one cannot expect students to independently come up with those historically developed rules. This explains why in schools, teachers' scaffolding is essential for meta-level learning to take place.

The teaching-practices that we identified demonstrate a possible way by which a teacher muddled-through the challenging task of promoting such learning: we learned how a teacher may lead her students in this rocky path of meta-level learning: explicating the relevant rules and objects in the discourse on numbers that the students are familiar with, identifying specific incompleteness that the new objects would resolve, introducing the new objects and convincing the students why and how adopting those objects resolves the incompleteness.

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SURFACING MISALIGNMENTS IN TEACHER LEARNING PROCESSES

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This paper focuses on learning processes of teachers in TEAMS (Teaching Exploratively for All Mathematics Students) professional development (PD) setting and on opportunities that encourage such learning. We conceptualize learning in this PD as a movement from Acquisition Pedagogical Discourse to Explorative Pedagogical Discourse. We show that learning opportunities could be characterized as friction between discourses, that is, the surfacing of misalignments between the two types of discourse. Friction occurred mostly when discussions arose around concrete teaching representations which offered opportunities for negotiation around the misalignment.

INTRODUCTION

Learning of ambitious, discourse-rich and student-centered teaching practices has been the focus of multiple studies (Sztajn, Borko, & Smith, 2017). Yet mostly, studies take a pre-post approach to the study of teachers' learning of such "reform-based" practices, focusing on teachers' "knowledge" and "beliefs" that have (or have not) changed as result of PD. In previous studies, we have shown the productiveness of adopting a discursive approach to teachers' adoption of explorative teaching practices. In particular, we have shown that teachers can be located on a continuum between Acquisition Pedagogical Discourse (APD), which supports and sustains teacher-centered, "traditional" forms of instruction, and Exploration Pedagogical Discourse (EPD), which values students' struggle, discussions, meaning-making and explicit attention to concepts (Heyd-Metzuyanim & Shabtay, 2019). In addition, previous studies (Heyd-Metzuyanim, Munter & Greeno, 2018) have shown that a discursive viewpoint can shed light on misalignments between teachers' and mathematics-educators' discourse in PD settings. Such misalignment can explain the often-detected problem of teachers adopting the explicit narratives of reform-based PD, while not enacting the PD-valued practices in their classrooms. In this paper we take this approach further, to examine what the detection of APD vs. EPD in a PD setting, along misalignments between the PD instructor and teachers' discourse, can tell us about opportunities for learning provided for teachers. Our goal is to examine the process of adopting the EPD, through examining the interactions between the PD instructor and the teachers as forming an interface or "friction" between the two Pedagogical Discourses.

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THEORETICAL BACKGROUND

In the present study, we adopt a discursive approach (Sfard, 2008; Heyd-Metzuyanim et al., 2018b) to teacher learning for conceptualizing two aspects: first - what is to be learned in a PD setting, and second - how this learning is supposed to take place.

Regarding what is to be learned, a view of learning as becoming a participant in a certain discourse (Sfard, 2008) operationalizes adoption of new teacher practices as becoming a participant in a certain pedagogical discourse (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2018; Heyd-Metzuyanim & Shabtay, 2019). Such a view is congruent with works that have conceptualized "reform" and "traditional" teaching as distinctly different "figured-worlds" (Ma & Singer-Gabella, 2011).

Heyd-Metzuyanm and Shabtay (2019) clearly delineated between teaching practices that are valued within these different "worlds", more accurately defining them as distinct Pedagogical Discourses. The Acquisition Pedagogical Discourse, which is often termed "traditional" or "teacher-centered", values actions that assume an acquisition model of learning - the learner is imagined as a vessel and the teacher "pours" knowledge into his brain. This Pedagogical Discourse has clear narratives regarding the actions that are most effective for learning - practicing and applying the acquired skills. It also has clear visions about what are the most effective teaching actions - demonstrating clearly the procedures, breaking them up to small chunks that can easily be repeated, and assuring that students reproduce the procedures exactly the way they were demonstrated.

The alternative Pedagogical Discourse, which in many ways has evolved as an antithesis of the APD, is the Explorations Pedagogical Discourse (EPD). The EPD views learning as participating in a certain social activity and as driven by students' construction of new knowledge, based on previous knowledge (often known as constructionism) (Heyd-Metzuyanim & Shabtay, 2019). We use Discourse with a capital D as inspired by Gee (2014), to stress that these Pedagogical Discourses precede the individual teacher. They exist in documents (such as the NCTM (2000) "Principles and Standards for School Mathematics") as well as in socially accepted narratives passed from one teacher to another. Teachers' pedagogical discourse (with a small d) is a private, individualized form of such Pedagogical Discourses, and is usually aligned partially both with the APD and the EPD. The differences relate to the extent to which a teacher's pedagogical discourse aligns with APD or EPD.

One of the trickiest parts of drawing teachers into the EPD is that some aspects of this Discourse are easily adopted while others often remain hidden. This results in a situation of "misaligned frames" (Heyd-Metzuyanim et al. 2018a), where teachers aligned with the APD interpret differently situations that are framed within the EPD as functioning for a certain cause. For example, Heyd-Metzuyanim and Shabtay (2019) showed that teachers aligned with the APD framed a situation in which a teacher posed a cognitively-demanding task, as one where the students were carelessly "thrown into the lion's den" to pointlessly struggle. Defining Pedagogical Discourses as Discourses

concerning with *what*, *how, when* and *who* to teach, they pointed to frequent misalignments between teachers who claim to adopt certain narratives from the EPD (such as stress on reasoning and conceptual understanding) yet differ from the canonical EPD in their narratives about *who.* Therefore, what it is that should be worked on, discussed, and justified, often remains in *who* can be taught in certain ways. EPD documents stress that *any* student, regardless of ability or previous knowledge, should receive opportunities to construct new knowledge based on their own reasoning. In contrast, teachers aligned with the APD often claim that constructivist, discussion-based teaching practices are appropriate for high-ability or "strong" students, while low-ability students should be taught by demonstrating and drilling tasks of low cognitive demand (Heyd-Metzuyanim & Shabtay, 2019).

Leaning on the view of learning as a form of becoming a participant in a certain Discourse (Sfard, 2008), we are now ready to conceptualize the activity of our TEAMS PD as an occasion where two Pedagogical Discourses interact. We, the PD instructors, mostly adopt the narratives of the EPD, and our framing of teaching-learning interactions are mostly consistent with it. The teachers that enter our PD mostly align, to varying degrees, with the APD. Even when they adopt narratives from the EPD, they often frame teaching and learning in ways that are aligned with the APD. Questions that remain to be inquired are - how does the interfacing between the APD and EPD occur? and what are the processes that may give rise to teachers' gradual adoption of the EPD (along release of the APD)? Specifically, we ask *what are the misalignments between teachers' and the PD instructor's discourses?* And *what is the discursive mechanism that affords change in teachers' pedagogical discourse, from alignment with APD to alignment with EPD?*

METHOD

The study took place in the context of the TEAMS PD program, a program that included 16 4-hours sessions over two years (2016-2017). The PD introduces teachers to the 5 Practices for Orchestrating Productive Mathematics Discussions (Smith & Stein, 2011) as well as to Accountable Talk™ tools (Heyd-Metzuyanim et al., 2018b). Teachers participating in this program were relatively experienced (ranging from 15-31 years of teaching). 14 of the participants were middle-school teachers and two were elementary school teachers. Nine teachers continued from the 1st to the 2nd year.

To address the above questions, we watched and partially transcribed all 16 PD sessions. Partial transcriptions were intended to capture the activities of the session, as well as important conversations and themes. Based on these partial transcriptions, we fully transcribed discussions about instruction from five sessions: the 2nd, 3rd, 9th, 14th and 15th. This allowed us to closely study changes in teachers' pedagogical discourse from initial stages of the PD to last sessions.

Our analysis included first delineating misalignments between the teachers' and the instructor's pedagogical discourses. A misalignment between APD and EPD was noted when discoursants declared (implicitly or explicitly) valuing similar actions yet the

reasons for those values differed (were aligned with different discourses). Next, we noted the object of misalignment, what the misalignment referred to: the *what*, *when*, *how* or *who* of the Pedagogical Discourse. During our analysis, a new category came up, which we named "friction". Friction was identified when the APD was questioned explicitly, and as a reaction, there was a certain move towards EPD. Non-friction was identified when misalignments occurred, yet they were not questioned or negotiated. Our final analysing step included documenting all misalignments and their object, and determining whether they included friction or not. The documentation was in a table similar to Table 1 (in Table 1 we only present one example).

FINDINGS

Our first finding is that there was a difference in the pedagogical discourse of the instructor and the participating teachers, as expected, and that multiple occasions of misaligned framings of the instructor and teachers occurred as result of these differences. An example of such a misalignment in framing is taken from the 1st session. In this session, the teachers worked on a rich mathematical task, which offered multiple solutions and opportunities for discussing mathematical concepts. After engaging with it as learners, the teachers discussed the task, their work on it, and how this way of working on a mathematical task could (or could not) be adapted to their classroom. One of the teachers said:

Teacher	This is nice for strong kids. Those that really have the motivation and patience to see more and more ways (for solving the problem). But there are students that are weak, that each additional way just confuses them more and more. They don't know how to solve. At a certain moment they understand something and they want to stop there. And all this discussion is just unnecessary for them.
Instructor	We will talk a lot about this issue. Now let's turn to….

We suggest that this is an example of misalignments of frames containing no friction. The teacher focuses on students' ability, and makes a direct connection between ability and the type of learning-opportunities that the students should be provided with. She stresses that posing a cognitively demanding task that invites multiple solutions, which is a highly valued teaching action within the EPD, is only appropriate for "strong kids". She adds that multiple solutions "only confuse" the "weak kids". In contrast, materials such as "the 5 Practices" or AccountableTalk, which reify the EPD, stress that cognitively demanding tasks and discussions that include multiple solutions are appropriate for *all* students. In this example the misalignment is about the *who* (who are the students that would benefit from discussions of multiple solutions) and the *what* (discussions are confusing for weak kids). The reason we claim the example above is of "non-friction" is that, despite the teacher clearly declaring narratives that are incongruent with the APD, the instructor does not contest these narratives, only signals that they may be under dispute ("we will talk about it").

The next example includes friction: this example took place during the 2nd session, while discussing the instructional actions that the instructor performed when the teachers were working on a mathematical task:

Teacher	And you passing between us, you directed "correct?" "not correct?" and then each group got up and presented on the board and we saw that there is no situation of something (a solution) being wrong. Each way (solution) is correct.
Instructor	Each way is correct?
Teacher	Almost. If you get the correct answer. There could be various… and the students love it. They don't like just your template, when you come and talk. They like it also on their own. They want to raise different creative ways. And they love coming and sharing.

In this example, the action which was valued by the instructor and the teacher is of "students presenting various solutions to a problem". The teacher explicated that she values this action since students get a chance to present their own solutions and that they are safe because "all solutions are correct". This narrative is misaligned with the EPD since according to the EPD, not *all* solutions are necessarily correct. On the contrary, some may be incorrect, providing a chance to discuss and refute them. The reasons for valuing various solutions, according to the EPD, are surfacing conceptual links between different solutions, as well as affording authority and agency for students to creatively come up with their own solutions. The teacher in this example aligns with the *how* (letting students present on the board), but she does so mostly because of valuing students' emotional enjoyment, not because of the *what* (the content to be learned through such a discussion). The evidence for friction can be seen in the instructor explicitly questioning this framing, asking: "Each way is correct?". This is followed by a correction ("almost") that brings the teacher's discourse a small step closer to the EPD (by implicitly acknowledging that not all solutions are correct).

We found other examples of misalignments during our PD sessions, which we documented in a table similar to Table 1. The instance shown in Table 1 is taken from the 14th session, when a teacher presented a short part of a video-taped lesson she taught, where she posed a cognitively demanding task and invited three students to present solutions on the board. The teacher talked of how important it was for each student to present her solution to the problem. She stressed that the students were feeling empowered through this process. However, when the instructor asked the teacher whether there was an attempt to link between the various solutions to allow for mathematics conceptualization, the teacher did not seem aware of the opportunities for making such links. The instructor then led the teachers to think together about links that could have been made between the various solutions presented on the videotaped board. Hence, this is an example of an instance with some friction: the teacher partially aligned with the EPD-valued action of presenting various solutions on the board, valuing it mostly for the agency it gave students to come-up with their own solutions. The instructor valued the action for that reason too, but stressed another reason –

making links between the different solutions to enhance conceptual understanding. Through the discussion around the imagined teaching action that could have been made at that point, friction occurred and the teachers' discourse moved towards the EPD-valued action of linking between solutions.

Valued pedagogical action	Teachers' reasons for valuing the pedagogical action aligned with APD	Instructor's reasons for valuing the pedagogical action aligned with EPD	Object of talk and friction
Present different ways of solving a problem	Linking between each student's ideas to the problem asked. Students feeling good.	Linking between ideas – linking between the three suggested solutions to help conceptualize a mathematical idea. Enables students' struggle	Discussing a part of the teacher's video-taped lesson The reasons are questioned: friction

Table1: Valued pedagogical actions, reasons for those values according to APD and EPD, the object of talk misalignments occurred.

Our findings also shed light on occasions that seem to give rise to friction between the APD and EPD: we found that friction occurred when the discussions related to clear and concrete representations of teaching that were in front of all participants' eyes, or at least in their memory. These occurred, for example, in (1) reflective discussions on the teachers' own work and the instructor's mediating remarks which took place earlier in that session, and (2) when teachers discussed clips of videotaped lessons. In contrast, most of the cases of misalignment with no friction occurred when such concrete teaching representations were not available. For example, when a teacher talked about her actions in the classroom that were not videotaped, other PD participants (including the instructors) could gain very limited access to the particularities of the valued actions and therefore no friction occurred.

DISCUSSION

Often teachers' learning is studied using questionnaires about initial and end conceptions and beliefs about teaching and learning. In this paper we suggest an alternative approach, which focuses on teachers' discourse. This discursive approach has become more common in mathematics education (Herbel-Eisenmann et al., 2017) but is less common in studying teacher's learning. In the past, we have defined the gap between the teachers' and PD instructors' conceptions of teaching and learning through the notion of Pedagogical Discourses and misaligned frames (Heyd-Metzuyanim et al., 2018a; Heyd-Metzuyanim & Shabtay, 2019). In this study we began to explicate the process that seems to enable promoting change in teachers' discourse, from discourse that is aligned with APD to one that is aligned with EPD. Our findings suggest that this process is best promoted by instances of friction – interactions or discussions where misalignments regarding valued actions of the EPD are surfaced and negotiated. Previous research (Heyd-Metzuyanim et al, 2018a) has shown that it is not easy to surface misalignments, and that these can often go unnoticed. This, because teachers

are quick to endorse (or at least declare alignment with) certain valued teaching actions of the EPD, such as giving students agency or promoting conceptual understanding. The more difficult narratives to change, as has been shown in this study too, are the narratives regarding *what* should be discussed and conceptualized (which mathematical concepts and ideas) and *who* can benefit from such teaching actions ("strong" students, or all students). Our findings hint that friction around the *what* is best achieved through highly concrete teaching representations such as focused discussions on the mathematical ideas presented in a videotaped lesson (Heyd-Metzuyanim & Nachlieli, 2019). Our findings also show that the *who* (who should be taught exploratively) is even more difficult to achieve friction on. This may be because teachers are quick to privatize their thoughts about the issue (deeming them as unacceptable according to the PD leaders, or not "politically correct"). It may also be a result of the difficulty to obtain strong evidence in the here-and-now of a PD discussion, regarding certain students' ability to learn exploratively. Future studies would be necessary to establish the conditions for friction on this difficult misalignment between the APD and EPD.

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EFFECTIVENESS OF A JAPANESE PRESCHOOL MATHEMATICS GUIDED PLAY PROGRAMME IN TEACHING MEASUREMENT

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This article, using statistical and interpretative analysis, discusses how Japanese preschool children (5–6 years old) performed mathematical tasks after a 9 month-long intervention programme. We included 35 children in 2 similar Japanese kindergartens in the study, conducting interventions for 9 months, and examining their performance in pre-/post-interviews. We conducted statistical analysis to reveal semantic similarities among interview items using correspondence and cluster analysis. We concluded that first, the comparison of numbers and quantities (length) improved, which was statistically significant, and second, 2-dimensional measurement was more difficult for children to understand than 1-dimensional measurement.

INTRODUCTION

Japanese preschoolers have not formally learned mathematics, but naturally acquire such skills in their daily life. Japanese studies related to mathematics education, therefore, have been limited. After drastic curriculum reform in 2018, the Course of Study stated that Japanese children should understand numbers and quantities before elementary school. We must now take action on preschool mathematics education.

Recent studies have focused on mathematics education for preschool and younger elementary school students (e.g., Brandt, 2013). Hachey (2013) states that mathematical conceptual change in young children is as effective as a substantial change in early childhood mathematical teaching practice, as a large body of developmental research advocates that young children are born mathematicians, and therefore, early childhood mathematics education is vital to foster their competencies. Several research papers examine the development of research-based preschool mathematics curriculum in different countries (Clements & Sarama, 2013; Chard et al., 2008; Vogt, Hausera, Steblerb, Rechsteinera, & Urecha, 2018). For instance, Clements and Sarama (2003) stated that a set of learning trajectories and framework comprising criteria and procedures for creating scientifically-based curricula using those trajectories makes it possible to develop an early mathematics curriculum. Vogt, Hausera, Steblerb, Rechsteinera, and Urecha (2018) developed play-based pedagogy and examined its effectiveness. Further, Worthington and van Oers (2016) note the importance of respecting children's cultural knowledge in play. Unfortunately, in Japan, with a different cultural and social environment than in the West, little is known about children's mathematical play and learning. To establish an effective mathematics education curriculum, we must understand the way our children learn. Thus, in this study, we examine if Japanese children develop their mathematical skills and concepts

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similar to children from other, primarily Western, studies. We use statistical and interpretive analysis of pre- and post-intervention interviews to reveal how Japanese preschool children (5–6 years old) respond to mathematics questions. Our research questions are: whether and how children improve their performances after the intervention; and how the intervention programme affects children's performance, focusing on some outstanding items from the statistical analysis.

DESIGN AND METHODS

In an earlier paper, we proposed a framework for constructing an early childhood mathematical curriculum (Matsuo, 2016). The theoretical and methodological underpinnings of our programme are social constructivism (Ernest, 1994), mathematical guided-play (Weisberg, Hirsh-Pasek, & Golinkoff, 2013) and the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982). This programme presents the scope for the content of preschoolers' activities, while the learning process is explained from the viewpoint of sequence, based on the relationship between age and modes of representation and focusing on activities for children to create a programme tailored to these concerns. We then proposed a mathematical education programme to enable a smooth transition from preschool to elementary school (Matsuo, 2017). The aim is for young preschool teachers to understand how to incorporate mathematical content into children's play and recognize activities to make children think and act mathematically. For the interventions, we developed 6 play-based activities on numbers, measurement, and shapes. For instance, regarding numbers, we developed a board game with a die for children to get to know about numbers and numerals from 1 to 20. Regarding shapes, we also developed a shape game, called 'making a fish', in which children were expected to make as many patterns of the congruent fishes as possible on a given frame, using congruent coloured shapes, such as right angle isosceles triangles, trapezia, parallelograms and squares.

2 kindergartens in an urban area of Kanagawa Prefecture of Japan participated in our pilot study. The children are all Japanese, coming from middle- and upper-middle-class families. There were a total of 35 children (28 boys, 7 girls). To evaluate the improvement of children's mathematical skills after these activities, we developed pre/post-interview questions. Pre-interviews were conducted in June 2017, and all 6 activities were implemented twice in the collaborative action research (Raymond & Leinenbach, 2000); therefore, 12 activities were implemented after the pre-interview. A total of 35 5- to 6-year-old children participated throughout the school year. After completion of the activities, a post-interview was conducted with each child in February 2018. Both interviews contained 15 common questions. The post-interview contained an additional 5 questions on cardinal and ordinal numbers. Among the common interview items, 10 were related to numbers, 4 to measurement, and 1 to shapes. We designed these questions based on a questionnaire for preschool and primary school teachers asking for mathematical contents to teach during preschool.

Methodologically, the authors conducted structured clinical interviews (Goldin, 1998). To alleviate children's anxiety, the teachers conducted the interview while one of the authors sat nearby, recording. The assigned teachers rehearsed the interviews to consistently use mathematical words and expressions. During the implementation, each child randomly came to the room. The teacher sat in front of the child. The pre-interviews took 15–25 minutes, and the post-interviews took 10–15 minutes. We analysed the data statistically and used it to observe the children's actions to determine improvement or where there was no qualitative change. The activities for interventions were also recorded by DVD camera for additional information to discuss the reason for the changes later in the article.

RESULT OF THE INTERVIEWS

The interview questions were scored; children received 1 point for every correct response and 0 points for incorrect responses. The total score of the common items between pre/post-interviews was 15 points. We calculated the participating children's total score in pre/post-interviews and the average percentage for each item. The average total pre-interview score was 10.14; that of the post-interviews was 11.40. The paired t-test shows statistical significance ($p < 0.01$, $t = -3.683$, effect size (d) = 0.780).

Table 1 shows each item and the average percent of right answers from pre/post-interviews.

		Pre	Post
S/N	Questions	Mean score (standard deviation)	Mean score (standard deviation)
1	'How many marbles do you have?', preparing 20 marbles in a container.	60.0% (0.497)	77.1% (0.426)
2	'Read the number shown on a die', randomly showing each face of a die.	97.1% (0.169)	100% (0.000)
3	'Take the same amount of marbles', showing 6 faces of a die.	91.4% (0.284)	100% (0.000)
4	'How many numbers are there altogether?', letting children throw 2 dice.	94.3% (0.236)	94.3% (0.236)
5*	'Which number on the die is bigger and what is the difference between the two numbers?'	25.7% (0.443)	68.6% (0.458)

6	'How many apples are there altogether?', showing 10 apples placed in a row.	91.4% (0.284)	88.6% (0.323)
7	'Circle the third apple from the far right', using the same diagram from no. 6.	71.4% (0.458)	71.4% (0.458)
8	'Circle the 4 left-most apples', showing the same diagram from no. 6.	37.1% (0.490)	48.6% (0.507)
9	'How many marbles are there altogether?', showing 2 and 3 marbles separately.	91.4% (0.284)	100% (0.000)
10	'How many marbles remain if you take 1 away?', showing 5 marbles in a container.	94.3% (0.236)	100% (0.000)
11a	'Take 20 blocks,' showing a number of them on their right hand.	-	91.4% (0.280)
11b	'Tell me what the third block from the right is', showing the result of question 11a.	-	74.3% (0.437)
11c	'Tell me the fourth block from the left', showing the result of question 11a.	-	57.1% (0.495)
11d	'Show me two blocks from the top'	-	74.3% (0.437)
11e	'Show me the fifth block from the bottom', showing the result of question 11a.	-	77.1% (0.420)
12	Prepare 4 pencils, each of a different length, as shown, and ask, 'Which do you think is the longest? Tell me the order from longest to shortest.'	91.4% (0.284)	100% (0.000)
13*	Show 4 pencils, each of slightly different length as shown, and ask, 'Which do you think is the longest pencil? Tell me the order from longest to shortest.'	34.3% (0.482)	60.0% (0.497)

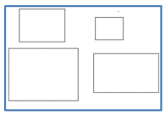
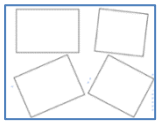
14	Prepare 4 different rectangles as shown, and ask, 'What do you think is the biggest rectangle? Tell me the order of the area from biggest to smallest.'		91.4% (0.284)	100% (0.000)
15	Prepare 4 different rectangles as shown, and ask, 'What do you think is the biggest rectangle? Tell me the order of the area, from biggest to smallest.'		42.9% (0.502)	31.4% (0.471)
16	'How many triangles are there in the diagram? Point them out', showing a diagram of the triangle.		0.0% (0.000)	0.0% (0.000)

Table 1: The interview items and percentages of the right responses

MacNemar's test shows that the result of Q5 and Q13 (marked with an asterisk in Table 2) showed a statistically significant difference ($p < 0.01$, χ^2=13.067, effect size (φ) = 0.398 in Q5 and $p < 0.05$, χ^2=4.267, effect size (φ) = 0.221 in Q13). We inferred that the improvement on these questions influenced the significant difference of the performance between pre- and post-interviews.

Statistical analysis

We investigated semantic similarities among interview items using correspondence analysis and cluster analysis, excluding 7 interviews items with average percentages of right answers of 0 and 1 (Q2, Q3, Q9, Q11, Q12, Q14, and Q16). Figure 1 shows the results of the pre-interview, the correspondence analysis is on the left and the cluster analysis on the right.

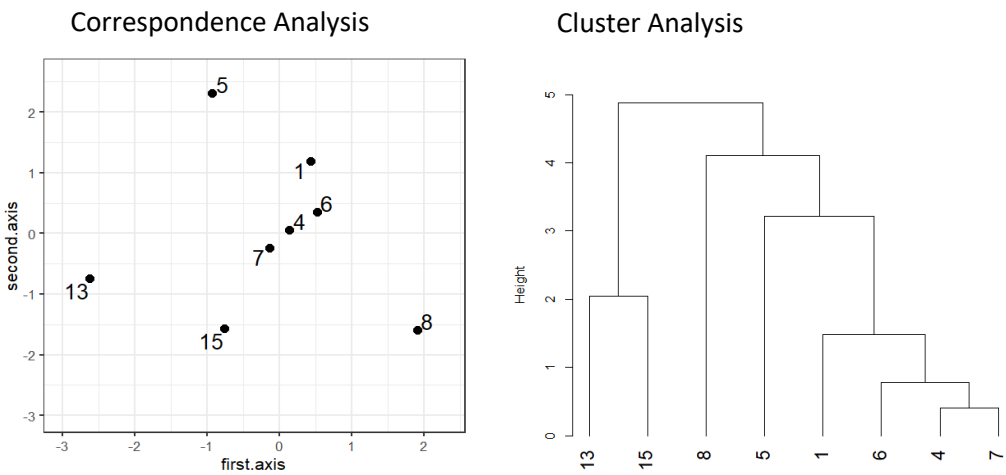


Figure 1: Pre-interview correspondence and cluster analysis

The numbers correspond to the numbers of the interview items. On the left of Figure 1, the contribution ratio of the first and second axes are 28% and 23%. In order to group

8 items on the left of Figure 1, cluster analysis was applied, which resulted in the figure on the right. One group is Q13 and Q15, and other clusters can be observed. Figure 2 shows the results of post-interviews with the same analysis of the pre-interviews.

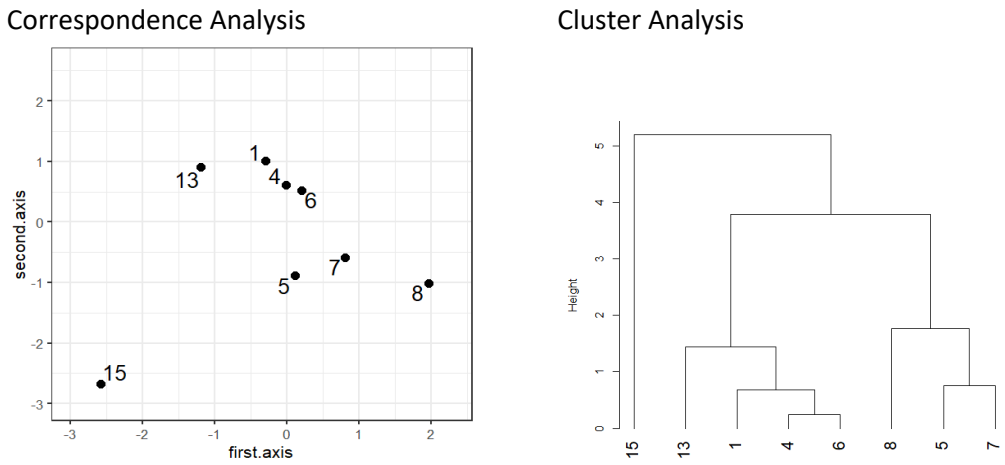


Figure 2: Post-interview correspondence and cluster

By comparison, on the left of Figure 2, the contribution ratios of the first and second axis are 32% and 19%. Cluster analysis was applied in the same way as before; the items are grouped into Q15 and others. Even though Q13 and Q15 are grouped together in the results of the pre-interviews, they belong to different groups in the results of the post-interviews. Q13 and Q15 are similar questions regarding measurement; however, these results indicate that Q15 may be more difficult for children or that the intervention regarding length comparison was effective for children's learning.

Specifics about Q13 and Q15

Q13 and Q15 are both related to measurement area; Q13 is 1-dimensional and Q15 is 2-dimensional. In the pre-interview, on Q13, only one child tried using paperclips to measure the pencils, but did not understand how to use them, while in the post-interview, 8 children used clips to measure the pencil, known as 'comparison by arbitrary unit'. In Q15, in the pre-interview, 5 children tried using some tools including transparent paper, a ruler, and paper, but only one correctly used them to measure the rectangles. On the other hand, in the post-interview, 11 children used these tools, including grid paper. Only 3 of those 11 reached the correct answer. The children who incorrectly measured compared the length and width of rectangles. We also observed that even though some children identified the differences of the areas of the rectangles at a glance, they incorrectly used the tools and ended up with the wrong answer. We inferred that the intervention programme to measure the length and area influenced children's responses in the post-interview, but they were confused about the difference between the length and area.

Measurement activities in the intervention

It seemed that the intervention programme made an impact on children's performance in the post-interview. Using examples of teaching from the intervention programme,

we will now discuss the relationship of the intervention programme to children's performance. In one preschool, a teacher (from the first course) taught the children how to measure the length of things carefully using paperclips. They measured different items by connecting clips, measuring the edge of the desk, the mat, the height of the teacher, and so on. Another teacher taught the second course of the same class; the children connected 8 paperclips and then used them to measure shorter objects than the ones measured in the first course. After the children used the connected paperclips to measure a building block or a box, the teacher asked the children to find objects with lengths equivalent to 8 paperclips. Next, they searched for objects with lengths equivalent to 16 paperclips. They lined up 16 red magnets and measured them using the paperclips. In the area measurement activity, the children measured items using arbitrary units. Although the activity was a code-breaking game, the comparison was not related to solving the code. When the code was solved, figures whose areas were objects for measurement were supposed to be arranged in descending order. A teacher directed the children to count the square grids covered in rectangular paper to measure their areas. The children copied the target rectangle onto tracing paper, put the tracing paper on top of grid paper, and counted the number of squares. The sizes of the 3 kinds of rectangles (32, 40, and 48 squares) were relatively large, and the children actively counted. Some children made mistakes when counting accurately, and other friends helped them. Finally, the paper with a number written on it corresponding to a letter was turned in, which solved code. Letters with meaning were arranged in order so that the numbers became larger. The same approaches of teaching were observed in both kindergartens.

CONCLUSION

First, the statistical analysis showed that children improved on the interview items regarding comparisons of numbers and quantities, which led to the overall improvement of performance in the programme. Second, the intervention programme indicated that comparison in 1 dimension was understood well by the children. Children learned to use tools to measure, but some incorrectly measured length to determine the area of rectangles. On the other hand, the comparison in 2 dimensions using arbitrary units, might not be appropriately taught by the teacher, or the content to learn measurement in 2 dimensions might not be suitable for the children's developmental stage. We need to investigate the reason carefully in the future. This paper has a few limitations: it did not cover the relationship between the intervention programme and the results of the interviews in topics other than measurement. In addition, the sample size should be larger to more fully understand the effectiveness of the programme.

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FIRST-YEAR UNIVERSITY STUDENTS' INTERPRETATION OF PRAXEOLOGY WHEN SOLVING AN INTEGRATION TASK

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From an institutional perspective, this study focuses on first-year engineering students at university and their use and justification of techniques when solving a task involving integration by parts. Students' discussions are analysed through observations of group work. Despite encouraging the students to justify their techniques, findings indicate that the discourse focused on rules and formulas without mathematical justifications. Hence, the applied techniques became almost exclusively of pragmatic value, focusing on whether or whether not they led to an answer. Further, this suggests that even for university students, techniques applied for simple routine tasks in integration, to a large extent become "de-mathematicised".

INTRODUCTION AND RESEARCH QUESTION

This study is a part of MatRIC's (*Center for Research, Innovation and Coordination of Mathematics Teaching*) research activity and focus on Norwegian engineering students at their first year at university. During this year, a calculus course is mandatory for all the engineering students, and integration is an essential topic in the course. In this paper, the goal is to investigate how students interpret the mathematical *praxeology*, when working with a routine task involving integration by parts. Sofronas et al. (2011) point to three significant "sub-goals of the integral" as essential for students understanding: 1) the integral as net change or accumulated total change, 2) the integral as area and 3) techniques of integration (p. 138). By focusing on "techniques of integration", this study is not primarily concerned with whether or whether not the students master the techniques, but rather with the discourse and students' justifications of methods and arguments. At the institution in focus of this study, mastering integration rules and techniques such as substitution and integration by parts, is an important part of students' curriculum and their final exam. On the other hand, the university teacher also expresses the importance of students obtaining an in-depth understanding of the fundamental concepts and ideas of integration. Hence, a relevant issue is if these two aspects imply a kind of contradiction, or if substantial mathematical arguments and ideas could be traced also in the work with routine integration tasks. Research in mathematics education has generally shown that students' experiences with integration often are limited to techniques and integration rules (Grundmeier, Hansen, &, Sousa, 2007; Mahir, 2009; Sherhan, 2015). These and similar studies are commonly rooted within cognitive perspectives. Through this study, an attempt is made to add to this body of research, by applying an institutional perspective, rooted within the anthropological theory of didactics (ATD). In line with the introduction above the research question relevant for this study is:

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What characterizes first-year university students' interpretation of praxeology when solving a task involving integration by parts?

The term "praxeology" will be explained in the next section, where the ATD perspective is accounted for.

THEORETICAL FRAMEWORK

The anthropological theory of didactics (ATD), as initiated by Chevallard in the eighties (Bosch & Gascón, 2014) offers the possibility of taking institutional aspects into account. The mathematical content is in the foreground of the analysis, and learning is modelled through students' activities as "thinking modes are […] seen as dependent on the social and cultural contexts where they develop" (Artigue, 2002, p. 248). In this perspective, a "praxeology" serves as a unit from which one can analyse human action, composed be the parts "praxis" and "logos" (Chevallard, 2006). "Praxis" consists of a certain type of tasks and a set of "techniques" to solve them, while "logos" contains two levels of descriptions and justifications of the praxis. The first level is called "technology" and concerns the discourse of the techniques. The second level is the "theory" which provides the basis for the technological discourse (Bosh & Gascón, 2014, pp. 68-69). Hence, a mathematical praxeology consists of four main components: A collection, T, of *types of tasks* which define the nature and goals of the activity, a corresponding collection, τ, of *techniques* available to accomplish each type of task, a *technology*, θ, that justifies these techniques and a *theory*, Θ, that justifies the technology (Hardy, 2009, p. 344).

In this study, the framework described above is combined with "pragmatic" and "epistemic" values of techniques. According to Artigue (2002) techniques is of pragmatic value if the focus is on their "productive potential" (p. 248). In the case of solving integral tasks this could be interpreted to mean a method that works and leads to an answer. Techniques associated only with pragmatic values, can hardly belong to "logos", since no attempt of mathematical justification are made. If techniques are of epistemic value, the techniques also serve as sources for insight into mathematical knowledge. These potential insights could belong both to the "technology" and "theory" depending on the content. Further, a "routine task" is here defined as a task situated within the context of pure mathematics and that is intended to be solved only by using specific procedures. In a mathematical praxeology such procedures belong to "techniques", although techniques in itself is a boarder term, involving all kinds of solution strategies. In addition, to be labelled as a "routine" task, this type of tasks should be included more than twice in the textbook and at least once on each exam for the last three years. In line with Artigue (2009), techniques in terms of being routines, run the risk of becoming "de-mathematicised", which implies detachment from mathematical discourses.

EDUCATIONAL SETTING AND RESEARCH METHODS

This study focuses on two so-called "study groups" consisting respectively of three and four students. One group contained students from the mechatronic engineering

program and the other group contained students from the computer engineering program. Both the groups attended the same lessons. Organized by the university teacher, all the engineering students (about 300) were divided into smaller study groups at the beginning of the semester, based on the engineering program that they attended. Lessons were held three times a week, each lasting for two hours, and once a week the study groups met to work with tasks. The mechatronic students and computer students met at different weekdays. One study group from each program was randomly selected among those groups who fulfilled the criteria of containing both male and female students. To be selected, the study groups also had to volunteer for the research.

In cooperation with the responsible university teacher, a set of tasks was designed and made as a mandatory assignment so that all the students should solve the tasks, preferable in their prearranged study groups. We estimated that it should be possible to solve the tasks during the two hours they had available in one study group session. The task involving integration by parts and that constitutes the basis of the analysis in this paper is: "Find the integral: $\int 2xe^{2x}dx$". In addition, the students were explicitly told to argue for their choice of technique. This task is in line with the definition of a "routine task" provided in the previous section.

To catch the details in students' arguments and solutions, the groups were observed with me present as a researcher. The observations were carried out using a voice recorder, recording throughout their two hours of work. A camera was used to take pictures of students' notes and drafts during their work. Except from guiding the students if they got stuck, I aimed at minimal intervention and took on the role as "observer-as-participant" (Wellington, 2015). Carrying out the analysis of the group work was done by coding the transcribed voice recordings in accordance with the theoretical framework presented in the previous section.

To be able to account for the mathematical praxeology of the institution, the lessons related to integration were videotaped and analysed. In addition, also the textbook and the exam tasks for the last three years were analysed.

ANAYSIS

This section is divided in two parts: First, based on the observed lectures, textbook and earlier exams, the mathematical praxeology of the institution, related to integration by parts, will be accounted for. In the second part, observations of students solving tasks will be analysed. This structure is to a large extent inspired by Hardy's (2009) research on limits of functions.

Mathematical praxeology at the institution

Since the focus of this study is on students' interpretation of praxeology related to a specific task on integration by parts, the overview of the mathematical praxeology at the institution will be limited only to contain a type of integration by parts tasks relevant to the actual task in focus of the study. The actual task, "Find the integral:

$\int 2xe^{2x}dx$", will serve as an illustrative example. Recapitulated, the mathematical praxeology is constituted by type of task, techniques, technology and theory.

Type of task, T (integration by parts): Find the integral $\int axe^{g(x)}dx$.

Description: g is a polynomial, and the quotient $g'(x)/x$ should somehow contain x, so that substitution cannot be directly applied. There are variants of integration by parts tasks, involving both logarithmic and trigonometric functions, but in the textbook and lessons the variant above seemed to be taken as the "simplest" type, since it came first in the line of examples. For the task in this study, $a = 2$ and $g(x) = 2x$.

Technique τ: From the expression $\int U\,dV = UV - \int VdU$, define $U = ax$ and $dV = e^{g(x)}dx$. From these expressions, find dU and V. For the task of this study, $U = 2x$ and $dV = e^{2x}dx$. Since $dU/dx = 2$, $dU = 2dx$. $V = \frac{1}{2}e^{2x}$. Hence $\int 2xe^{2x}dx = 2x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 2dx = xe^{2x} - \frac{1}{2}e^{2x} + C$. In the lessons, alternatives to the textbook notation above were offered, in terms of writing u' instead of du/dx. The equation then becomes $\int u\,v'dx = uv - \int u'vdx$.

Technology θ: From the textbook, in terms of various tasks and examples, the students could be expected to consider integration by parts if the integrand consists of two factors, where one factor is composed of two functions, involving for example Euler's number, e. Arguments for substitution *not* to be appropriate in these types of tasks, will also in some sense count as an argument for integration by parts due to the limited number of alternatives. Various arguments could be made for applying integration by parts and the textbook suggests to "look for a factor of the integrand that is easily integrated" (Adams & Essex, p. 335). Further, on the same page, it is argued that the choice of U and dV should be made so that VdU is easier to integrate than $U\,dV$. In lessons, and in line with the notation used there, it was suggested that u should be chosen so that u' "turns out to be simpler" than u, and that v should be chosen so that the integral of v does not become "more complex" than v.

Theory Θ: Integration by parts rests on the product rule for differentiation, which in turn depends on the fundamental theorem of calculus. The product rule and the fundamental theorem of calculus have been treated both in teaching and in the textbook.

Students' solutions

The task "Find the integral: $\int 2xe^{2x}dx$", forms basis of the analysis of students' discussions. The students were verbally told to argue for their techniques, and to justify why they were appropriate to use. Group 1, consisting of Chris, Megan and Sue immediately started the search for solution strategies.

Megan:	But if we put u as two x, then you have u times e to the power of u
Chris:	du equals two dx. Then we have e to the power of u, but when you have x in front here

Megan: You put u in front, because you have two x which is u

Magan's statement indicates substitution as "e to the power of u" is suggested. This is followed up by Chris in the next line, indicating that "du" equals "2dx", but this argument is not traceable further in the discussion, as it proceeds. Based on Megan's last statement, the expression ue^u was established as the new integrand, and further just treated as xe^x.

Sue: Here we have a sweet little expression. It says the integral of x times e to the power of x.

Chris: If you look at page 335 it is almost exactly the same expression.

Sue found this formula in the list from the textbook's formula section which states that $\int xe^x dx = (1-x)e^x + C$. Chris refers to an example in the textbook (at page 335) where $\int xe^x dx$ is solved. Since the textbook uses this as an example to illustrate integration by parts, Sue seemed to take this into account as she proceeded.

Sue: Yes, u times the derivative of v equals u times v minus the integral of the derivative of u times v.

Sue read the integration by parts expression directly from her own lecture notes and concluded that one had to find a u and a v. She then continued by giving an argument for what such choices should be.

Sue: The way NN writes this in lectures and such is that if it is stated like this, then you take two x as u, and then this [points to e^{2x}] is the derivative of v. Do you get it?

Sue's argument consisted only of a reference to how their university teacher normally chooses u and v'.

Group 2, the data engineering students, consisted of Steve, Seth, Tom and Emily. They approached the task in similar terms as group 1.

Emily: We have one, two, at least, so I think we must do such substitution by u and such.

Her argument seems to be related to the number of factors in the integrand. Although the group seemed to agree on this method, they were not sure what to substitute. The content of the discussions changed, as Steve looked to the formula section of the textbook for a suitable formula.

Steve: We have two x, can we then use 2?

Seth: We can

Steve: Because if we can, we can put 2 and then we get it, but I do not know if we can […]. Does it matter what we take it as the power of? Because the derivative of e is just e anyhow […] Does it then becomes two times x minus one, times e to the power of two x, plus c?

From the context, it seems like Steve's first comment referred to the exponent of e and whether or whether not the factor of two influences on the integration. Due to the exponent being $2x$, he was then applying the formula $\int xe^x dx = (x-1)e^x + C$ to achieve $(2x-1)e^{2x} + C$ as a possible answer. To check the answer, Seth put forward the following argument:

> Seth: I used that formula […] and I took the derivative and got what we started with.

Seth used the formula Steve came up with as a starting point, but he replaced the factor $(2x-1)$ with $(x-1)$. He then applied the product rule for differentiation $(uv)' = u'v + uv'$ on the expression $(x-1)e^{2x}$, letting $u = x-1$ and $v = e^{2x}$. This left him with $1 \cdot e^{2x} + (x-1) \cdot e^{2x} \cdot 2$. When the expression was simplified he performed an error when multiplying with the factor 2, as he treated the expression like $(1 \cdot e^{2x} + (x-1) \cdot e^{2x}) \cdot 2$. By differentiating his solution, he seemingly arrived at the integrand. This eventually also convinced the rest of the group.

DISCUSSIONS

In both groups the type of task initially was interpreted to concern substitution, and not integration by parts as intended. For group 1, the discussions were limited to investigate what happened if $2x$ was replaced by u, which led them to the expression ue^u as the new integrand. In their new integrand, u was replaced with x, based on the notation in the list of formulas. The contradictions in the double shift, $2x \to u$ and $u \to x$ were not considered. In group 2, Emily briefly linked the use of substitution to the number of factors being "at least two". Although further elaborations were lacking, this argument is categorized as technology, since it indicates elements of mathematical justifications. Another part of the discussion was concerning $2x$ as the exponent of e, and if the formula could be applied also in the case of $2xe^{2x}$ as the integrand, instead of xe^x. Seth's testing of the answer by differentiation is also categorized as technology, even though a calculation mistake led him to an incorrect conclusion. Sue and group 1, eventually arrived at a correct interpretation of the type of task but no mathematical arguments were put forward by Sue nor the others to substantiate this assertion. The choice of u and v' was justified only by referring to their university teacher. The figure below summarizes the findings for both groups.

Praxis		Logos	
Type of task: $\int 2xe^{2x}\,dx$ Group 1: -Initially interpreted to involve substitution, but eventually integration by parts Group 2: -Interpreted to involve substitution		Techniques of epistemic value	Technology: Group 1: -None Group 2: -At least two factors in the integrand to use substitution -The derivative of the integral results in the integrand
Technique: Group 1: -Incorrect shift of variables -Use of formula from formula list -Integration by parts Group 2 -Use of formula from formula list	Techniques of pragmatic value		Theory: Group 1: -None Group 2: -Differentiation and integration as inverse operation relates to the fundamental theorem of calculus

Figure 1: Students interpretation of praxeology in the case of the actual task

From the excerpts one can observe that students' use of techniques and their discourse are characterized by referring explicitly to sources in terms the textbook, the list of formulas and the lessons. With the exception of Seth's argument for applying the formula, the techniques appeared only to be of pragmatic value and the question of what might "work" or not dominated the discussions. Questions of why the techniques works were absent and the lack of mathematical justifications persisted throughout the dialogues.

CONCLUSIONS

In line with the claim of Artigue (2009), this study strengthens the assumption that techniques characterized by routines run the danger of becoming "de-mathematicised". Despite the "simplicity" of the task and the fact that integration by parts should be familiar to all the students due to required background knowledge, the type of task was incorrectly interpreted by both groups. Detachment from mathematical arguments and justification persisted, even though the students explicitly were encouraged to justify their techniques. Hence, the logos of students' praxeology suffered, both with respect to technology and theory. Since other types of integration tasks in the assignment (not focused on in this paper) created considerably more sophisticated mathematical discussions, it seems like the shortcoming of logos primarily is linked to the type of tasks. In other words, for routine tasks in integration, the students seem to hold a belief that mathematical justifications are not expected.

Since "techniques of integration" could be regarded as one of three significant areas in the topic of integration (Sofronas et al., 2011), the solution is probably not to omit these types of tasks completely. Changing institutionalized norms calls for awareness and

the willingness to modify certain kind of practices. One suggestion might be to slightly reformulate these types of tasks, for example from "Find the integral…" to "Find the integral and explain why your method works". Another possibility is to combine several routine tasks and ask for similarities and differences, or to look for patterns in the different solutions. If solutions of some types of tasks consequently are withdrawn from the expectancy of mathematical justifications, this might imply some educational drawbacks. For example, techniques without justifications could be perceived of as meaningless by the students. In turn, this might leave the impression of "having to do math just to get the diploma" as one of the students said as a joke during the work with these tasks. Further, such views could prevent engineering students to see mathematics as valuable for their future careers.

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AUTHENTIC HI-TECH PROBLEMS IN SECONDARY SCHOOL MATHEMATICS

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The current study investigates Authentic Hi-Tech – AHiT math problems which were retrieved from the Hi-Tech industry and simplified for formal secondary school math lesson. We first characterize the AHiT math problems according to Schoenfeld's framework of problem solving. We then investigate the perceptions of two stakeholder groups: math experts and policymakers and math teachers with respect to the characteristics of the AHiT math problems and their feasibility. Based on the Delphi methodology, we employed two stages for analysis: an open-ended questionnaire to interview ten stakeholders, and subsequently a Likert-type questionnaire for collecting data from 40 stakeholders. Main results suggest two aspects that underlie the use of the AHiT math problems: the benefit for learners and the problem characterization.

INTRODUCTION AND THEORETICAL BACKGROUND

One of the biggest challenges secondary school teachers face when teaching math, is keeping students interested and motivated (Harackiewicz, Smith, & Priniski, 2016). Too often, school math problems are disconnected from everyday experiences, and students frequently ask why they should learn math (Battin-Pearson, et al., 2000). Integrating authentic problems in class has the potential to improve students' motivation and understanding, as well as helping teachers create inspiring lessons (Dori, Avargil, Kohen, & Saar, 2018).

The advantages of using authentic problems in schools is well-established and is considered a successful pedagogy, commonly used in Science, Technology, Engineering and Mathematics – STEM studies (Habig et al., 2018). Yet, for most studies, this method is not focused on mathematics as a stand-alone body of knowledge. Moreover, when implemented, it is used in primary school math studies, or in informal settings (Van-den Heuvel-Panhuizen & Drijvers, 2014), mainly because when moving on to secondary school, the math content material becomes complex and hard to demonstrate using students' daily life scenarios (Imm, Fosnot, Dolk, Jacob, & Stylianou, 2012).

The current study characterizes authentic mathematics problems for secondary school based on the theoretical framework developed by Schoenfeld's (1985). The framework provides the knowledge and behaviour necessary for an adequate characterization of mathematical problem-solving performance. For the purpose of the study, we focus on two of the criteria that are related to the knowledge required for problem solving, nor the processes of problem solving and performance in class: a) *Resources* – which refer to mathematical knowledge that is brought to solve a math problem, and includes the

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following: Intuitions and informal knowledge regarding the domain; Facts; Algorithmic procedures; "Routine" non-algorithmic procedures; and Understandings (propositional knowledge); and b) *Heuristics* which are the strategies and techniques necessary for effective problem solving, including the following: Drawing figures and introduction suitable notation; Exploiting related problems; Reformulating problems; working backwards; Testing and verification procedures (Schoenfeld, 1985, p. 15).

However, the implementation of authentic math problems for secondary school might be captured differently by various stakeholders, in terms of the correspondence between the problems' characteristics to standard math curriculum, as well as their willingness to use such problems in class. For example, novice teachers tend to struggle most with the challenge of stick to the curriculum (Borko, 2004), thus might be seen the authentic problems as not corresponding with secondary school curriculum. Alternately, veteran teachers are more tradition and might be more resistant to changes and new materials, such as the use of authentic problem (Gainsburg, 2012; Remillard & Bryans, 2004).

RESEARCH QUESTIONS

The Research questions are: 1) What are the characteristics of authentic math problems, targeted for secondary school mathematics?; 1.1) Is there a difference between various stakeholders in their perceptions regarding the characteristics of authentic math problems?; and 2) Is there a difference between various stakeholders in their perceptions regarding the feasibility of authentic math problems integration in secondary class?

METHODOLOGY

The context of the study

The study investigates authentic mathematics problems that were developed as part of an extensive research conducted in the Faculty of Education in Science and Technology at the Technion. Taken from authentic scenarios from the Hi-Tech industry (hereafter: Authentic Hi-Tech – AHiT), we simplified problems that were solved by Hi-Tech employees (inventions as well as regular work problems), for use by teachers in formal secondary school math lessons. For example, engineers in the Hi-Tech industry who develop the platform for YouTube video clips try to ensure that the video flows smoothly and not get stuck while playing. Thinking of the two marks that appear at the bottom of the YouTube video screen, makes this problem analogous to the classical algebra question that starts with “One tap fills a pool. The other one empties it, what will happen?”, as the Video Loading mark is analogous to pool filling and the Video Watching mark is analogous to emptying the pool. Clearly, the classical problem taken from the math textbooks is inapposite to students' real-life experiences, compared to watching a YouTube video clip that students are exposed to everyday. See Figure 1 for illustration for this problem.

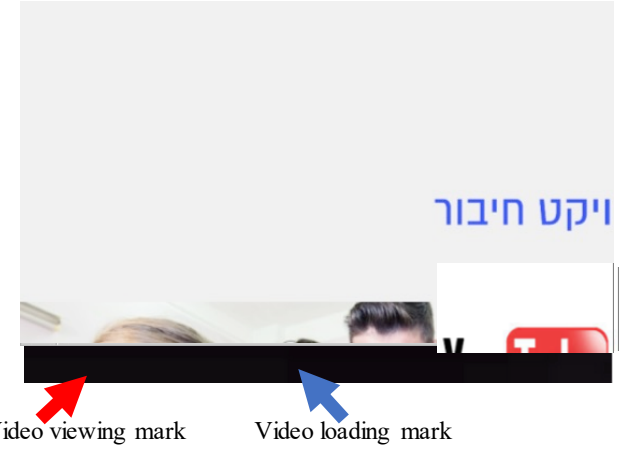

Figure 1: Illustration of an AHiT mathematics problem

The study focuses on four AHiT math problems from different fields of mathematics (algebra, geometry, and trigonometry) that were converted to teaching units for secondary school mathematics, in a format of a 15-min presentation that includes a short and simple explanation about the AHiT problem (simplification of key concepts), and mathematics exercises related to the AHiT problem, based on secondary school math curriculum. One of the teaching units in the field of Algebra was based on the YouTube problem that is illustrated above.

Participants and procedure

The study is based on the Delphi methodology (Murry & Hammons, 1995). This methodology is based on responses retrieved iteratively by anonymous group interactions. This methodology was chosen for the current study, since it is well suited for studies with incomplete knowledge about a problem or phenomenon (Skulmoski, Hartman, & Krahn, 2007), such as the characterization of AHiT math problems according to the problem-solving framework proposed by Schoenfeld (1985).

The iterative process for characterizing the AHiT math problems using the Delphi methodology was as followed. In the first stage, we used an open-ended questionnaire to interview ten stakeholders in the mathematical field: experts and policymakers ($N=5$), and teachers ($N=5$); In the second stage, we converted the collected information into a Likert-type questionnaire, which served as a survey instrument for collecting data from additional 40 stakeholders.

Research tools

The *open-ended questionnaire* was composed of two parts, aimed at responding the two research questions: the characterisation of the AHiT math problems, and the feasibility of the integration of AHiT math problems in secondary class. The first part includes questions based on the two criteria of Schoenfeld framework: *Resources*, e.g. relating to 'intuitions and informal knowledge regarding the domain', we asked: "what information is intuitive and what requires explanation and clarification?", and *Heuristics*?, relating to 'exploiting related problems', we asked: "is the analogy given

to the problem sufficient?". The second part of the questionnaire includes questions based on the SWOT (Strength, Weakness, Opportunity, and Threat) methodology, as it provides an objective tool for critical perspective (Thomas, Chie, Abraham, Jalarajan Raj, & Beh, 2014). For example: To what extent can this problem be of interest to students? [S]; What are the difficulties teachers may face when implementing the AHiT problems? [W]; What can help encourage teachers to use such problems during class? [O]; What external factors can impair the implementation of AHiT problems? (T).

The *Likert-type questionnaire* was composed of 30 questions on a scale from 1 (to a low extent) to 5 (to a high extent). The questionnaire was built according to the responses retrieved from the open-ended questionnaire and was phrased generally to adjust all AHiT problems as following detailed in the data analysis section. For example, with respect to the question of "is the analogy given to the problem sufficient?", we got responses such as *'In the YouTube problem, the analogy presented to a pool filling problem is excellent, but I would have preferred an analogy to a traffic problem (speed, time and distance calculation), which is more intuitive*'. Therefore, a quantitative related question was phrased: "*to what extent do you find the analogue that is presented in the AHiT problem as intuitive to students*'?"

Data analysis

For analysing the *open-ended questionnaire* responses, we used the thematic analysis (Braun & Clarke, 2006) for encoding each statement to various categories that represent general characterisation of the AHiT problems, i.e. by disconnecting the response from the specific problem context. A recursive process of this analysis was repeated by two experts in mathematics education until reaching 90% agreement for all statements, with inter-rater reliability of $\kappa = .83$. After the categories were determined and validated, we calculated the frequencies of statements associated to each of the categories. We then compared the frequency of statements within each category for examining the differences (if any) between the various stakeholders, regarding both the characterisation of the AHiT math problems, and the feasibility of the integration of AHiT math problems in secondary class.

For analysing the *Likert-type questionnaire* data (to be analysed and presented at the PME conference), One-Way MANOVA tests will be used to examine the differences between the perceptions of the various stakeholders, according to the two parts of the questionnaire.

RESULTS

For investigating the characterisation and feasibility of AHiT math problems, and difference between the various stakeholders from the mathematical field: experts and policy makers, vs. teachers, we present the classification of categories retrieved in this study for the different criteria, based on the two parts of the open-ended questionnaire. In order to focus our discussion of the findings, we present *the most frequent category* for each criteria as a representative example. Table 1 demonstrates the categories that arose for the characterisation of AHiT math problems, and Table 2 demonstrates the

categories that arose for the feasibility of the AHiT math problems for their integration in secondary class.

Criteria (by Schoenfeld, 1985)	Sub-criteria (by Schoenfeld, 1985)	Examples of categories retrieved in this study	
		Experts and policy makers	Teachers
Resources	Intuitions and informal knowledge regarding the domain	Suitable for high-level students	Suitable for high-level students
	Facts, rules and Algorithmic procedures	Be precise in sketches	Be precise in sketches
	"Routine" nonalgorithmic procedures	Suitable as a summary, enrichment or research question	Suitable as a summary or enrichment
Heuristics	Drawing figures; introduction suitable notation	The use of dynamic illustrations	The use of dynamic illustrations
	Exploiting related problems	Using a relevant analogy and a story that will motivate students	Using a relevant analogy and a story that will motivate students
	Testing and verification procedures	Using real scale in building the mathematical problem	Attach a student help page

Table 1: Characterisation of AHiT math problems, based on Schoenfeld's framework

According to Table 1, all stakeholders see the AHiT problems as being compatible with the sub-criteria that arose from Schoenfeld's theoretical framework. Additionally, it was found that the perceptions of the two stakeholder groups were consistent for most sub-criteria. Differences were found with regard to the *Testing and verification procedures* sub-criteria, in which teachers' perceptions were targeted to the aspect of learning at the classroom, as opposed to the experts' perceptions which were targeted to the characteristics of the mathematical problem itself. Another difference was revealed in the *"Routine" nonalgorithmic procedures* sub-criteria, when experts and policy makers saw the AHiT problems as encouraging research, as opposed to teachers who did not raise this point.

Criteria	Examples of categories retrieved in this study	
	Experts and policy makers	Teachers
Strength	Authenticity and relevancy	Increasing students' motivation

Weakness	Convincing teachers that they are able to use the AHiT problems	Teachers' lack of confidence to explain the related scientific knowledge of the AHiT problems
Opportunity	Teacher training sessions	Teacher training sessions
Threat	Availability of teachers	Conservative view of the education system and the teachers

Table 2: The Feasibility of AHiT math problems integration in secondary class, based on the SWOT methodology

According to Table 2, in general, the view of the various stakeholders regarding the feasibility of AHiT math problems was found to be different. With regard to *Strengths*, it was found that similar to the difference between the stakeholders described above (Table 1), teachers emphasized the learner's aspect, while the experts and policy makers emphasized the characteristics of the AHiT math problem itself. Another difference was found in both sub-criteria the *Weaknesses* and *Threats*, so that the teachers' attitude was negative (teachers' lack of confidence and lack of flexibility in the system), while the experts and policy makers demonstrated a positive attitude (belief in teachers' ability to assimilate the problems). However, the two stakeholder groups agreed on the need to train teachers so that the use of these AHiT math problems would indeed provide an opportunity for success.

DISCUSSION

The current proposal presents initial research results for the first stage of the Delphi methodology, which analysed the open-ended responses retrieved from ten stakeholders in the mathematical field [Extended results for the second stage of the Delphi methodology for a sample of 40 stakeholders from the mathematics field will be presented at the PME conference].

For the first research question, the results indicate that the AHiT math problems correspond with Schoenfeld's framework for problem-solving. Specifically, all stakeholders referred to two aspects that underlie the use of the AHiT math problems: the learner aspect and the characterization of the math problem. For the learner aspect, an agreement was revealed with respect to the importance of adjusting the AHiT math problems to the mathematics level of the students and using a relevant analogy for motivating the students; for the characterization of the AHiT math problem, an agreement was revealed with respect to the importance of preciseness in the definition of the problem and sketches, the use of dynamic illustrations, and the relevance of the AHiT problem to the learning unit. Yet, partially, the math teachers emphasized more the learner's aspect, while the math experts and policy makers emphasized more the characteristics of the AHiT math problem.

For the second research question, our findings indicate that all stakeholders view the Strengths and Opportunities positively, which means that they agree on the high feasibility of the AHiT math problems. However, the two stakeholder groups grasp the

Weakness and Threat of the AHiT math problems differently, as the teachers find the criteria mentioned as inhibiting factors, while the experts and policy makers find these factors as more of an opportunity.

Overall, according to various stakeholders from the mathematics field, the AHiT math problems have the potential to increase students' interest and motivation for mathematics learning, due to the authenticity and relevancy of these problems to students' everyday experiences (Battin-Pearson et al., 2000), thus can help teachers create inspiring lessons. However, as the participants in this study mutually agreed, teachers should be trained in order to efficiently use the AHiT math problems. This is a subject for future researches.

Moreover, due to research conducted in the STEM field regarding authentic problems and their effect on students' understanding (Dori et al., 2018), we suggest to focus on students as the main participants for further research. The use of AHiT math problems suggests an innovative method to demonstrate daily life scenarios for secondary school students, which are uncommonly used in regular secondary math classes (Imm et al., 2012), thus have the potential to attract more students to engage in high-level mathematics (Harackiewicz et al., 2016).

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A CONSIDERATION OF TECHNOLOGY WITH ROWLAND'S KNOWLEDGE QUARTET FRAMEWORK

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This paper reports on a study that seeks to examine changes in teachers' knowledge for teaching mathematics across the curriculum years using Rowland's Knowledge Quartet Framework (KQ). The observations reported here come from a Year 9 Australian classroom using integrated technology. The findings suggest that technology can impact on teacher knowledge within each of the four components considered in the Rowland's framework, with a particular focus on contingency. The study supports the potential of the KQ framework for deepening in-service teacher knowledge at the secondary level and suggests further research is needed to consider the extension of the model to incorporate the role of technology within its framework.

BACKGROUND

Teachers' knowledge for teaching mathematics has been a focus of attention in educational research for many years (e.g., Ball, Thames, & Phelps, 2008; Rowland, Turner, Thwaites & Huckstep, 2009). The project from which this paper originated was designed to extend the examination of teachers' knowledge for teaching mathematics from the primary context to all levels of the curriculum and to consider changes in the ways in which teachers drew on their subject and pedagogical knowledge as the subject became more specialised and advanced. The project had two phases and the previous phase, based on focus groups, has been reported earlier (Callingham, 2015; Chick 2015). These focus groups were followed by classroom observations in primary and secondary classrooms in New Zealand and Australia. The observations were video-recorded and field notes taken for later analysis. This paper reports on one such lesson in a Year 9 Australian class which made extensive use of technology.

THEORETICAL FRAMEWORKS

The Knowledge Quartet

Many studies have developed models to describe the components of pedagogical content knowledge (PCK) identified in Shulman's (1987) seminal framework for the specialised form of knowledge needed by teachers. Ball et al. (2008), for example, explored the relationship between mathematics subject matter knowledge and PCK, concluding that there was a need to map PCK in greater detail and not to take it for granted in teacher preparation courses. Although linked to Shulman's (1987) research, Rowland and his colleagues (Rowland et al., 2009; Rowland, 2013) developed a framework for the classroom analysis of teachers' behaviours and practices using grounded theory, which they have called the *Knowledge Quartet* (KQ). Based on observations of novice pre-service teachers of mathematics in English (UK) primary

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and middle school classrooms, they identified four components: *Foundation, Transformation, Connection, and Contingency. Foundation* is concerned with the extent of mathematical knowledge whereas the other three components focus on the processes of using that knowledge. A detailed description of each component can be found in Rowland (2013), but in brief:

- *Foundation* is based in the teacher's beliefs about and understandings of mathematics itself;
- *Transformation* is concerned with the ways in which teachers transform their foundation mathematical knowledge into pedagogical actions;
- *Connection* addresses the cohesion within the mathematical ideas, and the extent to which a teacher makes the links explicit;
- *Contingency* has a focus on unanticipated classroom events and a teacher's subsequent decisions, the teacher's capacity to "think on one's feet" (Rowland, 2013, p. 26).

These are not siloed but manifest in different and interacting ways; for example, as a result of contingent decisions, connection and transformation may be observed.

However, while we explicitly acknowledge that the *Knowledge Quartet* framework is innovative and that it has the potential to provide a "fresh lens" in the analysis of teacher's classroom practices, there is uncertainly of its generalizability outside of the context in which it was developed (novice-primary teachers in the UK). How might it be used in other curriculum areas? How applicable is it at higher secondary levels or what value does it have in examining the practice of expert teachers? A review of the analysis of lessons (i.e., Rowland et al., 2009) indicates that the pre-service teachers were operating within a traditional teacher led instructional paradigm with the teacher talk the main medium of instruction. This study seeks to question how the framework might change for example in a student-centred, technology-rich classroom.

Technology in the Mathematics Classroom

The effect and value of technology in the mathematics classroom has been widely investigated, but its consideration within the KQ-framework seems less well explored. Thomas and Hong (2013) have extended the PCK-model (Shulman, 1987) to develop a model for *Pedagogical Technology Knowledge* (PTK), also characterised elsewhere as *TPACK* (Mishra & Koehler, 2006). The model described in Figure 1 suggests that PTK might help inform the KQ, for example knowledge and orientations within *Foundation* and instrumental genesis (see Artigue, 2002) under *Transformation.* Thomas and Hong (2013) lend support to the potential of their PTK model to inform the components of the KQ, when they state that:

> A teacher's PTK applied to mathematics includes the principles, conventions and techniques required to teach mathematics through the technology (Thomas & Hong, 2013, p. 69).

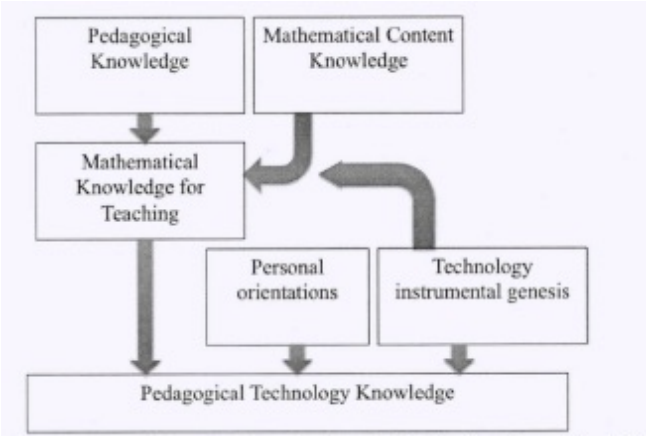

Figure 1: A Framework for PTK (Thomas & Hong, 2013, p. 70).

Stacey (2003) believed technology can impact significantly on the value of mathematical content and knowledge. She suggested, for example, that teachers should consider the *heuristic* value of content for making connections, especially to future concepts, reflecting what Ball et al. (2008) described as horizon knowledge and linking to the *Connections* component of the KQ (Rowland, 2013). Technology has also been shown to act as a *mediator* of learning, connecting students, teachers in the learning process (Goos, Galbraith, Renshaw & Geiger, 2003). Their findings suggest potential implications for the role of technology within all four components of the KQ.

This paper thus uses the KQ framework (Rowland, 2013) as the basis for its analysis, and seeks to identify the influence of technology within this framework. While this study shows evidence that technology may influence teacher knowledge within all four components of the Rowland's framework, this paper focuses predominantly on one teacher's contingency in a technology-rich classroom.

THE RESEARCH CONTEXT

The research for this paper took place in an independent (non-government, non-denominational) Australian school (boys only). It offers education from Years 5 through to 12 with the school identified as being in a high socio-economic band by Australian standards and making use of digital learning practices. Permission to conduct the research was provided by the relevant University and school ethics processes. The school administration selected the mathematics lessons to be observed based on timetabling availability of the day of the visit and the willingness of teachers to participate. Teachers were asked not to prepare a special lesson but to go about their regular teaching. Researchers took notes from brief conversations with the teachers before and after the lesson, to determine where this lesson fitted in with their overall teaching plan and the nature of their students, so the researchers might better interpret the progression of the lesson and the teachers' actions. Participating teachers were then given a lapel microphone that was wireless linked to a single portable tripod-based video camera set up in the back of the room. Two researchers were involved with the lesson recording, allowing one person to focus on the recording and the other on taking notes. Students were introduced to the researchers and the nature of the research briefly

described; there was no other interaction between the researchers and students. Recordings were transcribed identifying the teacher and generic student contributions.

The class on which this paper reports was Year 9 class of 25 boys, seated in groups of four or five students around circular tables. The teacher was an experienced registered teacher of secondary school mathematics with a degree in maths and science and a postgraduate university qualification in Education. She had taught mathematics at the school for more than 5 years and the lesson occurred towards the end of the school year so the teacher knew her students well. There was some flexible grouping of students by ability at each table; for example the teacher noted that a few boys were not confident in maths and tended to wait for confirmation before they progressed with the activity. She had placed a few boys who were stronger in maths who could act as informal peer helpers to the other boys at their table, and boys were encouraged to work in their groups, to talk about the problems and check their answers.

Each student used a tablet-laptop computer (i.e. a laptop with the facility to write on the screen), connected to the internet and the school's e-portal, which provided access to the particular lesson and enabled students to revise and review past content. The teacher used a tablet and a large large-screen television monitor at the front of the room as her instructional focus point, connected through the e-portal to students' lap-top computers. At different times she brought the class group together and used the screen to explain something to the whole group or to have the students review the problem under investigation and its "solution". Each student could access different packages of activities from their laptops depending on his ability and progress. Although the teacher could monitor all laptops from the front, she mostly moved around the room talking to students in groups and individually.

The lesson observed for this paper was a 60-minute part of a statistics unit the teacher had been developing for the last two weeks. After a brief review of previous work by the teacher, the students worked individually and in their groups through a set of prepared activities and problems. The problem the teacher intended was finding probabilities using two-way tables and Venn diagrams, which could be interactively manipulated using the software. For example, students were asked to find the probability a given student was studying Civil Engineering in an Engineering School, given that she was female. Each student received randomly generated numbers for the size of the school, classes and numbers of girls, so discussions were forced to be about procedures and concepts. Although formal statistical notation was used in the learning platform, at this early stage students' solutions were largely given through data-entry into tables and construction of the Venn Diagrams themselves. The software indicated if a student's answer was correct and the e-portal monitored their progress.

LINKING THE LESSON TO THE KQ FRAMEWORK

The transcripts provided rich examples of the teacher's practice in each of the four components of the KQ framework. Here we provide evidence within each but focus

mainly on the *Contingency* component as the examples here we found especially interesting in respect of the observable technology effects.

Evidence of Foundation

The teacher's foundation knowledge of where this lesson sat within the probability unit was clear and she made this explicit both to the students during class and the researchers in the pre- and post-lesson conversations. After the lesson, she talked about where she wanted the students to be with respect to the curriculum and students' understandings of probability, and she a clear goal in this respect. She was aware of the other curriculum areas involved such as proportional reasoning and fluency in converting fractions, ratios and decimals, and during the lesson realised she needed to revise probability notation. Her foundation knowledge also extended to her ability to adapt and work with the technology. A number of times she looked on her tablet-screen and was able to quickly navigate around each of the groups to alter her teaching and provide differential instruction to the diverse groups:

Group A	...look back on to last week's lesson where we set out the Venn diagram, use that as a model for this problem…
Group B	Student X, can you explain to Student Y how you found your answer?
Group C	…rather than do problem 2 and 3 try question 4 as it may be more interesting one for you.

The teacher articulated her deep foundation knowledge with respect to her philosophy of education; for example she talked about the long-term advantages of allowing the student to, at times, struggle but also encouraging them with "doable" problems that would, over time, make the students more resilient and enhance the students' own mathematics self-efficacy. She knew that some of the students were struggling with the task, and the temptation was to "rescue" some of the groups and do all the thinking for them. Rather, aided by the technology, she scaffolded the learning, for example by inviting students to go back and review the setting out from previous lessons, or adapting the task with simpler numbers.

Evidence of Transformation

This dimension was notably evident in several ways, for example in the preparation the teacher put into the lesson, with a focus on transforming the content to the specific needs of the different students. The technology operating in this maths lesson helped in this transformational process because the classroom portal enabled the boys to work on activities and review these at home in preparation for the next lesson. Technology also allowed the teacher to cater for individual differences in the learning needs of her students. Another element of transformation seen in this lesson was the development of the boys' skills to represent their answers in different visual and mathematical forms, for example constructing and interpreting Venn diagrams, and extending and transforming what students already know about fractions, ratios and decimals to their

use in probability contexts. Technology was used in this sense to scaffold the students' use of formal notation, building on their prior knowledge.

Evidence of Connection

The teacher made explicit references to connecting mathematical ideas, for example explaining to the students the value of this knowledge as a basis for more advanced future work. The purpose for doing the mathematics was connected to a wider understanding of mathematics as a discipline rather than just a set of unrelated activities. This awareness and modelling of connections by the teacher exhibit what Ball et al. (2008) described as *Horizon Knowledge,* and also demonstrated the teacher's perception of the *heuristic* value of the topic (Stacey, 2003). The e-learning platform and the group arrangements facilitated the students' connections with the mathematics, for example in using the "think-pair-share" strategy to encourage them to talk and reflect more on the task. The learning platform itself helped the students to make connections across multiple representations, by presenting these together in one window in a cohesive and interactive manner.

Evidence of Contingency

An important observation here is that aside from the introduction and wrap-up of the lesson (both less than five minutes), the teacher spent very little time speaking to the class as a whole which was common practice for the novice teachers with contingency in the Rowland studies (2013). The use of the technology made it easier for her to monitor students' progress, detect issues that might require decisions or be seen as *teaching moments,* and when appropriate, to deviate from an agenda set out when the lesson was prepared. She was thus able to deal with problems that arose more immediately and individually within the groups, for example in one instance suggesting the members of the group read the question more carefully when she observed they were using the universal set of all engineering students when the question was asking specifically about girls in the class.

Technology also informed the teacher about student progress which prompted several whole-of-class interventions, and an important unplanned decision on the teacher's part. At one point, the teacher realised that many students were struggling with the probability notation, so she brought up an example from a previous lesson using Venn diagrams on her large screen, and invited students to contribute responses to the probability values, prompting them to say how these might be written in formal notation. When asked, students could write their response on their tablet screen, and share it on the teacher's screen for the class. At another point early in the lesson, the teacher realised that the problem also involved conditional probability, which she had not noticed in her planning for the lesson. She made the decision to continue with this, but made it explicit to the students that this was harder than she had expected them to meet, and she modified her goals and expectations of them accordingly. Afterwards, she explained that she was comfortable that the students might not get as much done but preferred to make sure they gained a deeper understanding, rather than move on

with the lesson she had planned. Later in the lesson, she confirmed this with students, stating that "You have done really well. It was really hard, I totally apologise…" and she then spent five minutes consolidating their understandings of the "given" statements and the subsequent probabilities as a whole class using her tablet projected on the large screen. This was a clear response to an unplanned contingency.

DISCUSSION AND IMPLICATIONS

The discussion and evidence presented here has demonstrated there is indeed value in using the *Knowledge Quartet* framework (Rowland, 2013) to examine an experienced teacher's practice at the secondary level and provided support for generalising it to this extent. We have yet to consider how we might share this analysis with the teacher from a professional learning perspective, but we suggest such an approach may help identify the practices she sees as most valuable and possibly worthy of sharing with colleagues, and those she may wish to develop further.

With respect to technology, while on the surface in this e-classroom we might argue the teacher has just replaced the traditional "blackboard and chalk" with the e-screen and the students' writing pads with tablet-laptops. In reality the shift to an e-learning classroom was more sophisticated than this and the use of the Rowland's *Knowledge Quartet* framework has helped clarify this level of sophistication. Through the lens of the KQ, clear examples of the teachers' foundation knowledge with respect to technology were identified. She demonstrated fluency with instrumentation Artigue, 2002) and mathematical value of the technology (Stacey, 2003) in the design and implementation of the lesson, and her interactions with the students showed she had an acute awareness of potential technological obstructions for the students (Goos et al., 2003). This ability to integrate the technology within the teaching in a seamless way is a form of foundation knowledge that is not always recognised and reflects the high instrumental genesis and PTK (Artigue, 2002; Thomas & Hong, 2013) within this teacher's foundation knowledge.

The teacher was comfortable adapting her lesson to meet the contingency she confronted with unexpected conditional probability, and her response showed her foundation knowledge with respect to horizon knowledge (Ball et al., 2008) and the heuristic value of the topic (Stacey, 2003). It also showed her belief in the value of making connections and developing deeper understanding for her students.

The findings of this preliminary analysis support extending the use of the KQ framework across curriculum levels (i.e. from primary to secondary), and further suggest we should explore in more depth the role of technology within the KQ framework. There is an implication that the contributing codes within each of the four components may need some modification or extension to accommodate the impact of technology on teacher practice and the mathematics classroom. Finally, while a deeper exploration may well reveal other theoretical perspectives that may help inform such an analysis, we have shown that the PTK framework devised by Thomas and Hong (2013) has proved useful in our examination to date.

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ADULTS' STRATEGY USE IN COMPLEX FRACTION COMPARISON

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Several studies have found that when people compare fractions, they show a "natural number bias", that is, better performance on problems in which the larger fraction has the larger components (e.g., 7/8 vs. 2/3) than on problems in which the larger fraction has the smaller components (e.g., 2/3 vs. 4/9). However, more recent studies found the opposite pattern ("reverse bias"). To better understand what causes these patterns, we investigated adults' strategies in complex comparison problems with varying affordances. Adults reported strategies on a trial-by-trial basis. Overall, we found a reverse bias pattern. Strategy use varied by problem type, suggesting that participants used strategies adaptively. The study highlights variability in strategy use depending on problem type, which may account for diverse bias patterns in previous studies.

THEORETICAL BACKGROUND

Many people struggle with fractions (see Lortie-Forgues, Tian, & Siegler, 2015, for a review). One well-documented phenomenon is the "natural number bias" (Ni & Zhou, 2005), that is, people's tendency to overextend natural number reasoning to fraction problems. To assess the occurrence of the natural number bias, studies have often used fraction comparison problems, in which people are asked to choose the larger of two fractions (Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Dooren et al., 2016; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). One indicator of the bias is that people are more accurate and/or faster when the larger fraction has the larger components ("congruent" comparison problem e.g., 7/8 > 2/3) than when the larger fraction has the smaller components ("incongruent" comparison problem, e.g., 2/3 > 4/9). People may explicitly or implicitly rely on the fractions' natural number components (i.e., their numerators and their denominators) instead of the fraction' overall magnitudes when they make their decisions.

However, a number of recent studies on fraction comparison have reported the opposite response pattern. In these studies, people were more accurate and/or faster comparing fractions that are incongruent than fractions that are congruent (Barraza, Avaria, & Leiva, 2017; DeWolf & Vosniadou, 2015; Obersteiner & Alibali, 2018; for an overview, see Gómez & Dartnell, 2015). The studies that found such a "reverse bias" pattern differed from earlier studies in that the participants tended to be older, and the fraction comparison problems were often more complex. That is, the fractions were unfamiliar to participants, were composed of larger numbers, and had no common numerators or denominators (e.g., 19/24 vs. 25/36). Thus, the "reverse" bias pattern seems to be more representative of adults' performance pattern in fraction comparison than the typical bias. What can explain this reverse bias pattern? The present study

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addresses this question by investigating the strategies that people use to solve fraction comparison problems. Focussing on strategy use is relevant because some strategies may be more prone to the bias than others (Alibali & Sidney, 2015).

Strategy Use and the Reverse Natural Number Bias Pattern

DeWolf and Vosniadou (2015) explained the reverse natural number bias by suggesting that people relied on a "smaller number—larger fraction" heuristic to compare fractions. There is, however, limited evidence that people actually use such a heuristic unless fractions have common components. Instead, the reverse bias pattern documented in previous research might be explained by another component-based strategy: gap comparison. This strategy refers to considering a fraction as larger when it has the smaller difference ("gap") between its numerator and denominator (e.g.,19/24 > 25/36 because 24 – 19 = 5, which is smaller than 36 – 25 = 11). Although gap comparison is not a generally correct strategy, it always leads to a correct response in problems with fractions smaller than 1 that are incongruent, but it does not always yield a correct response in problems that are congruent. Thus, if people relied heavily on gap comparison, this could partially explain the reverse bias pattern.

In contrast to the gap comparison strategy, strategies that rely on holistic fraction magnitudes rather than on fraction components may be unlikely to yield a natural number bias response pattern. One such strategy involves using benchmarks (Liu, 2017). Benchmarks are common numbers that can serve as reference points. They allow easy access to approximate fraction magnitudes, which are often sufficient for solving fraction comparison problems. For example, to decide that 19/24 > 25/36, one can use 3/4 as a benchmark: 19/24 > 3/4 while 25/36 < 3/4, hence 19/24 > 25/36. If a fraction comparison problem can be readily solved using a benchmark, people may tend to use the benchmark strategy, thereby activating approximate holistic fraction magnitudes. On such problems, people should be less likely to display a natural number bias than on problems that cannot be solved using a benchmark strategy.

Little is known about which strategies people use to compare fractions. Clarke and Roche (2009) and Fazio, DeWolf, and Siegler (2016) found that school students and university students use a large variety of strategies, including benchmarking and also gap comparison and other generally incorrect strategies. However, these studies did not systematically address how strategy use may depend on problem features, and how strategy use relates to the bias response pattern.

Aim and Research Questions

The aim of the present study was therefore to investigate whether efficient strategy use (especially benchmarking) can reduce a potential (reverse) natural number bias in complex fraction comparison in adults. We addressed four research questions: (1) Do adults show a reverse natural number bias pattern in complex fraction comparison? (2) Which strategies do they use? (3) Do they adapt their strategies to the affordances of the problems? (4) Does encouraging participants to use efficient benchmark strategies affect their strategy use, their overall performance, and their potential bias?

METHODS

Participants

Participants were 72 university students (age *M* = 19.30 years, *SD* = 1.32) at a Midwestern university in the United States. Before the experiment started, 40 randomly selected participants (56%) received a tip that using benchmarks such as ½, ¼, or ¾ can be helpful in solving fraction comparison problems. The example "5/8 vs. 3/7" with 5/8 > 1/2 and 3/7 < 1/2 was provided to illustrate the benchmark strategy.

Procedure

Participants were asked to solve 28 fraction comparison problems on a computer as quickly and accurately as possible, and to explain, after each problem, how they solved the problem. On each trial, two fractions appeared on the screen next to each other, and participants indicated the larger fraction by pressing the left ("f") or right ("j") key on a regular keyboard. There was a 15 sec time limit for their response. Once participants gave their response, the prompt "How did you solve this problem?" appeared on the screen while the fractions remained visible. Participants then had a maximum of 40 seconds to provide their explanations. Participants wore headsets and spoke their responses into the microphones. The next problem appeared automatically after 40 seconds, or sooner if participants pressed the space bar. Between any two trials, a fixation cross appeared in the middle of the screen for two seconds. Response times, accuracy, and verbal responses were recorded using E-Prime software.

Comparison Problems

All fractions in the comparison problems were smaller than 1, and most fraction components were two-digit numbers. None of the fraction pairs had common numerators or denominators. Half of the problems were congruent and half were incongruent. The congruent and incongruent problem subsets were balanced in terms of the fractions' magnitudes relative to common numbers that people may use as "benchmarks" (i.e., reference points) for comparing fractions. There were three categories depending on the fraction magnitudes relative to the potential benchmarks 0, ¼, ½, ¾, and 1: In "straddling" problems, one fraction was smaller and the other larger than ¼, ½, or ¾ (thus "straddling" a benchmark). In "in-between" problems, both fractions were in between two adjacent benchmarks. A special case of "in-between" problems, and therefore a separate category, were problems in which fractions were both smaller than ¼ or both larger than ¾, because in these problems, one fraction was close to 0 or 1 ("0-1" problems), and 0 and 1 may be especially salient benchmarks. Fractions pairs were selected so that the average numerical difference between fractions and the closest benchmark was the same (0.06) for both congruent and incongruent problems, and the average numerical difference between fractions was the same (0.14) for congruent and incongruent problems. The problems were presented to the participants in random order. We initially created a set of 56 problems. The problem set was then split into two subsets of 28 problems each, both of which represented all of the problem features described above in the same way. Each

participant was randomly assigned to one of the two problem subsets. There were no differences in performance on the two problem subsets, so data were collapsed for analysis.

RESULTS

Accuracy and Response Times

Overall, accuracy was high (M = 86%, SD = 11%), and response times were low (M = 4813 ms, SD = 1432), suggesting that participants were well able to solve complex fraction comparison problems. To analyze effects of problem features, we report Wald X^2 statistics and estimated marginal means (M_{EM}) of a generalized estimation of equation (GEE) model, in which the within-subject factors were *congruency* (congruent/incongruent) and *benchmark* (0-1/straddling/in-between), and the between-subject factor was *tip* (yes/no).

Participants were more accurate and faster on incongruent than congruent problems (accuracy: M_{EM} = 94% vs. 81%, $\chi^2(1) = 38.86$, $p < .001$; response times: M_{EM} = 4285 vs. 4652 ms, $\chi^2(1) = 11.59$, $p = .001$), indicating an overall *reverse* bias. There was no main effect of *tip*, either for accuracy or response times. For response times, but not accuracy, the effect of *congruency* was qualified by an interaction with *tip*, $\chi^2(1) = 4.51$, $p = .034$. Overall, participants were faster on incongruent than on congruent problems, but this difference was smaller (and not significant) for participants who received a tip (M_{EM} = 4403 vs. 4542 ms) than for participants who did not receive the tip (M_{EM} = 4170 vs. 4766 ms, $p < .001$). Thus, although the tip did not affect overall performance, it reduced the reverse NNB in terms of response times.

There was also a main effect of *benchmark* (accuracy: Wald $\chi^2(2) = 24.63$, $p < .001$; response times: Wald $\chi^2(2) = 74.63$, $p < .001$), such that participants were most accurate and fastest on "0-1" problems (M_{EM} = 93%, M_{EM} = 3917 ms), followed by "straddling" problems (M_{EM} = 88%, M_{EM} = 4744 ms) and then "in-between" problems (M_{EM} = 83%, M_{EM} = 4789 ms); the difference between the latter two problem types was significant only for accuracy ($p = .014$) but not for response times. No other interactions were significant.

Strategy Use

The verbal recordings of participants' strategies were transcribed and coded by two coders (interrater reliability (N = 18 participants) was sufficiently high (Cohen's Kappa = 0.74). We collapsed strategies into five major categories (for strategy definitions, see Table 1). Two of these categories involved using benchmarks: straddling benchmark and close to benchmark(s). About one-quarter of all strategies described fell into one of these categories, although overall, the latter was more common than the former. A third category (holistic) consisted of strategies that involved reasoning about holistic fraction magnitudes (but not relying on benchmarks), such as converting both fractions into decimals. Almost one-third of all strategies fell into this category. A fourth category (component) consisted of strategies based on reasoning about fraction

components, but not on holistic fraction magnitudes. These strategies were generally incorrect but could yield correct responses on specific problems. Approximately one-third of all strategies fell into this category. Among these, the primary subtypes were gap comparisons (63%) or simple comparisons of denominator magnitudes (11%). It is notable that simple comparisons of both fraction components (i.e., "larger components—larger fraction", or "smaller components—larger fraction") were very infrequent overall (1%, and 0.3%, respectively). The final category (other) consisted of strategies that occurred infrequently (e.g., guessing) as well as unclear responses.

