

**Proceedings**  
**of the First Regional Conference**  
**of the International Group**  
**for the Psychology of Mathematics Education**

Rancagua, Chile

November 14-16, 2018

**Editor**

David M. Gómez



Cite as:

Gómez, D. M. (Ed.) (2018). *Proceedings of the First PME Regional Conference: South America*. Rancagua, Chile: PME.

Website: <http://www.uoh.cl/pme-regional/>

The proceedings are also available via <http://www.igpme.org/>

Copyright © 2018 left to the authors

All rights reserved

ISBN: 978-956-6031-01-7

Logo Design: Rocío Fuentes

Printed by Cirples Rancagua, <http://www.cirples.cl/>

# Preface

It is our pleasure to welcome you to the First Regional Conference of the International Group for the Psychology of Mathematics Education. PME is one of the most important international conferences in mathematics education, bringing together a multidisciplinary community of researchers from all over the world. In its continuous effort to enrich the diversity of this community, PME has decided to support the organization of a number of Regional Conferences in order to reach researchers from regions of the world that are underrepresented in PME. This Regional Conference, the first of them, was approved by the Annual General Meeting of PME members in the PME 41 conference, held in Singapore, and has a special focus on South America. This conference received submissions from 66 persons from 4 South American countries and 7 other countries worldwide, and about 60 people are expected to attend the conference.

This PME Regional Conference is organized in Rancagua, in the central area of Chile. Following the PME conferences organized in Brazil in Recife (1995) and Belo Horizonte (2010), with this scientific event PME comes back to South America in order to strengthen bonds with the local community of researchers in mathematics education.

The theme of the conference is *understanding and promoting students' mathematical thinking*, allowing us to reflect on one of the crucial goals of mathematics education from the diversity of perspectives and research traditions that is a hallmark of PME. The theme focuses primarily on students but has long reaching relations to, for instance, teacher training and professional development, and cognitive as well as sociocultural approaches to learning.

The PME Regional Conference, modeled after the series of PME conferences, includes the well-known personal presentation formats Research Report, Oral Communication, and Poster Presentation. Three Plenary Lectures, as well as PME sessions and Discussion Group sessions complete the conference program.

The organization of this Regional Conference is a collaborative effort involving many support staff of Universidad de O'Higgins (UOH), a newly created Chilean university located in Rancagua. The organization of the conference was also supported by the Program Committee, the International Group for the Psychology of Mathematics Education, the Center for Advanced Research in Education (CIAE) of Universidad de

Chile, and the Chilean Society of Research in Mathematics Education (SOCHIEM). We thank all those involved in making this conference possible, as well as the participants of this Regional Conference for coming to visit us in Rancagua to share their work.

We hope your visit to UOH will prove valuable scientifically and socially, as well as an opportunity for strengthening bonds with the South American and the PME communities.

*David M. Gómez and Wim Van Dooren*  
PME Regional Conference Chairs

## **SPONSORS**

We are grateful for the generous support received for this conference from:

The International Group for the Psychology of Mathematics Education (PME)  
Universidad de O'Higgins (UOH)

The Center for Advanced Research in Education (CIAE), Universidad de Chile  
The Chilean Society of Research in Mathematics Education (SOCHIEM)

# Table of contents

Preface	i
Table of contents	iii
The International Group for the Psychology of Mathematics Education (PME)	ix
The First PME Regional Conference	xii
Reviewing process	xiii
List of Reviewers	xiv
<b>PLENARY LECTURES</b>	
A review of research on understanding and promoting students' mathematical thinking published in Educational Studies in Mathematics 2014-2018	1
<i>Goos, Merrilyn</i>	
Class observation to enrich student thinking on mathematics inside the classroom	17
<i>Martínez Videla, María Victoria</i>	
Making sense of students' sense-making of mathematics	31
<i>Fusaro Pinto, Marcia Maria</i>	
<b>PME SESSIONS</b>	
An introduction to the International Group for the Psychology of Mathematics Education	47
<i>Van Dooren, Wim; Gómez, David M.</i>	
<b>RESEARCH REPORTS</b>	
Uncertainty and 3D dynamic geometry: A catalyst for the step from 2D to 3D geometry	49
<i>Echeverry, Armando; Camargo, Leonor; Gutiérrez, Ángel; Samper, Carmen</i>	
A model for pre-service teaching education focused on the development of Mathematical Knowledge for Teaching: Perception of future teachers	57
<i>Guiñez, Flavio; Salinas, Ricardo; Montenegro, Helena; Acevedo, Verónica; Radovic, Darinka; Martínez, Salomé</i>	
Tensions and dilemmas about what is "legitimate" in a geometry course	65
<i>Molina Jaime, Oscar Javier; Pino-Fan, Luis; Font, Vicenç</i>	
Conceptions of modeling by Chilean mathematics teacher educators: A phenomenographic study	73
<i>Montenegro, Helena</i>	

Prospective primary teachers' knowledge of pedagogical orchestration in problem-solving instruction	81
<i>Piñeiro, Juan Luis; Castro-Rodríguez, Elena; Castro, Enrique</i>	
Structures and generalisation in a functional approach: The inverse function by fifth graders	89
<i>Pinto, Eder; Cañadas, María C.</i>	
Theoretical advances in Mathematical Cognition	97
<i>Scheiner, Thorsten; Fusaro Pinto, Marcia Maria</i>	
Why women tend not to choose mathematically demanding careers: A systematic review of all time literature	105
<i>Radovic Sendra, Darinka</i>	
The Basic Ideas (BIs) of prospective teachers for the first class of teaching addition	113
<i>Reyes Santander, Pamela Alejandra; vom Hofe, Rudolf</i>	
The transition from mathematics teacher to facilitator: Challenges in scaling up professional development programs	121
<i>Valoyes-Chávez, Luz; Ruiz, Natalia; Espinoza, Carmen; Sepúlveda, Carmen; Fuenzalida, Nicole</i>	
Pedagogical problem solving knowledge of Chilean mathematics teachers and instructional reflection	129
<i>Saadati, Farzaneh; Chandia, Eugenio; Ruiz, Natalia</i>	
To add or to multiply? An in-depth study on primary school children's preference for additive or multiplicative relations	137
<i>Degrande, Tine; Verschaffel, Lieven; Van Dooren, Wim</i>	
<b>ORAL COMMUNICATIONS</b>	
Situated learning in an interdisciplinary prospective teacher course	145
<i>Acevedo Rincón, Jenny Patricia</i>	
Bridging mathematical modeling and argumentation in the classroom	146
<i>Aravena, Maria; Solar Bezmalinovic, Horacio; Goizueta, Manuel; Ortiz, Andrés; Ulloa, Rodrigo; Cardenas, Leonardo</i>	
The hypothetical learning trajectory as a teaching tool to promote mathematical thinking of students	147
<i>Cárcamo, Andrea</i>	
The gamification, and its use to teach numbers in the first school years	148
<i>Castillo Funes, Josefa Valentina; Ramos Rodríguez, Elisabeth</i>	
Linking mathematics and engineering through modelling: The case of civil engineering	149
<i>Cosmes Aragón, Saúl Ernesto</i>	
Statistics as a tool for decision-making: Two countries and one pattern	150
<i>Neto, Vanessa; Elicer, Raimundo</i>	

Relationships between mathematical creativity and motivation: Mathematical creativity workshops in high school student training <i>Fonseca, Mateus Gianni; Gontijo, Cleyton Hércules; Souza, Juliana Campos Sabino de; Zanetti, Matheus Delaine Teixeira</i>	151
Conceptions from freshmen engineering students on the concepts of absolute value and inequality <i>Fuentealba, Claudio</i>	152
Mind the gap(s): Self-concept as a mediator of math achievement differences in Chile <i>Gelber Nuñez, Denisse Andrea; Ortega, Lorena Constanza; Treviño, Juan Ernesto; Solar, Horacio; González, Alonso; Wyman, Ignacio</i>	153
Development of a questionnaire for teachers about teaching practices and motivational beliefs related to problem solving in mathematics <i>Giaconi, Valentina; Perdomo-Díaz, Josefa; Cerda, Gamal; Saadati, Farzaneh</i>	154
The construction of knowledge in the teaching of mathematics to students, for mathematics teachers in online debates <i>Guerrero Contreras, Oscar; Llinares, Salvador</i>	155
Comparative study on the changes and consistency of the Mathematics Curriculum in Mexico through the curricular reforms of 1993- 2009/2011 <i>Hoyos, Veronica; Navarro, Maria Estela; Raggi, Victor Javier; Rojas, Sandra</i>	156
Interactions of immigrant secondary school students during the resolution of algebraic problems <i>Lodhi, Atif; Cantero, Beatriz</i>	157
Problem posing as a tool for formative assessment: An experience with prospective teachers <i>Malaspina, Uldarico; Torres, Carlos; Campos, Magaly</i>	158
Representations of students' concept images about continuity of a function <i>Messias, Maria Alice de Vasconcelos Feio; Brandemberg, João Cláudio</i>	159
The role of analogy in the formation of early numerical representations <i>Navarrete, Jairo; Gomez, David; Dartnell, Pablo</i>	160
Citizenship notions in mathematics textbooks for rural primary schools in Brazil <i>Reis Pacheco, Débora; Neto, Vanessa Franco</i>	161
The effect of benchmarks on strategy use in fraction comparison problems <i>Obersteiner, Andreas; Marupudi, Vijay; Wagner Alibali, Martha</i>	162

Kindergarten and primary teachers' specialised knowledge in the context of classification	163
<i>Almeida, Alessandra; Zakaryan, Diana; Policastro, Milena; Oliveira, Marcos; Ribeiro, Miguel</i>	
Teaching reflection on student's performance in their mathematical learning assessment	164
<i>Rojas Bastias, Daniela; Galvez Perez, Grecia</i>	
Professional noticing of argumentation in mathematics classrooms	165
<i>Solar Bezmalinovic, Horacio; Cervantes, Jonathan; Ortiz, Andrés; Ulloa, Rodrigo</i>	
Understanding and promoting children's proportional reasoning	166
<i>Spinillo, Alina Galvão</i>	
Features of mathematics teacher argumentation	167
<i>Toro Uribe, Jorge Andrés; Castro Gordillo, Walter Fernando</i>	
How do Chilean math teacher educators prepare preservice teachers to achieve and develop argumentation skills?	168
<i>Ulloa, Rodrigo; Rivera, Sara; Solar, Horacio; Ortiz, Andrés</i>	
The education of urban mathematics teachers	169
<i>Valoyes-Chávez, Luz</i>	
Problem solving, an ability to be developed in primary teachers	170
<i>Vásquez, Patricia; Cerda, Mayra</i>	
<b>POSTER PRESENTATIONS</b>	
The development of visualization skills and processes	171
<i>Ojeda Guerrero, Daniela Camila; Acevedo Rincón, Jenny Patricia</i>	
Middle and high school students in Brazil and their beliefs about mathematics	172
<i>Comelli, Felipe Augusto de Mesquita; Silva, Wanusa Rodrigues; Manrique, Ana Lúcia</i>	
A glocal lesson study with prospective Brazilian mathematics teachers	173
<i>Acevedo Rincón, Jenny Patricia; Flórez Pabón, Campo Elías</i>	
Comparing fractions vs. comparing divisions: Different underlying processes in young adults	174
<i>Gómez, David Maximiliano; Dartnell, Pablo</i>	
Mechanisms underlying the gender gap in math performance at university admission in Chile	175
<i>Navarrete, Jairo; Sandoval, José; Espinoza, Juan; Moreno, Luis</i>	
Inclusive practices in the math class: The use of argumentation as a means to promote equity	176
<i>Peña-Rincón, Pilar; Solar, Horacio</i>	

Elements to observe and discern good educational practices in mathematics	177
<i>Ramos Rodríguez, Elisabeth; Reyes-Santander, Pamela; Soto-Andrade, Jorge; vom Hofe, Rudolf</i>	
Dividing equally - Results of a pilot study on the preconceptions of grade 1 to 2 children in Chile	178
<i>Reyes Santander, Pamela Alejandra; Peter-Koop, Andrea</i>	
Can we observe students' mathematical thinking?	179
<i>Soto-Andrade, Jorge; Yáñez-Aburto, Alexandra; Beltrán-Gabrie, Andrés</i>	
The game as a teaching resource in mathematics for youth and adult education	180
<i>Silva, Stephanie Machado da; Alvim, Brenner Gomes; Quadros, Matheus Procópio de; Souza, Juliana Campos Sabino de</i>	
Problem-posing strategy to stimulate the development of teachers' didactic analysis competence	181
<i>Torres, Carlos; Malaspina, Uldarico</i>	
Evolution in the design of school mathematical tasks for training teachers	182
<i>Valenzuela Molina, Macarena del Carmen; Ramos Rodríguez, Elisabeth</i>	



# The International Group for the Psychology of Mathematics Education (PME)

## HISTORY OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel  
Richard R. Skemp, UK  
Gerard Vergnaud, France  
Kevin F. Collis, Australia  
Pearla Nesher, Israel  
Nicolas Balacheff, France  
Kathleen Hart, UK  
Carolyn Kieran, Canada

Stephen Lerman, UK  
Gilah Leder, Australia  
Rina Hershkowitz, Israel  
Chris Breen, South Africa  
Fou-Lai Lin, Taiwan  
João Filipe Matos, Portugal  
Barbara Jaworski, UK

The current president is Peter Liljedahl, Canada, and the current president-elect is Markku Hannula, Finland.

## THE GOALS OF PME

The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

All information concerning PME and its constitution can be found at the PME website: <http://www.igpme.org/>.

## PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the PME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

### **WEBSITE OF PME**

All information concerning PME, its constitution, and past conferences can be found at the PME website: <http://www.igpme.org/>.

### **HONORARY MEMBERS OF PME**

Efraim Fischbein (Deceased)

Hans Freudenthal (Deceased)

Joop Van Dormolen (Retired)

### **PME ADMINISTRATIVE MANAGER**

The administration of PME is coordinated by the Administrative Manager:

Birgit Griese

Email: [info@igpme.org](mailto:info@igpme.org)

## **INTERNATIONAL COMMITTEE OF PME**

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference.

The IC work is led by the PME president who is elected by PME members for three years.

### **President**

Peter Liljedahl (Canada)

### **Vice-President**

David M. Gómez (Chile)

### **Secretary**

Einat Heyd-Metzuyanim (Israel)

### **Treasurer**

Laurinda Brown (United Kingdom)

### **Members**

Judy Anderson (Australia)

Richard Barwell (Canada)

Yiming Cao (China)

Man Ching Esther Chan (Australia/Hong Kong SAR)

Anika Dreher (Germany)

Anthony Essien (South Africa)

Mellony Graven (South Africa)

Maitree Inprashita (Thailand)

Berinderjeet Kaur (Singapore)

Maria Mellone (Italy)

Miguel Ribeiro (Brazil)

Lovisa Sumpter (Sweden)

Kai-Lin Yang (Taiwan)

# The First PME Regional Conference

Two committees are responsible for the organization of this PME Regional Conference: the Program Committee and the Local Organizing Committee.

## **THE PROGRAM COMMITTEE**

David M. Gómez	Universidad de O'Higgins (Chile)
Wim Van Dooren	University of Leuven (Belgium)
Manuel Goizueta	Pontifical Catholic University of Valparaiso (Chile)
Stefan Ufer	University of Munich – LMU (Germany)

## **THE LOCAL ORGANIZING COMMITTEE**

David M. Gómez, Patricia Contreras, Camila Bugueño, Héctor Díaz

## **HOSTING INSTITUTION**

The First PME Regional Conference is hosted by Universidad de O'Higgins (UOH), Chile.

<http://www.uoh.cl/>

# Reviewing process

## **RESEARCH REPORTS (RR)**

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The number of submitted RR proposals was 31: 5 of them were accepted and 10 of them were given the opportunity to re-submit a revised version. After revisions, the final number of accepted RR proposals increased to 13. Of those not accepted as RRs (whether in the first or the second round), 10 were invited to be re-submitted as Oral Communications (OC) and 5 as Poster Presentations (PP).

## **ORAL COMMUNICATIONS (OC)**

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted OC proposals was 39, and 24 of them were accepted. Of those not accepted as OC proposals, 3 were invited to be re-submitted as Poster Presentations (PP). In the end, considering re-submissions of Research Reports as Oral Communications, 30 OCs were accepted for presentation at the Regional Conference.

## **POSTER PRESENTATIONS (PP)**

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted PP proposals was 10, and 7 of them were accepted. In the end, considering re-submissions of Research Reports and Oral Communications as Poster Presentations, 13 PPs were accepted for presentation at the Regional Conference.

# List of reviewers

---

Judy Anderson	Australia
Nadia Azrou	Algeria
Irene Biza	UK
Orly Buchbinder	USA
Man Ching Esther Chan	Australia
Boon Liang Chua	Singapore
Csaba Csikos	Hungary
Pablo Dartnell	Chile
Cris Edmonds-Wathen	Australia
Ceneida Fernández	Spain
Marcia Fusaro Pinto	Brazil
Hedwig Gasteiger	Germany
Manuel Goizueta	Chile
David M. Gómez	Chile
Merrilyn Goos	Ireland
Ángel Gutiérrez	Spain
Markku Hannula	Finland
Matthew Inglis	UK
Pedro Ivars	Spain
Keith Jones	UK
María Victoria Martínez	Chile
Maria Mellone	Italy
Guri Norvedt	Norway
Miguel Ribeiro	Brazil
Silke Ruwisch	Germany
Stanislaw Schukajlow	Germany
Wee Tiong Seah	Australia
Nathalie Sinclair	Canada
Alina Spinillo	Brazil
Stefan Ufer	Germany
Wim Van Dooren	Belgium
Lieven Verschaffel	Belgium

---



# PLENARY LECTURES



# A REVIEW OF RESEARCH ON UNDERSTANDING AND PROMOTING STUDENTS' MATHEMATICAL THINKING PUBLISHED IN *EDUCATIONAL STUDIES IN MATHEMATICS* 2014-2018

Merrilyn Goos

University of Limerick, Ireland

*In this paper I look back over my five years as Editor-in-Chief of Educational Studies in Mathematics to review research published in the journal, from 2014-2018, that aimed to better understand and promote students' mathematical thinking. The review is guided by an analysis of conceptualisations of "mathematical thinking" proposed in the research literature, selected curriculum documents, and international assessment programs such as PISA. The review not only documents salient features of research studies, such as the country of origin of the authors, educational level of the participants, research aims, theoretical perspectives, and methodological approaches, but also identifies the contribution to knowledge made by this body of work as well as future research directions and opportunities.*

## INTRODUCTION

The theme of this first PME Regional Conference is *Understanding and promoting students' mathematical thinking*. According to the conference First Announcement, this theme

...emphasises the role of Math Education research in helping educators to foster mathematical thinking in their classrooms. It deals not only with how Math Education can be made more effective, but also more inclusive and equitable. (First PME Regional Conference: South America, 2018, p. 5)

My aim in this paper is to provide a focused review of recent research on mathematical thinking in order to address the conference theme and suggest future research directions. I have limited the review to papers published in *Educational Studies in Mathematics (ESM)* during my five-year tenure as Editor-in-Chief, that is, during the period from 2014 to 2018. This means that I cannot claim the review is comprehensive because I selected papers published in only one journal – albeit a journal that has been identified as one of the most highly cited and respected in our field (Williams & Leatham, 2017). *ESM* also has a strong historical connection with PME (Hanna & Sidoli), which makes it an appropriate source of papers for a review presented at this conference. Neither is it possible for the review to identify meaningful trends over time, given the five-year time frame I have selected. Nevertheless, the findings can be interpreted in terms of trends and research themes discussed in mathematics education

research handbooks published in the last fifteen years, giving an up-to-date snapshot of the field and how it might develop.

In conducting the review, I drew on my experience as a member of the ICME 13 Survey Team on “Teachers Working and Learning Through Collaboration” (Jaworski et al., 2017; Robutti et al., 2016) by adapting the Survey’s general framework and methodology. The review is organised around the following sections. First, I consider the meaning of “mathematical thinking” and delineate the parameters and particular aspects that guided the literature search. The next section outlines the methodology used for the review – how the sources were selected, organised, and analysed. The findings are then presented in response to the broad research questions I formulated to structure the analysis:

1. What were the different contexts and features of studies investigating how to understand and promote students’ mathematical thinking?
2. What theories and methodologies framed the studies?
3. What is the contribution to knowledge made by this body of work, and what future research directions are indicated?

## **MEANING OF MATHEMATICAL THINKING**

Mathematical thinking is considered to be an important goal of schooling across the world, but is difficult to define in just a few words. Mathematical thinking gives attention to process rather than content, although both are clearly important for learning mathematics and both are typically represented in school mathematics curricula. Insights into the nature of mathematical thinking can be gained from examining research frameworks and curriculum frameworks that attempt to delineate its salient features.

In the 1980s, research focused attention on the processes involved in mathematical thinking, particularly in relation to mathematical problem solving. For example, Schoenfeld (1992) developed a framework for mathematical thinking that included mathematical knowledge and heuristics, metacognitive knowledge and control to guide problem solving activity, beliefs and affects and how these are influenced by the instructional environment. Ways of engaging in, and promoting, mathematical thinking while solving problems were addressed in a practical way by Mason, Burton, and Stacey (1985), whose book titled *Thinking mathematically* identified two pairs of fundamental processes – specialising and generalising, and conjecturing and convincing.

While research on mathematical problem solving flourished in the 1980s, recent reviews have lamented “the lack of impact and cumulativeness” (Lesh & Zawojewski, 2007, p. 763) of research in this area, noting in particular that the literature on mathematical problem solving has not produced clear guidelines for school practice (English & Gainsburg, 2016). Many reasons are suggested for these disappointments and shortcomings, including: uncertainty over what might be the most fruitful

theoretical perspectives for understanding and promoting problem solving, the breadth of the domain and the consequent difficulties in defining what is meant by “problem solving”, cyclic trends in education policy that lead to shifts in emphasis between problem solving and basic skills, and lack of agreement about the overarching goal of including problem solving in the mathematics curriculum. Nevertheless, research interest in mathematical problem solving has been sustained over the decades since the 1980s. For example, research handbooks regularly include chapters on problem solving research (e.g., English & Gainsburg, 2016), sometimes expanding the field to include perspectives on mathematical modelling (e.g., Lesh & Zawojewski, 2007) or problem posing (e.g., Weber & Leikin, 2016).

Particular emphases in relation to mathematical thinking can be discerned in international assessment programs and curriculum frameworks. For example, the Programme for International Student Assessment (PISA) uses the term “mathematical literacy” in its assessment of the ability of 15-year-old students to apply mathematics in real world contexts. The PISA definition of mathematical literacy focuses on the processes of formulating situations mathematically; employing mathematical concepts, facts, procedures and reasoning; and interpreting, applying and evaluating mathematical outcomes (OECD, 2013). This definition gives attention to reasoning as an element of mathematical thinking alongside problem solving in applied contexts. Problem solving and reasoning are also given prominence in curriculum frameworks, such as the *Principles and Standards for School Mathematics* developed in the US by the National Council of Teachers of Mathematics (2000). In this document, conjecturing, justification, and argument are identified as fundamental reasoning processes that all students can learn, and that can ultimately be expressed in a formal way as mathematical proof. A contrasting approach to curriculum design for promoting mathematical thinking is illustrated by the Singapore mathematics framework, which since 1990 has been centrally focused on problem solving. In its current version the Singapore framework additionally identifies reasoning, communication and connections, as well as applications and modelling, as important processes for learning (Ministry of Education, 2012).