Percent Use	Strategy Category	Description/Examples
8	Straddling benchmark	One fraction is larger than a benchmark (i.e., ½), the other smaller.
19	Close to benchmark(s)	Both fractions are close to different benchmarks (i.e., ½ and ¼).
30	Holistic	One fraction is a small number, the other is a large number.
		Convert both fractions into decimals.
34	Component	Compare the differences between the numerator and the denominator of each fraction (gap comparison).
		One fraction has smaller denominator than the other.
10	Other	Guess.
		No or unclear response.

Table 1: Self-Reported Strategies.

We examined whether strategy use depended on problem type and whether participants who received the tip about the usefulness of benchmark strategies adapted their strategies. Figure 1 presents the results. Differences in strategy use were more pronounced between problem types than between the tip groups. On problems in which one of the fractions was close to 0 or 1, participants used generally incorrect component strategies much more often than on other problems. For the other two problem types, participants used component strategies less often and benchmark strategies somewhat more often. As expected, the increase of straddling-benchmark strategies was especially great for straddling problems, and the increase of close-to-benchmark(s) strategies was especially great for in-between problems. Although the shift from component to benchmark strategies was slightly more pronounced for the tip group than for the no-tip group, differences between the groups were generally small.

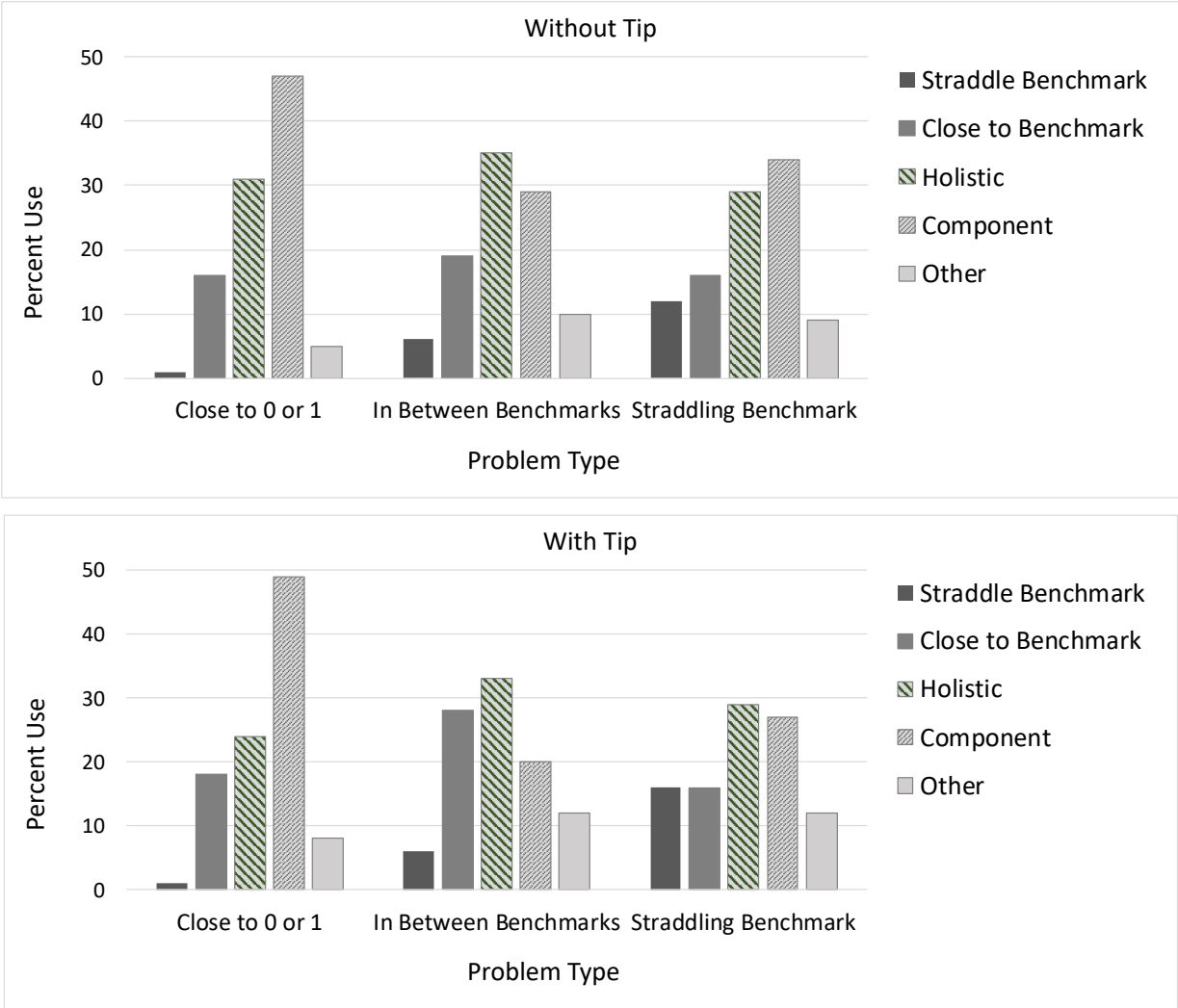

Figure 1: Strategy use for participants who did not (top) or did (bottom) receive a tip.

DISCUSSION

This study shows that adults can solve complex fraction comparison problems quickly and accurately. At the same time, adults display an overall reverse natural number bias. Even those who received a tip to use benchmarks displayed a bias in their accuracy, although the bias was not evident in their response times. This finding underscores the conclusions, drawn from earlier studies, that the bias can persist into adulthood.

Our data suggest that the reverse natural number bias may be associated with relatively strong reliance on component-based strategies, and that among such strategies, there is greater variety than has been suggested in previous research. The typical natural number bias may occur when people rely on comparisons between numerators or denominators, which people commonly use when they compare fractions with common components (Fazio et al., 2016; Obersteiner & Tumpek, 2016). In our complex fraction comparison problems, however, there were never common components, and participants almost never reported having used this strategy. Instead, people frequently used gap comparison, and frequent use of this strategy may explain the reverse natural number bias that we observed.

One may wonder why we did not find a stronger bias on "close to 0 or 1" problems than on other problem types, given that participants relied most strongly on component strategies in these problems. For these problems, gap comparison may have been a particularly efficient strategy, because the differences in gaps were most salient in these problems. When a fraction has an extreme value (close to 0 or close to 1), it also has a particularly large or small gap. This highlights the importance of understanding which features of fraction comparison problems are most salient to people—an important agenda for future research.

One limitation of this study is that immediately retrospective self-reports may not be a completely valid measure of strategy use. People may not be aware of all of the strategies they use for solving a mathematical problem, or they may struggle with verbalizing their strategies. Participants in our study solved the fraction comparison problems within a few seconds, and it may have been difficult for them to describe the cognitive processes that occurred within this brief time period. In addition, prompting participants to verbalize their reasoning may itself have influenced the problem-solving process. Participants may have been more cautious about using particular strategies, because they knew that they would have to explain their strategies after each trial. Further research with different methods is needed to assess the validity of immediately retrospective self-reports for fraction comparison.

In sum, this study revealed large variability in strategy use across individuals and across problem types. Participants used strategies adaptively in ways that made good use of the affordances of different problems; for example, they used benchmark strategies when fractions straddled a benchmark, and gap comparison when fractions had highly salient gaps. Our findings suggest that patterns of strategy use may at least partially explain the occurrence and direction of the natural number bias in fraction comparison. To better understand what it means to have a natural number bias, and why there is variability in the natural number bias across studies and samples, it is critical to understand the strategies people use in making specific fraction comparisons.

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THE COMPLEXITY OF INTERPRETING STUDENT THINKING AND INFERRING ITS POTENTIAL TO FOSTER LEARNING

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Research has shown that listening to and interpreting student thinking is challenging, yet critical for effective incorporation of student mathematical thinking (SMT) into instruction. We examine an exemplary teacher's interpretations of SMT, his inference of the potential of the SMT to foster learning, and the rationale for his responses to that thinking. Our findings reveal some reasons why teachers may fail to successfully act on SMT that emerges during whole class discussion. This study confirms previous research findings, that in order to incorporate SMT into instruction in a way that fosters learning, teachers must correctly interpret that SMT. The study also shows that even good teachers may need support in developing skills that will enable them accurately interpret SMT and its potential to foster learning.

The use of student mathematical thinking (SMT) to inform instructional decisions has been emphasized in current calls for reform as captured in US National Council of Teachers of Mathematics documents (e.g., NCTM, 2014). Using SMT in ways that can foster learning requires making sense of that thinking (Maher & Davies, 1990). However, listening to student thinking is challenging (Ball, 1993) and sometimes teachers incorrectly interpret that thinking (Maher & Davies, 1990). Understanding teachers' perceptions of SMT that emerges during instruction and their reasons for using that thinking in particular ways would enable teacher educators to be responsive to teachers in efforts to support them in enhancing their teaching practice around the use of SMT. To that end, the purpose of this exploratory study is to understand an exemplary teacher's interpretations of SMT and rationale for using the student mathematics available to him during whole class discussion in the way that he did.

LITERATURE REVIEW

Research has shown the instructional value of teachers making sense of SMT. The research of the Cognitively Guided Instruction (CGI) project has shown that providing teachers with opportunities to make sense of SMT has a positive impact on teacher learning and student achievement (Franke & Kazemi, 2001). Therefore, it is important that researchers and teacher educators understand teachers' processes of making sense of SMT so as to identify the kinds of support teachers may need in order to develop the skill of accurately inferring SMT. Studies that have examined teachers' interpretations of SMT include those that have examined a broad swath of classroom activity, of which SMT is only a part (e.g. van Es & Sherin, 2002), and those that have

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focused specifically on SMT (e.g. Crespo, 2000; Jacobs, Lamb & Philip, 2010). In this brief literature review we consider studies that have focused specifically on SMT.

Research has shown that interpretation of SMT is a skill teachers can develop. Jacobs, Lamb & Philip (2010) examined teachers' development of three skills—*attending* to children's strategies, *interpreting* children's understandings, and *responding* to those understandings—and found that these three skills can be learned. Crespo (2000) studied Preservice Teachers (PSTs) interpretations of SMT by examining how PSTs interpreted SMT as they exchanged letters with 4th grade students. The results showed that PSTs' interpretations changed from being evaluative to making sense of students' mathematical solutions and from making quick conclusions about student abilities to making more thoughtful interpretations. Crespo's (2000) study also highlighted the importance of attending to conversations that reflect teachers' processes of interpretation. Our work in this paper is informed by this finding on the importance of attending to such conversations. In this exploratory study we analyze conversations between the researcher and the teacher to infer the teacher's interpretations of SMT and their perception of the potential in the SMT to foster learning. Other studies that have provided further insight into teachers' development of the skill of interpreting SMT include van Es and Sherin (2008), which revealed that teachers follow different pathways as they develop the skill of interpreting SMT.

Some studies on teachers' interpretations of SMT have revealed factors that may influence teachers' inferences of SMT. Maher and Davies (1990) revealed that a teacher's limited understanding of a mathematical concept can impede the correct interpretation of SMT. Other factors that may influence a teacher's interpretation of SMT include the teacher's orientations towards listening to students. Davis (1996) described three such orientations: evaluative, interpretive and hermeneutic. Teachers with an evaluative orientation listen to SMT in order to determine the correctness of the SMT. Those with an interpretive orientation listen to SMT in order to get the student's understanding while those with a hermeneutic orientation listen to SMT by engaging with students in a process of negotiation of meaning and understanding. An understanding of the factors that influence teachers' interpretation of SMT could inform the design of professional development that would support teachers to accurately infer SMT and incorporate it in instruction in ways that foster learning.

Incorporating SMT in instruction in ways that foster learning while at the same time optimizing limited instructional time requires that incorporation of SMT be informed by the potential in the SMT to foster learning. Many studies that have examined teachers' interpretation of SMT have not explicitly addressed the potential of the SMT to foster learning, yet not all SMT has the same potential to foster learning (Leatham et al., 2015). Identifying the potential in SMT to foster learning is critical if teachers are to productively use SMT. In this study we explore a teacher's interpretation of SMT that occurred in his classroom with a view to better understand how he interprets SMT and his perception of the potential in that thinking to foster learning. We therefore seek to answer the following research questions: 1) How does an exemplary teacher

interpret SMT that emerges in-the-moment during whole class instruction? 2) What inferences does an exemplary teacher make about the potential of SMT, that emerges in-the-moment during whole class instruction, to foster learning of mathematical ideas?

THEORETICAL FRAMEWORK

Our examination of a teacher's interpretations of SMT and his perceptions of the potential of that thinking to support learning of important mathematical ideas is guided by the MOST Analytic Framework, where a MOST is defined as a "Mathematically Significant Pedagogical Opportunity to Build on Student Thinking" (Leatham et al., 2015, p. 90). We refer to the practice of taking advantage of MOSTs as *building* (Van Zoest et al., 2016) and define building as making an instance of SMT "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36).

MOSTs occur at the intersection of three critical characteristics of classroom instances: student mathematics (SM), significant mathematics, and pedagogical opportunity. For each characteristic, two criteria determine whether an instance of SM embodies that characteristic. For SM the criteria are: "(a) one can observe student action that provides sufficient evidence to make reasonable inferences about *SM* and (b) one can articulate a mathematical idea that is closely related to the SM of the instance—a *mathematical point (MP)*" (Leatham et al., 2015, p. 92). The criteria for significant mathematics are: "(a) the MP is *appropriate* for the mathematical development level of the students and (b) the MP is *central* to mathematical goals for their learning" (p. 96). Finally, "an instance embodies a pedagogical opportunity when it meets two key criteria: (a) the SM of the instance creates an *opening* to build on that thinking toward the MP of the instance and (b) the *timing* is right to take advantage of the opening at the moment the thinking surfaces during the lesson" (p. 99). The six *MOST Criteria* are considered linearly and an instance of SMT is classified according to the last criterion it satisfies (SM, MP, Appropriate, Central, Opening, and Timing). Those instances that appear mathematical, but for which the SM cannot be inferred, are designated *cannot infer* (CNI). When an instance satisfies all six criteria, it is a MOST.

METHODOLOGY

This study is part of a larger project focused on understanding teachers' in-the-moment responses to SMT during whole class instruction (see LeveragingMOSTs.org). The teacher who is the focus of this exploratory study was chosen because he regularly incorporates student thinking into his lessons and has been recognized by his school district and university mathematics educators as an exemplary teacher. The data for this study consisted of four videotaped math lessons from this teacher's classroom and seven corresponding follow-up interviews.

Our data analysis focused on four different units of analysis: instances of SMT, the teacher's in-the-moment responses to those instances, the teacher's retrospective interpretation of the SMT, and the teacher's retrospective reasoning for his responses. An instance of SMT consists of "an observable student action or a small collection of

connected actions" (Leatham et al., 2015, p. 92). The teacher's in-the-moment response to an instance consists of the teacher actions that begin "as a given instance of SMT ends and ends when that instance of SMT is no longer the focus of the observable teacher actions" (Peterson et al., 2017, p. 19). The teacher's retrospective interpretation of the SMT consists of interview instances in which the teacher explains his interpretation of an instance of SMT from the classroom video. The teacher's retrospective reasoning for his response to an instance of SMT consists of interview instances in which the teacher explains what he was thinking when he made a specific pedagogical move in class. We used video analysis software to segment each classroom lesson into the instances of SMT and the teacher responses to each individual instance. We used the same software to divide each interview into segments where instances of SMT from the classroom videos were discussed. Only those instances that the interviewer thought were likely to be MOSTs, appeared to be treated as MOSTs, or instances that the teacher wished to discuss, were addressed in the interviews.

We began coding by identifying instances of SMT that were specifically discussed in the interviews. These instances of SMT were located on the video recorded classroom lessons and were coded using the MOST Analytic Framework (Leatham et al., 2015), and the teacher responses to these instances of SMT were coded using the Teacher Response Coding (TRC) framework (see Peterson et al., 2017). For our third and fourth units of analysis, we categorized the teacher's responses in the interviews as either focusing on the students' thinking or the teacher's response. We used open coding to further categorize the instances dealing with SMT, and the instances dealing with the teacher's responses.

The goal of this study is to understand teachers' in-the-moment responses to SMT. Therefore, the teacher's interpretations of SMT relative to the MOST Analytic Framework are examined simultaneously with the teacher's reflections on his responses to that thinking. We examine the teacher's reflections on his responses to SMT and how those reflective thoughts align with the potential of that thinking to foster learning of important mathematical ideas.

RESULTS AND DISCUSSION

Using open coding on the teacher's interview responses focusing on students' thinking, three categories emerged: the teacher's inferences of a) the SM, b) the MP, and c) the instance's placement within the MOST Analytic Framework. Using open coding on the teacher's interview responses focusing on his in-the-moment responses to students' thinking, two categories emerged: the teacher's rationale for his responses, and his reflection on those responses.

From the four lessons and seven interviews we identified 34 instances of SMT where the teacher gave some insight into his understanding of, or reaction to the SMT. Thus, in addition to knowing what the student said and the teacher's in-the-moment reaction, we were also able to discern one or more of the following: how the teacher interpreted what the student said, how the teacher placed the SMT within the MOST framework,

the teacher's rationale for his in-the-moment response, or the teacher's reflection on his in-the-moment response. Through analyzing these factors, we saw that even in a classroom with an excellent teacher, barriers that may prevent building still exist.

Our analysis revealed that the majority of the time–18 out of 34 instances–the teacher's determination of the building potential either aligned with the MOST Analytic Framework, or we had no indication from the data that his determination differed from the framework. When the teacher recognized the building potential of SMT in an instance, his placement of these instances on the MOST framework were in line with the MOST Analytic Framework. Additionally, the teacher often explicitly spoke of the building potential that he saw in instances, and how this guided his in-the-moment responses to these instances. When both the teacher and the MOST Analytic Framework categorized instances as MOSTs, the teacher reported that his in-the-moment responses to SMT were meant to harness that building potential, which was corroborated by both the teacher's retrospective reflection on his actions as well as our coding of the teacher's response to that SMT.

For 14 of the 34 instances, the difference in building potential between the MOST Analytic Framework, and the teacher's explicit or implicit response can be grouped into four categories: a) taking up only part of the SMT because the teacher did not understand all of it, b) taking up only part of the SMT because the teacher focused on some aspect of that part of the SMT, rather than all of the SMT, c) the teacher considered additional context or thinking that was not part of the SMT because he wanted the conversation to move in a specific direction, a direction that the SMT alone would not likely have steered it, and d) the teacher did not see the importance in an instance of SMT. The first three of these were the most common; there was only one instance of d), so we only focus on the first three. Finally, of the 34 instances, there were two instances where the teacher made an inference of the SMT, which according to the MOST Analytic Framework, could not be inferred. This could have been due to classroom norms, or the teacher's insights to his students, but it was not clear to us that there was a shared understanding of what the student said by the rest of the class.

To help exemplify the three most common types of instances where the teachers did not recognize the building potential in SMT, we will look first at an instance where the teacher did not take up all of the SMT because he did not understand all of it. In one such instance, the class was working with geometric sequences, and in trying to understand the formula for a geometric sequence a student said: "every time we plug in a number it gives us, like, one term further than we wanted it to. So, if we subtract one from n then it puts us back one term every time." In-the-moment, the teacher did not fully understand what the student was saying, and the teacher interpreted the SMT as dealing with plugging values in to check the validity of an equation. After the teacher watched this portion of the class on video, he realized that the student was actually working towards understanding that altering the exponent in the formula of a geometric sequence will yield a different term in the sequence, which is something that could have been built upon to increase the understanding of the whole class. Thus, the teacher

was clearly able to see that a building opportunity existed when he had time to contemplate the SMT, but unfortunately, he was not able to see this in the moment, and a building opportunity was missed.

Next, we look at an instance where the teacher only took up part of the SMT because he focused on some aspect of that part of the SMT, rather than all of the SMT. In this example, the class was beginning to work with logarithmic equations, and a student said that "If you add $\log_2 2$, $\log_2 9$, and $\log_2 2$ you get the same thing as $\log_2(9 \cdot 2 \cdot 2)$.", and went on to note that there was a relationship between this idea and the exponential product rule. The teacher then focused only on the exponential product rule and asked the class "when you multiply exponents you add them together. What does that mean?" and after a student responded to this, the teacher dropped this instance of SMT, and moved on. Since the students in this class were already familiar with the exponential product rule, the teacher missed an opportunity to connect students' prior knowledge to the new material that they were currently grappling with. This differs from our last example because–as he explained in the interview–the teacher understood what the student was saying, but in the moment, he made the conscious decision to take up only a small portion of the SMT that was presented. In retrospect, the teacher was able to see that this was indeed a missed opportunity, and he noted that he should have further utilized this instance of SMT.

Finally, the third most common category of instances where the teacher missed an opportunity to build on SMT, were instances where the teacher considered additional context or thinking that was not part of the SMT because he wanted the conversation to move in a specific direction, a direction that the SMT alone would not likely have steered it. In one such instance, the class was working with geometric sequences and a student said, "Okay, um, we got $a_n = 2(4)^n$." Earlier in the lesson, other students had noticed that each term in the sequence that they were given was four times the previous term. Drawing on this earlier thinking and this newly presented equation, the teacher inferred that the SM was "you begin with 2 and continually multiply by 4." Unfortunately, by drawing on this earlier thinking, the teacher missed the opportunity to build on the thinking at hand, which was that a student had, for the first time in this class, presented an equation for a geometric sequence. Since the class had been trying to use their knowledge of exponents to help them construct the formula of a geometric sequence, the fact that after grappling with it, one student had finally presented such a formula would have been the perfect opportunity to build on that SMT to help the rest of the class understand how to derive this formula. However, since the SMT also lent itself to a point that another student had been trying to make earlier, the teacher decided instead to use the current SMT to exemplify an earlier student response.

CONCLUSION

This study confirms the results found by Maher and Davis (1990) that for teachers to productively use SMT it is important that they accurately infer that thinking. Our findings show that incorporating SMT in ways that foster learning requires that in

addition to correctly interpreting SMT teachers need to also correctly identify the potential of the SMT. This will help deepen or enrich the mathematical understanding of the other students in the class by making the SMT the object of consideration. Furthermore, the complexity of teaching is highlighted in this study as we find that even an exemplary teacher may not always be able to utilize appropriately the in-the-moment SMT that is available to them. As our study showed, possible reasons for this include: not completely understanding the SMT, choosing deliberately not to act on a portion of the SMT, or adding additional context to the SMT to make a specific MP. Regardless of the exact reason, not fully utilizing available SMT can lead to missed opportunities for learning when the instances are MOSTs–"Mathematically Significant Pedagogical Opportunity to Build on Student Thinking" (Leatham et al., 2015, p. 90). Our findings show that in two out of 34 instances the teacher inferred the SMT where according to the MOST Analytic Framework, SMT could not be inferred. Such inferences could result in unproductive engagement in situations where the class does not have a shared understanding of the SMT when the SMT is ambiguous and the teacher response does not seek clarification of the instance before allowing the class to engage with it. This study confirms findings of Cobb (1988) and Maher and Davis (1990), that it is important that teachers carefully listen to and seek to fully understand, and clarify if necessary, the SMT that occurs in their classrooms since not doing so can hinder students' construction of mathematical ideas as the teacher is then more likely to impose their own constructions on students. This suggests that there may be a need for professional development that is focused on supporting teachers to make sense of SMT that emerges in-the-moment during their instruction as a first step towards the development of the skill of building on SMT. Future studies could identify ways of supporting teachers' development of skills that would facilitate the accurate interpretation of SMT and its underlying potential to foster learning.

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EXAMPLES IN MATHEMATICS - DIFFERENCE AND REPETITION

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The availability of instructional examples in the mathematics classroom challenges the way teachers confront students with several aspects of the content, which can contribute to designating the generality of mathematical concepts. Using the theory of variation and Deleuze's philosophical concepts as the main interpretative framework, this study investigates what is involved in repetition and the nature of its interiority, what we understand by conceptual difference and difference without concept in a development study. The data consisted of 36 teachers' lesson plans. Qualitative analysis of these data led to the identification of events types and the characteristics of repetition.

INTRODUCTION

Krainer (2014) specifies that mathematics education deal with the learning and teaching of mathematics. Still, there is little consensus as to what role educational research should play in directly influencing educational practice (e.g. Vanderlinde & van Braak, 2010). Kieran, Krainer and Shaughnessy (2013) claim that a distinct gap between research and practice has been created because too often, in mathematics education, research is no connected to the needs of teachers and does not take into account teachers' reality. The purpose of this article is to investigate what in the use of examples enables a teacher's professional development. Also, the purpose is to investigate what is involved in repetition and the nature of its interiority, what we understand by conceptual difference and difference without concept in a development study. The word example is used in accordance with Watson and Mason (2005) definition, namely to be "anything from which a learner might generalize" (p. 3). In the teaching using examples, repetition can be used in order to help students be successful in their learning behaviour. The use of repetition in learning is a common finding in the literature (e.g. Marton et al., 2005). The following questions guide this paper: What are the characteristics of repetition in teachers' professional development? What is it that repeats in repetition?

DIFFERENCE AND REPETITION

An object of learning (the content that teacher intends to teach as well as how the students are expected to make sense of and make use of the content) is formed in the communication that occurs in different events between teacher and students (e. g. Olteanu, 2016). An event is a process that begins with one aspect of the content that one or more students have not yet discerned and ends with one or more students distinguishing that the aspect of the content (Olteanu, 2016). For example, if some

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students do not discerned what slope really means, an event begins with a sequences of examples aimed to developed the mathematical concept of slope. The event in the development study presented in this article is a problem or problematic idea rather than a truth or a solution. If the event is a sequence of mathematical examples, the problem or problematic idea is the students' capability to discern aspects related to the object of learning by experiencing them. If those aspects are not discerned they become critical aspects for the students (Olteanu, 2016). Deleuze (1994, p. 44) writes: "learning is the appropriate name for the subjective acts carried out when one is confronted with the objectivity of a problem ... whereas knowledge only designates the generality of concepts or the calm possession of a rule enabling solutions". When using examples in a mathematics classroom, students are confronted with several aspects of the content, which can contribute to generalizing mathematical concepts. For this reason, the analyses of the event as it changes from being specific (immanent) to a convergence of parts or elements (Deleuze, 1994), can provide a new way of understanding of what the characteristics of repetition are in teachers' professional development.

The things shared in an event having mathematical examples as a starting point generate a variety of aspects of the object of learning that do not have a fixed form but can be formed during the lesson. This ability to be formed can be expressed using different patterns of variation. Previous research (e. g. Marton, 2015; Olteanu, 2016) mentioned five patterns of variations, which can facilitate students' discernment of critical features or aspects of the object of learning. These patterns of variation are: (1) contrast (in order to experience something, a person must experience something else to compare it with); (2) separation (an aspect must vary while other aspects remain invariant); (3) fusion (several critical aspects need to be considered together); (4) similarity (the property of two or more expressions to adopt the same meaning); (5) generalisation (is to see variations in the use of the object to fully comprehend it and involves recognizing that some features are not critical to the identification of that phenomenon).

Stagoll (2005) explains that an event arises from a set of particular forces. These forces can emerge in relation to a set of dialectical opposites (e.g. whole/part) for which resolution is never fully granted. For example, a dialectical opposite related to an algebraic expression are: (1) a whole $ax + b$ and; (2) the parts that form the whole (e.g. a, x, b); (3) the relation between the parts (e.g. the operation); (4) the transformation between the parts (e.g. the rewriting $ax + b$ as $a(x + b/a)$, with $a \neq 0$); (5) the relation parts–whole; and (6) the relation between different wholes (e.g. the relation between $ax + b$ and $a(x + b/a)$). In such dialectical opposites there are qualitative differences. In the dialectical opposite "the transformation between the parts" a qualitative differences is to understand that two or more expressions are equivalent if they have the same value. Even two expressions have different symbolic forms; they represent the same quantity when each variable in both expressions has the same value. The expressions $2x - 3$ is equivalent to $\frac{2}{1}x - 3$ because x in both expressions have the same value for each value of the variable x. It is important to recognize that the linear functions $f(x) =$

$2x - 3$ and $f(x) = \frac{2}{1}x - 3$ have the same rate of change (or slope). Between the forces that can emerge in relation to a set of dialectical opposites, qualitative differences that are express by singularities can be identified.

Deleuze (1990) describes singularities as turning points and points of inflection in a topological way. In topology, a series of singularities can undergo a variety of transformations that allow the study of singular points. A singular point is simply a point where the differential of the function is not surjective. The differential function *f* is surjective (*f*: A → B be an arbitrary function with domain A and codomain B) if every element y in A has a corresponding element x in B such that $f(x) = y$. For example, the image can be connected to intended critical aspects (the aspects of the content that teachers intend to present in the classroom and the codomain can by connected to lived critical aspects (the students' distinguished critical aspects of the object of learning). If the image of the intended critical aspects is not equal with its lived critical aspects, then a singularity can be identified.

Difference is a singularity at the level of ideas and repetition is always affected by an order of difference. Deleuze (1994) specifies that the concepts of repetition and difference will intersect, "one concerning the essence of repetition, the other the idea of difference" (p. 31). Repetition is never the reproduction of the same, but the repetition of the different (Deleuze, 1994). A multiplicity is an idea; in other words, it is a set of differential elements, differential relations, and singularities (Deleuze, 1994). For examples, in the classroom the differential elements would be the teacher, the students, the mathematical content, and the examples. They are differential elements because they are defined only in relation to each other. The differential relations are what the students are able to do with the examples and with each other. They are differential in that they are relations of change in the elements: how students are able to discern important aspects of the content taken in the examples, to put these aspects together, and to use the examples to solve some tasks. These relations are strewn with singularities, or sensitive points, when the aspects of the content move across a certain threshold of the field. Changes in the elements, relations and singularities will change the teaching and learning in the classroom, which in turn enables the teacher to develop professionally.

METHODS

The results presented in this article are based on a three-year longitudinal study that involved eight secondary schools and 22 teachers. The teachers were voluntary participants, selected using maximum variation and typical case sampling (Patton, 2002). To facilitate the development study (McKenney, Nieveen & van den Akker, 2006), the design principles were: (1) interventionist: the aims were to designing an intervention in schools and in a real setting; (2) iterative: during the study the research team incorporated cycles of analysis, design and development, evaluation, and revision; (3) involvement of practitioners: the teachers participated in all the activities and lessons during the project; (4) process oriented: the focus was on understanding

and improving interventions by analysing the students' capability to discern aspects related to the object of learning after experiencing them; (5) utility oriented: the design was measured by its practicality for teachers in real contexts; and (6) theory oriented: the design was based on concepts from variation theory, namely the object of learning, critical aspects and patterns of variation.

The focal point for data analyses was the examples that teachers (two from elementary and two from upper secondary school) had planned to use and had has been used as the intended object of learning in three-year study of an elementary classroom and an upper secondary school. The teachers' planning is connected to the process of building generalization concerning the concept of first-degree functions in classes 7–9 (the students were 13–15 years old) and the first course in upper secondary school (students are 16 years old). 36 teachers' planning was analysed.

Drawing on Deleuze (1994) frameworks, we constructed a model for analysing the examples. The schema for codifying and analysing the information with which we obtained the results is based on a cyclical process that seeks to minimise discrepancies. The states of development that emerged from this process identified the combinations of values of the variables to which, as a whole, the observations for a given example were best adapted. This procedure enabled to characterise the singularities at the level of ideas (difference) and repetition affected by an order of difference. The analysis focused mainly on the set of differential elements, differential relations, and singularities (multiplicity) in the observed work of the teachers participating in the development study.

RESULTS

The results consist of a global inventory of the differential elements and the frequency distribution of the differential relations over the four teachers involved in the study.

The differential elements

The differential elements identified in the study are: the teachers, the students, the content and the examples. In the elementary school, the teachers had, in the first year of the study, students in grades 7. In the second year of the study these students were in grade 8 and the third year in grade 9. This means that the teachers and the students were the same during the three years of the study. In upper secondary school, the teachers were the same during the study, but taught new classes of students each year. The object of learning was the same for elementary and upper secondary schools during this period, namely to improve students' ability to discern relationships between variables.

The differential relations

Four types of event were identified through a combination of theory-driven and data-driven analysis.

Variables change (Event 1) is a basic situation in which functions are presented as an input-output relationship, and there is no variety from a variation theoretical

perspective. Most functions that teachers in secondary and upper secondary school choose as examples are described by an algebraic expression of first-degree. For example, the functions as $y = 4x + 3$ (characterized by a constant additive increase). Those functions belong to the same families of linear function, namely the rate of change is positive and is expressed by a positive integer.

Slopes (Event 2) are a transition state. There is some complexity in the conceptual structure, and variety begins to appear in the systems of representation. However, there is still no variety from a variation theoretical perspective. The examples in this situation are characterized by a positive rate of change (e.g. $y = 5x + 4$) and graphical representations, but they not focus attention on links between symbolic and graphical representations of linear functions. These functions belong to the same families of linear function as in Event 1.

Changing rates of change (Event 3) shows an advance in all types of first-degree functions except those of domain and codomain. The examples belong to different families of linear functions (positive and negative rate of change), and the teachers stressed the relationship between what happens with positive and negative slopes and how they are represented in conventional mathematics. There was variety in the systems of representation and the number of connections (Table 1).

Sequences 1	Sequences 1
$y = x$	$y = -x$
$y = 3x$	$y = -3x$
$y = \frac{1}{4}x$	$y = -\frac{1}{4}x$
$y = \frac{3}{4}x$	$y = -\frac{3}{4}x$

Table 1: Changing rates of change

Multiple ways to represent and analyse first-degree functions (Event 4) achieves full complexity from a variation theoretical perspective, and the information was used consistently for the completion of examples. This situation concerns a class discussion about what happens when the students justify their own generalizations focusing on how the constant rate of change in a linear function appears in different representations of the function and the relationships between those representations (similarity and generalisation as patterns of variation). For example, in a table the value of variable increase or decrease by a constant amount as the value of the other variable increases by a constant amount. In a graph the rate of change is represented by the slope—the steepness and direction—of the line, and in the symbolic form written in the form $y = ax + b$, a is the constant rate of change. In this way, the students have the opportunity to discern difference at the level of ideas and repetition that is affected by an order of difference.

Table 2 presents the final classification of the three years of the project of the two groups into each of the four events. Each row represents a group of teachers and their corresponding intended object of learning, organized chronologically. Thus, for example, the intended object of learning to the teachers in grades 7 was assigned successively to the following events: 1, 1, 1, 2, 2 and 2.

Between the first and the second year of the project, a turning point occurred for the groups in upper secondary school that generated a singular point at the level of ideas. For an example, the teachers used contrast, separation and similarity to describe differences between positive and negative slopes in both symbolic and graphical representations of a first-degree function (Table 1). Something similar occurred in the third year of the project in upper secondary school, with the notions of variety of phenomena, variety in systems of representation, complexity and systems of representation as an organizer of the conceptual structure.

	Intended object of learning					
Grades	1	2	3	4	5	6
7	1	1	1	2	2	2
8	1	2	1	2	3	3
9	3	3	3	3	4	4
Upper secondary school (year 1)	1	1	1	2	2	2
Upper secondary school (year 2)	3	3	3	4	4	4
Upper secondary school (year 3)	3	3	3	4	4	4

Table 2: Final assignment of intended object of learning

The analysis showed that what was repeated between the second and third year concerned the use of rich patterns of variation, that is to say an order of difference concerning the process of representing an idea using similarity and generalization as patterns of variation. In an example, the teachers used representations by keeping the meaning of an aspect invariant and varying the form of representation (words, symbols, tables and graphs). In this way, the repetition was not a reproduction of what was the same, but of things that were different. The characteristics of repetition consisted of: (1) qualitative differences in relation to the function as a whole; (2) the parts that formed the whole (slopes and y-intercept); (3) the relation between the parts (the ratio of the change in the output variable to the corresponding change in the input variable of the function); and (4) the relation of parts–whole.

In the secondary school, the teachers kept invariant the same family of a linear function and varied the situations that they modelled. For example, for instructional examples they used the change in distance that relates to a particular change in time or price per item. Those examples did not provide an opportunity for students to experience the

difference at the level of ideas. The intended object of learning was repeated in the second year of the project, and thereafter the teachers added some examples concerning the difference in the direction of line. In this way, the repetition was not a reproduction of what was the same, but a use of a different idea. By contrasting the slopes (positive and negative), the examples provided an opportunity to explore several different relationships and generalize the rate of change. By using similarity as a pattern of variation, the examples provided the opportunity to explore linear functions and identify linear patterns of change in tables, graphs, and symbolic rules. In the third year of the project, repetition occurred at the level of idea, but the examples provided the opportunity for students to experience a statement that described a general mathematical truth about some family of function, which can contribute to designating the generality of mathematical concepts.

CONCLUSIONS

The main result of the study reported in this article was the characterization of the events and classification of the intended object of learning based on an analysis of the instructional examples. This result confirms that the didactic knowledge of the groups of teachers evolved according to stable patterns. The groups progressed in the development of their choice of instructional examples at different paces and levels of advancement. Analysis of the discrepancies in each event shed light on which notions presented more difficulty to students. For example, the notion of connection presented a high number of discrepancies with positive difference. The numbers of examples, which were chosen, by the groups of teachers had a level of connection to each other higher than expected. The potential of repetition in the teaching and learning of school mathematics relies on the process of producing examples, which involves design and analysis. Design may include: (a) implementation of a known procedure to generate the successive terms in, for example, an arithmetic or geometric sequence or (b) modification of a given procedure to use a new but related example. Analysis is the process of determining the differences in repetition. To summarize, the potential of repetition lies in its ability to help students develop an understanding of mathematical concepts and. Spaced repetition is a learning technique that incorporates increasing the intervals of time between subsequent reviews of previously learned material in order to exploit the critical aspects effect.

The notions of repetition and difference are not well understood and research is needed to identify the ways in which mathematics teacher develop their own knowledge. The findings could inform development of a set of recommendations for supporting professional learning that is grounded in the practices of mathematics teachers who have varying levels of experience and work in a variety of different contexts. Such recommendations may stimulate further research on the design of examples in development studies with a focus on professional development of mathematics teachers.

Over time, mathematics education research has shifted its focus from consideration of mathematical content and curriculum development, to the mathematical learning of students, to interactions between students and teachers, to the learning of teachers, and most recently to the learning of teacher educators (Krainer, 2014). This paper argues that philosophical research is needed to enhance theoretical understanding of how mathematics teachers learn and develop. Research of this type is important because it acknowledges the complex forms of knowledge needed by teacher educators and the multiple social settings in which their learning takes place.

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EXPLORING DYNAMIC MEASUREMENT FOR VOLUME

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This paper describes our efforts for exploring students' thinking of Dynamic Measurement for Volume (DYME-V), an approach to volume measurement that engages students in dynamic experiences of extruding surfaces to create 3-D objects and assists students in reasoning about volume as a continuous structure that depends on the size of the surface being extruded and the height of the extrusion.

VOLUME: WHAT WE KNOW AND PUSHING FORWARD

In school mathematics, volume measurement focuses on packing the space within a three-dimensional (3-D) object using cubic units and quantifying that packing (Curry & Outhred, 2005). Students may use a variety of strategies to measure volume, such as counting each individual cubic unit, finding the number of cubic units in one layer and skip count or multiply with the number of layers, or using the *length* x *width* x *height* formula (Battista & Clements, 1996; Cullen et al., 2017; Curry, Mitchelmore & Outhred, 2006).

When using this *Volume as packing* approach to measure, students experience several difficulties, especially if they are asked to find the volume of a 3-D cube array or of a picture of a 3-D object on paper. In these situations, students often count the cube faces shown on the 3-D object or picture ignoring the third dimensionality of the object, or just count the cubic units that are visible ignoring the hidden portions of the shape (Ben-Haim, Lappan & Houang, 1985). Additionally, Battista and Clements (1996) stated that while calculating the volume of a given cuboid by enumerating unit cubes in arrays, students struggle in coordinating the separate views of arrays and integrate them to form a coherent mental model.

Volume as filling is a second approach to volume measurement used in math education that refers to filling a 3-D space with iterations of a fluid unit that takes up the space of the container (Clements & Sarama, 2009; Curry & Outhred, 2005). When Curry and Outhred (2005) asked students to find the volume of a jug by filling the jug using cupfuls of rice, they noticed that students treated the height of the rice in the cups as a unit length, which they iterated to fill the whole space inside the jug. Curry and Outhred (2005) described this approach as one-dimensional since students focused only on the height of the cups. This predominant use of a single dimension to make three-dimensional judgments was also noticed by Piaget and his colleagues (Piaget, Indelder & Szeminska, 1960) and later by Raghubir (1999) who found that these one-dimensional judgements prevail even when adults perceive volume in boxes at the grocery store.

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EXPLORING VOLUME DYNAMICALLY

The literature before comes in contrast with research stating that students already have a dynamic sense of volume. For instance, Lehrer, Strom and Confrey (2002) reported that students visualized volume "like pulling" the area through the height of the cylinder, in other words looking at three-dimensional objects as 2-D unfoldings. In exploring students' thinking of dimensions in geometry, Panorkou and Pratt (2016) discussed the dual nature of space stating that "one can see the space as incorporating objects; in this sense, the space contains the objects. At the same time, the space can be thought of as generated by the objects" (p. 213). The generation component of volume is also described by Lehrer, Slovin and Dougherty (2014) who gave the example of a volume being generated by sweeping an area through a length. To illustrate this generation, imagine extruding a two-dimensional rectangular surface (base) of area '*ab*' for a height or depth of '*c*' to generate a space of '*abc*' (Figure 1).

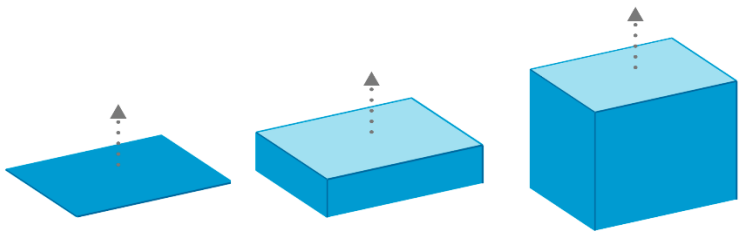

Figure 1: Volume as a continuous dynamic structure

To distinguish this dynamic generation approach to measurement from other approaches, we refer to it as *dynamic measurement* (DYME). DYME is defined as an approach to geometric measurement that focuses on how space is measured by the lower-dimensional objects that generate it. *DYME for Volume (DYME-V)*, involves extruding surfaces (a base) on a certain height and reasoning about volume as a continuous structure that depends on the area of the base and the height of extrusion.

This dynamic perspective of volume could be very powerful for reasoning about measurement for a number of reasons. This DYME approach involves reasoning about the *transformation* of quantities that underlie the volume formula. The packing approach to volume involves using a volume unit to measure volume (e.g. a number of 1 cm^3 cubes) while the volume formula of *area of base times the height* involves a multiplicative composition of two units (one linear and one area unit) to construct a new unit for volume. In other words, it involves a transformation in dimensions. Additionally, the DYME approach has the potential to introduce students from an early age to a study of *mathematics of change and variation* (Rochelle, Kaput and Stroup, 2000) by engaging students in reasoning about the quantities as continuous dynamic structures that can dynamically change. For instance, if I split the length of the area of base in half, the volume is split in half. Engaging in this type of reasoning characterizes what research refers to as *covariation reasoning*, which involves coordinating two quantities simultaneously, for example area of base (or height) and volume, as the

values of those quantities change (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Confrey & Smith, 1995). Covariational reasoning is seen as foundational for functional thinking (Confrey & Smith, 1995), therefore these dynamic experiences have the potential to connect volume measurement to advanced mathematical ideas in algebra and calculus.

Consequently, our goal was to test the conjecture that it is possible for children to visualize volume as a dynamic continuous structure and as a product of the area of base and height through careful task design. More specifically, we aimed to explore: (a) What type of tasks and tools may be used for developing students' DYME-V? and, (b) What forms of DYME-V reasoning are made visible and can be seen to develop as a result of students' systemic engagement in these tasks?

THE DESIGN STUDY

We used a design-based research methodology to engineer particular forms of learning and study how those forms of learning develop with the particular context of DYME-V (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003). To illustrate the dynamic nature of volume, we drew our attention to the power of technology. Ferrara and Mammana (2014) argued that students' physical manipulation of 2-D and 3-D shapes prevents them from discovering the crucial relationships between 2-D and 3-D figures, however the use of Dynamic Geometry Environments offer "the visual and cognitive potential of interlacing related but different figures…allowing for moving back and forth between plane and space" (p. 57). Consequently, for our task design, we used the dynamic features of extrusion and tracing of Geogebra to enable students to generate 3-D objects by extruding 2-D surfaces and reason about the measurements involved. We developed and refined our conjectures, tasks and tools by conducting design experiments (DE) (Cobb et al., 2003) with six pairs of fourth-graders. The students represented various abilities according to their teacher. We met with each pair of students for around 6-8 one-period (45-50 minutes) sessions.

FINDINGS

At the beginning of the design experiments, we asked the students to explain what volume means to them. All students responded that they learned how to measure volume in science by water displacement. Next, we present some DYME-V tasks and discuss the type of generalizations students made as they interacted with those tasks.

1. Exploring volume as generation

To help students develop an understanding of a 3D shape and its volume as an extrusion of a 2D surface, we asked students to extrude surfaces in different planes and reason about what they observe (Figure 2). As they dragged the surfaces, students reasoned that the software is "stretching them out" and 3D shapes are created "by pulling, by making them longer." While some students talked about extrusion as a transformation from 2D to 3D similar to the examples above, others discussed the extrusion as an iteration (creating identical copies) of 2D surfaces.

Figure 2: Extruding rectangular surfaces to create 3D shapes

Olympia's statement is an example of reasoning about an extrusion as an iteration:

Researcher: What does the software do?

Olympia: It like creates like…like…3D shapes kind of…with multiple 2D shapes. Like 2D square, 2D square and then it starts closing them together to look like…this is like 2D rectangles and then they close it together to look like a rectangular prism.

When we asked students who thought about volume as an iteration if they were able to count the number of those 2D shapes, they all stated that they cannot count the 2D shapes. For instance, Dan argued "that'll be hard. You have to make an estimate," showing an understanding of volume as a continuous rather than discrete structure.

2. Volume depends on the size of the base and height

Students initially noticed that the size of the 3-D shape created "depends on how far you stretch them out" or how much you "expand it." Next, we presented students various shapes to extrude, including rectangles, circles and hexagons (Figure 3). By extruding different shapes, students were able to generalize that the volume of the 3-D shape created depends on the size of the base (we introduced the term "base" to refer to the surface being extruded) and the height of extrusion. For instance, Molly argued "*it depends on how big the base is and it also depends on how much you can drag it*."

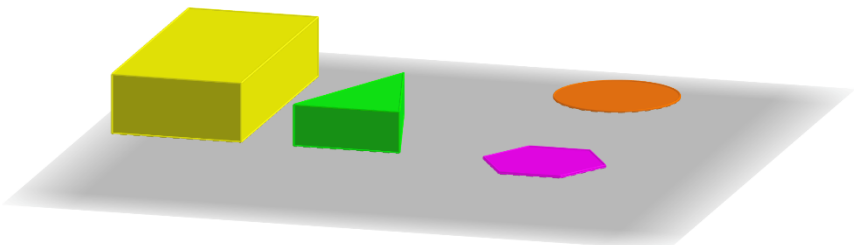

Figure 3. Example of a task where students extrude surfaces of various shapes

By providing students with opportunities to extrude various surfaces in different heights, they were able to notice how the changes in the size of the base and the dragging distance affects the size of the 3D shape generated. In other words, students recognized that measuring a 3D shape requires the coordination of two other quantities.

3. Volume as the product of the area of base and height

To help students recognize the multiplicative relationship between area of base, height of extrusion and volume of 3D shape, we made the design decision to just focus on one varying quantity and keep the other one fixed. We presented to students the same task as in Figure 3 but this time we included a value for the area of each base (Figure 4). We first asked them to extrude the shapes for a height of 1 inch and then 2 inches and reason about the space covered by each shape.

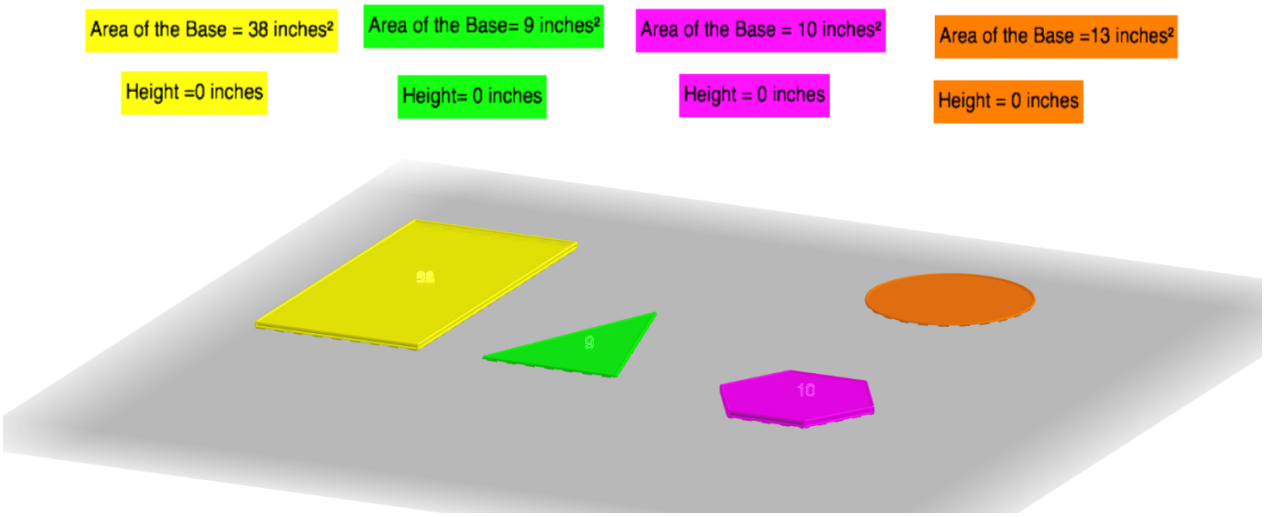

Figure 4. Extruding surfaces with various shapes that include values for area of base.

Orill: This one is 13, that is the highest 38, then second is 10 and 9 the lowest (Fig. 4).

Researcher: What are these numbers that are you telling?

Orill: The areas.

Researcher: The area of the base or area of the shape?

Orill: Area of the base.

Researcher: How much space did we cover in this one [the orange circle]?

Steve: 13.

Researcher: Why?

Steve: Because there is 1 and 13.

Researcher: How about this? [increases the height to 2]

Orill: That covers two 13s.

As students' thinking progressed, they were able to generalize that volume depends on the size of the base and the height, the number of drags you drag this base:

Researcher: How do we estimate the volume of something?

Lilly: We have to multiply the numbers of how much drags did you do, the height, and how much the base is to get the volume.

By generating 3-D shapes using 2-D shapes with various areas of bases and extruding those in various heights students were able to generalize that the volume of the 3-D shape can be found by multiplying the area of the base with the height of extrusion:

Orill: We have to know how much is the base.

Researcher: And what are you going to do if you know the base?

Steve: You are going to multiply with the height.

Subsequently, we presented students with a task in which the area of the base was not given. Instead we provided a value for length and width of two rectangles:

Steve: We have to multiply the length with the width to get the area of the base and then you have to multiply the area of the base with the height to get how much space in total is covered.

The excerpt above shows that students were able to recognize that when a rectangular surface of area ab is stretched to a height of c, then the volume of the cuboid is ab times c; in other words, volume is equal to the product of base times height. After students reached this generalization, we told them that, in the specific task, volume measured using this symbol "inches^3" and asked them to say what they think it means. Students argued that in measuring area we have a '2' on top of 'inches' or 'cm' because we measure a 2-D shape while now we have a '3' because we measure a 3-D shape. To them, the powers of 2 and 3 were distinguishing the type of space they were measuring.

4. Multiplicative coordination of length, area of base and volume measurement

Next, our goal was to explore whether students would use the reasoning they constructed about volume to reason about changes in volume when the area of the base changes multiplicatively. The tasks focused on changing the linear measures of rectangular prisms of a sculpture and reasoning about the change in volume (Figure 5). Aiming to help students to focus on the quantities and the relationships among them, we had the values of the dimensions of the sculptures hidden at the beginning so that students make a conjecture without any numerical calculations.

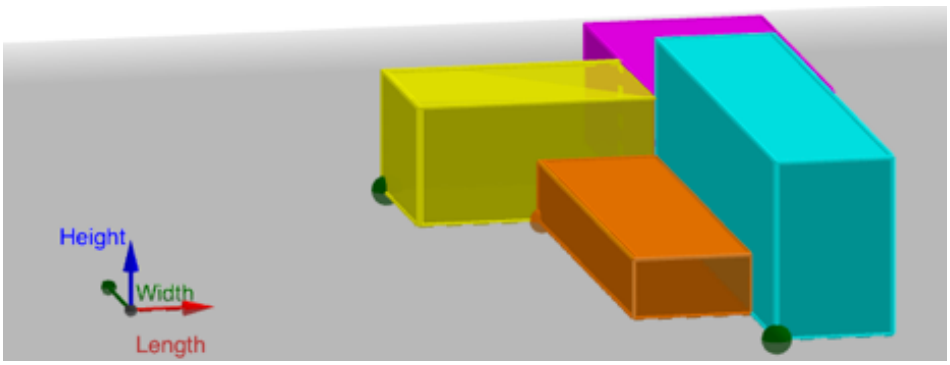

Figure 5. Students change the dimensions of rectangular prisms and reason about the change in their volume

[Task: Make the width of the Yellow block three times bigger.]

Researcher: If we change the width, what is changing?

Jayden: We are changing the area of the base.

Charles: Oh, yeah, the area of the base.

Researcher: How many times bigger is it going to be?

Jayden: Uh, three.

Charles: Three times, the area of the base is going to be 3 times bigger.

Researcher: What will happen to the volume?

Charles: It will be…

Jayden: The volume would be…bigger, if you just extend the volume ….

Researcher: How many times bigger?

Jayden: Three.

Students generalized that if we change the width of the shape, then the area of the base changes and as a result the volume of the shape changes with the same factor. By engaging with this type of tasks, they were able to recognize that to make the volume *n* times bigger, they need to make the area of the base or the height *n* times bigger.

CONCLUDING REMARKS

DYME-V is an exploratory study which examines how students reason about volume as a continuous quantity when they are exposed to dynamic Geogebra tasks. Our results suggest that the dynamic measurement approach has the potential to engage students in dynamic tasks that foster the development of volume as a dynamic continuous structure. Students' understanding of volume developed from perceiving volume as a generation of 3-D from 2-D, to coordinate both the area of the base and the height to make judgements about volume, to reasoning about volume as area of base *times* height and finally construct covariational relationships between the linear measures of 3-D shapes and volume measurement. These results suggest that exposing students to these dynamic tasks in earlier grades have the potential to guide their meanings about volume as a transformation of lower-dimensional objects and reason mathematically about change and variation.

Acknowledgements

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SUPPORTING STUDENTS' UNDERSTANDING GRAPHS AS EMERGENT TRACES: THE FAUCET TASK

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Students experience persistent difficulties creating and interpreting graphs intended to represent covarying quantities. In this paper, I describe ways of reasoning middle grade students (age 10-13) engage in as they construct and reason about graphs as representing emergent traces of covarying quantities. After describing relevant theoretical constructs, I present a seven-part task and describe how each part is explicitly designed to support students in developing meanings relevant to the theoretical constructs. For each phase, I present a student's activity highlighting the student's developing emergent thinking. I conclude by discussing implications of this study as well as areas for future research.

INTRODUCTION

Graphs represent information and relationships while also providing insights into students' thinking in ways that are difficult to express or determine in other forms (Arcavi, 2003). However, students experience persistent difficulties creating and interpreting graphs intended to represent covarying quantities (Leinhardt, Zaslavsky, & Stein, 1990). Researchers have identified a range of common misconceptions related to graphing including drawing graphs by connecting points without considering what happens between points (Castillo-Garsow, Johnson, & Moore, 2013) and coordinating one quantity while tacitly overlooking the other quantity (i.e. reasoning uni-variationally) when prompted to represent or interpret relationships (Leinhardt et al., 1990). Collectively, this research indicates instructional approaches have not provided students sustained opportunities to develop meaningful ways of understanding graphs as representing relationships between covarying quantities.

The purpose of this paper is to describe how a carefully designed task was productive in supporting middle grade students' (age 10-13) development of emergent thinking (Moore & Thompson, 2015). After describing theoretical constructs relevant to the current study, I present a seven-part task designed with these constructs in mind. I describe how each part of the task explicitly addresses key aspects of the theoretical constructs. I then present a student's activity addressing each phase to highlight the students' developing emergent thinking.

THEORETICAL BACKGROUND

Prior to describing emergent thinking, I first describe students' understandings of underlying coordinate systems. Lee (2016) distinguished between two uses of coordinate systems in students' thinking: spatial coordinate systems and quantitative coordinate systems. Spatial coordinate systems allow students to represent or

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mathematize a space or physical phenomena (i.e. using a GPS map to represent the location of a car on a trip). In order to represent and coordinate two non-spatial quantities (e.g., temperature, volume, etc.), students must construct a quantitative coordinate system. To construct a quantitative coordinate system, an individual must establish quantities within a situation, extract the quantities from the situation while maintaining an awareness of the situational quantities (Steffe & Olive, 2010), and project them onto the new space (i.e. the quantitative coordinate system).

With a quantitative coordinate system in mind, a student must conceive of a point corresponding to the two quantities' magnitudes as representing a multiplicative object (Thompson, 2011) to coordinate two quantities in this new space. Drawing on Piaget's notion of "and" as a multiplicative operator, Thompson and colleagues (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, as a multiplicative object, a point in a coordinate system simultaneously represents two quantities' magnitudes or values.

Drawing on previous descriptions of covariational reasoning (e.g., Thompson, 2011), Moore and Thompson described *emergent thinking* as conceiving a graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (2015, p. 785). Critical to such a conception is students understanding a point as a multiplicative object. The student can then conceive of the graphs in terms of an emergent, progressive, trace constituted by the point's movement in the quantitative coordinate system dictated by the covarying magnitudes of the quantities represented on the axes. Paoletti and Moore (2017) described how college students were able to leverage emergent thinking to conceive of a single graph as being producible by two different emergent traces. After producing a graph representing liquid height and liquid volume in a bottle as liquid entered the bottle, the students were tasked with representing the same quantities as liquid evaporated from the bottle. The students argued their first graph would represent this new situation *if* they imagined the graph tracing in the opposite direction; Paoletti and Moore (2017) concluded such reasoning was an indication of the students' thinking emergently.

The students in Paoletti and Moore's (2017) study leveraged the mental actions described in Carlson et al.'s (2002) covariational reasoning framework which was critical to their activity. The students coordinated *direction of change* (liquid volume *increases* as liquid height *increases*; MA2) and *amounts of change* (the *change* in liquid volume *decreases* as liquid height *increases in equal successive amounts*; MA3). Particular to their emergent reasoning, the students engaged in reasoning about increases and decreases in each quantity when reasoning about their graph as an emergent trace; this reasoning supported the students as they conceived of the same final graph as being producible by multiple emergent traces.

METHODS, PARTICIPANTS, AND ANALYSIS

To examine the potential of supporting middle-grades students' (age 10-13) development of emergent thinking, I conducted a design study (Cobb et al., 2003) consisting of three small-group (one to three students) teaching experiments (Steffe & Thompson, 2000). The teaching experiments occurred in a school, which is considered underperforming (as measured by standardized testing) that hosts a diverse student population (over 75% students of color and qualify for free or reduced-price lunch), in the United States. Students were recruited through teacher recommendations; I interviewed all students who returned consent forms.

To analyze the data, I performed conceptual analyses (Steffe & Thompson, 2000) to produce, test, and modify models of students' reasoning so these models provided viable explanations of their words and actions. I analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Specifically, I watched all videos identifying instances providing insights into each student's understanding related to my theoretical perspective. I then generated tentative models of each student's mathematics, which I compared to researcher notes taken during on-going analysis. I tested these models for viability by searching for supporting or contradicting instances in other activities. When evidence contradicted my models, I revised hypotheses to explain each student's meanings and returned to prior data with these new hypotheses in mind to modify previous hypotheses. This process resulted in viable models of each student's mathematics. For brevity's sake, I focus on my model of one student's mathematics, Amber, who is representative of other students' mathematics.

THE FAUCET TASK: BUILDING TO GRAPHS AS EMERGENT TRACES

For each part of the task, I describe the activity I prompted students to engage in and how I designed the part to support students developing emergent thinking including the prerequisites leading to such thinking. I then include examples of Amber's responses to highlight how the task has been effective in supporting students in thinking emergently.

In Part I, I present students with a GeoGebra applet intending to represent a faucet with water coming out (https://ggbm.at/rdxkrwek). Students can use red and blue sliders to turn the hot and cold knobs on and off. As they do so the color of the water changes according to the temperature (darker red for hotter, darker blue for colder). The goal of Part I is to present students with an experientially real context (Gravemeijer & Doorman, 1999) so they can establish quantities within the situation. After giving students the opportunity to explore the applet, I ask them what quantities they can measure, with the intention of discussing water temperature and amount of water leaving the faucet (e.g., flow rate, water pressure). In Amber's case, she quickly identified "how much water comes out" and "temperatures of the water" as quantities she could measure.

After students describe quantities consistent with water temperature and amount of water leaving the faucet, I present Part II, which includes a thermometer to gauge the water temperature. I present the thermometer for two reasons. First, it gives students a way to describe temperature changing without referring to the color of the water. Second, by using a vertical linear segment to represent temperature, I intend for students to be able to consider a similar representation in Part IV, during which I present them with what I intend to be a quantitative coordinate system with temperature represented by a red segment on the vertical axis.

For Part II, with both the hot and cold knobs turned half way on, I ask students to predict what will happen to water temperature and amount of water leaving the faucet for the following scenarios: (A) I turn the cold water all the way on (i.e. Figure 1), (B) I turn the cold water off, (C) I turn the hot water all the way on, and (D) I turn the hot water off. In each case the goals are to support students in engaging in reasoning about directions of change (Carlson et al., 2002) with either quantity increasing or decreasing as they make sense of the situation; by having students consider each quantity increasing or decreasing, I intend to foreshadow their anticipating graphs as being producible in different directions (i.e. support their emergent reasoning).

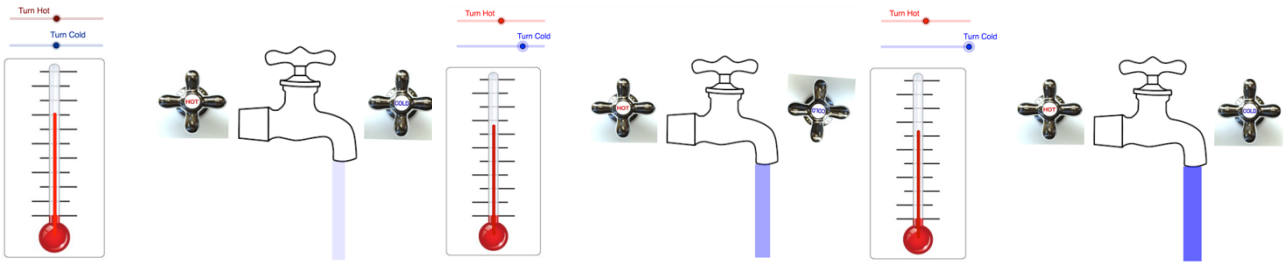

Figure 1: Several screenshots for Scenario A of Part II of the *Faucet Task* with the cold-water knob being turned all the way on.

Generally, students have little difficulty describing how amount of water leaving the faucet or water temperature changes as the knobs are turned. For instance, addressing Prompt A, Amber had no difficulty as she described "there's going to be more water coming out… [*I ask what is going to happen to temperature*] it's going to become colder". Addressing Prompt C, Amber wrote the temperature and amount of water would both go "Up" and when asked why, she said "since you're turning [*the hot water*] to the right, you're basically turning it on. And if you turn it slowly more, um more water comes out of it. And if, once you turn it fully on, all the waters that coming out, it would all become warmer." In each case, Amber accurately reasoned about directions of change of varying quantities as she anticipated turning each knob.

For Part III, I bring up an applet similar to Part II, with the only difference being the presence of a horizontal pink line segment corresponding to the width of the rectangle representing the amount of water leaving the faucet (Figure 2a). This pink segment varies in length as the amount of water varies. The goal is to support the students considering using a segment length to represent another non-spatial quantity, which is critical to their leveraging a quantitative coordinate system to represent a relationship

between non-spatial quantities. I ask students if they observe anything new (i.e. the pink segment) and then to describe what the pink segment is representing. For instance, when I asked what would happen to the segment "if the amount of water is getting bigger" Amber described "it's going to become longer" and when I asked if the amount of water is going to become less, Amber indicated the segment "is going to become smaller." Amber understood the pink segment varied with respect to the amount of water, which was critical to her eventually using the pink segment to represent the amount of water in a quantitative coordinate system.

After students describe what the horizontal pink segment represents, I move to Part IV, which no longer shows the pink segment below the water, instead showing a red segment I intend to represent water temperature on the vertical axis and a pink segment I intend to represent amount of water on the horizontal axis (Figure 2b/c). Hoping to support students in conceiving of a quantitative coordinate system, I ask them to describe what will happen to each segment for Scenarios A-D described above. The students describe quantities in ways consistent with Part II but extend this reasoning to describing how the segments vary. For example, when addressing Scenario A, Amber described, "the red line would go down…because you're adding more cold water, cold water would make it colder…and then the [pink] line…would become bigger…because you're adding more water." Amber leveraged her situational understanding to describe how segments, representing quantities extracted from the situation and imposed onto a new space, vary as the quantities changed.

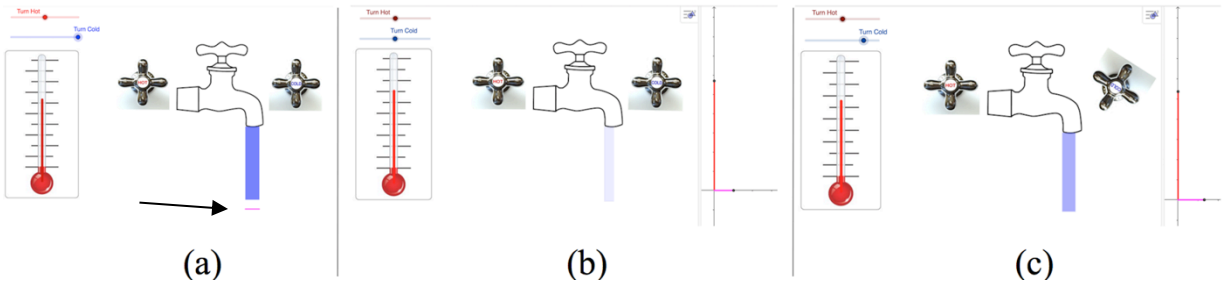

Figure 2: (a) The applet from Part III with arrow to highlight added pink segment, and (b)-(c) two screenshots from Part IV.

Part V is designed to support students in constructing a point as a multiplicative object simultaneously representing how water temperature and amount of water covary. In Part V, the applet includes a point in the coordinate system with a horizontal magnitude corresponding to the endpoint of the pink segment and a vertical magnitude corresponding to the endpoint of the red segment (Figure 3a). To support the students in conceiving the point as corresponding to both segments' magnitudes (i.e. as a multiplicative object), I ask them to turn the knobs and observe the movement of the point. For example, Amber described the point's movement by saying, "It's like if [the red segment] goes up [the point] would go up and if the width [*pointing to the pink segment*] is going…left, to the left, [the point]'d be moving with both of these [*motioning to the pink and red segments*], trying to stay equal with them." After describing the point as moving according to the endpoints of the two segments, Amber

used this understanding to describe how the point moved for Scenarios A-D. For example, addressing Scenario D, Amber described temperature and amount of water decreased, which corresponded to each segment getting shorter, then described, "since they're [*motioning to the segments on the axes*] both moving, it's going to go diagonally [*motioning diagonally down and to the left of the point on the screen*]." After conceiving of the point as a multiplicative object, Amber easily described the motion of the point in the quantitative coordinate system so that the point represented the covarying quantities magnitudes.

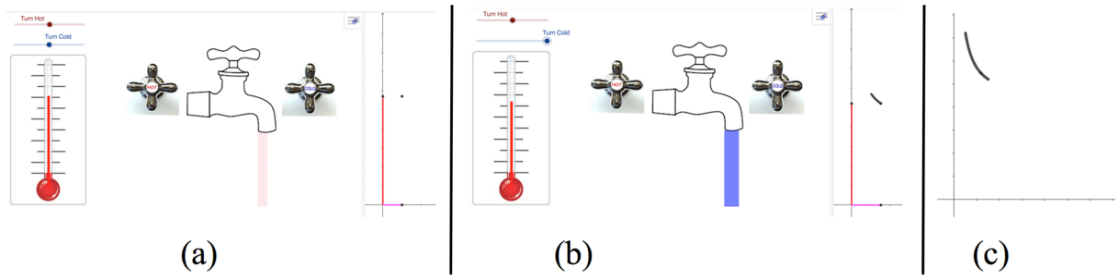

Figure 3: (a) The applet for Part V, (b) the applet for Part VI with one emergent trace resulting from turning the cold on, and (c) one example from Part VII.

In Part VI, I use the 'Trace' feature of GeoGebra to trace the point as we again go through Scenarios A-D (Figure 3b shows the resulting trace for Scenario A). My goal is to support students in imagining the graph as being produced by the trace left by the point as it moves (i.e. think emergently). Also, by having four scenarios that produce graphs in multiple directions, I intend to provide opportunities to consider the trace as emerging in various directions. For example addressing Scenario A, Amber first approximated where the point would end after turning the cold water on then described creating a line from the given starting point to her ending point to create a graph representing Scenario A. In this and other cases, Amber described the resulting graph as being produced by a trace of a point in motion.

In Part VII, I present students with several completed graphs (e.g., Figure 3c) and ask them to predict how the knobs began and what single action occurred to produce the graph. The goals of Part VII are to explore how students interpret a graph representing covarying quantities and to examine if students consider the possibility that different scenarios create the same final graph (i.e., reason about the graph as an emergent trace). If the students do not consider more than one possible scenario, I raise a second scenario as a hypothetical classmate's solution and ask students to comment on this solution. Addressing the first of these tasks (Figure 3c), and indicative of reasoning about a graph as an emergent trace, Amber immediately questioned, "it started down here [*pointing to the bottom right endpoint*] or up here [*pointing to the top left endpoint*]?" She then argued if the graph started at the top left endpoint, "turning cold on" would produce the given graph. Eventually, Amber argued if the graph started at the bottom right endpoint, "turning the cold water off" would produce the graph traced in the opposite direction. Hence, Amber reasoned about covarying quantities in the situation to describe two possible emergent traces producing the same final graph.

DISCUSSION AND IMPLICATIONS

I designed the Faucet Task to support students in constructing quantitative coordinate systems, understanding a point in this coordinate system as a multiplicative object simultaneously representing two quantities' magnitudes, then reasoning about graphs as emergent traces produced by multiplicative objects moving within a quantitative coordinate system. By describing the underlying principles I used when designing the Faucet Task, I intend to support other researchers and teachers in designing other tasks which can also support or extend students' constructing quantitative coordinate systems, multiplicative objects, and/or thinking emergently.

Additionally, I provide empirical examples showing that students as young as middle school are capable of constructing quantitative coordinate systems, reasoning about points as multiplicative objects, and thinking emergently. Whereas previous researchers have shown older populations (Lee, 2016; Moore & Thompson, 2015; Paoletti & Moore, 2017; Thompson et al., 2017) can engage in such constructions and reasoning, these results highlight students at least as young as 10 years old can construct similar understandings. Future researchers may be interested in examining ways to support middle grade students building other foundational school mathematics ideas after they have developed meanings consistent with those described.

I conjecture the experientially real nature (Gravemeijer & Doorman, 1999) of the Faucet Task supported students in constructing and reasoning about quantities; students have had a plethora of prior experiences dealing with faucets and water temperature. As a common phenomenon, this task has the potential to not only support students in developing understandings of quantitative coordinate systems and reasoning emergently but also other mathematical ideas including weighted averages, rational functions, and multivariable functions (i.e., f(Hot, Cold) = (Amount of Water, Water Temperature). Hence, I intend for teachers, curriculum designers, and researchers to leverage this common context to support students in developing their understandings of several (related) mathematical ideas.

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DESIGNING TASKS AND 3D PHYSICAL MANIPULATIVES TO PROMOTE STUDENTS' COVARIATIONAL REASONING

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Constructing and reasoning about covarying quantities is fundamental to students' modeling of dynamic phenomena. In this report, we describe how we designed and leveraged 3D-printed manipulatives in parallel with iterative task re-design to support middle-grades students (ages 10-14) constructing and reasoning about covarying quantities. We present data from a design experiment to highlight why we adapted the original task to incorporate 3D-printed manipulatives in order to support students' constructions of mental images of quantities in ways consistent with researcher intentions. In doing so, we highlight how 3D-printing has opened up new possibilities for manipulatives in mathematics education.

INTRODUCTION

Constructing and reasoning about quantities that covary is central to students' abilities to make sense of situations and model dynamic phenomena (e.g., Thompson 2008). Attending to students' images of quantities they are modeling is particularly important as researchers have shown that students can create mathematical representations (e.g., equation rules, graphs) of phenomena that are *consistent with the students' images* of the phenomena but *inconsistent* with ways researchers conceive of the quantities (Moore & Carlson, 2012; Paoletti, 2015).

The purpose of this paper is to describe how we adapted a task to support middle grades students (ages 10-14) constructing and reasoning about covarying quantities via new access to 3D design and printing technologies. Specifically, we describe results from a design experiment (Cobb et al., 2003) consisting of five small-group teaching experiments (Steffe & Thompson, 2000) addressing the *Cone Task*. The task initially presented students with a dynamic 3-dimensional (3D) situation represented on a two-dimensional (2D) tablet screen. With the goal of supporting students' covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), we asked them to describe how the height and surface area of a displayed cone covaried. Based on our on-going analysis, which raised concerns regarding whether students were constructing representations of the surface area of a 3D cone through their engagement with its 2D analog, we adapted the task by designing and 3D-printing new manipulatives to support students' construction of images of quantities and to examine how they reasoned about those quantities. We describe several iterations as we adapted the task to support students' covariational reasoning.

QUANTITATIVE AND COVARIATIONAL REASONING

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp 193-200). Pretoria, South Africa: PME.

Steffe, Thompson, and colleagues' (Steffe, 1991; Smith III & Thompson, 2008; Thompson 2008) stance on quantitative reasoning underscores that teachers and researchers can not assume students maintain understanding of quantities in ways compatible with their intentions as quantities are constructed by an individual in order to make sense of their experiential world (Glasersfeld, 1995). For example, Steffe (1991) described:

> Properties of concepts are introduced by the knowing subject's actions in the construction of the concepts. For example, the sweetness of sugar is introduced by the tasting individual just as the color of roses is introduced by the viewing individual. The idea that an activity introduced by the child in constructing a concept might be later abstracted as a property of the concept that could be called "quantity" is an important one. It changes the emphasis from viewing the sources of quantity as existing in what we take to be external to the child to taking quantity as emerging from the child's interactions with elements in his or her environment. (p. 62)

As conceptual entities, understandings of a quantity can and may differ from individual to individual. Therefore, it is critical to attend to *students'* conceptions of quantities, because while the mathematical representations of conceived phenomena they construct are consistent with their own conceptions, they can be inconsistent with quantities that the researchers intended for them to construct (Moore & Carlson, 2012; Paoletti, 2015).

In addition to attending to students' quantitative reasoning generally, we particularly attend to students' covariational reasoning. Carlson et al. (2002) described covariational reasoning as entailing a student coordinating two varying quantities with attention to the ways the quantities change in tandem. They specified five mental actions that allow for a fine-grained analysis of mental actions students engage in when coordinating covarying quantities' magnitudes or values. The mental actions include coordinating *direction of change* (surface area increases as height increases; MA2), *amounts of change* (the *change* in surface area *increases* as height *increases in equal successive amounts*; MA3), and *rates of change* (surface area *increases at an increasing rate* with respect to height; MA4-5). Whereas researchers (Johnson, 2012; Moore, 2016; Paoletti & Moore, 2017) have described productive ways high school and college students engaged in reasoning compatible with Carlson et al.'s mental actions, we are aware of no studies examining middle-grades students' enactment of these mental actions. In this study, we pay particular attention to MA3 as critical to students representing relationships between covarying quantities.

DESIGNING FOR MATHEMATICAL ABSTRACTION TO SUPPORT COVARIATIONAL REASONING

We report on students' activities addressing the *Cone Task*, which we iteratively re-designed to support them in engaging in MA1-3 as described by Carlson et al. (2002). Specifically, we adapted the *Cone Task*, which Stevens et al. (2015) used in clinical interviews to elicit college students' covariational reasoning, as an instructional task. By prompting middle grades students to describe the relationship between surface area

and height of the growing cone, we intended to support them in differentiating between the amounts of change in one quantity increasing, decreasing, or remaining constant for equal changes in the second quantity. The initial task involved a GeoGebra applet, which we intended to represent an infinitesimally thin, growing, cone in 3D (Figure 1). The applet allowed students to use the short slider to vary the cone's height (apparently) continuously or by discrete jumps in integer increments. Additionally, we had a handout representing the cone at integer heights from zero to five. By having specific equal changes in height in both the applet and handout, we intended to support students in considering how the amounts of change of surface area varied as the surface area increased for equal changes in height.

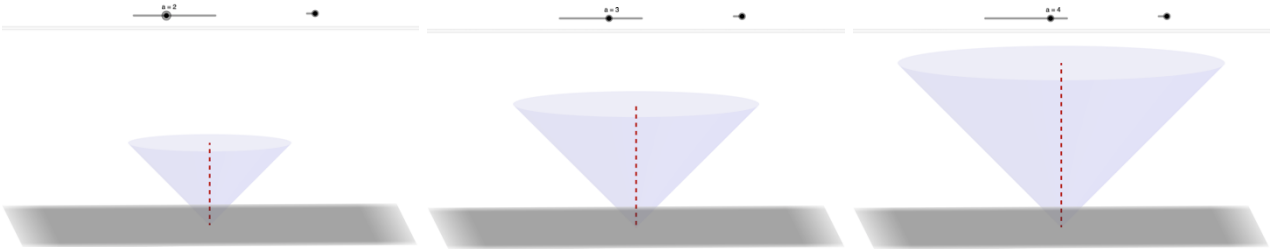

Figure 1: Several screenshots of the *Cone Task*.

After the first iteration of the task (described below), we leveraged 3D-printing to design physical manipulatives students could use to support their construction of quantities in ways consistent with our research focus. As Greenstein (2018) noted, "The faithful mental representation of objects is critical, because conceptual thought proceeds from representational thought and representational thought proceeds from perception" (p. 3). Hence, through a process of designing for mathematical abstraction (Pratt & Noss, 2010), we designed the physical manipulatives with the intention of making both surface area and amounts of change of surface area for equal changes of height available to students for abstraction through their sensorimotor engagement with them (Kamii & Housman, 2000; Piaget, 1970). Specifically, to support the students in explicitly constructing surface area as a two-dimensional quantity on a three-dimensional object, we designed and 3D-printed a set of five purple cones, each with an integer increase in height (see Figure 2). Further, we designed and 3D printed red truncated cones representing those changes in height (e.g., the part of the cone that would be added to the 3rd cone to get to the 4th cone). We hoped these red manipulatives would help students construct mental images to explicitly consider amounts of change of surface area as a quantity.

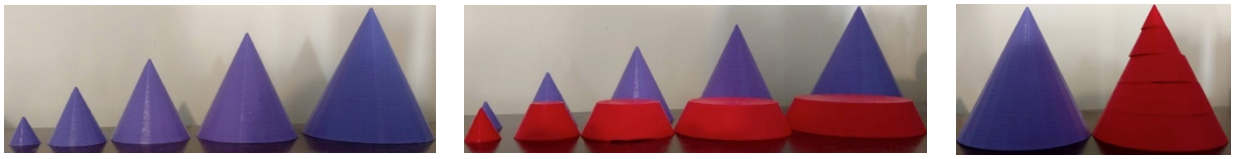

Figure 2: Photos of the 3D printed cone manipulatives

METHODS, PARTICIPANTS, AND ANALYSIS

In order to examine the potential of supporting students in constructing and reasoning about covarying quantities in ways consistent with our theoretical perspective, we conducted a design study (Cobb, et al., 2003) consisting of five small-group (one to three students) teaching experiments (Steffe & Thompson, 2000). The teaching experiments occurred in a middle school, which is considered an underperforming school (as measured by standardized testing) that hosts a diverse student population (over 75% of the population are students of color, and 75% of them are entitled to free or reduced-price lunch), in the northeastern United States. Students were recruited on a volunteer basis through teacher recommendations. Specifically, we asked teachers to recommend students who they believed could explain their thinking and reasoning, and we worked with all students who returned signed consent forms.

Adopting a radical constructivist perspective (Glasersfeld, 1995), we contend that a student's mathematics is inaccessible to us as researchers. Hence, to analyze the data (e.g., video recordings, students' written work), we performed conceptual analyses (Thompson, 2008) to generate and test models of the students' reasoning so that these models provided viable explanations of their observable words and actions. With the goal of building viable models of the students' mathematics in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Specifically, we watched all videos to identify instances that could offer insights into each student's understanding of quantities and relationships between quantities they conceived of in the *Cone Task*. Using these instances, we generated tentative models of each student's mathematics, which we compared to researcher notes taken during on-going analysis in each of the teaching experiments. We tested these models for viability by searching for supporting or contradicting instances in their other activities. When evidence contradicted our models, we revised hypotheses to explain each student's meanings and returned to prior data with these new hypotheses in mind to modify previous hypotheses. This process resulted in viable models of each student's mathematics and supported our adapting tasks from one iteration of the teaching experiment to the next iteration.

RESULTS: ITERATIVE REDESIGN TO SUPPORT COVARIATIONAL REASONING

We reiterate that the goal of the instructional task was to support students in constructing and reasoning about quantities as well as amounts of change of one quantity with respect to equal changes in the second quantity (MA3). In the first iteration of the task, which we used with two pairs of students, the students only had the GeoGebra applet displaying a growing cone and the handout with screenshots of the cone at integer height values. There were several features of the students' conceptions of the situation that we found to be problematic. First, when addressing the task, although each of the four students described the situation as happening in 3D, they tended to describe or reason about the cone as a 2D object (e.g., referring to the cone as a "triangle"). Second, three of the four students initially indicated that the "surface area" of the cone grew by equal amounts for equal changes of height, which

we take to indicate that they were not conceiving of surface area in ways compatible with our understanding of the relationship (i.e., for equal changes in height, the change in surface area increases by greater amounts). For example, one pair of students consistently drew what we infer to be changes in the slant height of the cone when describing that the amount of change of "surface area" was constant. Such a claim is viable as the students conceived of "surface area" as analogous to slant height but this conception was inconsistent with surface area as we intended.

Because each of these pairs had difficulty conceiving of surface area in ways compatible with our intentions, the teacher-researcher directed them to shade and color-coordinate surface areas and changes in surface area on their handouts (Figure 3). Although this activity, which took between 7 and 9 minutes, helped each of the four students eventually argue the amounts of change of surface area increased for equal changes of height (MA 3), the quantity with which they were operating was still often ambiguous from the researchers' perspective. For instance, when shading in the cones on the handout, one student described "how many [of the] first triangles [*referring to the shaded region in the first image of the cone*] will fit into the bigger triangles" and used this reasoning to determine the "space left" after fitting the smaller triangles into the bigger triangles. In addition to referring to "triangles," the student described the "space" as both volume and surface area. We infer that the students often conflated several quantities as they attempted to imagine 3D phenomena from 2D representations, and despite making claims which are normatively correct regarding how surface area and height covary, we are unsure of the extent to which they were reasoning about quantities in ways consistent with our intentions.

Figure 3: Photos of one pairs' color-coordinated work showing changes in surface area in red with the surface area of the previous cone circled in black.

As we conjectured the students were having difficulty constructing surface area as a quantity in ways compatible with our intentions, we designed the 3D manipulatives as described above. We asked two groups of students (one triple, one single) to discuss how the surface area and height of the cone varied; these students had the 3D manipulatives available with the GeoGebra applet and printed paper. There were several features of these students' activity that are notable. First, although the handout was available, the students rarely referred to it. Instead, they almost exclusively worked with the 3D-printed manipulatives and dynamic animation. Second, each of the students identified surface area in ways compatible with identifying a 2D quantity in a 3D situation; the students referred to the outer area of the cone as something we could measure. Despite this, when determining how the surface area changed for changes in the height of the cone, the specific quantity(ies) the students were reasoning about was

still often either ambiguous from our perspective or clearly *not* surface area in ways compatible with our intention.

As an example of a student who was reasoning with a concept of surface area incompatible with our intentions, Greta (the student working alone) consistently compared the relative size of the top faces of the red truncated cones to describe that the amounts of change of surface area increased for equal changes in height. Although Greta came to a normatively correct description of the relationship between outer surface area and height, she was reasoning about changes in the surface area of the top face of the cone, which does not appear in the applet.

Whereas Greta consistently described a surface area other than outer surface area as we intended, the group of three students seemed to rely on intuitive or explicit notions of volume when describing the amounts of change of surface area increasing for equal changes in height. For instance, the students consistently used the 3D-printed manipulatives to argue that the amounts of change would increase. However, when probed to describe what was getting bigger, one of the three students held two of the red manipulatives and explicitly described, "larger is like overall, like how big the mass is." Hence, as with the previous iteration, the students were able to make normatively correct claims regarding how surface area and height covaried despite their not necessarily reasoning about surface area in ways compatible with our intentions.

Reflecting across students' activity during on-going analysis, we noted that all of them readily identified volume as a quantity they could measure in the situation. We also noted that they tended to implicitly reason about changes in volume when we intended for them to describe changes in surface area. Additionally, because the manipulatives provided the students with solid objects for their sensorimotor engagement (versus an infinitesimally thin cone depicted by the GeoGebra animation), we adapted the task by asking the students to describe how *volume* varied for equal changes in height. This change in the quantity under consideration did not compromise the goals of our study or of the task (i.e., to support students' constructions of quantities and their reasoning about amounts of change of one quantity with respect to equal changes of a second quantity). Two groups of students (one pair and one single) engaged with this iteration of the task. We note that students reasoned with quantities that were compatible with our intention of volume (i.e., weight or amount of plastic inside the manipulative). As an example, in describing volume as weight, Amber determined that if the red manipulatives have "different sizes, their weights have to be different," and so the "weight is increasing" for consecutive changes in volume represented by the red manipulatives. Amber used this reasoning to claim that as the height of the cone increases by equal amounts, the volume of the cone increases by greater amounts (MA3). In doing so, Amber's reasoning is consistent with reasoning about what we would refer to as volume, amounts of change of volume, and height of the cone.

DISCUSSION AND IMPLICATIONS

There are several implications from this study. First, we emphasize students constructing and reasoning about quantities is non-trivial and re-emphasize researchers' calls to take students' construction of quantity seriously (Moore & Carlson, 2012; Thompson, 2008; Steffe, 1991). We highlight the extent to which students made viable claims regarding the situation *we intended* for them to be reasoning about (i.e. surface area) despite their reasoning about *a different* quantity altogether (e.g., volume). If we had only assessed students' understandings of surface area based on their conclusions (e.g., surface area increases by more for equal changes of height), we would not have been able to discern this disconnect. Hence, when analyzing students' reasoning, researchers need to pay careful attention to the quantities students are considering to ensure that they are, in fact, reasoning about the intended quantity rather than some surrogate quantity (e.g., Stevens et al., 2017).

Second, we highlight the extent to which 3D design and printing provided us with novel opportunities to support students' quantitative and covariational reasoning. By 3D-printing manipulatives with specific theoretical framings in mind (i.e., Carlson's framework and designing for abstraction), we were able to support students in imagining both the volume and height, and *changes* in volume and *changes* in height, in order to coordinate how these quantities covary. Put another way, the final task was productive in supporting students in imagining and coordinating quantities in ways consistent with mental actions described in Carlson et al.'s (2002) framework, which we conjecture is an important first step to their developing more sophisticated school mathematics ideas (e.g., rate of change). Hence, we intend for this research to be a starting point for future researchers interested in (a) exploring middle-grades students' development of covariational reasoning, and (b) examining ways emerging technologies, including 3D-printing, can support students in developing important mathematical ideas.

Acknowledgments

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IDENTIFYING EMERGING ALGEBRAIC REASONING

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Multiplicative thinking and generalisation are recognised as key to developing algebraic reasoning. In this exploratory study, conducted with 10-12 years old students from Australia and China, fraction tasks which require students to find the "whole", rather than the part, are used to identify students' progress towards algebraic thinking. Frameworks to classify their solution strategies, and degree of algebraic reasoning, allow teachers to chart student progress. While the Australian students demonstrated a greater range of methods the Chinese students favoured multiplicative methods with given numbers, but this did not always translate to generalisation. Students who generalised proficiently showed clear evidence of algebraic reasoning.

INTRODUCTION

Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra. The links between fractional knowledge and readiness for algebra have been highlighted by many researchers such as Wu (2001); Jacobs, Franke, Carpenter, Levi, and Battey (2007); Empson, Levi, and Carpenter (2011) and Siegler and colleagues (2012).

Our current research aims to identify the emergence of algebraic reasoning as middle-years students solve fraction tasks where algebraic reasoning has been defined by Kaput and Blanton (2005) as a process:

> … in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways (p. 99).

In this exploratory study, which is part of a larger study (Pearn & Stephens, 2017; Pearn, Pierce & Stephens, 2017), Australian and Chinese students from Years 5 and 6 completed paper and pencil tests that were designed to ascertain students' understanding of fraction concepts, their competence and fluency with fraction operations. The students were also examined for evidence of algebraic reasoning.

In this paper we describe the development of two frameworks designed to classify students' responses to specific fraction tasks. These frameworks were then used to analyse the responses from Years 5 and 6 students (10 – 12 years of age and in their last two years of primary school). Seventeen Australian students from one school and 12 Chinese students from three primary schools were involved. We compare Australian and Chinese students' written responses to three reverse fraction tasks (Figure 1) using the Classification Framework (Table 1) and their subsequent responses to a Structured Interview, using the Emerging Algebraic Reasoning Framework (Table 4). All student

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responses were examined for evidence of students' algebraic reasoning in an endeavour to answer the following research questions:

- Are similar strategies used by Australian and Chinese students in their written solutions to the three reverse fraction tasks?
- Does the Structured Interview provide clear evidence of students' ability to generalise their solutions based on variations of a set of particular instances?

LITERATURE AND FRAMEWORKS

Researchers such as Kieren (1980) and Lamon (1999) believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). Empson, Levi and Carpenter (2011) suggest that some strategies students use to solve fraction problems "are motivated by the same mathematical relationships that are essential to understanding high-school algebra" (p. 410).

Three reverse fraction tasks

Key to this study has been the development of three reverse fraction tasks that require students to find the number representing the whole using less common fractions of two-thirds, four-sevenths and seven-sixths (see Figure 1) and the number of objects designated to represent the given fraction (Pearn, Pierce & Stephens, 2017).

Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.	Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ *Show all your working.*	This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.

Figure 1: The three reverse fraction tasks

In these reverse fraction tasks students are given the fraction, and the number of objects representing the fraction, and then asked to find the number of objects representing the whole group. (In the Chinese version of these three questions, the term "circles" replaced "counters"). Here, the strategies enabling students to generalise from a set of particular examples require an understanding of equivalence, as well as an ability to transform quantities using equivalence.

Using a thematic analysis approach (Braun & Clarke, 2006) students' responses for the three reverse fraction tasks were classified according to the specific methods used in their written responses. Five categories were established and classified as illustrated in Table 1 which includes an explanation, with example, for successful responses for Reverse Fraction Task 2 (see Figure 1). A sixth category, *Incomplete,* was used for

responses that were left blank or were incorrect. Table 1 shows that students' written responses varied from the concrete (diagram dependent), strictly computational (additive, partly multiplicative) to the generalisable (multiplicative) and algebraic (advanced multiplicative).

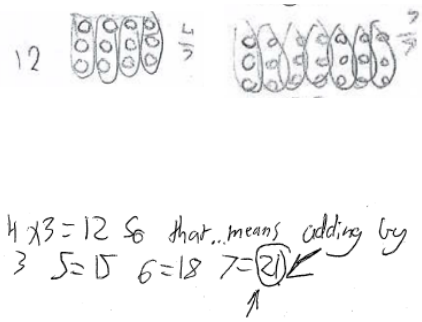
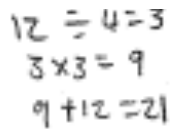
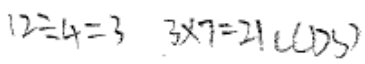
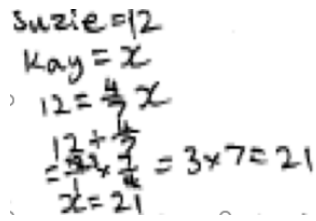
Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using additive or subtractive strategies.	
Additive /subtractive	Students use additive or subtractive methods without explicit partitioning of a diagram e.g. find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects in the whole.	
Partially multiplicative	Students use both multiplicative and additive methods e.g. they calculate the missing fractional part ($^3/_7$) and then add it onto the original quantity.	
Fully multiplicative	Students use fully multiplicative methods. They use division to find the quantity represented by the unit fraction and then multiply this quantity to calculate the whole.	
Advanced multiplicative	Students use more advanced multiplicative methods which include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction.	

Table 1: Classification Framework for strategies used for Reverse Fraction Task 2

The Structured Interview

The Structured Interview was designed to provide stronger evidence of generalised thinking (Pearn & Stephens, 2017). Task-based interviews have been used extensively by researchers to determine students' existing and developing mathematical knowledge and problem-solving behaviours (Maher & Sigley, 2014). The research of Marton, Runesson and Tsui (2004) demonstrated that a generalisable pattern can be fostered when numbers are varied. It was hypothesised that generalised thinking would be revealed if students were consistently able to use the same methods in similar

reverse fraction questions where the quantities were changed but the fractions remained the same. In the Structured Interview the fractions are the same as those used in the three reverse fraction tasks, but the quantities representing these fractions were changed.

While the same fractions were used for the Structured Interview no diagrams were included. For each fraction there are two variations. The initial variation is a change in the given number of counters (*Change of number 1)*. In the left-hand column of Figure 2 is Reverse Fraction Task 1 while the middle column shows Task 1 from the Structured Interview with the same fraction but a different number of counters. The second variation is another change in the given number of counters (*Change of number 2*) followed by another change of counters to 'any number' *(Unknown number).* An example of the second variation is Task 4 from the Structured Interview shown in the right-hand column of Figure 2. Questions involving *any number* were designed to make additive and subtractive strategies less attractive and less easy to use and encourage students to focus on the fractional structure.

Reverse Fraction Task 1	First variation Change of number 1	Second variation Change of number 2 unknown number
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain how you decided that your answer is correct.	Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain your thinking.	a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?

Figure 2: Example of variations for interview questions

If students successfully completed the second variation relating to 'any number' of counters for all three fractions, a third variation (Question 7, Figure 3) introducing 'any fraction' and 'any number' was designed to allow students to generalise their solution method, for example, by dividing the unknown quantity by the unknown numerator and then multiplying by its denominator, or by dividing by the fraction. This question was intended to provide an opportunity for students to use algebraic notation to represent the unknown quantity and its unknown fraction.

Think about the tasks you have just done.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

Can you tell me what you would do?

Please write your explanation in your own words.

Figure 3: Question 7 from the Structured Interview

While responses to each task from the Structured Interview could be classified individually according to the Classification Framework (Table 1), the overall results for each interview also needed to be classified in terms of the development of algebraic reasoning. Students' responses to the Structured Interview were analysed using a thematic analysis approach (Braun & Clarke, 2006). Students' responses varied from

computational to fully generalised which indicates algebraic reasoning. The Emerging Algebraic Reasoning Framework was developed and is shown in Table 2.

Level	Description
1. Computational fluency Partial	Solved only some questions with method restricted to given fractions and quantities.
2. Computational fluency Complete	Solved all questions with given fractions and quantities but unable to answer more than one question with 'any quantity'.
3. Generalising - Additive	Solved all questions with given fractions and quantities. Used additive or mixed methods to solve questions with 'any quantity'. No appropriate generalized multiplicative response for 'any fraction' and 'any quantity'.
4. Generalising-Multiplicative	Solved all questions with given fraction and 'any quantity' using multiplicative methods. No appropriate generalised response to 'any fraction' and 'any quantity'.
5. Algebraic generalisation - Verbal	Solved all questions with known fractions and 'any quantity' using consistent multiplicative methods. Students verbalised but did not symbolise full generalisation to 'any fraction' and 'any quantity'.
6. Algebraic generalisation - Symbolic	Solved all questions with known fractions and 'any quantity' and generalised using consistent multiplicative methods. Appropriate algebraic notation used to solve 'any fraction' and 'any quantity' task.

Table 2: The Emerging Algebraic Reasoning Framework

THIS STUDY

Table 3 shows the number of students from Years 5 and 6 who completed the three reverse fraction tasks and were interviewed. The Structured Interview was given to a selection of Australian and Chinese students who successfully solved and explained their solutions to at least two of the three reverse fraction tasks. Year 6 Chinese students are taught division by fractions, while this is not taught in Australia until Year 8.

	Australian students	Chinese students
Year 5	9	6
Year 6	8	6
Total	17	12

Table 3: Number of Australian and Chinese students in each year level.

RESULTS

The three reverse fraction tasks were in a format that would have been unfamiliar to both Australian and Chinese students. The responses for both Australian and Chinese students from Years 5 and 6 were analysed to determine the types of strategies students

used to solve the three reverse fraction tasks. These responses were then classified according to the classifications given in Table 1 and are presented in Table 4. There were 17 Australian students (AU) and 12 Chinese students (CH).

Response Type	Reverse Fraction Task 1		Reverse Fraction Task 2		Reverse Fraction Task 3	
	AU	CH	AU	CH	AU	CH
Incomplete	-	-	-	1	-	1
Diagram dependent	-	-	1	-	1	-
Additive/subtractive	-	-	1	-	-	-
Partially multiplicative	10	-	5	-	9	-
Fully multiplicative	6	5	9	4	6	4
Advanced Multiplicative	1	7	1	7	1	7

Table 4: Classification of students' responses to the three reverse fraction tasks

Successful Chinese students used either fully or advanced multiplicative strategies whereas Australian students tended to use a range of strategies including partially multiplicative strategies. Only one Australian student from this sample used advanced multiplicative strategies compared to seven Chinese students. Figure 4 shows a response deemed to be *Advanced Multiplicative* from an Australian Year 6 student in the left-hand column and a Chinese Year 6 student in the right-hand column.

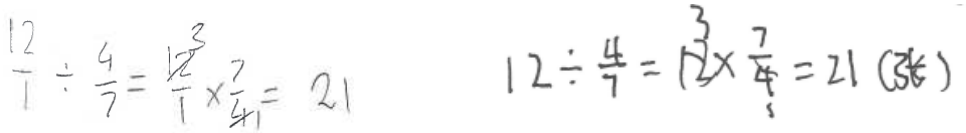


Figure 4: Advanced Multiplicative responses from Year 6 students

Even though some Chinese students used more sophisticated multiplicative strategies to solve the three reverse fraction tasks (Table 3) there were fewer differences between the two groups when the Structured Interview was analysed. Structured Interview responses (Table 5) demonstrated that students' strategies varied from strictly arithmetical (computational fluency), beginning to generalise (additive, multiplicative) to the fully generalised demonstration of algebraic reasoning (verbal, symbolic).

Level	Australian		Chinese	
	Year 5 (n = 9)	Year 6 (n = 8)	Year 5 (n = 6)	Year 6 (n = 6)
1. Computational fluency - Partial	-	-	1	2
2. Computational fluency - Complete	1	-	1	-
3. Generalising - Additive	1	3	2	1
4. Generalising - Multiplicative	1	2	-	1
5. Algebraic generalisation - Verbal	5	1	1	-
6. Algebraic generalisation - Symbolic	1	2	1	2

Table 5: Results for the Developing Algebraic Reasoning Framework

Table 5 shows that almost all students were able to calculate the number of objects in the whole group when given a specific fraction and the number of objects representing that fractional part. The first difficulty occurred when students were asked to think about the situation where the number of objects was not stated but there were 'any number of counters'. While some students gave no response when asked about 'any number', other students explained: "I can't do this because I don't know the number of circles." (Chinese student) or similarly: "I don't know how many circles so I can't solve it". There was no evidence of generalisation in this group.

Some students from both countries who demonstrated that they were *beginning* to generalise using strategies for the tasks with 'any number of counters' were unable to complete the third variation with 'any fraction' and 'any number'. The third variation identified students who used a range of explanations, verbal and symbolic, to fully generalise their solutions. For example, one Chinese student introduced a new fraction, five-ninths, in order to explain that: "We need to find out how many are in one piece, then multiply x (the quantity represented) by the denominator to get the total amount".

CONCLUSION

There were differences between Australian and Chinese students written solutions to the three reverse fraction tasks. Australian students used a greater variety of methods compared to Chinese students who tended to use multiplicative methods. However, the Structured Interview identified Australian and Chinese students who were computationally proficient but unable to generalise, as distinct from those who were beginning to generalise, and those who could fully generalise their solutions. The Structured Interview identified Australian and Chinese students in each of these three groups. Computational fluency, including the ability to divide by a fraction, did not guarantee generalisation. The Structured Interview with its three variations allowed researchers to identify students whose proficient generalisations showed clear evidence of algebraic reasoning.

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MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE ON DIVISION: A FOCUS ON KNOWLEDGE OF TOPICS AND STRUCTURES OF MATHEMATICS

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With a focus on teachers' Specialized Knowledge of division, we discuss some aspects related to the dimensions of teacher's knowledge, particularly Knowledge of Topic and Knowledge of Structures of Mathematics, both considered from the Mathematics Teachers' Specialized Knowledge conceptualization. The analysis of the teachers' interactions during a professional development program aimed at accessing and developing their specialized knowledge made it possible to elicit a set of indicators of teachers' knowledge. These indicators are perceived to be an effective tool for use in teacher education programs intended to develop teachers' knowledge and improve mathematical practices.

INTRODUCTION

Teachers' knowledge has been a focus of attention in teacher education from various viewpoints (e.g., Ball, Thames, & Phelps, 2008; Carrillo et al., 2018; Rowland, Huckstep, & Thwaites, 2005). Moreover, the extent of the impact of teachers' knowledge on students' learning and results is well known (e.g., Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2009), as this is one of the primary interests among international scholars seeking to understand which aspects related to the teacher's knowledge most influence their practice (e.g., Dooren, Verschaffel, & Onghena, 2002). Assuming the specialized nature of teachers' knowledge (in the sense of Carrillo et al., 2018) is perceived a premise for improving educational contexts and which requires still characterize the content of such specialization. Such characterization is particularly important in what are considered mathematical critical situations/topics, in terms of both learning and teaching. Division is such a case (e.g., Rizvi, 2007).

Among the problematic factors in the teaching and learning of division, is a focus on the algorithm without considering aspects such as the necessary discussion about the meanings assumed for the concept of division (Fischbein, Deri, Nello, & Marino, 1985); the crucial role of the different types of representations, and the relationships amongst them (e.g., Ainsworth, Bibby, & Wood, 2002); the connections between division and other topics (Young-Loveridge & Bicknell, 2018); and the principles associated with the counting and collection-based approaches to working with numbers – namely unitizing (Behr, Harel, Post, & Lesh, 1994) – or the intrinsic relationship with the foundations of measurement (Clements & Stephan, 2004).

Thus, aiming at exploring the content of teachers' knowledge of division that is associated with what and how they know (topics) and how to teach it/what they need

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to know (structure), here we address the following research question: *What knowledge of topics and structures of mathematics related to division do practising teachers reveal when discussing a task conceptualised with the aim of developing such knowledge?*

THEORETICAL FRAMEWORK

The topic of division is considered one of the most difficult, in terms of both learning and teaching (Rizvi, 2007). It is problematic for students as many foundational concepts are not employed to support the process of teaching, which has implications for students' understandings. For instance, although the concept of unit is crucial to understanding the role of each element involved in a division operation (Behr et al., 1994), it is not a focus of attention in teachers' practices (e.g., Zakaryan & Ribeiro, 2018), in particular when the division is taken in the *quotative* sense.

Division can be understood in two senses: *partitive* (*sharing*) and *quotative* (*measurement*). The *partitive* meaning is evoked in contexts in which a group of elements (dividend) must be divided (shared) into a number of (equal) sets (divisor) – this is the "typical" way division is perceived (One has an amount of "something" and one is going to share it amongst a certain quantity of people). The *quotative* meaning of division (*measurement* division), in contrast, is evoked in contexts in which a comparison between two quantities (dividend and divisor) is required – that is, when one needs to determine how many times the unit of measurement (divisor) is needed/contained to fulfil what needs to be measured (dividend). When aiming to clarify the *quotative* meaning of division, in addition to knowledge of the concept of unit of measurement, teachers also need to know the principles of the measurement activity (Clements & Stephan, 2004). These principles comprise: i) choosing the unit of measurement; ii) partitioning of the whole to be measured; iii) iteration of the unit; iv) accumulation of quantity; v) relation to number. In relation to the *quotative* meaning of division, core aspects concern (i) developing a perception of the divisor as the unit of measurement; (ii) the notion of decomposition of the dividend into equivalent parts; (iii and iv) the notion of the composition of a number that relies on the hierarchical inclusion and cardinality constructs; and (v) the perception of the quotient as a quantity obtained by a process of measurement.

Furthermore, in the process of learning division, two other aspects play crucial roles. One concerns the use of word-problems to contextualize the meaning (*partitive* or *quotative*) of division (Downton, 2009). The other relates to the different types of representation that help to give meaning to each of the associated mathematical constructs (Lesser & Tchoshanov, 2005). When focusing on developing students' knowledge of connections – one of the five school mathematics principles (NCTM, 2000) – establishing relationships between diverse forms of representations is another core aspect related to the process of learning division (Ainsworth et al., 2002).

In (and for) developing such knowledge of division, it is crucial to focus on teachers' knowledge of such topics, as this has significant effects on students' learnings (Boyd

et al., 2009). Amongst the diverse conceptualizations of teachers' knowledge, we assume the Mathematics Teacher's Specialized Knowledge (MTSK) conceptualization (Carrillo et al., 2018). Of the two domains considered in the MTSK conceptualization (Mathematical Knowledge – MK, and Pedagogical Knowledge – PCK), due to the scope of our work, we focus only on two of the subdomains of MK – Knowledge of Topics (KoT) and Knowledge of Structures of Mathematics (KSM).

The KoT includes teachers' knowledge of definitions (what is a division in a mathematical sense?); properties and foundations (e.g., the role of each element involved in a division); phenomenology and applications (e.g., *partitive* and *quotative* meanings of division and the contexts to evoke each meaning); procedures (e.g., knowing how, when and why to employ each procedure when solving a division); and systems of representation (e.g., the most adequate type of representation to be associated with each division meaning). As for the KSM, it concerns teachers' knowledge of connections between topics, considering four categories*: connections based on simplification* and *connections based on increased complexity*, both related to "temporal connections underlining the generative role of mathematical items in the construction of other items" (Carrillo et al., 2018, p. 8); *auxiliary connections* (e.g., the concept of unit to give sense to the role of divisor as an unit of measurement when considering quotative division); and *transverse connections* (e.g., the principles associated with measurement process to sustain the notion of *quotative* division).

CONTEXT AND METHOD

Being part of a broader research project, here we focus on a group of nine practising teachers (two from kindergarten, six from primary and one from secondary school levels) who participated in a professional development program (PD) in Brazil. The PD is a 2-year program focusing on developing teachers' Specialized Knowledge. It is comprised of seven courses (five meetings of 8 hours each), focusing on Numbers and Operations (two courses – integers and fractions), Algebraic Reasoning, Statistical Reasoning, Geometry (two courses), and Measurement. Here we focus on data gathered from the discussions held during one session of the course on Numbers and Operations, and particularly the session addressing teachers' knowledge on division.

Data was gathered through audio and video recordings of teachers' interactions (both when working in groups of three and the all-group discussions) and teachers' productions when solving the tasks aimed at developing their specialized knowledge. For the transcription we use pseudonyms and "Educator." Here we focus on three particular questions: (1) *What is division for you? What would you answer if a sixth grade student posed this question to you?*; (2) *Solve the expression 6÷3 describing the procedures employed*; (3) *Formulate two distinct problems that could be solved by the expression 6÷3=2, and then provide a solution for each of them, presenting two distinct types of representation that clarify the meanings of such solutions*.

Using the MTSK conceptualization as a theoretical and analytical lens, data analysis allowed a set of indicators of KoT and KSM to emerge. These indicators were named

following Zakaryan and Ribeiro (2018), so that, for example, KoTph1 corresponds to an indicator associated with the *phenomenology and applications* category, included in the KoT subdomain (see Table 1). The other indicators correspond to: KoTph2 and KoTph3, both related to the *phenomenology and applications* category; KoTrp1, related to the *systems of representation* category; KSMau1 and KSMtr1, related to the *auxiliary* and *transverse* connections categories, respectively. We emphasize that the given number does not represent any hierarchization, but only an identification.

RESULTS AND DISCUSSION

All groups revealed a knowledge of the *partitive* meaning of division (KoTph1), as is evident from the following excerpt from the plenary discussion about Question (1): "*What is division for you? What would you answer if a sixth grade student posed this question to you?"*:

Educator:	That's fine… And what would you say to sixth grade students?
Mila:	We would say that dividing is to share equally.
Will:	It is when you get the whole and you share it in a certain number of parts. We did not write down equivalent parts… But it must be equivalent parts.
Educator:	What else?
Norma:	We also answered that: sharing in a certain number of equal parts, since the algorithm is already known by the students.

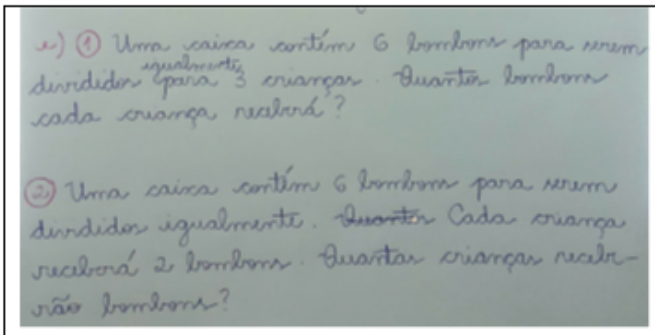
All the teachers revealed knowing the relationship between the *partitive* meaning of division and the notion of sharing. However, when asked to formulate two distinct problems that could be solved by the expression 6÷3=2, only one group was able to do so. It is worthwhile noting that a similar question had been posed in the tasks associated with addition, subtraction and multiplication, which also highlights what could be termed the non-transposability of teachers' knowledge – what is valid to a situation is not directly transformed to another, even when in a similar practice context – here involving operations. Below, an example of the small-group discussion, extracted from Group 2:

1	Mila:	It is that partitive one?
2	Mara:	It is (all) related to the expression six divided by three equals
3		two [writes down 6÷3=2], but probably it has (different) concepts here…
4		For example, as the addition, that we have the *add* and the
5		*putting together* meanings. Probably, here there are also two
6		meanings. Let's imagine that one is about sharing. And the other
7		one, is about what?
8	Marsha:	Wait! If it is not about sharing, what is it about?

It is important to highlight that the teachers reveal a knowledge associated with the role of word-problems (KoTph3) as a support to evoke distinct meanings of the operations (Downton, 2009). Although they assume the possibility of the existence of a distinct

meaning of division from the one they know (partitive), they were not able to provide a word-problem evoking such a meaning. This brings to front the need to focus on developing their knowledge of the quotative meaning (KoTph2) and a situation associated with it (KoTph3).

Only Group 2 was able to provide two word-problems that evoked the two meanings of division (KoTph3).



e) (1) *Inside a box there are 6 sweets to be divided equally among 3 children. How many sweets each child will get?*

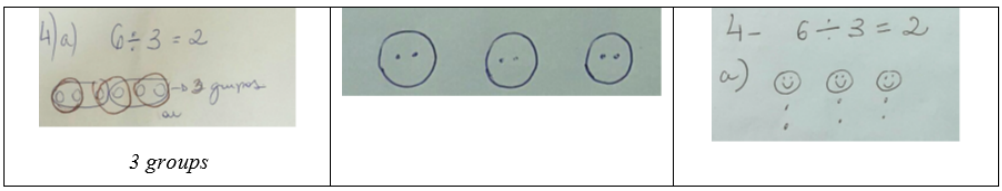
(2) *Inside a box there are 6 sweets to be divided equally. Each child will receive 2 sweets. How many children will receive sweets?*

Figure 1: Word-problems provided by Group 2 for the two meanings of division.

It is important to highlight that, although the group provided word-problems evoking the two meanings of division, one of the posed problems (2) was not associated with the expression 6÷3=2, but instead to 6÷2=3. Naturally, this misunderstanding could be avoided if the concept of unit of measurement (Behr et al., 1994) was mobilized to give sense to the role of divisor (KSMau1).

After the Educator intervention – aiming at elicit the meanings of division with a correspondence between the given expression and the posed problem (Ribeiro & Amaral, 2015) – the group recognized that the two word-problems provided were associated with two distinct reasonings. Nevertheless, they did not establish an explicit connection with the notion of measurement (KSMtr1) associated with the second problem provided (Clements & Stephan, 2004).

In relation to the most adequate type of representation to give meaning to a certain concept (the last part of the third question), the pictorial representations of the expression 6÷3=2 provided by all the groups associated with the notion of sharing:



3 groups

Figure 2: Pictorial representations provided by the groups.

However, when asked to provide a solution for each of the two distinct problems they formulated, only Group 2 provided two different pictorial representations (KoTrp1):

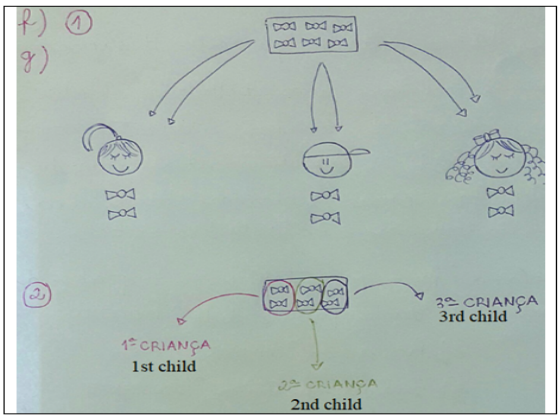


Figure 3: Pictorial representations provided by Group 2 associated to the word-problems.

As the other groups provided two problems both evoking only the partitive meaning, the focus of their representations was associated with sharing. On the one hand, this reveals their knowledge of this type of representation as being the most adequate to give meaning to partitive division (KoTrp1). On the other hand, it reveals the need to develop their knowledge, both in terms of the quotative meaning (KoTph2) and in terms of the adequacy of a type of representation to be associated with it (KoTrp1).

In order to synthetize the different aspects of the teacher's knowledge of topics and structures on the scope of division, according to the MTSK conceptualization, a set of indicators were elicited and identified in accordance with each subdomain (Table 1).

Categories of KoT and KSM	Indicators
Phenomenology and applications (KoTph)	KoTph1: Knowledge of partitive meaning of division. KoTph2: Knowledge of quotative meaning of division. KoTph3: Knowledge of types of problems/contexts created to evoke a certain meaning of division.
Systems of representations (KoTrp)	KoTrp1: Knowledge of the most adequate type of representation to give meaning to a certain concept.
Auxiliary connections (KSMau)	KSMau1: Knowledge of the concept of unit to give meaning to the role of divisor as a unit of measurement.
Transverse connections (KSMtr)	KSMtr1: Knowledge of the principles associated with the measurement activity as the foundation of the quotative division construct.

Table 1: Categories and indicators related to KoT and KSM subdomains on division

In this way, it was possible to highlight some evidence of teacher's knowledge of *phenomenology and applications*, *systems of representations* and *auxiliary* and *transverse connections*.

FINAL COMMENTS

This study focused on the characterization of teacher's knowledge related to the topics and the structures of mathematics, according to the MTSK conceptualization. The search for this characterization took as its starting point, on one hand, one of the most problematic topics on the scope of teaching mathematics, that is, division. On the other hand, we assumed the existence of indicators of teacher's specialized knowledge (Zakaryan & Ribeiro, 2018) also in the scope of division and analyzed the interactions of a group of practice teachers when discussing a task focusing on such topic. By scrutinizing the evidence of knowledge revealed by this, it was possible to elicit a set of indicators of specialized knowledge related to division.

The context of a PD program is conducive to access and develop teacher's specialized knowledge, particularly if the tasks implemented in teacher education are conceptualized with the aim of developing such specialized nature of teachers' knowledge. However, even if we had data gathered through three sources (audio and video recordings and teachers' productions to the tasks) it is worth highlighting the fact that it is desirable to investigate teachers' knowledge also in various contexts of practices, that is, for example, during class, focus group interviews or during preparation of tasks.

In that sense, this study points to at least two new directions of research. One is theoretical, since it is related to deepening the understanding of teacher's knowledge of the other MTSK subdomains – both in division as well as other topics. This focus also draws attention to the fact that, when providing new indicators of specialized knowledge, new categories associated with each subdomain may emerge, which can be perceived, therefore, as a refinement of the MTSK model and a contribution to the overall knowledge on teachers' knowledge. The other direction pointed out is related to teacher education. When characterizing teacher's specialized knowledge by identifying a set of indicators of such knowledge, it is possible to refine the conceptualization of courses and professional developments programs, focusing on developing the specificities of teachers' knowledge grounded on these indicators.

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PROFESSIONAL DEVELOPMENT FOR SECONDARY MATHS TEACHERS: EVIDENCE OF IMPACT ON TEACHERS' KNOWLEDGE AND ON LEARNER ATTAINMENT

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We report on the impact of a one-year mathematics professional development course on teachers' mathematical knowledge and their learners' attainment for two cohorts of teachers (2016 and 2017). In both cases teachers' mathematics test-scores improved significantly. We examined the learning gains of Grades 9 and 10 learners from 24 secondary schools taught by teachers who had attended the course. We compared these results with those of learners in 16 matched comparison schools taught by teachers who had not participated in the course. The intervention group made larger gains which were statistically significant. Learners taught by the 2016 teachers made greater gains than learners taught by the 2017 teachers. This is in line with the expectation that the impact of professional development on learners is delayed.

INTRODUCTION

Across the world a wide variety of programmes attempt to improve teachers' mathematical knowledge in order to improve learner attainment. In South Africa, many such programmes have been designed and implemented but there is little evidence of positive impact on learners' performance in mathematics. For the past nine years the Wits Maths Connect Secondary (WMCS) project has trialled different models of professional development for secondary mathematics teachers. In 2012 the Transition Maths 1 (TM1) course was offered for the first time to a small group of teachers in the broader Johannesburg area. A small study of learning gains (Pournara, Hodgen, Adler, & Pillay, 2015) showed that learners taught by teachers who had attended the course made more gains and out-performed learners in the same schools taught by teachers who had not attended the course. The results were treated as "evidence of promise" and a larger more rigorous study has been ongoing since 2016 to study the impact of further cohorts of teachers who have completed the course.

The notion of *learning gains* has proved useful as a measure of learner attainment where the gain is the change in test-score from pre- to post-test over one academic year. In the context of impact studies it enables us, to some extent, to attribute learning gains to the teaching received from a particular teacher in a given year. Nevertheless, we are well aware of a range of interventions that are taking place in secondary schools where we are conducting the research and so we make all claims with caution, knowing that no individual intervention at the level of the teacher can fully account for improvements in learner attainment.

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The TM1 course has been revised and refined annually since 2014 and has now been offered to five cohorts of teachers across the Gauteng province of South Africa. The Learning Gains II study (LG) seeks to answer the question: "what is the effect of teachers' participation in the TM1 course on their learners' attainment in Mathematics?"

In this paper we report on the initial findings of data collected in 2018 where we compared the impact of a group of TM1 teachers who completed the course in 2016 with the impact of the cohort who completed the course in 2017. This enables us to see whether the Learning Gains II intervention had an impact and identify key issues that need attention as the study continues and which have a broader impact for similar intervention studies.

TEACHER KNOWLEDGE AND LEARNER ATTAINMENT

Following from Shulman's (1986, 1987) distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK), many have attempted to elaborate this for mathematics teaching (Krauss, Baumert, & Blum, 2008). However, the boundaries between SMK and PCK are not sufficiently clear to be useful as analytical constructs and so we work with the notion of "mathematics-for-teaching" (MfT) (Adler, 2005; Adler & Davis, 2006) as an amalgam of mathematical and teaching knowledge.

Attempts to measure teachers' mathematical knowledge, whether SMK or PCK or both, have taken various forms across the world. In some instances, teachers have been given the same/similar test items to the learners they teach (e.g. Harbison & Hanushek, 1992; Mullens, Murnane, & Willett, 1996; Taylor & Taylor, 2013). More sophisticated measures have been developed in Germany and the United States (Baumert et al., 2010; Hill, Ball, & Schilling, 2008; Krauss et al., 2008). Both these studies have found associations between teacher knowledge and learner attainment.

PROFESSIONAL DEVELOPMENT AND LEARNER ATTAINMENT

Worldwide there is concern about the impact of professional development on learner attainment. A review of studies of professional development relating to school mathematics (Gersten, Taylor, Keys, Rolfhus, & Newman-Gonchar, 2014) showed that very few of the initiatives which met acceptable standards of rigour also led to positive effects on learner attainment. Sample-McMeeking, Orsi, and Cobb (2012) reported the effects of a middle school intervention in the US where teachers took university summer courses in mathematics lasting two to three weeks. They reported an effect size of 0.20 (Hedge's g) on learner attainment only for teachers who had attended two courses but no discernible effect size for those attending one course. There is a clear need for rigorous studies of the impact of professional development on learner attainment in mathematics, and this study makes a contribution in this regard.

THE TRANSITION MATHS 1 COURSE

The underlying assumption of all the WMCS project's professional development is that focusing on teachers' MfT will lead to better teaching which will in turn translate into improved learner attainment. We assume a direct effect of the TM1 course on teachers' MfT and an indirect and delayed effect (Clarke, 1994) on learner attainment.

The course is designed to support teachers currently teaching in Grades 8 and 9 (first two years of secondary school in South Africa), to address the mathematical transitions from the Senior Phase (Grades 7–9) to the Further Education and Training (FET) phase (Grades 10–12). It is offered over a ten-month period in eight two-day contact sessions. It deals with both mathematics content (75%) and mathematics teaching (25%). This 3:1 ratio is similar to the course described in Sample-McMeeking et al. (2012) above. The mathematical content includes algebra and number (6 days), functions (4 days), Euclidean geometry (3 days) and trigonometry (3 days). Teachers submit seven assignments and write two tests all of which include mathematics and mathematics teaching content.

We approach the learning of MfT through *revisiting* known mathematics (Pournara, 2013) and learning new mathematics. A revisiting approach to familiar mathematics problematizes taken-for-granted aspects, often drawing on extreme cases to deepen teachers' knowledge rather than merely redoing known mathematics to improve procedural fluency. For example, in working with linear functions we provide teachers with five representations of the same function, say $y = 2x - 3$ (verbal, table, function machine, equation and graph). We then invite them to consider questions about each representation that are likely new and unusual, such as: "where is the y-intercept in the function machine?", "where is *double* in the table?" and "where is *double* in the graph?"

The mathematical content extends into the Grade 11 curriculum in algebra, functions and trigonometry so that teachers are better prepared to teach at Grade 10 level and beyond in the future. As part of the course, we pay attention to common procedures in the senior secondary curriculum such as completing the square using algebraic and geometric approaches. We challenge teachers to solve geometry problems using more than one strategy, for example to use different definitions of a parallelogram to prove that a given quadrilateral is parallelogram.

The mathematics teaching component pays deliberate attention to aspects that are close to teachers' current practice with the assumption that proximity to practice increases the likelihood of take-up from the course and implementation in the classroom. The teaching aspects are built on the notion of teachers' *mathematical discourse in instruction* (Adler & Ronda, 2014; Adler & Venkat, 2014) and operationalised through the *Mathematics Teaching Framework* (MTF) which focuses on key elements common to all teaching practices: identifying and articulating a lesson goal, designing and selecting examples sets, selecting representations, selecting and designing tasks,

producing explanations and justifications, and building opportunities for meaningful learner participation.

RESEARCH DESIGN AND METHODS

We adopted a quasi-experimental design to assess the effect of the TM1 intervention on the attainment of the learners of the teachers participating in the study. Given that we had no way of influencing whether TM1 alumni would teach Grade 9 in 2018, we extended the sample to include Grade 9 and 10 learners in order to guarantee a larger sample of TM1 alumni. Teachers were invited to participate if they were teaching mathematics to at least one Grade 9 or 10 class in 2018. Having selected the sample of TM1 teachers across 24 schools, we then sought comparison schools. The selection criteria for the comparison schools were: same school district (geographical area), similar number of learners in the school, and similar performance of Grade 9 learners on national assessments in Mathematics and English language conducted in 2012. Despite the results of this national assessment being dated, the Department of Basic Education considered these results more reliable than the latest results of 2014. Ideally we wanted to match teachers in terms of number of years teaching mathematics at various grade levels. However, in the majority of participating schools, teaching allocations are only finalised after a few weeks of teaching and it is not possible to guarantee that a teacher who volunteers to participate in the study will indeed be teaching at a particular grade before data collection was to commence. We eventually selected 16 comparison schools.

There were 40 teachers who completed the TM1 course in 2016 and 39 teachers in 2017. To assess their gains on the course in terms of their MfT, tests were administered at the start and the end of the course. The test at the end of the course was more cognitively demanding than the test at the start and covered more topics. Both tests were developed by the project team.

We tracked 1716 Grade 9 learners and 565 Grade 10 learners taught by TM1 teachers over the 2018 school year. We refer to these as the TM1 learners. We also tracked 886 Grade 9 learners and 645 Grade 10 learners in the comparison schools. These learners are referred to as the comparison group. A test was administered to all learners in February/March and September/October 2018. The breakdown of learners per teacher group and per grade is given in table 1.

	Grade 9			Grade 10		
Teacher group	No. of teachers	No. of schools	No. of learners	No. of teachers	No. of schools	No. of learners
TM1 2016	12	11	991	9	8	375
TM1 2017	12	10	725	6	6	190
Comparison	23	15	886	24	16	645

Table 1: Teachers and learners participating in study

The test was designed by the project team for Grade 9 learners and was piloted in 2016 with Grade 9 and 10 learners in schools similar to those participating in the study. *It contains* typical curriculum items at Grades 8 and 9 levels that deal with number, algebra and function. A Rasch analysis showed that the test was fit for the purpose of testing learning gains at Grade 9 level. However, there were a few too many items that were difficult for many learners. Given this flooring effect with Grade 9 learners we expected that it would still be useful as a test of learning gains with Grade 10 learners.

Each test item was marked as correct, wrong or missing with only 1 mark being allocated for a correct response. Consequently, a learner's test mark is an indication of how many items s/he answered correctly. There is no provision for partial marks and no consideration of partially correct responses. We moderated 10-20% of coding and code-capturing for accuracy.

A repeated sample t-test was conducted on teachers' scores for TM1 to determine whether gains were statistically significant. A repeated measures ANOVA analysis was carried out see whether the interaction between pre/post gains in the learner assessment and the learners group (control and those taught by TM1 teachers) was statistically significant.

RESULTS

We compared the mean test scores for all TM1 teachers before and after the course for each full cohort, and for each of the groups in the LG sample. These results are reported in Table 2. A repeated sample t-test analysis showed that the increases in test scores were statistically significant at the 1% level for the 2017 LG sample and at the 0.1% level for the full cohorts in each year and the 2016 LG sample (see Table 2). We therefore conclude that the course had a significant impact on teachers' MfT. Moreover, we argue that the statistics under-report the impact of the course on teachers' MfT because the post-course test was substantially harder and covered more topics than the pre-test.

	Teachers completing TM1 course				*Teachers in Learning Gains study*			
	N	*Ave mk*	*SD*	*t-test*	*N*	*Ave mk*	*SD*	*t-test*
TM1 2016 pre-test	*40*	*61.58*	*15.79*	*t = 8.73*	*16*	*60.42*	*13.61*	*t = 5.17*
post-test	*40*	*75.91*	*18.25*	*df = 39* p < *0.001*	*16*	*76.43*	*15.08*	*df = 15* p < *0.001*
TM1 2017 pre-test	*39*	*65.39*	*17.77*	*t = 5.20*	*15*	*69.42*	*13.44*	*t = 3.48*
post-test	*39*	*76.83*	*19.85*	*df = 38* p < *0.001*	*15*	*81.39*	*17.50*	*df = 14* p < *0.01*

Table 2: Maths test scores and t-tests for teachers

Secondly, we examined in the results of the LG test scores for the pre- and post-tests, comparing the mean scores for the comparison group and the TM1 learners.

	Pre-test			Post-test			
Cohort	N	Ave mark	SD	N	Ave mark	SD	Change in Ave mark
Total TM1 group	1587	8.54	6.50	1587	11.90	8.44	3.36
Comparison group	1531	10.81	8.02	1531	13.58	9.27	2.78

Table 3: Pre- and post-test scores for Total TM1 and Comparison groups of learners

While the gains were generally small and the overall performance was low, the TM1 learners made greater gains than those in the comparison group. Carrying out a repeated measures ANOVA analysis, the interaction between pre/post-test and learner group was significant ($F = 9.321$, $df = 1$, $p < 0.001$). We therefore concluded that there was a statistically significant difference in the gains in the learner test scores pre to post between the comparison group and the TM1 group.

The effect size between the comparison and the combined TM1 group in the pre-test was of the order of 0.3. In the post-test, this was reduced to 0.1. So a change in the effect size of 0.2 took place in favour of the TM1 group. This effect size is similar to that reported by Sample-McMeeking et al. (2012) and in the first LG study.

We also disaggregated the TM1 learners taught by teachers in the 2016 cohort of the TM1 course and those in the 2017 cohort of TM1as shown in Table 4.

	Pre-test			Post-test			
Cohort	N	Ave mark	SD	N	Ave mark	SD	Change in Ave mark
TM1 2016	815	8.87	6.90	815	13.09	8.76	4.22
TM1 2017	772	8.20	6.03	772	10.64	7.89	2.44
Comparison group	1531	10.81	8.02	1531	13.58	9.27	2.78

Table 4: Disaggregating the TM1 2016 and 2017 learner scores

We can see that the gains made by the learners taught by the TM1 2016 cohort of teachers were considerably bigger than those taught by the 2017 cohort.

DISCUSSION

The findings show firstly that teachers' participation in the TM1 course had a significant impact on their own mathematical knowledge for teaching. However, our main interest was whether such gains translate into gains at the level of the learner. The learners' results show statistically significant gains with the greatest gains being made by the learners taught by the TM1 2016 cohort of teachers. This finding is in line with our expectations and the literature (e.g. Clarke, 1994) that there is a delayed effect of

professional development on learner attainment. Despite the statistical significance of the results, one cannot escape the fact that learner performance is well below the level expected of Grade 9. This is particularly worrying given that the sample included many Grade 10 learners. Further analysis will interrogate the impact of the proportion of Grade 9 to Grade 10 learners in each of the teacher samples.

Impact studies such as this cannot provide insights into the mechanisms which enable or constrain the desired change. While it is encouraging that the gains are higher for learners taught by teachers who attended the course in 2016, their scores are still lower than expected, even with Grade 10s on a test that was designed for Grade 9 learners.

Related qualitative studies are currently investigating teachers' take-up from the course in relation to their teaching practice. Other studies are investigating the nature of learners' errors and the extent to which these errors change between pre- and post-test. Taken together this suite of studies will provide deeper insight into the ways in which the professional development may (or may not) lead to gains for learners.

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DIFFICULTIES IN FORMULATING A GEOMETRIC SITUATION ALGEBRAICALLY: HINTS FROM A LARGE-SCALE ASSESSMENT

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Acquisition of algebraic language is one of the turning points that allow the student to develop advanced mathematical thinking. There are several theoretical schemes that analyse this process both in syntactical and semantic aspects. In all these schemes, language plays a fundamental role, as does the work on making explicit the processes through which one passes from situations to a first mathematical modelling, which uses letters and relations between them. In this empirical work, we analyse the answers given by a large sample of students in Grade 08 to a question that required to move from a geometric situation to its algebraic modelling. The use of a large amount of data coming from a large-scale survey allows an "ongoing" description of possible stages of development of algebraic thought and of maturity of symbolic thought.

INTRODUCTION

Large-scale assessment initiatives and surveys regarding mathematics learning are now commonplace practices worldwide, in terms both of national and international assessment (OECD-Pisa, IEA-TIMSS Advanced), and the projects contribute important data on various levels. The interest of both the general public and the researchers themselves is, however, focused mainly on emerging aspects of the system (see for example, Breakspear, 2012; Grek, 2009; Martens & Jacobi, 2010) or macro-phenomena regarding learning processes (De Lange, 2007; Bolondi, Cascella & Giberti, 2017).

The procedures of data collection, correction and codification implemented by the agencies who deal with these large-scale assessment projects also provide huge amounts of data which represent invaluable material for research. The use of this raw data in mathematics education research seems, however, to have been relatively restricted so far (Pozio, 2008).

In our opinion, this is due to two main factors. The first issue is that the questions used in the tests are generally not made public as they may be re-used later with the purpose of allowing longitudinal anchorage of the scales on which the results are organized (OECD, 2017). The second cause is probably due to the lack of validated methodological models that allow an integrated use of these materials with the qualitative and quantitative methods generally used in mathematics education (Battista et al., 2009; Lesh, 2002; Hart et al., 2009). This paper sets out to demonstrate, using an initial case study, how it is possible to use results collected during a large-scale national

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assessment project (the Italian national INVALSI tests) to study a classic problem of mathematics education: how students convert a geometric situation into an algebraic formula.

The data analysed are the answers given by 4,565 Grade 08 students to an open constructed response question (issued by INVALSI), included in a high-stakes test as part of the final examination of the first cycle of education. The initial goal was to use this data to classify the types of answers provided by students, identifying and quantifying recurrent behaviours, common errors, specific strategies, and results of didactic practices. The analysis of the data also revealed the importance of using various theoretical lenses, in a panorama of networking theories (Bikner-Ahsbah et al., 2010), in order to recognize the different phenomena and interpret the various situations that arose.

In the Discussion section we show how, in the light of the theoretical constructs used, the classification and analysis of answer sheets allows a "photograph" (with horizontal "revealing") of the students' behaviour and the creation of a detailed "map" of the difficulties they encounter.

This research study shows how the work on this large-scale assessment project can actually contribute, firstly in improving understanding of specific issues linked to teaching and learning, and secondly to highlighting the connections and interactions between different theory constructs used by the researchers. In this report, we present the main theoretical and operational steps of the research, leaving a complete discussion of results and the finer details for a later and longer paper.

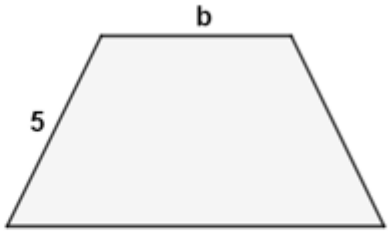
THE BACKGROUND AND CASE STUDY

The national system of learning assessment was introduced progressively around Italy from 2008 onwards. It involves more than 500,000 students per year for each school grade surveyed - currently (2019) grades 2, 5, 8, 10 and 13 are surveyed. The 8th grade test forms part of the final test at the end of the first school cycle (end of middle school) and its results are officially included in students' achievement records. In 2018, for the first time, the Grade 8 test was in CBT form, which allowed centralized digital data collection of all answer sheets.

The set of students involved 574,506 individuals, of whom 31,300 were part of a specifically selected sample group. The question being studied was administered to 4,565 sample students.

The item calls for the mathematical formulation of a geometric problem, and adopts two different question forms: a visual form with an isosceles trapezoid stating the length of the oblique side (5) and the value of the shorter base (b), and in wording, with a brief text which supplies the relationship between the shorter and longer base.

The longer base of the isosceles trapezoid shown is twice as long as the shorter base.



Write a formula expressing the trapezoid's perimeter *p* as a function of *b*.

Type your answer in the box here below.

Answer: *p* = ☐

Figure 1: the case study question (translated by authors)

The task given to students is in line with the National Guidelines, and thus with the official curriculum of Italian schools (MIUR, 2012; Kelly, 2009). In fact, one of the learning objectives for Grade 8 classes, in the content area of Relationships and functions, is "To interpret, construct and transform formulas that contain letters to express in general form relationships and properties".

Theoretical analysis of the questions allows the identification of different issues which may cause the students some difficulty: apart from the standard knowledge regarding the notion of perimeter, they will also need the ability to use letters in symbolic mode, checking the relationship between letters that express the relation of elements in the situation, and transforming verbal expressions while checking the meaning.

FEATURES OF THE THEORETICAL FRAMEWORK

The question considered in this case study requires the transformation of a geometric situation into an algebraic formula. To interpret and categorize the students' answers, various theoretical lenses are of help.

In fact, the issue of understanding algebraic reasoning has been widely studied in mathematics education and a copious literature is available: as to terminology and categorization, standard references are Arzarello et al. (1995) and Cai and Knuth (2011); for an appreciation of the issue in the framework of school curriculum, reference may be made to Usiskin (1988), Kieran (1992) and Van Amerom (2003); as to the problem of meaning control in the process of mathematisation by literal expressions, two relevant references are Capraro and Joffrion (2006) and the classic Arcavi (1994). Situations analogous to those we scrutiny have been analysed from a point of view similar to ours by MacGregor and Stacey (1993).

In the question studied, students are asked to move from a situation in a geometric context to its description in symbolic language. The dynamics of this transfer may be suitably described in the conceptual framework of the mathematisation cycle, at the heart of the framework of the OCSE-Pisa survey (OECD 2016). The question studied

is located in the process of formulating ("formulate") and involves various capabilities, particularly *Communicating, Mathematising, Representation* and *Using symbolic, formal and technical language and operations.*

In examining the processes behind answering this question, where the final goal is a symbolic representation, we use the idea of conversion transformation and treatment transformation, as stated by Duval (1993), and D'Amore (2006). After all, it is essential to consider the meaning of explicit representations in the light of educational practices (Sfard, 1991). Some clear traces of behaviour linked to the didactic contract (Brousseau, 1986) are also present.

RESULTS AND DISCUSSION

34% of the students gave the correct answer ($p = 3b + 10$ or an equivalent algebraic formula); 18% of students failed to provide an answer. The remaining answers were categorised according to the following table.

CATEGORY	Description	Frequency/ Percentage	Example(s)
1) Verbal	Students rewrite the problem in words	75 (3,5%)	Side + side + longer base + shorter base (half of the longer base)
2) Numerical	Students provide a numerical value	378 (17,6%)	5*2 +5*2+5+5
3) Wrong operation	Students make mistakes in choosing the operation(s) to be used in the formula; in particular, students use the expression b/2 instead of b*2	116 (5,5%)	5+5+(b*2b) 10*3b (5x2)+(b/2)+B (l*2)+b+(b/2)
4) Generic model	Students write a generic formula of a perimeter as the sum of 4 sides with no relation between the two basis	362 (17%)	(b+B)+5*2
5) Symbolic	Students do not use the numerical value 5 and indicate the oblique side with the letter l	61 (3%)	b+(b*2)+(l*2) bx3+2l
6) Wrong model	Students use a formula for the area	169 (8%)	(B+b)x5:2

7) Incomplete model	Students fail to consider all the elements of the situation; for instance, they take into account only the longer base as a function of the shorter, but do not add up the shorter base	270 (12,6%)	5*2+(b*2) 10+2b
8) Others	Incorrect answers that do not fall into any of the above categories	680 (31,6%)	

Table 1: The different response categories

In many answers (pertaining to different categories), the symbol B appears despite not being present in the proposed task – this symbol is often used in teaching practices in Italian schools to represent the larger base of a trapezoid. The letter l is the first letter of the Italian word "lato" (side).

As predicted, difficulties emerge in the Formulate process (OECD, 2016) when the student finds him/herself faced with the need to perform a conversion transformation (D'Amore, 2006) from visual/verbal format to symbolic form. The loss of meaning leads them to consider wrong or incomplete models (categories 6 and 7) or to apply the model too strictly (e.g. by using only numeric values or symbols but not both – categories 2 and 5). An incomplete mastery of the mathematical tools leads the students to make mistakes in the use of symbols which outline the relationship between the bases (category 3). The difficulty of converting the situation into symbolic language means that some students decide to abandon the use of this language altogether (category 1) or fail to use it to express the relationship between the sides (category 4).

The analysis of the answers organised into the various categories allows a more detailed overview of the processes (and difficulties) demonstrated by students, and in a certain sense also a quantification of them. As an example, we discuss briefly category 1. Here we find several students who do not use any symbols or numbers (Table 2). They try to re-formulate the situation textually, often verbally describing the requested formula in a procedural manner (in line with Sfard's theory, 1991, and according to the category *Rewrote the problem in words*, by Capraro et al., 2006). Some students correctly describe in words the procedure they would use to calculate the trapezoid perimeter, others describe a generic process for calculating the perimeter without taking into account the relationship between the two bases. Some students describe in words a generic formula of the area instead of the perimeter and this is probably due to the didactic contract. In fact, at school, students are taught the formula of the area, not that of the perimeter of the trapezoid. So, when they are asked for a trapezoid formula, they automatically think about the area formula. Generally, students belonging to Category 1 show that they give to the word "formula" only the meaning of "procedure" and not that of "relationship".

You have to multiply b by 2 to find the longer base, then add 5 plus 5, and finally add all the numbers to find the perimeter	Base1 plus base2 multiplied by the height, divided by two	All sides are added together, but you must first find 'b' and then make the sum of the sides
base1 + base2 + oblique side x 2	Side + side + base1+ base2 (half of the base1)	Side + side + longer base + shorter base (half of the longer base)

Table 2: Examples from category 1

A similar analysis was carried out on the other categories and on the correct answers, which differ between themselves in terms of different levels of *treatment* (Duval, 1993) of the algebraic expression.

CONCLUSIONS

From the analysis of students' answers, different types of errors emerged which highlight various misconceptions, procedural errors, formal errors, and the inability to coordinate different forms of representation. The biggest problems that students had with algebra learning can be traced back to the spread of didactic practices in which the construction of concepts tends to dedicate most attention to manipulative mechanisms while neglecting or awarding secondary importance to functional and semantic aspects (Malara & Navarra, 2003).

The research study has shown how it is possible to effectively use collected answer data (also from computer-based large-scale assessments) to study with education-research approaches the reasoning processes displayed by students. In particular, different theories can interact to describe the crucial and delicate process of formulation via a mathematical model (in this case, algebraic) of a geometric situation where aspects of visual representation (also established by educational methods) play a fundamental role.

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COMPREHENDING WORD PROBLEMS: EFFICACY OF AN INTERVENTION ON STRATEGIES AND SYNTACTIC AWARENESS

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Students' difficulties in reading and understanding word problems can be traced back to superficial reading strategies and a lack of awareness of syntactic language features. Therefore, an intervention was designed to foster subject-specific comprehension strategies and syntactic awareness using the design principles of strategic scaffolding and variation of syntactic features. This paper reports on a quasi-randomized controlled trial with n = 167 fifth graders that compared the learning gains between the intervention group and a control group who worked on the same word problems but without scaffolding and variation. The quantitative analysis of pre- and post-test scores (using ANOVA) showed that after six sessions of intervention, the intervention group had significantly higher learning gains than the control group.

BACKGROUND: APPROACHES FOR OVERCOMING STUDENTS' OBSTACLES IN UNDERSTANDING WORD PROBLEMS

Word problems are a specific task format used to help students apply mathematical concepts, models, and procedures to realistic situations and are often presented in written format, requiring several steps for solving them (Leiß, Schukajlow, Blum, Messner, & Pekrun, 2010). Based on the empirical identification of typical students' obstacles for mastering these steps, instructional approaches were developed to support students to overcome them:

- *Habitual obstacles* refer to habits of superficial reading that lead to constructing wrong situation models. Instructional approaches, which focus on increasing the authenticity of the problems and changed attitudes, have been successfully developed (Verschaffel, Greer, & de Corte, 2000).
- *Strategic obstacles* refer to misleading comprehension strategies or non-matching strategy use as the direct translation strategy (Hegarty, Mayer, & Monk, 1995; Leiß et al., 2010). Instructional approaches aiming at enhancing students' strategies have been suggested (see Reusser, 1997; Leiß et al., 2010; Prediger & Krägeloh, 2015) but have rarely been shown to be effective in controlled trials (except for Schukajlow, Kolter, & Blum, 2015).
- *Conceptual obstacles* refer to difficulties with the use of basic mental models in the steps of transferring the situation model into a mathematical model (vom Hofe, Kleine, Blum, & Pekrun, 2006). Instructional approaches that support the translation of mental text representations into mathematical models have been successfully evaluated.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp 233-240). Pretoria, South Africa: PME.

- *Language obstacles* refer to the noticing and understanding of lexical or syntactic language features that affect the process of understanding the problem text. Although the problems on the syntactical level have been well specified in different assessment studies (e.g. Haag, Heppt, Stanat, Kuhl, & Pant, 2013), interventions that prepare students for noticing syntactic features are still rare (Dröse & Prediger, submitted).

As interventions on strategic and syntactic obstacles are still required, the research presented in this paper focuses on these two kinds of obstacles. The authors developed an intervention for Grade 5 (presented in the next section). Previous qualitative research on the initiated learning processes (reported in Dröse, 2019; Dröse & Prediger, submitted) has revealed qualitative evidence that students seem to enhance their strategies and syntactic awareness in the intervention. This paper reports on a quasi-randomized controlled trial showing the effectiveness quantitatively.

DESIGN OF THE INTERVENTION ON STRATEGY USE AND SYNTACTIC AWARENESS FOR COMPREHENDING WORD PROBLEMS

Identifying the relevant learning content for fifth and sixth graders

By systematically comparing successful and non-successful students in their processes of cracking word problems such as the one in Figure 1, qualitative research has revealed indications for identifying the learning content most relevant for increasing fifth and sixth graders' competence in comprehending word problems:

- For overcoming the strategic obstacles, three comprehension strategies have been identified as helpful (Reusser, 1997; Prediger & Krägeloh, 2015): (S1) Focus on the question and find information relevant to answering the question, (S2) focus on pieces of information with their meanings, and (S3) focus on relations between the information (*relational processing* in Hegarty et al., 1995).
- Language obstacles in noticing and interpreting syntactic features occur for fifth and sixth graders, for example, in syntactically challenging word order: German grammar allows the use of different grammatical cases and pronouns to change the order of subject and object, but many students are not aware of these possibilities. Raising students' awareness of these syntactic subtleties is hence important learning content (Dröse & Prediger, submitted).

Design principles for fostering reading strategies and syntactic awareness

To foster both identified types of learning content in Grade 5, the three strategies and the awareness of syntactic features, an intervention was designed in several iterative design research cycles (Dröse, 2019). Two design principles were realized:

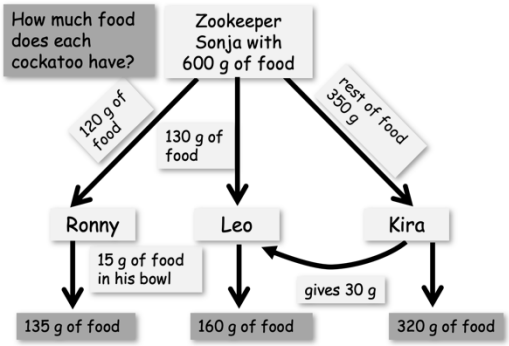
- The intervention follows the design principle of *strategic scaffolding,* in which students' strategy use can be supported by a strategic scaffold that is introduced and later faded out (Hannafin, Land, & Oliver, 1999). Figure 2 shows the developed strategic scaffold, a concept map with specific rules: While writing down the question supports Strategy S1, writing down the relevant information (always number + unit + explanation) supports Strategy S2. After extracting the relevant

information, Strategy S3 is supported by arranging the information cards and connecting them with arrows that display the relationships between the information cards (Dröse, submitted). Concept maps were chosen as a strategic scaffold, as they may encourage learning processes with their different functions and therefore "may facilitate cognitive representation of the information" (Nesbit & Adesope, 2006, p. 417). After the scaffolded construction of the situation model, students can use further approaches to mathematize and solve the word problem.

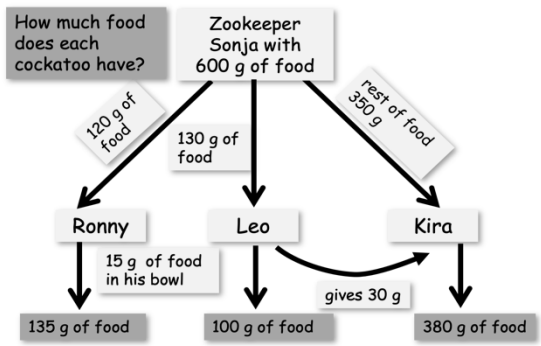
- The *variation principle* can generally be applied whenever students' attention should be focused to a specific aspect (Marton & Pang, 2006). In our design, we systematically vary the word problem formulations in the critical syntactic features (such as the subject and object positions; see Figure 1). This can enable students to notice the syntactic differences. The concept maps help to make the differences visible (as in Figure 2) and to interpret them in the situation model (Dröse & Prediger, submitted).

Cockatoo Food (Version A, English adaption)	**Cockatoo Food (Version B, English adaption)**
The class observes the feeding of the cockatoos. The three cockatoos, Ronny, Leo and Kira, live in one enclosure. Ronny still has 15 g of food in his bowl. The zookeeper Sonja enters the enclosure with a bucket. In the bucket is 600 g of food. Leo gets 130 g of food. Ronny gets 120 g of food and Kira gets the rest of the food from Sonja. Kira and Leo are friends. *His friend Kira gives him 30 g food as a present.* How much food does each cockatoo have?	The class observes the feeding of the cockatoos. The three cockatoos, Ronny, Leo and Kira, live in one enclosure. Ronny still has 15 g of food in his bowl. The zookeeper Sonja enters the enclosure with a bucket. In the bucket is 600 g of food. Leo gets 130 g of food. Ronny gets 120 g of food and Kira gets the rest of the food from Sonja. Kira and Leo are friends. *His friend Kira is given 30 g of food as a present by him.* (Literally: To his friend Kira gives he 30g for food.) How much food does each cockatoo have?
Syntactic case analysis of German original text: **Version A: <u>Seine</u> Freundin Kira schenkt <u>ihm</u> 30g Futter.** Nominativ (Subject) Dativ (Object)	**Syntactic case analysis of German original text:** **Version B: <u>Seiner</u> Freundin Kira schenkt <u>er</u> 30g Futter.** Dativ (Object) Nominativ (Subject)

Figure 1: Word problem variation with case analyses of German original texts (adapted from Dröse & Prediger, submitted)



Concept map for Version A in Figure 1



Concept map for Version B in Figure 1

Figure 2: Strategic scaffolding by concept maps (adapted from Dröse, 2019)

Based on these two design principles, an intervention was designed in which the strategic scaffold of concept maps was successively introduced for increasingly complicated word problem structures. The concept maps were also used as visualizations for making subtle syntactical differences visible and discussable when comparing slightly varied word problems. Word problems were varied in the positions of subjects and objects as well as other syntactic features. At the end of the intervention, the strategic scaffold was faded out so that students would internalize the three strategies without an external scaffold (Hannafin et al., 1999).

RESEARCH QUESTIONS

The previous qualitative analysis of small-group teaching revealed indications that both design principles and their realization in an intervention of six sessions could enhance students' strategy use and students' syntactic awareness (Dröse, 2019; Dröse, submitted; Dröse & Prediger, submitted). In order to also provide quantitative evidence for the efficacy of the intervention in whole classes and regular teachers, a quasi-experimental controlled trial was conducted with two research questions:

(Q1) Which are the learning gains of the intervention group compared to the control group?

(Q2) How do the learning gains differ between students with high and low general reading proficiency?

For research question Q1, we hypothesized that the intervention group would learn more than the control group (H1). Research question Q2 is relevant in terms of taking into account the most important general learning prerequisite for comprehending word problems: general reading proficiency (Leiß et al., 2010). We hypothesized that students with lower reading proficiency would profit more from the intervention than more highly proficient peers (H2), as they have more to learn by the intervention.

METHODS OF THE CONTROLLED TRIAL

Research design

The research was conducted as a quasi-randomized controlled trial with a pre- and post-test design with fifth graders in whole-class settings. The independent variable is the form of intervention, and the dependent variable is the learning gains in comprehending word problems.

Intervention forms

The intervention and control classes were taught by their regular teachers in 5-6 sessions of 90 minutes each. The intervention classes (n = 126 students) followed the designed intervention program. The two control classes (n = 41 students) were taught with the same word problems as the intervention group. The teachers of the control classes were sensitized for typical strategic and language obstacles (including the syntactic features in view) but were not introduced to the principles of strategic

scaffolding and the variation principle and their relevance for fostering strategies and syntactic awareness. The teachers of the intervention groups received the teaching material realizing these principles and were introduced to the underlying principles.

Measures for control variables and dependent variable

Four control variables were taken into account:

- General *reading proficiency* was assessed using the reading speed test ELFE, which captures reading on the word, sentence, and text level under strong time restrictions.
- *Age* and *language background* were captured by students' self-report of their ages and the languages they spoke at home.
- The *socio-economic status* was captured using the book scale.

In order to measure the learning gains between pre-test and post-test, students' achievements in comprehending word problems were assessed using two analogous standardized tests on word problems, which were optimized to assess students' strategy use and the identification and interpretation of syntactic language features. To assess comprehension, the right mathematization rather than the correctly calculated solution but was scored.

Sample

In order to control for the comparability in the quasi-randomized controlled trial, variances concerning reading proficiency, language background, socio-economic status, and age were analyzed (see Table 1). In the variance tests, no significant differences appeared between the intervention group and the control group concerning general reading proficiency, language background, and socio-economic status (with $p > .05$ in Wilcoxon tests and t-tests for the three ordinal or categorial variables). Significant (but nevertheless small) differences appear concerning the students' age in intervention and control group (with $p < .05$ in Wilcoxon tests).

	Reading proficiency (max. 120) m (SD)	**Age** in years m (SD)	**Language background** Percentages of mono-/multilinguals	**Socio-economic status** Percentages of low/ medium/high SES
Intervention group (n = 126)	60.03 (13.51)	10.90 (0.72)	53/47	28/25/48
Control group (n = 41)	57.29 (14.30)	11.22 (0.78)	66/34	37/34/29

Table 1: Descriptive data for comparable subsamples

Methods for data analysis

To test hypotheses H1 and H2, analyses of variance with repeated measures (Type III ANOVA accounting for different group sizes) were conducted using R software package ez. This allowed comparison of the differences in the learning gains of the intervention and control groups. To compare learning gains with respect to reading proficiency, a cut-off score of 60 split the intervention group into 50% with low and 50% with high reading proficiency (median split) and the control group into 49% and 51%, respectively.

EMPIRICAL RESULTS

Research question Q1 concerns comparing the learning gains of the intervention group (who were taught based on the two design principles of strategic scaffolding and the variation principle) with the control group (who worked on the same word problems but without using the two design principles).

The first two lines in Table 2 show the pre-test and post-test scores. Both groups had significant learning gains (F_{time} with $p < 0.05$ for both group with effect sizes of d = 0.86 and 0.58). However, the learning gains of the intervention group were significantly higher than those of the control group ($F_{time \times group}$ (1, 165) = 4674, $p <$ 0.05). Hence, hypothesis H1 that the design principles are effective can be confirmed.

	Pre-test m (SD)	**Post-test m (SD)**	**Learning gain m (SD)**	**Effect size**	**ANOVA-results**
IG (intervention group) (n = 126)	6.23 (2.48)	8.98 (3.16)	2.75 (3.21)	d = 0.86	F (1, 165) = 4.67 $p < 0.05$ $\eta^2 = 0.019$
CG (control group) (n = 41)	5.62 (2.92)	7.68 (3.59)	2.06 (3.54)	d = 0.58	
IG+ (IG with high reading prof.) (n = 63)	6.69 (2.73)	9.40 (3.37)	2.71 (3.31)	d = 0.82	F (1, 81) = 0.93 n.s. ($p > 0.5$) $\eta^2 = 0.01$
CG+ (CG with high reading prof.) (n = 20)	6.22 (3.01)	8.57 (3.75)	2.35 (4.19)	d = 0.56	
IG- (IG with low reading prof.) (n = 63)	5.76 (2.12)	8.56 (2.89)	2.80 (3.14)	d = 0.89	F (1, 82) = 4.94 $p < 0.05$ $\eta^2 = 0.04$
CG- (CG with low reading prof.) (n = 21)	5.05 (2.79)	6.83 (3.30)	1.79 (2.87)	d = 0.62	

Table 2: Learning gains for the intervention and control group

Research question Q2 asks for differences in learning gains between the subsamples with low and high general reading proficiency. The third to sixth lines of Table 2 show that all subsamples in the intervention and control group profited from the interventions (with effect sizes between d = 0.56 and d = 0.89). However, for the students with high general reading proficiency, there was no significant difference between intervention group IG+ and control group CG+ ($p > 0.05$), whereas for students with low general

reading proficiency, the learning gains between intervention IG- and control group CG- differed significantly ($F_{time \times group}$ (1, 82) = 4.94, $p < 0.05$ and $\eta^2 = 0.04$).

DISCUSSION

Hegarty et al. (1995) called for interventions that support students in overcoming strategic obstacles: "Thus, a first step in comprehension strategy instruction is to present students with problems that help them see that direct translation does not work well for some problems. A second step is to provide instruction in a method that emphasizes understanding the situation described in the problem" (p. 29). In order to enhance the students' comprehension strategies, strategic scaffolding has been shown to be useful (Schukajlow et al., 2015) and was included in the current study of the concept maps as a strategic scaffold. Compared to other strategic scaffolds, the concept map has a specific strength, as it scaffolds the students' relational processing, which had been shown to be crucial (Hegarty et al., 1995), especially for students with low reading proficiency who might not be aware of syntactic subtleties. By applying the variation principle to focus students' attention on syntactic subtleties, we extended the scope of the design principle that so far has mostly been used for conceptual purposes (Marton & Pang, 2006). To date, no empirical evidence has been provided that raising students' syntactic awareness is effective.

With the current research report, we can fill this research gap and provide empirical evidence that the combination of strategy training and raising syntactical awareness can significantly increase learning gains. The intervention group had significantly higher learning gains than the control group. The fact that the difference was not significant in the subsample of students with high reading proficiency might be the result of subsamples that were too small. However, it was significant in the subsample of students with low language proficiency, who have been shown by Leiß et al. (2010) to have the most needs in raising their abilities to solve word problems.

The major limitation of the study is that the control group had only 41 students. Future studies should extend the control group. However, the specific strength of this study is that the control group was not a "no-treatment group." Instead, it treated all word problems occurring in the main intervention, and the teachers were sensitized to the typical obstacles so they could talk about them in the classes. The only difference between the control group and the intervention group was that in the control group no strategic scaffolding took place and there was no explicit comparison of syntactic differences on the sentence level.

These results should have substantial consequences at the mathematics teacher professional development level. As the current controlled trial has shown, it is not enough to sensitize mathematics teachers to the typical obstacles. Instead, they should be given powerful designs to overcome these obstacles. Designs based on the design principles of strategic scaffolding and the variation principle promise higher learning gains. However, every transfer from one context and grade to another requires further design research in order to find the best ways of realizing these design principles.

Acknowledgment.

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CONDUCTING PD DISCUSSIONS ON LANGUAGE REPERTOIRES: A CASE STUDY ON FACILITATORS' PRACTICES

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Professional development (PD) programs need to qualify teachers to promote language learners by building upon and successively extending their language repertoires. Although PD facilitators play a crucial role in these PD programs, their facilitation practices have not yet been empirically captured very successfully. This case study investigates the facilitation practices of two facilitators in a PD on language-responsive mathematics teaching. The qualitative analysis identifies the goals and knowledge aspects that guide their practices. It shows that detailed pedagogical content knowledge on the classroom level (PCK-C) and PD level (PCK-PD) supports goal-oriented facilitation, in which PCK-PD is required to relate the teachers' ideas about first languages to the teaching approach of building upon languages repertoires. This study contributes to extending existing generic frameworks for facilitation practices to specific PD contents, in this case to language-responsive mathematics teaching.

Establishing new teaching approaches requires PD programs; this is also the case for language-responsive mathematics teaching approaches such as building upon students' language repertoires (Barwell, 2018; Planas, 2018). So far, PD programs on language-responsive teaching have mainly been studied with respect to their PD content and PD materials (Hajer & Norèn, 2017) and the teachers' learning processes (Prediger, subm.). Although the PD facilitators' practices are crucial for the success of the PD, research has not yet focused on them. In order to reduce this research gap, this research report presents a case study of two facilitators' practices conducting PD discussions on using students' language repertoires. For this, we adopt existing research frameworks on facilitation practices (developed for other PD contents) such that we can use them for identifying the underlying goals and knowledge resources (Borko et al., 2014; Tekkumru-Kisa & Stein, 2017) and pursue the following research question: By which practices do PD facilitators lead discussions on students' language repertoires, and by which goals and knowledge aspects are these practices guided?

THEORETICAL AND EMPIRICAL BACKGROUNDS

Background on the classroom level: Building upon language repertoires

Effective teaching approaches build upon students' individual resources and systematically connect new content to students' prior knowledge. For multilingual students, this general principle translates into the call to include students' first languages and all

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp 241-248). Pretoria, South Africa: PME.

informal, multimodal resources that can support their processes of meaning construction for new academic and technical language in math classrooms, i.e., students' language repertoires (Barwell, 2018; Planas, 2018). Figure 1 (Prediger, Clarkson, & Bose, 2016) depicts the components of these language repertoires and the need to repeatedly relate them. In order to unfold the principle in mathematics classrooms, teachers should learn the following content aspects of teachers' language-related pedagogical content knowledge (PCK-C on classroom level) in the PD program:

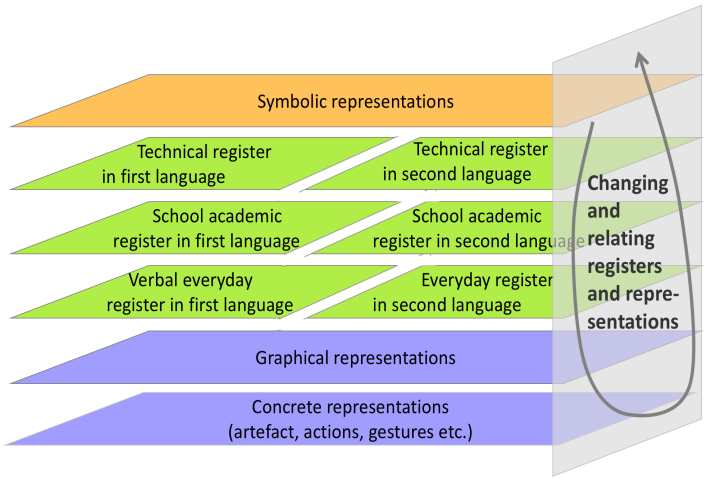

Figure 1: Relating registers and representations for building upon language repertoires

PCK-C1 Overcome *language policy concerns* against underprivileged first languages (Barwell, 2018; Planas, 2018)

PCK-C2 Challenge kids to produce output in all registers (*offensive approach* rather than reducing academic language demands; Prediger, subm.)

PCK-C3 Become aware of the *epistemic function* of registers and representations (rather than only a communicative function; Prediger et al., 2016)

PCK-C4 Support not only the change, but the systematic *connection* between registers and representations (Prediger et al., 2016)

PCK-C5 *Successively develop* students' language repertoires by extending the existing resources and connecting them (Planas, 2018)

Background on PD level: Goals and knowledge underlying facilitation practices

In the last decades, PD facilitators' practices (in PDs with other mathematical PD content) have been investigated with two different foci: the enacted facilitation moves and the underlying or necessary knowledge that facilitators should refer to. Some *facilitation moves* have been classified independently from the PD content (van Es et al., 2014): lifting up (identifying an important idea that a participant raised in the discussion for further discussion), countering (offering an alternative point of view), connecting ideas (making connections between ideas raised in the discussion), and standing back (allowing the group members time discuss an issue). In order to understand facilitation practices, the facilitation moves should be considered together with the underlying goals (already partly realized in the framework of van Es et al., 2014) and the knowledge aspects to which the facilitators implicitly or explicitly refer (in this perspective, we lift a research framework from the teacher PD level to the facilitator level, see Prediger, subm.).

For the *goals*, we adopt Schoenfeld's (2010) conceptualization and distinguish *content* goals (when facilitators pursue selected PD content learning goals) from *process* goals

(when facilitators strive for PD process qualities as active engagement of participants).

For the *knowledge*, we start from Borko et al. (2014), who show that facilitators' practices rely on the knowledge aspects the facilitators implicitly or explicitly refer to. The authors conceptualize relevant knowledge aspects (in what they call mathematical knowledge for professional development), comprising specialized content knowledge on "mathematical content and relationships" and pedagogical content knowledge including "the ability to engage teachers in the interpretation of students' mathematical ideas and the purposeful analysis of instructional practices" (p. 165). Hence, they relate content knowledge to the classroom level (here abbreviated by CK-C) and pedagogical content knowledge to the PD level (PCK-PD). To complete the components on both levels, we add CK-PD and PCK-C: CK-PD refers to the facilitators' knowledge about the teaching approaches and content in view, containing CK-C, and PCK-C refers to the teachers' existing and intended pedagogical content knowledge on the classroom level. We follow Borko et al.'s (2014) call to empirically disentangle the knowledge aspects that are most relevant in the facilitators' practices.

METHODOLOGICAL FRAMEWORK

The presented case study is embedded in a large PD research project on PD courses on language-responsive math teaching (Prediger, subm.). The video-recorded PD courses usually take four hours and are conducted by two facilitators each. To capture the facilitators' goals and the knowledge aspects they refer to, each PD session is triangulated using a video-based debriefing session in which both facilitators discuss with a researcher the critical incidents in the PD session (chosen by all three).

The data selected for the case study in this research report stems from Alice and Christin's first PD session on language-responsive math teaching with 16 unknown teachers, followed by a debriefing session with the facilitators. At the time of the study, Alice had 10 years of experience as a teacher and some experience as a facilitator and PD material developer (in other topics), while Christin had little experience as a teacher at school and as a facilitator but much as a researcher on language.

The qualitative data analysis followed deductive category-led analytic procedures (Mayring, 2015) in four steps: (1) Relevant sequences of discussing language repertoires were identified and transcribed; (2) the enacted facilitation moves were classified according to the framework of van Es et al. (2014); (3) the goals and knowledge aspects that were likely underlying were interpreted and classified as process/content goals as either PCK-PD, CK-PD, PCK-C, or CK-C (see above for the definition of categories); and (4) the interpretations were triangulated with the facilitators' reflections and their further perspectives in the debriefing sessions.

CASE STUDY: THE FIRST PD SESSION OF ALICE AND CHRISTIN

Alice's provocative move for the process goal of engaging teachers in discussions

The selected episode (occurred after a 2.5-hour of the PD session), emerged in a sequence dedicated to introducing the teaching approach of relating registers and

representations (visualized by a slide with Figure 1). Both facilitators felt a bit uncomfortable as they considered themselves as not sufficiently successful in engaging the teachers in vivid discussions. In this situation, Alice decides to deviate from the original PD schedule by addressing the additional topic of allowing students' first languages in (usually monolingual) mathematics classrooms which was planned for later in the PD.

S1 Al. It is suggested [*by the research*] that if two students have the same first language, for example Turkish, and sit side-by-side in the classroom, that they first have the opportunity to clarify certain things related to the task in Turkish.… So that the language isn't an obstacle.… We ourselves, we actually have the rule always to talk in German in class. But it [*allowing the use of the first languages*] seemed for me, somehow. I have tried it then. In the moment it was a success for me, because they could suddenly talk about things about which they couldn't talk about well in German. Have you had any experience with this?

She underpins her argument by referring to research and to own experiences in school (Turn S1: "I have tried it … it was a success for me"). Her question leads to a vivid discussion in which all contributing teachers disagree for various reasons:

S2 Te1 I'm just wondering if that isn't a contradiction to, that you have to challenge the students. I'm not sure.

… …

S4 Te2 Well, in our case, it is like that: Since they have already reached the eighth grade at the Gymnasium [*track for higher achieving half of students*], then they should be able to talk about an everyday problem. If they should not even use technical terms, then they should do it in German. Thus I don't like it if they communicate about the problem in Turkish….

S5 Te3 Exactly.

S6 Te2 In working on a task you can ask again…. I had the word "tachometer," which some didn't know.… for example… And for this [*answering the vocabulary question*], there are classmates and I'm still there [*to answer*], too.

The teachers' contributions reveal different pedagogical content knowledge aspects (from the list in the first section): Teacher Te1 (Turn S2) refers to PCK-C2, as she considers the use of first languages in mathematics classrooms as a contradiction to an offensive approach of pushing language demands. Teacher Te2's utterance (Turn S4: "They have already reached the eighth grade ... They should be able…") and the confirmation of Te3 (Turn S5) show that they share language policy concerns against underprivileged first languages (PCK-C1) but miss the epistemic function of the first language by focusing only on a deficit-compensating communicative function (PCK-C3). The second contribution of Te3 (Turn S6) again focuses only on the communicative function of registers and representations (limiting PCK-C3).

During the discussion Alice does not comment the teachers' contributions in a content-related way, thus she does not refer to the content aspects (but her co-facilitator does; see below). At the end of the discussion she closes with the following utterance:

S16 Al. …I put it [*topic of allowing first languages in mathematics classrooms*] here consciously on the table to discuss it controversially. You have done that well. Just to make suggestions about what is possible. Of course, you always have to weigh them for yourself: what makes sense and what does not make sense for you.

Alice knows that the idea of including first languages might raise teachers' objections. However, she addresses the topic of PCK-C1 (overcoming language policy concerns against underprivileged first languages, in the list of the first section), which usually comes later in the PD program. In Turn S16, she makes explicit that she has used the topic to bring up a provocative issue (a variant of countering in the category system of van Es et al., 2014). By this, she seems to pursue not a PD content goal, but the process goal of engaging teachers in a more vivid discussion. This interpretation is confirmed by her explanation and evaluation of her decision in the debriefing session:

D7 Res. What did you [*to Alice*] think in this moment? Which criterion did you use to decide, if it was right or wrong?

D8 Al. For me, the group seems so lifeless and I can't manage this though, because I feel that we overfeed them with content and they say afterwards: "It was nice, but I didn't get anything out of it." Then it was like that, that I thought, perhaps, …they have another orientation to this topic and I thought…it is provocative in this situation. And perhaps, I knew, that it could be difficult and that they face this negatively. But I was very aware of this and hoped that they get some way to see a new perspective for the thing.

D9 Al. And…when they discussed, I thought it was right to make this.…

To pursue her process goal of engaging teachers in a more vivid discussion, Alice draws upon her PCK-PD (Turn D8: "perhaps…they have another orientation to this topic and I thought…it is provocative in this situation.") that many teachers share language policy concerns against underprivileged first languages (PCK-C1). Her focus on a process goal could probably also explain why she does not intervene in the discussion and closes it (in Turn S12) without commenting on the PD content itself. Her goal is reached, as several teachers have contributed to the discussion (Turn S12: "You have done that well"; Turn D9: "when they discussed, I thought it was right to make this."). This focus on the process goal explains why Alice activated only very limited parts of her own CK-PD, reducing the use of first languages to its communicative function of compensating for deficits in the second language (in Turn S1 and by not reacting to Turn S4), without addressing PCK-C2 through PCK-C5.

Christin turns the process goal into a PD content goal

After the teacher's contribution in Turn S6 (see above), Christin tries to fuel the discussion by some framing using the theoretical construct of zone of proximal development, which refers to the idea of successively building and extending student language repertoire (PCK-C5):

S7 Chr. That is perhaps also a bit the question about what is the zone of proximal development of the students. Thus, if the language problem is perhaps so big that the given requirements are too big for the students to reach, it is perhaps helpful

if they initially speak in Turkish. If it isn't possible otherwise. But probably most of the students – I don't know how it is with your students – for them it is probably in their zone of proximal development.

Christin's enacted facilitation move can be classified as an instance of connecting ideas (van Es et al., 2014), as she tries to make connections between Alice's and the teacher's contributions and the PD content on a productive use of students' language repertoires. She introduces the construct of zone of proximal development relating to the longer-term development of language (PCK-C5), which serves here as a mediating construct to explain why some teachers believe the first language not to be necessary (because the zone has already been left behind) and Alice's experience of girl who used it productively. Mediating constructs are a key issue in PCK-PD. By these moves, she turns Alice's process into a content goal: Her facilitation goal is not only to have a vivid discussion, but she starts to work towards the PD content goals of teachers' PCK-C2 and PCK-C5. During the debriefing session, Christin explains why she made the comment:

D10 Chr. That was because I sat there and realized that you talked about different things. That they didn't understand what you meant. Instead, they perceived it as a contradiction too: We make it offensive [*i.e., we call for language learning opportunities*] and confront them with the requirements [*rather than allowing them to escape into their home language*].

… …

D11 Chr. In this situation I thought that I would try to make it explicit.

According to her debriefing reflections, Christin felt she made the comment because she noticed the participants' irritation with a felt contradiction between Alice's contribution (allowing the use of first languages in mathematics classrooms) and the offensive approach of language-responsive mathematics teaching (PCK-C2). Hence, she noticed the core of PCK-C2 in the utterance of Teacher Te1 (in Turn S2).

As Alice quickly closes the discussion about the unplanned topic, both facilitators could not yet exploit the emerging learning opportunities for PCK-C2 and PCK-C5. In the debriefing session, both facilitators reflect further on how to overcome teachers' concerns in order to also reach PCK-C2 and PCK-C5 for first languages.