The *Adding it up* report prepared by the US National Research Council (2001) introduced the notion of “mathematical proficiency” to propose a comprehensive view of what is necessary for all students to learn mathematics successfully. Mathematical proficiency is considered to have five interwoven and interdependent strands:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems
- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification

- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy. (p. 116)

The notion of mathematical proficiency has been taken up in mathematics curriculum frameworks in various countries as a way of specifying the process domain for mathematics learning (e.g., in Australia, see ACARA, n.d.; in Ireland, see NCCA, 2015). Although all five strands arguably have a bearing on mathematical thinking, it is the strands of *strategic competence* and *adaptive reasoning* that seem to be most closely related to mathematical thinking as conceptualised by previous research. It is noteworthy that strategic competence includes the ability to pose and formulate problems as well as solve them, and that adaptive reasoning refers to intuitive and inductive reasoning as well as informal explanation and justification, formal proof and other forms of deductive reasoning. Research interest in mathematical reasoning reflects this breadth of emphasis, from development of conceptual frameworks for understanding students’ conceptions of proof (e.g., Harel & Sowder, 2007) to investigations of instruction that supports students’ informal reasoning through explanation and justification (e.g., Yackel & Hanna, 2003).

In light of this overview of research into mathematical thinking, I decided to focus my review of *ESM* papers published from 2014-2018 on those that were concerned with *understanding* or *promoting* students’ mathematical *problem solving* and mathematical *reasoning*. These dimensions guided the literature search, further details of which are presented in the next section.

## METHODOLOGY OF THE REVIEW

Sources were identified from a manual online search of papers published in *ESM* from 2014 to 2018, that is, Volumes 85(1) to 99(2). (Volume 99(3) had not yet been published at the time of writing this paper.) Excluded from the search and subsequent analysis were Editorials, book reviews, errata, announcements, and introductory and commentary papers for special issues. I briefed a research assistant on searching the titles, keywords and abstracts of papers for a broad range of terms including problem solving, problem posing, reasoning, proof, argumentation, explanation, justification, generalisation, and abstraction. Information about the papers so identified was entered into a spreadsheet with columns that allowed for recording of citation details (including hyperlinks to the online versions of the papers), and organised information addressing the first and second research questions concerning contexts and features of the studies and the theories and methodologies that framed them.

In analysing contexts and features I was interested in capturing the geographical region in which the study was conducted (or the author’s country of affiliation if it was a theoretical rather than empirical study), the target educational level (primary school, secondary school, vocational education, tertiary study of mathematics, teacher education), the scale of the study (duration and number of participants), the overarching research aim (to understand or promote mathematical thinking), and the research focus (type of mathematical thinking, e.g., problem solving or reasoning). I

also classified the research orientation of each paper in response to a significant theme that emerges in recent research literature on mathematical problem solving (English & Gainsburg, 2016; Weber & Leikin, 2016). This theme distinguishes between research in which problem solving is the object of study and that which uses problem solving as a research tool to investigate other aspects of mathematics learning, such as understanding or teaching of mathematical concepts. I also used this classification to analyse papers about mathematical reasoning.

As the search progressed I met frequently with my research assistant to screen the papers that had been entered into the spreadsheet, removing any that did not seem to fit the search criteria and flagging those about which I was uncertain. This process of screening and discussion also enabled me to clarify the review's dimensions and articulate what seemed to be important distinctions, such as the difference in emphasis between investigating mathematical thinking processes as the object of study and using problem solving or reasoning as a tool for teaching mathematical content. When the initial search was completed, I skimmed the full version of every paper recorded in the spreadsheet and removed any that did not fall within the scope of the review. This process resulted in further clarification of its dimensions; for example, the focus on students' mathematical thinking rather than teachers' pedagogical reasoning, and removal of papers about concept formation via reflective abstraction.

The search and screening process yielded 55 papers, representing 20% of the 269 papers published in *ESM* in the five years from 2014 to 2018.

## THEMES FROM THE ANALYSIS

Findings are presented as sets of themes corresponding to my three research questions.

### Theme 1: Contexts and Features

The *geographical distribution* of papers is represented in Figure 1. This graphic excludes the two papers that reported on a comparison between two countries (in Europe/Asia and in North/South America). To understand whether this distribution was representative of all papers published in *ESM*, I carried out a similar calculation for papers accepted for publication in the years 2014-2017, using data from the annual publisher's report (Table 1; publication data were not yet available for 2018). This comparison revealed a slightly greater concentration of papers on mathematical thinking coming from Asia, Australia/New Zealand, North America, and South America, and a much lower concentration of papers from Europe.

The *research aims* were classified as being either to understand or to promote mathematical thinking, with the latter type of studies typically involving an educational intervention of some kind. The two types of research aims were fairly evenly distributed, with 32 studies (58.2%) classified as seeking to understand, 22 (40.0%) seeking to promote, and 1 (1.8%) aiming both to understand and promote mathematical thinking. As an example of the first kind of aim, the study by Heino (2015) sought to understand how Japanese secondary school students attended to and

compared multiple solutions proposed by their classmates in structured problem solving lessons. The study reported by Mata-Pereira and da Ponte (2017) exemplifies the second kind of aim, describing principles for design research where whole class mathematical discussions were conducted in order to enhance primary school students' mathematical reasoning processes.

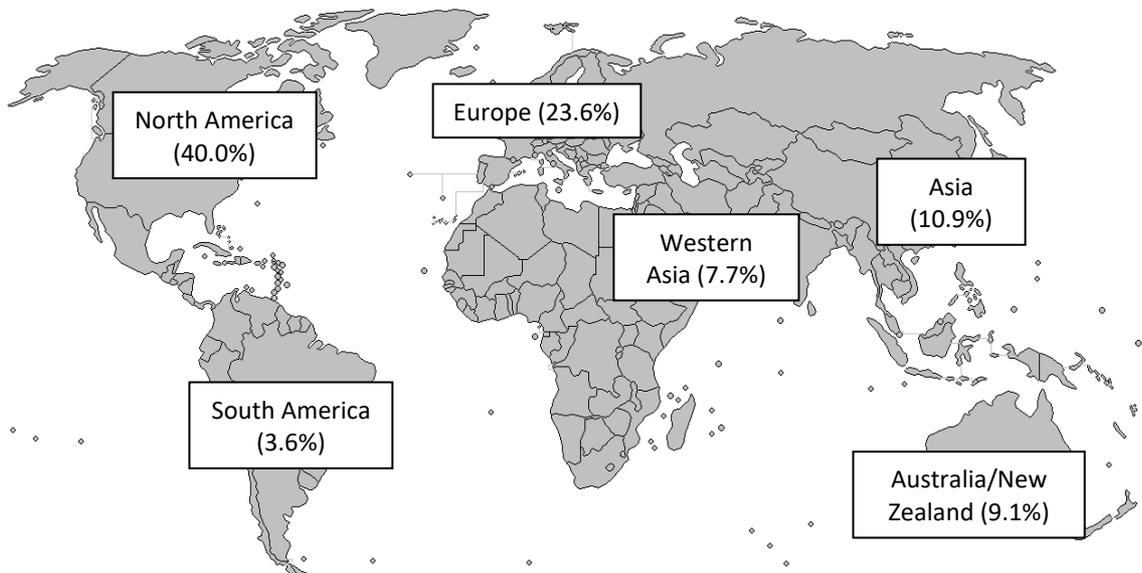


Figure 1: Geographical distribution of papers on mathematical thinking published in *ESM* from 2014-2018

Geographical region	Number and percentage of accepted papers 2014-2017
Europe	99 (41.1%)
North America	90 (37.3%)
Western Asia	18 (7.5%)
Australia/New Zealand	16 (6.6%)
Asia	11 (4.6%)
South America	4 (1.7%)
Africa	3 (1.6%)

Table 1: Geographical distribution of papers accepted for publication in *ESM* from 2014-2017

A visual impression of the *research focus* of the set of papers can be gained by inspecting the word cloud created from their titles shown in Figure 2. This graphic depicts words with frequencies greater than or equal to 3. The most frequent word used in the titles was “student” (appearing 20 times), followed by the words “problem”,



problem solving and problem posing and how these interactions supported pre-service primary teachers' conceptual understanding of fractions.

A different pattern with respect to research participation was observed in the studies that had reasoning and proof as their focus. Those classified as focusing on proof mainly involved secondary school students, university undergraduate students and lecturers, while primary and secondary school students were the most frequent participants in studies of mathematical reasoning. This finding is consistent with research themes identified in recent handbooks and curriculum frameworks, which give equal attention to informal and formal reasoning processes, and teaching approaches that develop children's reasoning capabilities. For example, at the primary school level, Downton and Sullivan (2017) conducted a study with 8- and 9-year-old children working on tasks that prompted multiplicative thinking. In contrast to this study, formal proofs were the subject of the research reported by Ramos and Weber (2014), who investigated how and why mathematicians read proofs.

I defined the *research orientation* of the papers in terms of whether mathematical thinking was the object of study or used as a research tool to investigate the learning or teaching of mathematical concepts (English & Gainsburg, 2016; Weber & Leikin, 2016). Papers were fairly evenly divided between these orientations, with 30 classified as treating mathematical thinking as the object of study and the remaining 25 papers using mathematical thinking as a research tool. Amongst the former category there was more of a research focus on proof (16 papers), with somewhat less attention given to problem solving (9 papers) and reasoning (5 papers) as objects of study. There were several theoretical papers in this group: for example, Simpson (2015) analysed a model solution to a proof question using Toulmin's scheme of argumentation in order to provide insight into what examiners might be expecting of students. In the latter category of papers, reasoning was overwhelmingly the focus (20 out of 25 papers): that is, reasoning tasks were used to investigate students' learning of mathematical concepts in different areas of the curriculum, such as statistics, algebra, number, and geometry. The large-scale professional development project conducted by Hilton, Hilton, Dole, and Goos (2016) exemplifies such studies. These researchers worked with middle school teachers to devise strategies for improving students' proportional reasoning abilities, and demonstrated the effectiveness of their approach through analysis of student pre- and post-test data.

Analysis of the *educational level* targeted by the sample of research papers revealed that almost one-third (17, 30.9%) reported on studies conducted with secondary school students, with 20% (11) targeting undergraduate mathematics students, nearly 15% (8) primary school students, and around 11% (6) both primary and secondary students. Smaller numbers of studies involved pre-service teachers (4, 7.2%), vocational education students (1, 1.8%), combinations of pre-service teachers and secondary students (1, 1.8%), or a mix of teachers and non-teaching adults (1, 1.8%). The remaining papers (10.9%) were theoretical pieces that did not draw on empirical data.

I analysed the *scale* of the research sample by recording the duration of the empirical studies and the number of participants. It was surprising to find that only two studies lasted longer than a year when a large proportion (40%) of the studies were aiming to promote mathematical thinking. This finding suggests that short-term interventions were a common approach, and raises questions about the enduring impact of such research. The number of participants varied widely: small studies involving up to 10 people (5, 9.1%), medium sized studies with 11-50 (19, 34.6%) or 51-100 participants (8, 14.6%), and large studies with more than 100 participants (14, 25.5%). This variation suggests that diverse qualitative and quantitative methodologies were being used to interpret the data gathered from participants.