The most critical issue seems to be that none of the teachers (nor the facilitators) address the epistemic function of first languages: First languages are not important for compensating communicative restrictions (especially not for native resident multilinguals) but contain valuable extended thinking tools relevant for the meaning-making process, even for students with good language proficiency in the language of instruction (PCK-C3). This epistemic potential for meaning-making unfolds not by switching between the languages, but by systematically connecting them in different registers and representations (PCK-C4). Connecting the teachers' ideas to these knowledge aspects would have required PCK-PD on further mediating constructs.

However, to understand Alice' decisions about her facilitation practice, the process goal in its own rationality must be taken into account. Due to time restrictions of PD sessions, nobody can exploit every learning opportunity that spontaneously emerges.

CONCLUSION AND OUTLOOK

What can we learn from the case study with its limitations in sample size, scope, and topic? With respect to the specific PD content of activating students' multilingual language repertoires (Planas, 2018; Prediger et al., 2016), we realize that not every hint to including first languages as a part of students' language repertoires is automatically productive for reaching the PD content goals. In particular, the analyzed discussion seemed to be hindered by reducing first languages to the communicative function for those who cannot sufficiently speak the second language. Barwell (2018) and Planas (2018) have pleaded for emphasizing the epistemic role for the meaning constructions and an extension from the first language or languages to other registers and representations. Christin's moves are first steps in this direction, but perhaps not yet sufficient for exploring the epistemic function of all registers with the participating teachers. In further work with these facilitators and in preparation of new facilitators, CK-PD should be strengthened to flexibly relate to all five aspects of PCK-C1 to PCK-C5.

On a more general level, the case study modestly contributes to successively developing a research framework for understanding facilitation practices in an empirically grounded way: Rather than only identifying the pedagogical tools such as facilitation moves (e.g. connecting ideas), understanding the facilitators' practices requires the interpretation of the goals and the knowledge aspects the facilitators explicitly or implicitly refer to or do not refer to in a certain PD situation. Methodologically, the video-recorded reflections from the debriefing sessions are essential for triangulating the interpretations of goals and knowledge resources. For these goals, we replicated Tekkumru-Kisa and Stein's (2017) and Borko et al.'s (2014) observations that pursuing the PD content goals is a demanding job for the facilitators. The distinction between PCK-C and PCK-PD allowed us to unpack not only the different PCK-C aspects to which Alice and Christin refer, but also to disentangle different ways in which PCK-PD can relate to these PCK-C aspects:

- Working with knowledge about typical obstacles in teacher learning for provoking discussion (Alice consciously brings up PCK-C1 for provoking objections),
- Noticing an occurring obstacle in teachers' learning pathways and supporting them to overcome the obstacle (Christin notices teachers' irritation at an apparent contradiction with PCK-C2 and tries to harmonize them),
- Pursuing PD content learning goals (Christin redirects the focus to PCK-C2 and the apparent contradictions), and
- Connecting two PD content learning goals (Christin introduces the construct of zone of proximal development to connect PCK-C5, but not yet PCK-C3 and -C4).

These distinctions should be followed up in future research in order to elaborate the conceptual frameworks for understanding PD facilitation practices. This is highly relevant as Borko et al. (2014) describe the specific importance of the flexibility and connectedness of facilitators' knowledge base for goal-oriented facilitation.

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MENTAL MATHEMATICS AS ORAL MATHEMATICS

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*In this Research Report, Nunes, Schliemann and Carraher's (1993) description of oral mathematics (*versus *written mathematics) is used as a way to conceptualize about mental mathematics strategies. Meaning-preservation and meaning-making, as characteristics of oral mathematics strategies, are used to scrutinize examples of mental mathematics strategies and their underlying processes. These analyses raise sensitivities about differences between mental strategies and paper-and-pencil work.*

INTRODUCTION – WHAT DOES MENTAL MATHEMATICS ENTAIL?

Most if not all of the existing literature somehow defines mental mathematics along Hazekamp's (1986) long-standing definition as the solving of mathematical tasks through mental processes without paper-and-pencil or other computational (material) aids. To this, one can add that there are some time constraints to produce an answer, as well as the fact that questions are often asked orally. It is thus mostly the environment (e.g., time constraints, no paper-and-pencil or material aids, questions asked orally) that appears to define what mental mathematics is and implies. However, although of significance, this view of mental mathematics in terms of environment says little about what doing mental mathematics entails, the kinds of meanings it produces, the nature of the mathematical activity engaged in by solvers, and so forth.

It is often reported that the strategies used to solve mental mathematics problems differ from those usually referred to in a paper-and-pencil context. Butlen and Pézard (1992), for example, report that the practice of mental calculations can enable students to develop new and economical ways of solving that paper-and-pencil contexts rarely afford, because the latter are often focused on techniques that are too time-consuming for mental calculations. Others, like Threlfall (2002), have discussed the on-the-spot nature of mental calculations, where strategies are said to be developed and tailored for the problem at hand, and not pre-scripted in advance of solving. This is also discussed by Plunkett (1979), who refers to mental strategies as "*fleeting* and often difficult to catch hold of [as] *variable* [and] *active*" (p. 3, emphasis in original), which sensitizes one to the emerging production of strategies for *that* task, at *that* moment, in *that* context. Referring to Plunkett's work, Murphy (2004) explains that mental mathematics strategies are "seen as 'active' as they are created by the user to suit the numbers involved" (p. 4). Based on her review of curricular documents, Poirier (1990) adopts similar views. She posits that mental mathematics strategies have their own processes, that differ from regular written calculations seen mostly as the application of known procedures; hence raising the issue that mental mathematics strategies are of a particular nature, different from written processes that are the usual focus of paper-and-pencil contexts. This idea is extended in Reys and Nohda (1994), where it is

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asserted that mental calculations strategies are more than the mental application of algorithms, and thus need not necessarily be linked to paper-and-pencil strategies because each has its own processes. Mental calculations strategies are then discussed in terms of "higher-order thinking process[es] where the generation of a strategy is as important as the execution of the strategy" (p. 12).

On this basis, mental mathematics seems to engage in the production of specific ways of solving that are of a different nature than paper-and-pencil work. Understanding the particularity of mental mathematics solving processes appears important for gaining a better understanding of how mental mathematics fits into, and can contribute to, the actual mathematical education landscape, dominated mostly, if not entirely at times, by paper-and-pencil work: Are mental mathematics strategies simply an extension of paper-and-pencil work, or are they a different practice? In what ways of solving do solvers engage? What are their characteristics? These questions orient this Research Report, to investigate and conceptualize what doing mental mathematics implies, the kind of mathematics being produced within its activity and the nature of that mathematical activity itself. These issues are first addressed conceptually through using Nunez, Schliemann and Carraher's (1993) distinction between oral and written mathematics. They are then addressed empirically through an analysis of strategies taken from mental mathematics studies, offering illustrative and concrete examples for scrutinizing solvers' mathematical activity in mental mathematics. In that sense, this paper is *partly theoretical*, offering a perspective for conceptualizing about mental mathematics strategies, and *partly empirical*, using data from my own studies and the research literature to examine and make sense of the conceptual issues highlighted.

MATHEMATICAL ACTIVITY IN ORAL MATHEMATICS

As mentioned above, it is mostly the environment (no paper-and-pencil, time constraints, questions asked orally, etc.) that defines mental mathematics. Because of this focus, and its usual contrast with paper-and-pencil work in schools, one way of conceptualizing mental mathematics is to relate it to Nunes et al.'s (1993) distinction between oral and written mathematics. In their seminal work on street mathematics, Nunes et al. argue that street and school mathematics are supported and structured by different types of symbolic representations, respectively oral and written. Their investigation of both forms led them to associate street mathematics with oral mathematics, and school mathematics with written mathematics. Nunes et al. use these associations as one source of explanations for the major difference in performances and forms of activities undertaken by solvers on street-related and school-related tasks.

In addition to their difference in form (one oral, the other written), Nunes et al. show, through their scrutiny of each practice, how *meaning is preserved* in oral mathematics solving. For example, faced with the computation 200 – 35, one student answered orally: "*If it were thirty, then the result would be seventy. But it is thirty-five. So it's sixty-five; one hundred sixty-five*" (Subject L, p. 41). They contrast these kinds of oral practices with written practices present in schools, where for the same task one solver

underwent the same oral procedure after failed attempts at the school-prescribed algorithm. The following is taken directly from Nunes et al.:

> [The child writes 200 – 35 in vertical arrangement. Then he writes the result from units to tens to hundreds, computing out loud, and obtaining 200 in the following way.]
> R: Five, to get to zero, nothing. Three, to get to zero, nothing. Two take away nothing, two.
> Experimenter: Is it right?
> R: *No. So you buy something from me, and it costs thirty-five, you pay with a two-hundred-cruzeiro note and I give it back to you?*
> E: Do it again, then.
> [R. writes down 200 – 35 in the same way, writes the result from units to tens to hundreds, counting aloud and obtaining 235 as follows.]
> R: *Five, take away nothing, five. Three, take away zero, three. Two, take away nothing, two. Wrong again.*
> E: Why is it wrong again?
> R: *Now you buy something and its costs thirty-five. You give me two hundred and I give you two hundred, and thirty-five on top?*
> E: Do you know what the result is?
> R: *If it were to cost thirty, then I'd give you one seventy.*
> E: But it is thirty-five. Are you giving me a discount?
> R: *One hundred and sixty-five.* (p. 46, Subject R)

This leads Nunes et al. to assert that "Street mathematics is oral and preserves much of the meaning of the situations at hand. Mathematical practice in school is written and leaves out as much of the specifics of situations as possible in striving for generality." (p. 49). They explain how in written mathematics the solvers were often couched in a role reminiscent of a detective, attempting to decipher through the problem which rule to apply for solving. Written mathematics in this sense aligned with a mechanical activity, focused on applying school-prescribed routines to solve the right problems. And in the process, Nunes et al. show how meaning was left behind in the application of general procedures, which brought in many mistakes and much confusion for these solvers because they applied these procedures mechanically. In contrast, their oral practices were said to preserve meaning all through the computations. But there is even more: street vendors showed that in oral mathematics they not only preserved meaning, but *generated meaning* in the action of solving, as it unfolded. Oral mathematics practices were thus not scripted but enacted in the midst of solving. For example, they show how M., a 12-year-old in Grade-3, engaged with the following task:

> Customer: How much is one coconut?
> M.: *Thirty-five*.
> Customer: I'd like ten. How much is that?
> M.: [Pause] *Three will be one hundred and five; with three more, that will be two hundred and ten*. [Pause] *three hundred and fifteen…I think it is three hundred and fifty*. (pp. 18-19)

The real-time unfolding of the solving process makes oral mathematics strategies appear as meaning-making practices, where sense is made *through* solving as it unfolds. In contrast to their written mathematics, where solvers applied rules mechanically, in their oral mathematics practices meaning was produced through the process of solving *per se*. To insist on this dimension, Nunes et al. addressed the adapted nature of the oral strategies, tailored to the problem under scrutiny, as well as varying at times for the same task and even for the same solver. For example, some computation problems were often addressed from left to right (from hundreds to units) in oral mathematics, and these same numbers were at other times decomposed (57 became 52 and 5) or grouped (as in the coconut example), and even parts of numbers were left out (like Subject L, where one of the 100 of 200 is left aside to first subtract 35 from the other 100). These adaptations in oral strategies, and their variations from problem to problem, were not observed among solvers when they engaged in written strategies. When these solvers referred to school-based written procedures, it resulted in the application of a series of pre-decided steps to obtain the answer. For Nunes et al., written mathematics appeared related to conforming to a set of predefined rules. Oral mathematics, on the other hand, transcended this by being much freer and adaptive to the problem and the data at hand. Thus, in the oral practices, mathematics was engaged in as a process, as something produced in the action of solving, as a meaning-making endeavor. In contrast, for written mathematics strategies, mathematical processes were tamed down, reified as rules, as a tool one searches for and attempts to use and apply. Hence, one main distinction between oral and written mathematics that came out of Nunes et al.'s work is not only about preserving meaning in oral mathematics, but also about making meaning in the process of solving.

This analysis of Nunes et al. in relation to oral mathematics, as a meaning-preservation and meaning-making process, appears fruitful for investigating the nature of the mathematical activity with/in which solvers engage when solving mental mathematics tasks. Although it would be imprudent to generalize Nunes et al.'s results to any or all written work done in schools (as it mainly relates to their studies and its solvers), establishing parallels between oral and mental mathematics is of theoretical interest; if only, as mentioned, because of similarities in the environment into which solvers are plunged in mental mathematics contexts (time constraints, no paper-and-pencil or material aids, questions asked orally, etc.) and the usual contrast drawn with paper-and-pencil work in schools. This parallel is investigated in the following through the presentation and analysis of strategies taken from mental mathematics studies.

INVESTIGATING MENTAL MATHEMATICS SOLVING PROCESSES

The illustrations of students' ways of solving mental mathematics tasks offered here come from studies conducted in various classrooms. The classroom organization in these studies is quite simple: students sit at their desks, the researcher offers tasks orally and/or on the board and allows 15-20 seconds to solve them. Students are then asked to share and explain their strategies. As the analyses show, strategies students engage in are closely tied to the problem and its "information", where students focus on aspects

directly part of, and characteristic to, the problem. This leads to an analysis in terms of *meaning-preservation* and *meaning-making* in solving.

Meaning-preservation strategies

As Nunes et al. insisted, when enacting oral mathematics practices solvers stay close to the task at hand, offering ways of solving directly linked to and using the data of the problem. Subtracting 35 from 200 engaged in working with the 35, at times splitting it as 25 and 10, as 30 and 5, etc., to treat the 200 as a hundred plus a hundred or as the double of a hundred. Or again, in 252 – 57, where 57 is split into 52 and 5 to match the 52 of 252. These strategies contrasted with general algorithms used in written mathematics, where the numbers of the problem (be it a 117 or a 51) did not matter, and the method, the rule, or prescribed routine was central. The idea of staying close to the data of the problem is also strongly present in mental mathematics strategies. The following examples offer illustrations of how mental mathematics leads one to focus directly on aspects characteristic of the problem. In one mental mathematics study, Grade-10 students had to solve a variety of statistical tasks (e.g., central tendencies, graphical representations and interpretation, standard deviation; see Proulx, 2017, for details). For example, the following task was given to students:

> Here are four distributions. Which distributions have the same mean?
> (a) 41, 42, 44, 45; (b) 41, 42, 43, 44, 45; (c) 41, 42, 47; (d) 40, 42, 44, 45.

To solve this, students scrutinized the values in each distribution: some paid attention to the closeness of numbers in each distribution, others that values were mostly distributed "around" 43, others analyzed the distance between each value, and so forth. Here is a sample of the strategies engaged in:

> Strategy 1 – *Near values*. Some students explained that data in A and D distributions are very close, almost identical, except for 40 and 41. So, their means are very close. Other students asserted that C and D have the same mean. They explained that C has one fewer data than D, but that it is possible to make subtractions, for example 41-1, also to get 41 in D or to take out 47 to give back to some of the other data in D.
>
> Strategy 2 – *Analysis of data and compensating*. Some students asserted that B and C distributions have the same mean, because 47 in C is equivalent and compensates for "43, 44, 45" of B to "pull it toward" the same mean. Other students said that distributions A and C have the same mean, explaining that C has the biggest data of all distributions while also having the smallest number of data. And, that the total of distribution A is bigger but needs to be divided by a larger number.
>
> Strategy 3 – *Discarding the 40s*. Some students explained paying attention only to the number units in the distribution, because 40 could be taken out of each datum. They then added the units and divided by the number of data, obtaining the same mean for A and B.
>
> Strategy 4 – *The balance*. Some students mentioned that A and B have the same mean, which is 43. They explained that data in distribution A are ordered. And, that on one side there is 41 and 42 with one of difference in between, and that on the other side there is 44 and 45, also with a difference of one, these numbers being before and after 43. For B, it is the same thing, but 43 is also part of the distribution as the middle number.

These strategies are not simply about calculations, but about scrutinizing the entire data distributions: students stay close to the problem and its data, preserving its meaning as they solve. These strategies are an expression and arise out of their efforts to make sense of the problem, to consider the nature of its data. In these, students enter into a statistical endeavor focused on analyzing the data and inferring from them: they are about analyzing the entire distribution as well as considering it as a whole, about transforming the data and assessing the effect of these transformations, about establishing resemblance between different distributions, and so forth. The arithmetic means obtained are more than numbers resulting from a calculation: at times, it is an order of magnitude that represents the data in the distribution offering a view of its central tendency—without actually calculating it—and at other times it is an exact (or almost exact) number, not obtained by calculations but by reasoning through the actual data (compensating, subtracting, discarding information, etc.). In this sense, these are far from a purely arithmetic school exercise, as Gattuso (1997) critiques, where the arithmetic mean is often treated as a calculational labor of adding and dividing. The strategy about discarding the 40s is a good example of how students stayed close to the data in the problem, because this strategy would hardly have been proposed for a distribution without data in the 40s (e.g., 29, 37, 39, 54, 56). As well, the regular written algorithm, requiring one to add the values and divide by the number of data, does not distinguish between values like 29, 42 or even x in its calculation (it is indeed the strength of an algorithm to be that way, as Lockhart, 2017, explains). Not being a critique of the written algorithm, it illustrates how close strategies were to the data, and used them, preserving meaning like for oral mathematics strategies.

Meaning-making strategies

As with oral mathematics strategies, meaning is not only preserved in mental mathematics strategies, but also generated in the making. Time constraints and the absence of material aids force the solver in mental mathematics to find a way into the task, creating a need to grasp something, to draw out aspects of significance for solving the problem while solving it: it becomes as much a solving *process* as the *production* of the solving itself. There is a constant articulation between making sense of the problem and using this sense to continually pursue its solution. In another mental mathematics study, Grade-8 students were asked to solve for x a variety of algebraic equations of the form Ax+B=C, Ax+B=Cx+D, Ax/B=C/D (see Proulx et al., 2017). For example, for solving the equation $\frac{2}{5}x=\frac{1}{2}$ for x, students produced the following:

Transforming to equivalent forms. One student explained transforming ½ into 10/20, making the ½ divided by 2/5 simpler, then repeating it for 2/5, obtaining $\frac{20}{50}x=\frac{10}{20}$. This for him was equivalent to 0.4x=0.5, and thus the response became x=0.5/0.4.

Inversing and transforming. One student explained inversing the equation to $\frac{5}{2}x=2$. Then, transforming into decimals to 2.5x=2, and dividing by 2 gave 1.25x=1, so that x=1.25.

Cross multiplying. After transforming the equation to $\frac{2x}{5}=\frac{1}{2}$, one student explained having cross multiplied, where 5 times ½ gave $2x$= 2.5 and thus x=1.25.

Halving. One student explained that half of 5 is 2.5, but here it was 2 over 5 and not 2.5. Because he is looking for ½, then x had to be 1.25 so when multiplied by 2 one gets 2.5. Similarly, one student looked for the value of x that would make 2/5 equals to ½. Placing fractions over 10, he explained that x is 1.25 because 4 times 1.25 is 5 and 5/10=½.

Finding common denominator and adding. One student explained placing fractions over 10, obtaining $\frac{4}{10}x=\frac{5}{10}$. Subtracting 4/10 and 5/10 gave –1/10, so then x is worth 1/10.

These strategies are not necessarily adequate, but illustrate an emergent solving process geared toward making sense for finding a value for x satisfying the equation. These examples raise a diversity of entries for solving: about decimals, about inversing, about halving, and so forth. The "mental" dimension provokes the search for an entry point, a way of getting into the problem, of making a first step in it; far from the mechanical application of a procedure, it is solving in the making, meaning-making through the act of solving. As in oral mathematics, these strategies are responsive to themselves, being constituted along the way, one step leading to its consideration in the following one, and the next. The strategy unfolds in relation to itself, contingent on its own development, influenced and constituted by it. This unfolding and enmeshment of meaning is well illustrated in this example, taken from a study on mental calculations with preservice elementary teachers (Proulx et al., 2013). When asked to solve 741–75, Amy explained:

(a) 741 – 75 is like 700 – 75 + 41.
(b) 700 – 75 is like having 7 dollars and subtracting 3 quarters. I am left with $6.25.
(c) 6.25 is six-twenty-five. I add 41 to 625. 5+1 is 6, 4+2 is 6, and I have 600, so 666.

When 741–75 was given, Amy's first step was to find a way of entering the task, through decomposing it, leading her to decompose 741 in 700 and 41 in order to subtract 75 from 700. This decomposition produced in return a new step, that is, to solve 700–75, for which she referred to a monetary context (7 dollars minus 75 cents). This led to another step, 625+41, which Amy decomposed again for each digit in each number in relation to their position (hundreds, tens, units) and its successive addition. Each solving step led to or produced another, while producing meaning through it.

Hence, these mental mathematics strategies, like oral mathematics ones, stay close to the problem at hand, preserving its meaning along the way, but also generating meaning through solving as steps unfold. These strategies differ from predefined and general mechanisms to be used for any kind of number or equation, as they adapt to each new "step" developed along the way, and generate what Threllfall (2002) calls entire solution paths. Seen like this, mental mathematics strategies become, like oral mathematics ones, meaning-making endeavors.

FINAL REMARKS

Like Nunes et al.'s oral mathematics, mental mathematics strategies embody an aliveness that maintains mathematics as a dynamic activity filled with meaning. The concept of oral mathematics is here useful and invites one to see mental mathematics strategies as being far from a simple application of learned rules and procedures, and more toward meaning-making and meaning-preservation endeavours as solvers engage with the problem and the mathematical ideas. Mental mathematics, seen here as a form of oral mathematics, appears to distinguish itself from usual paper-and-pencil, written, mathematics. There is in fact a sense that the nature of the mathematical activity engaged with in mental mathematics creates a space of mathematical exploration that is more about engagement with the task to go forward, than about zeroing in on a single answer for the task to be done with. Solvers develop strategies to get answers, obviously, but all the while by giving and preserving meaning for the task at hand. The opening of this space of exploration, of meaning, engaged with in mental mathematics is one worth studying and paying more attention to, in order to better understand its potential contribution to the mathematical education landscape.

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SUBJECT MATTER KNOWLEDGE AND THE QUALITY OF MATHEMATICS MADE AVAILABLE TO LEARN: SOME HYPOTHESES

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We offer here some hypotheses about how teachers' subject-matter knowledge is implicated in instruction through the lens of mathematical discourse in instruction (MDI) framework (Adler & Ronda, 2015).

INTRODUCTION

It is a common view that teachers' knowledge of mathematics relates to the mathematics made available to learn in instruction. However, there are only a few studies that provide empirical evidence and explanation of how this knowledge is implicated in instruction. Hill, Blunk, Charalambous et al. (2008) investigated the relationship between mathematical knowledge for teaching (MKT) and the *mathematical quality of instruction* (MQI) seen through the various tasks of teaching. Their in-depth and extensive study of five telling cases, while confirming an overall positive correlation between the teachers' MKT and their MQI, illuminates the complexities of this relationship, and so the challenges in all-encompassing claims. More recently, and with a larger study, Munter & Correnti (2017) examined not only the relation of teachers' MKT to the *quality of their mathematics instruction* but also to teachers' vision of high quality instruction and doing this over time using a tool for examining the presence of ambitious mathematics teaching. Their results illuminate that "teachers' MKT was related to their "current" instructional practice but not to its growth; rather, growth in instructional quality was linked to teachers' vision of quality instruction.

The study we report here aims to contribute towards this emerging field. However, it differs in at least four aspects from these earlier works. First, our 'measure' of teachers' mathematical knowledge does not include pedagogical content knowledge (PCK) which is part of MKT. Our focus is teachers' subject-matter knowledge (SMK). Second, our study is located in a context where traditional or direct teaching is predominant. Third, our study involves a particular and narrowly focused domain of mathematics. We focus on teachers' algebra knowledge and even more specifically to knowledge that can be classified as transformational activities of algebra (Kieran, 2004) as this is the dominant type of activities in our algebra classes. Fourth, possibly a result of our mathematical focus, our study differs in terms of our descriptions of mathematical quality and what we consider as mathematical in instruction. Our description of mathematical quality of instruction is a function of our view of mathematics as a form of scientific knowledge (Vygotsky, 1978) and we contend that mediating this is the critical task of the teacher. Our tool for determining the

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mathematical quality in instruction is the mathematical discourse in instruction (MDI) analytic framework (Adler & Ronda, 2015) which we developed by identifying culturally established instructional tools in the contexts in which we work through which mathematics is mediated, together with those features of mathematical discourse that each tool makes available for learners' engagement. This framework which we present below enables description and evaluation of observable aspects mathematics in and through various mediational tools of instruction.

The level of specificity of the study in terms of its content focus and the dominance of direct teaching is a double edged sword: on the one hand it limits the study's generality particularly in content terms. On the other, it affords a sharpened illustration of a case of how teachers' SMK plays out during instruction. We thus set out to answer the question: *How, if at all, is teachers' subject matter knowledge in algebra related to the algebra they make available to learn in their lessons?*

Transformational algebraic activities

School algebra consists of three interrelated activities: *generational*, *transformational* and *global/meta-level* activities (Kieran, 2004). While these are interrelated activities, each mediates particular aspects of algebra. Each may thus require different mediational means and processes.

Much of the initial-meaning making for school algebra lies in generational activities (Kieran, 2004). The key algebraic object here is the variable and the various relationships between quantities variables represent. School algebra also involves processes where algebra is used as a tool. Global-meta level activities includes problem solving, modelling, studying change, noticing structures, noticing and generalizing relationships, justifying, proving, etc. The multiplicity of processes and ideas that need to be coordinated in global-meta level activities, together with the generational or transformational activities embedded in them, may be cognitively demanding both for the teachers and her learners, but one that is considered to afford more powerful mathematical experiences for learners.

Transformational activities involve various types of algebraic actions or procedures on algebraic objects. These are often referred to as 'rule-based' activities. We assert here that this does not imply mindless learning of procedures and techniques. Transformational activity has its own meaning-making power and this lies in the development of the notion of *equivalence* (Kieran, 2004). Moreover, it provides the general context for applying arithmetical structures e.g., the distributive property or for generalizing operations e.g. subtraction as addition. However, without deliberate foregrounding of the fundamental concept of equivalence beyond numbers to operations and expression; and without deliberate attention to noticing structures and to making explicit the mathematical principles that legitimize the transformational processes, transformational activity can become an exercise for memorizing unrelated techniques. In contexts where transformational activities are dominant, it is thus important to examine whether there is such deliberate attention that leads instead to the

development of these notions and an appreciation of general and coherent nature of mathematics, and whether and how this relates to teachers' SMK.

METHOD

Data was sourced from teacher participants in a 20-day subject-matter focussed professional development course spread over one year for Grade 8-10 teachers in disadvantaged schools in the Johannesburg area in South Africa. Out of thirteen teachers who completed the course, seven were teaching transformational activities of algebra when we collected video data at the end of the course. These teachers, the video records and the end of course test results formed our data archive.

Test items for measuring SMK. The end of course test assessed what the teachers learned from the SMK-focused PD the main goal of which was to strengthen teachers' relationship with mathematics, particularly algebra. There were 24 algebra items in all, half of which were purely on transformational tasks, four generational and global meta level tasks where transformational tasks were implicit and eight were on interpreting the properties of function in symbolic form and graphical form. All these items assessed fluency with algebraic expressions in various ways. The seven teachers' results on these test items made up our SMK data.

The video lessons. Our second set of data was the teachers' lessons collected sometime after they took the test. These were video recorded and transcribed. Each transcript was then carefully checked by one of us closely examining and refining the transcript as we watched the video again, and again. Each author individually analysed the lessons at first using the MDI analytic tool and then we compared and discussed our codes. Towards the final round of coding we invited another colleague to do an analysis of the lessons for further confirmation. It is this process of successive video watching, transcript development and analysis that improves the rigour of the analysis.

Data production – constructing mathematical episodes as unit of analysis. A critical task for any lesson analysis is how to divide the lesson into analytic units. Our unit of analysis is a *mathematical episode* so named as our goal is to describe the mathematical 'story' of the lesson as it unfolds. We began by watching the video-recording and simultaneously (re)reading the transcript to identify the intended object of learning that we know is not synonymous with what is enacted (Marton & Tsui, 2004). We then divided the lesson transcript into *mathematical episodes* each identified by a shift in focus of attention with respect to content, typically marked by a new task.

Our Analytic tool. The tool we used to analyse and evaluate the quality of mathematics made available to learn in the lessons is the mathematics discourse in instruction (MDI) framework (Adler & Ronda, 2015). The starting point of the framework is that "teaching and so learning is always about 'something' and bringing this into focus – its mediation - is the teacher's work" (Adler & Ronda, 2015, p. 238). This "something", is the *object of learning* (Marton and Tsui, 2004). For the object of learning to be visible and be in focus during instruction, it needs to be exemplified and experienced by leaners. The most common way this is done is through *tasks*. Embedded in tasks is the

mathematical object which is made visible through its *examples*. However, what is being mediated also needs to be *named*, elaborated and explained necessitating use of words and notations to communicate and convey meaning. In addition, what counts as mathematics is also communicated implicitly or explicitly during the lesson reflecting the authority from which mathematical procedures and claims are *legitimated*. In the MDI framework, we claim that these cultural tools of instruction - tasks, examples, naming/word use and legitimations – are means for mediating various aspects of the object of learning and it is in and through these tools that we analyse what is made available to learn. The framework has both descriptive/analytic and evaluative features for each of these mediational means which we present below.

Tasks. The most common and visible mediational means teachers use to engage learners in mathematics is the *task*. A mathematical task has at least two basic components: the object and the action on the object. The MDI framework classified tasks in terms of its cognitive demand. In one of the lessons we analysed here (Teacher 6) the tasks in the main part of lesson were presented and solved one-by-one:

(1) $5x+5y-3x+2y$; (2) Add $(8x^2+4x-6)$ and $(-3x^2+6x+4)$; (3) Subtract $(7x^2+4)$ from $(15x^2-2)$; (4) Subtract $(-3x^2+6x+4)$ from $(8x^2+4x-6)$.

The objects in focus are algebraic expressions and the action required is to add/subtract these. Task (1) involves known facts and procedures because combining similar terms comes before the topic on adding and subtracting algebraic expressions. Task (2) is an applying task because learners are expected to use their knowledge on combining similar terms to add the two multi-term expressions. The same can be said of Task (3) and Task (4) which are also applying tasks. In the MDI analytic framework, Task (1) is coded *R* for recall of facts and procedures and Task (2) is coded *A* for application. Of course, Task (2) may be experienced differently, for example, when the task is stated as *Write 2 algebraic expressions one of which has a negative leading coefficient that give a sum of* $5x^2+10x-2$. This version of the second task embeds the transformational activity within a generational activity and for some learners this could be a problem to solve. This latter version is an example of a task that we code as *C/PS* (connections/problem solving). In this task type, learners can generate many correct answers and potentially encourage them to consider a more systematic and faster way of getting the answers. In this case they would need to see the underlying structure of each term and of the expression itself.

Examples. Examples for us are instantiations of the mathematical object(s) or the object of learning. MDI is interested in two dimensions of example use. First is how the set of examples (and with it the choice of representation) can make visible features that are key to the object of learning. Second is how the set of examples build towards generality. In our analysis of examples, we draw on different patterns of variation proposed in variation theory (Marton & Tsui, 2004). We claim that each pattern of variation and combinations of them within a lesson offers different degrees for experiencing generality of the object that is exemplified. A pattern called similarity

(S) enables local generalization. Similarity occurs when one dimension of the object (its examples) is varied and the rest invariant. If a set of examples brings attention to difference or *contrast* (C), opportunities are made available to recognise boundaries between classes of examples, and so further opportunities to generalise. When examples are *fused* (F), with simultaneous variance/invariance across an example set or through several example sets, opportunities for generalisation is further enhanced. Fusion does not refer to simply presenting examples at random but selecting and sequencing them to make visible the different dimensions of the object that are being varied. In Teacher 6's lesson for instance, what is being exemplified are the various forms in which learners encounter adding and subtracting expressions and the forms of expressions itself. That is, the set of adding and subtracting tasks is the exemplification itself of the object of learning and so we analyse these 'objects' accordingly. Examples 1 and 2 (that is tasks (1) and (2)) differ in one feature: the context for adding expressions. In example 1, adding is in the context of simplifying one expression and so involves looking at individual terms. In example 2, adding is in the context of two algebraic expressions. Note that the expressions in example 2 were bracketed to enable learners to see these as two expressions. Examples 3 and 4 are subtraction tasks with the same structure but the sign of the first terms in the expressions to be subtracted differs. Examples 2 and 4 involve similar expressions but differ in operation. Taken together, the juxtaposition of addition and subtraction operations highlights the contrast between operations. The examples are also sequenced so that there is only one dimension varying at a time. The example set in this lesson is thus coded F for fusion.

Naming. Words or names are signifiers. Each carries a potential meaning and hence its importance in learning and knowledge development. As we stated earlier, what is being mediated needs to be the subject of the talk, and elaborated, examined and explained, necessitating the use of words and notations to communicate and convey meaning. Thus, *what* is named (or not) and *how*, the choice of name to refer to other words, symbols, images, procedures or relationships in the tasks and during elaboration, matters. Naming may involve a range of word use from colloquial nonmathematical words to use of mathematical words as names. We use the following codes to analyse the word use in the lesson. We also exemplify it from some texts in T6 lesson:

1) *Colloquial* (C*) e.g.* everyday language and/or ambiguous pronouns such as this, that, this, to refer to objects in focus; 2) Ms for *math words used as name or labels only* e.g. in reading string of symbols or in referring to the surface feature of the object such as "*constant* is the number beside x"*;* and, 3) Ma for *mathematical words* used in talks about objects, procedures, properties (Ma) e.g *constant is the numerical coefficient of x* or *constant is the number factor in an algebraic expression*.

Legitimations. Part of teaching mathematics is about communicating criteria about what counts as mathematics. Thus, the authority from which the mathematical procedures and claims are *legitimated*, matters. Legitimations ranges from nonmathematical criteria and reliance to teacher authority to use of specific examples and use of mathematical principles. Legitimating criteria may include 1) *Non*

mathematical e.g. cues are iconic or mnemonic; *a* statement or assertion, typically by the teacher, as if 'fact' or 2) *Mathematical criteria: which may be Local* e.g. a specific or single case (real-life or math) or established shortcut or convention or *General* such as definition, previously established generalization; principles, structures, properties. We identify them when teacher ask why questions or prompt learners with 'because?' or similar prompts.

Having elaborated the elements of the framework and their descriptive coding, we now present the evaluative feature of the framework in Table 1 below. We used this for the summative judgment of the lesson – our 'measure' of mathematical quality in instruction. In the table, the subscripts from 1-4 indicate progression in quality towards the privileged scientific knowledge.

Object of Learning (OoL)			
Tasks	***Example set***	***Naming***	***Legitimations***
T1–only include recall (R) type tasks **T2** – at least includes an application (A) tasks in the main lesson **T3** – A and includes at least a problem solving or task requiring multiple connections (PS/C)	**E1** - example space provides contrast (C) or similarity (S) **E2** - example spaces provide both C *and* S or two example spaces involving S **E3** – example space shows all fusion	**N1** – talk consist mostly of colloquial (C) words; if there are mathematical words (Ms), it is used as label. **N2** - movement between C and Ms but mostly Ms; some MA **N3** – movement between C, Ms and Ma but mostly Ma words	**L0** – no justification provided or if there is it is nonmathematical **L1** – most claims made are legitimated using a single example (Local) **L2** - criteria extend beyond NM and local and attempt to include generality, but this is partial **L3** - General full math legitimation of a concept or procedure, is principled and/or derived/proved

Table 1: MDI indicators of the quality of mathematics made available to learn.

RESULTS

Having illustrated our coding, and how these accumulate across episodes into levels, we can now present the overall results of our study for all seven teachers (Table 2).

Teachers	**Object of Learning**	**SMK (%)**	**MDI =** (Examples, Tasks, Naming, Legitimating)
T1	Evaluating algebraic expressions	23	(E1, T1, N1, L0)
T2	Factorizing Monomials and binomials	39	(E1*, T2->T1, N1, L0)
T3	Multiplying two binomials	57	(E1, T2, N2, L0)

T4	Solving linear and quadratic equations	66	(E3, T2->T1, N2, L1)
T5	Solving quadratic equation	78	(E3, T2->T1, N3, L3)
T6	Adding and subtracting algebraic expressions	78	(E3, T2->T1, N2, L3)
T7	Simplifying/Dividing Algebraic expressions	83	(E3, T2->T1, N2, L2)

E1* - Correct with the example set that can only be coded E1.

Table 2: Teachers' SMK and what is made available to learn in instruction (MDI).

The results in Table 2 indicate that the teachers' SMK is related to quality of mathematics they make available to learn – to some extent – that is only through use of examples and legitimating criteria. The same cannot be said as strongly for the kind of tasks set forth for learners to engage nor in teachers' naming. The teachers that scored below 60% in the test used examples that varied in one dimension only while those who scored above 60% use examples that varied on two or more dimensions. In addition, teachers with scores below 60% did not provide legitimations that were mathematically authorised. In contrast, those with scores above 60% provided mathematical legitimations ranging from giving a specific example to drawing on mathematical principles that were either complete or partially complete.

DISCUSSION

The first hypothesis that we can offer from our focused study is that teachers' subject matter knowledge does matter in how they choose the features of examples critical to the object of learning and that could then offer opportunities for generalizing. Selecting and sequencing of examples is not trivial. The teacher needs to be able to identify those critical aspects of the object of learning that need to be foregrounded. Our second hypothesis is that the domain of knowledge that teachers draw from to substantiate mathematical claims made during the lesson (legitimations) also has strong links to teachers' knowledge of mathematics. It appears appreciating the need to mathematically substantiate the claims made in classroom talk is linked to stronger SMK. Our third hypothesis relates to word use/naming. Teachers' SMK seems related to the quality of mathematical talk in the lesson although we cannot state this hypothesis as strongly as the other first two above. The lessons with predominance of colloquial names were by the two teachers with the weakest SMK. The other teachers used mathematical names in the lesson but this mostly entailed reading of strings of symbols. We can think of two explanations for this. First is a weakness in our coding – that it is not sufficiently differentiated to make visible more nuanced differences across T3-T7. Also, our N2 level covers a wide spectrum of *values* between colloquial and more formal talk and would benefit from further categorization or disaggregation. We have pursued this in our analysis of textbook lessons (Ronda & Adler, 2017) where we differentiated between how objects and procedures are named. A second explanation may be because the lessons were on transformational activities of algebra where predominance of the reading of symbols in the talk is typical. It thus would be interesting to study if the same result will be obtained if the object of learning of the

lesson were more on generational or meta-level activities. Our last hypothesis relates to task demand. Tasks that involve problem solving and multiple connections which are features of reform-oriented teaching did not figure in any of the lessons we investigated. Is it possible that this has to do with the rule-based nature of transformational activities or to the predominant traditional teaching of mathematics from which we draw our lessons? Has it to do with teachers having limited exposure to *rich* mathematical tasks or has it to do with teachers vision of quality mathematics teaching (Munter & Correnti, 2017). Is it that teachers position learners as unable to engage with complex processes typical of meta-level algebraic activities? These questions merit further study, particularly whether the hypotheses we presented will hold for algebra lessons that involve generational and/or meta-level activities.

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USING GRUNDVORSTELLUNGEN AND CONCEPT IMAGE THEORY FOR ANALYSING AND DISCUSSING STUDENTS' CHALLENGES IN THE TRANSITION FROM SCHOOL TO UNIVERSITY – THE EXAMPLE OF EXTREME POINT

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This study is concerned with the conceptual understanding of mathematical notions taught at school and extended at university. We therefore concentrate on the secondary-tertiary transition that falls - in Germany - together with the transition from calculus at school to analysis in the first semester at university. In particular we focus on the concept of extreme point. To analyse and interpret possible difficulties that learners face in relation to these transitions, we draw on two theoretical planes: a normative plane that consists of mathematically correct facts and conceptions and a descriptive plane that contains the students' actual thoughts. For this we use the approach of concept image as well as Aspects and Grundvorstellungen (GVs). Out of their comparison we try to get insight into difficulties and their possible reasons.

INTRODUCTION

Many studies have shown that the transition from school to university in mathematics brings along a variety of difficulties for students. Gueudet (2008), for example, gives a broad overview of these studies and categorizes them into three main perspectives that go from individual behaviours, over socio-cultural perspectives to an institutional viewpoint. The approach we present is related to Gueudet's first category. Within this category, the high complexity of the mathematical contents at university, individual thinking modes, and knowledge organization play a prominent role. We are particularly interested in the subject of analysis and want to analyse university students' conceptual understanding within the transition to higher mathematics in order to determine challenges in the learning process. Therefore, we focus on mathematical objects already discussed at school but formally introduced at university. One of these concepts in analysis is the mathematical concept of extreme point, which we define here as follows:

> Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ a real valued function. The value of f in $x_0 \in I$ is called a *local minimum* if there exists a neighbourhood U of x_0 such that $f(x_0) \leq f(x)$ for all $x \in I \cap U$. Similarly, the value of f in $x_0 \in I$ is called a *local maximum* if $f(x_0) \geq f(x)$ for all $x \in I \cap U$. In both cases the point $(x_0, f(x_0))$ is called a *local extreme point* of f.

THEORETICAL BACKGROUND

Traditionally, studies that are focusing on the conceptual understanding, especially in higher mathematics, have been based on the concept image/concept definition approach (Tall & Vinner, 1981). This theory distinguishes between the *formal concept*

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definition that are the words used to define a concept accepted by the mathematical community, and the *concept image* that is described as follows:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes (Tall & Vinner, 1981, p. 152).

A closer look reveals that the formal concept definition is hereby used as a normative orientation, while the concept image is a construct that helps to specify the students' ideas by offering both a model and language to describe the learners' thoughts.

An alternative approach developed within the German tradition of subject matter didactics, are Grundvorstellungen (GVs) (cf. Straesser, 2014). The notion *Grundvorstellung* could be translated as "basic idea" or "basic mental model". It has its roots in the ideas of Pestalozzi at the beginning of the 19th century, however, it gained importance in the second half of the last century with its main advocate vom Hofe (e.g. 1995). He characterizes GVs as having three main features: 1) giving meaning to a concept by connecting it with already known facts or experiences, 2) bringing along the development of corresponding representations and 3) having the ability to use the concept in a problem. In this study we use the construct of GVs in a normative way, that is, a way to demonstrate a guidance of how learners' conceptions could ideally look like. Lately Greefrath et al. (2016) introduced the notion of *Aspects*, a construct that is built on a close connection to GVs. From their perspective "A Grundvorstellung of a mathematical concept is a conceptual interpretation that gives meaning to it" (Greefrath et al., 2016, p. 101), whereas "An Aspect of a mathematical concept is a subdomain of the concept that can be used to characterize it on the basis of mathematical content" (Greefrath et al., 2016, p. 101). An Aspect could either be a definition, theorems related to the concept, properties or connections to other concepts. Thus, Aspects could be seen as a supplement to GVs in that this notion can be used to structure the normative plane in a more detailed way. Against the background of transition we differentiate between *general* and *partial* GVs: A *partial* GV is an idea that gives meaning to a concept in a limited context. It is not generally valid but loses its scope of application through restrictive premises. A *general* GV, however, is valid in the whole context. Regarding this distinction it is obvious that the context plays an integral role for the classification as general or partial GV. For example, the idea "multiplication makes bigger" is a general GV within the set of natural numbers (bigger than 1), but a partial GV in the real numbers.

Both theories, concept image and GV, have different advantages when it comes to the analysis of conceptual understanding. While the concept image theory delivers a strong model and language to analyse the descriptive plane (actual thoughts of the students), it does not offer many possibilities to structure the normative plane, just the formal concept definition, which we think is for our purposes not enough. As, however, a comparison of the descriptive plane and the normative plane and the resulting differences could give integral insights in the learners' conceptual understanding and their possible difficulties, we conclude that the normative plane should be structured

with the aid of Aspects and GVs. Especially during the transition from school to university we want to explore the potential of using Aspects and GVs normatively to gain insight into concepts students bring from school into mathematics studies at university. Out of this motivation the following research questions arise.

RESEARCH FOCUS

Q1: Which specific GVs and Aspects characterize the concept of extreme point?

While this question aims at a construction of a normative plane, the objective of the second question is to structure the descriptive plane.

Q2: Which concept images do students have regarding the concept of extreme point?

To get a deeper understanding of these images and to find possible reasons for learners' difficulties, a comparison between both planes seems helpful.

Q3: Which differences exist between the normative and descriptive plane and what could be possible reasons for these differences?

METHODOLOGY

Our response to Q1 is based on normative considerations. We attend to Q2 through an empirical study. Finally, we address Q3 via a comparison between the responses of Q1 and Q2 and refer to the theories explained before. Consistent with the demands of Q2 we went into the last tutoring exercise of the Analysis I course at our university and handed out a questionnaire to eighty-nine students. The students were either in their first year, if they were mathematics undergraduates, or in their second year, if they were enrolled in a degree for becoming a mathematics teacher at high school. Thus, all participants studied mathematics at mathematics faculties. The questionnaire consisted of 5 items (true/false) about extreme points that were designed by a mathematics professor. The format followed two-tier-tests that are used in science education to analyse learners' misconceptions (e.g. Chou, Chan & Wu, 2007; Svandova, 2014). In first tier, students had to decide if they think a given statement is true or false. For the second tier, students were asked to explain their decision (drawings were also supported). The two-tier-test helped us in analysing the given solutions in a structured way, as the chosen answer possibilities showed first patterns (like the person thinks the statement is false but gives a wrong explanation). An illustrative short example of one of these items is presented below:

> **Item 1**: If the function $f: [a, b] \to \mathbb{R}$ is continuous on (a, b), then f takes its extreme values on $[a, b]$.

The questionnaire was completed during the regular exercise slot. Fifty-nine students provided their email addresses as a sign that they are willing to take part in further interviews. Out of the fifty-nine questionnaires we invited twenty-five students, based on the analysis of their written solutions, to a follow-up interview. We chose the participants to cover the whole variety of difficulties shown in the written explanations of the questionnaire. We were especially interested in gaining more information to

responses we found interesting or unclear. Between February and April 2016 ten interviews were conducted and transcribed. During interviews the students were presented again their questionnaire and had to explain their given answers in more detail. Furthermore, the interview protocol contained questions based on graphical representations that were collected from several written solutions of the students' questionnaires. These graphs as well as the correct answer were given to the interviewee masked as an answer of another student and the participants should take a stand to them. Based on the transcripts categories of concept images concerning the concept of extreme point as well as possible reasons for difficulties were build following Mayring's qualitative content analysis (Mayring, 2010) by using the program MAXQDA.

RESULTS

With regard to the research questions we will first present the normative plane (Q1). Out of a subject matter analysis and discussions with lecturers, three Aspects and four GVs of the concept of extreme point could be determined.

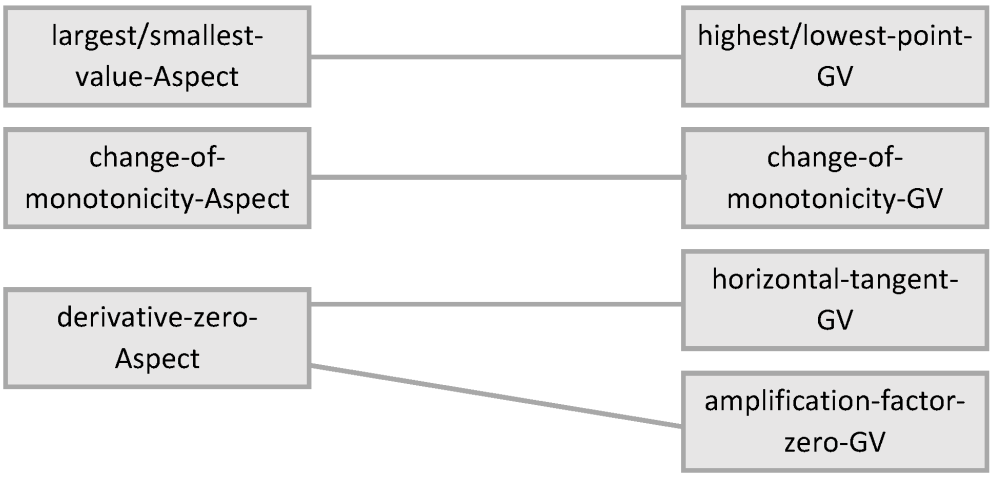


Fig. 1: Normative plane of the concept of extreme point

Here, because of space restrictions, we will concentrate on the three Aspects:

<u>largest/smallest-value-Aspect</u>

Extreme points are points (x,y) with the largest/smallest y-value with respect to a neighbourhood of x.

This Aspect refers directly to the definition of an extreme point. It illustrates the connection of an extreme point (of a graph of a function) with the maximum/minimum of a (totally ordered) set: extreme points are the points, whose *y*-values coincide with the maximum/minimum of the range (on a given domain).

<u>derivative-zero-Aspect</u>

Extreme points can be located at points where the derivative becomes zero.

In this connection, however, one has to be careful and take into account two important factors. First, there are points, which are not extreme points, although the derivative becomes zero at these points (saddle points). Second, since this Aspect refers to the necessary condition for the existence of (local) extrema, one has to impose two additional premises: the point must be an interior point of the domain, and differentiability of the function is required. By dropping one of these premises, one can easily find examples of extreme points where the derivative is not zero (e.g. boundary points of an interval). Consequently, this Aspect leads to partial GVs.

<u>change-of-monotonicity-Aspect</u>

Extreme points are located at points where the sense of monotonicity of a function changes, i.e., from increasing to decreasing or vice versa.

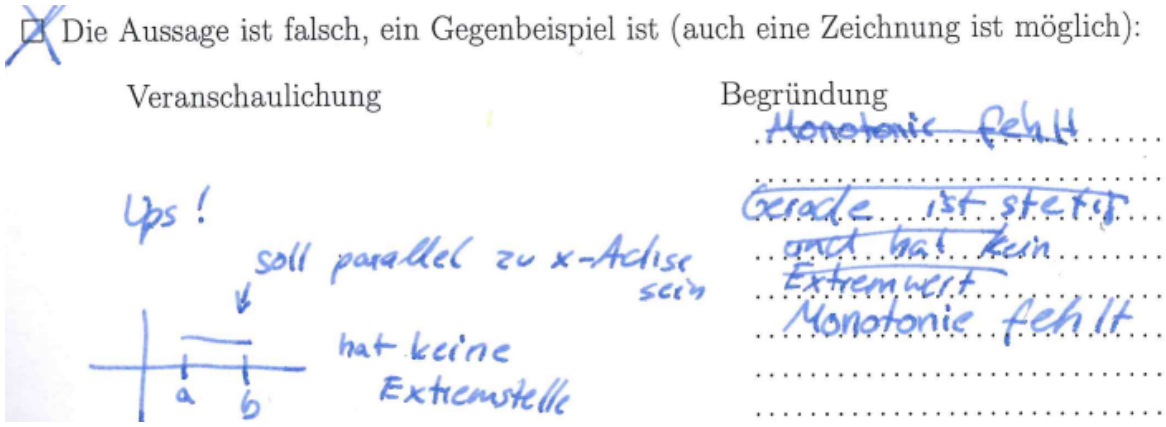
It is, in fact, easy to see that a change of monotonicity always implies the existence of an extreme point. The converse, however, is false: not every extreme point induces a change of monotonicity (consider e.g. constant functions). Hence also this Aspect leads to partial GVs.

While these Aspects represent the mathematical basis of the concept of extreme point the associated GVs are strongly connected to graphical representations, like a horizontal tangent with respect to the derivative-zero-Aspect (for further details see Roos, 2017). Taking the above structuring into account we demonstrate now some of the discovered concept images (Q2) and compare them with the described Aspects of the normative plane to get insight into possible reasons for the students' difficulties with extreme points (Q3).

<u>Category 1: Constant functions</u>

Difficulties in this category include ideas such as constant functions do not have extreme values. This category has been previously identified by Fischer and Malle (1985, p. 148-150) who discuss the challenges faced by teachers during professional learning with constant parts of functions. Constant functions are often seen as non-prototypical cases for extreme points. However, mathematics at university level should concentrate on a precise and overall understanding of the formal concept definition that results in adequate concept images of the learners, including different varieties of examples. An example of this category was provided by the student Ali (3rd semester) via a counterexample to Item 1:

If the function $\boldsymbol{f}: [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is continuous on $(\boldsymbol{a}, \boldsymbol{b})$, then $\boldsymbol{f}$ takes its extreme values on $[\boldsymbol{a}, \boldsymbol{b}]$.



English translation: This statement is false. A counterexample is given by the drawn function (constant function). The function has no extreme point - monotonicity is missing.

Fig. 2: Extract of Ali's questionnaire

During the interview we wanted to know more about his idea of "missing monotonicity". When asked to further explain his answer, Ali declares:

> Ali: …A straight line with no slope. Thus there is nowhere an extreme point. Because it has no ups and downs. It is always flat. Always the same.

And when asked about the definition of an extreme point Ali answers:

> Ali: And I think, that my teacher in Grade 11 said 'A minimum that decreases and increases' and this is still in my mind.

The example of Ali shows that a concept image focussing only on the change-of-monotonicity-Aspect could be an obstacle for the acceptance of extreme values regarding constant functions, hindering rather than helping the students. This is because the Aspect leads to only a partial GV in the sense that if there is a change of monotonicity there is an extreme value. This is in contrast to Ali's argument that if there is no change of monotonicity that there could not be an extreme value.

Ali explains in his interview that his part of the concept image that connects the change-of-monotonicity-Aspect with extreme points is based on experiences from school.

Category 2: Borders of an interval

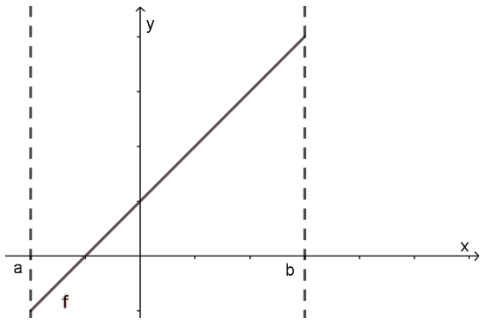


Fig. 3: Graphical representation in the interview

When Svenja (1st semester) is given the graph of Fig. 3 in the context of the discussion of Item 1 she explains:

Svenja: f', the first derivative, is not zero, neither in a nor in b. Insofar it shouldn't be an extreme point.

She activates here part of her concept image that is close to the derivative-zero-Aspect. However, in this situation it is not valid, as a and b are no inner points.

The strong connection of extreme points with the derivative could have its roots in experiences at school. There the abscissas of extreme points are mainly found by the same process: Find the zeros of the first derivative.

The two examples show that an overgeneralization of Aspects that lead to partial GVs, that is that are only usable under certain premises, could entail difficulties for students. In the context of transition, an analysis of German school books used in the region of our study showed, however, that with regard to the concept of extreme point at school a concentration on both the change-of-monotonicity-Aspect and the derivative-zero-Aspect predominates tasks and explanations in comparison to the largest/smallest-value-Aspect.

DISCUSSION AND CONCLUSION

After constructing a normative plane and a descriptive plane with regard to the concept of extreme point, a comparison between these perspectives on students' responses to specific analysis based tasks provided insights into learners' difficulties: The usage of Aspects that refer to partial GVs could promote false overgeneralizations, i.e. premises were not taken into account. Furthermore, the fact that at school mainly differentiable functions were analysed had a particular impact on the students' concept images, what categories like "differentiability is a premise for extreme points", suggest. We conclude for our case that it could be meaningful to emphasize a discussion about the validity of the change-of-monotonicity-Aspect and the derivative-zero-Aspect at school. But also for the teaching at university we see some implications. It is important to build on the conceptions students bring from school and to enlarge them. Neither is it helpful to treat knowledge from school as completely false, nor to treat university mathematics just as addition to already fixed acquired knowledge from school. The idea of reconstructing conceptions surely plays a prominent role especially in the field of real functions that are an important topic in school as well as in university mathematics. While "pathological cases" (extreme points at constant functions, extreme points at boundaries of an interval) have been (mostly) excluded at school, they must find their place in the learners' concept images at university. At large, students' concept images

seen in our study indicate the huge impact experiences from school even after attending mathematics courses at university.

In general, the use of GVs and Aspects to structure a normative plane and the use of concept image for the descriptive plane has been a fruitful approach to analyse the conceptual understanding of the participants. Their combination offered the opportunity to structure both planes in more detail than with just utilizing one of them. Especially in the context of transition, our introduced construct of partial GV has proven itself helpful. Some GVs that were valid for the functions discussed at school (mainly differentiable functions on $\mathbb{R}$) led to errors respectively of overgeneralizations when working at university. Our construct of partial GV helped to describe the challenges students have when using "old" conceptions attained in new and broader contexts.

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INSIDE TEACHER TENSIONS: EXAMINING THEIR CONNECTION TO EMOTIONS, MOTIVES, AND GOALS

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This paper examines tensions faced by mathematics teachers and their effect on teachers' actions using constructs from activity theory. Findings suggest that emotionally laden tensions can reveal motives, and impact teachers' goals by altering, prioritizing, or strengthening them. Therefore, in the relationship between emotions, motives and goals, tensions can be understood as drivers of teachers' actions.

INTRODUCTION

Reflective conversations with mathematics teachers invariably contain what I have come to call 'I…, buts':

> I want to implement problem solving in my classroom, but I don't know how.
>
> I know collaboration is a good thing, but some of my students work better alone.

For those working in professional development, these conflicts are familiar refrains. Some teachers want to make changes, but do not know how. Others are trying to make change but are encountering challenges. Conflicts such as these are endemic to teaching and are commonly referred to as tensions (e.g., Berry, 2007; Carr, 1998; Jaworski, 1999; Mason, 1988). Studies in mathematics education have produced lists of tensions that impact mathematics teaching (e.g., Liljedahl, Andrà, Di Martino, & Rouleau, 2015; Mason, 1988; Page & Clark, 2010) however there has been less focus on their mechanisms of action. Liljedahl et al. (2015) argue that "better understanding of these tensions would allow us, as mathematics education researchers, to better understand the intentions and actions of mathematics teachers — and to better respond to their needs in the crafting and delivery of professional development opportunities" (p. 200). My aim with this study is to further develop that "better understanding" of tensions. I move beyond the identification of tensions to examine more closely how it is tensions act, and how they are acted upon, by using constructs based in activity theory.

THEORETICAL UNDERPINNING

One of the findings from Liljedahl et al. (2015) was that tensions are tied to teachers' needs. For this reason, I turn to Leont'ev (2009) to understand the complex relationship between tensions and needs. Consider again, for example, the opening refrain:

> I want to implement problem solving in my classroom, but I don't know how.

We see that underlying the tension (I don't know how) that causes the teacher to actively seek professional development is an unfulfilled goal (implementing problem solving) that is impeding an unspoken need (e.g., being seen as a good teacher perhaps

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or improving student learning). This is the essence of Leont'ev's (1974) notion of activity: "Behind the object, there always stands a need or a desire, to which [the activity] always answers" (p. 22). He adds that a motive is an object that meets a certain need, and that generally, motives go unperceived by the subject:

> The paradox lies in that motives are revealed to consciousness only objectively by means of analysis of activity and its dynamics. Subjectively, they appear only in their oblique expression, in the form of experiencing wishes, desires, or striving toward a goal. (Leont'ev, 2009, p. 171)

Goals, however, are conscious; we are typically aware of what it is we are striving to achieve and can pinpoint our aims. If visualizing activity as a hierarchy, we would see unconscious motives/needs driving the activity, with the activity being directed at the conscious goals and their related actions (see Figure 1). For Leont'ev, activities are composed of actions, which are, in turn, composed of operations. These three levels correspond, respectively, to the motive, goals, and conditions. As indicated by the bi-directional arrows, all levels can move both up and down, e.g., goals can become motives, actions can become operations.

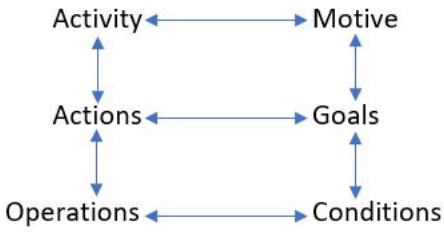

Figure 1. The activity hierarchy of Leont'ev (2009).

To understand the dynamic nature of the components within the activity hierarchy, Leont'ev (1974) described a situation in which the activity is learning to drive a car with a manual transmission. When first learning, shifting gears is a conscious action with the goal of smooth coordination of the clutch and gear-stick. Later, the action of shifting gears becomes operational as the learner no longer has to think "How do I move my hand or foot?"; these are now unconscious operations determined by the conditions (e.g., the speed of the car or the position of the gear-stick). Eventually, driving the car is no longer an activity in itself, it becomes an action in another activity such as getting to work (for which the unconscious need for status may be one possible motive).

For the purposes of this study, only the top two levels of Leont'ev's activity hierarchy will be considered, where activity can be seen as comprising actions related to associated goals. Engeström (2009) argues that tensions are essential for understanding the motivation behind particular goals and actions. For example, a subject can have conflicting needs, or there can be a conflict between a need to be satisfied and the actions that are allowed to be done in order to satisfy it. In both cases, the subject feels a tension that can drive the subject to action. Tensions, thus, belong to the subject's consciousness, while motives and needs can be subconscious — even if they cause the tension.

This impacts analysis as Kaptelinin and Nardi (2006) point out the difficulty in establishing motives or needs in human activity: "The link between what an individual is doing and what she is trying to attain through what she is doing is often difficult to establish" (p. 58). Noting that, for Leont'ev, emotions were signals of the actualization of a motive or need, Engeström (2009), suggests that "to gain access to motives, one must proceed along a "round-about way," by tracing emotionally marked experiences. In other words, the study of action-level emotional experiences is an avenue to an understanding of activity-level motives" (p. 7).

Emotions, then, play an important role in activity theory where they reflect relations between motives (needs) and the real or potential success of an activity. Referring to emotions as "internal signals" in that they arrive from lived experiences rather than intellectual reflection, Leont'ev (2009) states, "they appear as a result of actualization of a motive (need), and before a rational evaluation by the subject of his activity" (p.166). Leont'ev places emotion at the level of activity rather than at the level of actions or operations but suggests that both can be affected by the emotions experienced by the subject.

This suggests a potential connection between tensions, emotions, and motives, discernible at the action-goal level. This leads to my research question: How do emotionally laden tensions affect goals and motives?

METHOD

In this study, I adopt an exploratory and qualitative approach that focuses on documenting the presence of a phenomenon rather than quantifying its prevalence. Data for analysis was taken from a larger study involving six teachers whose teaching experience ranged from 5 to 16 years. The data used was obtained during semi-structured interviews that ranged from 40 to 60 minutes. The interviews were audio-recorded and then fully transcribed. The structure of the interview aimed at letting tensions and emotions emerge through a narrative rather than by direct questioning. For example, the teachers were asked to describe their school, their relationship with their colleagues, and with parents, without explicitly asking them to describe the tensions they lived. This allows for richer descriptive data of personal experiences that leading questions may inhibit. The transcripts were then scrutinized for utterances with emotional components such as "I wasn't happy…" and for utterances that conveyed doubt or uncertainty such as "I wasn't sure, but…". The identified emotions and tensions were then re-examined for their potential connections to goals and motives.

FINDINGS

In the following, I highlight three ways in which tensions impact goals. I characterize the instances with excerpts from the interviews and explicate their interconnections with emotions and motives.

Changing goals

Excerpts in which the mathematics teachers expressed tensions frequently began with some variation of "this never used to bother me, but…". This piqued my interest and these instances were explored to examine what had changed. For instance, Eric mentioned that he used to assign, collect, and mark homework. He would give zeros for unfinished work and felt it was a good use of both his own, and students', time. His implicit goals were gathering evidence of learning and work habits. As his actions worked cohesively towards achieving his goals, there was no apparent tension. This changed when he encountered a student who took no notes, handed in no homework, sat at the back of the class and yet engaged fully in the lesson:

> He'd sit at the back and say "No, you're wrong" or "I disagree" or "What about this?". And I loved it because there was this back and forth, and like this is good! So, I think zeros, forget that! And man did he bring something to the [class]. I loved it. So that really changed my philosophy on taking in homework. Because he just sat there, but he was into it. I thought this was great! A lot of the students, all they do is just hand me homework, I like this better.

I argue that the emotion Eric experienced indicates a perception of motive. He recognized that, beyond his goal of collecting data for assessment, was the deeper desire to engage his students. This becomes apparent when he further explained:

> I used to collect the homework and mark it and there was no engagement with the students.

In recognizing his motive, he marked it as his new goal — to engage with his students. This shift in goals is accompanied by a tension, as he is not yet certain how to proceed.

> I do know one thing—after 10 years of collecting homework and marking, I don't want to do that anymore. And so I'm trying to… I want to change. I'm trying to fix that, but I'm still struggling with that. So I'm going to make that my focus for the next year.

This is an ongoing tension for Eric, coloured by uncertainty, as he searches for actions that will help him achieve his new goal.

Prioritizing goals

There were several instances where tensions experienced by teachers caused them to rearrange the priority of their goals. Although the teachers valued both goals, the tensions they experienced made them realize that they had been favouring one goal at the expense of the other. We see this in Lacey who explained that, when she began teaching, she was fine with having her students learn mathematics by completing worksheets. She was teaching a combined class of grade one and two students and felt that assigning individual worksheets helped with her dual goals of managing the behavioural issues in her classroom and fostering student learning.

> I would give them worksheets and then get them to sit at the tables and work on it independently. And then I would teach the grade ones, like whole group.

These actions worked well for her until her participation in professional development brought her to the realization that her students were developing very little mathematical

understanding. Behavioural issues were under control, but at the expense of student learning. This resulted in a tension for Lacey as she explained:

> I knew now that it wasn't working. I wasn't happy with my math program, but I didn't know how to change it either.

In Lacey's later description of the recognition as a *"horrible feeling"*, we see a strong emotional response as she comes to an unwelcome realization: her need for classroom management superseded student learning. It also suggests that her unconscious motive may have been to be a good teacher. Unlike Eric, the tension does not cause Lacey to shift from one goal to a new goal. She continued to value her goal of classroom management but has given higher priority to finding complementary actions to aid her goal of developing her students' mathematical understanding.

Strengthening goals

I found instances where the ongoing tensions experienced by a teacher served to strengthen their resolve to achieve a goal. This was exemplified in an excerpt from Mia who views herself as a progressive teacher who wants students to problem-solve and think mathematically. She further explained that she wants to teach in a way that makes learning mathematics *"an enjoyable experience for students and meaningful for kids"*—her professed goal. To assess their learning, she relies heavily on formative assessments that lead to mastery. She experienced tension when she was forced to measure her students' learning in standardized assessments:

> It was really frustrating in that I had this idea of how I wanted to teach and how I thought students should learn. And especially after what I would consider a successful unit or a successful lesson and then I would give them this formalized test that was the same as all these other classes and then it actually meant little, because if the average mark of the class wasn't 75%, then there was something wrong with my teaching or my marking. So, if my average was 78%, that probably meant that I was marking too easy and if my average was too low, I wasn't teaching them good enough. I found it really difficult, because I had this idea that if I taught my students well, then they would succeed at what they were learning and if they were above average that maybe meant that I had done something right, that I had taught them well. So that was just stifling.

Mia's strong emotions evidence the tension she experienced when her teaching style was threatened by the imposed assessment. Not wanting to lose sight of her goal, Mia's response was to push back:

> I did [pushback], and that kind of eventually settled and you find enough working terms, but it was an issue.

She goes on to explain that she continued experimenting to find teaching methods that suited her students and did not let the threat of standardized testing interfere with her goal.

We see this same tension regarding her teaching style appear again for Mia when she recounts colleagues questioning her practice:

> So, they're like, this isn't right, they [students] need to know these steps. And I'd be like, why? They understand what they're doing and they're getting the right answer… like get over it.

Mia likened this to *"harassment"*, an emotional response that reflects the ongoing tension surrounding her pedagogical choices. However, her final comeback, "*like get over it"* indicates Mia's determination to continue with her actions of working towards her goal. Rather than weakening her goal, both instances of tension appear to solidify it. Leont'ev (2009) suggests that the emotion that accompanies goals is short-lived and does not bring awareness of underlying motive.

Therefore, I find a third instance referencing this tension enlightening. In this instance, Mia had begun preparing her students for their year end standardized exams. However, this time Mia experienced tension when she found she had to defend her teaching style to her *students*:

> They were starting to stress out, "We have never seen this, what are you talking about? How are we supposed to do this? You never taught us anything". They started to get angrier and angrier as the exam got closer. And I don't know, like that really bothered me—in that okay, for my students, like for their sakes, they do have an exam at the end and I wanted them to be prepared for it. And even though I could see that they had done some amazing math, they never were aware of what they had done.

Here we see again Mia's emotional response. She explicitly mentioned feeling bothered, but implicitly there is also a sense of uncertainty and disappointment. As with the other instances, her teaching style played a role in her tension, but this time the results are different. Rather than solidifying her goal, this time the tension caused Mia to alter her goal and thus her actions:

> And then what happened the following year when I had them [same students] again, they started the year with some of that tension that was still there, even though it sort of had settled over summer. It was new curriculum, a new year. But now they were like, "Well, what's this going to be about?" And then I ended up teaching them very traditional. And, like on my scale of traditional too. So like, I taught them very traditional this year in comparison to last year. But as I said earlier, my traditional is still not a typical traditional.

The emotion and accompanying tension Mia experienced with her students caused her to reorient her goals. Mia may have expressed her need to teach in ways that make mathematics meaningful for her students, but I argue that in backing down, Mia revealed her true motive — building relationships with her students. And like Eric, recognition of her motive marks it as her new goal. Tensions with assessment and colleagues did not cause her to change her goal or actions, but tensions with her students did.

DISCUSSION AND CONCLUSION

For the teachers in this study, tensions arose that, I argue, made self-evident their motives, and subsequently required them to adjust their goals. According to Leont'ev (2009), the level of actions and goals is most readily understood, while the level of

activity and its motive is less accessible to individuals. It is through the emotional colouring of their actions that an individual's motive for the activity is revealed. However, it is important to note that emotions are not a reason to act, they are a result of activity (Leont'ev, 2009). Therefore, I suggest that while emotions are effective in revealing tensions, it is the tensions themselves that are vital to the subsequent changes in goals and actions. It was only through experiencing tension that the teachers were motivated to change their actions. This resulted in the participants changing, prioritizing, or strengthening their goals. This leads to two interesting conjectures.

First, context creates tension. The participants in this study were content in pursing their goals until confronted with a new context. We see this in Eric who happily collected homework until he met with a student for whom homework had no purpose. Likewise, Mia held strong to her goal of student engagement despite pushback from colleagues, but this changed when she met resistance from her students. This suggests the tensions teachers feel subjectively are there for a reason; they are the objective result of goals clashing with a (new) context. The resulting emotionally laden tension makes apparent the teachers' motive, which, in Eric's case, becomes a new goal (see Figure 1). In Mia's case, the goal was strengthened until it met yet another context (see Figure 2).

Figure 2. (a) Eric's Goal Transformation (b) Mia's Goal Transformation

Related to this is that tensions are also useful in delineating primary and secondary goals. This is evident in Lacey, who, through participation in professional development (new context), comes to the realization that she values student learning even more than classroom management. This suggests an image of tandem goals functioning in parallel until they hit a context that creates tension (see Figure 2). What emerges is a prioritization of goals where both are in play, but one is given higher priority.

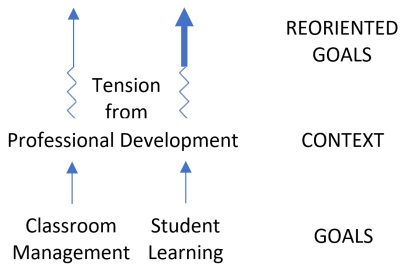

Figure 3. Lacey's Goal Transformation

In this dynamical framing of the relationship between emotions, motives, and goals, tensions can be understood as drivers of teachers' actions. I see tensions as complex collections of opposing forces between possible actions and contrasting motives. Tensions give rise to emotional responses that, in turn, make teachers conscious of their motives. Tensions have an emotional nature and, consequently, they act as signals; the teacher feels, through tensions, that the motives of her actions are contradictory. As such, tensions drive the teacher to action.

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DIFFERENCES IN QUANTITY RECOGNITION OF STUDENTS WITH AND WITHOUT MATHEMATICAL DIFFICULTIES: ANALYSIS THROUGH EYE-TRACKING AND AI

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Difficulties in mathematics learning are an important topic in practice and research. In particular, researchers and practitioners need to identify students' needs for support to teach and help them adequately. However, empirical research about group differences of students with and without mathematical difficulties (MD) is still scarce. Previous research suggests that students with MD may differ in their quantity recognition strategies in structured whole number representations from students without MD. This study uses eye-tracking (ET), combined with Artificial Intelligence (AI), in particular pattern recognition methods, to analyze group differences in gaze patterns in quantity recognition of N=164 fifth grade students.

INTRODUCTION

Learning difficulties in mathematics are an important topic in practice and research and have attracted increased interest not least since inclusive education has gained significance. Researchers and practitioners aim to understand knowledge and learning in a fine-grained way, and to foster students with MD individually and adequately (e.g., Moser Opitz et al., 2016; Scherer et al., 2016).

Previous research has indicated that students' strategies in quantity recognition can be used to identify difficulties in mathematics learning (e.g., Schleifer & Landerl, 2011; Schindler et al., 2019). An important question is how students' quantity recognition strategies can be observed. A promising method for investigating students' strategies in whole number representations such as the abacus or dot field is analyzing students' eye movements through ET (Lindmeier & Heinze, 2016; Rottmann & Schipper, 2002). Qualitative eye movement analyses in such representations may even outperform thinking aloud analyses in precision and level of detail—especially for students with MD (Schindler & Lilienthal, 2018). However, the qualitative analysis of eye movements is laborious and potentially subjective, and it is not yet sufficiently clear how eye movements in quantity recognition tasks differ between students with and without MD. Even though initial studies report group differences in students' strategy use (Schindler et al., 2019), the statistical analyses and the explanatory power of the results are limited due to small sample sizes of students. Therefore, this study aims to investigate group differences in students' gaze patterns when determining quantities in structured whole number representations—with a computer-supported, i.e., automated, evaluation method. We pursue the research question *Do the gaze patterns of MD vs. non-MD students in quantity recognition in structured whole number representations differ?,* which contributes to our overall purpose to investigate whether we can

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automatize the analysis of group differences in ET data and the evaluation to which group a student belongs.

Automatization of the analysis of ET data is desirable because analyzing eye movements for identifying student strategies is extraordinary time-consuming and demanding, given that there are—depending of the framerate of the device—100 or even 1000 frames per second recorded by the eye-tracker, all of which need to be analyzed if strategies are to be identified in videos (like in Schindler & Lilienthal, 2018, 2019). The effort required to manually analyze ET data is also prohibitive if ET based methods are to be used routinely by practitioners, e.g., school teachers.

Motivated by these considerations, we apply a methodology that makes use of a set of pattern recognition methods from AI. In order to compare eye gazes of the groups of MD vs. TD (typically developing) students, we analyze differences in students' gaze patterns on digital task sheets. In general, gaze patterns between the groups of students could differ in a myriad of ways that arise from combinations of where, when, for how long and in which sequence the students look at the stimuli. To render the subsequent analysis feasible, we first select a reduced representation: heat maps, i.e., visual representations displaying all gazes for each task. Simply put, we then investigate whether the heat maps of the two groups can be separated well on task level by our pattern recognition system, indicating significantly different gaze patterns. Our approach even allows us to semantically interpret group differences: The analysis of group-averaged heat maps (displaying all MD (vs. TD) students' gazes for the tasks, Fig. 2) allows us to identify differences that are meaningful for mathematics education research and hint at how strategy use might be different between the groups.

MATHEMATICAL DIFFICULTIES

To date there is no common definition or term describing the group of students having difficulties in mathematics (Scherer et al., 2016). Terms such as *mathematical learning disabilities*, (severe) *mathematical difficulties*, or *developmental dyscalculia* are used—depending on different educational contexts and research traditions. Medical models label a *disorder* (e.g., WHO, 2018) and support an IQ-discrepancy model. However, recent research suggests not to distinguish between students with MD depending on the discrepancy between their IQ and their math performance, since cognitive patterns of all students with MD, e.g., in counting, subitizing and magnitude comparison do not differ qualitatively (Kuhn et al., 2013). In our research, we address students with MD following Moser Opitz et al. (2016) and Scherer et al. (2016) as those students who encounter difficulties with a certain set of mathematical problems both on a conceptual and procedural level, including, e.g., basic arithmetic such as counting (also counting principles and counting by groups), (de-)grouping, the base-10 system, understanding place values, and basic arithmetic operations.

QUANTITY RECOGNITION

To determine quantities—i.e., to grasp a set of items and say how many they are—is a crucial skill for children to learn. Whereas young children typically already have the

ability to grasp numbers of small sets of items in one glance ("subitizing", Clements, 1999), they later on learn to count and to subitize conceptually, i.e., to make use of patterning abilities and to structure sets into subsets when determining numbers (ibid.). For students to apprehend the number range up to 20 or 100, teachers commonly use external representations such as the abacus (frame) or dot field (Gaidoschik, 2015, see Fig. 2): These representations both visualize substructures (10s, 5s, 50s)—in slightly different ways (e.g., through change of colors or gaps)—for the students to understand the base-10 system and to develop mental representations of the structures (Wartha & Schulz, 2012). Whereas investigating students' strategies in such representations (identifying what structures they use and how) is a challenging task (Obersteiner et al., 2014), researchers have found that ET may be useful to analyze students' quantity recognition strategies in structured whole number representations (Lindmeier & Heinze, 2016; Rottmann & Schipper, 2002; Schindler & Lilienthal, 2018). Lindmeier and Heinze (2016) concluded that ET data are useful to infer student strategies, and Obersteiner et al. (2014) point out that a combination of tasks on computerized versions of structured whole number representations together with ET appears to be a promising approach to assess students' strategies. For investigating students' quantity recognition strategies through ET, researchers particularly analyzed qualitatively students' scanpaths (e.g., Lindmeier & Heinze, 2016) or gaze-overlaid videos (i.e., augmented videos of the scene with the gaze visualized as point, e.g., Schindler & Lilienthal, 2018), which reveal where the students looked at and indicate student strategies. However, as Schindler and Lilienthal (2019) point out, the qualitative analysis of such gaze patterns is demanding and time-consuming. Therefore, this study uses AI, in particular pattern recognition to (partially) automate the analysis of gaze patterns and focuses on the spatial distribution of gazes over the task sheet.

THIS STUDY

Students. For answering the research question, we use data from a research project with 164 (92 males, 72 females) fifth-grade students in a German comprehensive school ("Gesamtschule"). The mean age was 10;9 (SD = 0;7) with ages ranging between 9;10 and 12;6. The participating school was in a town of 80,000 inhabitants, situated on the edge of a German urban area. The study took place in the first weeks of fifth grade.

We conducted a standardized arithmetic paper-pencil speed test (HRT; Haffner et al., 2005) with all 164 students in classroom settings. Only the first part of HRT, which can be used solely for diagnosing MD (at percentile rank (PR)<11; Haffner et al., 2005), was administered (similar to Schleifer & Landerl, 2011). The six subtests address mental addition, subtraction, multiplication, division, magnitude comparison (e.g., 7 _ 6; correct response: >) and completion tasks (e.g., _ - 2 = 6). We identified MD (at PR<11) and TD (PR>25) following the test's instructions, resulting in 69 MD students and 59 TD students. Percentile ranks between 11 and 25 are considered "at risk zone" (AR, Haffner et al., 2005, p. 20), which applied to 36 students. For the analysis of group differences, we focus on the groups of MD and TD students,

disregarding the AR students (see Fig. 1). Their mean t-values on the test were 32.1 (SD = 6.3) for the MD group, and 49.7 (SD = 7.5) for the TD group.

Tasks. We used a computerized version of the 100-bead abacus and the 100-dot field. The numbers were systematically chosen so that all ones and tens were included once. We also included 100, which led to eleven tasks (arranged according to size: 7, 15, 20, 31, 43, 54, 68, 76, 89, 92, and 100). The tasks were presented in randomized order (different randomization for each representation, i.e., abacus or dot field).

Procedure and eye-tracker. The students were tested individually in a quiet room. They were seated in front of a 24'' full HD computer monitor. We used the remote eye-tracker Tobii x3-120, which allows for video-based binocular tracking at a sampling rate of 120 Hz. Looking like a black stick, it was attached to the bottom frame of the monitor and hardly noticeable. Its presence and function was explained to the students, yet, it did not interfere with the students' work on the tasks. For adjusting the eye-tracker, a nine-point calibration was conducted. Then, before the students started working on the tasks, they first saw a picture of the respective representation (100-bead abacus, 100-dot field) and were asked to describe it. This was followed by two practice tasks with numbers that were not used in further test tasks. The students were instructed to correctly name the number of dots in every task as fast as possible. In between the tasks, the students were instructed to fixate a star in the middle of the screen before the next task appeared. The students received no response whether their answers were correct. Verbal answers were recorded through an audio-recorder.

Heat maps and spatial information. ET provides rich information and a large amount of data. The obtained gaze patterns can differ in many ways, including, in our case, where, when, for how long and in which order the students looked at the quantity recognition tasks. In order to identify group differences, we needed to choose an intermediate representation of the recorded gazes to allow for a feasible subsequent analysis. This intermediate representation should lower the dimensionality of the problem (loosely speaking, it should reduce the amount of data to be handled by the pattern recognition system) while preserving the relevant features of the gaze patterns. Following previous research that indicated that students' gaze distributions on the task sheets might differ on group level (Schindler et al., 2019), we decided for heat maps that show how gazes were spatially distributed over the presented digital task sheets. We thus disregard information about the order in which the students looked at the task and consider how long the students paid attention to certain areas only relative to the total duration of the task. To compute the individual students' heat maps, we use the Tobii Pro Lab Software and aggregate all gazes (not only fixations). We only include heat maps of correctly or inversely (common mistake in German) solved tasks for further analyses (e.g., for 68: "sixty eight" or "eighty six"), since we intend to sort out instances where students guessed rather than perceived the given information.

AI and pattern recognition methods. In order to assess which tasks' heat maps allow separating TD students from MD students, a Multivariate Analysis of Variances

(MANOVA) (Morrison, 2005) is performed. MANOVA is closely related to Linear Discriminant Analysis (LDA) (Izenman, 2013). Both methods are based on the same mathematical transformation but offer different interpretations of the results.

Unfortunately, it is not possible to use the heat maps directly for the analysis as the dimensionality of the input data that can be processed applying this method is limited by the available number of samples. Each heat map image represents a single point in a 2764800 (=1280*720*3, width*height*color) dimensional space. Therefore, it is necessary to compress the data contained in the heat maps into a lower dimensional space (fewer dimensions than students) before applying the MANOVA.

To this end, grayscale images are used, reducing dimensionality by one third. Then a Principal Component Analysis (PCA) (Abdi & Williams, 2010) is performed. PCA generates a new orthogonal coordinate system along the directions of high variance in the original data and achieves compression by dropping dimensions with the lowest variances. Intuitively speaking, those parts of the heat maps that look the same for every student, i.e., show low variance, are removed as they contain no information about group discrimination. 50 dimensions are chosen as target dimensionality in order to have at least 50 examples for both classes available, since the number of samples per class has to be higher than the dimensionality for some statistical tests in the MANOVA space. Through the use of 50 dimensions, 91% of the information was preserved on average in the compressed representation.

The heat maps, including those from AR students in the HRT test, are used to calculate the compression as the aim is to preserve all possibly occurring heat maps in the reduced space. Before the next step, the actual LDA/MANOVA, the heat maps of the AR students are removed from the data set to investigate differences between MD und TD students. The LDA/MANOVA can be understood as another compression method with the goal to maximize linear separability between the classes, i.e., TD and MD students, in the reduced space. Since we are considering a two-class problem, this step reduces to a single dimension (see the processing pipeline in Fig. 1).

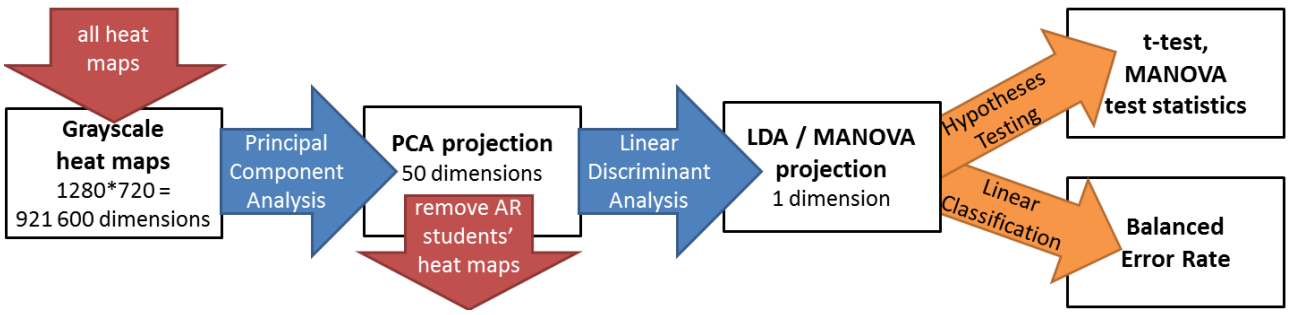


Figure 1: Pattern recognition system.

RESULTS

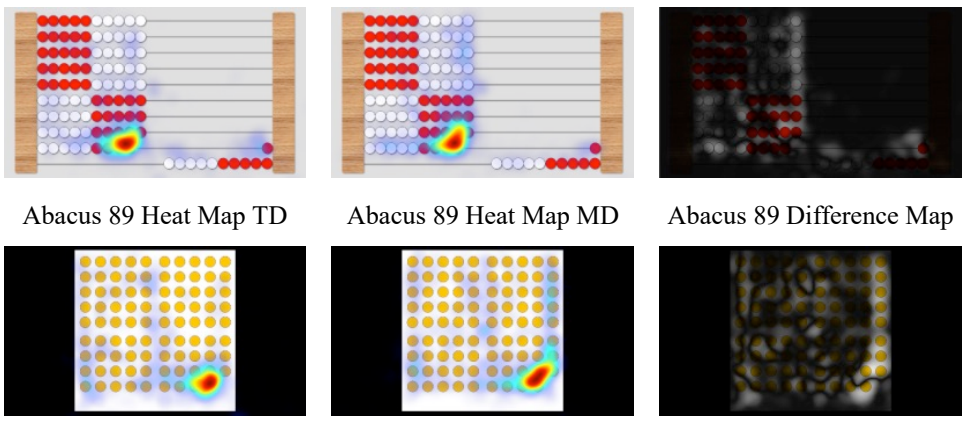
For pursuing the question if the spatial gaze patterns in quantity recognition differ between groups on task level, we used t-tests on the remaining dimension after PCA and LDA (Fig. 1). The group differences (MD vs. TD) were significant on a $p<.01$

level for all tasks, confirming significant differences between the groups' spatial gaze pattern data for each task. Yet, significant group differences do not guarantee for linear separability, since the actual performance of a classifier depends on the overlap of both data distributions. It is common practice for pattern recognition systems to report error rates as a practical assessment of how well groups can be distinguished, not just whether the differences are significant from a statistical point of view. Hence, we performed an actual classification on the given data to assess with what error rate it is actually possible to discriminate between the two groups. We report the Balanced Error Rate (BER) for a linear classifier using 3-fold cross validation (Kohavi, 1995). The BER is calculated according to $BER = \frac{1}{2}\left(\frac{fn}{fn+tp} + \frac{fp}{fp+tn}\right)$, with *fn* – number of false negatives, *tp* – number of true positives, *fp* – number of false positives, and *tn* – number of true negatives. A BER of 5% means that the average number of students wrongly classified is 5% of all samples. Any classifier that would assign classes by chance achieves a BER of 50%. The BERs are provided in Table 1.

task	**7**	**15**	**20**	**31**	**43**	**54**	**68**	**76**	**89**	**92**	**100**
dot field	15.93%	18.20%	28.26%	10.19%	8.96%	16.71%	12.99%	9.42%	5.96%	18.95%	18.93%
abacus	18.05%	17.40%	14.14%	16.41%	17.32%	21.32%	4.88%	12.11%	9.73%	13.79%	17.91%

Table 1: Balanced Error Rates per task separating TD and MD students.

Certain quantities have low BERs in both representations: For these tasks, the percentage of wrongly classified students is low and the students' gazes differ substantially on group level. For 43, 68, 76, and 89 we found BERs of below 10% in at least one representation. For other quantities (e.g., 20), BERs are higher, indicating less pronounced, but still relevant differences between the groups of students in these tasks.



Abacus 89 Heat Map TD | Abacus 89 Heat Map MD | Abacus 89 Difference Map

Field 89 Heat Map TD | Field 89 Heat Map MD | Field 89 Difference Map

Figure 2: Average heat maps and difference maps for 89

Through average heat maps, the group differences in students' spatial gaze patterns can be visualized (Fig. 2; Note: Heat maps have warmer colors where the students looked at more often, and difference maps visualize the group differences of the gazes: the brighter the bigger the differences in the respective area; maps self-produced by authors). As can be seen in the examples, MD students' gazes appear to be more on the right edge of the dots/beads.

DISCUSSION

The aim of this paper was to investigate group differences in students' gaze patterns when determining quantities in structured whole number representations. We used pattern recognition from AI to find differences in students' spatial gaze patterns on the 100-abacus and dot field—in eleven tasks per representation.

Looking at statistical comparisons, we found that the spatial gaze patterns in quantity recognition on the abacus and dot field differed significantly between MD and TD students for all tasks. In every task, when determining quantities of 89, 54, or 7, the groups' gaze distributions on the (digital) task sheets were significantly different. Calculating furthermore error rates (BER in particular), we found that every task contains exploitable information to separate MD from TD students (i.e., for none of the tasks the classifier came close to 50%, i.e., guessing). In summary, through AI we found that the groups' gazes differed—substantially in some tasks. This result may hint at different strategy uses of the groups of students. This is in line with results from previous explorative, qualitative studies which revealed that MD students tend to use other strategies on such representations than TD students (Rottmann & Schipper, 2002; Schindler et al., 2019). In our study, the visualizations of average heat maps (cumulative heat map of all MD/TD students, see Fig. 2) helped to understand group differences. These visualizations shed light on the students' spatial gaze distributions on the task sheets and indicate that MD might count rows more often than TD students.

Besides these empirical findings, our results indicate what tasks (quantities) might be most adequate for identifying students with MD. Our results do not suggest that the abacus is better suited than the dot field or vice-versa, but instead certain tasks produce the lowest errors of our pattern recognition system—often in both representations. While the lower error rates (e.g., for 89) are promising, the results indicate that a highly reliable classification based on a single task is hard to achieve—and also not reasonable from a pedagogical perspective. Future research should investigate what a reasonable set of tasks may look like to perform a classification with high confidence. This would help identifying students' needs in order to support them adequately.

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LATENT STRUCTURES OF MEANING – A PREREQUISITE FOR INCLUSIVE LEARNING IN MATHEMATICS CLASSROOMS

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This paper focusses conditions for successful inclusive mathematics lessons. The special education concept of elementarization for the development of inclusive learning materials is introduced. On the basis of these materials, teaching units were performed and analyzed. An inclusive learning environment demands the participation of all students in classroom interaction. In order to provide this, the relevance of latent structures of meaning is pointed out as a crucial aspect. The theoretical considerations about latent structures of meaning are exemplified by the interpretation of two scenes on elaborating mathematical concepts.

INTRODUCTION

Since the ratification of the UN-CRPD (2008) a lot of research on inclusive mathematics education has been done (cf. Häsel-Weide & Nührenbörger, 2017; Scherer, Beswick, De Blois, Healy & Moser Opitz 2016). Starting point for socially relevant discussion on inclusion is the demand of the UN-CRPD (2008) to grant every learner access to high quality education (Art. 25, 2a).

In our work we use inclusion-focusing principles such as "egalitarian difference" (Prengel 2006) in order to achieve the aims of the UN-CRPD (2008): Differences between the pupils were and are always present in mathematics classrooms – even without a learner with a specific diagnosis of special needs. Those differences should not be hidden but adopted and productively turned. Equal educational opportunities for all learners do not mean the equalization of all learners – just as no striving for equal treatment is meant – but (existing) hierarchies should be broken up (Prengel, 2001). Considering these principles, the opportunity of "learning to know, to do, to be and to live together" (UNESCO, 2009, p. 19) could be given to all pupils.

In German-speaking traditions in Special Education Research, the concept of elementarization is a common access to assure access to high quality teaching content for all students with and without special needs (cf. Terfloth & Bauersfeld, 2015). On the basis of the concept of elementarization math, lessons for inclusive classrooms were conducted. In this paper, the concept of elementarization will be described and crucial scenes from the empirical study will be analyzed with interpretative methods.

THEORETICAL BACKGROUND

The theoretical background of the presented research traces back to the concept of elementarization and the concept of latent structures of meaning. To address academic competences ("learning to know, to do"), participative competences ("learning to live together") and personal competences ("learning to be") in mathematics classrooms, the

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concept of elementarization forces to identify the elementary structures of the learning content on content level, methodological-medial level, cultural-social level, personally significant level and development-psychological level (Terfloth & Bauersfeld, 2015). The starting point of all educational considerations is the analysis resp. creation of a teaching theme in terms of those five directions presented below. In the field of special education, this approach has been proven to be a suitable tool to enable education based on cultural and social participation, especially for learners with (severe) disabilities (Heinen & Lamers, 2006).

In the first direction of elementarization *Elementary Structures*, the fundamentally constituting characteristics of the concrete mathematical content underlying the teaching are sought. Turning to mathematics lessons, this dimension seeks the irreducible mathematical kernel that has to be learned in order to gain a rough idea of the content. At this core, all other considerations depending on teaching should be aligned.

The second direction *Elementary Life-guiding Basic Assumptions* focuses on the cultural-social relevance and justification of the content: What is the significance of the content for society in general and therefore does it also apply in particular to (future) life of learners in society?

The third direction *Elementary Experiences* considers the concrete learning prerequisites of the pupils. Which challenges of everyday life may be explained by the subject of learning or can be mastered by the means of acquired strategies? Or in the opposite direction: Which everyday experiences of the pupils can and should be taken up in relation to the learning object? Considering the heterogeneity in a mathematics classroom, different background knowledge (and, thus, contexts) should be regarded in the tasks. Consequently, different tasks for introducing content are reasonable.

In the context of the fourth direction of *Elementary Approaches*, developmental psychological and biographical points of reference for pupils in the respective subject of learning are taken into consideration. Which developmental competences in the areas of cognition, language, motor skills, emotion etc. and which learning strategies have been acquired by the pupils?

In the fifth direction *Elementary Arrangement* concrete possibilities of learners to deal with the content are focussed: Which methodical-medial approaches are feasible? Which are already known to the learners?

By using the different directions, a possible learning environment can be created. We performed them for two content areas in two fourth-grade classrooms: introducing preformal algorithms of multiplication and introducing the concept of surface areas.

The second background, the concept of latent structures of meaning, can be considered as a key concept in the theory of "Objective Hermeneutics" by Ulrich Oevermann (e.g. 1987). Oevermann introduced this theory as a methodology for the interpretation of texts of interactions (e.g. transcripts of classroom discussions) in sociology:

> "The reconstructive interpretation of interaction texts permits the discovery of rules which constitute interaction texts as *objective structures of significance* [*Bedeutungsstrukturen*], which reflect the *latent structures of meaning* [*Sinnstrukturen*] of interaction itself. The objective structures of significance of interaction texts (which are prototypes of objective social structures itself) are real and have some permanence. Analytically (though not empirically) they are independent of any specific and conscious representation of the meaning of interaction on the side of the participating subjects. […] The latent structures of meaning of a single interaction or utterance (as the structure of situationally and contextually possible relations of significance) permits, as a rule, different 'ways of reading.' Participants in the original situation of action produce only *segments* of these readings intentionally." (Oevermann, 1987, pp. 438f).

Latent structures of meaning trace back to the existence of "objective" structures: In every situation, different rules and principles enable the partners of interaction to understand each other. A simple example: In a mathematics classroom, the word "times" is mostly used by referring to an operation. The context makes it possible to gain a possible meaning of the word. In every situation, different backgrounds for interpreting words or utterances might be useable – especially in a heterogenic group. Cicourel (1974, pp. 85ff) used the "et cetera principle" for this phenomenon in situations of interaction: Implicit rules must be used to understand explicit utterances.

Latent structures of meaning do not need to be intended by the speaker. He even might not be aware of them. In other words: By using the concept of latent structures of meaning, different "objective" meanings of a statement are seen as existing meanings – irrespective of the intended meaning. Thus, the latent structures of meaning have to be differentiated from the subjective structures of meaning.

The description above shows that latent structures of meaning do not only concern the process of analyzing transcripts. They also belong to the interaction itself, in which the partners have to make meaning of former statements in order to make the interaction pass smoothly along. If different latent structures of meaning exist, it depends on the partner of interaction which meanings are going to be manifest. This implies, that the partner must have realized the former latent meaning.

METHODOLOGY

The analysis of case studies on the genesis of empirically founded theories continues to be a major branch of research – even with international impact (Blum, Artigue, Mariotti, Sträßer & van den Heuvel-Panhuizen, 2017, p. 294). In German-speaking traditions in Mathematics Education Research the interactionist approach initiated by Heinrich Bauersfeld (e.g. 1980) and his colleagues (e.g. Voigt, 1995) has provided detailed insights into the understanding of learning mathematics by using an interpretative framework.

In order to gain the explicit and latent meanings in the texts (cf. the theoretical part above) we followed the described tradition and used the method of objective hermeneutics for analyzing transcripts of videotaped classroom communication. By

interpreting them step-by-step, latent meanings – not only from a mathematical point of view but also from an interactionist's one – will be considered.

EMPIRICAL RESULTS

Classroom Interaction

Students of a fourth-grade classroom had been solving tasks created by using the method of elementarization in order to introduce the concept of surface area. While the elaboration of solutions has been made by the students in cooperative small groups, the whole class is discussing the solutions. The following discussion (all transcripts have been translated and smoothened) took place concerning the working sheet in Fig. 1:

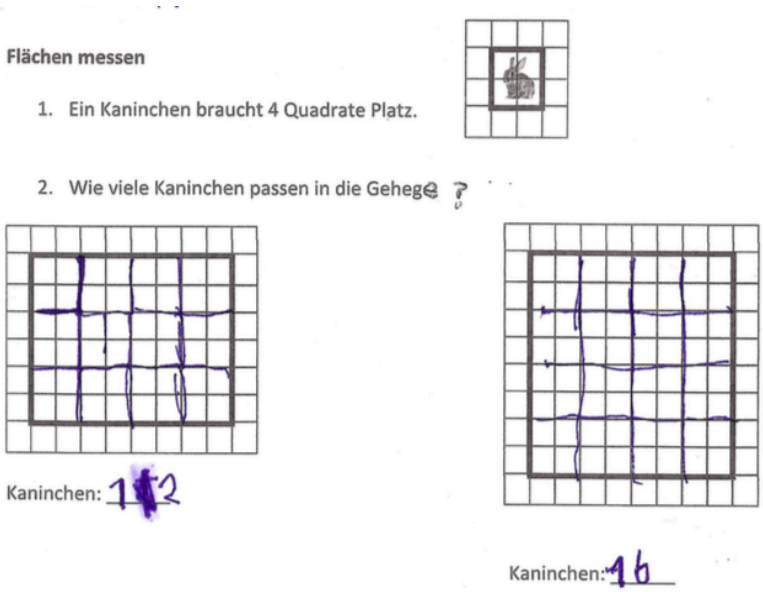
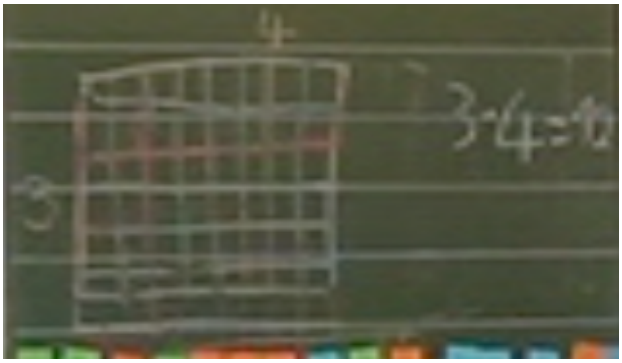


Figure 1: Part of Gereons working sheet. (Translation: measuring areas 1. A rabbit is in need of an area of four squares. 2. How many rabbits fit in the enclosures?)

Raja	I put the rabbits in eeh in the array.
Teacher	Yes .. and afterwards? … There is no sign on the rabbits saying we are 12 (*points to the rectangle on the left side on the worksheet*), or? The rabbits. *(laughs)*
Raja	*(smiles)* I counted the rabbits. [...]
Gereon	I multiplied.
Teacher	Multiplied? Why that?
Gereon	Ehm, because it is faster. [...]
Teacher	*(Draws a 8x8 rectangle at the blackboard)* Explain us how you have done it. [...]
Gereon	So, I calculated here *(follows the first row by his finger)*
Teacher	Say it loud, so everyone can hear you.

Gereon	I calculated here. *(repeats his movement)* and calculated so *(follows the first column by his finger)*
Teacher	Ok, so you first draw them, always here. Always 4? *(boarders four squares for a bigger square)* And you put in the rabbits.
Gereon	Afterwards I have here- *(uses red chalk to make a partition of two squares in the first row and the first column)*
Teacher	And afterwards you calculated? Write your calculation on the blackboard.
Gereon	three *(writes „3“)* times four. *(writes „* 4“)* [...] 
Leo	*(in the meantime, the classroom discussion goes on by the next task. Leo pipes up and points at the blackboard)* four times four! And not four times three.
Teacher	*(looks to the sketch)* Yes, the sketch is not really well, that is right. … Below there is one too much, yes. *(nods)*

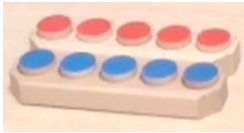
Raja presents a first solution (counting the 2x2 squares). Afterwards Gereon (a student with learning disabilities) presents another way in order to determine the number of 2x2-squares by using a multiplication which he presents as a faster method. The solution of Gereon is correct considering the tasks on the worksheet. Nevertheless, the teacher draws another rectangle on the blackboard, and Gereon fills the whole rectangle by squares, forgetting to adapt his solution for the new situation.

The latent structure of meaning of his solution – multiplying the number of 2x2-squares in the rows and columns in order to determine the number of rabbits by multiplication – seems not to be problematic for the class. At least Leo realises it and makes his understanding manifest by adapting it to the given situation correctly.

Interview with Laura

Tasks introducing preformal algorithms of multiplication (e.g. the grid method) created by using the concept of elementarization were given to another Grade 4 class. Laura – a classmate with down’s syndrome in the inclusive classroom – was excluded from the tasks from her teacher. Hence, a follow-up interview considering some of the tasks her classmates performed earlier in the study was conducted. In the initial discussion concerning the given mathematical manipulative, called Rechenschiffchen (cf.

Transcript), Laura correctly determines the number of tiles within a Rechenschiffchen by counting them. Now the following dialog starts:

I	If that together, *(taking the two Rechenschiffchen in front of Laura)* if that's ten, *(puts the two Rechenschiffchen asides)* how many *(puts two other Rechenschiffchen in front of Laura)* are that?	
Laura	ey *(looks at the Rechenschiffchen that were put aside, afterwards looks at the Rechenschiffchen in front of her)*	
I	(5 sec) look Laura, these were- what did you say? how many together? *(pushes the two Rechenschiffchen from the beginning towards Laura)*	
Laura	*(grabs the presented Rechenschiffchen, shoves them next to the Rechenschiffchen in front of herself, then sorts all of them directly)* five. *(points at the top Rechenschiffchen)* one, *(points at the second one)* two, *(tapping the third one)* three, *(points at the lowest one, indistinctly)* four	

Apart from the use of number words, Laura hardly speaks in the interview. However, Laura's solutions allow many interpretations that could have been taken up in classroom interaction. A pessimistic interpretation of the scene is that the number range exceeds Laura's counting skills. In order not to fail the last task by acting in an uncomfortable number range, she counts a countable amount of objects for her.

An optimistic interpretation of the scene is that Laura wants to determine the number of tiles of the Rechenschiffchen in front of her. After the replacing of the Rechenschiffchen by the Interviewer, she is irritated. She accepts the re-offered Rechenschiffchen from the beginning and sorts them for the easier numbering exactly one above the other. Through this alignment, Laura states that there are "five" tiles in each row. To determine the total number, she counts the number of bundles of five, systematically proceeding from the top to the bottom. The determination of the result remains in the interaction. To put it bluntly, the calculation of the total number of tiles (with 5 as the number of tiles per Rechenschiffchen and 4 as the number of Rechenschiffchen, leaving the supposed count error in the last turn untouched) leaves Laura to the interviewer or the experienced readership of this article.

FINAL REMARKS

The method of elementarization indicates the advantage of using different tasks in order to activate different elementary experiences of the students. Thus, not all students should work on the same task. Nevertheless, students working on different tasks must be able to follow the classroom discussion afterward. The opportunity of working together without solving the same tasks belongs to the existence of latent structures of meaning: By working on different working sheets the students elaborated and used mathematical knowledge (e.g. rules) which had to be adapted to the solutions of other tasks (concerning the same mathematical background) given by other students (at least the method of elementarization demands the connection to one specific mathematical kernel).

The interpretation of classroom discussions showed the use of the latent structure of meaning. The latent rules of Gereon have not been explicated by Leo. The use of the rule by Leo indicates the use of Gereon's rule and, thus, latent structures get meaning by their use.

Regarding the interpretation of the second transcript, different latent structure of meaning had been elaborated. These structures could also be realised by students in an inclusive mathematics class. By manifesting them, every student has the opportunity to take part in the classroom discussion. In other words: Latent structures of meaning constitute the possibility of participation of all students in a mathematics classroom. An inclusive mathematics education for the realization of an "egalitarian difference" (cf. Prengel 2006, 2001) can take place.

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I ENJOY MAKING DRAWINGS! ENJOYMENT, KNOWLEDGE ABOUT DRAWINGS, USE OF DRAWINGS, AND STUDENTS' PERFORMANCE

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Positive emotions in mathematics have been found to be important for problem solving. In the present study, 196 ninth- and tenth-graders filled out a questionnaire about how much they enjoyed making drawings and took tests that measured their knowledge about drawings, intra-mathematical performance, and modelling performance. Because students had to calculate the length of one side of a right-angled triangle to solve the modelling problems, some students made drawings. We confirmed the expectation that enjoyment of drawing was positively related to the use of drawings to solve problems. However, enjoyment of drawing was not related to knowledge about drawings, intra-mathematical performance, or modelling performance. We suggest that more attention should be paid to emotions toward strategies in future studies.

INTRODUCTION

Emotions are important for learning and problem solving (Hannula, 2015; Pekrun, 2006). One positive emotion that students frequently experience in the classroom is enjoyment. Consequently, the enjoyment of mathematics was found to affect problem solving and even more specifically students' modelling performance and students' interest in mathematics (Schukajlow & Rakoczy, 2016).

However, we do not know much about emotions toward strategies. As the use of strategies has been found to be important for successful task processing (Hembree, 1992), more attention should be paid to the affective factors (e.g., emotions) that can affect students' use of strategies. In the present study, we aimed to take the first step toward clarifying the role of emotions in the use of strategies and investigated the relation between enjoyment of drawing and achievement-related factors.

THEORETICAL BACKGROUND

Enjoyment

According to the control-value theory of achievement emotions, enjoyment is an activating positive emotion (Pekrun, 2006). Students who enjoy an activity feel arousal when engaging in this activity and are expected to outperform students who do not feel enjoyment. Research on emotions has revealed that enjoyment of problem solving arises in students if they have high control appraisals (the confidence to master the problem) and high value appraisals (they value the problem-solving activity) (Buff, 2014).

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Similar to other affective constructs, enjoyment can refer to different objects. The object of emotions varies from general (enjoyment of life) to specific (enjoyment of solving a problem). An analysis of the structure of the emotion of enjoyment regarding different objects with 513 students in Grades 5 to 10 confirmed that the enjoyment of strategies could be statistically separated from the enjoyment of life, school, and problem solving (Goetz, Hall, Frenzel, & Pekrun, 2006).

Self-generated drawing and modelling

Weinstein and Mayer (1986) defined strategies as behaviors and thoughts that learners engage in and that are intended to influence their learning and problem solving. Strategic knowledge makes up part of the static knowledge component of metacognition as opposed to the dynamic process component of metacognition, which includes strategy use (the application of a strategy during problem solving).

Modelling problems are problems with a connection to reality, and their solutions require demanding transfer processes between reality and mathematics (Niss, Blum, & Galbraith, 2007). Modelling performance can be clearly distinguished from students' performance in solving problems without a connection to reality, called intra-mathematical performance. An important goal of research in mathematical modelling is to clarify factors that are related to modelling performance and can be addressed in interventional studies (Schukajlow, Kaiser, & Stillman, 2018). One of these factors is students' strategies, such as making a drawing.

Self-generated drawing has been identified as an important strategy for problem solving (Hembree, 1992). It can be defined as the process and the product of generating an illustration that corresponds to the objects and relations described in a problem (Rellensmann, Schukajlow, & Leopold, 2017). Students' strategic knowledge about drawing comprises students' knowledge about the characteristics of a drawing that fits a given problem. In a prior study, we confirmed the importance of strategic knowledge for solving modelling problems (Rellensmann et al., 2017). Apart from strategic knowledge about drawing, students' strategy use was found to predict students' performance in problem solving. Students who spontaneously applied a drawing strategy were found to demonstrate higher performance than students who did not apply a strategy (Hembree, 1992). However, this finding was not always confirmed when students were asked to construct a drawing (De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003). Students who were asked to make drawings for geometric word problems were found to perform the same or even worse than students who were not asked to make drawings. Thus, it is important to identify the factors that influence the spontaneous use of drawings. One reason why students do not always spontaneously generate drawings might be their affective perceptions of this strategy such as how much they enjoy making a drawing to solve a problem.

Enjoyment of drawing and achievement-related outcomes

To the best of our knowledge, the relations between enjoyment of strategy use and achievement-related outcomes has yet to be investigated. However, research has found

that affective factors that are related to enjoyment might be important for the spontaneous use of the drawing strategy. In a correlational study, Uesaka and Manalo (2017) found that students made drawings less often when they perceived that making a drawing would cost them a lot of effort and when they were not confident they could perform this strategy well. Given that in the control-value theory of achievement emotions, high efficacy beliefs were suggested to accompany students' enjoyment, and the two constructs were found to be positively related (Buff, 2014), we expected that enjoyment of drawing would also be positively related to the use of drawings to solve modelling problems.

The relation between enjoyment of drawing and intra-mathematical performance cannot be clearly derived from prior research. As intra-mathematical performance is an important part of modelling (Niss et al., 2007) and enjoyment is positively related to modelling (Schukajlow & Rakoczy, 2016), enjoyment of drawing might be positively related to intra-mathematical performance.

Further, students who enjoy applying strategies might apply this strategy more often and might thereby gain deeper knowledge of this strategy. Thus, a positive relation between enjoyment of drawing and students' knowledge about drawings could be expected.

As enjoyment of strategies was found to be related to enjoyment in mathematics (Goetz et al., 2006) and enjoyment in mathematics was found to affect students' modelling (Schukajlow & Rakoczy, 2016), we expected a positive relation between enjoyment of drawing and modelling. We expected that this positive relation would hold even after we controlled for strategic knowledge about drawings and intra-mathematical performance.

RESEARCH QUESTIONS

RQ: 1. Is enjoyment of self-generated drawing related to the use of drawings, intra-mathematical performance, strategic knowledge about drawings, and modelling performance? We expected a positive correlation between enjoyment and learning outcomes.

RQ: 2. Is enjoyment of self-generated drawing related to the use of drawings and modelling performance after strategic knowledge about drawing and intra-mathematical performance are controlled for? We expected that the relation between enjoyment and modelling would remain positive even after the strategic factor and the achievement factor were controlled for.

METHOD

Sample and design of the study

Two hundred twenty German ninth- and tenth-graders from 10 classes in middle- and high-track schools (German Gesamtschule and Gymnasium) participated in the present study (mean age about 15 years, 109 female students). The study consisted of two test sessions on two different days. On the first day of the study, students filled out a

questionnaire on enjoyment of drawing, age, and other constructs. They also worked on a test on strategic knowledge about drawing. On the second day, at least 2 weeks after the first day, they took a modelling test and an intra-mathematical test. Because each student worked on the tests on two different days, some data were missing (3% on the first day and 8% on the second day). In the presented analyses, we analyzed students with complete data only. This means that we excluded students from the analysis if they missed the first or second test session, and we performed the analysis on 192 students. Students were not instructed to make a drawing while working on the modelling problems because we were interested in their natural use of the drawing strategy. Their spontaneous use of drawings was coded by analyzing students' solutions to the modelling problems.

Measures

The scale for the assessment of enjoyment of self-generated drawing was adapted from the enjoyment Likert scale from the Achievement Emotions Questionnaire (Pekrun, Goetz, Frenzel, Barchfeld, & Perry, 2011) by changing the scope of the items from problem solving to the drawing strategy while solving word problems. It consisted of three items that were rated on a Likert scale ranging from 1 (*not at all true*) to 5 (*completely true*). One sample item was: "I enjoy making a drawing when solving a difficult word problem." The reliability (Cronbach's α) was .766.

We developed and validated the strategic knowledge about drawing scale in a prior study (Rellensmann, Schukajlow, & Leopold, under review). It comprised eight real-world problems. Each problem was followed by an item that included situational drawings and an item that included mathematical drawings (see Figure 1). On the 16-item test, students were asked to evaluate the utility of the drawings provided for each problem by comparing the drawings with regard to their helpfulness in solving the presented problem. Each item comprised three drawings (a correct and complete drawing, a correct but incomplete drawing, and an incorrect drawing) that students rated on a scale ranging from 1 (*not helpful at all*) to 5 (*very helpful*). The scores for each item ranged from 0 to 3. The number of points depended on the sequence of drawings concerning their usefulness for solving the task. If students identified that the correct and complete drawing was more helpful than the correct but incomplete drawing and that the correct and incomplete drawing was more helpful than the incorrect drawing, they were given 3 points (test reliability .76). If a student suggested the reverse order of the drawings, he or she received 0 points.

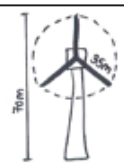
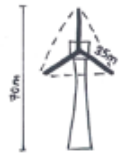
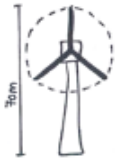
The modelling performance test comprised eight problems that all required students to identify a right-angled triangle and could be solved by applying the Pythagoras theorem (see examples of the tasks by Rellensmann et al., 2017). Students' solutions were scored by two raters on a scale ranging from 2 (correct problem solution) to 0 (incorrect solution or a missing solution). The interrater reliability (Cohen's κ) was > .81. The test reliability of the modelling performance test was .772.

Wind

The rotor blade of a 70 m high wind turbine has a length of 35 m and a so-called wind area of about 3,800 m². The wind area is the area that is swept by a rotor blade during a complete rotation. Can the information on the swept area be correct?

You do not have to solve the task!

First compare these three drawings with one another. Evaluate how useful each drawing is for solving the task.

Drawings	Not helpful at all	Not helpful	Rather not helpful	Rather helpful	Helpful	Very helpful
	☐	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐	☐

Next compare these three drawings with one another. Evaluate how useful each drawing is for solving the task.

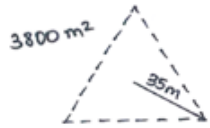
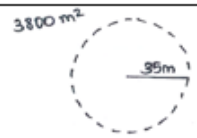
Drawings	Not helpful at all	Not helpful	Rather not helpful	Rather helpful	Helpful	Very helpful
	☐	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐	☐

Figure 1: An item from the knowledge about drawing scale (Rellensmann, Schukajlow, & Leopold, under review)

Students' use of drawings was assessed via the number of drawings they constructed while solving the eight problems on the modelling test. If students constructed a drawing for a modelling problem, they received a score of 1; if they did not make a drawing, they were given a score of 0. The test reliability (Cronbach's α) was .866.

Intra-mathematical performance was assessed with the test that included 10 items on applying the Pythagorean theorem or solving quadratic equations (e.g., $x^2 = 3.82$ –

2.52). Students received 1 point for the correct solution and 0 points for an incorrect or missing solution. The test reliability was .760.

RESULTS

Our first research question referred to the relation between enjoyment of self-generated drawing and achievement-related outcomes. By computing Pearson correlations, we found a positive relation between enjoyment of drawing and the use of drawings to solve modelling problems (see Table 1). This result confirmed our expectation that students who enjoy making drawings would construct more drawings to solve the problems than students who do not enjoy this strategy.

		UD	SKD	IM	MOD
Enjoyment of self-generated drawing	*r*	.159*	.09	-.016	.046

Note. * *p:* two-tailed, $p < .01$, UD: Use of drawings, SKD: strategic knowledge about drawing, IM: intra-mathematical performance, MOD: Modelling performance.

Table 1: Correlations between enjoyment and the achievement-related measures

However, we did not find relations between enjoyment and the other achievement-related factors. Enjoyment of self-generated drawing was not correlated with strategic knowledge about drawings, intra-mathematical performance, or modelling performance. We found that students with high and low enjoyment of drawing had similar knowledge about drawings and that they did not differ in their performance in solving intra-mathematical and modelling problems.

The second research question was aimed at investigating the relation between enjoyment and modelling performance while controlling for important factors that have been found to influence modelling performance in prior studies (knowledge about drawings and intra-mathematical performance). Partial Pearson correlations did not support our expectations because we did not find a positive relation between enjoyment of drawing and modelling after we controlled for knowledge about drawing and intra-mathematical performance ($r = .050$, $p = .488$).

DISCUSSION

The goal of our study was to explore the role of enjoyment toward strategies in students' learning outcomes. We found that enjoyment of drawing was positively related to the spontaneous use of the drawing strategy. This result confirmed the importance of affective factors for students' strategic behavior. Apart from the costs of strategy use (Uesaka & Manalo, 2017), students' emotions might influence their decision to apply a strategy to solve a problem. In future studies, the role of other emotions and more broadly the role of other affective factors for strategy use should be explored. A practical implication of this finding might be that improving students'

enjoyment of strategy use might increase the frequency with which they use these strategies.

However, students' enjoyment of self-generated drawing was not related to students' knowledge about drawings. We could not confirm our expectation that students who enjoy a strategy and thus apply this strategy more often gain deeper knowledge of this strategy. One reason for this finding might be that teachers do not often discuss when a student has applied a strategy appropriately. In the case of drawings, teachers should not only present the correct solution, but they should also discuss why a drawing is helpful and how a student can make a helpful drawing (Rellensmann et al., 2017).

Enjoyment was not related to intra-mathematical performance. For some tasks, students with high or low mathematical knowledge can be confident about making a drawing, and they can value making a drawing, whereas for other tasks, they might feel less confident about making a drawing and might not value this strategy. Because of the similarity between students' control and value appraisals assumed in the control-value theory (Pekrun, 2006), enjoyment did not differ for students on different levels of intra-mathematical knowledge. However, the importance of control and value appraisals regarding strategies for students' emotions toward these strategies had not been investigated until now and should be addressed in future research.

Given that enjoyment of strategies and enjoyment of mathematics were previously found to be related (Goetz et al., 2006) and enjoyment of mathematics was found to affect modelling (Schukajlow & Rakoczy, 2016), we expected a positive relation between enjoyment of drawing and modelling. However, we could not confirm this expectation. Other factors that are related to the strategies (e.g., knowledge about a strategy or the quality of the strategy use) were previously found to be more important to modelling performance than enjoyment (Rellensmann et al., 2017). Further, in our study, enjoyment was not related to strategic knowledge about drawings. However, it might be different for other emotions. One of our previous analyses revealed that anxiety about drawing was related to modelling performance.

In the present study, we focused on making drawings because this strategy was found to be helpful for solving modelling problems with a spatial structure of the mathematical model (Rellensmann et al., 2017). However, from other studies on modelling, we know that students also use other cognitive (e.g., highlighting, forward/ backward strategy) and metacognitive (e.g., planning, monitoring) strategies (Stillman & Galbraith, 1998). It will be interesting to determine whether emotions influence the spontaneous use of these strategies.

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THE IMPACT OF A CONTENT FOCUSED VIDEO-BASED PD ON TEACHER KNOWLEDGE, INSTRUCTIONAL PRACTICE, AND STUDENT ACHIEVEMENT

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Video is a powerful tool in professional development (PD) that can support teacher learning of content and skills. In the mathematics field, PD aligned with high quality standards is especially needed to help teachers gain mathematical knowledge for teaching and to develop fluency in making instructional decisions that support students' learning of challenging content. The Learning and Teaching Geometry intervention consists of well-specified PD materials that engage teachers in learning complex mathematical concepts through video cases. This paper describes a group-randomized efficacy study examining the effectiveness of this intervention. The findings suggest that the intervention strongly impacted teachers' instructional practice, leading to some student learning improvements.

INTRODUCTION

The power of using video in professional development (PD) to elicit critical reflection and to support teachers' learning of new content and skills has been widely documented (e.g., Borko, Koellner, Jacobs & Seago, 2011; Brophy, 2004; Harford & MacRuairc, 2008; Rich & Hannafin, 2009; Rosaen et al., 2008; Santagata et al., 2007; Sherin, 2007). Video allows for the complexities of classroom practice to be stopped in time, unpacked, and thoughtfully analysed, helping to bridge the perpetual theory-to-practice divide and support instructional improvement. While in the classroom teachers must constantly make individual in the moment decisions, viewing video during PD allows teachers the opportunity to collectively deconstruct and discuss familiar experiences and to actively generate new understandings about content, pedagogy and student thinking (Cullen, 1991; Korthagen, Kessels, Koster, Lagerwerf, & Wubbels, 2001).

An accepted theoretical trajectory of teacher learning from effective professional development is that increased teacher knowledge leads to subsequent changes in classroom instruction which, in turn, support improved student learning (Desimone, 2009). Therefore, to measure the effectiveness of a professional development intervention, it is critical to document changes in knowledge and instruction, along with more distal student outcomes (Kennedy, 2016; Wayne et al., 2008).

This paper examines the overall effectiveness of the Learning and Teaching Geometry mathematics professional development intervention (LTG PD) on teacher participants in an effort to understand the impact on teacher learning, classroom practice and student achievement.

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Theoretical Perspective

The LTG PD is conceptually grounded in the development of professional knowledge that consists of deep and connected mathematical content knowledge, the knowledge of students thinking and how students learn the content, and knowledge of pedagogical practices and norms to support student learning. Goals for mathematics teachers' learning should include gaining proficiency in the content they teach, improve their understanding of student thinking and learning, and improving their instructional practices to meet the needs of diverse learners (NCTM, 2000). Ball and colleagues have identified and elucidated "knowledge of mathematics for teaching"—the professional knowledge that mathematics teachers must have in order to do the mathematical work of teaching effectively (e.g., Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). This conception of knowledge of mathematics for teaching is multifaceted and includes both content and pedagogical content knowledge.

"Common" and "specialized" knowledge of mathematics comprises the content knowledge that mathematics teachers need. Common content knowledge is the basic understanding of mathematical skills, procedures, and concepts acquired by any well-educated adult. Specialized knowledge involves a deeper, more nuanced understanding of mathematical skills, procedures, and concepts. It enables teachers to evaluate mathematical representations and solution strategies; to analyse (rather than just recognize) errors; to give mathematical explanations; and to make connections among mathematical strands. It is what Ma (1999) characterized as "profound understanding of fundamental mathematics" (p. 120).

The pedagogical content knowledge (PCK) that mathematics teachers' needs include a sophisticated understanding of effective instructional practices and student thinking related to specific mathematical content and comes into play during all phases of teaching. For example, during instructional planning teachers draw upon their PCK to select curricular materials and sequence content to facilitate student learning, to predict how their students will approach specific mathematical tasks, to consider the needs of linguistically and culturally diverse students, and to anticipate student errors. As they conduct lessons, PCK enables teachers to recognize instructional affordances and constraints of different representations, to interpret incomplete student ideas, to anticipate opportunities to address language or cultural references, and to consider how to respond to various correct or incorrect pathways students explore. After completing a lesson, PCK is central to teachers' reflections on the learning that did or did not take place, and to their consideration of how to plan for and improve future lessons.

The Learning and Teaching Geometry PD Intervention

The Learning and Teaching Geometry (LTG) program is a video-based mathematics professional development intervention, targeted for teachers serving grades 6-12. The intervention consists of 54 hours of PD focused on improving teaching and learning of mathematical similarity based on geometric transformations (Seago et al., 2017). The

program is designed to be implemented by a knowledgeable facilitator, using a set of provided resources to engage teachers in a specified learning trajectory (Seago et al., 2013). The intervention includes a 30-hour Foundation Module followed by four 8-hour Extension Modules that explore related topics such as using appropriate representations and tools and supporting English Language Learners. The LTG PD intervention targets teachers' mathematics knowledge for teaching geometric similarity, meaning that it is intended to inform both their content and pedagogical content knowledge in ways that foster student learning of this topic.

METHOD

The LTG Efficacy Study (NSF award #1503399) aims to determine the effectiveness of the LTG PD intervention using a group-randomized experimental design. Participants were 103 secondary mathematics teachers serving Grades 6-12 (47% middle school, 53% high school) from two diverse geographic locations. Randomization was conducted at the school level, with 23 schools and 49 teachers assigned to the treatment group and 18 schools and 54 teachers assigned to the comparison group. No statistically significant differences were found by treatment and control groups.

Professional development workshops for treatment teachers in both locations began in Summer 2016 and continued throughout the 2016-17 academic year. Nine full days of professional development were offered to teachers in each location, beginning with the five-day Foundation Module, followed by four days of Extension Modules. On average, treatment teachers attended 7 workshops. Control teachers were offered the same workshops during the 2017-18 school year, after pre- and post-data collection was completed. The same facilitator led all of the workshops, after completing an extensive facilitator preparation process that included a multi-faceted assessment of fidelity. Based on this assessment process, facilitator was deemed capable of using the PD materials as intended and making appropriate context-based decisions and adaptations (Jacobs, Seago & Koellner, 2017).

Pre and post data collected on the effectiveness of the LTG PD included four teacher knowledge assessments: (1) Diagnostic Science Assessments for Middle School Teachers (DTAMS), a general assessment of teachers' geometry and measurement content and pedagogical content knowledge; (2) Horizon Research, Inc's assessment of teachers' content and pedagogical content knowledge of geometric similarity, developed in alignment with the LTG PD materials; (3) a similarity task embedded assessment, and (4) a video case analysis task embedded assessment. In addition, teachers were rated on the quality of their mathematics instruction captured in videotaped classroom observations using the Math in Common Observation Protocol (Perry et al., 2015). The classroom observation ratings included eight individual ratings and three sub-scores. A student geometry knowledge assessment measured student gain in five content areas over the academic year. In addition, the treatment teachers provided daily reflections on their workshop experiences.

RESULTS

Impacts on Teacher Content Knowledge

Repeated measures analyses of variance were used to detect patterns of change in teachers' content knowledge after completing the LTG PD, relative to the comparison group. Significant (p<.05) interactions were found for all embedded assessments indicating greater gains for the treatment group compared to the comparison group. Significant (p<.01) group and time effects were found for knowledge of transformations on the Horizon assessment and a significantly (p<.001) larger gain for the treatment group. Of the 8 DTAMS measures, significant and positive time effects were found for knowledge of reasoning / problem solving (p<.001) and Measurement (p<.001), and a significant and negative effect was found for Geometry 3D (p<.05). Within the groups of teachers, both groups gained significantly in the areas of reasoning and measurement, but only the treatment group declined in Geometry 3D. No other effects were found knowledge of facts, conceptual understanding, pedagogy, Geometry 2D, or transformations/coordinates. These results confirmed that teachers made some knowledge gains, but they were mostly captured by the highly targeted assessments. Additional analyses revealed that within the treatment group, teachers made significant gains in all content areas except scaling.

Impacts on Classroom practice

Teachers were rated on the quality of their mathematics instruction captured in videotaped classroom observations. Both pre- and post-PD ratings were available for a subset of 92 teachers (43 treatment, 49 control). No background differences were found between this subsample and the remaining sample of teachers.

The classroom observation ratings included eight individual ratings from the Math in Common Observation Protocol. Three quality of instruction averages were created: Richness of Mathematics (linking representations, multiple solution methods and making sense), Student Engagement in Mathematical Practices (explanations and reasoning), and the Mathematics (math, access and agency). Repeated measures ANOVA identified a significant (p<.05) time effect for Richness of Mathematics and a significant (p<.05) interaction effect for Student Engagement (see Table 1). Treatment teachers made significant gains in both averages while the control group did not change. On average, treatment teachers improved their ability to engage students in mathematical reasoning (p<.01) and to provide accurate, coherent, and well-justified mathematical content (p<.05).

	Pre PD		Post PD		*p*-value		
Instruction Rating	Treatment	Control	Treatment	Control	Group	Time	Group X Time

Richness of mathematics	1.95	2.25	2.26	2.43	ns	.014	ns
Student engagement in mathematical practices	1.81	2.31	2.16	2.24	ns	ns	.011
Mathematics	1.87	2.16	2.02	2.05	ns	ns	.053

Table 1: Results of Repeated Measures Analyses of Variance on Teacher Quality of Instruction Ratings, by Group (*N*=92)

Impacts on Student Knowledge

The Horizon Research geometry knowledge assessment was given to students of both the treatment and control groups of teachers. Both pre-test and post-test scores were available for 758 students (373 treatment, 383 control) in 56 classrooms and 41 schools. Students' mean scores for the total assessment and five content areas are presented in Table 2. A series of hierarchical linear models was used to estimate the impact of school participation in LTG PD on student knowledge. We found a significant and positive effect of LTG PD on student gains in knowledge of transformations ($p<.05$; $ES=.23$). LTG PD explained a significant proportion of variance in gains in transformations knowledge with the treatment group making almost twice the gain of the comparison group.

Knowledge Outcome	Pretest		Posttest	
	Treatment	Control	Treatment	Control
Total percent correct	48.51	40.81	56.55	48.66
Total number correct	9.70	8.16	11.31	9.73
Dilations	1.54	1.27	1.73	1.42
Scaling	2.81	2.29	3.30	2.76
Ratio	2.68	2.16	3.03	2.72
Proportion	1.20	1.06	1.24	1.11

Transformations	1.57	1.53	2.09	1.80

Table 2: Means of Student Knowledge Pre-test and Post-test, by Group (N=758)

CONCLUSION

Teachers' participation in the LTG PD intervention supported critical gains in both knowledge and practice, with particularly large gains in instructional practice. Improvements in practice were then directly related to student learning in the area of transformations. Most researchers in the field theorize that knowledge gains precede practice gains. The fact that the teachers made substantial gains in practice without showing similarly robust gains in knowledge warrants further investigation. Importantly, the observations of classroom practice were not limited to content areas covered by the PD; rather, teachers were filmed teaching whatever topic areas they would normally cover. Therefore, the impact on practice appears to be quite widespread and readily discernible by the observation instrument used in the research study. Lastly, the finding that the largest impact on student knowledge was in the area of geometric transformations suggests this topic was intentionally and heavily targeted by the treatment teachers in their instruction, perhaps due to its relevance across grades and mathematics content areas.

This efficacy study presents important implications for others designing, implementing, and studying efforts to support teacher learning. The findings underscore the importance of a well-specified year-long intervention to improve critical aspects of teacher knowledge and instructional practice, leading to gains in student learning in a targeted content area. The study further points to the importance of using measures that detect these gains, especially in instructional practice.

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PREDICTORS OF EXACT ARITHMETIC AND COMPUTATIONAL ESTIMATION IN KINDERGARTEN

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The current study investigates which general cognitive functions and early numerical skills are predictive of kindergartners' exact arithmetic and computational estimation performance six months later. Results show that intelligence, verbal counting, and number ordering skills are important predictors for both exact arithmetic and computational estimation. Secondly, symbolic comparison was only predictive of exact arithmetic, whereas verbal working memory only played a role in computational estimation. These results suggest that kindergartners' exact arithmetic and computational estimation do not rely on exactly the same processes.

INTRODUCTION

In the last decades, there has been much research interest in the learning of mathematics before the onset of formal schooling, and more specifically, the relationship between whole number arithmetic and early numerical skills (Clements & Sarama, 2007). Clements and Sarama (2007) distinguish six types of early numerical skills: subitizing, verbal and object counting, comparing and ordering, beginning addition and subtraction, composing and decomposing, and numerals and mathematical language and symbols. Research in mathematical cognition has mainly focused on the role of the first three types in exact arithmetic. However, whole number arithmetic is broader than exact arithmetic and also comprises computational estimation (Verschaffel, Greer, & De Corte, 2007). Much less is known, however, about the role of these three types of early numerical skill in computational estimation.

Concerning the relationship between these early numerical skills and exact arithmetic, it has been found that kindergartners' subitizing abilities (i.e., the ability to enumerate small sets) have been shown to be a concurrent predictor of their exact arithmetic skills (Gray & Reeve, 2014). Secondly, kindergartners' counting skills were predictive of their arithmetic scores right before the start of first grade (Tobia, Bonifacci, & Marzocchi, 2016). This indicates that children who are familiar with a larger number range are better in exact arithmetic. Thirdly, the relative contribution of non-symbolic and symbolic comparison to exact arithmetic is not clear yet. But based on recent meta-analysis (Schneider et al., 2017), the symbolic task seems to be a more robust predictor for exact arithmetic than the non-symbolic task. Lastly, research on number ordering in children shows that number ordering only becomes a predictor of arithmetic skills from third grade onwards (Lyons, Price, Vaessen, Blomert, & Ansari, 2014).

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As mentioned earlier, very little is known about the relationship between early numerical skills and computational estimation. One skill that was suggested as an important predictor is number comparison (Sowder & Wheeler, 1989). However, in a study with 13-year-olds, number comparison did not seem to be a concurrent predictor for computational estimation (Rubenstein, 1985).

In addition to the early numerical skills, general cognitive functions also play a role in whole number arithmetic. Several studies addressed the role of working memory in both exact arithmetic and computational estimation. In adults, exact arithmetic relies more on working memory resources in comparison to computational estimation (Kalaman & LeFevre, 2007), while in 3rd and 4th graders, computational estimation is more demanding of working memory resources than exact arithmetic (Caviola, Mammarella, Cornoldi, & Lucangeli, 2012). This discrepancy might be because children's strategies for computational estimation are less developed in comparison to their strategies for exact arithmetic. In addition to working memory, children with higher intelligence scores tend to be better in both exact arithmetic (Geary, 2011) and computational estimation (Reys, Bestgen, Rybolt & Wyatt, 1982).

The current study investigates the relationship between early numerical skills, general cognitive functions and both exact arithmetic and computational estimation before the onset of formal schooling. Data from the first two waves of a large-scale longitudinal study on the development of young children's early mathematical competencies will be used (https://ppw.kuleuven.be/o_en_o/CIPenT/wis-co-start). Firstly, we are interested in which general cognitive functions are important for exact arithmetic and computational estimation and whether these are the same or different. Secondly, we are interested in which early numerical skills are predictors of kindergartner's exact arithmetic and computational estimation performance and whether these are the same or different.

METHOD

Participants

A group of 407 kindergartners from 17 schools from various regions in Flanders, Belgium, participated in the study. A first wave of data (T1) was collected in the spring of 2017 when children were in their second year of kindergarten (N = 407, M = 58.14 months, SD = 3.51 months, 213 boys). A second wave (T2) took place in the fall of 2018 when the children were in their third year of kindergarten (N = 389, M = 63.79 months, SD = 3.48 months, 200 boys). The study was approved by the social and societal ethics committee of KU Leuven (G-2016 07 591) and all parents approved through informed consent.

Materials and procedure

Kindergartners' exact arithmetic performance was assessed by means of eight addition problems, which were divided into four difficulty levels: sum between 1-5, 6-10, 11-15, and 16-20. The task was administered using concrete materials with support of

verbal number words (Jordan, Kaplan, Ramineni, & Locuniak, 2009). The experimenter said, for instance, "Here are two pieces of candy", showing them on a horizontal line in front of the child and puts them in a box. Then one piece of candy was placed in front of the child and put in the box and the child was asked: "How many pieces of candy are in the box?" Children had to answer by placing the correct number of pieces of candy on the table. The total score was the number of correct answers.

Kindergartners' computational estimation performance was assessed using eight addition problems, which were just outside 5-year-olds' level of exact arithmetic and could, therefore, be solved only by estimating the outcome (cf. Dowker 1997). These problems were divided into four difficulty levels: sum between 11-15, 16-20, 21-25, and 26-30. Administration of the task was similar to that of the exact arithmetic task but now with cows and a stable instead of candies and a box. At the end of each trial, the experimenter asked this time: "About how many cows are there in the stable?" and children answered by placing an appropriate number of cows on the table. Estimation accuracy was calculated as the percentage absolute error (PAE) of the answers of the children relative to the exact answer (PAE = $\frac{absolute\ deviation\ of\ the\ exact\ answer}{exact\ answer} * 100$).

We used a subset of tests from a larger battery of basic numerical competencies that was developed within the framework of the larger research project (Bakker, Torbeyns, Wijns, Verschaffel, & De Smedt, 2018). Following early numerical skills were assessed. *Dot enumeration*: children were presented random dot arrays ranging from one to nine dots and were asked to say, as quickly as possible, how many dots they saw. *Verbal counting*: children were asked to count as far as they could starting with 'one'. *Object counting*: this competence was assessed by means of the "Give me n"-task. Children had to provide a certain number of stones varying from 3 to 19 using a set of 20 stones. *Non-symbolic magnitude comparison:* children were presented two dot arrays ranging from 1 to 9 dots and had to indicate which array contained most dots. *Symbolic magnitude comparison:* children were presented two Arabic numerals ranging from 1 to 9 and had to indicate which of both was the largest. *Ordering*: children were asked to name the number that came immediately before and after a given Arabic numeral.

With regard to working memory (WM), the Corsi Block Tapping task and a word recall forward task were administered to assess visuospatial and verbal short term working memory span, respectively. The Corsi Block Tapping task started with tapping a sequence of two until nine blocks, while the forward word span task required recalling a word string from one until five words. As an intelligence measure, the Block Design Test from WPPSI-III was administered. Children were asked to recreate a constructed model from a stimulus book using one- and two-colored blocks within a limited time.

Data collection

Children were tested individually in a quiet room by trained experimenters. In the first wave the early numerical skills were assessed in one session of approximately 30

minutes. During the second wave exact arithmetic, computational estimation, working memory, and intelligence were assessed separately during three sessions of approximately 30 minutes.

RESULTS

A multilevel analysis to determine the influence of the classroom and school context on children's performance revealed that these two factors explained less than 5% of the variance in exact arithmetic and computational estimation abilities between children. As a result, data were collapsed across classes and schools. Estimates that deviated more than three standard deviations from the mean PAE for each item were considered as outliers and therefore removed from further analyses (44 out of 3056 estimates or 1.4%). Descriptive statistics can be found in Table 1.

	Theoretical maximum	*M*	*SD*	Range	α
Visuospatial WM (number correct)	24	6.02	2.17	0 – 11	.64
Verbal WM (number correct)	15	9.16	2.11	3 – 15	.52
Intelligence (scaled score)	20	9.17	3.12	1 – 19	-
Exact arithmetic (number correct)	8	2.34	1.73	0 – 7	.73
Computational estimation (PAE)	-	32.70	16.40	0 – 86.31	.74
Dot enumeration (number correct)	9	5.57	2.12	0 – 9	.72
Verbal counting (largest number counted)	40	18.81	10.72	0 – 40	-
Object counting (number correct)	8	3.89	2.07	0 – 8	.71
Non-symbolic comparison (number correct)	12	9.43	1.98	4 – 12	.52
Symbolic comparison (number correct)	12	8.49	2.47	0 – 12	.64
Number ordering (number correct)	8	2.89	2.12	0 – 8	.74

Table 1: Descriptive statistics for the different measures that were included in the study.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1) Visuospatial WM										
(2) Verbal WM	.210***									
(3) Intelligence	.322***	.238***								
(4) Dot enumeration	.320***	.139*	.266***							
(5) Verbal counting	.331***	.224***	.388***	.403***						
(6) Object counting	.348***	.249***	.328***	.536***	.587***					
(7) Non-symbolic comparison	.277***	.153**	.287***	.248***	.321***	.323***				
(8) Symbolic comparison	.212***	.039	.221***	.244***	.349***	.369***	.263***			
(9) Number ordering	.293***	.190**	.255***	.417***	.501***	.596***	.348***	.401***		
(10) Exact arithmetic	.284***	.219***	.305***	.300***	.403***	.393***	.248***	.312***	.417***	
(11) Computational estimation (PAE)	-.289***	-.356***	-.325***	-.244***	-.345***	-.300***	-.212***	-.226***	-.341***	-.311***

Note. *$p < .05$, **$p < .01$, ***$p < .001$

Table 2: Partial correlations between the different measures that were included in the study, controlling for age at T2.

As a first step, we calculated all partial correlations controlling for age at T2 (see Table 2). All of the early numerical skills were significantly correlated with both exact arithmetic and computational estimation.

In a second step, two hierarchical multiple regression analyses were conducted to unravel the respective contributions of the general cognitive functions and the early numerical skills on exact arithmetic and computational estimation performance, respectively (see Table 3). In both regressions, age at T2 was entered in the first step. The general cognitive measures were entered in step 2 and the early numerical skills in step 3. Collinearity checks revealed no multicollinearity problems.

In the first regression, a total of 31% of the variance in exact arithmetic is explained by the full model, $F(1, 302) = 7.17$, $p < .01$. The inclusion of the different predictors in the respective steps always leads to a significant increase in explained variance. Inclusion of the early numerical skills in step 3, reduces the influence of both types of working memory to non-significance. Results show that intelligence, verbal counting, symbolic comparison, and number order are predictors of exact arithmetic. In the second regression, a total of 28.6% of the variance in computational estimation is explained by the complete model, $F(1, 296) = 5.96$, $p < .05$. Again, inclusion of the different predictors in the respective steps explains a significant additional portion of the variance. In contrast to exact arithmetic, verbal working memory remains significant when adding the early numerical skills in the step 3. Only verbal working memory, intelligence, verbal counting, and number ordering are predictors of computational estimation.

	Exact arithmetic			Computational estimation (PAE)		
	Step 1 (R^2 = .023**)	Step 2 (ΔR^2 = .149***)	Step 3 (ΔR^2 = .138***)	Step 1 (R^2 = .020*)	Step 2 (ΔR^2 = .208***)	Step 3 (ΔR^2 = .058**)
Age (T2)	.152**	.111*	.017	-.140*	-.100	-.046
Verbal WM		.147**	.092		-.270***	-.245***
Visuospatial WM		.188**	.060		-.167**	-.091
Intelligence		.207***	.114*		-.205***	-.153**
Dot enumeration			.071			-.053
Verbal counting			.158*			-.138*
Object counting			.039			.071
Non-symbolic comparison			.018			.002
Symbolic comparison			.112*			-.067
Number ordering			.181**			-.159*

Note. *$p < .05$, **$p < .01$, ***$p < .001$

Table 3: Hierarchical regression on exact arithmetic scores and computational estimation performance, standardized coefficients (β).

DISCUSSION

The current study focused on the predictive power of general cognitive functions and early numerical skills of kindergartners' exact arithmetic and computational estimation performance. In the first research question we were interested in which general cognitive functions are important for exact arithmetic and computational estimation and whether these are the same or different for both types of whole number arithmetic. We observed that working memory played a different role in both types of arithmetic skills. Verbal working memory remained a significant predictor after adding the early numerical skills for computational estimation but not for exact arithmetic. These results confirm those reported by Caviola et al. (2012), who showed that computational estimation places greater demands on working memory than exact calculation in third and fourth graders. In addition, our results expand Caviola et al.'s (2012) results by showing that this is also true in younger children on tasks administered with concrete materials. Surprisingly, despite the fact that both tasks were measured using concrete materials with verbal support, it is verbal working memory rather than visuospatial working memory that is an important predictor of kindergartners' computational estimation performance. These findings suggest that children rather rely on the verbal labels given by the experimenter than on the amount of objects being presented.

In the second research question, we were interested in which early numerical skills of 4-year-olds are predictive of exact arithmetic and computational estimation performance six months later. Results show that verbal counting and number order skills are significant predictors of both exact arithmetic and computational estimation performance. Children who can count higher are familiar with a larger number range and being familiar with a larger set of numbers facilitates both exact arithmetic and computational estimation performance. Furthermore, children who are better at ordering symbolic digits have more insight in the decimal number system which might help them to understand that, for example, the sum of the arithmetic problem must be larger than both addends. An unexpected result is that comparison is not a significant predictor of computational estimation in young children, suggesting that kindergartners do not simply rely on their sense for order of magnitude when solving the computational estimation task.

In sum, the importance of verbal working memory and verbal counting and the absence of an influence of comparison skills suggests that kindergartners do more than simply relying on their non-symbolic number sense and might already use a verbal code scheme when estimating. Secondly, our results confirm and expand previous research that enlarging kindergartners' counting range and paying attention to ordinality is not only beneficial for exact arithmetic, but also for their computational estimation performance. Thirdly, the fact that symbolic comparison was only a predictor for exact

arithmetic and verbal working memory only for computational estimation suggests that these abilities do not rely on the same processes.

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SEMIOTIC POTENTIAL OF INQUIRING-GAME ACTIVITIES

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We integrate the Theory of Semiotic Mediation with a phenomenological perspective to analyse the semiotic potential of GeoGebra when inquiring-game activities are implemented. These activities aim at developing a theoretical approach towards geometry in secondary school students, starting from game situations. An inquiry-game activity based on the property of isosceles triangle is presented and experimental data from a Grade 9 classroom is analysed. Results show how the inquiring-game activity contributes to the production of artefact signs that can evolve into mathematical meanings, and of different layers of meaning developed within the contexts evoked by these signs.

INTRODUCTION AND THEORETICAL FRAMEWORK

Research has shown that the use of Dynamic Geometry Environments (DGEs) can contribute positively in students' harmonization of theoretical and empirical aspects of geometry (Laborde et al., 2005). In particular, dragging tools have been proven to allow students experiencing the variation hinted in static diagrams, and so improving their ways of seeing and conceiving them as different instances of the same geometrical objects (Leung, 2003), beyond the prototypical cases that are usually considered in classical teaching (e.g. a rectangular with a longer horizontal side and a shorter vertical side). Since the creation of the first DGEs, researchers and teachers devoted their attention to *robust diagrams* (Healy, 2000), i.e. diagrams constructed according to geometrical properties of the figures. In these diagrams, the geometric properties of the figure remain invariant when its base-points are dragged. On the contrary, in *soft diagrams* properties are empirically constructed: hence dragging cannot be used for testing the construction and discerning the invariants.

In our study, we focus on *soft diagrams* within *inquiring-game activities* aimed at introducing students to the theoretical aspects of geometry (e.g. distinguishing between hypothesis and thesis in a theorem, reasoning on properties, formulating conjectures, etc.). Inquiring-game activities (Soldano & Arzarello, 2018) are inspired by games of verification and falsification, known as *semantical games*, used by the logician J. Hintikka (1998) to establish the truth of statements in his Game Theoretical Semantic. According to this theory, in order to establish the truth of statements expressed in the form $\forall x\ \exists y \mid S(x, y)$, we can imagine a game in which a player, called Falsifier (F) controls variable x and has the goal to show that the statement is false and another player, called Verifier (V) controls variable y and has to show its truth. In order to win, F will look for the worst values of variable x. If, even in the worst-case scenarios, the verifier is able to find a suitable value of y, then the statement is true; otherwise it is false. Within a DGE we conceive x and y as base-points of a *soft diagram* and $S(x, y)$

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as a geometric property (e.g. being an isosceles triangle). During the game, F typically tries to create troubles for the verifier by proposing non-prototypical diagrams. V, on the contrary, will try to show the possibility to achieve his goal in all situations, also in the non-prototypical ones.

Although each game is designed on a geometric theorem, it is not granted that this foreseen mathematical meaning does emerge when the students play the game. This aspect is framed within the Theory of Semiotic Mediation (TSM) with the notion of *semiotic potential of an artefact*: it refers to the double semiotic relationship occurring on one side, between the artefact and the personal meanings elaborated by the students when they use it, and on the other side, between the artefact and the mathematical meanings foreseen by an expert (Bartolini Bussi & Mariotti, 2008). Through the analysis of signs produced by students and teacher, it is possible to document how this double relationship emerges and evolves. The basic signs for the development of the semiotic potential are produced when the students use the artefact and are called *artefact signs;* also thanks to the teacher's intervention, these signs can evolve, first into *pivot signs* relating both to the specific instrumented actions and to the mathematical domain, and then into *mathematical signs* that refer to the mathematical domain as culturally and historically developed.

Also phenomenological perspectives on mathematical teaching-learning processes points out how a delicate process is to guide students to "see and recognize things according to 'efficient' cultural means" and to convert their "eye (and other human senses) into a sophisticated intellectual organ" (Radford, 2010, p. 4). As Radford stresses, it is necessary to promote a "lengthy process of domestication" (p. 4) of the way they are looking at things while learning mathematics. Depending on the ages and the backgrounds of the students, a given mathematical situation may evoke different contexts and lead to different sense-making (e.g. an increasing and decreasing continuous graph in a Cartesian plane may be seen as the picture of a mountain by young students). According to the mathematician and philosopher G. C. Rota (1991), the starting point for understanding how mathematical meaning arise and evolve is the key assumption that there is "no such thing as true seeing", but "there is only *seeing as*" (p. 239). In Rota's account, this process is referred to as *disclosure*. Disclosure is a Husserlian concept that indicates the process by which people make sense of and interpret the various situations of the world in the contexts in which they are exposed. Different contexts may provide different meanings to the objects of our knowledge; furthermore, these different contexts are not isolated, on the contrary they are *layered* upon one another, and the layers can generate *different (layered) meanings* in the flow of time.

Adopting Rota's phenomenological account, Arzarello, Ascari, Baldovino, and Sabena (2011) showed how suitable didactic techniques made by the teacher may promote different layers of meanings in students' disclosures of calculus concepts, highlighting the role of *making present* things that are absent, and of *prompting* students' *attention* on some specific aspects of the contexts (Mason, 2008).

Our study is grounded on the research hypothesis that using *soft diagrams* within the inquiry game can initiate the students' disclosure of the meaning of a geometric theorem because the necessary/sufficient condition for the thesis are not hidden in the construction steps but are produced by intentional moves made on *soft diagrams* within DGEs. Under this hypothesis, in this paper, we investigate which artefact/pivot/mathematical signs emerge in the students' disclosure processes based on inquiring-games and how the contexts in which these signs are produced contribute to the development of different layers of meanings.

METHODOLOGY

The study grounds on a pilot teaching experiment on geometric inquiring-games carried out in an Italian Grade 9 classroom. Twelve students were involved for about two hours in inquiring-games and in answering some guiding questions, and for another two hours in a classroom discussion under the guidance of the teacher. We took part in the experiment as participant observers, in collaboration with the teacher. Collected data consists of the videorecording and screen capture of two pairs of students while playing the game and answering the worksheet questions, the completed worksheets from all the students, and the videorecording of the classroom discussions. Videorecorded students were chosen by the teacher among those who felt self-confident in front of video cameras. Data are firstly analysed according to the lens of TSM, in order to seize the evolution from artefact signs to mathematical signs produced within inquiring-games activities. In order to deepen the interpretation of the evolution of signs, the analysis is then integrated with a phenomenological lens.

We present some results from the first inquiring-game activity, which is played on the following theorem (T_{MB}): "*if the median and the angle bisector drawn from the same vertex of a triangle coincide, then the triangle is isosceles*".

Within your pair, establish a verifier who moves point B and a falsifier who moves point C. Each match is made by two moves and the first one is always made by the falsifier. The verifier's goal is to make segment CD and line b coincide; the falsifier's goal is to prevent the verifier from reaching his/her goal.

The winner of the match is the player who reaches the goal at the end of the verifier's move.

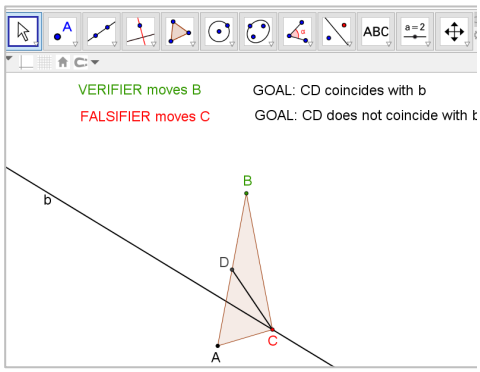

Table 1: Initial configuration and rules of the game (English translation)

The game is played in a GeoGebra file containing a dynamic triangle ABC (A is fixed, while B and C are free points), a segment CD robustly constructed as median from the vertex C and a line b robustly constructed as angle bisector from the vertex C (see Table 1). By moving B and C, the triangle changes and consequently also the positions

of the median and of the angle bisector change. However, being *robust diagrams*, they preserve their constructive nature, i.e. they always remain median and angle bisector. On the contrary, isosceles triangles may be obtained as *soft diagrams*. In particular, the falsifier may produce triangles in which the median and the angle bisector do not coincide (non-isosceles triangles, Fig.2a), and the verifier may produce isosceles triangles in which the median and the angle bisector coincide to the eye (Fig.2b).

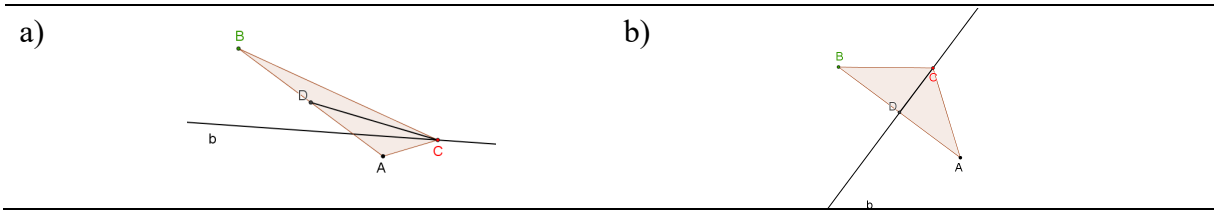

Figure 2: Standard examples of the falsifier's and the verifier's moves

The soft nature of the isosceles triangle and the game context are designed so to make the students *see* the link between the hypothesis of the theorem "the median and the angle bisector drawn from the same vertex of a triangle coincide" and its thesis "ABC is an isosceles triangle" *as* a conditional link. In fact, V's moves are guided by the intention of making the median CD and the angle bisector b coincide and the output produced by this type of move is an isosceles triangle. Once the students notice the isosceles triangles configuration, they can use it to guide the move and to provide the coincidence between median and angle bisector, reversing the conditional link. In order to foster this disclosure, questions are provided to students in a worksheet (Table 2). Q1 focuses students' attention on the geometric nature of CD and b (median and angle bisector), in the cases in which ABC is a triangle. Q2 guides students to observe the type of triangle produced by V's moves (isosceles). Finally, by exploiting the game dynamics Q3 is meant to disclose the relationships between the geometrical objects and to formulate T_{MB}.

Q1) What are CD and b with respect to the triangle ABC?
Q2) Which are the properties of the geometric figure when the verifier reaches his goal?
Q3) From the facts observed during the game and the answers given to the previous questions, formulate a geometric conjecture.

Table 2: Guiding questions on the game

DATA ANALYSIS

Game phase. All the pairs except one attribute the winning of all the matches to the verifier. Videorecorded students S1 and S2 follow this expected trajectory. We report an excerpt showing their first observations after playing several matches in which the verifier has always won:

1	S2:	Impossible that the falsifier wins
2	S1:	Moving after you *(dragging C in Fig.3a)*, it's always possible to create triangles *(making Fig.3b by moving B)*. C is always on the straight-line b, hence in order to move D, which is the midpoint, it is sufficient…

		make b coincide… then the verifier always wins. How is it possible not to make them coincide? *(moving C on AB, see Fig.3c)*

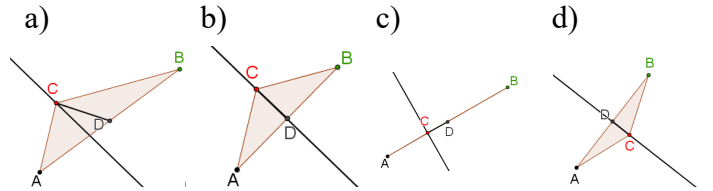

Figure 3: Configuration explored by S1 (fictitious game)

3	S2:	You can't!
4	S1	Even in this case you manage to make it *(moving B so that CD and b coincide, Fig.3d)*

The verifier's winnings are justified by S1 (who plays both the verifier's and the falsifier's role, ll. 2 and 4) through two fictitious matches: one played on a standard configuration (Fig.3a), the other played on a degenerate case (Fig.3c). These fictitious matches and their description can be interpreted within TSM as *artefact signs*. As a matter of fact, although geometric terms are used ("triangle", "midpoint"), students' attention is mainly focused on the game: they are disclosing what happens in the inquiring-game, which constitutes a first layer of meaning in the given context.

Question 1. All pairs of students easily recognize CD as the median of triangle ABC; on the contrary, they face more difficulties in discovering the geometric nature of line b, due to the confusion between the *soft properties* that line b can assume by dragging it and the *robust properties* that it always has because of its construction. Specifically, some students *see b as perpendicular to AB*: this is a *soft property* for line b, observable only in verifier's winning configuration (when the triangle is isosceles as a *soft diagram*). What is more striking is that students are disclosing a mathematical meaning for this line (a *mathematical sign* within TSM), which was *not expected* by the designers of the activity. When the teacher notices that S1 and S2 are working within this unexpected layer of meanings, he decides to intervene:

5	T:	Do you see any right angles there? *(pointing to the scalene triangle on the screen, configuration similar to Fig.3a)*
6	S2:	No, only when it [b] coincide with CD
7	T:	Ah interesting! Good point! But in general, you can't say it's perpendicular to AB, in general it has some other properties, try to understand it.
8	S1:	I don't know... I confess that I have no idea... *(starts dragging C)*

In his intervention, the teacher prompts students' attention on the line b when the triangle configuration is non-isosceles and b is not perpendicular (l. 5, configuration similar to Fig. 3a). In this way, S1 and S2 become aware of their mistake (l. 6) but they need more investigation to provide the right answer to question 1 (l. 8), which entails *seeing* line b *as something else* than a perpendicular line. A second teacher's intervention which prompts a double focus of attention ("on the property which remains true while moving the points" and "on the known elements of triangles") is necessary.

Question 2. Four pairs of students out of six give the expected answer (the triangle is isosceles). One pair writes that all points are aligned on line b. S1 and S2 write that the configuration is "always a straight-line and sometimes an isosceles triangle". This answer is a compromise between two different ways of seeing the situation: S2 is focused on the isosceles triangle, S1 is focused on the straight line that appears in the degenerate cases. The teacher decides to intervene:

9	T:	If you read question 2 here, [reads] 'which are the properties of the geometric figure when the verifier reaches his goal'... if you want, which are the properties of the triangle each time the verifier reaches the goal? That is, how would you describe that triangle?
10	S1:	Scalene
11	S2:	Isosceles
12	S1:	Why do you see it as isosceles *(looking Fig.4a)*? Maybe it is isosceles *(making Fig.4b by moving C)* Do you really see it as isosceles? *(He moves C upside down and obtains Fig.4c)* Yes, ok. It's isosceles *(making Fig.4d)* So every time they coincide it is an isosceles triangle.

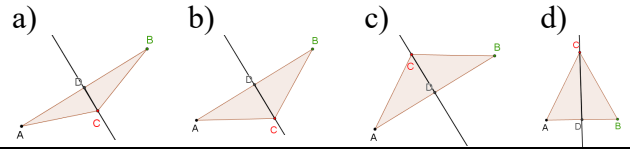

Figure 4: Getting a prototypical configuration in order to answer to question 2

The teacher reformulates the question: instead of speaking of "properties of geometric figure", he makes explicit reference to the "triangle" (l. 9). In this way, he is prompting the students' attention towards the envisaged mathematical signs involved in T_{MB}. In order to discern which kind of triangle is ABC when the verifier wins, S1 drags the points until AB is horizontal: in this prototypical configuration she is able to *see* ABC *as* an isosceles triangle (l. 12).

Question 3. Three pairs of students write the statement of the theorem using "if… then…" formulation, i.e. the mathematical sign corresponding to the expected outcome of the activity. One pair produces a pivot sign mixing artefact signs (referring to the game) with mathematical signs: "If the verifier reaches the goal then an isosceles triangle is formed". Two pairs do not get to the expected formulation of the theorem. In one case, they focus on a degenerate configuration ("If CD is the median of the triangle then aligned with the angle bisector, it makes a straight line"); in the second case, they do not mention the triangle ("If CD is perpendicular to AB then CD coincides with straight line b"). The latter answer is written by the second video-recorded pair of students. They played the game in an unexpected way, using the zoom tool to check the coincidence between b and CD (e.g. Fig. 5c) and to fix the position of point B (e.g. Fig. 5d). Since by zooming the coincidence is never accurate enough the students get lost in fixing/zooming cycles of the triangle configuration. The produced artefact signs develop a layer of meaning which cannot evolve to the expected mathematical one.

The overall data analysis shows that some of the extreme configurations produced by students in playing the game (*soft diagrams*) prompts them on unforeseen paths in which the produced artefact signs do not evolve in the expected mathematical ones. In Fig. 5 we report on such extreme configurations, produced with the dragging tool and/or the zoom tool:

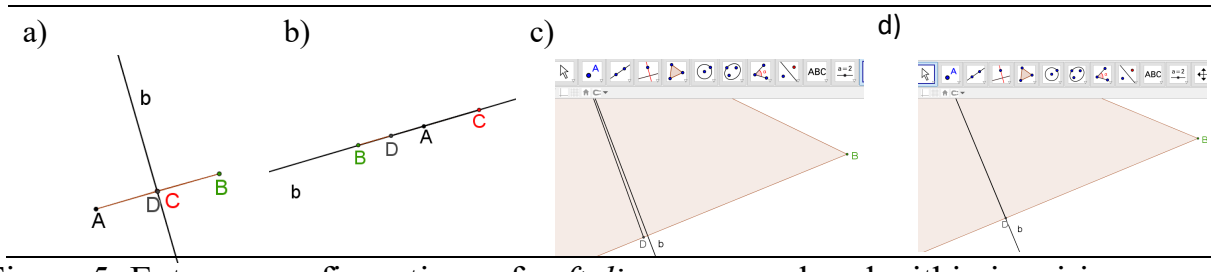

Figure 5: Extreme configurations of *soft diagrams* explored within inquiring-games

Fig.5a-b are degenerate cases in which V's goal is reached (b and CD coincide) but the artefact signs cannot evolve in the expected mathematical sign since the triangle ABC is collapsed. Fig.5c shows a zoom-in configuration in which V's goal is reached but the triangle ABC it is not visible because of its big dimensions.

CONCLUSION

The analysis provides evidences that through guiding questions and the teacher's interventions, most of the produced artefact/pivot/mathematical signs evoke expected layers of meanings related to the game and to the mathematical contexts, initiating students in the disclosure of the geometric theorem on which the game is based. Our hypothesis that the use of *soft configurations* allows students to experience the conditional link on which the theorem is based is confirmed (4 pairs out of 6 formulate the expected statement of the theorem as answer to question 3). Furthermore, results indicate that the game dynamics push the students in a deep exploration of both prototypical and non-prototypical situations (e.g. zoomed configurations and degenerate diagrams). The letter situations can produce unforeseen artefact signs and develop unexpected mathematical layers (e.g. seeing the angle bisector as a height; seeing the degenerate configuration as general case).

We believe that the exploration of these configurations should not be prevented to students since they can enrich their theoretical reflections and understanding of geometry. Degenerate figures can be exploited by the teacher to define the boundaries of the theorem subscribed by its hypothesis, hence they can be used to reflect deeply on the role of the hypothesis in the statement of the theorem. In our case, if ABC is not a triangle, CD cannot be the median drawn from vertex C and b cannot be the angle bisector drawn from the same vertex, since their existence depends on the triangle configuration, hence we cannot make sense of the theorem using these configurations. Our current research is investigating how, in order to discuss these situations, a class discussion on the verifier and falsifier dynamic and on the transformations produced on *soft diagrams* with the dragging tool in DGE may be managed.

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ANALYSING THE ROLE OF LANGUAGE IN STUDENTS' CONCEPTION OF PARALLEL LINES

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The concept of parallel lines is foundational to much of the work that students need to do in high school geometry – especially in reasoning tasks involving parallel lines and transversal and in developing a relational understanding of different quadrilateral classes. There appear to be a few exceptionally tenacious alternate conceptions related to parallel lines amongst students who learn mathematics in Hindi. This is likely to add to the already considerable difficulty that students in state-run secondary schools in India have in learning Euclidean geometry. This paper examines data from a large-scale research project to get a nuanced understanding of students' concept of parallel lines and also explores whether a language-related issue might be challenging concept formation.

INTRODUCTION

The teaching and learning of Euclidean geometry at the secondary level is a complex challenge that several researchers and frameworks have tried to address (Battista, 2007; Shaughnessy & Burger, 1985) over the past few decades. In the context of learners studying in state-run secondary schools in India, formal geometric reasoning and proofs, which constitute over a third of the secondary mathematics curriculum, pose an even greater challenge. Additionally, there are complexities related to the multilingual context of India. A module on Geometric Reasoning (GR) designed for the Connected Learning Initiative (CLIx) project aims to address some of these by designing learning experiences that factor in the complexity of the challenge, making selective use of technology to do so. The GR module attempts to help students progress from lower to higher van Hiele levels of reasoning.

Some diagnostic studies done prior to module development had revealed several learning challenges – some of them expected, but also a few unexpected ones – like some extremely tenacious alternate conceptions in students' understanding of parallel lines (Srinivas, Khanna, Rahaman, & Kumar, 2016). It was important to understand the nature of the challenge faced by learners and address it, since a robust understanding of parallel lines is a foundational concept in geometry. It was also somewhat curious, since no other study that we came across had stressed these specific alternate conceptions previously, even those specifically on parallel lines (Mansfield and Happs, 1992). A recent research had discussed the influence of language on students' relational understanding of squares and rectangles (Bussi & Baccaglini-Frank, 2015). This led us to examine whether something in the language of instruction – Hindi, might be influencing students' concept formation of parallel lines in some

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 327-334). Pretoria, South Africa: PME.

manner. It seemed liked an interesting and less-explored problem to study and report at the PME.

THEORETICAL ORIENTATION

In the secondary classrooms where this study is situated, students are typically introduced to formal concept definitions from the textbook and expected to use those for reasoning tasks. But contrary to this expectation, it is not the concept definition, but a very personalised mental image of the concept that is evoked in each student when they have to retrieve a concept for a task. This is what some researchers have called "concept image" (Vinner, 1983). Vinner argued that concept definitions "remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked." While teachers at the secondary expect concept formation to be a one-way process – a precise formal definition leading directly to a clear and accurate concept image, this in fact is not so. The final concept image, which accommodates all features present in the formal concept definition and discards the non-essential features, is formed and refined over time. Watson and Mason (2002) discussed the idea of 'personal example spaces' and propounded 'extending the example spaces' as an important aspect of concept formation. We use the idea of concept images and the extension of students' personal example spaces to reveal students' existing and developing understanding of 'parallelness'.

SAMPLE, TOOLS AND METHODOLOGY

Our research is a sub-study done as part of a large-scale learning outcomes study done in CLIx. The study on geometry learning, which provided us data for our sub-study was undertaken with Grade 9 students in 10 Intervention schools (INT) and 9 non-Intervention schools (non-INT) in the Dhamtari district of Chhattisgarh, a state in east-central India. The official language of learning at the secondary level here is Hindi. The tools used in the larger study from which we have drawn data for our sub-study are mentioned in the table below.

Tool	Description and Purpose	Sample
Pre-test and Post-test	Written assessment tools based on van-Hiele levels with 8 MCQs + constructed response items	INT: 466 (91.9% of the cohort, paired) non-INT: 499 (88.3% of the cohort, paired)
Student Interviews	Interviews done with pairs of students immediately after the Pre, and then again after the Post-test	4 pairs from each school – 2 pairs each of high-performing and average-performing, ensuring equal representation of boys and girls
Observation Freewrite	At least 2 classroom observations in each school	(All students present in class on observation days)

Table 1: List of tools used in the sub-study on parallel lines

OBSERVATIONS AND ANALYSIS

According to the data from the MCQ items of the written pre and post-tests, the INT group (Gain score 0.93) showed significantly higher ($p<0.001$) overall learning gains than their peers in the non-INT group (Gain score 0.19). This was true even on individual items, including those on difficult concepts like understanding hierarchical class relationships amongst quadrilaterals. However, both groups had difficulty on the item on identification of parallel lines even in the post test. In the following sections, we discuss in detail the students' concept images of parallel lines that were encountered and explore a few probable explanations.

Data from the Pre-test and Post-test

In this sub-section, we discuss the MCQ items on parallel lines in the written tests.

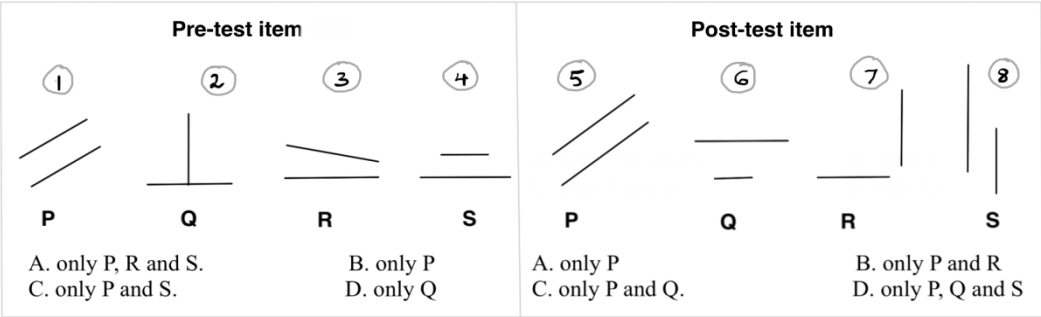

Figure 1: Test items on parallel lines: 'In which of these figures are the lines parallel?' (Translated to English, figures numbered here for easy referencing. The figures will be referred to as Img.1 to Img.8 in the text)

	Pre-test				Post-test			
	A	B	C	D	A	B	C	D
INT	13.3	48.3	21.9	7.3	51.1	12.0	7.8	23.8
non-INT	11.2	52.3	24.7	7.4	53.7	12.6	6.2	25.3

Table 2: Performance data on the Pre-test and Post-test items on parallel lines

In the pre-test, only 21.9% of the students in the INT group, and 24.7% in the non-INT selected the correct option. Just over 10% in each group selected option A, perhaps intuitively, based on overall appearance. Nearly half the students in each group selected option B -indicating, perhaps, that they couldn't identify unequal parallel line segments (Img.4) as a valid example. In the interviews that followed the pre-test, students' concept images were explored through further probing.

In a bid to understand whether they had a concept definition in place, students were asked to explain what they meant by lines being 'parallel'. Many students could retrieve some form of mathematically acceptable definition of parallel lines. Most

students who tried to give a formal definition used the 'lines that never intersect' construct in their definition rather than the 'maintaining a constant distance' construct. This was especially interesting for two reasons:

- In Hindi, the language of learning and teaching in these schools, the term for 'parallel' is *samaantar* (or *samaanantar*)– which is a conjugation of *samaan*[equal] + *antar* [distance], a direct statement of the 'maintaining a constant distance' construct, and
- their grade 8 textbook defines parallel lines using the 'constant distance' construct and in the context of state-run schools in India, the textbook is the primary (and in most cases, the only) resource and considered sacrosanct.

The idea of lines that 'never intersect' was articulated with varying degree of sophistication by the students – ranging from the very mathematical "*kabhi pratichhed nahi karti* [never intersect]", to the informal "*ek doosre ko kaat ti nahi hai*" [don't cut each other] or "*ek doosre se takrayengi nahi*" [won't collide with each other]. One common colloquial phrase used by some students for parallel lines was '*sojh*' [straight] or '*sojh-sojh'* indicating lines that move 'straight' and don't intersect. A few students produced incorrect concept definitions – for instance, one boy who used the term *sojh* interpreted it as 'horizontal' and mentioned in the interview that he would choose only Img.4 (pre-test) as parallel if that had been an option.

Interestingly, not even half the students who could produce mathematically acceptable definitions in the pre-test could identify Img.4 as parallel. This seemed to indicate a gap between the concept definition and the evoked concept image. Data from the pre-test and interviews revealed that a concept image that students frequently evoked was that 'parallel lines are necessarily equal'. While we foresaw this happening at the pre-test stage, a considerable change was expected post teaching - especially in the INT group, as the GR module had several tasks specifically designed to reveal students' temporary concept images, and 'extend their example spaces'. Despite this, in the post-test item too over 50% of students in the INT group chose only Img.5 (Option A), the equal parallel lines. This led us to examine whether the Hindi terms being used was somehow informing this concept image. Examples from the student interview data and the class interactions provided some insights about how the terms used were influencing student thinking about these concepts and are discussed in the next sub-section.

A classroom snapshot: *The story of sam, samaan, samaantar and smaanaantar*

This sub-section reports an excerpt from a classroom discussion between the teacher (T) and a group of girls (G1 to G3) that happened in INT-10, one of the INT schools.

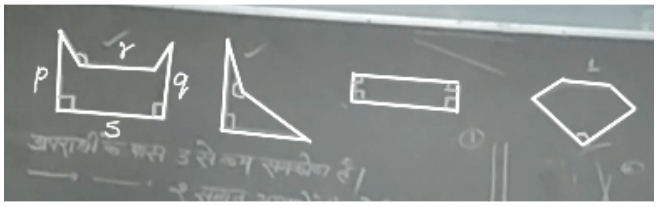

Figure 2: A snapshot of the blackboard during the classroom discussion on '*samaan*' [equal] vs '*samaantar*' [parallel]

The class is discussing properties of shapes and T asks whether the first shape (see Figure 2) has equal sides. The students reply 'yes' in chorus. T points to sides *p* and *q* and asks the class if those were the sides that are *samaan*[equal]. A few students say 'yes', while others, including student G1, disagree. She indicates sides *r* and *s* and says those are equal. T asks whether they appear to be equal. Some students say 'no', but G1 and some others say 'yes'. T ignores those who said yes and confirms that *r* and *s* are not equal and asks students to justify. A discussion follows:

G1: Par sir, yahan *baraabar* nahi bol rahe, *samaan* bol rahe hain!/ [But sir, they aren't saying *baraabar* here, they are saying *samaan*l!] (*'Baraabar'* is the colloquial word for 'equal' in Hindi, while '*samaan*' is the formal one.)