## **Theme 2: Theories and Methodologies**

One of the criticisms of research in this field is the lack of a strong theoretical base (Lesh & Zawojewski, 2007). The proliferation of theories in mathematics education in general has also been identified as a challenge for our research community (Prediger, Bikner-Ahsbals, & Arzarello, (2008). This diversity was evident in the 55 papers selected for analysis, and so it is difficult to draw conclusions about the major theoretical perspectives that supported researchers to study mathematical thinking. Although it was possible to recognise broader families of theories emanating from cognitive or constructivist or sociocultural standpoints, it seemed that the theories in use were developed for specific purposes and often combined with other specific theories in order to illuminate a particular phenomenon. For example, Alberracin and Gorgorio (2014) drew on several theoretical ideas and bodies of literature about problem solving (Polya's framework), estimation and representational models, and Fermi problems to ground their study of the plans made by secondary school students for solving Fermi problems involving large numbers. Two papers, by Fiallo and Gutierrez (2017) and Johnson and McClintock (2018), showed evidence of an explicit attempt to network theories in the manner proposed by Prediger et al. (2008) in order to combine different theoretical ideas in a principled way. This approach shows promise of bringing some coherence to a very diverse theoretical landscape.

It was an easier task to classify the methodologies used in the selected papers. Nearly one-third (17, 31%) used a form of classroom intervention, design experiment, or teaching experiment that involved analysis of lesson video-recordings, interviews with teachers and students, or students' written work. A further 6 studies (10.9%) adopted an experimental or quasi-experimental design. Studies that aimed to promote mathematical thinking typically used one of the latter two methodological approaches. Other common approaches, most often associated with the aim of understanding mathematical thinking, included analysis of student written work (without being part of a classroom intervention; 7 studies, 12.7%), clinical or task-based interviews (6 studies, 10.9%), other interviews or surveys (3 studies, 5.5%), and case study (3 studies, 5.5%). Two studies (3.6%) were cross-national comparisons: one investigated problem solving strategies used by Chinese and Singaporean students (Jiang, Hwang, & Cai, 2014) and the other compared the nature of proof taught in secondary school

geometry classes in France and Japan (Miyakawa, 2017). One interesting observation arising from my analysis was that for many papers it was often difficult to discern either the theoretical or methodological approach just from reading the abstract.

### **Theme 3: Contribution to Knowledge and Future Research Directions**

The foremost impression I gained from surveying the set of papers selected for this review was of the sheer diversity in research focus, scale, educational level, theoretical perspectives and methodological approaches. One could view this diversity as productive, suggesting a broad interest around the world in understanding and promoting mathematical thinking in all its guises, and for all students. Alternatively, too much diversity makes it difficult to synthesise findings, identify the most useful theories, and generate guidelines for classroom practice – adding fuel to existing criticisms concerning the lack of impact and cumulativeness of research in this field (English & Gainsburg, 2016; Lesh & Zawojewski, 2007).

Despite these concerns, I can identify some theoretical, methodological, and practical contributions to knowledge emerging from the research I reviewed for this paper. The first theoretical contribution comes from frameworks that organise ideas in the field in new ways in order to improve our understanding of mathematical thinking and how it can be promoted. One example of such a framework is provided in the study by Jeannotte and Kieran (2017), which developed a conceptual model of mathematical reasoning (MR) for school mathematics. These authors noted that although curricula around the world identify mathematical reasoning as an important goal of schooling, the way in which reasoning is described in these documents “tends to be vague, unsystematic, and even contradictory from one document to the other” (p. 2). Jeannotte and Kieran conceptualised mathematical reasoning as a discursive activity, using the commognitive framework of Sfard (2008) to construct

...a coherent theoretical model that synthesizes and builds upon the convergences to be found in the main types and characteristics of MR described in the mathematics education research literature and that can thereby serve as a conceptual tool for both teachers and researchers. (p. 4)

The resulting model highlights the dialectical relationship between structural and process aspects of mathematical reasoning, with the former aspect foregrounding deductive, inductive and abductive modes of inference and the latter aspect identifying processes of searching for similarities and differences, validating, and exemplifying. Interestingly, Jeannotte and Kieran were clear to state that they did not create their model in order to provide practical advice on classroom tasks to encourage development of mathematical reasoning. Instead, the aim was to improve communication within the field by promoting a common discourse.

My second example of a new theoretical framework incorporates the dual aims of understanding and promoting mathematical thinking. Dawkins and Weber (2017) developed a framework for conceptualising proof in terms of mathematical values and norms. Motivated by the observation shared by many researchers that it is difficult to

foster classroom proving practices, they drew on philosophical and sociocultural writings to identify epistemic values held by the community of mathematicians and discuss how norms with respect to proof and proving can work to uphold these values. The four values identified were expressed as follows:

- (1) Mathematical knowledge is justified by a priori arguments.
- (2) Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
- (3) Mathematicians desire to increase their understanding of mathematics.
- (4) Mathematicians desire a set of consistent proof standards. (p. 128)

Norms claimed to uphold each of these values were also identified. The authors' central argument was that "students are being asked to adopt mathematicians' proof norms, but students may not perceive the mathematicians' values that those norms are intended to uphold" (p. 133). They went on to discuss possible reasons for students' misunderstanding of or resistance to mathematical norms, and for mathematicians' views that students are incapable of producing proofs. They concluded that many of the challenges of proof instruction could be understood by acknowledging that teachers and students in these classrooms are engaging in cross-cultural interactions underpinned by distinct, and differing, sets of values.

An innovative methodological contribution to advancing research on mathematical thinking was found in the study by Bruce et al. (2017). The study used network analysis to understand current patterns of communication across the fields of Education, Mathematics, Psychology, and Neuroscience in research on spatial reasoning. The analysis identified connection gaps, that is, blockages or other limitations on communication between the disciplines, that the authors – a multidisciplinary team – suggest might be frustrating efforts to understand and promote spatial reasoning. The methodology used citation analysis of 7200 articles to create a visual representation of the distance-based citation network connecting the articles. The analysis pinpointed weak bidirectional information flow between Education and the other three disciplines, and the authors presented case studies of research related to each discipline to illustrate some negative consequences of this lack of flow for development of the field of mathematics education, and especially spatial reasoning. They argued that transdisciplinary research is needed to close these connection gaps so that researchers can effectively address complex issues in the teaching and learning of mathematics.

I selected two papers from my sample that illustrate the potential for research on mathematical thinking to make practical contributions to knowledge and impact positively on classroom practice. Both feature interesting approaches to assessing students' reasoning or problem solving. The first study, by Hilton et al. (2016), has already been mentioned. This was a two-year professional development study involving more than 130 primary and secondary school teachers and their students, with the focus on improving students' proportional reasoning skills. An innovative

assessment approach involving development of a two-tier diagnostic instrument was used to collect baseline data on students' proportional reasoning skills at the start of the study and at the end of the first year, for students in participant ( $n=1026$ ) and control ( $n=277$ ) classes in Years 5, 6, 7 and 8. The first tier of each item required a true-false response, while in the second tier students were asked to choose from four possible reasons for their first tier response. The options in the second tier were based on research literature concerning students' proportional reasoning errors and the findings from previous studies undertaken by this research team. While there were some differences between results for the various Year levels, in general the participant and control classes had similar pre-test scores while the participant groups recorded statistically significant higher scores than the control groups for the post-test. In addition, mean post-test scores for the participant classes were beyond the pre-test scores of control group students at least two years older. This was an example of design-based research with multiple cycles of design, enactment and evaluation to promote change in teachers' knowledge and classroom practices. As well as presenting evidence from the diagnostic assessment instrument the paper provides a detailed account of the professional development approach. The study therefore contributes to the limited literature on the effect of teacher professional development on mathematics students' learning.

The second example of potential for practical impact in research on mathematical thinking comes from a study by Jones and Inglis (2015). These authors pointed out that traditional examination papers comprise mainly short, structured items that are not suited to assessing students' reasoning or problem solving. Such examinations have high reliability (via a specified marking scheme) but low validity when it comes to assessing problem solving activities. The authors worked with four experienced examination paper writers to produce a paper that aligned with mathematics curriculum expectations but deliberately did not have a marking scheme. The paper was administered to 750 secondary school students, whose work was assessed by 20 mathematics education professionals with teaching experience ranging from one to more than ten years. The markers had been trained to use comparative judgment, an alternative to traditional marking involving pairwise global judgments of the quality of students' work. The study found that the examination paper writers were able to design more open-ended, less structured questions and that the comparative judgment approach yielded assessments that were both reliable and valid. Jones and Inglis speculated that, in addition to impacting on the design and assessment of written examination papers, comparative judgement has potential as a teaching tool if used to encourage discussion about what makes a good solution to an unstructured problem.

While the discussion of examples in this section has focused on contributions to knowledge, possibilities for future research directions are also suggested by the findings summarised here. However, additional possibilities for research arise from what is *not* represented in my five-year *ESM* sample of 55 papers. The conference First Announcement refers to making mathematics education "more effective, but also more

inclusive and equitable”. It would be fair to say that all the papers I identified were concerned with effective learning and teaching, but I am not sure that my review strategy and its design parameters were able to identify studies of mathematical thinking with a primary aim of making learning inclusive and equitable. Neither were studies of “applied” mathematical thinking captured by my review methodology. By this I mean studies of mathematical modelling, statistical literacy, numeracy (all topics of papers recently published in *ESM*), or so-called 21<sup>st</sup> century competencies needed for dealing with the ill-defined problems of modern work and life (English & Gainsburg, 2016). Studies in these fields are also introducing socio-political concerns with respect to critical thinking in workplace settings and everyday life, the role of digital technologies in affording new problem solving and reasoning strategies, and interdisciplinary problems that require synthesis of knowledge across the STEM domains (science, technology, engineering, and mathematics). These emerging trends towards studying mathematical thinking in real world contexts might move researchers to focus more explicitly on issues of inclusivity and equity in preparing learners for lives beyond school.

## CONCLUDING COMMENTS

This review of research on mathematical thinking published in *Educational Studies in Mathematics* from 2014-2018 is necessarily brief and limited by my methodological choices and my own understanding of the field. In addition to the tentative conclusions I have offered about the contributions and future directions of research in this field, I conclude with some practical observations directed at authors of research papers. A review of research literature begins with a search of the titles, key words, and abstracts of published papers to identify sources of possible interest. It was surprising to find that there was often little alignment, and sometimes even contradiction, between these three important features that together communicate what a paper is about. It was also common to find little or no information in the abstract about the theoretical perspective informing the study or the methodological approach that was taken. Not only does this observation remind me, as a journal editor, of the importance of checking the quality of abstracts of submitted manuscripts, but it should also encourage authors to ensure that they create informative titles, key words, and abstracts so readers – and reviewers – can easily judge the relevance of papers to their own research interests.

## Acknowledgement

I am grateful to Sila Kaya for her assistance in searching for and organising the papers that informed this review.

## References

Alberracin, L. & Gorgorio, N. (2014). Devising a plan to solve Fermi problems involving large numbers. *Educational Studies in Mathematics*, 86, 79-96.

- Australian Curriculum, Assessment and Reporting Authority (ACARA) (n.d.). *Mathematics proficiencies*. Retrieved 12 October 2018 from <https://www.australiancurriculum.edu.au/resources/mathematics-proficiencies/>
- Bruce, C., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., ... Woolcott, G. (2017). Understanding gaps in research networks: Using “spatial reasoning” as a window into the importance of networked educational research. *Educational Studies in Mathematics*, 95, 143-161.
- Dawkins, P., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, 95, 123-142.
- Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, 95, 303-328.
- English, L., & Gainsburg, J. (2016). Problem solving in a 21<sup>st</sup>-century mathematics curriculum. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3<sup>rd</sup> ed.) (pp. 313-335). New York: Routledge.
- Fiallo, J., & Gutierrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics*, 96, 145-167.
- First PME Regional Conference: South America (2018). *First announcement*. Retrieved 12 October 2018 from <https://drive.google.com/file/d/18S3CPSpinBDec2vKxPR7IosLuwpKeXGM/view>
- Hanna, G., & Sidoli, N. (2002). The story of ESM. *Educational Studies in Mathematics*, 50, 123–156.
- Harel, G., & Sowder, L. (2007). Towards comprehensive perspectives on the learning and teaching of proof. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805-842). Reston, VA: NCTM.
- Heino, K. (2015). Comparing multiple solutions in the structured problem solving: Deconstructing Japanese lessons from learner’s perspective. *Educational Studies in Mathematics*, 90, 121-141.
- Hilton, A., Hilton, G., Dole, S., & Goos, M. (2016). Promoting middle school students’ proportional reasoning skills through an ongoing professional development programme for teachers. *Educational Studies in Mathematics*, 92, 193-219.
- Jaworski, B., Chapman, O., Clark-Wilson, A., Cusi, A., Esteley, C., Goos, M., ... Robutti, O. (2017). Mathematics teachers working and learning through collaboration. In G. Kaiser (Ed.), *Proceedings of the 13<sup>th</sup> International Congress on Mathematical Education, ICME-13 Monographs*, DOI 10.1007/978-3-319-62597-3\_17 (pp. 261-276). Springer Nature.
- Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics*, 96, 1-16.

- Jiang, C., Hwang, S., & Cai, J. (2014). Chinese and Singaporean sixth-grade students' strategies for solving problems about speed. *Educational Studies in Mathematics*, 87, 27-50.
- Johnson, H., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics*, 97, 299-316.
- Jones, I., & Inglis, M. (2015). The problem of assessing problem solving: Can comparative judgement help? *Educational Studies in Mathematics*, 89, 337-355.
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Reston, VA: NCTM.
- Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically* (rev. ed.). Wokingham, UK: Addison-Wesley.
- Mata-Pereira, J., & da Ponte, J.-P. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics*, 96, 169-186.
- Ministry of Education (2012). *Mathematics syllabus secondary one to four: Express course, normal (academic) course*. Singapore: Ministry of Education.
- Miyakawa, T. (2017). Comparative analysis on the nature of proof to be taught in geometry: The cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics*, 94, 37-54.
- National Council for Curriculum and Assessment, Ireland (2015). *Leaving certificate mathematics syllabus: Foundation, ordinary and higher level*. Retrieved 10 October 2018 from [https://curriculumonline.ie/getmedia/f6f2e822-2b0c-461e-bcd4-dfcde6decc0c/SCSEC25\\_Maths\\_syllabus\\_examination-2015\\_English.pdf](https://curriculumonline.ie/getmedia/f6f2e822-2b0c-461e-bcd4-dfcde6decc0c/SCSEC25_Maths_syllabus_examination-2015_English.pdf)
- National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Research Council (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Washington, DC: National Academy Press.
- Organisation for Economic Cooperation and Development (OECD) (2013). *PISA 2015 draft mathematics framework*. Retrieved 12 October 2018 from <https://www.oecd.org/pisa/pisaproducts/Draft%20PISA%202015%20Mathematics%20Framework%20.pdf>
- Prediger, S., Bikner-Ahsbals, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education*, 40, 165-178.
- Ramos, J., & Weber, K. (2014). Why and how mathematicians read proofs: Further evidence from a survey study. *Educational Studies in Mathematics*, 85, 161-173.
- Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., ... Joubert, M. (2016) ICME international survey on teachers working and learning through collaboration. *ZDM Mathematics Education*, 48, 651-690.

- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: McMillan.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.
- Simpson, A. (2015). The anatomy of a mathematical proof: Implications for analyses with Toulmin's scheme. *Educational Studies in Mathematics*, 90, 1-17.
- Weber, K., & Leikin, R. (2016). Recent advances in research on problem solving and problem posing. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 353-382). Rotterdam, The Netherlands: Sense Publishers.
- Williams, S., & Leatham, K. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education*, 48, 369–396.
- Xie, J., & Masingila, J. (2017). Examining interactions between problem posing and problem solving with prospective primary teachers: A case of using fractions. *Educational Studies in Mathematics*, 96, 101-118.
- Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 227-236). Reston, VA: NCTM.

# CLASS OBSERVATION TO ENRICH STUDENT THINKING ON MATHEMATICS INSIDE THE CLASSROOM

María Victoria Martínez Videla

Center for Advanced Research in Education, Universidad de Chile

*We present the design of an observation protocol as a tool to observe mathematics classes, work on teacher feedback and elaborate teacher professional development plans. This is part of the work made with mathematics teachers from public schools in Chile, and along a bilateral project developed with a Mexican team. We share the construction and validation processes of an observation protocol for mathematics classes, detailing the steps and giving an example of use.*

## INTRODUCTION

This paper aims to share part of the development made on mathematics class observation by different projects carried out by a group of researchers at Universidad de Chile's Advanced Education Research Center (CIAE, after its acronym in Spanish) and other associated centers.

Most of such development relates to the feedback coming from the class observation process; however, in this paper we will focus on the development, construction and validation of certain tools that allow mathematics classroom observation, and the way these tools may define a basic structure for teacher accompaniment.

We will base this on the work developed by two CIAE's projects:

- Mejor Matemática (Better Mathematics) (2015-2018). This teacher accompanying program was developed by the Mathematical Modeling Center at Universidad de Chile and the Ministry of Education to improve mathematics class practices by accompanying teachers at the school.
- "The challenge of teaching mathematics to primary and secondary school students in the first years of the teaching career in Chile and Mexico" (2016-2018). This Project was developed by the Mexican Education Evaluation Institute (INEE, after its acronym in Spanish) and the Chilean Ministry of Education and was financed by the Chile-Mexico Joint Cooperation Fund. Its main goal was to develop tools that allow for the classification of initial mathematics teacher's performances in the classroom in Chile and México, having context in mind.

## CLASS OBSERVATION AND TEACHER ACCOMPANIMENT

Class observation has historically been a crucial instrument for approaching the classroom. Teachers' professional training always considers an observation period to

learn from other teachers: “First, I watch; then I do” (Grossman et al., 2009; Anijovich, 2009).

On the other hand, over the last decades, research has approached what students learn by directly observing other teachers. Several tools have been developed to describe the work in the classroom. We can mention CLASS (Pianta et al., 2004; Pianta et al., 2012), Framework for Teaching and Tripod 7Cs (Ferguson & Danielson, 2014), which are some of the tools created to analyze effective practices in teaching in general, while other tools allow for the observation of specific classes, such as PLATO for language class (Mihaly & McCaffrey, 2014), MQI for mathematics class (Hill et al., 2008; Hill et al., 2012) , and QST for sciences class (Schultz & Pecheone, 2014). Part of the advances in building such instruments has been possible because of the research efforts to identify those characteristics of teaching and pedagogical interactions inside the classroom that make children have more and better learning.

Another goal of class observation is evaluation. In Chile a standard example is the Evaluación Docente (Teacher Evaluation System). One of the requirements is to record a class of about 40 minutes, which is assessed considering the explanations, feedback, monitoring, beginning and ending, among others (Manzi et al., 2011).

At CIAE, we approach the classroom only for research, to understand and describe the work inside the classroom. We use such tools as CLASS and MQI (Martínez, Godoy, Treviño, Varas and Fajardo, 2018; Godoy, Martínez, Varas, Treviño and Meyer, 2016). We consider class observation as an important element that allows us to work in a formative and thoughtful way with teachers to improve the work in the classroom.

Mathematics classrooms are complex. Teachers have the responsibility to work on certain contents that have been specified by the curriculum and must get involved in developing competences that allow the students to carry out their present and future lives in a constructive, committed and thoughtful way (OCDE, 2004). Therefore, it is important to constantly work on strategies that allow us to achieve that goal, especially when most of the teachers have been formed under a traditional and focused-on-the-content paradigm, instead of a skill developing paradigm.

According to the literature, one of the most effective elements is feedback of teachers' work in the classroom. In this context, from the comparative evidence by the World Bank (2012), two critical policies relating to the teachers' professional development based on individual accompaniment and formation of learning communities have been identified: i) to observe teaching and learning: to monitor how teachers are teaching is essential (Grossman et al., 2010); ii) to support teachers to improve instruction, to move to a professional development that exposes them to better practices and offers them clear guidance on how to implement those practices (Rockoff, 2008). These results are consistent, as per the evidence gathered by Hattie (2012) regarding the most relevant factors to improve teachers' work and learning. Within the framework of several meta-analysis studies, giving teachers formative evaluation and feedback and

the conformation of effective teacher communities are among the greatest impact strategies.

Given the infinity of elements that define the teaching and learning processes within the classroom, and understanding the importance of the practice feedback in improving teachers' work, the following questions are crucial:

- What do we have to observe in the mathematics class to narrow up class observation and feedback?
- How do observed elements allow us to enrich classroom work and learning improvement?
- How can we define such elements to create a common language with teachers, so that they identify such elements in their practice and can think about them?

These are key questions because different authors underline the importance of defining a starting point, an ideal, and especially a route, so that the feedback process is effective. That's why observation instruments play a key role in reflecting on the mathematics classroom, especially when such work aims to improve the practices, to enrich the students' learning. In this paper, we will pay attention to the construction and validation of a class observation instrument that allows us to address these questions.

## **HOW TO OBSERVE CLASSES? CONSTRUCTION AND VALIDATION OF AN OBSERVATION PROTOCOL**

The use of class observation instruments is a way to define the work of observing a classroom. It defines what we will consider important and a way to watch it, by using a checklist or a rubric. On the other hand, the content of such an instrument may vary. It can focus on a specific area of knowledge or on elements relating to the classroom working, regardless the class.

The following is a review of the importance of defining a class observing instrument, within the context of teacher feedback and we describe the construction and validation of an instrument as an example.

Recognizing that mathematics learning is linked to multiple factors, one of the most important factors without a doubt relates to the role teachers play in such a process. The Chile-Mexico project focuses on the study of teachers' work in the classroom, specifically on beginning teachers (0 to 5 years of expertise). Why only these teachers? To get to know them better and accompany them in their professional development on the base of evidence. It is well known that beginning teachers test what they have learnt during their starting formation almost always alone (Marcelo, 2009) and, besides, those who are just starting their professional lives are sent to schools who show difficult conditions for teaching, such as geography, infrastructure, and the characteristics of the population attending to it (Toledo & Valenzuela, 2012).

The class observation protocol is based on two main ideas. The first one is that teaching practices do not stay still during the years; on the contrary, they constantly transform

(Llinares, 2000; Perrin-Glorian, Deblois & Robert, 2008). What practices, and how much they change depends on personal and institutional factors. All we know is that changes do not address only one direction, and that they happen at different rhythms (Reis & Climent, 2012; Sandoval, 2009; Marcelo, 2009; Borko, Koellner & Jacobs, 2014). The second idea was taken from the evidence developed by didactics of mathematics: There are more relevant teaching practices to promote student's superior knowledge and thinking skills. These practices require teaching knowledge and skills very specialized in mathematics and, besides, they result from experience, and conform knowledge aspects for mathematics teaching that are more difficult to develop, even more than the ability to solve mathematics problems (Hill, 2010)

Having this in mind, the observation protocol attempts to achieve the objectives of the research team: to be mathematics specific, that can be used on all basic level years, that allows us to get information about the mathematics teaching practices of initial teachers and to identify their evolution, the same as to be in accordance with the Chilean Framework of Good Teaching and with the Mexican Profiles, Parameters and Indicators.