T: *Samaa*n aur *baraabar* mein kya antar hai? [What's the difference between *samaan* and *baraabar*?]

G1: (indicating sides *r* and *s*) *'Baraabar*' matlab, jaise yeh 2 cm hai toh yeh bhi 2 cm Aur *samaan* mein na sir, usko aage bhi badha sakte hain… (trails off, appearing unsure)/ ['B*araabar*' means if this one is 2 cm then this one is 2 cm too. And sir, in *samaan,* they can be extended further…]

T then asks G1 and another girl G2 to come up to the board and there is a discussion with an isosceles trapezoid drawn on board where G2 uses the construct of 'lines that never intersect' to explain 'parallelness'. G1 repeatedly points at the pairs of parallel lines during her explanations, using the word *samaan* for them. At this point T specifically asks her to define what '*samaan*' means.

G1: *Samaan* woh hotein hain 'jo ek doosre ko kabhi nahi kaat ti./ [*Samaan* means those (lines) which never cut each other.]

T: Usko *samaan* bolte hain?/ [Are those called '*samaan*'?]

G1: *Samaantar* bolte hain sir…/ [They are called *samaantar*(parallel), sir…]

T: Toh phir…?/ [Then…?]

G1: *Samaantar* matlab *samaan* hota hai. / [*Samaantar* means *samaan*]

T: Kaun bolta hai?/ [Who says so?]

At this point, T comes to the board and unpacks the terms used by explaining their conjugation: *samaanaantar* as *samaan* + *antar*, and *samaantar* as *sam* (also meaning equal) + *antar* (distance*),* explaining how this actually defines the notion of parallelness. The discussion continues and by now many students are seen discussing this in their own groups. After a while when T asks if the distinction between *samaan*

and *samaantar* was now clear, about half the class says 'yes'. When T is about to move on, G1 and G3 stop him and ask him something (inaudible), and again there is a discussion on *samaan* and *samaantar*. This happens twice. T explains the derivation of the terms again and finally asks students to classify a few pairs as examples and non-examples.

Post-test interview responses

In the interview following the post test, G1 was paired with student G3, and an attempt made to reveal their 'personal example spaces' and the extent of concept formation through some extension tasks that had not been part of the interviews done after the pre-test.

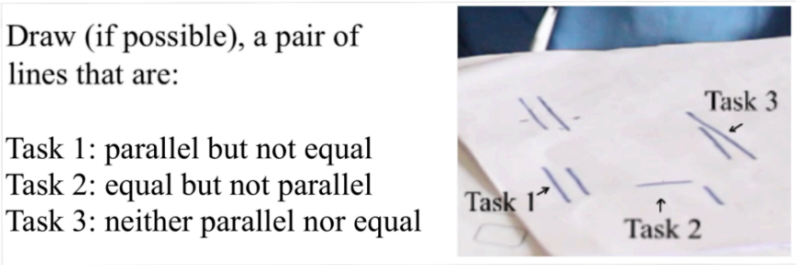

Figure 3: Extension tasks 1-3 of the Student Interviews and G1's responses

On Task 1, both G1 and G3 could individually produce correct examples. However, when presented with Task 2, G1 looked puzzled, and her immediate response was "*...jo samaan hai ussey samaantar bol sakte hain hum*" [lines which are equal 'can be called parallel']. G3 interrupted her to counter this by pointing out Img.7 of the post-test as an example that fits the task. G1 pondered over this, and subsequently, both students produced correct examples that resembled Img.7, with G1 agreeing that it is indeed possible. Further on, in Task 3, G1 could produce an example, and justify why her example fitted the task. But her initial response to Task 2 showed that her concept images of '*samaan*' vs '*samaantar*' were still not quite robust, and she was not able to isolate one concept from the other with consistency.

DISCUSSION

Based on class observation data and the student interactions, it appeared that a few students (like G1, at the initial stages of the reported classroom discussion), and possibly those who chose the lines in Img.7 too as 'parallel', were unable to discern the difference between the two terms and often used them interchangeably. In both pre and post intervention interviews, many such students expressly stated that there is no difference between '*samaan*' and '*samaantar*'. However, a much higher percentage of students (51.1% in INT group) had a different issue – while they understood the idea of 'parallelness' as lines or line segments that would never intersect even if extended, they *assumed equality of length* as an essential feature of the concept. This was a notion that they found (as the data showed) extremely difficult to discard. This seemed to be the case with G1 during the post-test interviews.

Several possibilities were explored while trying to figure out the reason why for so many students the evoked concept image of parallel lines was of *those that would never intersect, but were also necessarily equal in length*, and why this particular concept image was so unshakeable. The examples that students have had prior exposure to, was deemed to be a possible influence. However, it did not seem like the most compelling one, especially because their previous class (Grade 8) textbook had roughly 20 examples each of equal and unequal parallel lines (or line segments).

A more plausible explanation seems to be the linguistic challenge related to the Hindi words for the two concepts – '*samaan*' and '*samaantar*'. This seems plausible in the light of this particular notion not having been reported as being widely prevalent, or persistent, in hitherto reported studies on students' concept of parallel lines (Mansfield and Happs, 1992). We discuss a few possible ways in which the terms used could be impeding students' concept formation.

The words *samaan* and *samaantar* are close to each other in sight and sound and might be causing students to substitute the usage of one for the other. This is compounded by the fact that there is a third word in Hindi, called '*sam*', also meaning equal (among other things), which is also used in mathematics, especially in the context of geometry. So *sam* and *samaan* both mean the same thing (equal), while *samaantar* (*sam* + *antar*) and *samaanantar* (*samaan* + *antar*) both mean the same thing too (parallel). In this context, we expect students to understand that *samaan* and *samaantar* represent two different concepts. One more compounding factor here could be that the everyday word used by the students for 'equal' is not *samaan*, but *barabar*.

Another challenge could be that the word '*samaan*' is nested in the word '*samaantar*' in its entirety. It is possible that in the students' mental schema, this results in the concept of *samaan* (equal) being subsumed within the concept of *samaantar*(parallel). It's important to note that both *samaan* and *samaantar* are attributes applicable to the entities in question – line segments. This might offer a plausible explanation for students' extreme difficulty in dissociating the equality feature from the concept image of parallel lines.

In their study, Bussi & Baccaglini-Frank (2015) had suggested that the *inclusive* sequence of ideograms in the representation of 'squares' and 'rectangles' in Chinese is perhaps more effective than the separate, everyday names 'square' and 'rectangle' learnt by English speaking students for developing an understanding of their inclusive class relationship in later grades. In our context, the inclusion of one unrelated concept (*samaan*) within another in the Hindi term for parallel (*samaantar*) might perhaps be impeding concept formation, making it more difficult for students to separate the essential and non-essential attributes of the concept of parallel lines. This is something that needs to be studied further. Also, it is not to say that this is the only challenge to students' understanding of parallel lines – there are many others, a few of which have been discussed here. It is important for teachers to understand these challenges and take up tasks that help in revealing students' concept images, and weave a discourse

around it to reduce the gap between the concept image and the concept definition, in order for them to be used effectively in reasoning tasks.

Additional information

Study done with Connected Learning Initiative (CLIx). Seeded by Tata Trusts, Mumbai; Founding Partners: Tata Institute of Social Sciences, Mumbai and Massachusetts Institute of Technology.

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TEACHERS' INITIAL RESPONSES TO HIGH-LEVERAGE INSTANCES OF STUDENT MATHEMATICAL THINKING

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We investigate teachers' initial in-the-moment responses to instances of high-potential student mathematical thinking (SMT) during whole class discussion to understand what it means to productively incorporate SMT into instruction. Teachers' initial responses were coded using the Teacher Response Coding scheme, which disentangles the teacher action, who the response is directed to, and the degree to which the SMT is honored. We found that teachers incorporated students' actions and ideas in their response, but tended to address the SMT themselves and did not fully take advantage of the SMT. We consider the productivity of teachers' initial responses in relation to principles of productive use of SMT and compare the results to those of a previous study of teachers' hypothetical initial responses to SMT in an interview setting.

The incorporation of student mathematical thinking (SMT) into classroom instruction has been a consistent focus of recommendations for effective mathematics instruction (e.g., National Council of Teachers of Mathematics, 2000, 2014) and has been recognized as a central component of what has been referred to as *ambitious* (Lampert et al., 2013) or *responsive teaching* (Robertson, Atkins, Levin, & Richards, 2016). Research has begun to help us understand how to effectively incorporate students' thinking about a high-level task that they have worked on (e.g., Stein, Engle, Smith, & Hughes, 2008), but much less is known about how to effectively respond to and incorporate SMT that emerges during mathematics classroom discourse. Research has shown, however, that teacher responses matter; the ways in which teachers respond to SMT affects student learning in the classroom and can support very different types of instruction. For example, Kazemi and Stipek (2001) identified teacher responses that contributed to high-press and low-press interactions, with the high-press interactions supporting classroom instruction driven by SMT. Ing et al. (2015) studied teacher responses that encouraged students to engage with each other around SMT (e.g., asking students to explain each other's strategies and to discuss differences among strategies), finding such responses to be correlated with increased student participation and higher achievement. Correnti et al. (2015) identified the use of *uptake* moves—a move to "extend, deepen, clarify, or elaborate the discussion" (p. 308)—as a key difference in the responses of two teachers who enacted instruction aligned with traditional versus ambitious mathematics teaching, with this move accounting for four times as many responses in the classroom aligned with ambitious instruction. This suggest that the use of this move could be a key difference between such classrooms.

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One issue related to responding to SMT is that not all student thinking has the same potential to provide leverage for accomplishing mathematical goals, and thus, does not all warrant the same response. We are interested in understanding how teachers might most productively respond to particular instances of student thinking that have high potential to advance student learning, those we have identified as MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking (Leatham, Peterson, Stockero, & Van Zoest, 2015). In prior work we have developed the MOST Analytic Framework, a tool to identify instances of student thinking that are MOSTs (Leatham et al., 2015), but we are only beginning to understand effective teacher responses to MOSTs. In Stockero, Van Zoest, Peterson, Leatham, & Rougée (2017) we investigated teachers' descriptions of how they would respond to a common set of MOSTs in a scenario-based interview. This study provided some important insights into initial teacher responses, including that such responses most often were directed to the student who had contributed the MOST, aimed to develop or justify the student contribution, and incorporated the students' words and ideas. However, we do not know the extent to which these hypothetical responses that teachers described in the scenario interview reflect teachers' actual responses during instruction. Thus, the study reported here addresses the question: What are the characteristics of teachers' initial in-the-moment responses to MOSTs that surfaced during their instruction? We use these characteristics to discuss the productivity of various teacher responses. We also compare teachers' responses during instruction to a set of hypothetical responses during a scenario interview. Better understanding teacher responses will contribute to our understanding of the current state of the practice of responding to MOSTs, as well as what it means to productively incorporate high potential student thinking into instruction.

THEORETICAL FRAMEWORK

MOSTs are instances of SMT that are particularly worth noticing and acting upon since they have high potential to help students better understand important mathematical ideas if made the object of consideration by the class. As we have described elsewhere in greater detail (Leatham et al., 2015), MOSTs simultaneously satisfy three critical characteristics of student contributions: student mathematical thinking, significant mathematics, and pedagogical opportunity. We see these instances as those worth *building* on—that is, "student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). As elaborated elsewhere (Stockero et al., 2017), our conception of productive use of MOSTs is grounded in four core principles of quality mathematics instruction (Figure 1) that we distilled from current research and calls for reform (e.g., NCTM, 2014). Thus, to determine the productiveness of a teacher response to a MOST, we focus on the extent to which the response effectively coordinates these core principles.

Mathematics Principle: The student mathematics of the MOST is at the forefront.
Legitimacy Principle: Students are positioned as legitimate mathematical thinkers.
Sense-making Principle: Students are engaged in sense making.
Collaboration Principle: Students are working collaboratively.

Figure 1: Principles underlying productive use of MOSTs (Stockero et al., 2017)

METHODOLOGY

This study is part of a larger project focused on understanding what it means for teachers to build on SMT during classroom instruction (see BuildingOnMOSTs.org). The larger MOST project analysed 11 videotaped mathematics lessons from 6-12th grade mathematics classrooms from different geographical regions of US that reflected diversity of teachers, students, mathematics topics, and curricula. None of the teachers had training specific to responding to MOSTs.

The data analysis of these lessons for this study focused on two different units of analysis: an instance of student thinking and the teacher's initial response to that instance. In prior work, we analysed each instance of student thinking that occurred during whole-class interaction to identify those that were MOSTs. In the current study, we analysed the initial teacher response, our second unit of analysis, to each MOST found in the data. We operationalized a *teacher response* as *the collection of observable teacher actions that begins as a given instance of SMT ends and concludes when the initial teacher turn ends or there is a clear shift to a different activity*.

In the 11 video-taped classroom lessons, we identified 251 MOSTs for which a teacher response was inferable. We applied the Teacher Response Coding Scheme (TRC) (Peterson et al., 2017) to each of these responses. Figure 2 provides the TRC coding categories, definitions and codes.

RESULTS

Our analysis revealed that some aspects of teachers' initial responses to MOSTs aligned well with the principles underlying productive use of MOSTs while others were less aligned with these principles. We begin by discussing the Student Recognition aspects of teacher responses, which generally showed good alignment with the principles. We then discuss the most common Actor and Action codes in the data and discuss how, in many cases, the alignment of these aspects of teacher responses to the principles could be enhanced with variations in the responses.

Student Recognition of Actions and Ideas

We capture the extent to which teachers honoured student thinking in their initial response to a MOST using the categories Recognition-Student Action and Recognition-Student Idea. Recognition-Student Action (as described in Figure 2) is the

Category	Coding Category Description	Codes
Actor	Who is publicly given the opportunity to consider the instance of student mathematical thinking	teacher, same student(s), other student(s), whole class
Student Recognition	Recognition-Student Action: The degree to which the teacher response uses the student action, either verbal (words) or non-verbal (gestures or work)	explicit, implicit, not
	Recognition-Student Idea: The extent to which the student is likely to recognize their idea in the teacher response	core, peripheral, other, not applicable
Action	What the actor is doing or being asked to do with respect to the instance of student thinking	adjourn, allow, check-in, clarify, collect, connect, correct, develop, dismiss, evaluate, justify, literal, repeat, validate

Figure 2: Teacher Response Coding Scheme (TRC).

extent to which the teacher response uses the student action, either verbal (words) or non-verbal (gestures or written work). About 70% of the teacher responses to MOSTs used either the student's specific actions (coded as explicit; see example below) or referred to the student's actions using pronouns or other referents (coded as implicit; see Table 1). The remaining 30% of the teacher responses did not explicitly or implicitly use the student actions (coded as Not).

		Student Ideas				
		Core	Peripheral	NA	Other	Total
	Explicit	95	15	3	2	115 (46%)
Student	Implicit	54	5	5	1	65 (26%)
Actions	Not	23	14	31	3	71 (28%)
	Total	172 (69%)	34 (14%)	39 (16%)	6 (2%)	251

Table 1: Recognition of student actions and ideas.

In the Recognition-Student Idea category, about two thirds of the teachers' initial responses were coded as core to the student mathematics of the MOST (see example below). This means that the teacher response focused on a main idea of the MOST in a way that the student who contributed the instance would likely recognize the idea as their own. Only 2% of the teacher responses for Recognition-Student Ideas focused on an idea that did not seem to be related in any way to a main idea of the MOST.

It is worth noting that about 82% (95 out of 115) of teacher responses to MOSTs that were coded as *Explicit* for Recognition-Student Action were also coded *Core* for Recognition-Student Idea. Similarly, 83% (54 of 65) of the responses that were coded as *Implicit* for Recognition-Student Action were also coded *Core* for Recognition-Student Idea. As an example of a response that is *Explicit* and *Core*, consider an instance where students were finding the lengths of the sides of similar triangles. A student said, "Oh wait, you can have any number for side A. You just have to use the corresponding… you just have to use the rules to get sides B and C." In response, the teacher asked the student who contributed this idea, "So what are the rules to get sides B and C if you start with side A?" This teacher response used the student's words and focused on a main idea of the MOST in a way that the student would likely recognize the idea as their own. Our results indicate that teachers are honouring students' thinking to a great extent in their responses by using the student actions and focusing on the students' ideas. This is important because such responses align with the first principle of productive use of MOSTs (see Figure 1), the Mathematics Principle, since the mathematics of the MOST is at the forefront.

Actor and Action

Six moves occurred most frequently in the data (clarify, develop, dismiss, evaluate, literal and repeat), collectively accounting for two-thirds of all teacher initial responses (Table 2). Three of these actions—dismiss, literal and repeat—clearly do not position students to make sense of the SMT because they either let the instance pass by (dismiss), or require only a minimal response (literal, repeat). Yet, they made up about 38% of the data. For example, in a calculus class a student said that they had used the quotient rule on $\int \frac{e^x}{1+e^x}\,dx$ and got $\frac{e^x}{1+e^x}$. The teacher responded to this incorrect answer by directing the whole class, "OK, the quotient rule. Tell her when you use the quotient rule." Rather than directly asking the other students to engage with the student's mathematics, the teacher instead asked them to give a literal response—factual information about the use of the quotient rule. Additionally, of the 95 response that fell into these three categories of actions, 56 (59%) also had a teacher actor, meaning the teacher did not use the action to engage students in the instance at all. Given that the instances of SMT in our data were all MOSTs, the abundance of such responses results in a lot of lost opportunities for students to make sense of important mathematics, and thus do not support the Sense-making Principle.

The other three predominant actions in the data have more potential to support building. Clarify actions (11% of the data) seek to make the instance of SMT more precise. Develop actions (10%) provide or ask for an expansion of the instance that goes beyond a simple clarification but does not require a justification. Evaluate actions (9.3%) provide or ask for a determination of the correctness of the SMT. These actions have potential to position the class to make sense of the SMT (and thus support the Sense-making Principle), either by making clearer the student thinking they are going to consider or by asking them to engage with the thinking in some way. As with the

	Actor			
Move	Same Student	Whole Class	Teacher	Total in Data Set
Literal	8	20	17	46 (18%)
Clarify	10	1	16	28 (11%)
Repeat	8	2	16	26 (10%)
Develop	11	5	9	25 (10%)
Dismiss	0	0	23	23 (9%)
Evaluate	3	14	6	23 (9%)
Total in Data Set	56 (23%)	69 (27%)	116 (46%)	251

Table 2: Summary of subset of actor and moves codes discussed in the paper.

actions discussed previously, however, clarify and develop most often had a same student or teacher actor, meaning that the engagement that would result from the action would likely be quite limited. Evaluate moves were the only one of these potentially-productive moves that most often engaged the whole class.

To illustrate why it is more productive for teachers to direct their initial responses to MOSTs to the whole class, consider the previously-discussed instance related to finding the lengths of the sides of similar triangles. The teacher response, “So what are the rules to get sides B and C if you start with side A?" was directed to the student who contributed the idea. In directing the response in this way, it engaged the contributing student in developing their idea, but it limited the opportunity for other students in the class to engage with the important mathematics of similarity, and thus did not support the Collaboration Principle. This broader engagement could have easily been achieved had the teacher instead directed the same question to the whole class.

DISCUSSION AND CONCLUSION

The high level of honouring student thinking that we observed in the use of the student actions and the incorporation of student ideas aligns with the principles underlying productive use of MOSTs (Figure 1) by putting the student mathematics of the MOST at the forefront, and by positioning students as legitimate mathematical thinkers. Also, with nearly half of the initial responses to MOSTs engaging either the same student or the whole class, the data indicates that students are being given the opportunity to engage with each other’s thinking—a component of the Collaboration Principle. Since the instances to which the teachers are responding to are MOSTs, however, they are instances that the whole class could engage with, making the move of going back to the same student a missed opportunity for students to work collaboratively and thus for the response to fully align with the Collaboration Principle.

A comparison of the results of the actor coding in this study to the hypothetical responses to the scenario interview discussed in our previous work (Stockero, et al., 2017) provides additional insight into the extent to which teachers' initial responses align with the principles underlying productive use of MOSTs. In both settings, teacher responses were directed to the whole class about one fourth of the time. The teachers' hypothetical responses had a teacher actor 6% and a same student actor 65% of the time, but in the actual classrooms reported in this study, the teacher was the actor 46% and the same student was the actor 23% of the time. In both settings, we only examined teacher responses to MOSTs—instances which could be productively turned over to the whole class. Thus, regularly having the same student as the primary actor in the hypothetical study or the teacher as the primary actor in this classroom study leaves the whole class actor as a distant second choice in both situations, implying many missed opportunities for the class to engage in collaborative sense making (Collaboration Principle). Additionally, having the teacher as the actor in nearly half of the instances in this study misaligns with the Legitimacy Principle, since responses with this actor do not position students as legitimate mathematical thinkers.

Also, in our hypothetical work, over half of the initial teacher responses asked students to develop or justify their thinking–two actions that are of importance when attempting to engage students in sense making. Unfortunately, in this study, only about a quarter of responses included these two types of actions. The three most common actions in the classroom study, which account for just over half of teacher responses, were literal, dismiss, and evaluate–moves that do not align with the Sense-making Principle.

While the results of this study have some positive aspects in the ways SMT is used, we can see that, as a whole, the initial teacher responses in our data are not as well aligned with ambitious or responsive teaching as the responses that teachers gave to interview scenarios in our prior work. We acknowledge that this weak alignment may have resulted from our methodological choice of only examining the initial teacher response and that subsequent teacher turns may align better—an area for future research. This weak initial alignment, however, suggests that the ways that teachers want to or think they should respond to SMT are difficult to enact in the moment that decisions about responding are made, highlighting the complexity of enacting ambitious teaching. An area for future research could be to examine possible reasons for this disconnect.

Acknowledgements

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TACTILE CONSTRUCTION OF MATHEMATICAL MEANING: BENEFITS FOR VISUALLY IMPAIRED AND SIGHTED PUPILS

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Tactile perception in inclusive mathematics education has been associated mostly with visually impaired (VI) pupils. We endorse an alternative perspective: that tactile perception can be of high relevance to the mathematical learning of sighted pupils too. Here, we report from a study in which we explore the impact of universally designed mathematical practices upon the mathematical learning of VI and sighted pupils in class. In the episodes we report here, we invited the VI pupil and the sighted pupils of a Year 5 (Y5) class to construct meaning of shapes through touch. We found that tactile perception led not only to better inclusion of the VI pupil but also brought benefits to sighted pupils too. We conclude with our study's aim to contribute to inclusion and challenge ableism in the mathematics classroom.

INTRODUCTION

Inclusive education has been an issue of international consideration especially since the Convention on the Rights of Persons with Disabilities (CRPD) (United Nations, 2006). Article 2 of the CRPD defines two different ways in which inclusion can be achieved: "reasonable accommodation" and "universal design". The former denotes modification and adjustments which are done to ensure that people with disabilities enjoy all human rights and fundamental freedoms on an equal basis with others, without imposing a disproportionate or undue burden. The latter denotes the design of environments, services and tools that can be used by every person, to the biggest extent possible, without the necessity for adaptation or specialised design.

In the study we draw from in this paper, we focus on the inclusion of VI pupils in mathematics lessons. We investigate how inclusion and disability are constructed in the discourses of teaching staff and pupils in mainstream mathematics classrooms; and, we examine how collaboratively designed mathematics lessons impact upon teaching staff's and pupils' (both VI and sighted) discourses on inclusion and disability. Our work resonates with studies (e.g. Nardi, Healy, Biza, & Fernandes, 2018) which challenge "ableism" in mathematics education – the "network of beliefs, processes and practices that produce a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human" (Campbell, 2001, p. 44) and which objects to a perspective on disability "as a diminished state of being human" (p. 44).

A conjecture that our study explores is whether, and if so how, universally designed mathematical practices lead not only to better inclusion of VI pupils but also bring benefits to all pupils. In this paper, we discuss the impact of one tactile mathematical

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task upon the mathematical learning of both VI and sighted pupils. We present evidence from one classroom task. We include evidence of a mathematical contribution, made by a sighted pupil, which involves tactile as well as visual perception of a shape that is a circle minus a circular segment. We then include evidence of a mathematical contribution, made by a VI pupil, which involves tactile perception of the same shape and of a circle. We highlight the benefits that the pupils' tactile constructions of mathematical meaning have generated and we conclude with a broader discussion of these benefits as they emerge from the study's ongoing analyses.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Vygotskii (1993) suggested that, in VI pupils, the substitution of their eyes with their hands may result in the emergence of perspectives that differ from those of sighted pupils, due to the difference in the sensory tool through which they access mathematics and construct mathematical meaning. In the context of mainstream mathematics classrooms with VI and sighted pupils, tactile perception has been associated mostly with VI pupils (e.g. Argyropoulos & Stamouli, 2006; Leuders, 2016).

This association of tactile perception with visual impairment often results in the inclusion of VI pupils through accommodations of visual materials typically used for sighted pupils. While in some cases accommodations are successful – e.g. design of tactile shapes for the VI pupil while the rest of the class used the visual shapes of the school textbook (Argyropoulos & Stamouli, 2006) – in other cases such accommodations have limitations. The limitations can be technical or affective and social. Technical limitations constitute, for example, interpretation of mathematical notations as images by a screen reading software (JAWS) as well as errors and missing elements in Braille textbooks (Bayram, Corlu, Aydın, Ortaçtepe, & Alapala, 2015). An example of affective and social limitations is found in the adaptation of a visual task with linear patterns represented with dots of two colours. The provision of counters with different textures instead of different colours may lead to processing overload for the VI pupils and also to difficulties in their communication with sighted pupils on the mathematical task (Leuders, 2016).

Acknowledging the limitations of the above adaptation, Leuders (2016) suggests the implementation of the linear pattern task through a universally designed practice: the transformation of the visual task to an auditory task, which will be the same across the entire class. Leuders' suggestion, alongside suggestions of other researchers, e.g. Healy, Fernandes and Frant's (2013) work on multimodal mathematical tasks, are starting to shift the long-established perspective on tactile perception from the VI pupils to the entire class. Healy, Fernandes and Frant (2013) argue that the multimodal nature of mathematical representations, which meets the sensory needs of every pupil in the classroom, benefits not only the pupils with sensory impairments but also the pupils with no sensory impairments in that it allows them to develop a range of ways to think mathematically (Healy, Fernandes, & Frant, 2013).

The enrichment of sighted pupils with mathematical opportunities provided through multimodal tasks, in combination with the affective and social limitations frequently caused to VI pupils through problematic accommodations, have led us towards designing tactile mathematical tasks and trialling them with the entire class.

Our study's theoretical framework is sociocultural and draws upon: Vygotskian sociocultural theory of learning (Vygotskii, 1993); Sfard's discursive perspective, known as the theory of commognition (Sfard, 2007); the social model of disability (Oliver, 2009); and, the theory of embodied cognition (Gallese & Lakoff, 2005). In this paper, we focus on the discursive activity of a sighted pupil and of a VI pupil from the same class. We use the theory of embodied cognition (Gallese & Lakoff, 2005) to analyse the two pupils' mathematical contributions which are constructed through bodily tools. We draw upon Vygotskii's theory of mediation (1993) to explore the different mathematical meaning making within each pupil.

In what follows, we present the study's context, participants and methods. We then sample from the data with two episodes which evidence tactile construction of mathematical meaning about shapes by the pupils. We conclude with highlighting the benefits of tactile meaning making upon all pupils and make the case that these benefits are a key feature of inclusive mathematics classrooms.

METHODOLOGY

The study we draw from in this paper constitutes part of the first author's doctoral thesis [see (Stylianidou & Nardi, 2018) for further information on the study]. Data collection was conducted in four UK mainstream primary mathematics classrooms (one Year 1, one Year 3 and two Year 5 classes; pupils' ages: 6-10). The VI pupils' presence and the willingness of teaching staff and pupils to participate in the study constituted our criteria for the selection of the classrooms. We collected data after securing ethical approval by our institution's Research Ethics Committee and ensuring participant anonymity, confidentiality and right to withdraw from the study.

We collected data through observations of 29 mathematics lessons (33.5 hours in total); individual interviews with 5 class teachers (6 interviews, 2 hours and 10 minutes in total); individual interviews with 4 teaching assistants (6 interviews, 2 hours and 15 minutes in total); focussed-group interviews with 35 pupils (16 interviews, 2 hours in total); 2 ten-minute individual interviews with one pupil; written transcripts of the teaching staff's contributions in the design of the three lessons that constituted an intervention phase of the study; photographs of the pupils' work in the three intervention lessons; and, pupils' evaluation forms of the intervention lesson in two classes.

During observations, written notes were kept in all lessons. 21 lessons were audio-recorded and 14 lessons were also video-recorded. All interviews were audio-recorded, except four, following interviewee requests. For these, written notes were kept instead. Data analysis is ongoing and consists of analysis of inclusion and disability discourses of teaching staff and pupils in the mathematics classroom, both before and after the

design and implementation of intervention lessons. Our analysis focuses on the impact of classroom practices upon the participating pupils and teaching staff. It is with this focus that our study's conjecture – outlined in the Introduction – is explored.

The mathematical task that is the focus of the episodes we present in this paper is as follows. The teacher asks the class to close their eyes and describe two shapes (see Figure 1, hereafter we call the second shape "Shape X"), both of which were constructed with Wikki Stix. The shapes were constructed on the same white A4 paper and copies of the paper were given to the class. Wikki Stix is a flexible teaching tool made of a wax and yarn combination and can be used for VI pupils' learning. In this paper, we focus on Shape X. At some point during the pupils' engagement with the task, the teacher also gives circles of various colours and sizes to the class and asks them the difference between Shape X and these circles.

In this task, we wanted to invite the entire class to explore mathematics through touch. We saw this invitation as potentially beneficial for both the VI and the sighted pupils: it would make the VI pupil feel that he is no more the only child in class who accesses mathematics differently from his peers. It would increase the sighted pupils' familiarity with a sense which is mostly associated with VI pupils and often under-used by sighted pupils. Furthermore, being aware of the characteristics of vision and touch – vision is wholistic and touch is gradual, allowing the exploration of an object from its individual parts to its whole (Ochaita & Rosa, 1995) – we wanted to explore whether vision may generate a misinterpretation of the two shapes (Shape X, for example, at first glance may be perceived as a circle) and whether touch may generate a more accurate interpretation of the shapes. In any case, we have argued that inviting the entire class to explore mathematics through touch could possibly lead to broadening everyone's perspectives on what constitute valid mathematical practices. We use pseudonyms for the pupils.

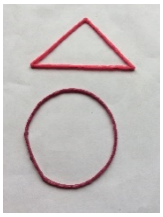

Figure 1: The two shapes made with Wikki Stix. Shape X is a circle minus a circular segment.

Figure 2: Luke holding the yellow circle.

TWO Y5 EPISODES: TACTILE CONSTRUCTION OF SHAPES

The following episodes come from a recapping lesson on mathematical topics in a Y5 class. One of these topics was shapes. Luke is the VI pupil of this class, is blind in one eye and has reduced vision in the other. The class has a general teaching assistant who supports pupils that need help at particular instances and whose role is not on supporting the VI pupil specifically. On the day of the lesson implementation, the teaching assistant was not in the classroom and the teacher was the only member of

teaching staff in there. We will first present the contribution of a sighted pupil (Zak), made while in conversation with the first author, on Shape X. His contribution consists of two parts: in the first part he accesses the shape through touch and in the second part he accesses it visually. We noted that Zak's tactile contribution was common among pupils in the class, including the VI pupil, and we have selected Zak as typical. We will then present the contribution of the VI pupil (Luke), to the entire class, on the difference between Shape X and a circle (the yellow circle seen in Figure 2).

A factual account of the episodes

Shape X: Zak's contribution

During the entire lesson, the first author was sitting next to Luke. Zak was sitting in Luke's table too. At some point during the pupils' engagement with the shapes in Figure 1, Zak talks to the first author and tells her that he feels a straight line segment on Shape X when he has his eyes closed while that it is not that clear for him that there is a straight line segment when he actually opens his eyes and sees the shape.

Difference between Shape X and the circle: Luke's contribution

When the teacher invites the class to share their experiences of the two shapes in Figure 1, the conversation turns to Shape X. Luke then proposes the following comparison with the yellow circle.

> "With the normal circle like this *[he shows and grabs the yellow circle]* feels like, feels like it's gonna roll more *[he positions the yellow circle as if it is ready to roll]*. That one [*he shows Shape X]* feels like it's just gonna bob up and down."

A preliminary analytical account of the episodes

Zak's contribution

Zak constructs different meanings of Shape X through touch and through vision. While he confidently states the existence of a straight line segment when he feels the shape with his hands, he does not clearly see a straight line segment when he sees the shape with his eyes. Drawing upon Vygotskii's (1993) theory of mediation, according to which material, semiotic and sensory tools impact upon the construction of meaning, we attribute Zak's different mathematical constructions to the different sensory tools through which he accesses the shape each time: his hands and his eyes.

Drawing upon the theory of embodied cognition, according to which concepts – and therefore mathematical constructions – are embodied (Gallese & Lakoff, 2005), we relate each of Zak's mathematical constructions to the characteristics of the corresponding sensory tool. Zak clearly feels the straight line segment with his hands, possibly because tactile perception is characterised by gradual perception of an object, allowing the exploration of the object from its individual parts to its whole. So Zak had to trace Shape X gradually and successively to be able to perceive it as a whole. On the other hand, Zak does not clearly see the straight line segment with his eyes, possibly because visual perception is wholistic, namely allows the perception of the entire object

at once. Since the straight line segment covered only a small part of Shape X – and the curved part dominates the shape – Zak's uncertainty regarding the existence of a straight line segment when he sees the object with his eyes is understandable. We may even surmise that Zak would not have noticed the straight line segment at all with his eyes if he had not firstly perceived the shape with his hands.

We have evidence that Zak appreciated tactile construction of mathematical meaning about shapes. In the evaluation form of the lesson, Zak wrote that he liked "the hidden facts on the shapes" (Figure 3).

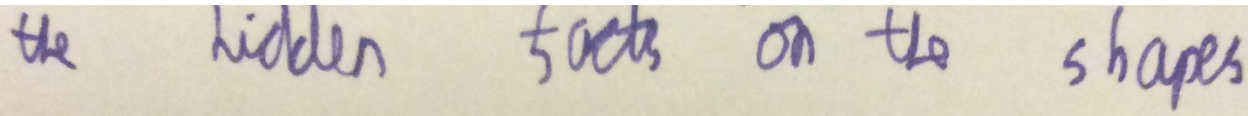

Figure 3: Zak's response to what he liked on the shape task

We see Zak's statement as evidence in favour of the conjecture our study explores. He benefited mathematically through touch and he also appreciated a form of perception that is mostly associated with VI pupils – we note that he is aware of Luke's learning Braille at school, and his own experience with tactile meaning making in mathematics may indeed help him develop a non-ableist (Campbell, ibid) perspective on the mathematical learning and capabilities of VI pupils.

Luke's contribution

Luke makes different meanings of the circle and of Shape X through touch. He feels that the circle is going to roll more – while Shape X is not; it is instead going to "bob up and down". Drawing upon Vygotskii's (1993) theory of mediation, we attribute Luke's different mathematical constructions to the different material tools which he accesses each time: the circle and Shape X. Therefore, while in Zak's case the material tool is the same in both cases and the sensory tool changes each time, in Luke's case it is the other way around: the sensory tool is the same in both cases and the material tool changes each time. We see Luke's use of the word "normal" as acknowledgement that Shape X is not a circle. We also note that Luke's description of Shape X does not come from an actual practical implementation of the rolling of the shape but from imagining the shape doing so. We see his imagining the rolling of the shape as an instance of what Gallese and Lakoff (2005) label as "embodied imagination" (p. 456).

We see Luke's meaning making about Shape X and the circle ("rolling", "bobbing up and down") as quite different from those of the other pupils who – as evidenced in the data we collected from this lesson – mostly draw upon properties of shapes and whose contributions relate to the description of shapes in school textbooks. We see Luke's contribution not only as different but also as refreshingly practical. For example, before Luke, Zak said that in a circle "you need everything to have a slight edge". We also note that Luke's contribution, especially his description of a circle, was highly appreciated by the teacher, who integrated it into his lesson and initiated its further discussion with the entire class. We see the teacher as "attuning" (Nardi et al., 2018, p. 154) his lesson to the mathematical productions of a VI learner and we argue that

inviting the class to participate in tactile construction of mathematical meaning about shapes not only resulted in Luke's better inclusion – he takes centre stage in parts of the lesson – but also benefited others in class.

DISCUSSION AND CONCLUSION

In this paper, we see evidence in favour of our conjecture: tactile perception may lead not only to better inclusion of VI pupils but can also bring benefits to sighted pupils, too. We see these benefits as a key feature of inclusive mathematics classrooms. Similar benefits emerged in other mathematical topics, e.g. numbers, in which we invited the entire class to access mathematics through touch. More examples of mathematical tasks and of generated benefits will be included in the first author's doctoral thesis and subsequent papers. Based on our findings with regard to the benefits of tactile perception upon both VI and sighted pupils, we can argue that tactile perception of both VI and sighted pupils has the potency to generate multiple benefits to all pupils in inclusive mathematics classrooms. We envisage that more tactile tasks will be designed and trialled in inclusive mathematics classrooms with all pupils.

Our evidence and analyses resonate with Healy, Fernandes and Frant's (2013) argument about the impact that universally designed tasks may have upon the entire class: tactile construction of mathematical meaning benefits both the VI pupil and the sighted pupils, in that it allows them to develop a broader repertoire of ways to think mathematically. We see the invitation of the entire class to experience mathematics through the same sensory tool as a way to challenge ableism in the mathematics classroom and also as a way to create more inclusive mathematics classrooms.

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BEYOND A SHADOW OF A DOUBT: DO FIVE TO SIX-YEAR-OLDS RECOGNIZE A SAFE BET?

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The present study investigated 5- to 6-year olds' ability to distinguish a certain from an uncertain outcome when both were simultaneously presented in a probability game context. Three hundred and eighty-two children had to decide which of two boxes had the highest probability of blindly drawing a desired element. In five items, one box contained desired and undesired elements (uncertain outcome), while the other box contained only desired elements (certain outcome). The results showed that children within the same year of kindergarten strongly differed in their ability to distinguish a certain from an uncertain outcome. Age and sex were not predictive for these differences. Children's performance on this task was predictive for their performance on a task about comparing probabilities of only uncertain outcomes.

INTRODUCTION

Uncertainty is omnipresent in our everyday lives, which in turn justifies why we would want students to learn about probability (Gal, 2005). The lack of probability as a subject in many elementary school curricula (Langrall, 2018) might be explained by the fact that Piaget and Inhelder (1951/1975) concluded that the complete understanding of the probability concept can only be achieved in the formal operational stage, starting around the age of 11 (Jones & Thornton, 2005).

However, views of Piaget & Inhelder (1975) are challenged by more recent studies that investigated children's ability to identify the best of two sets (i.e., the best sample space) to blindly draw a desired element from. In such studies, children had to select the best set by comparing the probabilities of a favourable outcome in two different sample spaces that contained more than one possible outcome. Falk, Yudilevich-Assouline, and Elstein (2012) found that children already as young as 8 were generally able to choose by the greater probability, which is the only correct rule. Denison and Xu (2014) even suggested that infants show sensitivity to proportions when comparing the likelihood of an outcome in two different sample spaces. Given these examples, it is no surprise that more and more authors advocate for the introduction of probability as a study domain in the early grade mathematics curricula (e.g., Langrall, 2018).

Despite of the growing body of evidence suggesting that young children have some notion about chance, there is little agreement regarding the age at which children understand the different key ideas in probability (Lajoie, 1998). In order to develop instruction for young children, it is therefore important to learn more about the onset and (natural) development of these key ideas. Jones, Langrall, Thornton, and Mogill (1997) suggested that the recognition of certain or impossible events (as opposed to

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uncertain but possible events) might form the starting point for the development of probabilistic reasoning. If the recognition of (un)certainty is, indeed, a precursor for other components of probabilistic reasoning, it might be of interest to know the age of onset of comprehending the difference between certain and uncertain events.

Development of the understanding of (un)certainty

Children's understanding of the distinction between certainty and uncertainty has been assessed in various ways. A well-known paradigm is the box task of Piéraut-Le Bonniec (1980). Children receive a box with two holes (one larger than the other) and two objects (a pencil that fits through both holes and a marble that only fits through the larger hole). They are then informed about the hole (larger or smaller) that was used or about the object (pencil or marble) that entered the box and they are asked whether or not they are able to know for sure which object is in the box or through which hole the object entered the box. In this type of task, children are presented with information and are then asked whether or not the information provided suffices to know what happened. Children thus need to reflect on *what could have happened* and use their logical reasoning abilities. Horobin and Acredolo (1989) attributed the difficulties that young children experienced in an adapted version of the Piéraut-Le Bonniec box task to their confusion between possibility and probability. Children are for example, able to identify multiple possibilities in an uncertain situation. Thus, they are able to answer the question: 'What can happen?' However, they are perhaps still unable to assign (equal) probabilities to the different possibilities: every possibility has a certain (or the same) chance to occur. This might lead children to close on one single possibility, when they are asked 'What happened?'.

To the best of our knowledge, there is little research on the understanding of the difference between certainty and uncertainty in probabilistic situations similar to those presented in Falk et al. (2012) and Denison and Xu (2014). This is remarkable, given that (degrees of) (un)certainty are inherent in reasoning about probabilities and given the popularity of binary choice tasks in research. In one of Piaget and Inhelder's (1975) experiments, children received two sets each containing blank and/or crossed counters and were asked which set would yield them the better chance to blindly draw a crossed counter. In contrast to the previously mentioned box task, children had to adopt a more anticipatory way of reasoning and think about "*What will happen*?". They had to make predictions when all possible outcomes were directly visible. In the 'Possibility vs. Certainty' items children received one set only containing crossed counters, while the other set contained both types. In these items, choosing the set only containing crossed counters implied a safe bet. Piaget and Inhelder (1975) described that children in the preoperational stage (until the age of 7-8) were unable to systematically identify the 'certainty set' as the more opportune set. They explained that many of the children in this stage are "not at all concerned with the possible cases and consider only the number of favourable cases as if that were an absolute number" (p.139).

These findings contrast sharply with a more recent study by Girotto, Fontanari, Gonzalez, Vallortigara, and Blaye (2016). Not primarily interested in children's ability to distinguish a certain from an uncertain outcome, they nevertheless presented two 'Possibility vs. Certainty' items to 3- to 5-year olds. In the first item, the 'Certainty box' contained a larger number of target chips than the 'Possibility box'. In the second item, the 'Certainty box' contained a smaller number of target chips than the 'Possibility box'. Girotto et al. (2016) explained that, for the latter item, children have to inhibit the absolute number heuristic (i.e., choosing by the greatest number of desired elements). They found that 63%, 91% and 84% of 3-, 4- and 5-year olds, answered this first item correctly. The second item, which could be considered more difficult, was correctly answered by 54%, 76% and 88% of 3-, 4- and 5-year olds. These findings cast doubt on the conclusions of Piaget and Inhelder (1975). With the aim of elucidating these differing findings, the current study was set up.

The present study

The main aim of the current study was to investigate 5- to 6-year olds' ability to distinguish a certain from an uncertain outcome and whether age and sex predict differences in this ability. Furthermore, we were interested in the link between the ability to distinguish a certain from an uncertain outcome and the ability to compare the probabilities of uncertain outcomes.

METHOD

Participants

The current study is part of a larger longitudinal research project in which we map the development of different key mathematical abilities in children from kindergarten until the third year of elementary school. Children from 17 different schools in Flanders, Belgium participated in the longitudinal research project. Different schools were selected to ensure the representation of different socio-economic backgrounds. Parents of 410 children consented for participation in the overall project. For the current study, data were used of the first wave in which the probability tasks were introduced to the children. At that time, children were in the second semester of their third year in kindergarten. It is important to bear in mind that during previous waves of the longitudinal research project none of the mathematical abilities assessed were related to any form of probabilistic reasoning. Because of unavoidable dropout (6.83%) due to children moving away and changing schools, 382 children were tested during this third wave (194 boys, 188 girls). At the moment of testing, the average age of participants was 5.88 years (range 5.15 years to 6.58 years).

Design and materials

During this wave of data collection, children were individually tested in three sessions of 30 minutes. In each session, children received a collection of tasks assessing the different key mathematical abilities that form the focus of the larger longitudinal research project. For the current paper, we only focus on the assessment of children's

probabilistic reasoning abilities, and more specifically on the binary choice task assessing children's ability to 1) distinguish a certain from an uncertain outcome and to 2) compare probabilities of two uncertain events. Due to the length restriction of this report, results of other measures assessing components of children's probabilistic reasoning abilities are not included.

For the binary choice task, children sat in front of a laptop, operated by the experimenter. Children were first introduced to a blindfolded bird on screen and were told that the bird loves black berries but hates white and green berries. In every trial, the bird could blindly pick a berry from one of two boxes which were filled with a different number of berries of the desired and undesired colors. Unlike the bird, participants could see the content of every box and they were asked to help the bird by deciding which of two boxes was best for the bird to blindly pick a berry from. So for each item, children were asked to help the bird by deciding which box would yield the bird the highest probability of blindly picking a desired berry. Children had to point at the box of their choice. It is important to note that no actual draw took place and that children did not receive feedback on the accuracy of their decisions.

The binary choice task consisted of 29 items of two main categories. The first category consisted of five items in which one of two boxes contained only desired black berries, while the other box contained desired as well as undesired berries. We will refer to these items as the *C-items (referring to certainty)*. To solve these items, participants had to notice that one of two boxes contained only desired berries and that blindly picking from that box meant certainty of success. For all these items, the number of desired black berries was always larger in the 'uncertain' box and thus, application of the absolute number heuristic would lead to the incorrect choice. The second category of items consisted of 24 items (based on the study of Falk et al., 2012) in which children, again, had to decide which of two boxes was the best option for the bird to blindly pick from. Now, both boxes contained desired and undesired berries, thus, children had to compare the probabilities of two uncertain events. We will refer to these items as the *P-items (referring to possibility)*. An example for both types of items is shown in Figure 1. Items from both categories were mixed.

Figure 1: Example of one item for each category.

RESULTS

Our first interest was children's performance on the C-items. Internal consistency for these items was good (α = .84). On average children solved 2.37 (sd = 1.92) out of five C-items correctly. Figure 2 shows the distribution of accuracies. Remarkably, children's scores followed a U-shape: A large group of children (44.76%) solved none or just one item correctly and a large group of children (37.44%) solved four or five items correctly. A small group of children (17.80%) responded rather inconsistently, solving two or three out of five items correctly. Exploring answering patterns of children that solved only one item, indicates that in 84% of cases it concerns the same item, in which the relative difference between desired elements in the certain and uncertain set was considerably smaller than in other items. This item possibly caused children who were answering by the largest number of desired berries to 'accidentally' choose the correct box. Children that solved four out of five items did not consistently fail the same item.

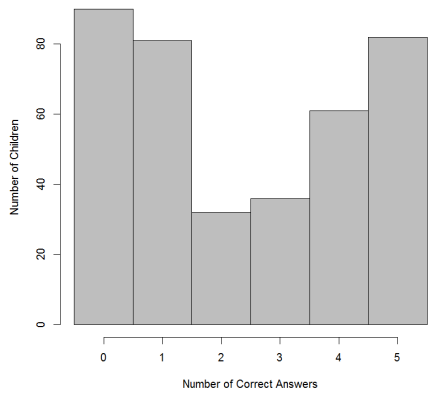

Figure 2: Distribution of total scores on C-items for all participants

Second, we tested whether age and sex predict variability in children's scores on the C-items. Intraclass correlation coefficient (ICC < .01) and design effect (*deff* =1.11) were first calculated to verify the proportion of variability in scores on the C-items that was attributable to the school that children were in. This design effect smaller than two indicated an ignorable effect of clustering within schools (Lay & Kwok, 2015). Therefore, a regression model with only fixed effects (age and sex as predictors) was fitted to the data. Variance explained by the model seemed negligible, with $R^2 < .01$ and no significant evidence for the overall model was found, $F(2, 379) = 1.07$, $p = .34$. No significant evidence was found for age or sex predicting scores on the C-items.

Third, we investigated whether children's performance on the C-items is predictive for their performance on the P-items. Again, due to children being nested in schools, the intraclass correlation (ICC = .01) and design effect (*deff* = 1.23) were first calculated, now for the P-items. A regression model with only fixed effects was run, with performance on the C-items and age as predictors. Significant evidence for the model was found, $F(2, 379) = 94.97$, $p < .001$, with $R^2 = .33$. Children's scores on the P-items increased with $B = .95$ ($p < .001$) for each C-item solved. Comparable to performance on the C-items, age was not a significant predictor of performance on the P-items ($B < .01$, $p = .13$).

CONCLUSION AND DISCUSSION

Whereas Girotto et al. (2016) found that most children of this age perform well on the

'Possibility vs. Certainty' item, Piaget and Inhelder (1975) suggested that children in the preoperational stage are unable to consistently identify the set in which the desired outcome was absolutely certain. Findings in the current study provide a more nuanced picture of 5- to 6-year olds' ability to distinguish certain from uncertain outcomes and relying on it in their predictions in probabilistic contexts. Results suggest that 5 to 6-year-old children either did or did not understand the difference between certain and uncertain events, as a large group of children performed very well while a second large group performed very badly.

Based on these results, it seems premature to draw general conclusions on 5 to 6-year-olds' abilities to identify certainty. However, a considerable number of children seem to consistently identify certainty, while another considerable number of children consistently fail to do so. Possibly, around the age of 5 to 6 a transition takes place: Children move from not understanding to understanding the implications of certainty, and once this shift took place it seems to be an all-or-nothing issue. It is, however, unclear whether the children performing around chance level, (1) are to be included in the 'not understanding group', (2) represent children that were experiencing this transition right at the moment of testing, or (3) represent children whose performance is explained by measurement error. Longitudinally following the same children on this ability, will shed more light on the nature of this transition: Do children move rather abruptly from one group to another or does this happen more gradually?

Comparing our results to those obtained by Girotto et al. (2016), it is important to note that in their study, children received the 'Possibility vs. Certainty' items after an item with both sets containing a sample space with only one possibility (only desired elements in one set and only undesired elements in the other set) and before items with both sets containing a sample space with different possibilities. Furthermore, concerning the 'Possibility vs. Certainty' items children first received an item that was congruent with the above-mentioned absolute number heuristic. Only afterwards they received an item that was incongruent with this heuristic. In the present study, all C- and P- items were mixed and the absolute number heuristic would cause children to choose the incorrect set for all C- items. Possibly, the first tasks that were presented in a set order of complexity in the study of Girotto et al. (2016) (unintentionally) functioned as training items, resulting in improved performance on subsequent items. If this is true, our results can be interpreted as demonstrating that 5- to 6- year-olds differ strongly in their ability to distinguish a certain from an uncertain event when the context is less explicit, perhaps even misleading.

Furthermore, results of the present study provide some first empirical support for an association between the ability to differentiate certainty from uncertainty and the ability to reason about probabilities (Jones et al., 1997). Future waves of data collection within the current longitudinal research project will shed light on the nature of the link between both abilities. Does the ability to distinguish certainty from uncertainty indeed precede the ability to compare probabilities, or are both abilities two sides of the same coin: making the connection between the number of possibilities within a set and the

corresponding probability of an outcome (Horobin & Acredolo, 1989)? For example, Bryant and Nunes (2012) stated that in many probability problems, the most important step is working out the sample space. Perhaps, the ability to imagine all the possibilities in simple events explains the link between distinguishing certainty from uncertainty and comparing probabilities.

In future studies, it may help to generate more in-depth insight in differences in children's understanding of the distinction between certain and uncertain outcomes. For example, children could be asked to imagine for each item that an actual blind draw would take place in the set of their decision. For each item, a scale could be added with which children can assess their degree of certainty about the desired element being drawn (Horobin & Acredolo, 1989). If scoring well on the C-items, indeed, means that these children are able to distinguish certainty from uncertainty, we might expect them to indicate that they are 100% sure that the desired outcome will arise in the C-items, while they would indicate that they are not absolutely sure about this in the P-items. For children that fail C-items, we would not expect such an answering pattern. However, before implementing this, we have to know whether children of such young age are able to correctly interpret and use this kind of scale. Furthermore, we have to be aware that judgements about certainty could also be interpreted as judgements of confidence in the correctness of their answer (Piaget & Inhelder, 1975).

Finally, we are aware that the mere reliance on one type of tasks (i.e., binary choice tasks) might have led to an underestimation of children's probabilistic reasoning abilities (e.g., Acredolo, O'Conner, Banks, & Horobin, 1989). The current study was not intended to draw more general conclusions. The measures we reported (as well as other measures within the longitudinal research project) are intended to inform us on specific components of children's probabilistic reasoning abilities. In this case, their ability to identify the certain outcome when two sample spaces, one with only one possibility and one with two possibilities, are simultaneously present. However, our findings do suggest that some children experience difficulties in putting aside deterministic ways of thinking when confronted with uncertain situations. Something Falk et al. (1980) recommend to take into account when including probability in the mathematics curricula that generally contains topics that are deterministic in nature. Or how Falk et al. (1980) put it: "One of the aims of teaching about probability in the first grades should be to restore the balance in favour of indeterminism" (p. 202).

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CONTINGENT MOMENTS IN UNIVERSITY MATHEMATICS LECTURING

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This paper explores the contingency dimension of university mathematics lecturing using the theoretical framework of the knowledge quartet (Rowland et al., 2015). Semi-structured interviews and lecture observation from eight research mathematicians are used to understand and analyse contingent lecturing moments in undergraduate mathematics. Findings suggest that research mathematicians deviate from their planned agenda for the lesson according to the contingent moments in lecturing. Student responses were one of the triggers for contingency in lectures.

INTRODUCTION

Contingent moments in mathematics teaching described in past research (Mason & Davis, 2013; Rowland, Turner, Thwaites & Huckstep, 2009; Rowland, Thwaites & Jared, 2015) suggests, is important for professional development in teaching. Contingency is an unplanned event that can happen in mathematics teaching. Contingency in mathematics teaching refers to the teacher's ability to 'think on their feet' (Rowland et al., 2015) to an unexpected event during teaching. Rowland et al. (2015) describe contingency in teaching as the moments where teachers 'encounter unexpected responses or events in the classroom. There could be events, agents and actions that can lead to contingent moments in teaching (Rowland et al., 2015). Because of the unexpected event, the teacher had to deviate from their planned agenda for the lesson. The unpredictability in mathematics teaching is revealed in a contingent teaching moment. It is critical to understand how lecturers respond to those contingent moments, as it demands considerable knowledge in mathematics and its pedagogy. Schoenfeld (1998, p. 18) described these contingent moments as bringing 'surprises' to the teacher. This paper uses the Knowledge Quartet Framework (Rowland et al., 2009) to describe and analyse contingent moments in mathematics lecturing at undergraduate mathematics courses in a university. The research question addressed here is: what is the nature of contingent moments in university mathematics lecturing when analysed using the framework of knowledge quartet?

THEORETICAL FRAMEWORK - THE KNOWLEDGE QUARTET

Rowland, Huckstep and Thwaites (2005) have used the framework of 'Knowledge Quartet' as a tool to identify Mathematical Content Knowledge for teaching. Their empirical work with primary mathematics teacher trainees led them to conceptualise the 'Knowledge Quartet'. The four pillars of the 'Knowledge Quartet' are foundation, transformation, connection and contingency. Foundation refers to the teacher's knowledge. Transformation is this knowledge in action, and Connection refers to the

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connected knowledge required for teaching. The fourth pillar of contingency is the ability respond to students' questions. The focus of this paper is on this dimension of the contingency component of the knowledge quartet. This empirical work has its basis in the notion that the mathematical content knowledge required for teaching is realised through its practice. It gives the prospect for reflecting on teaching and teacher knowledge (Rowland et al., 2009).

Rowland et al. (2015) suggest that teachers have a planned sequence in their mind for mathematics teaching. Leinhardt (1993) used the term 'agenda' to describe a planned lesson structure, its context and underlying pedagogy. Contingent moment arises when the teacher has to deviate from the planned lesson or agenda. According to Rowland, Turner and Thwaites (2014, p. 320) "teacher's intended actions can be planned, but students' responses cannot." Mason (2015) describes how unexpected events in mathematics classrooms can be idiosyncratic and situation dependent. This means that what could be unexpected for a teacher in a particular situation may not be the same for another. However, Mason (2015), using the discipline of noticing, emphasises that the ability to notice and respond to unexpectedness needs a teacher's attention. Mason and Johnston-Wilder (2004) use the term 'knowing-to-act' in the moment, which is explained as something relevant has to 'come to mind' and then 'come-to-action' (Mason, 2015) and this needs attention to direct action. It is the ability to cope with unexpected needs more than the experience of different actions and mathematical knowledge. Mason and Davis (2013) maintain that awareness is important for making in-the-moment decisions. They use the term 'being' to suggest it as an aspect that is sensitised and responsive to what emerges at the moment based on the teachers' personal experiences of mathematics learning and teaching. The unpredictability in mathematics teaching is described as inflight decisions (McNair, 1978-1979). Contingency is also seen as a decision-making moment in teaching as it is unplanned and unexpected (Rowland et al., 2015; Schoenfeld, 1998, 2011). Paterson, Thomas and Taylor (2011) studied lecturers' decision-making in a university using the framework of Resources, Orientations and Goals. This paper uses the knowledge quartet framework to understand the nature of university research mathematicians' contingent moments in mathematics lecturing.

The contingency dimension of the knowledge quartet is constructed from four aspects of lesson observations. They are: Responding to students' ideas, Deviation from the agenda, Teacher insight during instruction and (un)availability of resources (Rowland et al., 2014). According to Rowland and Zazkis (2013), unexpected responses from students created most of the contingent moments. Rowland et al. (2015) identify three types of triggers for contingent moments. The student initiates the first type, the teacher insight during the lesson initiates the second and the third is by some cultural, pedagogical tool. They identify three types of contingent responses from teachers, which are to ignore, to acknowledge but put aside, and to acknowledge and incorporate. Contingent decision making partly depends on teachers' subject and pedagogical content knowledge. Ball and Bass (2000) maintained that teaching in the real-time need

knowledge, mathematical insight and sensibility depending on the situation. Rowland and Zazkis (2013) suggest that teacher responses to problematic contingent decisions depend on their mathematical knowledge in deciding and implementing appropriate pedagogical decisions. They suggest that contingent moments give teachers' an opportunity to learn to improvise their knowledge and skills in mathematics teaching.

METHODOLOGY

This paper reports on data from eight research mathematicians in a university mathematics department in New Zealand. The data collection followed pre-lecture interviews, lecture observations and post-lecture discussions with research mathematicians (RM1, RM2, RM3, RM4, RM5, RM6, RM7 & RM8). All research mathematicians had more than 15 years of teaching experience and had a PhD qualification. Additionally, research mathematicians RM4 and RM8 had a post-doctoral qualification. None of them had any formal teaching qualification.

All the interviews and discussions were semi-structured. An observation schedule was used for lecture observations. Research mathematicians explained their planning for the lecture in the pre-lecture interview. Most of them had clear ideas about their lecture for the class and followed a course book as a guideline for topics in the lesson plan. Lecture observations are used to identify contingent moments in the classroom. Post-lecture discussions were used to understand and confirm the contingent moment in the observed lecture. Classes observed were in undergraduate mathematics including calculus, linear algebra, statistics, applied mathematics and number theory. The lectures were of one-hour duration. The pre-and post-lecture interviews were audio recorded, and lecture observations were video recorded and transcribed. The interviews lasted for 30 minutes to one hour. The interview questions were: Can you recall a contingent moment in lecturing where you had to decide in the moment? I observed this moment in your lecturing as a contingent moment - do you agree with this? Can you describe that contingent moment? Inductive thematic analysis (Braun & Clarke, 2006) was used to analyse the data from the transcripts. The knowledge quartet was used to identify the themes corresponding to contingent moments in research mathematicians' lecturing.

RESULTS AND DISCUSSION

The lecture observations and post-lecture interviews gave evidence for mathematicians' in-the-moment decision making. The triggers of contingent moments are identified using the knowledge quartet framework and are illustrated using examples from the data.

According to RM1, students misinterpret contingent decision making in lectures as they think he is unprofessional.

> I just try to think really hard and fast. To the students, it appears that I don't know what I'm doing. And perhaps I am unprofessional. [RM1, Post-lecture discussion]

When asked how RM1 can be sure of this, he replied that student evaluation indicates this.

> That's the evaluation feedback. [RM1, Post-lecture discussion]

For research mathematician RM2, a contingent moment in lecturing is related to the timing of the material to lecture. Making a decision during lectures about when to stop a topic was a concern for RM2. Contingent moments are there when an exercise takes longer time than expected and when there is an unexpected classroom discussion during the lecture.

> Timing is a simple one if I got to a certain point in lecture and my plan was to finish a complete handout, but I'm only going to get halfway through, I will stop at some point because there is a better place to stop then, in the middle of something. So I do that to some extent… The biggest changes really are how I am covering the materials in the lecture, and if there is an exercise which takes longer than I was expecting, or ends up with a discussion that takes longer, usually I will take that and do it and continue the material on to the next time. Some of the diagrams I use in lectures are preplanned, some of them will be like *'I'm saying some words, but just listening to myself*; *I can tell that not being very clear*. *So here is a picture of it that may be of help'.* And if that happens sometimes in lectures, I will then try to take that actually to do a proper diagram for the next time. [RM2, Post-lecture discussion]

RM2 also mentions that his contingent decisions might not be very clear at the very moment in the lecture. However, he takes it seriously to improvise for the next time. This is an example where contingent moments are used for improving pedagogical content knowledge (Rowland & Zazkis, 2013).

According to RM2 contingency in lectures occurs when he has to think of appropriate words to use in the context of the lecture and when he has to consider a particular problem from different perspectives. However, it happens rarely in his experience.

> Sometimes, I will think of especially a way of using metaphors or different ways of looking at the problem, sometimes I will think of those in the lecture and pursue that at the time, but that's relatively rare. [RM2, Post-lecture discussion]

Mathematician RM3 was asked if limited mathematical knowledge will lead to difficulty in appropriate decision-making during contingent moments in mathematics lecturing. RM3 replied as follows:

> Not in this course. I had experiences like that before. It is usually either because you are teaching something more advanced and then maybe a student asks a question, or the question occurs while you are teaching. And you think, *I wonder if that was true* and then you realise that it is not clear. *How do I answer that?* I had this experience when teaching advanced materials and I also have this experience when teaching a subject that I wasn't really very knowledgeable about. [RM3, Post-lecture discussion]

RM3 reveals that lack of mathematical knowledge may make decision-making in contingent moments difficult.

Mathematician RM4 described a contingent moment in his lecture of Laplace transforms on wave equations. To explain the given wave equation, RM4 used the example of an infinitely long string and a wave that travels to the right along an infinitely long string.

> I was able to compare [the solution of the wave equation] with what students should have been able to experience themselves; for instance when they wobble around a string or something and see the waves move along a string and or a rope… Now they have seen the mathematics that describes that. [RM4, Post-lecture discussion]

Explaining the contingent moment, RM4 said:

> That was more of a spontaneous thing because when I was thinking about the actual lecture beforehand, I was thinking about the mathematics of solving the example, just reminding myself of doing that, but I wasn't actually thinking in terms of the solution. It was when I got to the solution that I realised, *Ahh, this is a good thing to remind students*, well to tell students about how to connect the actual solution to something that they would have observed themselves, yeah. I made the decision at the time, almost we didn't have enough time to do that and it was right at the very end in fact. [RM4, Post-lecture discussion]

Mathematician RM5 describes a contingent moment in her lecture on mathematical modelling using ordinary differential equations. The lecture was on modelling populations and the example used was a predator-prey system. The physical significance of the system was explained, and a phase portrait was constructed modelling the given model of predator-prey population. RM5 had a contingent moment during the lecture where the model did not produce an expected outcome.

> One of the places where I had to make a decision that I'm not sure worked out as I wanted, was when we had looked at the first model and we had seen that it had produced some kind of unusual behaviour. Umm, that did not completely model what we wanted, and then there was a discussion about how we could modify the model. [RM5, Post-lecture discussion]

During the contingent moment, RM5 invited students' suggestions.

> I had some suggestions on ways we could modify the model that I put up on the board, and I was also asking students for suggestions and trying to show them what happened in the MATLAB. Some of that worked. [RM5, Post-lecture discussion]

RM5 did not ignore students' ideas, but responded and acknowledged them (Rowland et al., 2014, 2015). However, it did not produce the required results.

> One of the two students sitting up the front had some suggestions of how we could model certain changes, and the suggestions they had, I mean they were good suggestions, but they had limitations like other ones, and there wasn't a lot of time to explore with MATLAB. So, I could put them in, and we could see it didn't quite work, then I just say, well, it doesn't do this, or it does do this and I didn't feel like there was really time to do that properly. And you know when they made suggestions, it was not very sensible, it was hard to get something useful out of it because there was not really the time to explore the suggestions properly. So I think that if I were going to teach that lecture again, I would change the way; the direction with the class a little bit so that it could go in a more useful direction. [RM5, Post-lecture discussion]

We can see that RM5 has taken this opportunity to learn and improve mathematical and pedagogical content knowledge (Rowland & Zazkis, 2013). Rowland et al., (2015) identified a teacher's insight during lecture as another trigger for contingency. Mathematician RM6 described his insight during mathematics lecturing leading to a contingent moment.

> I also decided not to go through and prove in detail the fact that between every two rational number, there is an irrational number and between every two irrationals, there is a rational number. In my lesson planning, I was planning on doing that, but during the lecture, I saw that the class was interested in what I was talking about, I didn't want to lose that interest, so I decided not to do that. In fact, I said in the class that I am going to put that in the tutorial question. I put it on the tutorial question that they had to do it today. Hopefully, they work through that and figure that out that for themselves. And I do think that was the right decision at that time in the class because I think that they remained interested and intrigued by the things I was talking about right to the end and they were not distracted by the boring calculations and stuff. [RM6, Post-lecture discussion]

However, this was initiated by student response during the lecture. This contingent decision was clearly based on the whole class interaction and engagement which made RM6 deviate from his planned agenda for the lesson as evident from the below quote.

> If there had have been no response or if they [students] had been very unresponsive, then I would have just gone through the details and this often will force them to pay attention because 'oh, he is writing all the stuff down which they [students] have to write down' and they think as soon as you write down calculations, they have to start paying attention. If they had been not interactive, that's what I would have done. But once they start interacting and you get lots of talking from the class, then you have got their attention already, then you can go on doing interesting, bigger picture things. [RM6, Post-lecture discussion]

RM7 described a contingent moment in his lecture on calculus when he used the idea of summing series in solving a problem on finding the limit for a sequence.

> The point that I was trying to make then which I had not planned on making until then, was that you have this method of knowing there is a limit... having the confidence of knowing there is a limit then you can try to find what that limit is. And this is often the case that you process it as a two-step process where you deduce that something you are looking for actually is there. Then you try coming up with some sophisticated way of finding what it actually is. Solving differential equations could be similar to that in which you have a general theorem for the existence of a solution of the differential equations that tell you that there are solutions, then knowing that you might go hunting for solutions for a particular equation. You might not find them because you might have these numerical methods. But at least you've got some idea. So I brought that example up because it seemed at the time in the lecture a good way of illustrating how this theorem could be useful. But it was not something I planned in advance; I hadn't thought about this until that moment. [RM7, Post-lecture discussion]

RM8 during the interview talked about how she was keen on using real-life examples to reduce the abstract nature of some statistical concepts in her lectures. RM8 identified her use of real-world examples in lectures as contingent moments.

> Often such things [use of real-world examples] do arise during lectures, and I suddenly think of 'well it is like this'. And then I refine that, and I write it down in my notes and remember to use it again in the future. [RM8, Post-lecture discussion]

These are some of the contingent thoughts that occur to mathematicians amid the lectures '*being in the situation*' and '*be fully mathematical with and in front of students*' (Mason & Davis, 2013, p. 188).

CONCLUDING REMARKS

In summary, research mathematicians in this study described their contingent moments in mathematics lecturing. Triggers for contingent moments are identified using the knowledge quartet framework, RM2 reported that unexpected classrooms discussions, thinking of using appropriate words to use to describe in the context of lectures and considering a problem from different perspectives in the lectures led to contingent moments. RM4 and RM8 mentioned thinking and using real-world examples in lectures as contingent moments. Students' responses in lectures triggered RM4's contingent moments, which was acknowledged and incorporated in the lecture but did not produce the required results. Mathematician RM5 had his insight leading to contingent decision-making in a lecture in which he deviated from the planned agenda for the lesson as supported by knowledge quartet study (Rowland et al., 2015). Research mathematicians in this study experienced different contingent moments in their lectures. However, student response was a common factor leading to contingency in RM5 and RM6's lectures. Time was mentioned as a constraint in contingent decision-making as it pressed mathematicians to deviate from their planned lesson which seemed a concern for them (for example, RM2, RM4 & RM5). This study has shown that knowledge quartet is an effective tool in analysing university lecturers' practice, specifically contingent moments in lecturing. It enabled us to understand mathematicians' reflective practice. Research mathematicians (RM2, RM3, RM5, RM8) see contingent moments as providing learning opportunity and use it to improvise their pedagogical content knowledge (Rowland & Zazkis, 2013) which has implications for research mathematicians' professional learning and development.

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USING LEGITIMATION CODE THEORY TO EXAMINE MATHEMATICS CURRICULUM MATERIALS

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This paper uses the concepts of semantic gravity and semantic density from Legitimation Code Theory to examine the knowledge-building evident in a set of curriculum materials taken from an Australian project. It shows how the materials shift between various semantic modes, emphasising to a greater or lesser degree the link between mathematics and context, and capturing meaning in forms that have greater or lesser degrees of condensation. The paper suggests that it is the shifts between semantic modes rather than the mode of presentation itself that promotes robust knowledge-building.

INTRODUCTION

Questions relating to the importance of context and how best to promote transfer of learning to new situations abound in the mathematics education literature (e.g. Boaler, 1993; Carreira, Evans, Lerman, & Morgan, 2002). However, discussions of whether mathematics is best learned through concrete or abstract situations grossly oversimplify the question. Rather, what promotes robust knowledge-building may well not be the learning situation itself but rather the *shifts between* learning situations (Maton, 2014). Legitimation Code Theory (LCT) (Maton, 2000) provides a framework that enables the researcher to examine curriculum materials and teaching in practice to identify and describe such shifts.

This paper analyses a set of mathematical modelling materials from an Australian mathematics curriculum project, reSolve: Mathematics by Inquiry, in which students first encounter a challenging real-world problem, explore the situation, unfold the mathematics to seek meaning, make conclusions, and ultimately relate it back to the original situation.

Specifically, the paper asks:

- What semantic codes are evident in the curriculum materials; and
- How do the materials vary along the semantic gravity and semantic density dimensions of LCT?

The study simultaneously informs both the mathematics education research and design community and the LCT community. It provides a new perspective on mathematics curriculum and knowledge-building that remains unexplored but offers much potential in mathematics education research. At the same time, it adds to the body of work currently examined using LCT and sheds light on how LCT can be used to examine written curriculum materials that are designed to be educative for teachers.

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LEGITIMATION CODE THEORY (LCT)

Most mathematics education research constructs "knowledge" as a state of understanding characteristic of an individual developed through a process of individual and social construction. Research studies therefore focus on the cognitive and emotional processes involved in a person "coming to know", leaving aside questions relating to knowledge itself. Knowledge as an object of study in its own right—that is, the forms knowledge takes, the modalities that characterise different forms of knowledge, and how these are realised in the artefacts and processes of teaching—is seldom considered in educational research. Such knowledge-blindness (Maton & Howard, 2018) takes for granted what we mean by knowledge and thus becomes an obstacle to addressing the knowledge-building that is intended in mathematics curriculum resources.

Legitimation Code Theory seeks to address knowledge-blindness through a careful examination of the principles underlying the practices and relationships evident in learning, teaching and associated interactions and resources. It builds on Bernstein's (1990) notions of classification and framing to develop a set of codes relating to the organising principles of specialisation, semantics and autonomy embodied in an educational practice. In each of these dimensions the principles are organised along two independent axes to create codes characteristic of the knowledge relations evident in a given practice.

Specialisation refers to the degree to which knowledge is classified and framed along epistemic and social dimensions. Specialisation codes were used by Thornton (2008) to examine the responses of mathematicians and mathematics educators to an Australian Government review of numeracy education in schools. The textual analysis of the responses suggested that "knowledge within the disciplines of mathematics and mathematics education relies on different epistemic devices, and hence that debates surrounding mathematics education arise, at least in part, from differing ways of viewing knowledge" (p. 523).

Semantics explores how depth of meaning and complexity are encoded within an educational practice and how dependent that meaning is on the context within which the practice is embedded. The semantics dimension has been used to explore how mathematics is used as a tool in physics to condense meaning while maintaining a close connection to the empirical world (Doran, 2017). Mathematics provides a platform on which hierarchies are built through the creation of scientific propositions and theories that can be generalised across a range of phenomena.

Autonomy explores how constituents such as people, ideas and artefacts are positioned within a disciplinary category and how strongly they are related to constituents in other categories. Maton and Howard (2018) used autonomy codes to examine knowledge-building practices in school history and science, identifying pathways that either inhibited or promoted knowledge-building among students.

This paper uses the semantics dimension of legitimation code theory to examine the codes embedded in published curriculum resources and how these vary through the course of the intended use of the resources.

THE RESOLVE: MATHEMATICS BY INQUIRY PROJECT

reSolve: Mathematics by Inquiry is an Australian Government Department of Education and Training funded project that aims to "transform the teaching and learning of mathematics in Australian schools". The project was designed for coherence, scale and sustainability through the development of an underpinning philosophical framework, the reSolve Protocol, and through the parallel activities of resource development and the recruitment and training of more than 250 Champions across Australia. The Protocol highlights the importance of purposeful mathematics, inclusive and challenging tasks, and classrooms that have a knowledge-building culture (Thornton, 2017). The resources include teaching resources written for teachers from Kindergarten to Year 10 that exemplify the elements of the Protocol and professional learning resources that engage teachers in sustained and targeted discussion of pedagogic features of mathematics classrooms that enact the elements of the Protocol. The Champions, volunteer teachers from all geographic areas and all levels of schooling across Australia, are the mechanism through which the project embeds scale and sustainability.

A key feature of many of the teaching resources is a focus on appropriate imagined, mathematical or real-world contexts to make mathematics meaningful and purposeful for students. In some cases, such as a Year 3 resource imagining that a rook has escaped from a chessboard onto a number chart, the context acts as a springboard for developing and consolidating powerful mathematical ideas, in this case place value. In others, such as a Year 10 resource examining competing claims to be the world's steepest zipline, it is the context itself that is the focus, and mathematics is a powerful tool through which to investigate the context. In every case, mathematics and context are intertwined so that each supports the other.

The next section of the paper examines one Year 10 reSolve resource, *Cornering*, using the semantics dimension of legitimation code theory.

SEMANTICS IN LCT: EXAMINING A RESOLVE RESOURCE

The semantics dimension of legitimation code theory "conceives social fields of practice as *semantic structures* whose organising principles are conceptualised as *semantic codes* comprising *semantic gravity* and *semantic density* (Maton, 2014, p. 182, italics in original). Semantic gravity refers to the degree to which meaning relates to context and can be stronger or weaker depending on the strength with which meaning is related to context. Stronger or weaker semantic gravity are denoted respectively as SG+ or SG-. Semantic density relates to the degree of condensation of meaning within practices and can similarly be coded as SD+ and SD-. Together the codes form a semantic plane in which the strengths of each semantic code can vary independently (Figure 1). Maton (2014) terms the modalities in the four quadrants of the plane

rhizomatic (SG-, SD+), *prosaic* (SG+, SD-), *motif* (SG-, SD-), and *figurative* (SG+, SD+).

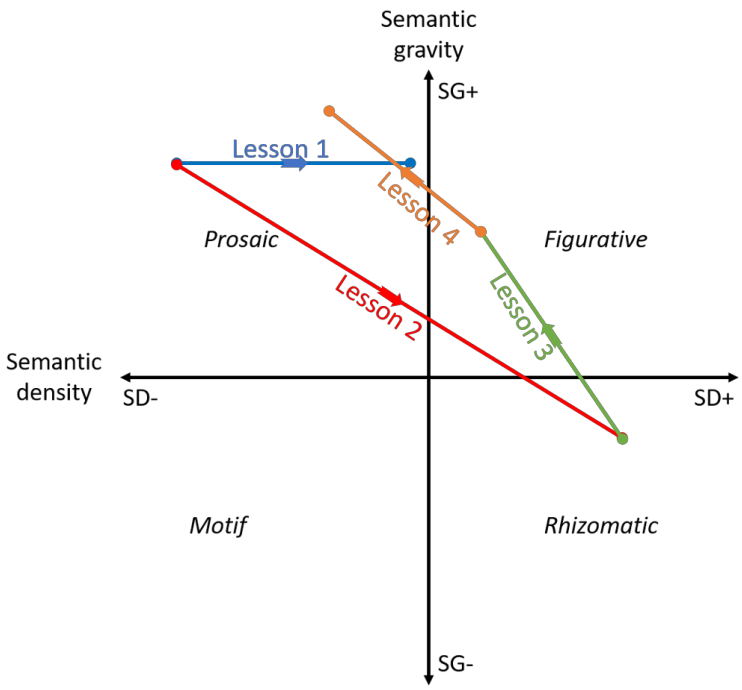


Figure 1: The semantic plane (adapted from Maton, 2014). Note that in Maton's original diagram SG- was placed on the positive y-axis as it involves greater abstraction.

The embedding of these codes in practice is illustrated using the reSolve mathematical modelling activity Cornering, designed to investigate how long vehicles negotiate corners and roundabouts in roads. The semantic shifts described below are illustrated by the directional arrows included in Figure 1. Cornering includes five distinct learning experiences, each designed to introduce or develop important mathematical ideas related to the practical problem.

Lesson 1: Sharing the Road

The lesson commences in a strongly prosaic (SG++, SD--) mode. Students discuss the meaning of the "Do not overtake turning vehicle" sign found on some trucks and buses and recall instances when a truck or bus has had difficulty getting around a corner or roundabout. The meaning is thus highly dependent on context (SG++) but neither condensed nor abstracted as there is no attempt to examine the situation mathematically (SD--). The lesson then moves towards a figurative code using a simple paper scale model to visualise what happens when long vehicles turn corners (Figure 2). The meaning remains highly context dependent (SG++) but the use of a scale model for the corner and vehicle to answer questions such as how much wider the lane would need to be to avoid the truck moving outside the lane strengthens the semantic density (SD-). The strength of semantic gravity is maintained through a discussion of the limitations of the model and the need for a model with steerable wheels.

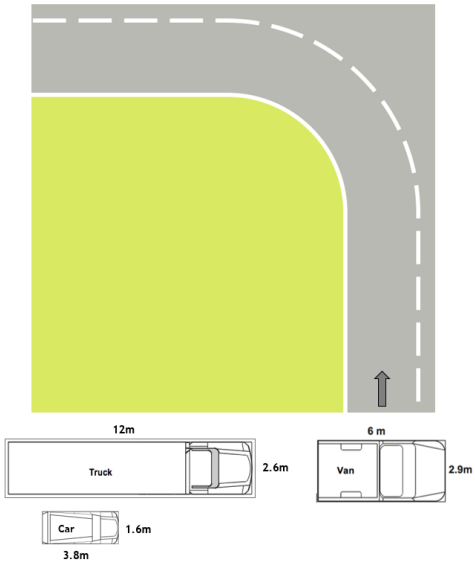
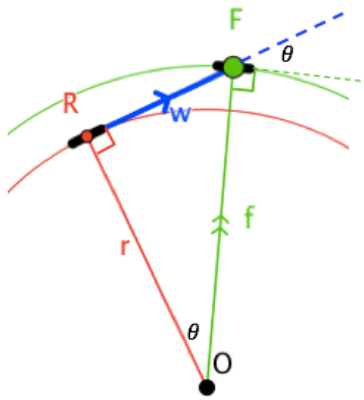


Figure 2: Using a fixed paper model

Lesson 2: A Better Model

The lesson again commences in a strongly prosaic mode (SG++, SD--) as students walk a bicycle or scooter in an arc, discovering that the rear wheel travels along an arc of smaller radius than the front wheel. The semantic density is strengthened when students draw a diagram of the experiment from above (Figure 3) to understand the geometry of the situation. It is further strengthened, and the semantic gravity weakened, by applying trigonometry to calculate the distance by which the radius of the arc traced by the front wheel exceeds the radius of the arc traced by the back wheel (the cut-in distance, $f - r$ in Figure 3). This part of the lesson therefore has a rhizomatic mode (SG-, SD+), embedding strong semantic density but weaker semantic gravity in that although the calculation relates to the context it is a specific instance of a result that could be applied to any two concentric circles.



Angle FOR is equal to θ (exterior angle of a triangle is equal to sum of 2 interior angles). Hence values of f and r (and the cut-in distance) can be found using just w and θ.

$\frac{w}{r} = \tan\theta$ and $\frac{w}{f} = \sin\theta$, so that

$r = \frac{w}{tan\theta}$ and $f = \frac{w}{sin\theta}$

Figure 3: A bicycle or scooter model

Lesson 3: Software Models

Lesson 3 moves towards the figurative mode through the use of a dynamic geometry model to relate the trigonometric results from lesson 2 to the real world of a bicycle.

Students investigate the relationship between the length of the wheel base, the angle through which the wheel can turn and the cut-in distance (Figure 4).

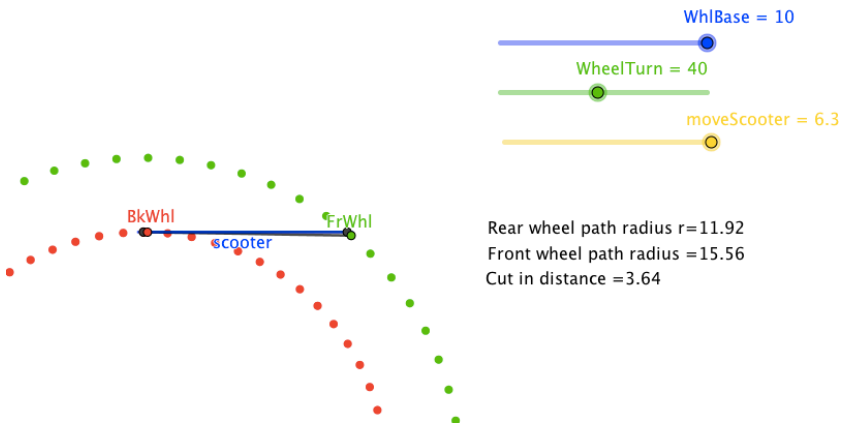


Figure 4: Software used to investigate the bicycle model

In this case a bicycle of length 10m (this is not totally unrealistic, students are shown a picture of a bicycle built for 10) with a wheel that can turn through an angle of 40° (about the maximum for a car) could travel around a roundabout with outside radius 15.56m with a lane at least 3.64m wide. The use of dynamic geometry weakens the semantic density and at the same time strengthens the semantic gravity through the interpretation of the results in terms of the radius of the roundabout and the width of the lane.

Lesson 4: Modelling Four Wheels

The bicycle model is extended to a car or truck model in lesson 4 using new software models, first to investigate the effect of front and rear overhangs on space required for turning, and then taking vehicle width into account (Figure 5). Stronger semantic gravity is evident as students are presented with real data relating to a range of vehicles. Given the sophistication of the mathematical calculations involved, only software is used to model and investigate the situation, further weakening the semantic density. The lesson therefore moves back towards the prosaic mode (SG++, SD-). Note that the strength of semantic gravity is greater than in the initial investigation as the investigation now relates to a four-wheeled vehicle rather than a bicycle.

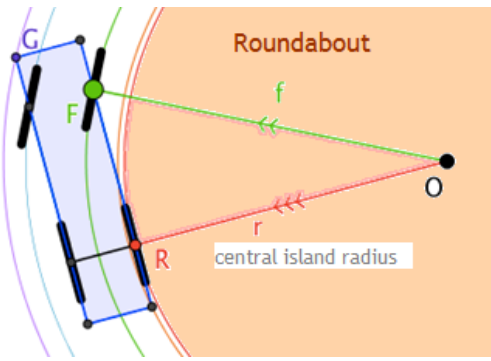


Figure 5: A four-wheel model

Lesson 5: Presentations and Parking

In the final lesson of the unit the students and teacher reflect on the modelling process and recap the findings, strengths and limitations of each of the preceding four lessons. Although not, of course, using the language of semantic codes, this reflection summarises the various modality shifts that occur throughout the process. These shifts are shown in Figure 1. Students bring this together through presentations to peers, writing a report to the local council and possibly extending their work to designing a car park.

DISCUSSION AND CONCLUSION

Legitimation code theory provides a powerful framework through which the semantic shifts between learning situations can be investigated. Significantly, the semantics dimension of LCT not only addresses the relation of meaning to context but also addresses the degree of condensation of meaning. The example of the Cornering unit shows that each of these aspects of semantics can vary independently. The resources intentionally move students between prosaic modes in which meaning is discussed informally and tightly linked to context, through rhizomatic modes where the context remains important but is represented mathematically, towards a figurative mode where the context is backgrounded and finally back to a prosaic mode where students interpret their results and present them to a non-mathematical audience. There is no suggestion that one mode is preferable to another; each provides a different and important perspective on the problem.

This contrasts sharply with traditional textbook exercises, which often commence in a rhizomatic mode showing students how to solve a particular type of mathematical problem and (perhaps, pretend to) move towards a figurative mode through the use of (often artificial) word problems. It also contrasts sharply with resources that assume that abstract understanding develops naturally through the use of concrete resources through some kind of infallible genesis of mathematical knowledge, a phenomenon that Brousseau (1997) terms the Dienes Effect.

The example of the Cornering unit shows how knowledge can be built through deliberate shifts in semantic modalities. Of course, it is relatively easy to identify semantic gravity shifts in mathematical modelling units as the point is to move between the real world, the mathematics and back. However, the semantics dimension of LCT adds the further element of semantic density, enabling us to examine the semantic shifts in a much more nuanced way. Intentionally designing such semantic shifts into other mathematics curriculum resources has the potential to promote more robust knowledge-building and to enhance the degree to which they are educative for teachers as well as students (Davis & Krajcik, 2005).

Any curriculum or teaching resource can, of course, be subverted or enhanced by a teacher. As Maton (2014) has shown, the semantics dimension of LCT can be used to examine the complex features of teaching in practice, including the semantic range, ideal entry and exit points, and the semantic flow or connectedness between points in

a teaching and learning episode. How teachers realise the semantic shifts embedded in teaching resources, or how best to design resources to embed productive semantic shifts are empirical questions beyond the scope of this paper. However, the paper has shown that the semantics dimension of LCT can shed light on how a particular set of curriculum resources moves between different degrees of condensation and context dependence.

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PAPER PLATE PATTERNS: PRE-SCHOOL TEACHERS WORKING AS A COMMUNITY OF PRACTICE

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Young children's patterning, defined as 'finding a predictable sequence', has been identified as significant for later mathematical achievement, while proving amenable to amelioration. This study examines how English teachers of three- to five-year-olds engaged in a collaborative project designed to develop their children's pattern awareness. The teachers and researchers formed a community of practice that enabled the teachers to share and develop their pedagogical practice. This paper illustrates one example of how the teachers worked together to change their practice around border patterns; as their collaboration developed they produced interim stages in the teaching sequence that fostered children's reasoning.

INTRODUCTION

Recently pre-schoolers' pattern awareness has been identified as significant for later mathematical achievement (Rittle-Johnson, Fyfe, Hofer, & Farran, 2017). Fortunately, it can be taught, with positive effects relating to number and pre-algebraic thinking, which has particular relevance for low-achieving or disadvantaged children (Papic, Mulligan, & Mitchelmore, 2011).

Two years ago, we began a collaborative project with English teachers of three- to five-year olds, drawing on the work of Papic et al. (2011) and Mulligan and Mitchelmore (2016). In this paper we consider the following research question: How do teachers work together as a community of practice to develop their children's understanding of repeating patterns?

LITERATURE

Professional Development

In PME 42 Tirosh, Tsamir, Levenson and Barkai (2018) identified that pre-school teachers receive little preparation for teaching mathematics to very young children and that there is therefore a need for professional development in mathematics for this group of teachers. Although Tirosh et al. (2018) were discussing the Israeli context, this need for professional development in early mathematics education is equally present in the English context (AAPG, 2014).

Many of the design principles that Tirosh et al. (2018) engaged in as they designed their professional development were the same as the design principles we used: playful learning, flexible activities, activities where the child is active but the teacher guides the child's learning, and objects that are familiar to the children and readily available to the teacher. Most of our teachers were trained in Froebelian principles and therefore

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child-led, playful learning was an important component of our design. We designed flexible activities and encouraged the teachers to adapt them to their own contexts using objects that were readily available in their classroom and thus familiar to the children. A difference with the design of our study and that of Tirosh et al.'s (2018) is that we engaged our teachers in the design of the curriculum and evaluation of the scope and sequence of the activities we proposed. Thus, we offered professional development in the model found to be most effective, by engaging teachers in curriculum development and evaluation (NCETM, 2009).

Repeating patterns

Patterns are composed of discernible regularity, whether a regularity involving repetition or systematic change. Mulligan et al. (2015) define a pattern as "some regularity observed in a mathematical context and the description of this regularity is its structure" (p. 1). Repeating patterns are the first type of patterns that are explicitly taught to children, often being introduced during preschool (Clements & Sarama, 2008). These patterns involve a unit of repeat, which is the smallest unit which when repeated makes the entire chain/surface/structure, for example, the pattern ABAB's unit of repeat is AB.

It is not clear why knowledge of repeating patterns should predict later mathematical achievement (Rittle-Johnson et al., 2017). Papic et al. (2011) argue that their project was effective partly because teachers encouraged children to look for structural similarities and differences. However, Threlfall (1999) pointed out that identifying the unit of repeat is key, as children can produce simple repeating patterns by visual matching or alternating actions, without understanding that it can be infinitely repeated. Papic et al. (2011) suggest that recognising a composite single unit which can be counted leads to multiplicative reasoning and functional thinking. Teaching about repeating patterns must therefore focus on the unit of repeat.

Border patterns are an extension of linear repeating patterns, because they add the complexity of changing direction and introduce the necessity of thinking of equal groups if the spaces in the border are defined (Papic et al., 2011). Our introduction of border patterns in our professional development addresses Tirosh et al.'s (2018) request that professional development for early years' professionals widen their understanding of what of repeating patterns can consist of.

THEORETICAL FRAMEWORK

Communities of practice is a social theory for investigating how people work and learn collaboratively (Bannister, 2018). Wenger (1998) identifies a community of practice as a group whose members are mutually engaged in an activity, have as joint enterprise and have a shared repertoire of customs of practice. Mutual engagement of the participants is an essential part of a community of practice, because there is no practice without participation. This mutual engagement creates "relationships among people" (Wenger, 1998, p. 76) that can result in deep interconnections. Joint enterprise is a goal that has been mutually negotiated by the members of the community but has not

necessarily been reified as a vision or mission statement; as the community of practice evolves, the joint enterprise may evolve too. Over time, as the community works together, they will develop a shared repertoire of participation and reification structures. These shared repertoires may include routines, use of words, tools and ways of talking.

Wenger (1998) defines practice as "meaning as an experience of everyday life" (p. 52) and further defines meaning as located in the process of negotiation of meaning through the dual processes of participation and reification. Participation is the "experiential process of taking part and sharing in communities" (Bannister, 2015, p. 249), whereas reification refers to the "process of giving form to our experiences by producing objects that congeal our experience into 'thingness'" (Wenger, 1998, p. 58). As we analyse what learning has occurred in our community of practice we need to examine both the participants' change in participation in the practices and any reification that occurs.

METHODOLOGY

Context and participants

This project extended over two years. In the first year we worked with six teachers of three- to five-year olds in four socially diverse state-funded schools in London, UK, which expanded to 11 class teachers in six schools in the second year, with five teachers continuing throughout. The schools had volunteered to join the project and the teachers were mostly highly experienced.

The teachers attended 14 after-school meetings, at which we offered a flexible core teaching programme, which we encouraged the teachers to adapt for their children and settings. At these meetings we provided a brief background to the research, introduced the next month's topic and the teachers shared how they had been teaching pattern. While four main areas of pattern were examined across the year, in this paper we focus on repeating patterns, in particular border patterns.

We also visited the schools three times each year to observe teaching and learning, to assess the children's pattern awareness and to discuss the children's progress.

Data collection

The data reported here derive from three sources: field notes from teachers' reports at project meetings and informal interviews, our observations of teaching and learning in schools, and semi–structured group exit interviews. Our findings rely on teachers' stated reflections on the progress made by children and what helped to develop this. We asked the teachers at each meeting to report on the children's responses and any progress the children had made in their pattern awareness. When we visited the teachers at school, we asked what had made a difference to the children's progress in pattern development and what they had struggled with. All teachers kept "learning journals" with photos relating to the children, which they discussed with us. More generally we asked how pattern had related to the children's mathematical development and, if positive, why they thought that. In the exit interviews we asked what impact the project

had on the children and their teaching. We analysed our notes and transcriptions to identify common themes and conclusions in relation to our focus on learning in a community of practice.

FINDINGS AND ANALYSIS

Our group of teachers consisted of a community of practice in that they were mutually engaged in a joint enterprise and had shared repertoires for engaging in this community. This group consisted of early years teachers who were engaged in the joint enterprise of improving their young children's engagement in mathematics, though pattern activities. While engaged in this joint enterprise the teachers were mutually engaged in curriculum development in pattern instruction for the early years and refining the current developmental progressions. These teachers also had a shared repertoire in that they met every six weeks and at these meetings they had norms for sharing how they had implemented and adapted the programme.

As members of this community of practice, the teachers developed their subject knowledge about patterns and changed their teaching practice by developing several convergent pedagogical approaches, such as integrating patterning into whole class routines and encouraging children's independent "co-working", which included the children challenging each other to continue patterns or spot errors. These are all examples of the teachers' learning within this community of practice.

In this paper we will investigate one specific example of the teachers' learning within the community of practice, examining how they changed their participation in the practice of teaching their children about border patterns through the reification of paper plates. In Papic et al.'s (2011) original border task the children were asked to complete a 14 square border pattern when given two colours, identify whether the pattern has a beginning and end, and justify whether the pattern could be completed. This task presented young children with numerous challenges. Some children found the fine-motor task of fitting the blocks into the given spaces difficult and consequently ended up with an AB pattern that was not continuous around the rectangle (see Figure 1). Other children could make a linear AB pattern but had difficulties continuing the pattern around a corner (see Figure 2), while others came up with more innovative patterns, such as AABB, but were then frustrated when they found that their unit of repeat would not fit into the fixed number of squares (see Figure 3).

During the second year, Sam (all names are pseudonyms) reported that his children "made patterns with objects going round a paper plate, which gave them a template for a circle" and enabled them to make continuous circular border patterns (see Figure 4). The paper plates had several advantages over the rectangular border because on the paper plate there was no prescribed space for an object to be placed so children could squeeze another object in or adjust the spacing between the units to fit. The paper plate was the reification of an open circular border, with no corner to negotiate.

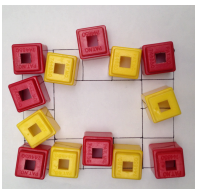

Figure 1: Blocks in spaces

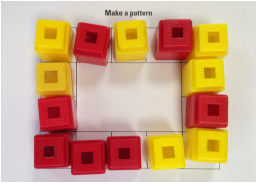

Figure 2: Turning a corner

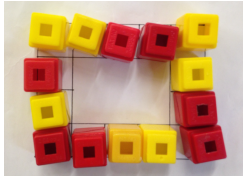

Figure 3: Unit of repeat

Figure 4: Paper plate

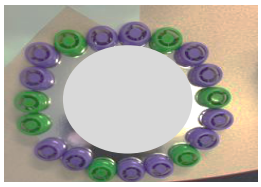

Figure 5: Mirror border

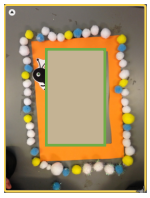

Figure 6: Open rectangular

The other teachers took up this idea in their practice and commented on it in subsequent meetings, with both Tina and Pam remarking that it is "easier to do circular borders than square" and that her children "enjoyed the circular patterns". In this community of practice, the open circular border pattern had become an interim stage between linear repeating patterns and the rectangular border patterns proposed by Papic et al. (2011).

However, several teachers noted disadvantages in using plates and made further adaptations: for example, Pam said, "Dave wanted to put another circle inside the first one, maybe because he thought that he had to fill in the plate. He wanted to match the inside and outside circle but didn't realise that the numbers couldn't match. So I coloured the inside of the circle so that the children didn't feel like they had to fill in the middle." Kathy also found that the children tended to want to fill in the entire plate (see Figure 4), so she got her children to make a border around a circular mirror (see Figure 5). As the other teachers engaged with Sam's practice they negotiated the meaning of the paper plate and adapted its form to meet their understanding of the purpose of the plate.

Kathy and Kim developed a further interim stage of making a border around rectangular frames, which involved turning corners, but without a fixed number of squares (see Figure 6). We observed two children fitting pom-poms around a rectangular border. At first the pattern was white, white, white, yellow, yellow, blue but there was an error where it joined up with itself, which they fixed, with some prompting, by adding more pom-poms. When we asked if they could remove some pom-poms because the border was "a bit wavy", one child said, "I might change the pattern" and made it fit better by removing one yellow from every unit of repeat. The two children jointly described this pattern as "white, white, white-3 whites" and "…blue, yellow and start over and over". The changes in their teachers' practice and the reification of these changes (e.g. the open border with and without corners) seem to have enabled these children to analyse and identify the unit of repeat in a continuous pattern.

CONCLUSION

The early years teachers in this study worked together in a community of practice to improve their children's engagement and knowledge of mathematics through developing curriculum. As they collaborated together and shared their observations and pedagogical innovations they also improved our knowledge of the developmental progression from linear repeating patterns to border repeating patterns.

As these teachers participated in the community of practice they realised that although their children found linear repeating patterns relatively simple, many children found border patterns more complex. One teacher came up with the idea that children found it easier to make a border pattern in a circle because this avoided the difficulty of going around corners, and he reified this innovation in the form of a paper plate. Other teachers in the group took up this idea and tried it out in their classroom, adapting the reification to address the new complications and to add further interim stages between linear repeating patterns and border patterns.

The strength of this community of practice lay in the participants' willingness to share and develop their pedagogy, which resulted in the teachers creating significant interim stages in the trajectory of developing pattern awareness. As a consequence of this group's work the current developmental progression between linear repeating patterns is: 1) linear repeating patterns; 2) circular repeating border patterns; 3) repeating border patterns around shapes that have corners; 4) repeating border patterns around shapes with corners and a defined number of spaces to fill.

The study shows that when early years teachers are given the opportunities to work collaboratively together in a community of practice on a regular, on-going basis, they may not only improve their own subject knowledge and pedagogical practice but may add to our understanding of how young children learn and develop mathematically.

Acknowledgement

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TEACHER BELIEFS AND PRACTICE WHEN TEACHING MATHEMATICS WITH TECHNOLOGY – DOES GENDER MATTER?

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Teacher gender has been identified as a differentiating factor in the context of teaching mathematics with technology. However, there is also evidence that gender differences are becoming less pronounced, potentially due to the omnipresence of technology in everyday life. The present study investigates gender differences in the context of teaching mathematics with technology by focusing on teacher's self-efficacy, teacher's technology related beliefs and teacher's frequency of technology use. Analyzing the data from n=198 upper secondary school teachers in Germany, we observe no gender differences with respect to teachers' beliefs about teaching with technology and frequency of technology use. However, women show significantly lower self-efficacy beliefs with respect to teaching with technology.

INTRODUCTION

Research in the last decade has shown, that teaching with technology like function plotters, geometry packages and computer algebra systems (so called "Mathematics Analysis Software") can support the learning of mathematics in many different ways (e.g. Drijvers et al 2016). In this context it is important for teacher educators to understand the factors associated with technology integration and teacher gender has been discussed as a differentiating factor influencing technology uptake (e.g. Forgasz, Griffith, & Tan, 2006; Buabeng-Andoh, 2012). For example, research indicates that female teachers and female students may be less positive and less confident in using technology and make use of technology less often than male teachers and students. However, there is also evidence that gender differences are diminishing, possibly due to the fact that technology use has become ubiquitous in peoples private and professional life. In order to get a nuanced picture of gender differences with respect to technology related constructs of in-service mathematics teachers, we take a multidimensional approach and use differentiated multi-item scales to scrutinize gender differences with respect to teacher's self-efficacy beliefs, teacher's technology related beliefs and teachers' frequency of technology. The results can help to design professional development programs that take gender differences into account by providing tailored support for male and female teachers.

THEORETICAL BACKGROUND

Technology in the mathematics classroom

The use of digital technology in the mathematics classroom can comprise a plethora of different technologies. These range from general technology that can be used across

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 382-389). Pretoria, South Africa: PME.

different subjects (e.g. word processing software like MS-Word) to subject-specific technology like digital learning environments, function plotters, geometry packages and computer algebra systems (CAS) that are specifically used in mathematics education and are also known under the term "Mathematics Analysis Software" (MAS; Pierce & Ball, 2009). Research shows, that technology can support student learning in many different ways. For example, these tools can facilitate constructivist teaching approaches like discovery learning by giving pupils the opportunity to explore mathematical links on their own. In addition, digital tools can enhance conceptual understanding of mathematics by providing easy access to multiple and linked representations (Drijvers et al., 2016). For this reason, policy makers, researchers and teachers advocate the use of technology in the mathematics classrooms.

Gender differences

With the introduction of technology in schools, teacher and pupil gender has been discussed as a factor affecting technology use in the classroom (e.g. Forgasz, Griffith, & Tan, 2006; Buabeng-Andoh, 2012; Wong, Teo, & Russo, 2012; Sang, Valcke, Van Braak, & Tondeur, 2010; Pierce, Stacey, & Barkatsas, 2007; Antonietti & Giorgetti, 2006). In particular, some results indicate that female teachers (as well as female students) have more negative beliefs towards technology use and perceive themselves as less competent using technology. For example, Meredyth, Russell, Blackwood, Thomas, and Wise (1999) examined teacher gender differences in an Australian national sample and found that:

> "Male teachers were more likely to possess advanced information technology skills than female teachers and significantly more likely to indicate that they enjoyed using computers a great deal. They were more likely to say that they felt able to keep abreast of new programmes and educational applications. Information technology skills were more likely to be self-taught for male teachers than for female teachers. Male teachers appear to be more likely than female teachers to engage with a range of information technology uses, independently of the support provided by their employers." (Meredyth et al., 1999, pp. 281-282)

In addition, there are results indicating that women have lower self-efficacy beliefs with respect to teaching with technology compared to male teachers (e.g. Sang et al., 2010, p.104; Scherer & Siddiq, 2015). For example, Pierce and Ball (2009) find that significantly more female teachers agree to the statement "if there are unexpected problems caused by technology, this will be very difficult for me". The authors therefore hypothesize "that female teachers may be less confident than males about using technology" (Pierce & Ball, 2009, p. 312). However, Scherer and Siddiq (2015) caution that comparisons among female and male teachers must be based on a multidimensional construct. They measure self-efficacy related to computer use in the classroom in four areas, namely "Self-efficacy in basic operational skills", "Self-efficacy in advanced operational and collaborative skills" und "Self-efficacy in using computers for instructional purposes". Analyzing data from n=1208 Norwegian secondary school teachers they find gender differences in favor of male teachers only

for the first two scales. An explanation for this could be "that the instructional factor may additionally involve aspects of teachers' pedagogical content knowledge" (Scherer & Siddiq, 2015, p. 54).

According to Cooper (2006) the observed gender differences are fundamentally a problem of computer anxiety rooted in the socialization pattern of boys and girls. In addition, Huffman, Whetten, and Huffman (2013) show that gender roles like masculinity can be the source of gender differences and not biological sex alone. Furthermore, there is also evidence that unfavorable attribution patterns cause the gender differences. For example, Koch, Müller, and Sieverding (2008) show, that women attribute technical problems when using technology to their own failure whereas males attribute problems to external factors.

However, there are also studies finding no significant gender differences. For example, Forgasz (2002) finds no gender differences with respect to beliefs, perceived levels of competence or confidence in using computers for the teaching of secondary mathematics. Forgasz et al. (2006) asked n=38 teachers in an open-ended question to describe the impact that CAS calculators will have on teaching, student learning and curriculum. They could not find any statistically significant gender differences. With regard to teacher beliefs about technology in the mathematics classroom Pierce and Ball (2009) develop the "Mathematics with Technology Perceptions Survey". Gender differences were only observed for the item: „If I use more technology, my students will be more motivated to work on their math". Handal, Cavanagh, Wood, & Petocz (2011) investigated weather gender is related to the stage of adoption of graphing calculators. In a sample of n=587 teachers they could not find a significant relation between gender and teachers' stage of adoption of graphing calculators.

In total, it is unclear to what extent the "digital divide" (Cooper 2006; Vekiri & Chronaki, 2008), which refers to gender differences in access, intensity and nature of technology use, is still present. For example, the gender gap might be narrowing due to the increasing availability and penetration of technology in everyday life (e.g. Teo, 2010; Wong et al., 2012; Sang et al. 2010). Hence, Siddiq and Scherer (2016) make the point that the

> "[…] conflicting findings require a continued focus on whether differences across gender exist for the specific [technology related] constructs […]" (Siddiq & Scherer, 2016, p. 6).

RESEARCH QUESTION AND METHODOLOGY

The study aims to explore gender differences with respect to the use of technology with a focus on Mathematical analysis software (MAS) and addresses the following question:

How is gender related to

- self-efficacy beliefs in the area of teaching mathematics with MAS
- beliefs about the use of MAS
- frequency of the use of MAS.

To answer this question, it is necessary to measure teachers' self-efficacy beliefs, teacher beliefs about teaching mathematics with MAS as well as teachers' frequency of MAS use. In the following we give an overview about the measures used in this study to capture these constructs.

To measure teachers' self-efficacy beliefs, we used a questionnaire that assessed self-efficacy in two domains: (s1) Self-efficacy related to task design & task selection when teaching with MAS (4 Items), (s2) Self- efficacy related to lesson design & lesson implementation when teaching with MAS (4 Items). Response category was chosen according to Banduras "Guide for constructing self-efficacy scales" (Bandura, 1997) which recommends that teachers rate the strength of their belief in their ability on a scale from 0-100. In addition, we used a single item to capture the perceived competence in operating the GC. Teachers were asked to rate their competence on a five-point Likert-scale ranging from "very low" to "very high".

To measure teacher beliefs we used a questionnaire which consisted of the five following scales (Thurm, 2017): b1) beliefs that MAS supports discovery learning, b2) beliefs that MAS support multiple representations, b3) beliefs that MAS is too time consuming (as there is a general concern that there is not enough time to cover the MAS and the required curriculum), b4) beliefs that MAS has a negative impact on computational skills (as there is a common concern that pen-paper-skills may be lost in the presence of MAS), b5) beliefs that MAS leads to mindless working, b6) beliefs that students must master concepts and procedures prior to MAS use. Responses were given on a five point response format ranging from 1= "strongly disagree" to 5= "strongly agree". Hence high values on the scales b1) and b2) with simultaneously low values on the scales b3) – b5) reflect positive beliefs about the use of MAS in mathematics education.

To capture teacher's MAS use we focused on frequency of MAS use which can be more validly captured by self-reports than the quality of teaching (Mayer, 1999). We assessed the frequency of MAS use with a Likert-scale questionnaire (Thurm, 2018) that differentiates the use of MAS in the following areas: (f1) use of MAS for discovery learning, (f2) use of MAS for linking multiple representations, (f3) use of MAS to support individual learning, (f4) use of MAS when practicing. In addition, a category capturing how often the use of MAS was subject to critical reflection was included (f5). Items in this category asked for example on how often limitations of MAS were discussed or on how often there was a critical reflection about when to use pen & paper skills and when to use MAS to solve a given task.

Data was gathered in 2014 from n=198 upper secondary school teachers within a larger research study (Thurm, Klinger, & Barzel, 2015) in North Rhine-Westphalia, Germany. In this state the use of MAS is compulsory in the final examination in upper secondary since the schoolyear 2014/2015. The average age of the teachers was 43 years, which is comparable to the average age of 45 years for all teachers in North

Rhine-Westphalia. Teaching experience with mathematics was distributed as follows: 0-5 years (27%), 6-11 years (19%), 12-17 (16 %), 18-23 (11%), more than 23 years (25%). Due to the fact that MAS had been made compulsory only shortly before the study took place, there was a large number of teachers without any previous experience in teaching with MAS. In total, 52% had no previous experience teaching mathematics with MAS and only 12% had more than 5 years' experience in teaching mathematics with MAS.

RESULTS

Table 1, Table 2 and Table 3 show the correlations between self-efficacy, beliefs about teaching with MAS and frequency of MAS use, respectively. No significant relationship between gender and beliefs about MAS use was found. There was also no significant correlation between frequency of MAS use and gender. However, gender differences were found with respect to teachers' self-efficacy beliefs. Female teachers had significant lower self-efficacy beliefs with respect to task design & task selection (s1) and operating MAS (s3). Self- efficacy beliefs related to lesson design & lesson implementation (s2) were also negatively correlated with gender, but in this case the correlation was not significantly different from zero.

Self-efficacy	ρ
s1) Self-Efficacy with respect to task design & task selection	**-0.19***
s2) Self- efficacy related to lesson design & lesson implementation	-0.12
s3) Operating MAS	**-0.22***

Table 1: Correlation between self-efficacy and gender

(* p<0.1, ** p<0.01, *** p<0.001)

Beliefs about technology use	ρ
b1) MAS supports discovery learning	0.02
b2) MAS support multiple representations	0.07
b3) MAS is too time consuming	0.05
b4) MAS has a negative impact on computational skills	0.08
b5) MAS leads to mindless working	0.10
b6) students must master concepts and procedures prior to MAS use	0.13

Table 2: Correlation between beliefs about MAS use and gender
(* p<0.1, ** p<0.01, *** p<0.001)

Frequency of technology use	ρ
f1) use of MAS for discovery learning	0.00
f2) use of MAS for linking multiple representations	0.05
f3) use of MAS to support individual learning	-0.07
f4) use of MAS when practicing.	-0.06
f5) reflection of MAS use	-0.10

Table 3: Correlation between frequency of technology use and gender (* p<0.1, ** p<0.01, *** p<0.001)

DISCUSSION

The study aimed to scrutinize the link between gender, self-efficacy beliefs, beliefs about teaching with MAS and frequency of MAS use. The results indicate that there is no differential effect of gender with respect to beliefs about MAS use and frequency of MAS use. These results are in line with the results of Forgasz (2002), Forgasz et al. (2006), Pierce and Ball (2009) and Handal et al. (2011). However, gender differences are apparent with respect to teacher self-efficacy beliefs which supports the hypothesis of Pierce and Ball (2009), claiming that women have lower self-efficacy beliefs with respect to teaching with technology compared to male teachers. In addition, the multidimensional approach taken in this paper shows that differences in self-efficacy beliefs are more prominent for operational skills and task design and less pronounced for designing and implementing lessons when teaching with technology. This seems in line with the results of Scherer and Siddiq (2015) who show that gender differences in self-efficacy beliefs depend on the extent to which technical skills play a role in the different self-efficacy domains that are captured. In this study, the scale that focuses on lesson design and implementation is not only capturing technical skills but also aspects of teachers' pedagogical content knowledge.

In summary, this study could not find a gender gap for technology related beliefs and frequency of technology use. However, gender differences with respect to self-efficacy beliefs seem to remain in place. For example, Scherer and Siddiq (2015) found that professional development programs can be particularly effective for woman to increase their self-efficacy beliefs:

> "[…] for the subsample of female teachers, participating in courses on ICT integration into teaching and learning was positively and significantly related to CSE [computer self-efficacy] in the advanced and instructional use of computers" (Scherer & Siddiq, 2015, p. 52).

Hence if we want to close the remaining gap designers of professional development need to be aware of potential gender differences with respect to self-efficacy and provide means to strengthen female teacher's self-efficacy beliefs.

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A TEACHER'S CONCEPTUAL 'AHA' ENABLES REAL-TIME ADAPTATION TO STUDENTS' MULTIPLICATIVE REASONING

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We examine how conceptual progress in an elementary teacher's mathematical knowledge for teaching seen as a units coordination 'aha moment', enabled real-time adaptation of her teaching to fit students' available conceptions. We analyze the case of Mona, a grade-5 teacher who planned a lesson to foster students' reasoning with the Unit Differentiation and Selection (UDS) conception. In a pre-lesson interview, Mona realized why the Same-Unit Coordination (SUC) conception, which involves additive operations on composite units, serves as a conceptual requisite for UDS. This seemed to underlie a real-time adaptation to her goal for students' learning during the observed lesson. We discuss the key role a units coordination lens can serve in deepening teachers' mathematical knowledge to benefit students' reasoning.

To explicate our research question, we first illustrate the mathematical conceptions involved. Consider a situation involving 40 packs, 6 cans each (40x6=240 cans). A pack is thought of as a composite unit, as it is composed of smaller units—here, cans thought of as 1s (Steffe, 1992). A logical anticipation is that the total of cans in the entire set remains the same even if organized into two sub-sets (e.g., 21+19). Consider further that Store A gets 21 packs and Store B gets 19 packs. Two numbers of interest could be: the total of *cans* in both stores, and how many *more cans* Store A has than Store B. The logical anticipation entails a general procedure: find sub-totals of cans (21x6; 19x6) and add or subtract them (126±114). Yet, reorganizing the sets of composite units (packs) could be much more efficient. For the total we add 21+19=40 packs and multiply 40x6=240 cans; for the difference we subtract 21-19=2 packs and multiply 2x6=12 cans. The latter is implied by the logical anticipation: 21=19+2 packs. Such reorganization relies on (a) differentiating the two types of units (packs=composite units, cans=1s) and (b) selecting the composite units (packs) as a start. It also underlies the coordination of multiplicative and additive operations known as the distributive property (Tzur et al., 2009): (21+19)x6=21x6+19x6 and (21-19)x6= 21x6-19x6. We think teachers can realize that operating only on composite units (e.g., 21-19=2 packs) is a conceptual prerequisite for finding the total or difference in 1s (e.g., 2 packs x 6 cans each = 12 cans). Accordingly, our study addresses the question: How may such a realization ('aha moment') enable a teacher to adapt her teaching so it fits with students' available conceptions?

This question is situated at the nexus of four lines of work, which we endeavor to connect further: teachers' mathematical knowledge for teaching (MKT; Hill, Rowan, & Ball, 2005), teachers' noticing (Jacobs, Lamb, & Philipp, 2010), units coordination

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as a lens to articulate children's mathematical reasoning (Norton et al., 2015), and student-adaptive pedagogy (AdPed; Tzur, 2013). MKT pertains to teachers' intertwined understanding of particular mathematics and of how students think about and learn it. MKT has become a central construct, informing both assessment of and PD for teachers, not the least because it has significant impact on student learning and outcomes (Hill et al., 2005). Noticing, an important component of MKT, pertains to teachers' awareness, propensity, and capacity to discern specific students' behaviors and strategies when engaged in doing mathematics (e.g., counting on fingers, drawing diagrams, etc.). We draw on both lines of work, while stressing the need to augment teachers' MKT and noticing with conceptual models that enable: (a) going beyond students' observable behaviors to articulate their available conceptions, and (b) adapting goals/activities for students' learning to those conceptions.

CONCEPTUAL FRAMEWORK

We draw on Piaget's (1985) core notion of assimilation, which entails one's available conceptions govern her mathematical behaviors. We further build on Steffe's (1992) units coordination lens, which articulates children's numerical conceptions in terms of operations on units—distinguishing 1s from composite units. We focus this study on three conceptions in a multiplicative reasoning progression (Tzur et al., 2013). The first, *Multiplicative Double Counting* (mDC), entails distributing items of one composite unit (e.g., 3 packs of gum) over another composite unit (e.g., 5 pieces of gum in each pack), and then finding the total of 1s (e.g., 1-is-5, 2-is-10, 3-is-15). MDC is multiplicative, as this distribution *transforms the units* (Simon, Kara, Norton, & Placa, 2018). The second, *same-unit coordination* (SUC), entails being cognizant of and operating on sets of composite units without losing sight of the 1s in them (e.g., 3 packs and 6 packs, each with 5 pieces of gum). One's goal may be to find the sum of or difference in those sets (e.g., 6+3=9 packs, 6-3=3 packs). SUC is an *additive, unit-preserving operation* (Schwartz, 1991) on sets of composite units. We explained the third conception, *Unit Differentiation and Selection* (UDS), in the Introduction. It involves operating additively on two sets of composite units and then multiplicatively on the sum/difference of composite units and the 1s in them. That is, UDS is a coordination into a single conception of SUC followed by mDC.

The notion of assimilation underlies our student-adaptive pedagogy (AdPed; Tzur, 2013) stance, which we apply in our work with teachers and students. AdPed builds on and goes beyond noticing by inferring available conceptions that plausibly underlie students' noticeable behaviors. Goals/activities for students' learning are adapted to fit with those inferred conceptions. In this study, our work with teachers thus fostered not only their reasoning with mDC, SUC, and UDS but also explicating similarities and differences in units and operations each conception entails. If a teacher indicates she can assimilate and pose mDC and SUC tasks, we promote her linking of those with UDS—particularly SUC as its first step. We see such a linkage as a necessary component in a teacher's MKT, as it enables going beyond noticing to inferring students' conceptions and adapting teaching accordingly.

METHODS

This case study with Mona, a 5th grade teacher at a school within a US metropolitan area, was part of a project[1] to promote and study elementary teachers' professional development (PD). About 85% of the students in that school (and her class) identified as students of color, with ~70% learning English as an additional language. Mona, along with her 3rd, 4th, and 5th colleagues, participated in the 2-year PD program of our larger project. Using the units coordination lens, the PD program focused on advancing three facets of teachers' knowledge. One facet, the teachers' MKT, focused on their multiplicative and fractional reasoning. Another facet, focused on the teachers' capacity to (a) notice specifics of students' work and (b) explain it in terms of units (1s, composite units) and operations (additive, multiplicative) that underlie students' observable behaviors. A third facet focused on teachers' ability to adapt their teaching so it fits with and explicitly draws on what students do know.

We mingled work on all three facets in four components of our PD program, while using playful activities and tasks that could both foster teachers' learning and be easily modified for their own classrooms. First, we engaged all teachers in a 5-day Summer Institute. Next, over one school year we used a job-embedded PD consisting of four half-day workshops and monthly reciprocal visits by partner teachers to ("buddy-pair"), an adaptation of the Lesson Study (Corcoran & Pepperell, 2011) approach to the US context. Then, we conducted a second, 5-day Summer Institute in which observing, reflecting on, and discussing video segments of teachers' work with students in their own classrooms served as a key tool. A similar job-embedded PD effort during the following school year concluded the PD program.

For data collection and analysis, we followed Simon and Tzur's (1999) *Account of Practice* (AOP) strategy. An AOP data set involves observing a teacher's class while she teaches new mathematical ideas, and pre/post interviews about the observed lesson(s). This study focuses on a video recorded set (Intrv-Obsrv-Intrv) with Mona 1.5 months before the PD program ended. Glaser and Strauss' (Glaser & Strauss, 1967) grounded theory methodology underlies analysis of AOP data. Using the units coordination lens to discern Mona's understanding of operations on 1s and composite units that underlie the mDC, SUC, and UDS conceptions, we scrutinized, discussed and transcribed video segments. First, team members working individually and in small sub-teams, observed the entire set while jotting logs of main events. The entire team then met to analyze data segments that demonstrated Mona's understanding of the three conceptions, her teaching moves, and her thinking about those moves. Once discerning Mona's 'aha' (pre-interview) that deepened her MKT and noticing of students' operations on 1s to solve an SUC task, we could analyze her rationale for adapting the lesson as she assimilated their work into her novel, SUC-UDS linkage.

RESULTS

We analyze data to support a twofold claim about augmenting a teacher's MKT with units coordination: It deepens her MKT and noticing, and thus enables conceptually-

driven adaptation to her teaching. We first demonstrate how Mona realized a novel link between the UDS conception—her original goal for students' learning—and the SUC conception as a conceptual foundation of UDS. Then, we show how this link enabled recognizing (a) the conceptual source of students' lacuna when they solved an SUC "review" task and (b) a need for and actions to change her lesson so it nurtures the prerequisite, additive operations on composite units.

Deepening MKT: Conceptualizing SUC as UDS Requisite

To start the pre-lesson interview, the researcher asked Mona to describe her plan for the upcoming lesson. Mona said she would begin with an mDC and and SUC review tasks. Then, she would turn to her main goal: students' learning to reason with UDS. To illustrate the meaning of each conception, Mona pointed to specific examples of tasks she prepared on large posters. These illustrations, which Mona further explained to the researcher, indicated she could clearly articulate each of the three conceptions (mDC, SUC, and UDS). Mona said this plan drew on her assessment, through individual interviews of each student (two months earlier, before the class shifted attention to the state annual test), that most students could reason with mDC and SUC. Excerpt 1 presents Mona's response to the researcher's probing about the link between mDC and SUC, after M ("M" stands for Mona, R for the "Researcher").

<u>Excerpt 1</u> (Pre-Interview): Mona's 'aha moment' of the SUC-UDS link.

R: So, can you talk to me about the goal for today, UDS? What's different about UDS?

M: [In UDS] Comparing our composite units, but then determining the difference in the single units. So, you're still tracking the Multiplicative Double Counting. And a lot of the kids in this one, can do it, but they do it in a not very efficient way.

R: Can you explain that? … [Have you] already done some UDS with them?

M: Yeah, well, I asked them questions … A majority were still just confused. But a lot of them will figure out [their] single units, and my single units, and then subtract them versus being able to see we have the same exact composite units. [But in UDS we] just have to figure out what's the difference in composite [units] … OH! ('aha' facial expression) So that's why Same Unit Coordination is important.

R: Can you tell me what you're thinking?

M: (Brings a piece of paper) So, in single unit (she draws 6 lines, writes 5 above each line, and writes '6T' and '5c'), I do not know why I keep going to 5. It doesn't really matter. (She draws 3 more lines, writes 5 above each line, and writes '4T' and '5c'). So, in Same Unit [Coordination], you're concentrating really only on the towers (circles '6T' and '4T' in her drawing). And, Unit Differentiation [and Selection], you really only want to concentrate on the towers … I have two more towers, so I am still only back to that Same Unit Coordination. Cubes don't matter until I realize that I have two more towers, [and] 5 [cubes] in each would be 10 cubes altogether, [with this last step being Multiplicative Double Counting.

We see three key points to understanding Mona's novel units coordination, leading to her 'aha' about the SUC-UDS linkage. First, she distinguished mDC ("pay attention to

composite units as well as the single units") from SUC ("only asked about the composite units"). This indicated her MKT already included a focus on units that underlie different solutions and operations on those units. Second, she genuinely stated her inability to explain ("I don't really think I know") the importance of this distinction in linkage to her UDS lesson goal. We infer the missing linkage was between SUC and UDS based on Mona's utterance right after her 'aha' ("So that's why Same Unit Coordination is important"). Third, she then brought forth her conceptualization of UDS ("just have to figure out what's the difference in composite [units]"), followed by her reflection on students' difficulties through the lens of units coordination ("A majority were still just confused"). Reflecting on those experiences led to her realization ("Oh!") of the SUC-UDS linkage, which she explained in terms of units and operations ("And, Unit Differentiation [and Selection], you really only want to concentrate on the towers … I have two more towers, so I am still only back to that Same Unit Coordination."). She finished this explanation with a numerical example, subtracting 6-4 (towers) to find that difference and then multiplying (using mDC) to find there are additional ten 1s in the larger set of 5-cube towers. Heading into the lesson, we infer Mona's MKT was deepened: She could consider students' learning of and struggles with UDS in light of SUC as a conceptual prerequisite.

Conceptually-Driven, Real-Time Adaptation of Teaching

To start the lesson, Mona engaged all students in solving the mDC review task (total of cubes in 7 towers with 4 cubes each). She looked for their coordination of the composite units (a finger raised for each tower) and 1s in each (uttering multiples of 4). While some students appeared facile with mDC, others seemed to struggle. Mona asked a student to share a solution, which seemed to meet Mona's expectation for both the correct answer and the double-count (1-finger-is-4, 2-are-8, etc.).

Letting students work for over a minute on the SUC task (3 packs and 6 packs, each with 5 pieces of gum; how many packs), Mona noticed they got stuck. She first asked them to talk with a partner, noticing most answered, "45." Then, she asked the entire group who could explain. Only one student raised her hand. To Mona's question, "Ok, what's your answer?" the student responded, "Nine. Because 3 (packs of 5) is 15, and six is 18." Thus, four minutes into the lesson, Mona decided to adapt it. While flipping the poster with review tasks and the poster for teaching UDS, she told the class: "I'm changing what we were originally going to do." As Excerpt 2 from the post-lesson interview shows, she decided to focus on SUC instead of on UDS.

<u>Excerpt 2</u> (Post-Interview): Mona's rationale for adapting the lesson.

R: After you did the mDC, you went into the SUC question. Talk to me a little about what you observed then, and why you chose [to change the lesson].

M: (Appears perturbed) Umm. I mean (pause). When we went into the SUC, they reverted back to mDC. They did not see that as a different question at all. They saw it as 3 packs of gum. [A little later, after clarifying the numbers in the SUC task] But even if I put towers and cubes [in place of packs and guns] I don't think it would have changed the

outcome. I think that had I asked them how many pieces of gum, that would have demonstrated they understood mDC. But they totally saw that as an mDC, not as an SUC, and not one group said the correct answer.

R: What were the groups saying? 'Cause you [kept asking] What did you say?

M: [As their answers] I heard 15; I heard 18. I didn't hear 9; ever. Student C was the only one who kept going, "wait, why?" [And he] kept going "I don't know if that's the answer." But he couldn't articulate why it wasn't the answer anyway. So they were 45, that was a [common] one. So they were just figuring out how many pieces of gum [in each set] and then adding them together.

[A short exchange here focused on Mona's choice of numbers for the SUC task.]

M: But as soon as I heard all those answers, I was like, we are at the very beginning stage of this. Because they don't see the composite (units). They only see that as…they did 3 times [5]. The people who got 45, that tells me that they understood 3, the packs were [like] towers, the pieces were [like] cubes, and they figured out the cubes altogether.

R: So that's when you chose to stop, and you didn't move on to UDS (original goal)?

M: Noooo (body expression indicating UDS was absolutely improper in her mind).

R: Why?

M: Because to do UDS, you have to be able to see it as composite units first, then single units. [R: Why?] Because your ultimate goal is to [first] compare your composite units, figure out the difference of composite [units], and then know the left overs, then [it goes back] to mDC.

R: So why stop? Why not just keep going [to UDS]?

M: (Looks at the researcher as if the answer is obvious) Why go through the motions when they wouldn't have understood what they were doing?

R: (A bit later) So, you see it very important that they have abstract understanding of mDC and SUC before moving into UDS?

M: Yeah, and I really have never realized how much … until I did that and listened to answers, I was like, I now totally get it … (R asks: Get what?) That they have to be able to differentiate, and understand the question, of composite units versus single units.

Data in Excerpt 2 support our claim that Mona's deepened MKT enabled adapting her lesson in real-time. Her noticing that students reasoned with mDC to solve the SUC task could be rooted in her prior distinction between the focus on 1s (mDC) and on composite units (SUC). Similarly, rejecting that the issue was context ("… had I asked them how many pieces of gum, that would have demonstrated they understood mDC") might have also been supported by her previous MKT. However, her noticing ("So they were 45, that was a [common] one") and acute awareness that students operated on 1s, not on composite units, seemed to then be assimilated into her novel SUC-UDS link (e.g., "Because they don't, they don't see the composite units").

When asked of her decision not to go to UDS, her rejection ("Noooo") showed it contradicted her stance of setting a sensible goal for students' learning. She realized

her students were yet to develop the conceptual prerequisite for UDS, a linkage she made through the 'aha' moment in the pre-lesson interview. Thus, changing the lesson to focus on SUC seemed to her a more suitable path ("Why go through the motions when they wouldn't have understood what they were doing?"). Her explanation of the conceptual basis for her adaptation reflects the conceptual linkage she made in the pre-lesson interview, between SUC and UDS. That is, her deepened MKT included units coordination for UDS that requires operating on composite units before operating on 1s ("Because to do UDS, you have to be able to see it as composite units first, then single units."). Thus, to her it made sense to focus first on students' identification of and additive operations on composite units, that is, SUC.

DISCUSSION

We see two key contributions of this study. First, it stresses two roles that a units coordination lens can serve in promoting teachers' MKT and, hence, their practice. One role is in promoting teachers' learning through teaching (Tzur, 2010), as they make sense and use of research that explicates units and operations underlying their own (MKT) as well as their students' mathematics. Mona's case is telling. She participated in a 2-year PD program that constantly emphasized units and operations linkages between mDC, SUC, and UDS. Through further probing of her thinking about those linkages during the pre-lesson interview, she conceptualized the SUC-UDS linkage for herself (see Excerpt 1). The second role, once a teacher deepens her MKT with an explicit units coordination linkage—is driving her (a) noticing of students' abilities and struggles, (b) inferring conceptual sources of what she notices, and ultimately (c) adapting her teaching to fit learning goals to students' reasoning (not fit students to a pre-set plan). Again, Mona's case is telling. It illuminates how a teacher's MKT, once deepened with a units coordination lens, afforded conceptually-driven planning and adaptation to her planned lesson.

A second contribution is in using Mona's case to elucidate a reorganization of four constructs: teacher MKT (Hill et al., 2005), teacher noticing (Jacobs et al., 2010), a units coordination lens, and student-adaptive pedagogy (Tzur, 2013). Here, a conceptual 'aha' rooted in coordinating units and operations that help link SUC and UDS deepened her MKT. In turn, her explicit units coordination expanded what she could notice in her students' work (e.g., attributing their responses to the SUC task to irrelevant operations on 1s). Thus, her deepened MKT afforded going beyond noticing to explicate conceptual causes for students' behaviors (Hodkowski, 2018). Being committed to students' learning of what is conceptually reachable for them, these new capacities afforded Mona's conceptually-driven adaptation to her teaching. Drawing on Mona's case, future research can further articulate complexities of the interplay between a teacher's MKT, noticing, and adaptive teaching.

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MEASURING SPATIAL ABILITY – WHY THE PROJECTION METHOD MATTERS IN THE PSVT:R MENTAL ROTATION TASKS

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Spatial ability is an important component of mathematical competence. There are multiple ways to represent three-dimensional figures in a two-dimensional way. This paper reports a study that systematically compared the effect of the projection method used in a validated questionnaire (PSVT:R) on performance in conducting mental rotations. It was found that accuracy and efficiency was lower when items were presented in isometric projection as compared to a trimetric or oblique parallel projection, and this was the case for a sample of Grade 6 students as well as Grade 12 specializing in mathematics. Implications for research and practice are discussed.

THEORETICAL AND EMPIRICAL BACKGROUND

Strengthening learners' spatial ability is an important goal of nowadays' mathematics education, both at the primary and secondary level. This ability includes understanding two-dimensional representations of spatial situations and using such representations in problem solving. For instance, one of the educational standards in the Flemish secondary curriculum states that learners should be able to illustrate that information can be lost when representing three-dimensional situations by means of a two-dimensional representation (Vlaams Ministerie van Onderwijs en Vorming, 2014).

There are multiple ways to represent spatial situations on a two-dimensional plane, based on different projection methods. In Figure 1, the same geometrical object is represented in four different ways. Only the differences between figures 1b and 1c (orthogonal projections) are based on taking a different viewpoint in relation to the figure. The other differences are intrinsically related to the specific properties of the projection method itself. For instance, the vertical edges in Figure 1a (central projection) are not shown in parallel. In Figure 1d (oblique parallel projection), the front side can be seen in its true shape (lengths and angles are shown truthfully), which is impossible in Figures 1b and 1c. All these projections are used commonly in school contexts, professional contexts and daily life.

An important question is how mathematics education should deal with these different representations. Each of the representations has specific properties that could be used in education to enhance spatial ability. The central projection in Figure 1a is the

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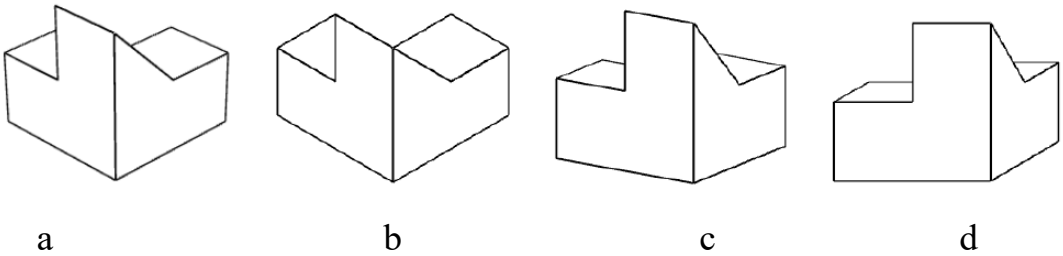

Figure 1: Central (a), orthogonal (b and c) and oblique parallel (d) projections of the same figure.

representation that best connects to the way we observe objects, but is too complicated to use for constructions in mathematics classes, whereas the oblique parallel representation in figure 1d may be the easiest to use and construct, although it is the furthest away from the way we observe (van de Craats, 1986). In educational practice, this complexity is typically not addressed, and also in research, representations are used inconsistently (for a clear example, see Baki, Kosa, & Guven, 2011).

This paper reports about one of the first studies that systematically investigates the impact of the projection method that is used to represent spatial objects, specifically in the measurement of an important component of spatial ability, i.e. mental rotation (Borella, Meneghetti, Ronconi, & De Beni, 2014). A widely used and well validated test instrument to measure mental rotation ability is the (revised) Purdue Spatial Visualisation Test: Rotations (PSVT:R). It was originally developed by Guay (1976), and revised and psychometrically validated by Yoon (2011; Maeda, Yoon, Kim-Kang, & Imbrie, 2013). The test contains 30 items in which one given object is rotated, and

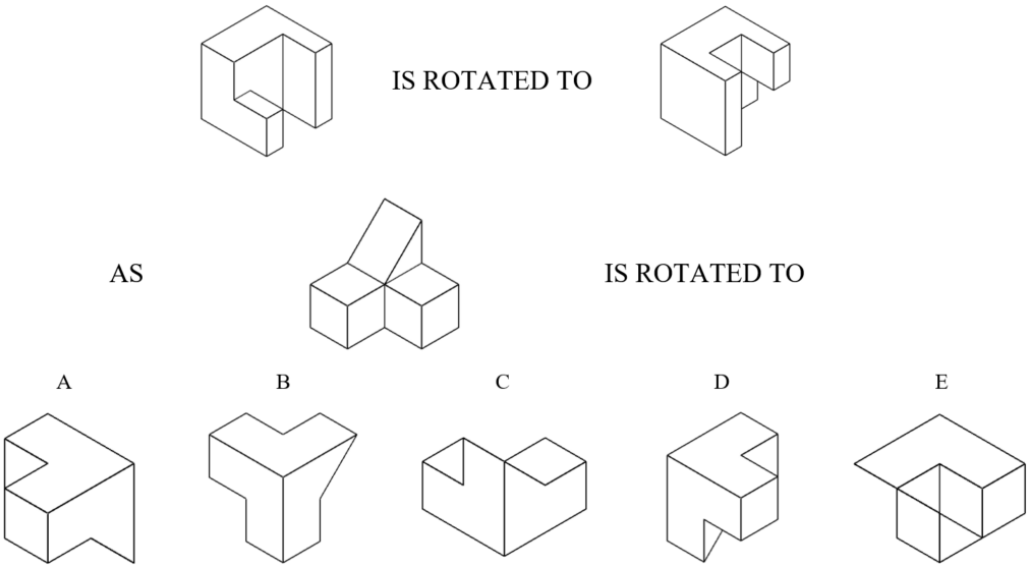

Figure 2: Item 10 of the PSVT:R (Yoon, 2011, p. 30).

respondents have to indicate how another object would look like if it is rotated in the same manner. An example item is given in Figure 2.

The PSVT:R uses an orthogonal projection, and more specifically the isometric projection method (as also used in Figure 1b). The question remains, however, to what extent the performance on this questionnaire is affected by the use of this projection method. Branoff (2000) has already criticized this choice based on the fact that the isometric projection method leads to figures that are difficult to interpret: "isometric projections of three-dimensional objects create accidental instances where the three-dimensional objects may be interpreted as two-dimensional patterns" (p. 15). A clear example is answer alternative E in Figure 2. Branoff (2000) and Yue (2008) conducted some exploratory studies in which they respectively used a test variant with trimetric projections and central projections, and they found indications for shorter test times and better performance. However, differences were not significant.

RESEARCH QUESTIONS

The present study was conducted to systematically investigate the impact of the projection method that is used in the PSVT:R. We compared the original PSVT:R with a variant that used a trimetric projection as well as a variant that used an oblique parallel projection. We were particularly interested in the effect of the projection method on the performance on the questionnaire, as well as the efficiency (i.e. the performance taking into account the time used).

A further question was whether the effect of the projection method would be similar across different levels of expertise. We therefore conducted the study in two populations: Grade 12 students in study programmes with a strong mathematical focus (the age and expertise level for which the PSVT:R is intended) and Grade 6 students. If certain projection methods would be the cause of difficulties, one could expect these to occur more prominently in students with lower levels of expertise.

We also systematically investigated which item characteristics could be the underlying cause of students' errors, and of differences in performance between the projection methods. Particularly the fact that in some items the isometric projection caused edges or vertices to coincide largely contributed to the difficulties we observed. Due to space limitations, these findings cannot be reported in the current paper but they will be presented at the conference. An illustration of this issue are the coinciding edges and vertices in the isometric representation of answer alternative A in Figure 2 as compared to the trimetric and oblique representations of answer alternative A in Figure 3.

METHOD

Instrument

The original PSVT:R questionnaire was developed in isometric projection. For the purpose of the current study, all objects were first reconstructed using Geogebra 2013 (version 4.2.30.0). Next, variants of the questionnaire were made in trimetric projection and in oblique parallel projection. For reasons of illustration, Figure 3 shows the answer alternatives of Item 10 of the PSVT:R in these projections (the original isometric alternatives are shown in Figure 2).

Sample

The test was administered to two samples in Flanders, Belgium. The first sample was a group of 187 Grade 6 primary school students (nine class groups from five different schools). As there are no study tracks in primary school, this sample can be considered representative for the population of that age (11-12-year olds). The PSVT:R is originally intended for participants starting from the age of 13, so it could be assumed that the test would be quite challenging for this first sample.

The second sample consisted of 161 students from Grade 12 of secondary school (ten class groups from 6 different schools). They were following a track in general secondary education, preparing for higher education studies and with an above-average focus on mathematics, following 6 to 8 hours of mathematics per week. Due to selection mechanisms throughout the Flemish secondary education system, this group cannot be considered representative for the general population of 17-18-year olds. This sample has a specific interest, prior knowledge and ability in mathematics.

Although we will refer to these two samples as two "age groups", it will be clear that they differ in several ways beyond their age. We intentionally included these two different groups in order to gain a better understanding of the effect of the projection method. It might be that the projection method played a different role in students with different levels of expertise.

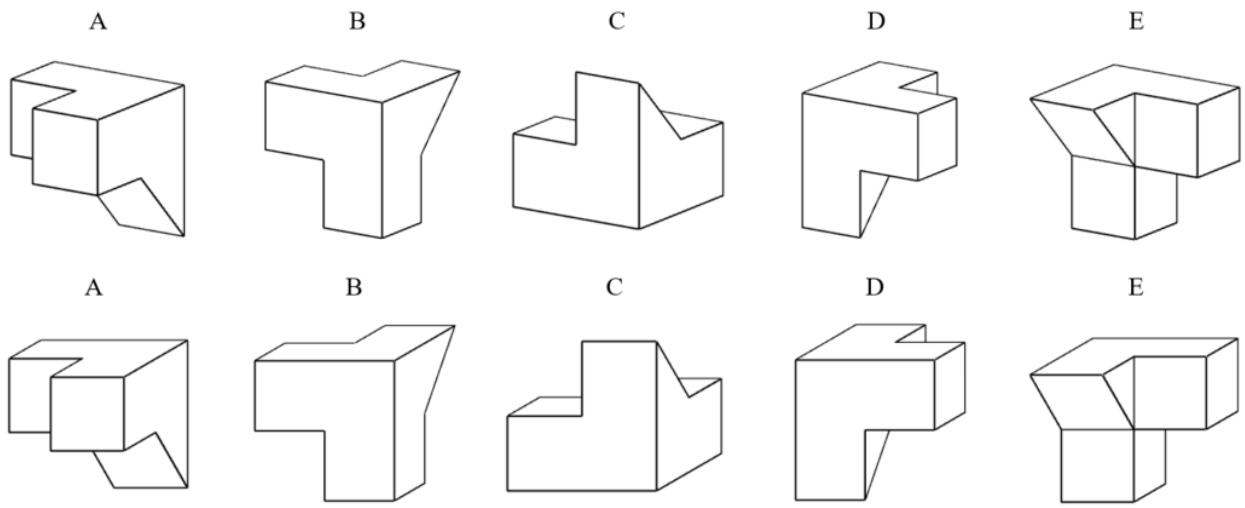

Figure 3: Answer alternatives of Item 10 of the PSVT:R. Figure above in trimetric projection. Figure below in oblique parallel projection. (example of isometric projection in Figure 2)

Procedure

Test administration was done collectively in class groups. The three test variants were randomly distributed to students within each class group. Students were not aware that different test variants were used.

Because of the young age of our Grade 6 sample, we introduced the test by means of two self-developed practice items, in which real spatial objects were shown and rotated at the front of the classroom. In this way, we clarified the reasoning that was expected in the test itself. The same instructions were also given to the Grade 12 school sample. Students could work as long as they wanted, but they were asked not to return in the booklet to items previously solved. In order to calculate the efficiency, the time it took students to complete the test was also registered on the test booklet.

RESULTS

Accuracy

An overview of the accuracies in both age groups and in the three conditions is provided in Table 1. In order to test the effect of the projection method on accuracy, in each age group a one-way ANOVA was conducted in SPSS23. Post hoc tests to test for significant differences across conditions were conducted using Tukey's HSD. Outlier scores were removed. Scores for Grade 12 students were then transformed (using a square root transformation) in order to meet assumptions of normality. Levene's tests indicated that homogeneity assumptions were met.

In the group of Grade 6 students, a significant effect of projection method was found, $F(2, 182) = 4.035$, $p = .019$, *partial* $\eta^2 = .042$. The post hoc tests indicated that accuracy in the isometric projection was significantly lower than in the other two projections, with no difference between the latter two.

A similar trend was found in the accuracy scores of the grade 12 students, but the effect was not significant, $F(2, 156) = 2.461$, $p = .089$, *partial* $\eta^2 = .031$. Again, accuracy in the isometric projection was lower than the other two.

	Grade 6		Grade 12	
	Mean	SD	Mean	SD
Isometric	12.4	5.19	22.4	4.35
Oblique parallel	13.8	4.40	24.0	4.16
Trimetric	14.0	4.06	24.1	3.48
Total	13.4	4.57	23.5	4.07

Table 1: Mean accuracy and standard deviation, by age and projection method

	Grade 6		Grade 12	
	Mean	SD	Mean	SD
Isometric	1.0	0.38	1.2	0.51
Oblique parallel	1.1	0.50	1.4	0.52
Trimetric	1.2	0.49	1.6	0.50
Total	1.1	0.47	1.3	0.52

Table 2: Mean efficiency score and standard deviation, by age and projection method

Efficiency

Efficiency was operationalised as the ratio between the number of correct responses and the time (in minutes) taken to complete the test. An overview of the efficiency scores in both age groups and in the three conditions is provided in Table 2.

Similar to the analysis of accuracies, in each age group a one-way ANOVA was conducted with condition as a predictor and using Tukey's HSD for post hoc comparisons. Scores for Grade 6 and Grade 12 students were then transformed (using a log10 transformation) to meet assumptions of normality. Levene's tests indicated that homogeneity assumptions were met.

In the Grade 6 students, a significant effect of projection method was found, $F(2, 184) = 4.535$, $p = .012$, *partial* $\eta^2 = .047$. The post hoc tests indicated that efficiency in the isometric projection was significantly lower than in the other two projections, with no difference between the latter two.

The same effect was found in the Grade 12 students, $F(2, 156) = 6.094$, $p = .003$, *partial* $\eta^2 = .072$. Again, efficiency in the isometric projection was lower than the other two projections, while there was no difference between the trimetric and oblique parallel projection.

CONCLUSIONS AND DISCUSSION

In general, we found that the accuracy on the PSVT:R in the isometric projection was lower than in the oblique parallel and trimetric projection. This effect was significant in the Grade 6 sample but not in the Grade 12 sample. The same pattern was found in the efficiency scores, and this time, differences were significant both in the Grade 6 and Grade 12 sample.

The original PSVT:R was developed by Guay (1978) using isometric projection, and also the validation studies happened using this original version (e.g., Maeda et al., 2013; Yoon, 2011). We now have observed that this projection method makes the test more difficult, as compared to when other projection methods would be used, and this seemed particularly the case in younger students with less expertise. If the goal of the

test is to measure students' spatial ability and more specifically their mental rotation skills, one can wonder whether the choice for the isometric projection is the best possible choice. Even when students have managed to conceive the spatial figures and to rotate them in the same way as the given example, they may still encounter difficulties finding the correct answer alternative merely because of confusions that occur due to the choice of the projection method.

At first sight, it may look as if this issue merely relates to a psychometric discussion about how mental rotation ability can be measured reliably and validly. This is important in its own right, but the choice of a projection method, has also specific relevance from a mathematics education perspective. The choice of a method to represent three-dimensional figures two-dimensionally is a far from trivial one, but as far as we know, there is little or no systematic research on this topic, and educational practice often overlooks this issue as well. For example, the use of powerful software for visualization in the teaching process may enhance the learning of students (Ferrara & Mammana, 2014; Widder, Berman, & Koichu, 2018), but the projection method used in this software seems to remain largely unproblematized. Various projection methods may have their place in educational practice, in textbooks, in software, in tests, etcetera. Each of these projection methods may have its own advantages, such as the ease of creating a drawing oneself, the mathematical properties that can be deduced from a two-dimensional drawing (comparing lengths, comparing the size of angles, finding parallel lines and planes, finding right angles, …). Teachers' explicitly discussing and comparing various projection methods may also lead to a deeper understanding in students of the process of creating two-dimensional representations of three-dimensional figures and the fact that this process always involves a loss of specific information. A good understanding of spatial ability, its development and its measurement is moreover not only relevant for the domain of geometry; it is of crucial importance for mathematics education more generally. Spatial abilities play an important role on mathematical performance also in other domains, such as algebraic reasoning (Chimoni & Pitta-Pantazi, 2018).

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MATHEMATICS CURRICULUM REFORM IN THE U.S., FINLAND, SWEDEN, AND FLANDERS: REGION-WIDE COHERENCE VERSUS TEACHER INVOLVEMENT?

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This paper zooms in on the curriculum systems for elementary mathematics education in the United States, Finland, Sweden, and Flanders (Belgium), which are all in the midst of curriculum reform. Our analysis builds on a framework that focuses on curriculum policy, design and enactment in each of these regions and draws on interview data with teachers in all four regions, documents composed to support the international research team's prerequisite understanding (sample case descriptions of curriculum use per region, context descriptions), and available descriptions of mathematics education in these four regions. This leads to a nuanced understanding of the four curriculum systems, which sheds light on a challenging balance concerning a curriculum reform that is both coherent across a region and supported by teachers.

MATHEMATICS CURRICULUM REFORM: A DELICATE PROCESS

Curriculum reform is a delicate process because multiple factors influence implementation, and, ultimately, student performance. If a curriculum is to promote region-wide reform, it should be coherent across that region. Further, there is evidence that the teacher has a crucial role, in that teachers should embrace and advocate the underlying vision (e.g., Tarr et al., 2008).

This paper aims to add to a better understanding of the mathematics curriculum systems in the U.S., Finland, Sweden, and Flanders (Belgium). All four regions have recently undergone or are in the midst of mathematics reform. We draw on Stigler and Hiebert's (1999) assertion that comparing educational systems allows bringing to the surface otherwise hardly noticed aspects from within a system. By describing and comparing the curriculum systems of these four regions, this study will consider consequences for teacher involvement in, and region-wide coherence of the curriculum.

CURRICULUM POLICY, DESIGN, AND ENACTMENT FRAMEWORK

We understand teachers' use of resources to be situated in a broader school system and draw on the curriculum enactment process as conceptualized in Remillard and Heck (2014) (See Figure 1). Remillard and Heck differentiate between an official and operational curriculum. The official curriculum is authorized by governing agencies and includes curricular aims and objectives; assessments; and the designated curriculum – a set of instructional plans specified by a governing agency. The operational curriculum captures the enactment process. It acknowledges the central

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role that teachers have in interpreting and mobilizing curriculum resources and differentiates between a teacher-intended and enacted curriculum, and student outcomes. The location of instructional resources outside of the official and operational curriculum allows to fit both (centralized) systems in which the instructional resources are part of the official curriculum, and other systems in which they are not.

We use the term instructional resources to refer to the resources used to support curriculum enactment. These resources include curriculum resources that sequence a particular content such as student textbooks and teacher's guides, but also other resources such as digital (online) applications.

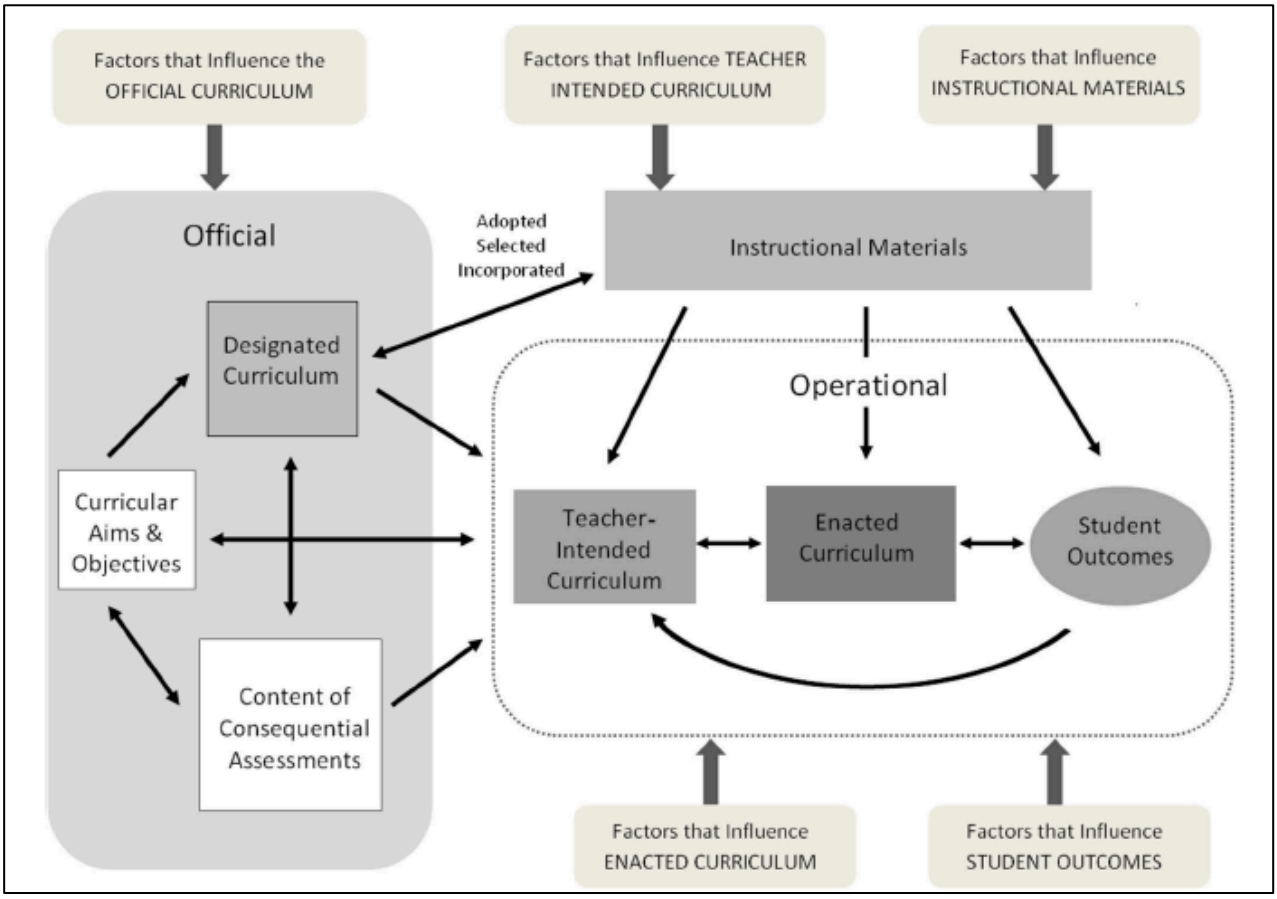

Figure 1. Visual model of the curriculum policy, design, and enactment system (Source: Remillard & Heck, 2014, p. 709)

CONTEXT AND METHOD OF STUDY

This study is part of a larger cross-cultural study on elementary school teachers' use of printed and digital instructional resources in the U.S., Finland, Sweden, and Flanders (the northern part of Belgium, which has its own educational system). Although largely an opportunity sample, the selection of these four regions addresses both constants and contexts (Osborn, 2004) comprising a sound rationale for comparison. Talking to the constants, all four regions value local educational authority, emphasize similar aspects as to the mathematical curriculum, and teachers rely on (printed) curriculum resources when teaching mathematics. Our previous analyses of printed curriculum resources

also shed light on differences in provided teacher support, surfacing context-specific assumptions of teaching and learning mathematics (e.g., Remillard, Van Steenbrugge, & Bergqvist, 2016).

When designing and analyzing interviews on resource use with teachers in the four regions, we were faced with challenges of equivalence, validity, and comparability (Clarke, 2013; Osborn, 2004), and with challenges related to the undertaking of such a study in a cross-cultural team of researchers. To develop the research team's prerequisite intersubjectivity (Andrews, 2007) needed to fully understand the completed interviews as situated in their specific context, we developed case descriptions illustrating curriculum use for one teacher per context, and context descriptions. This paper draws primarily on these four case and context descriptions, but also on interview data specifically relating to the selection of instructional resources, and additional readings on mathematics curricula in these four regions (i.e., Hemmi, Krzywacki, & Partanen, 2017; Remillard & Reinke, 2017; Van Steenbrugge & Ryve, 2018; Verschaffel, 2004).

Ten teachers in Finland, the U.S., Flanders, and Sweden were interviewed in the fall of 2017 and again in the spring of 2018 on their use of resources when planning and teaching mathematics. The first interview was general and addressed teacher and school backgrounds, what resources teachers used, teachers' views on the curriculum resources being used, and teachers' general beliefs on teaching and learning mathematics. The second interview focused in more detail on teachers' actual use of both print and digital resources, and centred around a walk-through of planning, decisions, and enactment of a lesson that the teacher taught recently.

Each case description was prepared by a team member who is a cultural insider, written in English for shared use. We first applied low-inference codes to the interviews to index excerpts of the interviews. These codes identified, for instance, teachers' descriptions of resources, how they were used, and reasons for use. The process of coding was tried out individually, discussed in, and refined by the team. Having coded two interviews for one teacher per region, we gathered similarly-coded statements and applied the following structure to the cases: a) teacher education and teaching background, b) information about school and class, c) selection process of the resources, d) use of resources and purposes for use, e) teacher beliefs and conceptions, f) changes in resource use.

The process of writing and reading cases made us aware that significant insider knowledge was necessary to make sense of them, which is why we also developed context descriptions. Context descriptions are organized according to the following structure: a) school system-structure, b) pathways into teaching elementary mathematics, c) school environment, d) financial resources for organizing education, e) decision-making mechanisms in schools in relation to mathematics education, f) student assessment, and g) monitoring and quality assurance of education.

An important step in the process of developing the cases and context descriptions was full-team review and discussion of them. In fact, we arrived at a common structure and approach through incremental development, review, and discussion.

Building on the curriculum policy, design, and enactment framework (Remillard & Heck, 2014), we came to the following analytical structure to compare the four educational systems, based on case and context descriptions, interview data, and the abovementioned additional readings:

- Educational jurisdiction and school funding;
- Most recent central mathematics curriculum, including name and launching date, novel aspects, requirement of adoption;
- Role of assessments;
- Curriculum specification in addition to central curricular aims and objectives;
- Instructional resources and influential factors, including resource market and acceptance criteria.

Curriculum systems in the U.S., Finland, Sweden, and Flanders

Table 1 includes our descriptions of the curriculum systems of the U.S., Finland, Sweden, and Flanders. Looking across the table helps highlight similarities and differences related to a) regulations and incentives to steer local authority, b) role of curriculum resources in curriculum reform, and c) curriculum interpretation. We discuss these aspects below and relate them in a final section to two crucial aspects of curriculum reform: coherence and embracement of the reform by teachers.

Regulations and incentives to steer local authority are present to different extents. The curriculum systems of the U.S. and Flanders have the most explicit mechanisms to steer local authority. In the U.S., states possess authority in relation to educational policy and, sometimes, delegate policy to school districts, but the Government by means of applying specific funding mechanisms influences policy and curriculum use at state, district, and school level. In Flanders, schools are in principle free to determine how to work toward the attainment targets, but the Government, through regulations such as school inspectorates and the requirement to adopt a learning plan, and through targeted funding, sets the framework of the curriculum system and influences curriculum policy and use at the local school level. In Sweden, the Government also sets the framework of the curriculum system, but influences curriculum use at a more implicit level, through rolling out a nation-wide professional development program following the curriculum reform. From our study, it appears that central regulation is the least manifested in Finland. The Finnish National Board of Education commissions on a regular interval-base an expert group to develop a new curriculum. Schools and teachers are provided with guiding documents and regulations but are not checked upon regarding the application of the guidelines and regulations.

	U.S.	Finland	Sweden	Flanders
Educational jurisdiction and school funding				
Jurisdiction	School districts with oversite from states	Finnish government	Swedish government	Flemish government
School funding	State and local taxes	National and local funding	Government to municipals to schools	Government; also targeted funding
Most recent central mathematics curriculum				
Name & launching date	Common Core State Standards (CCSS), 2010	National Core Curriculum (NCC), 2016	Läroplan (LGR 11), 2011 – revised 2018	Attainment targets; 1998/under development
Novel aspects	Emphasis on visual models and conceptual understanding	Cross-curricular competences (e.g., digital competence)	Mathematical competences, digital competence	Structured around 16 key competences (e.g., digital competence)
Adoption	Not required; Federal government incentivizes states toward CCSS & assessment adoption	Required, but not checked	A nationwide professional development program supports adoption; checked by school inspectorates	Required, checked by school inspectorates
Assessments				
	Consequential for student promotion, teacher employment, school funding	Individual teachers are responsible for assessment	National tests check performance on school and population level	Tests assess mastery of attainment targets on population level

Continued curriculum specification				
Level & content	Districts and schools often specify instructional resources to be used, and issue pacing guidelines	National board of education hosts a website with available resources; School teams and/or local authorities concretize NCC into a central school-level curriculum	/	Three umbrella organizations issue learning plans, which break down attainment targets per grades, add goals and didactical suggestions
Instructional resources and factors that influence resources and use				
Instructional resource market	Commercial enterprise, limited number of publishers	Commercial enterprise, limited number of publishers	Commercial enterprise, limited number of publishers	Commercial enterprise, limited number of publishers
Acceptance criteria	By some states/districts	No	No	No
Selection instructional resources	Main curriculum resource & larger digital platforms: districts & schools; Digital resources: teachers	Main curriculum resource & digital resources: teachers	Main curriculum resource: schools; Digital resources: teachers	Main curriculum resource: schools; Digital resources: teachers, schools, school groups, umbrella organizations

Table 1: The curriculum systems of the U.S., Finland, Sweden, and Flanders

In all four regions, curriculum resources served as interpreters of the official curriculum, hereby serving as mediators between the intended curriculum and the classroom (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). Additionally, and talking to the systems of Finland and Flanders, curriculum resources can also potentially influence curriculum making. In Finland, teachers at times rely on the

learning sequence in commercial curriculum resources to design their crucial school-level curriculum. Currently in Flanders, new curriculum resources, often complemented with digital applications, are published before the actual launch of the new attainment targets, hereby possibly influencing the novel aspects of the mathematics reform related to digital competence.

Our comparative analysis also reveals differing levels where significant curriculum interpretation happens to reside. In Sweden, the bulk of interpretation happens at the individual teacher level. In Finland, significant interpretation is applied to compose a school-level curriculum, whereas in Flanders, major interpretation of the attainment targets is located above the school-level, by the umbrella organizations issuing learning plans. In the U.S., significant curriculum interpretation resides in the assessments.

CURRICULUM REFORM: A DELICATE BALANCE BETWEEN REGION-WIDE COHERENCE AND TEACHER APPROVAL

The findings suggest that a curriculum reform that is both region-wide and supported by teachers, is a challenging balance. Both Flanders and the U.S., through their layered curriculum system, succeed most toward a region-wide curriculum coherence, but this goes at the cost of teacher involvement in the reform process. In Finland, teachers are most involved in reform through the design of a school-level curriculum, but this goes at the cost of a nation-wide curriculum-coherence. Sweden stands out in that teachers were asking for reform and that the Government answered the call by rolling out nationwide professional development. It still has to be seen to what extent that results in curriculum coherence. In all four regions, commercially published curriculum resources are a central aspect in a region-wide curriculum reform. Given this significant position, it is remarkable that only in the U.S., sometimes quality criteria are issued that curriculum resources have to pass.

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EARLY PROPORTIONAL REASONING ABILITIES: THE ROLE OF RELATIONAL PREFERENCE

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Recent literature suggests that children develop proportional reasoning abilities at an earlier age than traditionally assumed. It also indicates that children's preference for additive and multiplicative relations plays a role in this development. In this study we investigated four- to nine-year olds' proportional reasoning abilities in missing-value proportional reasoning problems, and how these are related to the extent of their preference for additive and multiplicative relations. Having a multiplicative preference was a significant predictor for proportional reasoning abilities, but surprisingly an additive preference was a significant predictor too. It seems that at this young age, an additive preference is not a barrier for proportional reasoning abilities. Rather, along with a multiplicative preference it is a facilitator for this ability.

INTRODUCTION

Proportional reasoning is an important goal in mathematics education. Proportions are characterized by the equality of two ratios (mathematically written as $a/b = c/d$) or put differently as a multiplicative relation between two co-varying measures (Vergnaud, 1988).

The traditional Piagetian stance on the development of proportional reasoning considers additive reasoning as a precursor. Inhelder and Piaget (1958) state that children first tend to focus on the "equality of differences' (p. 177), and only later on the equality of ratios. Put differently, early in development, children are only able to quantify relations in an additive way, and only later when they have reached the formal-operational level of cognitive functioning, they can think of relations in a multiplicative way, and thus understand proportions. Many other authors supported the traditional assumption of an additive phase preceding a proportional phase (e.g., Lamon, 2008). This assumption is strengthened by numerous studies revealing that children erroneously use additive reasoning to solve proportional reasoning problems (e.g., Kaput & West, 1994).

However, recent studies started to question whether the sequential development of additive and proportional reasoning is as straightforward as has been assumed traditionally. First, Van Dooren, De Bock and Verschaffel (2010) discovered that besides the inappropriate application of additive reasoning in multiplicative missing-value problems, the inverse mistake occurs as well. Children also over-use multiplicative reasoning in additive missing-value word problems, while they do master additive reasoning. Second, other recent studies suggested an early

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development of proportional reasoning abilities, starting from the beginning of primary school (e.g., Boyer & Levine, 2012).

It seems that children's inappropriate use of proportional reasoning in additive missing-value word problems and vice versa cannot be explained solely by their *abilities* to reason additively or multiplicatively. Degrande, Verschaffel, and Van Dooren (2018) proposed an additional explanatory element, namely children's *preference* for additive and multiplicative relations. Preference expresses that children are inclined to use one type of relation above the other, in situations where they can choose and where both types are equally valuable. Degrande et al. (2018) investigated children's preference for additive and multiplicative relations in schematic missing-value problems that were open to both types of reasoning. Firstly, they explicitly found the existence of a preference for additive relations in some children and a preference for multiplicative relations in others. Secondly, they showed that such a relational preference offers an additional explanation for the over-use of additive and proportional methods in solving world problems, over and above calculation and modelling skills (Degrande, Verschaffel, & Van Dooren, 2017). This suggests that besides their skills, children's preference for additive and multiplicative relations plays a unique role in the type of reasoning they exhibit when solving proportional problems.

The present study

In this study we focused on children at an age before the start of instruction on proportional reasoning. We investigated how four- to nine-year olds' abilities to solve missing-value proportional reasoning problems are related to the extent of their preference for additive and multiplicative relations. Is children's multiplicative preference associated with their performance on proportional reasoning ability tasks? And does an additive preference act as a barrier to proportional reasoning abilities, or can it also be a facilitator?

METHOD

Participants

Data were collected in two elementary schools in Flanders, Belgium, resulting in a sample of 185 children: 43 kindergartners (M=5.06 years, range=5.00-6.03 years), 44 first graders (M=6.07 years, range=6.01-7.07 years), 40 second graders (M=7.07 years, range=7.02-9.00 years), and 58 third graders (M=8.08 years, range=8.01-10.01 years). 43.2% of the participants were boys. All children participated voluntarily with informed consent of their parents and teachers.

Design and materials

All children were individually tested during 30 minutes with tasks using manipulatives and oral instruction. Two aspects were addressed: children's preference (P) for additive and multiplicative relations and their proportional reasoning abilities (A). All tasks were non-symbolic, because the youngest children were not yet familiar with symbolic number presentations.

Preference (P) – To identify children's preference for additive and multiplicative relations, a preference task consisting of four open missing-value items was developed. These open problems did not contain any inherent mathematical model. Different ways of relational reasoning, including additive or multiplicative reasoning, could be used and led to equally valuable answers. Thus these items revealed what kind of relations a child preferred. Two items were in a "magic" context and two in a "growth" context. For each context there was an item with discrete quantities (i.e., pills in a magic hat and caterpillars) and an item with continuous quantities (i.e. wands in a magic cauldron and snakes). Figure 1 shows the instruction and the presented materials for the two discrete items.

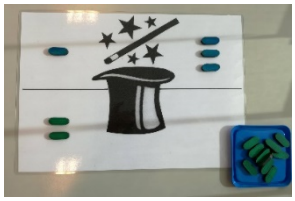
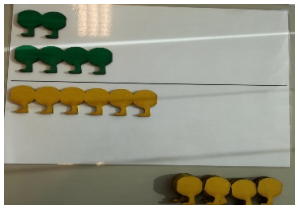
a) "Martijn has a magic hat. When he puts this pill (point top left) in the hat, these pills (point top right) come out. He has more pills now. How many pills will come out of our hat, when he puts these pills (point bottom left) in the hat? You can put the right number of pills next to the hat." (additive answer: 4 pills, multiplicative answer: 6 pills)	
b) "Yesterday Fien saw two caterpillars in her garden, a green one and a yellow one. Yesterday the green caterpillar was this long (point at short green caterpillar), today the caterpillar is this long (point at long green caterpillar). The yellow caterpillar was this long yesterday (point at yellow caterpillar). The caterpillar has grown. How long will the yellow caterpillar be today? You can make the yellow caterpillar with your pieces." (additive answer: 8 pieces, multiplicative answer: 12 pieces)	

Figure 1: Oral instruction and presented materials for the discrete preference items: a) pills in a magic hat (magic context) and b) caterpillars (growth context).

Children's answers were classified as *additive*, when the numerical answer was obtained by calculating the difference between two numbers and applying it to the third number, or as *multiplicative*, when the numerical answer was obtained by calculating the ratio between two numbers and applying it to the third number. Answers were classified as *other* either when the given numbers were combined in another way than specified above or when the problem was left unanswered. Children could only give one answer per problem, therefore it was not possible to answer both additively *and* multiplicatively within one problem. However they could answer additively in some problems and multiplicatively in others.

Ability (A) – Children completed two tests addressing their abilities in proportional reasoning. One eight-item test measuring the ability to reason proportionally with two discrete quantities (A-DD, puppets and grapes) and one eight-item test measuring proportional reasoning with a discrete and a continuous quantity (A-DC, puppets and chocolate bars). All items were missing-value problems, in which children had to construct a set B equivalent to a set A by putting the elements in set B in the same ratio as the elements in set A. Figure 2 shows an example item for both tests.

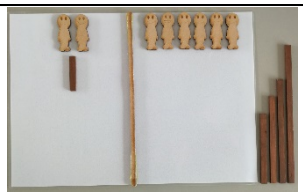
A-DD a) "All puppets are equally hungry. If I give four grapes to these puppets, how many grapes do you have to give to these puppets for it to be fair?"	0
A-DC b) "All puppets are equally hungry. If I give this chocolate bar to these puppets, which chocolate bar do you have to give to these puppets for it to be fair?"	

Figure 2: Example item with instruction and materials for both ability tests: a) with two discrete quantities and b) with a discrete and a continuous quantity.

RESULTS

First, results on the preference task showed a steady increase with grade in the mean number of both additive and multiplicative answers at the expense of other answers (Table 1). Kindergartners solved on average 0.98 out of 4 problems additively, while third graders solved on average 2.62 out of 4 problems additively. This increase was less steep for the number of multiplicative answers, and generally multiplicative answers remained quite rare in this young age group. Kindergartners solved on average 0.42 out of 4 problems multiplicatively, while third graders solved on average 0.78 out of 4 problems multiplicatively. "Other" answers decreased. A multivariate analysis of variance showed that there was a statistically significant effect of grade for each of these answer types ($F(6,360)=18.05$, $p<0.001$; *Wilk's* $\Lambda = 0.59$, *partial* $\eta^2 = 0.23$).

	K3	E1	E2	E3
P – Additive	0.98 (0.96)	1.64 (1.06)	2.30 (0.99)	2.62 (1.01)
P – Multiplicative	0.42 (0.70)	0.30 (0.55)	0.47 (0.78)	0.78 (0.80)
P – Other	2.60 (1.18)	2.07 (1.07)	1.23 (0.97)	0.60 (0.77)

Table 1: Mean number (and SD) of additive, multiplicative or other answers on the preference task (four items) per grade.

Second, we also observed a gradual increase with grade for the A-DD and -DC test (Table 2). For the A-DD test, kindergartners solved on average 2.23 out of 8 items correctly, while third graders solved on average 6.71 out of 8 items correctly. For the A-DC test, kindergartners solved on average 2.09 out of 8 items correctly, while third graders solved on average 5.47 out of 8 items correctly. A multivariate analysis of variance showed that there was a statistically significant effect of grade on both ability tests ($F(6,360)=26.76$, $p<0.001$; *Wilk's* $\Lambda = 0.48$, *partial* $\eta^2 = 0.31$).

	K3	E1	E2	E3
A-DD	2.23 (2.05)	3.32 (2.33)	5.73 (2.17)	6.71 (1.63)
A-DC	2.09 (1.77)	2.77 (1.51)	3.93 (1.72)	5.47 (1.59)

Table 2: Mean number (and SD) of correct answers on A-DD and A-DC per grade.

Third, a medium positive correlation was found between the total score on the proportional reasoning ability tests and the number of multiplicative answers on the preference task (r=0.30, p<0.01). Contrary to our expectations there was an even higher positive correlation between the total score on the proportional reasoning ability tests and the number of additive answers on the preference task (r=0.51, p<0.01). Table 3 shows the correlations between the scores on the ability tests and the number of additive and multiplicative answers on the preference task.

	1	2	3	4	5
1. A-DD	--				
2. A-DC	0.69**	--			
3. A-Total	0.93**	0.90**	--		
4. P-Additive	0.50**	0.39**	0.51**	--	
5. P-Multiplicative	0.24**	0.36**	0.30**	-0.19**	--

** Correlation is significant at the 0.01 level (2-tailed).

Table 3: Correlation matrix of A-DD, A-DC, A-Total, P-Additive and P-Multiplicative.

To further investigate the associations between children's proportional reasoning abilities and their preference for additive and multiplicative relations, hierarchical linear regression analyses were performed. The first two had the A-DD score as outcome and grade as predictor in the first step. The difference between the two regressions was the order in which the two preference measures were entered as predictors: P-Additive was entered before P-Multiplicative (Step 2a), or vice versa (Step 2b). In the third and final step both preference measures and grade were entered simultaneously. The assumptions for the regression analyses were examined and not violated in in any of the analyses. The regression diagnostics showed no multicollinearity (VIFs < 1). The results in Table 4 indicate that the two preference measures and grade together explained half of the variance in the A-DD test. The number of additive and multiplicative answers on the preference task uniquely explained respectively 6.0% and 3.0% of the variance in the A-DD test, over and above grade. Both were significant additional predictors.

		Predictors	*B*	*t*	Unique R^2
Step 1	$F(1,183) = 145.37$**	Grade	1.57	12.06**	.44
	Total $R^2 = .44$				
Step 2a	$F(2,182) = 82.64$**	Grade	1.30	8.65**	.22
	Total $R^2 = .48$	P-Additive	0.50	3.40**	.04
Step 2b	$F(2,182) = 75.02$**	Grade	1.30	8.65**	.40
	Total $R^2 = .45$	P-Multiplicative	0.50	3.40**	.01
Step 3	$F(3,181) = 62.32$**	Grade	1.09	6.88**	.13
	Total $R^2 = .51$	P-Additive	0.70	4.55**	.06
		P-Multiplicative	0.73	3.44**	.03

Note. * $p < .05$, ** $p < .01$

Table 4: Regression analyses with A-DD as an outcome measure.