### Construction and validation phases

Constructing a class observation protocol requires the development of a series of actions so that the tool can be statistically validated and, thus, it requires a methodological design that must be thorough and planned. It also must be a trustworthy instrument in terms of usefulness and sense.

For the development of this protocol, that we have called Promate, we have carried out a series of actions organized in construction phases (Table 1), for 2 years, so that we got a valid and trustworthy instrument, that represents a contribution to the educational system.

Year	Phase	Description
2017	Checking literature and observation protocol	Checking literature on mathematics teaching and learning and beginning teachers. Analysis of observation protocols of classes in general, mathematics classes, and others.
	1 <sup>st</sup> Version	
	Panel of experts to validate the content	2 panels in Chile and 2 panels in Mexico. One of them was formed by experts in works from the initial formation, and another by mentor teachers in the system. They worked on content validation, from which modifications were added.

2 <sup>nd</sup> Version		
Training and certification of encoders	Training for encoders is designed and validated, based on knowledge from the protocol and on training to encode using videos. Then, they get certification to encode.	
Panel with encoders	Interview to encoders to get feedback about the use of the protocol, as they are experts in the use of the instrument.	
Analysis of the results of coding	The analysis of the coding was based on G Theory, and the results were complemented with the panel of encoders to make a third version.	
3 <sup>rd</sup> Version		
2018	Panel 2 of experts to validate content	Panels of experts validate the content of 3rd version.
	2nd training and certification of encoders	Encoders are trained again on the changes made to the instrument, and they get certification in the use of a new version of the protocol
	Analysis of the results of coding	Repetition of the analysis and adjustment of dimensions.
4 <sup>th</sup> Version		

Table 1: Actions to construct and validate the mathematics class observation protocol Promate.

### Methodology and Statistical Validation

To carry out statistical validation of the protocol, 2 consecutive classes of 60 mathematics teachers of different levels (Table 2) from primary schools who had between 1 to 5 years of experience (Table 3) were recorded on video. Except for two cases, two classes of each teacher with a duration between 45 and 90 minutes were recorded. Tables 2 and 3 show the distribution of such recordings:

	Chile	Mexico	Total	
Recorded level	1 <sup>st</sup>	2	4	6
	2 <sup>nd</sup>	5	5	10
	3 <sup>rd</sup>	6	5	11
	4 <sup>th</sup>	3	1	4
	5 <sup>th</sup>	0	1	1
	6 <sup>th</sup>	2	0	2
	7 <sup>th</sup>	7	10	17
	8 <sup>th</sup>	4	5	9
Total	29	31	60	

Table 2: distribution of recordings according to level and country

	Chile	Mexico	Total	
Years of experience	1 year	8	7	15
	2 years	3	4	7
	3 years	6	8	14
	4 years	4	5	9
	5 years	8	7	15
Total	29	31	60	

Table 3: distribution of recordings according to years of experience of the teachers

Every class was divided into 15-minute segments (+/- 1 min) and were encoded three times in a first stage, and two times in a second stage. Classes and encoder teams from each country were randomly assigned, and each encoder assessed two complete classes of one teacher.

We analyzed the reliability of the instrument with a study of *generalizability* to estimate the variance components in only one analysis (Shavelson & Webb, 1991; Brennan, 2001; Hill, Charalambous & Kraft, 2012). By considering the multiple sources of variability, we could establish how many observations per teacher, in how many classes and how many observers are necessary to get more precise scores and reduce the unwanted variance components.

The observation protocol is organized in two domains: General management of the class, and Management of mathematics teaching. The following is the variance breakdown in both domains, in Tables 4 and 5.

In addition, an analysis was conducted to determine the level of reliability of each dimension according to the number of observers and classes. In this analysis all the dimensions, except Mathematical errors and Mathematical closure, with 3 observers and two classes surpass a reliability of 0.5, enough for this type of instrument in this stage of its construction. Also, there is a tendency to increase reliability when the number of classes or observers increases.

For the two dimensions that do not reach the level of acceptable reliability, modifications have been proposed that we hope to validate in the short term.

Dimension / Component	Monitoring	Classroom climate	Use of time
Teacher	34.74	40.42	12.50
Observer	5.18	10.40	2.31
Segment (Class)	1.76	6.06	16.66
Class (Teacher)	6.55	1.57	6.82
O*T	13.41	13.95	1.56
O*(S(C))	0.05	4.71	0.00
O*(C(T))	4.14	3.59	0.00
Residual Not attributable	34.16	19.31	60.15

Table 4: percentage of variance explained by the different component in a generalizability study for the General management of the class domain

Dimension / Component	Mathematical language promotion	Diversity of representations	Diversity of procedures or strategies	Thinking promotion	Use of errors	Mathematical Errors	Use of productions*	Mathematical closure
Teacher	22.60	13.67	12.69	21.44	17.33	0.74	26.42	4.87
Observer	16.27	1.44	1.27	12.21	2.10	6.92	15.25	1.69
Segment (Class)	4.12	4.74	2.21	6.47	0.41	3.13	0.20	14.61
Class (Teacher)	7.57	11.56	12.44	12.68	6.10	14.76	2.69	4.77
O*T	12.35	16.35	9.46	9.42	17.64	8.18	5.42	1.17
O*(S(C))	0.41	6.10	12.00	0.00	7.78	2.67	0.00	0.00
O*(C(T))	1.49	4.76	4.57	0.00	2.54	3.40	28.59	0.00
Residual Not attributable	35.20	41.39	45.36	37.78	46.10	60.20	21.42	72.90

Table 5: percentage of variance explained by the different component in a generalizability study for the Management of mathematics teaching domain (\*Only with Mexican data)

## OBSERVATION PROTOCOL. A DIMENSION AS AN EXAMPLE

Promate protocol is organized into 2 blocks called domains. The first domain of *General management of the class*, includes aspects of teaching practice that set a minimum operating basis in any class and allows the promotion of opportunities for participation and integration of all students. The second domain of *Management of mathematics teaching*, focuses on aspects that allow the promotion of mathematical competences that are an objective for mathematics education in the Chilean and Mexican curriculum, as well as in an international framework such as the PISA test (OCDE, 2004).

On this logic the observation protocol is composed of a total of 11 dimension as we show on Table 6.

Each dimension is defined in the format of the rubric, in which there is a homogeneous format that distinguishes: name of the dimension, description of the aspects of the observation, the relationships of the observed elements, levels and criteria. In Table 7, we show one of the dimensions of the Management of Mathematics Teaching as an example.

<b>Domain</b>	<b>Dimension</b>
General management of the class	Monitoring student work
	Classroom climate
	Use of time for teaching and learning mathematics
Management of mathematics teaching	Mathematical language promotion
	Diversity of representations
	Diversity of procedures or strategies
	Thinking promotion about mathematics
	Use of errors and difficulties as a learning instance
	Mathematical errors
	Use of the student's mathematical productions
	Mathematical closure of the activity

Table 6: Domains y dimensions observation protocol Promate.

---

Thinking promotion about mathematics

---

On this dimension we observe if the teacher promotes thinking around the mathematics in his/her students.

It is considered that the teacher promotes thinking around mathematics when asking questions or requirements that meet any of the following characteristics:

- › They provoke students to think and analyze ideas, concepts or procedures.
- › They demand elaboration and communication of arguments and conjectures.
- › They involve the extension of a situation (example: change conditions, generalize, analyze invariants, etc.).
- › It relates the content that is being addressed with the previous knowledge (pointing out similarities, differences, examples used at one time or another, conclusions reached, etc.).
- › Uses everyday knowledge and common sense as a support or as a starting point to address or interpret a mathematical situation.

In addition, the thinking promotion includes that the teacher gives students enough time to think, work or elaborate; and that his/her interventions do not interrupt the work of the students, reformulate when it is necessary and do not accept short answers without argumentation.

---

Incipient	Medium	Competent
<p>The questions or requirements do not respond to any of the listed characteristics.</p>	<p>Questions or requirements have at least one of the listed characteristics. However, one of the following situations occurs. The teacher:</p> <ul style="list-style-type: none"> <li>- It does not give the students a reasonable time for their elaborations.</li> <li>- Accept short answers, without argumentation.</li> <li>- Interrupt the students' answers, answer their own questions or take short or partial answers from the students and he/she completes them.</li> <li>- Fails to reformulate a problem or question that remains unanswered, even though it has given students time to respond.</li> </ul>	<p>Questions or requirements have at meet one of the listed characteristics. In addition, the following happens. The teacher:</p> <ul style="list-style-type: none"> <li>- Gives the students a reasonable time for their elaborations.</li> <li>- If the students give short answers or without argumentation, the teacher asks that they expand and explain them.</li> <li>- Allows students to elaborate their answers without intervening.</li> <li>- If a problem or question is left unanswered, reformulate it so that students can approach it from another perspective.</li> </ul>

---

Table 7: Dimension Thinking promotion about mathematics. Domain Management of mathematics teaching, Promate protocol.

### CLASSROOM SITUATION EXAMPLE

The use of protocols to observe classroom practices allows identifying situations that can be analyzed in the framework of the instrument to generate reflection on the improvement of practices. Also, it is interesting to identify situations that allow us to give life to the dimensions defined as desirable. Below we describe a classroom

situation, that allows us to exemplify the dimension of *Thinking promotion about mathematics* that we have described previously.

Andrea teaches in a 10th grade. She is working on the Pythagorean theorem.

The class have remembered that the Pythagorean Theorem works on a right triangle, and they complement this information with the classification of triangles, establishing that the right triangle can be scalene and isosceles.

At that moment the following dialogue occurs:

Student: Miss Andrea, can the triangle be equilateral for example?

Teacher: Analyze it yourself, what is the characteristic of an equilateral triangle?

Student: Ah! Of course ... it cannot, because if all sides of the triangle measure the same, their angles also have the same measure between them, and therefore they all measure  $60^\circ$ . So, the triangle cannot have a right angle.

The previous situation relates to a brief episode in a mathematics classroom, in which a student asks a question to his teacher and she, instead of giving the right answer in a direct way, generates a space for the student to analyze the situation and give his own response. In terms of the dimension of *Thinking promotion about mathematics*, the teacher generates a question that *requires the student to relate the content that is being addressed with previous knowledge*.

Through the example, we want to show how the *Thinking promotion about mathematics* is present in daily life within the classrooms, in which way, based on small questions, teachers allow students to generate mathematically valid explanations.

## FINAL COMMENTS

As we described at the beginning of this work, class observation and feedback are a key tool for the enrichment of classroom work. Now, throughout the time we have been working on this subject, on the construction of observation instruments as well on the use of these for teaching accompaniment, we have been able to verify some elements, which we find valuable as apprenticeships throughout of said processes.

First, it is very important to build class observation instruments that obey the context of the work we do. In our case, placing the construction of an instrument in the Latin American context has allowed us to collect relevant information from our classrooms, information that we could not build with other instruments such as MQI and CLASS, which have been constructed and validated in other kind of classrooms (Martínez et al. al., 2018).

On the other hand, the validation processes of observation instruments require a series of stages, which go beyond statistical validation. Content validation by expert on the area of study is indispensable, as well as analyzing the potential use of the instrument.

Also, from the training work of coders and the diffusion of the instrument that we have shared in this paper, it is revealed that the work of designing and validating an instrument, but also training to do class observation, is highly relevant, as a way of developing a common language that allows us to understand the work in the classroom and to communicate with teachers to build together real and situated alternatives to do inside of the classroom. Similarly, the importance of working on understanding, characterization and training in the feedback process is highlighted.

It is very important to work on the elaboration of observation instruments, but also to carry out the work of disseminating them, to contribute to the educational system with ways, ideas and examples of where to move forward, thus providing examples of how to materialize and operationalize those aspects that are declared as desirable.

## References

- Anijovich, R. (2009). *Transitar la formación pedagógica*. Buenos Aires: Editorial Paidós.
- Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *The Journal of Mathematical Behavior*, 33, 149 – 167.
- Brennan, Robert L. (2001). *Generalizability Theory*. New York: Springer-Verlag.
- Ferguson, R. F. & Danielson, C. (2014). How Framework for Teaching and Tripod 7Cs Evidence distinguish key components of effective teaching. In: Kane, T. J., Kerr, K. A., & Pianta, R. (Eds.). *Designing teacher evaluation systems: new guidance from Measures of Effective Teaching Project*, 98 – 143. San Francisco, CA: Jossey-Bass. A Wiley Brand.
- Godoy, F., Varas, L., Martínez, M., Treviño, E. & Meyer, A. (2016). Interacciones pedagógicas y percepción de los estudiantes en escuelas chilenas que mejoran: una aproximación exploratoria. *Estudios Pedagógicos*, XLII(3), 149 – 169.
- Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching Practice: A cross-professional perspective. *Teachers College Record*, 111(9), 2055 – 2100.
- Grossman, P., Loeb, S., Cohen, J., Hammerness, K., Wyckoff, J., Boyd, D., & Lankford, H. (2010). *Measure for Measure: The Relationship between Measures of Instructional Practice in Middle School English Language Arts and Teachers' Value-Added Scores*. NBER Working Paper 16015. National Bureau of Economic Research (NBER), Cambridge, MA.
- Hattie, J. (2012). *Visible learning for teachers. Maximizing impact on learning*. Abingdon and New York: Routledge editor.

- Hill, H, Blunk, M., Charalambos, Y.Ch., Lewis, J.M., Phelps, G., Sleep, L., & Ball, D. (2008). Mathematical Knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430 – 511.
- Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 513 – 545.
- Hill, H., Umland, K., Litke, E., & Kapitula, L. (2012) Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education*, 118(4), 489 – 519.
- Hill, H. C., Charalambous, C. Y, & Kraft, M. A. (2012). When rater reliability is not enough: Teacher observation systems and a case for the generalizability study. *Educational Researcher*, 41(2), 56 – 64.
- Llinares, S. (2000). Intentando comprender la práctica del profesor de matemáticas. In Da Ponte, J.P. & Serrazina, L. (Eds.). *Educação Matemática em Portugal, Espanha e Italia*. (109 – 132) Lisboa: Secção de Educação Matemática da Sociedade Portuguesa de Ciências de Educação.
- Manzi, J., Gonzalez, R. & Sun, Y. (2011). *La evaluación docente en Chile*. Santiago de Chile: MideUC, Centro de Medición.
- Marcelo, C. (2009). Los comienzos en la docencia: un profesorado con buenos principios. *Profesorado, Revista de Currículum y Formación del Profesorado*, 13(1), 1 – 25.
- Martínez, M., Godoy, F., Treviño, E., Varas, L. y Fajardo, G. (2018). ¿Qué nos revelan los instrumentos de observación de aula sobre clases de matemática en escuelas con trayectoria de mejoramiento?. *Educação e Pesquisa*, 44, e165144.
- Mihaly, K. & Mccaffrey, D. (2014). Grade Level Variation in Observational Measures of Teacher Effectiveness. In: Kane, T. J., Kerr, K. A., & Pianta, R. (Eds.). *Designing teacher evaluation systems: new guidance from Measures of Effective Teaching Project*, 9 – 49. San Francisco, CA: Jossey-Bass. A Wiley Brand.
- OECD, (2004). The PISA 2003 Assessment Framework: Mathematics, Reading, Science and Problem Solving Knowledge and Skills. USA: OECD Publishing.
- Perrin-Glorian, M.; Deblois, L. & Robert, A. (2008). Individual practising mathematics teachers. Studies on Their Professional Growth. In K. Krainer & T. Wood (Eds.), *Participants in Mathematics Teacher Education*, 35 – 59. Netherlands: Sense Publishers, Netherlands.
- Pianta, R., Laparo, K., & Hamre, B. (2008). *Classroom Assessment Scoring System™ (CLASS™)*. Baltimore, Maryland: Paul H. Brookes Publishing Co., Inc.
- Pianta, R., Hamre, B., & Mintz, S. (2012) *Classroom Assessment Scoring System. Upper Elementary Manual*. Charlottesville, Virginia.: Teachstone.
- Reis, P., & Climent, N. (2012). *Narrativas de profesores: reflexiones en torno al desarrollo personal y profesional*. España: Universidad Internacional de Andalucía.
- Rockoff, J. E. (2008). *Does Mentoring Reduce Turnover and Improve Skills of New Employees? Evidence from Teachers in New York City*. New York: Columbia University Business School.

- Sandoval, E. (2009). La inserción a la docencia. Aprender a ser maestro de secundaria en México. *Profesorado: revista de currículum y formación del profesorado*, 13(1), 183 – 194.
- Schultz, S. E., & Pecheone, R. L. (2014). Assessing quality teaching in Science. In: Kane, T. J., Kerr, K. A., & Pianta, R. (Eds.). *Designing teacher evaluation systems: new guidance from Measures of Effective Teaching Project*, 444 – 492. San Francisco, CA: Jossey-Bass. A Wiley Brand.
- Shavelson, R. J., & Webb, N.M. (1991). *Generalizability Theory: A Primer*. Newbury Park: SAGE Publications.
- Toledo, G., & Valenzuela, J. P. (2012). Ordenamiento de profesores y estudiantes entre y dentro de los establecimientos escolares: el caso de Chile. Documento de Trabajo CIAE y Departamento de Economía, (5).
- World Bank (2012). Learning from the best: improving learning through effective teacher policies. Education Notes 67345

# MAKING SENSE OF STUDENTS' SENSE-MAKING OF MATHEMATICS

Márcia Maria Fusaro Pinto

Federal University of Rio de Janeiro, Brazil

*Students' sense-making strategies in mathematics learning, or how individuals come to know mathematics and how they evolve over the time, have been recurrent in mathematics education as a focus of research. On the other hand, research rationales have not been unique. For instance, the last decade's revival of research upon the topic reveals a movement from the nineties' motivation of informing alternatives to classroom teaching to the development of quantitative studies to inform educational policies. This scenario is explored and includes a discussion on a new view on abstraction that conceives specific cognitive processes underlying mathematical concept construction interrelated with a particular strategy of sense-making.*

## INTRODUCTION

*Understanding and promoting students' mathematical thinking is the theme of this conference. It is a fortunate convergence with the long-term research interest that drove me, as a university teacher, from the initial practices as a professional mathematician to embrace other research questions, other research methods, other professional motivations. Such gradual growth of interest on the phenomena of teaching and learning mathematics and on how different individuals come to know mathematics are neither rare stories nor are they new in the community of mathematics researchers. As most mathematicians share the activities of doing mathematics with teaching mathematics, it sounds natural that questions would emerge on how to improve teaching of certain mathematical contents, and on how individuals come to know mathematics and evolve over the time, to support learning. No less important is how to inform other teachers about critical findings and experiments. Broadening the immediate context of schools and classroom practices we are immersed in, reasons of other nature may emerge from demands of living and society, as rationales to promote both knowledge and understanding, and to address teaching.*

Of particular interest, and surprising, is the discussion presented by Shulman (1986) on the medieval educational practices when “content and pedagogy were part of one indistinguishable body of understanding” (p.6). For the author, such interpretation is reflected upon the denomination of the highest degrees attained in the academy, of master and doctor, which actually mean just teacher. Shulman refers to Ong (1985) and claims that, in the medieval times, an individual proved his understanding by

demonstrating ability to *teach* the subject. Perhaps the final oral examination of candidates to attain the master or doctor academic degrees, being still a tradition in many universities, could be a trace of considering content and pedagogy as inseparable.

Around three centuries later, and earlier than the acknowledged leadership of Felix Klein as setting the seeds of mathematics education, some of the late nineteenth century mathematicians' initiatives explicitly addressed teaching amongst other professional intentions. In Furinghetti and Radford (2002) one finds, for example, that Cajori and Zeuthen were concerned with teachers when they wrote their books on history of mathematics, in 1894 and 1902 respectively. Cajori even devised and suggested a pedagogical use of history of mathematics interweaving "our knowledge of past conceptual developments with the design of classroom activities" (Furinghetti; Radford, 2002, p.632) with the goal of developing students' mathematical thinking. Such initiatives promoted and were promoted by an intense debate, once the historical conceptual mathematical development and the development of students' mathematical thinking belong, respectively, to historical and psychological domains; thus, having different research questions and methods to investigate. A diversity of perspectives support the variety of proposals on the use of history of mathematics for teaching. At first, mathematicians re-signified the biological recapitulationist view and scholars conceived a 'psychological recapitulation' or 'linear evolutionism perspective' (Miorim; Miguel, 2011), approaching the intellectual development of an individual as naturally following paths that human kind had once gone through. These perspectives were re-examined through the lenses of biological evolutionary frameworks that followed, putting in evidence the role of the environment and culture in the intellectual development. Till today there are controversies on the networking of the two research areas (Furinghetti; Radford, 2002, p.633), and the use of history of mathematics in mathematics education constitutes itself a research area within mathematics education. In Miorim and Miguel (2011, p.82) we found Poincaré (1908)'s declaration in favour of such a perspective to support a pedagogical approach to the teaching of mathematics:

Zoologists claim that the embryonic development of an animal sums up in a rather short time the whole history of its ancestors of geological times. It seems that the same can be said about the development of the mind. The educator must cause the child to go back where his or her ancestors passed; more quickly, but without omitting steps. For this reason the history of science must be our first guide. (Poincaré, 1947, p.135)

In the same year, Felix Klein (1908) emphasises

I should like to bring forward the *biogenetic fundamental law*, according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. Such thoughts have become today part and parcel of the general culture of everybody. Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. *Taking into account the native*

*ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge.* It is necessary to formulate this principle frequently, for there are always people who, after the fashion of the mediaeval scholastics, begin their instruction with the most general ideas, defending this method as the “only scientific one”. And yet this justification is based on anything but truth. *To instruct scientifically can only mean to induce the person to think scientifically, but by no means to confront him, from the beginning, with cold, scientifically polished systematics.* (Klein, 1908, p.292)

Miorim and Miguel (2011, p.82) analyse Klein’s attempts to overcome the different methods of production and teaching and learning of mathematics, although restricting their focus on the “positivist” perspectives taken by both Klein and Poincaré to support their arguments. Other than reflecting on the earlier nineteenth century evolutionist perspective embraced by both scholars, I find interesting to comment that both statements presented here indicate that (at least the two) working mathematicians were searching for theoretical foundations for improving the teaching of mathematics. In fact they made an attempt to re-signify areas of knowledge – biology, psychology, epistemology and mathematics – to refine and contrast methods of teaching – such as those inherited from mediaeval scholastics. Their scientific posture differs for sure from those of scholars who believe, as it is commonly shared in the contemporary mathematics community, that teaching practices are a sub-product of acquiring an increasing amount of content knowledge.