Finally, two similar hierarchical regression analyses were performed, but this time with the A-DC score as outcome. The results in Table 5 indicate that the two preference measures and grade together explained almost half of the variance in the A-DC test. The number of additive and multiplicative answers on the preference task uniquely explained respectively 2.0% and 6.0% of the variance in the A-DC test, over and above grade. So, both were significant additional predictors.

		Predictors	*B*	*t*	Unique R^2
Step 1	$F(1,183) = 119.03$**	Grade	1.15	10.91**	.39
	Total $R^2 = .39$				
Step 2a	$F(2,182) = 60.01$**	Grade	1.08	8.68**	.25
	Total $R^2 = .40$	P-Additive	0.12	0.10	.01
Step 2b	$F(2,182) = 71.45$**	Grade	1.06	10.25**	.32
	Total $R^2 = .44$	P-Multiplicative	0.63	3.85**	.05
Step 3	$F(3,181) = 51.93$**	Grade	0.85	6.64**	.13
	Total $R^2 = .46$	P-Additive	0.34	2.77**	.02
		P-Multiplicative	0.81	4.69**	.06

Note. * $p < .05$, ** $p < .01$

Table 5. Regression analyses with A-DC as an outcome measure.

DISCUSSION AND CONCLUSION

In this study we administered two tests to measure children's proportional reasoning abilities, together with a task to assess their preference for additive and multiplicative

relations. This made it possible to investigate whether there is an association between children's proportional reasoning abilities and the extent to which they prefer additive and multiplicative relations. Since numerous studies revealed that children erroneously use additive reasoning to solve proportional problems, one could expect that a preference for multiplicative relations is a facilitator for proportional reasoning abilities whereas a preference for additive relations may act as a barrier.

As expected, a medium positive correlation was found between the number of multiplicative answers on the preference task and the total score on the proportional reasoning ability tests. However, contrary to our expectations, we found an even higher positive correlation between the number of additive answers on the preference task and the total score on the proportional reasoning ability tests. Linear regression analyses showed that *both* preferences were significant predictors for proportional reasoning abilities with two discrete quantities. An additive preference was a significant predictor and it explained even more of the variance on the A-DD test than a multiplicative preference. Another surprising result was that we did not find a similar pattern for the proportional reasoning ability test with a discrete and a continuous quantity. Again both preferences were significant predictors, but this time a multiplicative preference explained more of the variance on the A-DC test than an additive preference.

Contrary to what we expected, an additive preference does not seem to be a barrier for proportional reasoning abilities. It even seems that having a relational preference, regardless of whether it is an additive or a multiplicative preference, facilitates proportional reasoning abilities. It may be that at this young age, children who already spontaneously exhibit a kind of focus on any quantitative relation underlying our preference tasks, be it in an additive or multiplicative way, are a step forward in their thinking about mathematical relations as such, and benefit from this when thinking about proportional situations. Our findings are in line with recent research that has shown that a spontaneous focus on quantitative relations has a beneficial effect on the development of certain mathematical concepts such as rational numbers (McMullen, Hannula-Surmonen, Laakonen, & Lehtinen, 2016).

Unexpectedly, at this young age the additive preference is an even better predictor than the multiplicative preference for performance on the proportional reasoning test with two discrete quantities, while this was not the case for the test with a discrete and a continuous quantity. Indeed, in proportional situations with discrete quantities children can use specific strategies, such as equal distribution or sharing, to directly "model" the situation at hand and create exact numerosities. Such approaches are not possible in situations with a discrete and a continuous quantity, where counting and manipulating the continuous quantity is no longer possible. The latter seems to be more directly relying on multiplicative reasoning.

Children's preferred type of relations is associated with their performance on missing-value proportional reasoning ability problems. The role of this preference seems to be less straightforward than we previously hypothesized. Longitudinal-follow up research

would be useful to map the development of this preference throughout primary school, and to see whether preference at an earlier age predicts children's proportional reasoning abilities at a later age, as well as the development of these abilities.

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SHIFTS IN EARLY NUMBER LEARNING IN SOUTH AFRICA

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In this paper, we report on outcomes from the Structuring Number Starters intervention project focused on early grades' number learning in South Africa. Using tests drawn from the Maths Recovery programme, a cross-attainment sample of Grade 2 students in six schools took part in oral interview assessments early in 2011, 2014 and 2018, with professional development activities for their teachers occurring across this period. Student outcomes point to increasing proportions, over time, of students able to use more sophisticated counting strategies, and to work with number structure for more efficient calculation.

INTRODUCTION

Low performance in mathematics at all levels of education continues to be a feature of the South African education system. In this context, a public-private partnership was set up in the late 2000s that established a small number of research and development Chairs, based in South African universities, charged with the explicit aim of developing and studying the implementation of research-based interventions to improve learning outcomes across the system. The interventions had to be developed with potential for working at a larger scale if outcomes proved positive. In this paper, we report on the outcomes from an intervention focused on early grades' number learning – the Structuring Number Starters project – that was run by one of these Chairs within the Wits Maths Connect-Primary (WMC-P) research and development project, and carried out in ten partner schools.

BACKGROUND

Poor performance in mathematics in South Africa has led, over the course of the current decade, to a groundswell of interest in primary mathematics teaching and learning. When the WMC-P project began in 2011 key problems that had been written about in primary mathematics were related to two issues. Firstly, there was wide evidence of primary school students' approaches to number work being marked by counting based approaches, and often, highly inefficient concrete counting-in-ones or skip counting in multiples (Schollar, 2008). Second was evidence of poor understandings of mathematical progression among primary teachers. One response to this had been the development of a more closely specified curriculum with attempts to highlight the growth of concepts across terms, grades and phases (DBE, 2011), but there is limited evidence that standards have improved as a result of this. Writing across several developing countries also reports on this evidence of low student performance and limited progression, together with the limited success of close curricular prescription (e.g. Pritchett & Beatty, 2012). This evidence indicates that studies of progression in

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early numeracy in the South African context are likely to be of broader international interest.

We began our work in South Africa with ten partner primary schools, all serving historically disadvantaged populations. Baseline observations in classrooms across these schools confirmed the prevalence of concrete counting-based approaches to early additive relations problems, which suggested a need for intervention approaches, as described in the international literature, for supporting students to move towards more sophisticated approaches when working with number. In particular, the aims and approaches of the Maths Recovery programme (Wright, Martland & Stafford, 2006) were in line with our goals, and additionally, provided interview-based assessments that we could use to build a profile of the sophistication of strategies Grade 2 students (grade appropriate students aged 6-7) used to answer problems. This was especially useful in a national assessment terrain where tasks were marked as correct or incorrect, without attention to the approaches used to produce answers.

Our broader intervention model was based on working with grade cohorts of students across the school year, focusing on one grade in each academic year. Professional development and intervention activity comprised a termly workshop with WMC-P team members presenting and discussing, with teachers, tasks, activities and resources that focused attention on number structure and relations. The teachers then tried out these tasks, activities and resources with their classes, with one lesson observed and/or co-taught by a WMC-P team member each term. This programme is now in its 8th year. Table 1 summarizes our tracking of cohorts in the years of empirical data collection in the six schools that have worked with the project across this whole period.

Year and grade cohort worked with during intervention							
Phase 1 (2011-15)					Phase 2 (2016-date)		
2011	2012	2013	**2014**	2015	2016	2017	**2018**
G2	G3	G1	**G2**	G3	No data collected	GR&1	**G2**

Table 1: Cohorts tracked across years, assessment data collection years highlighted

As part of our study of the effectiveness of intervention, we administered two parts of the Maths Recovery assessments to a cross-attainment sample of students early in Grade 2 in each of 2011, 2014 and 2018. Grade 2 represents the middle grade of the Foundation Phase in South Africa (Grades 1-3), with some Reception Grade classes coming on stream in the partner schools only in recent years. The broader study of learning has included a written assessment at the start and end of Grade 3. In-depth assessment interviews early in Grade 2 provided us with snapshot data over time of students' approaches to additive problem-solving, with our intervention work drawing diagnostically from the profiles of performance seen in each of these test administrations. On the research side that we focus on in this paper, this assessment

dataset provides a quasi-longitudinal profile of performance in Grade 2 over time in the context of ongoing intervention activity.

Below, we discuss the literature and theoretical models of progression that underpin these assessments, and then summarise the profile of learning outcomes seen across the quasi-longitudinal test administrations in 2011, 2014 and 2018. We analyse these outcomes, and discuss the implications of shifts in profiles of attainment for policy-making in the broader terrain and for our ongoing work.

LITERATURE & THEORY

In the context of additive enumeration, a broad swathe of international literature clusters around learning trajectories related to increasing efficiency of counting and calculating strategies. One trajectory begins in the terrain of enumerating quantities, drawing on Gelman & Gallistel's (1986) seminal work setting out the principles that have to be mastered in learning to count (1-1 correspondence, stable order of the number words, cardinality, counts as abstractions, and order irrelevance). Within this trajectory students move into increasingly efficient counting via what can be described as 'count all', 'count on' or 'count down from' strategies, prior to beginning to work with number structure and properties such as commutativity, place value and inverse relationships.

A body of literature questions this trajectory, arguing that instruction should start in number structure and properties set within measurement contexts, rather than start with counting (Davydov, 1990). The argument for this 'straight for structure' trajectory is that counting fails to communicate a sense that counts essentially represent a ratio between a unit quantity and the entity being measured by that unit. This, it is argued, creates artificial and unhelpful boundaries between natural numbers and fractions: in the measurement-based approach, rational numbers arise 'naturally' alongside whole numbers.

The South African curriculum context largely works with the former – a counting into calculating trajectory – a perspective firmly embedded in the practices of our partner schools. Reflecting this focus on counting, studies of the 2011 project dataset pointed to students getting correct answers to additive questions in lower number ranges merely by using concrete counting strategies rather than through any combination of calculating and reasoning based on number structure (Weitz & Venkat, 2013).

Our intervention work in the Structuring Number Starters project has, however, moved increasingly towards a 'straight for structure' model. This has involved the development of task sequences and resources/representations underpinned by attention to numerical relationships. Examples include a focus on using single digit number bonds and base ten related number combinations to develop non-count-by-one calculation strategies, using semi-structured number lines (see https://www.wits.ac.za/wits-maths-connect/wits-maths-connect-primary/structuring-number-starters/term-1-resources/ for further examples). Developing familiarity with

'five' and 'ten' as useful benchmarks for efficient calculation is a feature that recurs prominently in these tasks.

While moving towards the 'straight for structure' approach in our development work, the broader curricular context, and the underlying agreement between the counting into calculating and the straight-for-structure advocates on appreciation of structure as a key goal in early number learning, led to our working with the 'counting into structure' trajectory for analysis of counting and calculating efficiency among students. We chose to work with Wright, Martland and Stafford's (2006) Learning Framework in Number (LFIN) which deals with a range of number aspects (including forward and backward number counting, and numeral recognition), but is centred around a spine in which strategic counting efficiency leads into further efficiencies involved in working with number relationships. Wright and colleagues describe the hierarchies in this spine as the 'Stages of Early Arithmetic Learning' (SEAL), with the label 'stages' marking what they see as the substantial conceptual reorganizations involved in the moves between steps. These stages, in paraphrased terms, are outlined in Table 2.

Stage	Title	Description of counting/calculating strategies
0	Emergent count	Cannot count a visible collection of counters
1	Perceptual count	Can count perceived items (seen, heard or felt) and solve additive tasks involving displayed collections
2	Figurative count	Can solve additive tasks involving one or two screened quantities using the 'count all' strategy
3	Initial number sequence	Can solve addition tasks using 'count on', 'count-up-to' and 'count-down-from'
4	Intermediate number sequence	Can use 'count-down-to' to solve missing subtrahend tasks and can choose the more efficient of 'count-down-from' and 'count-down-to' for task at hand
5	Facile number sequence	Can use a range of 'non-count-by-one' strategies involving calculation-by-structuring (doubles, near doubles, compensation, 1010, N10 and bridging-through-ten) and known and derived facts

Table 2: The Stages of Early Arithmetical Learning

In Stages 1 and 2, the enumeration of quantities relies on unit counting of all the quantities presented in the problem – each part, and then the whole, have to be counted out from one. 'Count on' and 'count down from' strategies at Stage 3 shift this triple count to a single count – on or down from the starting number. Hence the shift from Stage 2 to Stage 3 in Wright et al.'s model is important in that it marks the first move to any reification of counting processes into number objects (Gray, 2008). Whilst Stage 3 represents an important shift in efficiency terms, it remains, as van den Heuvel-Panhuizen (2008) notes, within the terrain of 'calculating-by-counting' strategies,

rather than the 'calculating-by-structuring' strategies which involve both attending to relationships between numerical quantities and also deriving efficient strategies based on number fluencies that present themselves as recalled facts. It is in Stage 5 that such calculation by structuring, using number relations and properties, comes to the fore, with Stage 4 laying the groundwork for this move.

METHODOLOGY

Each of our partner schools were asked to select six Grade 2 students from across the mathematics attainment range: two under-attaining students, two average attaining students and two higher-attaining students. Following the Math Recovery model, in-depth 1-1 assessment interviews were conducted with each of the selected students (with informed consents from the children, their parents/carers, teachers and school principals). The interviews, conducted with student first language translators where needed, were video-taped in order to capture gesture-based counting strategies (in particular, finger counting) which, as we noted earlier, were widely prevalent.

Subsequent coding of each child's assessment responses using the LFIN led to the development of individual learner profiles. In this paper, we focus on the SEAL aspect only, described in Wright et al's work as the single most important aspect in relation to supporting early number progression in the overall LFIN profile. Although we worked with ten partner schools each year, six schools were involved across each of the 2011, 2014 and 2018 test administrations, so we focus here on the profile of SEAL stages seen across the 108 students interviewed: 36 in each year.

FINDINGS AND ANALYSIS

In Table 3, we summarize the SEAL Stage profiles across 2011, 2014 and 2018 of the cross-attainment sample of 36 learners from the six schools participating across all three test sittings.

SEAL Stage	2011	2014	2018
0	4 (11.1%)	1 (2.8%)	1 (2.8%)
1	8 (22.2%)	4 (11.1%)	1 (2.8%)
2	15 (41.7%)	9 (25%)	10 (27.8%)
3	8 (22.2%)	20 (55.6%)	15 (41.7%)
4	1 (2.8%)	2 (5.6%)	7 (19.4%)
5	0 (0%)	0 (0%)	2 (5.6%)

Table 3: SEAL profile across 2011, 2014 and 2018 (n=36 in each year)

We considered the data quantitatively and more qualitatively to understand SEAL profiles over time. Drawing from the literature noting that the move from Stage 2 to 3 marks an important transition into beginning to see numbers as reified structures, we

carried out a chi-square test of independence to examine the relation between grouped SEAL stages 0-2 and 3-5 and the year of assessment. This grouping ensured that all cells contained five or more students, allowing for a valid chi-square test application. The null hypothesis here was that the distribution of students across these two combined levels was independent of the year of assessment. Contradicting this hypothesis, the relationship between these variables was highly statistically significant: χ^2 (2, N = 108) = 14.75, p <.001, indicating that differences between the years were highly unlikely to be due to chance alone.

Returning qualitatively to the individual SEAL stages, a graphical representation of the proportions of students using the strategies associated with each of the SEAL Stages illustrates the shifts in the profile of strategy use over time (see Figure 1).

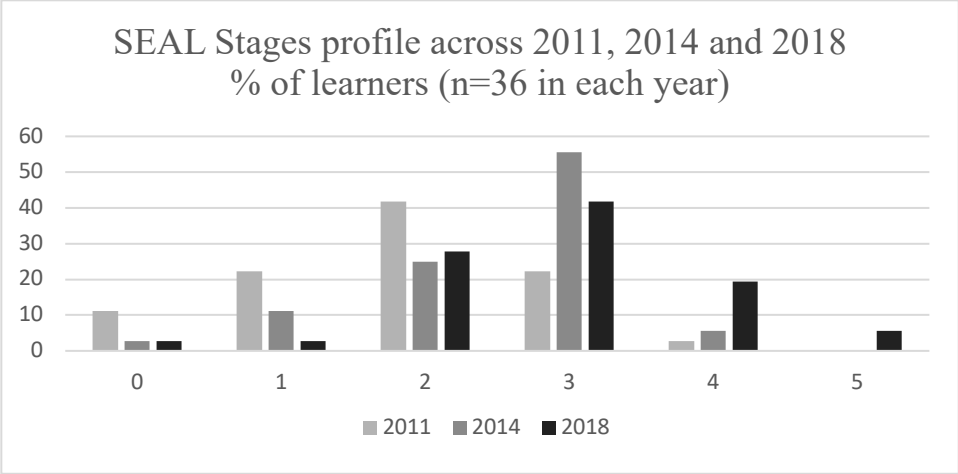

Figure 1: SEAL profile over time

Several features of the shift in profile over time illustrated in Figure 1 are noteworthy. Most importantly, in 2011, 75% of the student cohort in our cross-attainment range sample were assessed as using the concrete unit counting strategies associated with Stages 0-2, but by 2014, this proportion had dropped to 38.9% of the student cohort, and by 2018, had decreased further to 33.4% of the student cohort. The key shift seen in the 2014 attainment profile is a large increase in the students assessed as being at Stage 3 – the 'count on'/'count down from' stage – which, as we noted earlier, marks the start of reifications of counting processes into number objects. There is also a shift in 2014 to over 60% of the students exhibiting strategies related to SEAL Stages 3 to 5, in contrast to only 25% of the student cohort being in this range in 2011.

The key shift in 2018 is evidence of growing proportions of students able to work with number structure and relationships in ways that move them beyond counting-in-ones strategies. In 2018, 25% of the cross-attainment sample used SEAL Stages 4 & 5 in their responses – the stages in which 'calculating-by-structuring' comes into play making use of relationships between numbers for increasingly efficient working. This proportion is substantially higher than the 2.8% of the cohort at these stages in 2011 and the 5.6% at these stages in 2014. This large increase points to successes associated

with teachers' work with tasks underpinned by the 'straight for structure' approach in the Structuring Number Starters project.

CONCLUDING COMMENTS

These shifts in the profiles of student performance across the years point to teaching and learning that is moving forwards in relation to strategic efficiency and a stronger sense of number structure and relationships. Such a move is important given the literature showing that this is necessary for subsequent working in expanded number ranges and for algebraic thinking (Kieran, 1996). There is also though, further work to be done with the stubborn third of students who remain in the concrete counting from one stages, and the roughly 40% of the sample working with 'count-on' and 'count-down-from' strategies at Stage 3 in 2018. This work goes hand-in-hand with simultaneously working to encourage further moves into the more efficient relational strategies associated with SEAL Stages 4 and 5.

This analysis provides us with useful insights into changes over time in strategic efficiency, in the context of our broader development activity. The terrain of our work was one in which multiple initiatives were at play – national curriculum changes, the rollout of national workbooks, and provincial initiatives focused on supporting curriculum coverage and pacing through the provision of scripted lesson plans. What demarcated our intervention from these broader issues was specific attention to instruction focused on number structure and strategic efficiency, rather than more generic attention to curricular coverage. Additionally, our move into a 'straight-for-structure' approach appears significant in its association with the substantial increase in the proportions of students now displaying approaches in the upper, structural, SEAL stages. The analysis suggests that the Structuring Number Starters project is paying dividends in supporting both pedagogic attention and also learning outcomes towards efficient working with number based in a sense of number structure and relationships. As such, these findings are of broader interest given the evidence from contexts of disadvantage and poverty in several parts of the world where many students' poor performance in number in the early grades remains a live concern.

Acknowledgments

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OPPORTUNITIES FOR SAMING REALIZATIONS IN DIFFERENT TASKS

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The Realization Tree Assessment (RTA) tool offers a graphical presentation of the mathematical ideas students got exposed to and engaged with through a certain task. It depicts the mathematical object together with its different realizations, the extent to which opportunities for saming those different realizations were given to students during a lesson and the extent to which students had authority to produce narratives about the object. Five mathematics lessons which were based on a middle-school task dealing with linear functions were analysed using the RTA. The results were compared to RTAs of lessons based on a pattern generalization task. We discuss the similarities and differences between the RTAs in terms of opportunities for explorative participation as well as exhausting the potential of a task.

INTRODUCTION AND THEORETICAL BACKGROUND

The Realization Tree Assessment (RTA) tool (Weingarden & Heyd-Metzuyanim, 2017, 2018; Weingarden, Heyd-Metzuyanim, & Nachlieli, 2017) is designed to explicate the mathematical object that appeared through engagement with a task together with its different realizations. Based on ideas from the commognitive framework (Sfard, 2008), the RTA displays graphically the extent to which opportunities for saming different realizations of the mathematical object were given to students during a lesson. Uniquely from other evaluations tools, the RTA allows exposure to the mathematical content of the lesson, including the mathematical objects that students could be exposed to through engaging with the task. However, the uniqueness of the mathematical content of each task means that each lesson needs a new and unique "skeleton" of an RTA. This "skeleton" includes the main mathematical object that could be exposed by the task, as well as its different realizations. This exclusiveness enables a relatively straightforward comparison between lessons based on the same task, yet it limits the comparison between lessons based on different tasks. This limitation is crucial when one wishes to use the RTA for evaluating various lessons, and planning and analyzing lessons together with teachers.

Until now, we applied the RTA only on lessons based on one particular task – the Hexagons task, to examine how different lessons offer different opportunities for saming realizations of a mathematical object (Weingarden & Heyd-Metzuyanim, 2017, 2018). In the present study, we continue the development of the tool. Our goal is to examine the similarities and differences of RTAs of lessons based on two different tasks.

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 430-437). Pretoria, South Africa: PME.

THE RTA

The Realization Tree Assessment (RTA) tool (Weingarden et al., 2017) has been developed in the context of the TEAMS (Teaching Exploratively for All Mathematics Students) professional development program. The RTA was developed as an answer to the insufficiency of previous tools to capture the mathematical aspects of explorative instruction – instruction that offers students opportunities for explorative participation.

In explorative participation, the learner establishes new mathematical narratives based on formerly established ones, while objectifying – talking about mathematical objects as existing by themselves (Sfard, 2008). In order to objectify, the student needs to 'same' the different realizations of the mathematical object. The process of *saming* is described by Sfard (2008) as "assigning [the signifier] to a number of things that, so far, have not been considered as in any way 'the same' but are mutually replaceable in a certain closed set of narratives" (p. 170).

The RTA was initially developed to examine students' opportunities for explorative participation, while focusing on the opportunities for saming different realizations of the mathematical object. This is done by graphically illustrating (1) the different realizations of the mathematical objects that are presented during the lesson; (2) the extent to which links between realizations are made; and (3) the extent of students' authority (who produces the mathematical narratives).

In our former works with the RTA, we applied it on one particular task (the Hexagons task) and used it to analyse 10 lessons of different teachers (Weingarden & Heyd-Metzuyanim, 2017, 2018). From these studies we learned about the various ways in which teachers implement the Hexagons task, the different levels of exposure to the mathematical objects afforded to students in different classrooms, and the connections between level of explorations and characteristics such as grade level and track.

In the Hexagons task students are asked to describe the perimeter of a general "train" in a pattern of hexagons "trains" (see Figure 1).

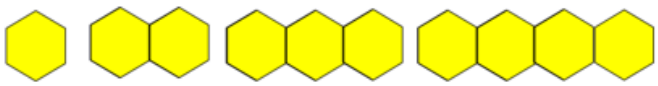

Figure 1: The Hexagons Pattern

This task's richness lies in its affordance to connect different algebraic expressions to a single visual mediator (the perimeter of the Hexagons' train), as there are different algebraic expressions that express how this perimeter can be counted. Therefore, the task provides opportunities for saming the different algebraic expressions that describe different procedures for counting the Hexagon's sides. For example, 4x+2 describes counting upper and bottom sides, then the 2 edges external edges; 6x-2(x-1) describes counting all the hexagon edges, then omitting the internal ones; and there are many more.

The comparison of 10 different lessons that were all based on the Hexagons task revealed interesting differences in the extent to which students got exposed to the

various realizations of the "algebraic expression" object. Some lessons included only very little exposure to different realizations, yet those realizations that were mentioned, were explained by students. Other lessons contained many realizations, yet most were authored by the teacher. The lessons we evaluated as most "explorative" were those that contained multiple realizations, multiple links drawn between them, and highest students' authority.

Comparing the "level of exploration" in a lesson is useful for several reasons. First, it enables studying the effects of interventions such as professional development. Second, it may offer insights into factors that contribute to explorative instruction (such as teachers' expertise, school/curriculum variables, tracking and more). Finally, it can serve as a tool for teachers to reflect upon lessons that have implemented a cognitively demanding task.

However, all these potential benefits will only be possible if the RTA can be flexibly applied to a variety of tasks. Therefore, our research question is: to what extent can the opportunities for students' explorative engagement be identified, compared and contrasted by the RTAs of lessons implementing two different tasks?

METHOD

As indicated above, the study reported here was performed in the context of the TEAMS project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms. As part of the professional development, the teachers were asked to implement and videotape a lesson based on a task they had experienced as learners in the professional development. One of those tasks is the Calling-Plans task (originally designed by the Institute for Learning, University of Pittsburgh) which deals with the intersection of two linear functions (see Figure 2). The Calling-Plans task's richness lies in its potential to expose, discuss and link the four realizations of the function object taught in middle-school: verbal, algebraic, graphic and ordered-pairs. This provides opportunities for saming the different realizations of the "intersection of functions" object. In particular, since the Calling-Plans scenario includes two different "calling plans" (each which can be described as a linear function), we denoted as the main object at the root of the RTA to be: intersection of two functions.

> Long Distance Company A charges a base rate of 45 NIS (equivalent of dollars) per month plus 5 agoras (equivalent of cents) per minute that you are on the phone. Long Distance Company B charges a base rate of only 20 NIS per month but they charge you 30 agoras per minute used. Which company would you choose, why?

Figure 2: The Calling-Plans task

The analysis of lessons using the RTA is based on watching only the whole-classroom discussion. It does not require transcription, but rather a careful design of the "skeleton" of the tree, based on theoretical knowledge of the potential of the task. This knowledge is derived through discussion with mathematicians and mathematics educators. The

design of the "skeleton" is followed by shading and marking realizations and links whenever they are spotted during the lesson. In the case of the Calling-Plans task, the skeleton of the RTA depicts the mathematical object "intersection of two functions" at the top of the tree and its different realizations as nodes in the tree (see Figure 3). We code the tree according to two criteria: (1) Coloring the realizations that were exposed to students during the lessons based on who articulated the realization (dark color = student; light color = teacher.) (2) Drawing arches between the realizations that were linked during the discussion (continuous line = link made by students; dashed line = link made by the teacher).

Two elements of explorative lessons are examined by the RTA. The first – opportunities for saming realizations - describes the extent to which the lesson exposed students to the different realizations and offered opportunities to same them, that is, view them as representing the same mathematical object. This element is pictured in the RTA by the fullness of the tree: multiple realizations are shaded and multiple links between realizations are drawn. The second element of explorative lessons – students' authority, describes the extent to which *students* (rather than the teacher) articulate the realizations and links. Students' authority is identified in the RTA by the darkness of the tree and by the continuity of the lines.

FINDINGS

Level of explorations in the RTAs of the Calling-Plans task

We start by describing the RTAs of three lessons based on the Calling-Plans task. This is done to exemplify the method and to display contrasting levels of opportunities for explorative participation. Due to space limitations, we only present full RTAs of one Calling-Plans and one Hexagons lesson. However, our analysis is based on unique RTAs drawn for each lesson. The first lesson took place in 8th grade and is called Calling-Plans lesson 1 (CP-Ls1, see Figure 3). In this lesson, students articulated an algebraic realization of the Calling-Plans problem: “y=0.05x+45 , y=0.3x+20", and made some connections to the possible verbal realizations (underlined in the verbal realization box). They described briefly how they found that x=100 ("I subtracted [the two functions]”, “I compared them [the two functions] to find the intersection”) and mentioned the verbal realizations when they explained: “[when the calling time is] less than 100 minutes, it's more profitable to choose Company B… and above 100 it's better to choose Company A”. No other realizations were made explicit during CP-Ls1 and no other connections to the various realizations of the object were drawn. CP-Ls1 is thus identified by a high level of students' authority but opportunities for saming realizations were scarce.

Quite a different picture of explorative participation was found in the second lesson (CP-Ls2). Here, opportunities for saming realizations were found throughout the lesson, but the level of students' authority was low. In CP-Ls2, three types of realizations were exposed to students during the whole-classroom discussion: the verbal realization, several algebraic realizations and the ordered-pairs realization. In

addition, links between the realizations were made. However, the light color of most of the realizations (teacher-authored realizations) and specifically, the dashed arches between realizations (teacher-authored links) show that although students were exposed to different realizations of the mathematical object and were provided with opportunities for saming realizations during the lesson, they did not author narratives about the links between the realizations and, consequently, no new narratives about the main mathematical objects were constructed by the students.

In the 3rd lesson (CP-Ls3), both elements of explorations were found: opportunities for saming realizations and students' authority. Students articulated all three types of realizations that were discussed during the lesson: verbal, graphic and inequality

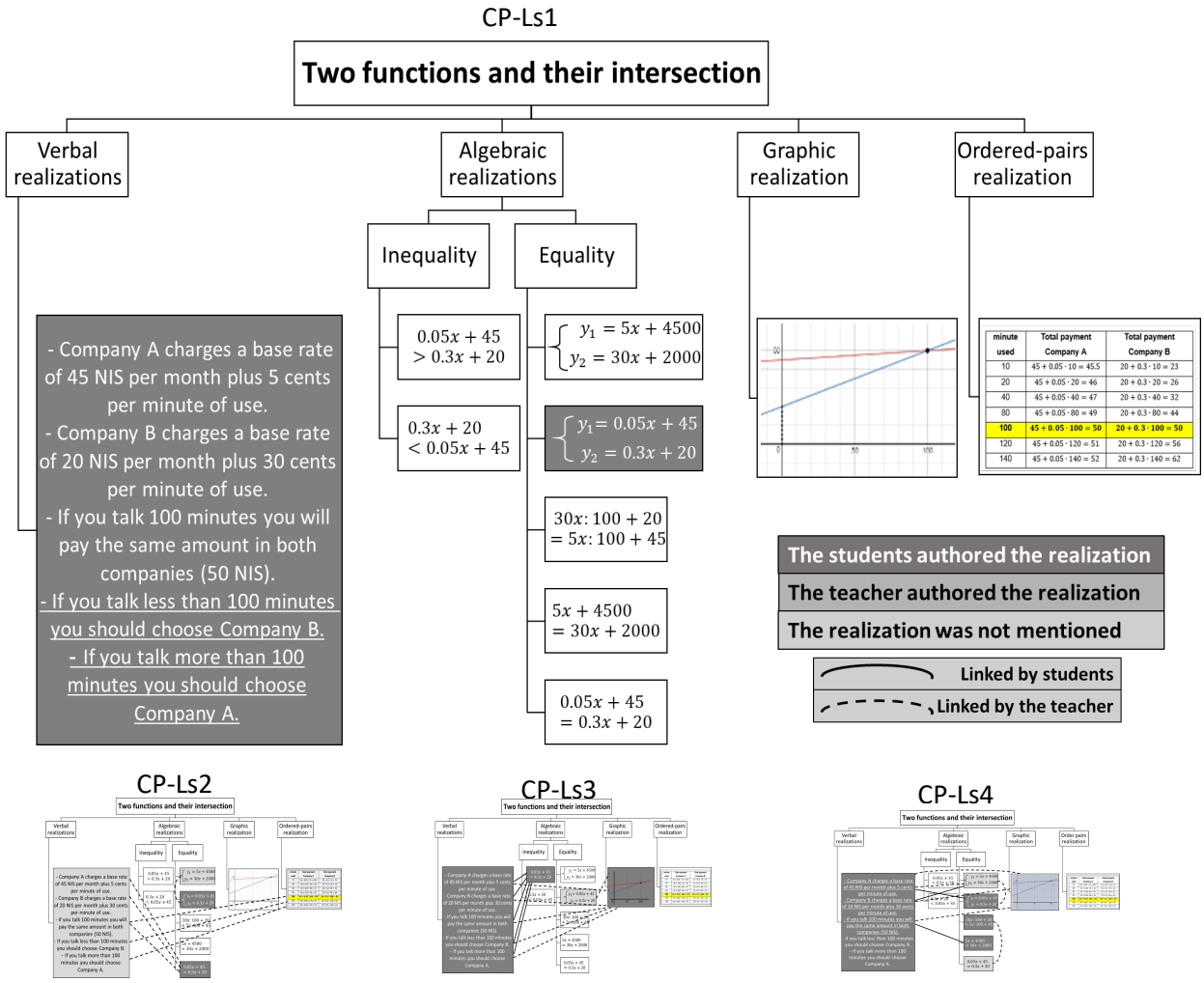

Figure 3: Calling-Plans RTAs

algebraic realizations. One student explained the inequality that she built as she was making links between the verbal realization (underlined) and the inequality algebraic realization (in bold):

> "<u>we did the rate of Company A that is 45 NIS (equivalent of dollars) per month and 5 agoras</u> and then we basically did: **45 + 0.05x**. And then we actually did (an) **inequality** of

the rate of Company B, which is 20 NIS per month and 30 agoras per call [minutes], which is: **0.3x+20**. And then actually you do the inequality and then find the [intersection]"

Similarities and differences between the RTAs of the two tasks

We now move to compare and contrast the Calling-Plans RTAs with the Hexagons-Task RTAs. At first glance, the skeleton of the Hexagons RTA (Figure 4) is significantly different from the Calling-Plans RTA. The main difference is that each task offers engagement with a different mathematical object.

While the Calling-Plans task deals with the 'intersection of two functions' object, the Hexagons task deals with the 'perimeter' object (perimeter of the Hexagon's general train). Each task affords opportunities for highlighting and saming its main object's different realizations.

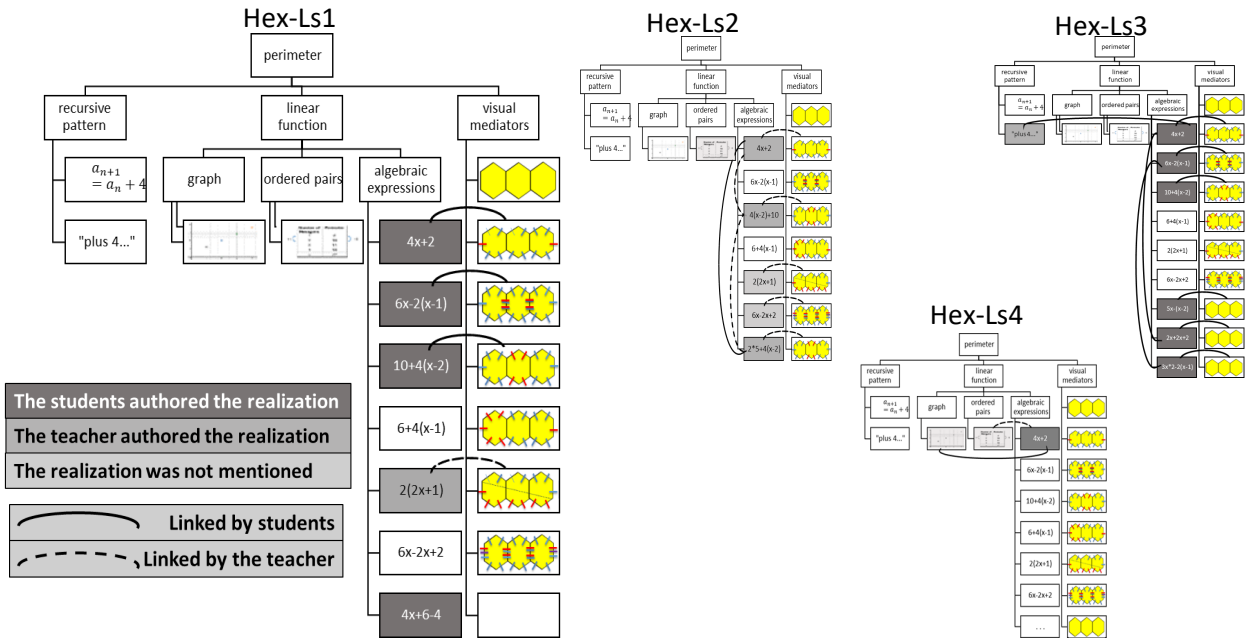

Figure 4: The Hexagons' RTA

Despite these differences, more general similarities can be found between the RTAs of the two tasks. For example, in each of the RTAs (CP and Hex), there are lessons that can be characterized as relatively "weak" or "strong" in students' authority. This is relatively simple to observe. For example, the RTAs depicting CP-Ls3 and Hex-Ls3 are strong in students' authority, while CP-Ls2 and Hex-Ls2 are weak. Another observation that can be made, across the tasks, relates to the opportunities for saming different realizations. Unlike CP-Ls3 and Hex-Ls 3, which contain multiple realizations and multiple links, CP-Ls1 and Hex-Ls1 contain relatively little opportunities for saming different realizations. In CP-Ls1, there are only 2 realizations shaded, and no links. In Hex-Ls1, there are several realizations but very few links, and none of these relate to the saming of the algebraic expressions. We call lessons depicted by such RTAs "show and tell" lessons, where multiple students present their different solutions, but no links are made between them.

The comparison of the two RTAs (CP and Hex) highlights that not all links are necessarily productive for achieving the mathematical goal of the lesson. Take for example the links drawn in different RTAs of the CP task. In CP-Ls4, one teacher devoted much attention to the question of how to convert NIS (equivalent of $) to agoras (cents) and what that would mean for the algebraic expressions y=0.05x+45 which would turn into y=5x+4500. This shifted the discussion to the two *instantiations* of the function objects (y=0.05x+45 & y=0.3x+20; y=5x+4500 & y=30x+2000). Although these two pairs of functions are useful for solving the same problem, this issue was peripheral to the object of the lesson, which was saming the different realizations of the two functions and their intersection.

A similar situation occurred in some Hexagons lessons (e.g Hex-Ls4), where links were made between the algebraic, ordered-pairs and graphic realizations of the 4x+2 expression. Paradoxically, such links would have been appropriate for the Calling-Plans task. In the Hexagons' task, however, the main object to be samed were the algebraic expressions, as embedded in the visual realization of the perimeter of the trains (being "the same" for all the various ways in which it can be counted and expressed algebraically).

DISCUSSION AND CONCLUSION

Our goal for this paper was to examine the extent to which opportunities for students' explorative engagement can be identified, compared and contrasted by the RTAs of lessons implementing two different tasks, the Calling-Plans and the Hexagons task.

Our findings indicate similarity in certain elements of the RTAs and differences in others. The element of students' authority can easily be identified in the RTAs of both tasks. This element is identified by the darkness of the tree (students' realizations) and by the continuity of the lines (students' links between realizations). However, the fullness of the tree (multiple realizations and multiple links), does not necessarily indicate the level of explorative instruction. There are instances where multiple realizations appear in a lesson, yet the lesson does not amount to a substantial mathematical idea. These occur in two main types of lessons: (1) "Show and tell" lessons (Stein, Engle, Smith, & Hughes, 2008), where multiple realizations are shaded, yet no links are drawn between them. (2) "Concepts-gone-wrong" lessons, where multiple links are made, yet they are not the important links that should be highlighted by the task.

The important conclusion drawn from the addition of examining the RTAs of the Calling-Plans task is that links do not always produce opportunities for saming. In certain cases, classroom discussions diverge into making links that do not same realizations of the central mathematical object underlying the task. We claim that the RTA, especially when applied to different tasks, is useful for exposing the relation between the goal of the task and its enactment. This should be useful not only for researchers attempting to understand classroom instruction, but also as a "representation of practice" (Grossman et al., 2009) by which teacher educators could

discuss with teachers the various opportunities for engaging students with objectification that arise from a task.

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INTEREST AND EMOTION PREDICTORS OF MOTIVATION IN SECONDARY MATHEMATICS CLASSROOMS

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This paper uses data from a large study of US high school mathematics engagement to quantitatively examine how different aspects of affect—interest, positive emotions, and negative emotions—influence different aspects of motivation—mastery goal orientation, performance goal orientation, and self-efficacy—in the context of mathematics classrooms. The results of a latent path analysis suggest that whereas interest was significantly associated with each of the different types of motivation, positive and negative emotions were only associated with self-efficacy. Implications for differentiating between the influence of different types of affect in learning contexts are discussed.

INTRODUCTION

Students' motivation to learn and do mathematics in the US is volatile during the first few years of high school. For example, in a recent study, many ninth graders interested in pursuing math-intensive career paths were found to be no longer interested two years later. By contrast, some who were previously uninterested became interested by eleventh grade (Mangu, Middleton, & Lee, 2016). However, the reasons behind such volatility are not well understood. Given the importance of mathematics for many careers, and that secondary education may be the last opportunity for students to opt in to mathematics-related pursuits, it is important to understand the complex nature of motivation for mathematics during this phase of their development.

Motivation in the Classroom.

Three types of motivation variables have consistently been shown to impact student engagement in mathematics: mastery goal orientation—in which students are motivated to understand the content; performance goal orientation—in which students are motivated to gain favourable judgement and avoid unfavourable; and self-efficacy—the belief that one has the capability to succeed. Mastery goal orientation is associated with more effective learning strategies, such as adapting one's strategies and seeking out help, whereas performance goal orientation is associated with somewhat less effective learning strategies, such as avoiding the appearance of incompetence (e.g., Elliot & Dweck, 2005, Harackiewicz et al., 2002). Mathematics self-efficacy, for its part, is associated with increased math performance and with choosing a mathematics-related major (Hackett & Betz, 1989).

Although these three commonly studied components of motivation are somewhat distinct, they are also related and may influence the expression of each other. For instance, mastery goal orientation and high self-efficacy together appear to be related to students' ability to muster the resources necessary to persevere in mathematics, as

2019. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 438-445). Pretoria, South Africa: PME.

learners with this combined motivational profile tend to be more adept at coping with failure (Meyer, Turner, & Spencer, 1997).

Affect and Motivation.

Affect is also a key aspect of students' experiences with mathematics and has important relationships with motivation. Specifically, positive emotions (such as relief, hope, and joy) improve motivational outcomes such as effort and self-efficacy, whereas negative emotions (such as shame, hopelessness, and anger) typically lead to avoidance responses such as decreased self-efficacy (Pekrun et al., 2004, Akin & Kurbanoglu, 2011). Yet negative and positive emotions are rarely considered simultaneously with another component of affect, interest, which is associated with increased attention, academic goals, and higher levels of learning (Hidi & Renninger, 2006). Notably, math interest includes not only positive emotions in the moment (i.e., *situational interest*), but also a stored sense of value and desire to seek out repeated engagement with mathematics over time (i.e., *personal interest,* Hidi & Renninger, 2006).

The Present Project.

In the present project, we examine the simultaneous relationship between each of the three components of affect in the mathematics classroom—positive emotions, negative emotions, and math interest, which are conceptualized as predictors and as mediators between classroom experiences and students' academic history—and the influence of these affective variables on motivation—mastery goal orientation, performance goal orientation, and self-efficacy, which are conceptualized as outcomes in the present study. The study is part of a longitudinal approach to measuring students' mathematics engagement. As such, it reports early results on the cross-sectional relationships among these variables. Longitudinal relationships will be modelled in subsequent years.

Hypotheses.

From the research, we identified three hypotheses that are tested in this study: (1) significant, negative relationships should exist between *negative emotions* and mastery goal orientation, self-efficacy, and performance goal orientation, (2) significant positive relationships should exist between *positive emotions* and mastery goal orientation, self-efficacy, and performance goal orientation, (3) significant positive relationships should exist between *math interest* and mastery goal orientation, self-efficacy, and performance goal orientation. Correlations are hypothesized to exist among affective variables (specifically, math interest was hypothesized to be positively correlated with positive emotions, but both interest and positive emotions were expected to be negatively correlated with negative emotions), and motivation variables (specifically, mastery goals were hypothesized to be positively related with self-efficacy, and negatively related to performance goals. Performance goals were expected to show a low-to-moderate correlation with self-efficacy).

METHOD

Participants

As part of a larger study on mathematics engagement in secondary classrooms, 296 students enrolled in first-year mathematics classes across three high schools each in Delaware and Arizona were recruited to take a 91-item survey in Spring 2018.

Instrument

The survey used was developed by a team of five researchers with expertise in mathematics learning. They adapted items from previous surveys to reflect four major conceptual dimensions of mathematics engagement: cognitive engagement, behavioural engagement, affective engagement, and social engagement. Preliminary psychometric analyses on this instrument using the pilot data from the present study suggest that the instrument has a seven-factor structure for the Likert items, and a two-factor structure for the categorical emotion items (see Zhang, in prep).

Measures

Emotions were measured using a checklist in which a student was asked to select from a list of 28 emotions, which he or she commonly felt during mathematics class. N*egative emotions* comprised thirteen binary response items, *positive emotions* comprised ten binary response items. *Math interest* was comprised of thirteen 7-point Likert scale items. Motivational constructs were measured on a 7-point Likert scale: mathematics *mastery goal orientation* comprised thirteen items, mathematics *performance goal orientation* comprised five 7-point Likert scale items, and mathematics *self-efficacy* comprised eighteen 7-point Likert scale items (see Zhang, in prep for the wording and source of all items). Each of these constructs was treated as a latent variable in our analyses, thus helping to correct for attenuation in the estimates of our path coefficients due to measurement error.

Analysis

The pertinent variables for this study comprised 69 items, 23 binary and 46 Likert scale. Because of the difference in level of the variables, we treated all responses as ordinal, thus using polychoric correlations and WLSMV estimations with theta parameterization. We used pairwise deletion to handle missing data in our analyses, yielding a final sample size of 296 in each analysis.

First, we estimated the fit of our measurement model using a confirmatory factor model to examine whether our expected latent variables adequately captured common variance in our measured variables (see Figure 1).

Then, we examined the fit of our structural model using a hypothesized latent variable path model to see whether the expected relationships between each of the affective latent predictors (math interest, negative emotions, and positive emotions) and each of the motivational latent outcomes (mastery goal orientation, performance goal orientation, and self-efficacy) fit the data well (see Figure 2). However, because our

hypothesized path model was saturated, we did not expect to see a difference in fit from our CFA model. Instead, we were interested in examining and interpreting theoretically relevant path coefficients.

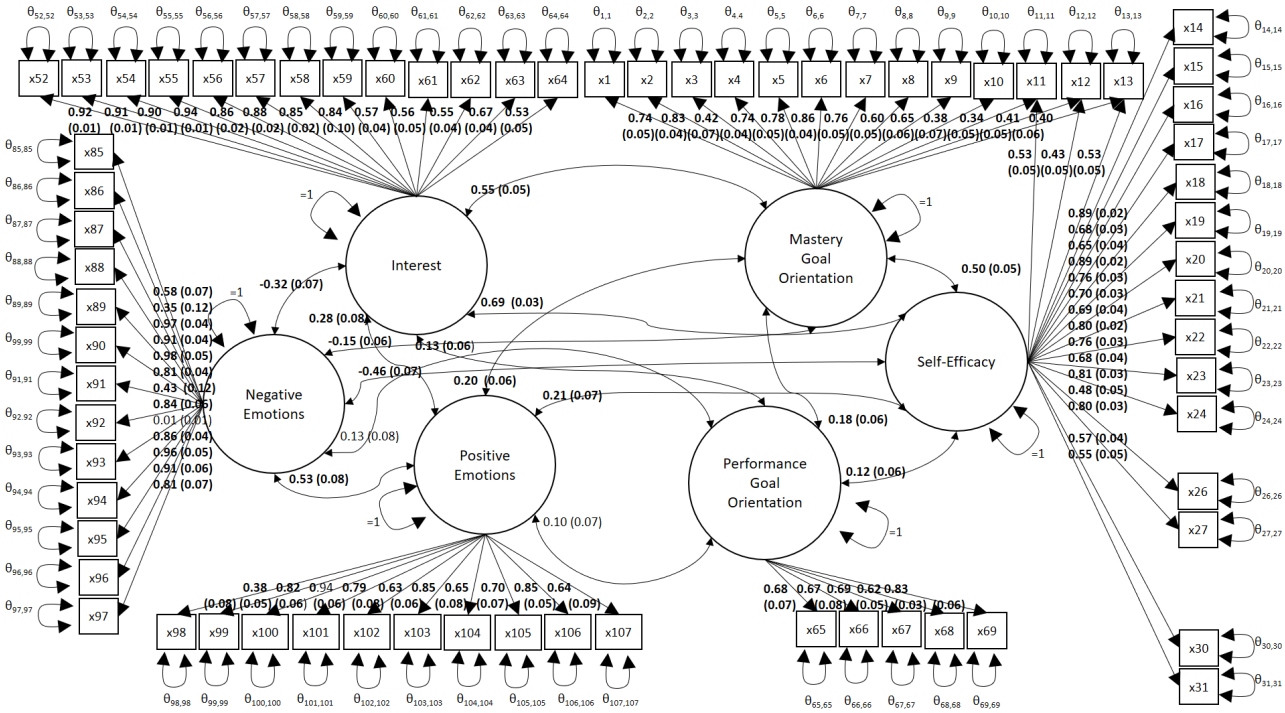

Figure 1: Parameter Estimates for the Confirmatory Factor Model.

Bolded values are significant at the $p < .05$ level. Standard errors are in parentheses. Results reflect standardized output.

Note also that in the latent path model, there were some significant expected correlations between each of the affective latent predictors (math interest, negative emotions, and positive emotions) and between each of the motivational latent outcomes (mastery goal orientation, self-efficacy, and performance goal orientation), consistent with theoretical conceptualizations of these constructs. All other possible correlations between latent variables were fixed to zero.

For ease of interpretation, we reported the fully standardized results from our output. All models were estimated in MPlus Version 8 (Muthen & Muthen, 2017).

RESULTS

Confirmatory Factor Model

Overall, the confirmatory factor model fit relatively well (see Table 1 for fit statistics). Although our chi square was significant [$\chi^2(2259) = 3800.56$, $p < 0.001$], the rule of thumb that χ^2/df be less than 3 shows that our model has adequate fit. The RMSEA was 0.048 with a 90% confidence interval between 0.045 and 0.051, which indicated relatively good model fit (Browne & Cudeck, 1993; Hu & Bentler, 1999). Moreover,

CFI and TLI were also good by standard cutoffs (0.90 - 0.95) (Brown, 2006). However, note that given our use of WLSMV estimation, these estimates may be somewhat artificially high for CFI and TLI and artificially low for RMSEA (Xia & Yang, 2018). Nevertheless, overall, our fit indices suggested that our measurement model fit the data reasonably well. Because the factor structure of these items is addressed elsewhere (Zhang, in prep), we do not interpret factor loadings and parameters estimates here. We instead fit the hypothesized latent variable path model, interpreting parameter estimates.

Model	Goodness of Fit Index	Estimate
CFA	χ^2	3800.56 (p<.0001)
	df	2259
	RMSEA	0.048 (0.045, 0.051)
	CFI	0.91
	TLI	0.91
Path Model	χ^2	3800.56 (p<.0001)
	Df	2259
	RMSEA	0.048 (0.045, 0.051)
	CFI	0.91
	TLI	0.91

Table 1: Goodness of fit indices for the CFA and Latent Path Models

Hypothesized Path Model

Overall, the hypothesized latent variable path model fit the same as the CFA model (see Table 1 for fit statistics, and Table 2 and Figure 2 for parameter estimates). Together, this suggests that the proposed structural model fit the data well. This is not surprising given the saturated nature of the model at the path (but not measurement) level.

Accordingly, we interpret standardized regression coefficients and correlations between the latent variables, which are theoretically relevant to our research questions (See Table 2 and Figure 2).

The relationships among latent variables are presented at the top of Table 2. In the hypothesized model, there was a significant, positive standardized regression coefficient between math interest and all three motivation variables (mastery goal orientation, performance goal orientation, and self-efficacy). This suggests the potential importance of math interest in terms of supporting multiple components of motivation in secondary math classrooms.

There was also a significant, strong, *negative* standardized regression coefficient between negative emotions and self-efficacy, and a moderate positive coefficient between positive emotions and self-efficacy. Standardized regression coefficients between both emotion factors and the mastery goal and performance goal motivation outcomes were not statistically significant.

Next, we discuss the correlations among the affective variables. There was a significant negative correlation between math interest and negative emotions and a significant positive correlation between math interest and positive emotions, as might be expected. However, contrary to expectations, there was a large significant *positive* correlation between negative and positive emotions.

For the motivation variables, we found a significant, small, positive correlation between mastery goal orientation and self-efficacy, and between mastery goal orientation and performance goal orientation. However, the correlation between and between self-efficacy and performance goal orientation was not significant.

Standardized Regression Coefficients			
	Interest	*Positive Emotions*	*Negative Emotions*
Mastery Goals	0.53***	0.05	-0.01
Performance Goals	0.25**	-0.11	0.27
Self-Efficacy	0.42*	0.37*	-0.52**
Correlations Among Affective Variables			
	Interest	*Positive Emotions*	
Positive Emotions	0.28***		
Negative Emotions	-0.32***	0.53***	
Correlations Among Motivation Variables			
	Mastery Goals	*Performance Goals*	
Performance Goals	0.14*		
Self-Efficacy	0.22**	0.07	

Table 2: Relationships Among Latent Variables

DISCUSSION

In this project, we examined the relationship between three components of affect in the mathematics classroom: positive emotions, negative emotions, and math interest, and three commonly measured aspects of motivation: mastery goal orientation, performance goal orientation, and self-efficacy. Although we expected that negative emotions would be significantly negatively related to mastery goal orientation, performance goal orientation, and self-efficacy, whereas positive emotions and math interest would be significantly positively related to mastery goal orientation, performance goal orientation, and self-efficacy, our hypotheses were only partially supported. Specifically, we found that math interest was significantly positively associated with each of the three motivational outcomes—mastery goal orientation, self-efficacy, and performance goal orientation—as predicted.

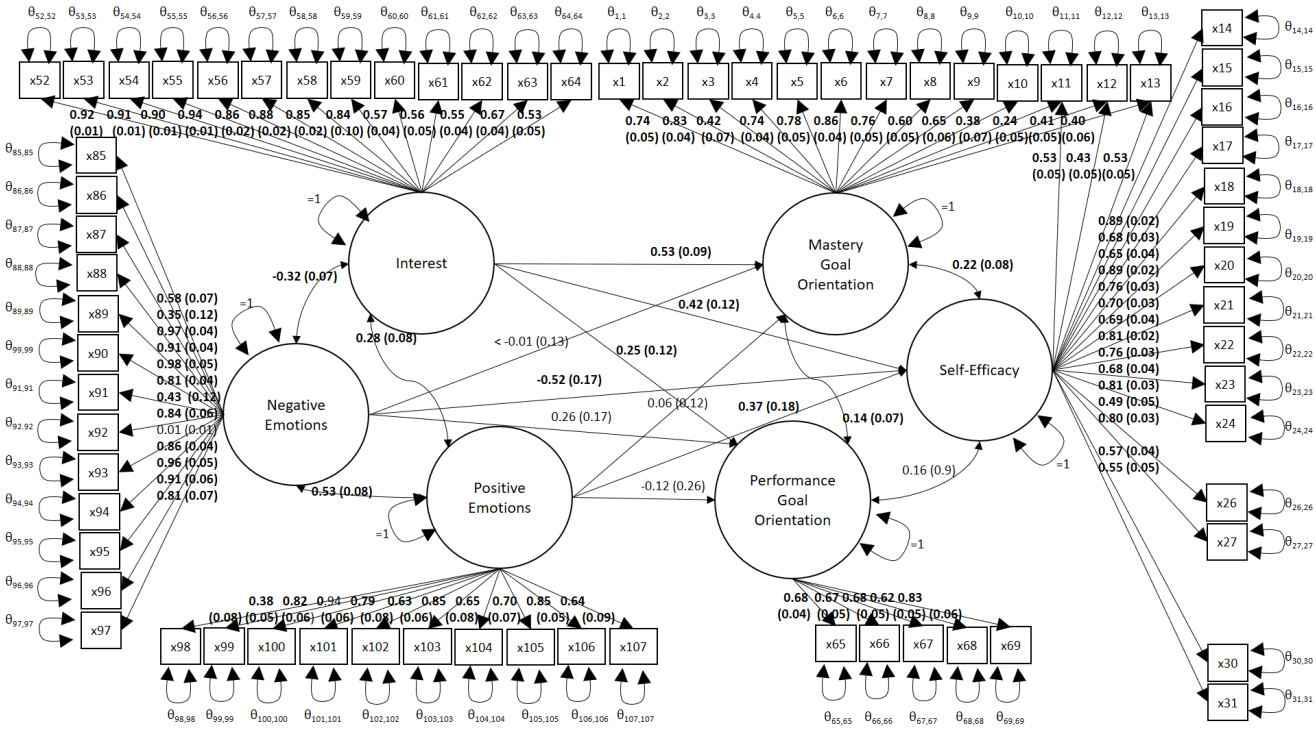

Figure 2: Parameter Estimates for the Hypothesized Latent Variable Path Model

Bolded values are significant at the p < .05 level. Standard errors are in parentheses.

However, our two emotion variables did not quite perform as expected—positive and negative emotions only showed significant relationships with self-efficacy. Specifically, in our sample, positive emotions were positively associated with self-efficacy, whereas negative emotions were somewhat more strongly negatively associated with self-efficacy. Although this is consistent with some prior research (Pekrun et al., 2004, Akin & Kurbanoglu, 2011), it is notable that math interest, which involves a math-specific knowledge structure (Hidi & Renninger, 2006) appeared to be more broadly related to motivation in math classrooms than did positive and negative emotions in our study.

This may be because our measures of emotions were relatively general. Specifically, we operationalized emotions using an item that asked participants to "think about a typical math experience" and indicate which among a list of emotions they felt during that experience. In responding to this question, students may have reflected not only on their feelings about math, but also on their feelings about other agents in the math classroom, such as teachers or classmates, which may have had conflicting influences on mathematics motivation. Indeed, if students were considering a variety of possible sources of math affect, this may help make sense of the unexpected positive correlation between positive and negative emotions in our sample (r = 0.53, p < 0.001). For instance, it may have been that when reflecting on their feelings about a typical math class, students had negative emotions about the math but positive emotions about their peers, or vice versa, thus allowing both positive and negative emotions to correlate when measured in the form of a single, undifferentiated self-report measure.

Whereas our results suggest the importance of disentangling different aspects of math affect, our findings also suggest that future research may benefit from taking this even further by disentangling whether positive and negative emotions felt about the math task, teachers, or classmates also have differing effects on motivation in the classroom. Such work may help us better understand how to foster richer affective and motivational secondary math experiences.

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WHICH EARLY PATTERNING ACTIVITIES COUNT THE MOST?

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Young children's performance on activities with repeating patterns is found to contribute to their later mathematical development, but it remains unclear whether specific patterning activities contribute more than others. The current study systematically analysed 369 four-year olds' performance on three different patterning activities (i.e., extending, translating, and identifying the unit) in relation to their future numerical ability. Performance on all three activities contributed uniquely to children's numerical ability one year later above age, spatial ability, and visuospatial working memory. Performance on two activities remained a significant predictor when controlling for the autoregressive effect of prior numerical ability, suggesting that these patterning activities are critical to children's numerical development.

INTRODUCTION

Children's ability to handle repeating patterns recently became an important topic within early mathematics research (Wijns, Torbeyns, De Smedt, & Verschaffel, in press). Pre-schoolers' performance on activities with repeating patterns has been found to contribute to their later mathematical development, even above other domain-general and domain-specific measures (e.g., Lüken, Peter-Koop, & Kollhoff, 2014; Rittle-Johnson, 2017). Researchers have used a broad range of activities to measure children's ability regarding repeating patterns, including extending (i.e., continuing the following elements of a pattern), translating (i.e., copying the pattern using different materials), and identifying the unit (i.e., indicating the smallest part that repeats). These various types of activities are assumed to contribute differently to children's mathematical development. However, empirical evidence regarding these assumptions is still missing. The current longitudinal study looked into this matter by systematically analysing four-year olds' performance on three types of patterning activities in relation to their future numerical ability.

THEORETICAL BACKGROUND

Repeating patterns and patterning activities

A repeating pattern (e.g., ABAB, Δo□Δo□) is a predictable sequence that contains a unit (e.g., AB, Δo□) that repeats indefinitely. The insight that a repeating pattern consists of a unit is mathematically important, because it is necessary to predict or generalize a repeating pattern (Rittle-Johnson, Fyfe, Loehr, & Miller, 2015; Threlfall, 2005). Several researchers made assumptions on whether or not this insight is required to solve a specific patterning activity (Sarama & Clements, 2009; Threlfall, 2005). For

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example, for some activities pre-schoolers are required to identify the unit of repeat explicitly (i.e., circle the smallest part that repeats) or implicitly (i.e., memorize the pattern), whereas this is not assumed to be required for extending. Some researchers claim that children also need to recognize the unit in a translation activity (Rittle-Johnson et al., 2015), but this claim has been questioned recently (Lüken, 2018; Rittle-Johnson, Zippert, & Boice, 2018).

Association between patterning ability and mathematical ability

Pre-schoolers' performance regarding repeating patterns is found to relate to their later mathematical ability. In a first study, five-year olds who could copy, extend, and explain an ABCC pattern, had significantly better mathematical abilities one and two years later (Lüken et al., 2014). In two other studies, four- to six-year olds' performance on activities including copying, extending, interpolating, translating, and unit identification, predicted mathematical abilities in fifth grade, above other early math measures such as counting, geometry, and calculation (Nguyen et al., 2016; Rittle-Johnson, Fyfe, Hofer, & Farran, 2017). More recently, four-year olds' performance on extending, interpolating, and translating repeating patterns was a unique predictor of their mathematical ability concurrently and seven months later (Rittle-Johnson et al., 2018). This association remained while controlling for verbal and spatial measures, and, importantly, prior math knowledge as measured by the same measure as the outcome variable.

CURRENT STUDY

Several studies have recently identified performance on activities with repeating patterns as a unique predictor of later mathematical competence, above domain-general cognitive measures and prior mathematical competence, even when measured by the same instrument as the outcome variable. These studies suggest that patterning activities are important for children's mathematical development. It remains however unclear whether specific patterning activities contribute more to later mathematical development than others. Although previous studies included different types of patterning activities, children's performances on the different types of activities were combined into one overall patterning score in all analyses. Therefore, the association between different types of patterning activities and later mathematical development is unclear. The current longitudinal study addressed this issue by analysing pre-schoolers' performance on three types of activities in relation to their numerical ability one year later. Performance on each activity was included as a predictor of numerical ability one year later in a regression analysis, while taking into account spatial ability and visuospatial working memory (given that the patterning activities were spatial in nature), as well as prior numerical ability.

METHOD

Participants

This study is part of a large-scale project on the development of young children's early mathematical competencies (https://ppw.kuleuven.be/o_en_o/WisenCo). We received an active informed consent from the parents of 410 children from one year group of the 17 schools (31 classrooms) that agreed to participate in the project. Schools were selected in order to include children from a range of socio-economic backgrounds. Data on one or more tasks were missing for 41 children, due to moving away, technical issues, and serious attentional or motivational problems, ultimately leading to a group of 369 participants (182 boys, 187 girls) with complete data. Their average age was 4 years, 10 months (range 4 years, 3 months to 5 years, 7 months) at the first time point.

Procedure

We assessed children's patterning ability and numerical ability during two test sessions of 30 minutes each in Spring 2017, i.e., when the children were in the last semester of their second preschool year (T1). Session 1 involved tasks with repeating patterns, whereas session 2 involved the numerical tasks. Spatial ability and visuospatial working memory were measured after the summer break of 2017. Finally, we administered the set of numerical tasks for a second time in Spring 2018, i.e., when the children were in the last semester of their third preschool year (T2).

Materials

We selected three activities that are assumed to contribute differently to later mathematical development, namely extending, translating, and identifying the unit. Each of these three activities was presented with six repeating pattern items (i.e., AB, AAB, ABC, AABB, AABC, and ABCD), leading to 18 items. The exact instruction, as well as an example item for each activity, is provided in Figure 1. The internal consistency was good for the overall measure (Cronbach's $\alpha = .79$) and acceptable to good for the different activities separately (Cronbach's $\alpha_s = .60$-$.72$).

Our measure of numerical ability was created based on recent research with children of a similar age range (see Andrews & Sayers, 2015; Purpura & Lonigan, 2013). At T1 the measure consisted of 88 items covering various aspects of numerical ability comprising verbal counting, dot enumeration, object counting, symbolic and non-symbolic comparison, number order, number recognition, and verbal arithmetic. The internal consistency of this measure was high ($\alpha = .93$). At T2 we used the same set of tasks, but we added a symbolic arithmetic task and extra items of the dot enumeration task, leading to a measure consisting of 105 items with a high internal consistency ($\alpha = .94$). We used a sum score of the Z-scores of performance on each of the aspects in our analyses.

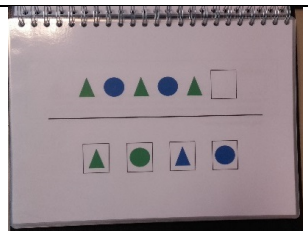
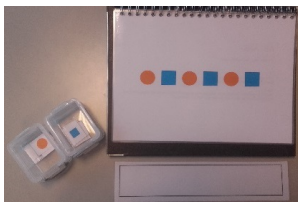
Activity	Instruction	Example item
Extending	"Look carefully at this row. There is a pattern in it. At the end there is something missing. One of these figures has to be placed in the empty spot. Do you know which of these has to be on the empty spot?"	
Translating	"Look carefully at this row. There is a pattern in it. Please make the same pattern with your figures on the paper strip."	
Identifying the unit	"You will soon see a row with a pattern in it. You will have to look very closely at it and try to remember the pattern. After a short period I will hide the pattern and you will have to copy it."	

Figure 1: Instruction and example of a repeating pattern item for each patterning activity.

The Block Design subtest (α = .76) of the WPPSI-III-NL was used to measure spatial ability (Hendriksen & Hurks, 2002). Children are asked to make a certain construction of red and white blocks within a given time limit. We used a Corsi span task (α = .77) to measure visuospatial working memory (De Smedt et al., 2009). Children need to reproduce a sequence of block that the experimenter had tapped in a random order.

RESULTS

We present the means and standard deviations of all variables, as well as correlations between patterning and numerical tasks in Table 1. Significant medium to large correlations were found between the patterning and numerical tasks, and these correlations held while controlling for age, spatial ability and visuospatial working memory (see partial correlations below the diagonal in Table 1).

We further explored the association between the different patterning tasks and later numerical ability by means of a regression analysis using numerical ability at T2 as the outcome variable. In the first model we included age, spatial ability, visuospatial working memory, and the three patterning activities as predictors. In the second model we also included an autoregressor (i.e., numerical ability at T1) as a predictor in order to examine the unique contribution of patterning in the development of numerical ability. We examined the assumptions for regression analysis, and these were not violated (e.g., VIFs < 2.0).

Variable	Raw score	Correlation							
	M (SD)	1	2	3	4	5	6	7	8
1. Age (in months)	70.10 (3.49)	-	.14**	.17***	.11*	.22***	.23***	.29***	.23***
2. Spatial ability	22.10 (5.53)		-	.40***	.38***	.34***	.38***	.43***	.40***
3. Visuospatial WM	6.05 (2.16)			-	.23***	.38***	.37***	.45***	.38***
4. Extending	3.00 (1.69)				-	.36***	.29***	.45***	.35***
5. Translating	2.99 (1.78)				.25***	-	.47***	.48***	.46***
6. Identifying the unit	2.65 (1.60)				.15**	.33***	-	.55***	.51***
7. Numerical ability T1	0.32 (5.54)				.33***	.31***	.38***	-	.69***
8. Numerical ability T2	0.13 (4.90)				.21***	.31***	.36***	.57***	-

Notes. * $p < .05$, ** $p < .01$, *** $p < .001$. Zero-order correlations are presented above the diagonal, partial correlations taking into account age, spatial ability, and visuospatial working memory, are presented below the diagonal.

Table 1: Descriptive statistics of all variables and correlations between patterning and numerical tasks

The results in Table 2 show that in the first model the two spatial tasks and the three patterning tasks all had a unique contribution to numerical ability at T2. When we included prior numerical ability in the second model, the spatial tasks and the extending task were no longer significant predictors. Translating and identifying the unit remained significant and unique predictors.

	Predictor	B (SE)	β
Model 1	Age	.11 (.06)	.08
$F(6,362) = 36.80$***	Spatial ability	.12 (.04)	.13**
Total $R^2 = .38$	Visuospatial WM	.26 (.11)	.11*
	Extending	.34 (.14)	.12*
	Translating	.49 (.14)	.18***
	Identifying the unit	.87 (.15)	.28 ***
Model 2	Age	.02 (.05)	.01
$F(7,361) = 54.38$***	Spatial ability	.07 (.04)	.08
Total $R^2 = .51$	Visuospatial WM	.05 (.10)	.02
	Extending	.01 (.13)	.00
	Translating	.29 (.13)	.11*
	Identifying the unit	.44 (.14)	.14**
	Numerical ability T1	.45 (.05)	.51***

Note. * $p < .05$, ** $p < .01$, *** $p < .001$.

Table 2: Regression models predicting numerical ability at T2

DISCUSSION

Recent evidence suggests that children's ability to handle repeating patterns contributes to their later mathematical development (e.g., Lüken et al., 2014). Studies thus far have used a general patterning measure including various patterning activities, yet several researchers claim that not all patterning activities are equally important for children's later mathematical development. The current study explored the contribution of different patterning activities to mathematical development by means of correlational and regression analyses in which different patterning activities were included as separate predictors and numerical ability one year later as an outcome variable. Our findings indicate that performances on the three patterning activities (i.e., extending, translating and identifying the unit) were associated with numerical ability one year later and that these associations were unique above age, spatial ability, and visuospatial working memory. When prior numerical ability was also included in the regression analysis, only translating and identifying the unit remained unique predictors.

This study addressed an important gap in research on early math education by looking into the contribution of specific types of patterning activities to children's mathematical development. Our results suggested that patterning activities are related

to children's mathematical development. More importantly, translating and identifying the unit contributed uniquely to the development of children's numerical ability. One possible explanation for the latter finding might be that children need to recognize the unit of repeat of the pattern in order to complete these two patterning tasks and that understanding this concept of a unit supports numerical tasks such as counting (e.g., counting in pairs).

Since the present study is only the first to focus on different patterning activities separately, more research is necessary to validate our findings. Longitudinal studies following children throughout preschool and primary school are needed to further explore the role of performance on different patterning activities for later mathematical development. Future intervention studies could provide evidence for a causal association between specific patterning activities and (specific) mathematical tasks.

Overall, the current study supports the idea that early childhood settings should include patterning activities in their daily practices. We suggest that early childhood educators and caregivers should focus more on activities such as translating and identifying the unit, which are currently underrepresented in early educational settings.

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STRUCTURAL THINKING AND ITS IMPACT ON SOLVING LINEAR EQUATIONS: A QUANTITATIVE STUDY IN SECONDARY SCHOOLS

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This paper is focused on the relationship between structural and operational thinking in the context of prior arithmetic and algebraic knowledge and the subsequent ability to solve linear equations. A test was administered to 469 students consisting of six item groups both with conventional operational items and with newly developed tasks to measure structural thinking in arithmetic and algebraic contexts. The influence of structural thinking on solving linear equations was significant and substantial. Nevertheless, the impact of operational skills is nearly twice as large as the impact of structural skills. Additionally, investigating mean differences of the six scales shows gender differences especially related to structural thinking and gives some support for the thesis that structural thinking does not grow according to the grade level.

INTRODUCTION: SFARD'S THEORY OF STRUCTURAL THINKING AND ITS ADAPTION TO THE FOLLOWING STUDY

Sfard (cf. 1991) introduced a theory on the ontological development of mathematical concepts that has become influential in many parts of mathematics education. She expands upon a widely held view of how mathematical concepts can be conceived. She denotes the two distinct approaches as an operational and as a structural understanding. A structural understanding can be characterised as the ability of "treating mathematical notions as if they referred to some abstract objects" (Sfard, 1991, p. 4). Its underlying idea can be recognised at a glance and its static nature permits to consider properties of mathematical concepts in the same way as properties of real objects. An operational conception however is about processes, algorithms and actions rather than about objects and their properties. In contrast, it is dynamic (Sfard, 1991, p. 4, p. 33). Sfard stresses that a structural understanding is the more advanced stage. Nevertheless, she considers this distinction less as a dichotomy, but as a duality: "The thesis […] implies that one ability cannot be fully developed without the other: on one hand, a person must be quite skillful at performing algorithms in order to attain a good idea of the 'objects' involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects, since without them the processes would seem meaningless and thus difficult to perform and to remember" (Sfard, 1991, p. 32). For the transition from an operational to a structural understanding of mathematical notions, she assumes three steps a student has to go through whenever he gets acquainted to a new mathematical concept: interiorisation, condensation and reification (cf. Sfard, 1991, pp. 18f.). This means that some operational skills are the initial points to gain a structural understanding and the latter cannot emerge without a sound operational base

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(Sfard, 1991, p. 32). Since Sfard regards the major feature of a structural understanding as a handling with objects that have properties, we follow this approach to design our items: This kind of understanding manifests itself in the same treatment of different expressions as representatives of the same mathematical object with equivalent properties.

Present research regards the role of the equal sign as central to the ability of solving linear equations (cf. Knuth et al., 2006, p. 297, cf. also McNeil & Alibali, 2005). We, however, consider a structural perception of arithmetic and algebra expressions to be the crucial point. With this idea in mind, we designed a test to investigate the impact of a structural understanding of previous knowledge, namely the understanding of arithmetic and algebraic expressions, on the ability to solve linear equations. Nevertheless, with respect to Sfard's idea of duality, it would be too simple to assume a monocausal connection between a structural understanding of prior knowledge and the ability of solving linear equations. It seems to be more appropriate to include some background variables that are not connected to structural thinking but are supposed to have a positive impact on solving linear equations. Knuth et al. used the score of a national standardised test (cf. Knuth et al., 2006, p.301). We decided to act on the stage model Sfard proposed and used different groups of items that measure operational skills that are supposed to have a positive influence on solving linear equations as well, although not presupposing any kind of structural understanding.

To summarise our intention: 1) We used Sfard's theory and adapted her stage model to investigate the impact of an operational and a structural understanding of prior knowledge on the ability to solve linear equations. 2) We started from Knuth's hypotheses that a structural understanding had a positive impact on solving these equations and that this understanding did not grow according to the grade levels (Knuth et al., 2006, p. 303). In contrast to Knuth et al., we do not regard the equality symbol on its own to be of central importance, but rather to what extent a student is able deal with arithmetic and algebraic expressions structurally. 3) To isolate the influence of a structural understanding, we included some operational items that are supposed to have a positive impact on the ability to solve linear equations.

THE TEST: ITEM GROUPS AND THEIR UNDERLYING CONCEPTS

Our test consists of six groups of items (cf. Table 1): Eight items are related to the tasks of solving linear equations ("lineq"). These items are intended to form the dependent variable that will be explained by all of the other item groups as independent variables or predictors. The predictor variables are represented by two classes consisting of two and three item groups, respectively: Two groups ("op_al_1" and "op_al_2") measure basic skills to deal with arithmetic and algebraic expressions being useful (or necessary) to solve linear equations. They have an operational nature, since they can be solved by only executing actions. Three groups are related to a structural understanding of arithmetic (str_ar_a and str_ar_m) and algebraic (str_al) expressions: Since we consider a structural understanding as the ability to recognise expressions as

equivalent on the base on its underlying concept (e.g. 5x), we used the idea of (de-)composing (formal) expressions to measure this ability. In Table 1, Cronbach's alpha is reported based on the data collection that will be described subsequently. The table shows that only one scale (op_al_1) has a questionable value, whereas all the other scales have good to excellent internal consistencies (cf. Cronbach, 1951).

abbrivia-tion	Cronbach's alpha	No. of items	Example(s)
op_al_1	0.67	6	Simplify: x + 66 + 4x = _____
op_al_2	0.93	8	Expand: 4(x+1) = _____
			Factor out: 5x – 15 = _____
str_ar_a	0.82	4	Enter the correct numbers: 479 + 75_7 = 7588 + 47_
str_ar_m	0.85	4	Enter the correct numbers: 82 · _30 = 82_ · 73
str_al	0.94	6	Determine the solution set by cleverly disassembling:
			$5x = 20 =$ __ · __ $\mathbb{L} = \{$ ____ $\}$
lineq	0.91	8	Determine the solution set of the linear equation:
			$10 = 5x - 7$ $\mathbb{L} = \{$ ____ $\}$

Table 1: Test items and reliabilities of the scales

DATA COLLECTION AND ANALYSIS

Our data were collected as a convenience sample of 469 students in autumn 2018. The German school system consists of three types of schools, separated according to the (expected) capabilities of the students: Level I is called "Hauptschule" (a high school open to all students, focused on developing hands-on, practical skills); level II is called "Realschule" (a selective high school for middle-tier students, providing an all-rounded education); and level III is called "Gymnasium" (an academically-oriented, selective high school for strong students). Additionally, there exists a so-called "integrated comprehensive school" (ICS) that provides classes or courses on all three levels.

gender	type of school	Grade
male: 242	level I: 89	grade 8: 108
female: 227	level II: 228	grade 9: 214
	level III: 93	grade10: 147
	ICS: 59	

Table 2: Composition of the sample

Table 2 shows the composition of our sample. In Table 1, the item groups are described, presupposing that there exist six different skills – each measured by one item group. Whether this assumption empirically holds, was tested both by explorative (EFA) and confirmatory factor analyses (CFA). The explorative analysis consisted of two steps:

A parallel analysis according to Horn was performed to determine the optimal number of factors to extract (cf. Horn & Engstrom, 1979). This procedure suggested six factors (as expected). Second, an exploratory factor analysis (with a promax rotation) was performed to check if the factor loadings correspond to the assumed six factors, i.e. that each of the items essentially loads to just one factor. This was in general fulfilled – with one exception: Three of the eight items of group "lineq" has the form $5(3+2x) = 39$. Comprehensively, they also had smaller, but substantial loadings on the factor "op_al_2" besides their main loadings on the factor "lineq". The calculations were performed using R (R Core Team, 2018, version 3.5.2) with the psych package (Revelle, 2018, version 1.8.10).

The explorative results gave support for our theoretical assumptions and, additionally, suggested some refinements by allowing double loadings for the three items discussed above. Therefore, we performed a confirmatory factor analysis, using the R package lavaan (Rosseel, 2012). We used a DWLS estimator. This is the standard method that is currently advised in case of dichotomous data (cf. Beaujean, 2014). The fit indices of this model were excellent (CFI 0.983, RMSEA 0.38, SRMR 0.39, cf. Beaujean, 2014, for interpreting the model fit).

	op_al_1	op_al_2	str_ar_a	str_ar_m	str_al	lineq
op_al_1	1	0.688***	0.398***	0.365***	0.396***	0.651***
op_al_2		1	0.371***	0.338***	0.326***	0.646***
str_ar_a			1	0.571***	0.409***	0.469***
str_ar_m				1	0.378***	0.377***
str_ al					1	0.419***
lineq						1

Table 3: Latent correlations between latent variables

In addition to the fit indices, the model used for the CFA also provided the latent correlations between the six factors defined by our item groups. Using latent correlations has one important advantage compared to calculating Pearson's product-moment correlations between the row sums of each item group: The product-moment correlations contain measurement errors and underestimate the "true" correlations between the underlying latent variables, whereas latent correlations are adjusted for the measurement error and express the relationships between the latent variables more accurately (cf. Beaujean, 2014, pp. 100-103). Table 3 shows the latent correlation between the six factors of our test (the asterisks here and in the following denote the usual significance levels: * stands for $p < 0.05$, ** for $p < 0.01$, and *** for $p < 0.001$). It is noteworthy that the latent correlations are substantial, but not as high as that the phenomenon of multicollinearity would appear (cf. Kock & Lynn, 2012, e.g. if the correlations are greater than 0.8 or 0.9). In terms of content, this finding means that the

six item groups measure six skills that are related to each other, but they are empirically clearly distinguishable.

The next step of our analysis was focused on group differences defined by mean differences of the six factor scores. As Table 2 shows, the internal structure of our sample, we used the three subcategories of our sample (gender, type of school, and grade level) as covariates in the following manner: for each subcategory, a reference group was chosen (male, school level I, and grade 8), its mean was set to zero and the means of the other group(s) of the same subcategory were expressed in terms of Cohen's d, i.e. as the standardised mean difference between the reference group and the other group(s) relative to the pooled standard deviations (cf. Cohen, 1988). All calculations were performed using R with the lavaan package. Cohen proposed the following rule to interpret the values of Cohen's d: 0.2 indicates a small, 0.5 a medium, and 0.8 a strong effect. This rule was expanded by Sawilowsky: 1.20 stands for a very large effect and 2.0 for a huge effect (cf. Sawilowsky, 2009).

	ref. gr.: male	*reference group: level I*			*ref. gr.: grade 8*	
abbr.	female	ICS	level II	level III	grade 9	grade 10
op_al_1	-0.09	0.76***	1.56***	2.63***	0.13	0.37**
op_al_2	0.08	0.61***	1.29***	1.46***	0.51***	0.20
str_ar_a	-0.33**	0.09	0.69***	1.39***	0.45***	0.29*
str_ar_m	-0.22*	0.10	0.50***	1.10***	0.34*	0.39**
str_al	-0.14	0.23	0.31*	1.14***	0.11	0.10
lineq	-0.09	0.39**	1.21***	1.87***	0.75***	0.50***
whole test	-0.08	0.53***	1.32***	2.02***	0.62***	0.42***

Table 4: Mean differences expressed in terms of Cohen's *d*

Table 4 contains the mean differences of the six factors discussed above. Additionally, the score of the whole test was regarded. Even though the six factors are clearly distinguishable and an one-factor score has a significantly worse model fit than the six-factor model, the fact that the score of all items leads to Cronbach's alpha of 0.96 may justify that the whole test can be analysed using a one-dimensional model.

The values of Table 4 are interpreted as follows: In general, there are no significant gender differences observable – except for two of the structural factors (str_ar_a and str_ar_m). The differences are small, but significant. Papers like Knuth et al. (2006) has not dealt with gender differences. Hence, it would be an interesting question if our observation is a singular case or if gender differences concerning structural thinking can be found in other contexts. The massive mean differences related to types of schools are mainly interesting for the German community. The schools have very different levels; and especially the integrated comprehensive school (ICS) does not satisfy its self-imposed goal to be an adequate alternative for students on all possible

levels. In particular, it is remarkable that the results of the ICS students do not significantly differ from level I students in case of the three factors related to structural thinking and, additionally, that only level III students achieve large values of Cohen's d in these cases. This finding could be an indicator for the hypothesis that structural thinking might be an ability predominantly reachable for high-performance students. The mean differences related to grade levels seem to support the findings of Knuth et al. (2006). The increases from grade 8 to 9 are partly not significant and – if so – they are rather small. From grade 9 to 10, the means even decrease in some cases. This observation fits to Knuth's finding that structural perceptions do not increase according to the grade level. But our findings are ambiguous: The non-structural item groups vary quite consistently with respect to the grade levels.

The last part of our analysis was committed to the core question of our research: Does structural thinking have a positive impact on solving linear equations? We decided to use a linear model to investigate this question (cf. Searle & Gruber, 2016). We declared the factor related to linear equations to be the dependent variable and all the other factors to be the predictors. To be more precise, the formula for this model is as follows:

$$lineq = \beta_1 \cdot op_al_1 + \beta_2 \cdot op_al_2 + \beta_3 \cdot str_ar_a + \beta_4 \cdot str_ar_m + \beta_5 \cdot str_al + \varepsilon$$

The regression coefficients β_i can be regarded as the equivalents of covariances and the standardised regression coefficients $\hat{\beta}_i$ as the equivalents of correlations. However, the role of regression coefficients is different. A covariance is a measure for the whole variance two variables have in common, but a regression coefficient corresponds only to the amount of variance a predictor has in common with the dependent variable that cannot be explained by other predictors. This is the reason why regression coefficients are typically smaller than the corresponding covariances or correlations – especially, if the predictors are correlated with each other (like in our case, cf. Table 3).

	β_i	$se(\beta_i)$	$\hat{\beta}_i$	$se(\hat{\beta}_i)$	$\hat{\beta}_i$ male	$\hat{\beta}_i$ female
op_al_1	0.982***	0.259	0.305***	0.080	0.350***	0.261**
op_al_2	0.303***	0.056	0.371***	0.068	0.275***	0.398***
str_ar_a	0.199**	0.064	0.170**	0.054	0.170**	0.150*
str_ar_m	0.017	0.068	0.013	0.052	0.030	-0.012
str_ al	0.115*	0.047	0.116*	0.048	0.149*	0.081

Table 5: Unstandardised (β_i) and standardised regression coefficients ($\hat{\beta}_i$)

Table 5 shows the regressions coefficient of our linear model with their standard errors $se(\beta_i)$ and $se(\hat{\beta}_i)$. The standardised regression coefficients can be directly compared with each other. The linear model estimated with the lavaan package produces good fit indices (CFI 0.981, RMSEA 0.42, SRMR 0.39). The total R^2 of variance explained equals 0.545. That is a considerable proportion of explained variation. The outcome can be interpreted as follows: The basic operational skills (op_al_1 and op_al_2) have

a greater impact on the ability of solving linear equations than the factors related to structural understandings. However, two of the structural factors (str_ar_a and str_al) have a significant impact. Their standardised regression coefficients are approximately half as large as those of the operational factors, but they provide a substantial contribution. Only the structural factor related to multiplicative contexts (str_ar_m) does not have a significant influence. Remarkably, the newly invented item group concerning a structural understanding of an algebraic expression (str_al) possesses a significant impact. This finding may indicate that these items could be used as a class of suitable exercises to empower students both to gain structural insight in numbers and algebraic contexts and to solve linear equations. The gender differences reported in Table 5 are not easy to interpret. The operational item groups (op_al_1 and op_al_2) swap their role between male and female students; two structural item groups (str_ar_a and str_ar_m) behave in the same way; and the third one (str_al) has a less important role for female students.

CONCLUSION

Our research results expand the picture that already exists concerning the question of, if and to what extent a structural conception of previous knowledge has an impact on future learning. Previous knowledge and competence of being able to deal with arithmetic and algebra expressions structurally has a significant and substantial impact on solving linear equations – but with two limitations: 1) The impact of basic operational skills is twice as large (op_al_1 and op_al_2); 2) only structural understanding in additive contexts (str_ar_a) and in the form of decomposing numbers according to algebraic expressions to find solutions of linear equations (str_al) provide a significant influence. An example for the latter is "Determine the solution by cleverly disassembling the number 20 in: $5x = 20 = __ \cdot __$". Tasks like this may be used to establish a new class of exercises as a pre-stage to solving linear equations algorithmically. In addition to our main result, our study detects gender differences concerning structural understanding both in arithmetic and algebraic contexts: Female students earned significantly lower results on the related scales than their male classmates, but the effect size is small. Quite similar than in previous research (cf. Knuth et al., 2006), there is some evidence that a structural perception does not increase substantially according to the grade level.

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