At an institutional level, another important initiative in the nineteenth century within the mathematicians community to an inceptive mathematics education research were the editions of journals directed to those who taught mathematics. Furinghetti (2003) analysed the first years of the *L’Enseignement Mathématique*, which was founded in 1899, and then became the official journal of CIEM (Comission Internationale de L’Enseignement Mathematique), later on termed ICMI (International Comission on Mathematics Teaching) and still published today. The term “enseignement” always had the broader meaning as “the teaching of pupils and the teaching of teachers” (ibid., p. 23) and the journal intended to be an international contribution to the debate on the teaching of mathematics; an intention which undoubtedly it attained. According to Furinghetti’s analysis, some hints at psychology as a possible link between mathematics and the pedagogical issues appeared in some publications of the journal, and an interest emerged on the links between psychological themes such as conditions for creativity and invention and mathematical themes such as axiomatisation, rigour and intuition (p.34). This provided a context for the promotion by the journal of an investigation on ‘methods of working’ of professional mathematicians. The journal attended the suggestion of a reader’s letter published in a ‘Correspondence’ section of the journal in 1901, where secondary and university teachers had the opportunity of joining the debate on the teaching of mathematics and presenting their own questions and ideas. The editors launched the project by elaborating a questionnaire, including questions sent by the journal’s readers, with an

aim of “collecting advice useful to researchers in mathematics, and on the other hand to contribute to research in the field of psychology of professions.” (p.35). A mathematician and two psychologists analysed more than a hundred answers to the thirty questions of the questionnaire. The results were published in 1905, 1906, 1907 and 1908, acknowledging the difficulties in drawing a conclusion. In Furinghetti’s words, the project “constitutes an early example of making explicit the feeling mathematicians have about the nature of their discipline, of their work, of their being mathematicians” (p.35). Furinghetti considers that this project resulted in the famous works that followed, such as those by Hadamard (1954), Hardy (1940) and Poincare (1913), to mention a few and the most well known. As further developments, the author observes that even the initial focus of investigation being mathematical research, the method and the results of the project launched by the journal, suggested contrasting and reflecting on students’ mathematical performances at school from psychological perspectives. (Furinghetti, 2003, p.36)

These are, inside the community of mathematicians, core initiatives of the origins, institutionalisation and initial developments of mathematics education as an academic field. In what followed, the proposal of a commission on mathematics education, initially named *Internationale Mathematische Unterrichtskommission* and *Commission Internationale de L’Enseignement Mathématique*, later on named International Commission on Mathematics Instruction (ICMI), was due to David Eugene Smith in 1905, in a paper published in the journal *L’Enseignement Mathématique* (Kilpatrick, 1992). Its first president was Felix Klein, elected in 1908, and the question to be addressed by the commission was “how the teaching of pure mathematics might be improved” (ibid. p. 2). The initial focus on secondary schools and the adoption of comparative studies on methods and plans of teaching were gradually expanded to include other school levels and methods, such as those used in psychology.

Now one may step back and think about the motivations, or the interests, behind a movement of the community of mathematicians towards a discussion of a different research object, namely the teaching and learning of mathematics. Other than the possible (and natural) self-preservation as a lively community, it is important to mention the demands for technical schools arising from the growing use of technology in society, and the industrialisation effect. (Schubring, 1989). According to Schubring (1989), in the German case, mathematicians were unable to present in a comprehensive way pure mathematics now approached in technical colleges in their rise of status to the tertiary level of education. Strong reactions from prospective engineers refused mathematicians as teachers in those schools. As part of the response to this movement, Klein proposed his well-known programme of writing books for teachers and revising mathematics curricula. His project was dedicated to coordinating the mathematics curricula in the three types of secondary schools and in the two types of tertiary institutions that coexisted in Germany, so that the transition from secondary to tertiary would be smooth, enabling to continue there better

prepared. Klein's conception was to fill the mathematical gaps between the secondary school and the technical school level, in order that mathematicians there would be able to address the technical school students. The entire plan included teacher education to provide secondary schools students the necessary mathematical background, based on the famous key word of functional thinking.

This initial historical account reveals a one-sided motivation, origin and initial growth of mathematics education. In particular, it reveals aspects of the initial research on the psychology of mathematics education within the community of research mathematicians, which could be a surprise for many of us. In fact, this interpretation does not contemplate the full story. For Schubring (2012), for instance, till the 1970s, "the pertinent research was effected by psychologists; doing research themselves was still outside the horizon of mathematics educators." (p.221). In fact, the International Group for the Psychology of Mathematics Education was created in 1976, during the Third International Congress on Mathematical Education organised by ICMI, in Karlsruhe, Germany. And no less important than the 'empirical turn' for mathematics education research in the seventies, and complementary to the institutionalized initiatives of the mathematics community, were those initiatives emerging from teachers' and psychologists' practices at school. In his investigation on these roots of the 'experimental psychology' perspective in mathematics education, Schubring (2012) reports on studies in Germany investigating students' mathematical errors with the aim of informing mathematical practices at school. Here scholars' interests differ from those reported before. Mathematicians and secondary school teachers had a focus on creativity and invention, and proposed a method to investigate mathematicians' work with the aim of approaching mathematically gifted students at secondary school and college. In contrast, the school teachers, supported by psychologists, were aiming at mathematical comprehension and the inclusion of students in the elementary school classroom, expressing a children-oriented approach to the question. Schubring reports on a journal from 1896 focusing on 'pedagogical pathology' and 'therapy', showing the authors as "practitioners, not questioning established theories." (p.223). In this case, experimental psychology is gradually introduced as a research method with a "common basic assumption about cognitive abilities that was of stable predispositions assumed as largely innate qualities." (p.225). Individual differences were admitted, and a "range of health" is identified, with deviations and extremes such as giftedness and disabilities of some kind. For example, the psychologist David Katz (1884-1953) who presented the first paper ascribing types to each individual "according to his/her proper perception of the outer world." (p.230), hypothesizing a pedagogical use of his distinction in adapting teaching to individual profiles, confirming a teaching and learning centred focus on children.

In addition to the 'experimental psychology' research roots with initial focus upon students' errors, a hypothesis is that the access to the introspective analysis of mathematicians on their mathematical activities raised an interest and suggested other

qualitative methods to investigate the different kinds of mathematical minds and cultures; thus being at the early steps of the psychology of mathematics education when referring to the teaching and learning of undergraduate mathematics. For instance, in the introductory chapter of the *Advanced Mathematical Thinking* book, Tall (1991) sustained the importance for the psychology of mathematics education at advanced levels to consider the mature professional mathematicians' perspectives side by side with research on students' conceptual development, to "seek insights of value to the mathematician in his or her professional work as researcher and teacher" (p. 3).

Thus, given that the research theme is not new in mathematics education and that the research methods and focus of current investigations are varied, one may question the motivations for the initial interest in this research strand and how it evolved over time.

My aim is to reflect on some research perspectives on how adult individuals come to know mathematics that followed the 'empirical turn' for mathematics education research, in the seventies. Rather than the methods used in empirical investigations, those used by Schubring (2012) are adapted to organize a review unraveling features of research group profiles, which explored this research theme.

From the earlier qualitative research on mathematicians' practices and on gifted students' strategies of producing/reproducing knowledge, mathematics education research moved to explore the undergraduate mathematics students' sense-making strategies, with an expectation of getting to know students' perspectives to inform alternative teaching approaches to the subject. In the last years, the revival of the research on students' sense-making strategies in mathematics suggests a movement towards the development of broad quantitative studies to inform alternative educational policies. A qualitative approach to the diverse perspectives of students with special needs broadened an educational scope firstly restricted to giftedness, while a new view on abstraction conceives specific cognitive processes underlying mathematical concept construction interrelated with a particular sense-making strategy in mathematics.

## **METHOD**

The method to organise this presentation is adapted from historical studies to investigate the shared views and common characteristics of particular social groups. A technique, known as prosopography (see Stone, 1971), is adapted to delineate a collective profile of groups of researchers, in order to reveal roots of common interests underlying their research actions and to indicate the degrees and nature of research movements within the groups. It consists of establishing the universe of individuals to be studied through homogeneous questioning, identifying variables that emerge as correlated with their characteristics as a group of actors. Here, the universe is constituted by those researchers who investigated sense-making strategies in mathematics from the psychology of a mathematics education perspective. The

focus is on the late 80's constitution of the PME research group on mathematics thinking and learning at post-secondary level, denominated Advanced Mathematical Thinking group (AMT group), which congregated “researchers from different origins reflecting on these issues [who] established a pattern of regular exchanges and collaborative work” (Artigue et al, 2007, p. 1011), and the related research production in the nineties and early 21st century. A revival of the research theme at ICME13 and last PMEs is included to conjecture on its nature and the possibilities opened for further developments.

In this discussion, my hypothesis is that the sense-making strategies in mathematics is a research theme at the roots of the psychology of mathematics education at post-secondary level. Amongst the documents selected as pertinent sources for investigating the underlying research motivations on this theme and the dynamics within the research groups is the *Advanced Mathematical Thinking* book (Tall, 1991), a publication in this research area which represents well “the state of research on mathematical thinking and learning at the post-secondary level in the early nineties” (Artigue et al, 2007, p.1013); in particular, its First Chapter. For juxtaposing and contrasting researchers' perspectives, a second document is the *Second Handbook* chapter by Artigue, Batanero and Kent (2007), presenting a perspective on the evolution of the research area on post-secondary level mathematics education.

A set of research papers and thesis are case studies of further developments related to the AMT group on sense-making strategies in mathematics. Presentations at ICME13 and at recent PMEs are included to draw on the recent evolving research.

### **REASONS TO ACCOUNT FOR SENSE-MAKING STRATEGIES IN MATHEMATICS AND MATHEMATICIANS' PRACTICES**

The main theme of interest of the AMT working group at the beginning of its constitution was to unravel the nature of advanced mathematical thinking in relation to other elementary forms of thinking, and to investigate the cognitive difficulties faced by students when developing the mental processes required by such a form of thinking. (Artigue et al., 2007, p. 1013). On the other hand, the first chapter of the *Advanced Mathematical Thinking* book is opened with a quotation from Hadamard (1945) on the difficulties to be faced by a research area on the psychology of advanced mathematical thinking, as it involves “two disciplines, psychology and mathematics, and would require, in order to be treated adequately, that one be both a psychologist and a mathematician.” (p.1). A reflection follows the quotation:

Exponents of the two disciplines are likely to view the subject in different ways – the psychologist to extend psychological theories to thinking processes in a more complex knowledge domain – the mathematician to seek insight into the creative thinking process, perhaps with the hope of improving the quality of teaching or research. (Tall, 1991, p.3)

Such a requirement of improving post-secondary level mathematics teaching was already a concern for Skemp (1971), who warned against the traditional form of teaching undergraduate students as he considered they were being taught the “final

product of mathematical thought rather than the process of mathematical thinking” (Tall, 1991, p. 3). In Artigue, Batanero and Kent’s (2007) words, they considered that

... the gap between the logic of the mathematical edifice and the logic of cognitive processes explained the observed inefficiency of university teaching strategies based on the former, for the majority of students. (p. 1013)

Therefore, in this sense, access to the diversity of mathematicians’ perceptions on their own practices were welcome as a means to “heighten ones’ awareness that his or her conceptual views will differ from others” (Tall, 1991, p. 5), and as a means to propose alternative approaches to the logical presentation of mathematics which “fail to give the full power of mathematical thinking and may not be appropriate for the cognitive development of the learner” (ibid, p. 3). In his characterization of the processes of advanced mathematical thinking, Dreyfus addressed “the diversity of mathematical thinking modes” (Artigue et al., 2007, p. 1013) considering the classical distinctions between styles of doing mathematics which were in general distinguished in mathematicians’ reflections as the logical-analytical and the visual or intuitive-geometric ones (see e.g., Hadamard, 1945, p. 86; Poincaré, 1913, p. 212). Tall (1991) had already acknowledged that “of course, there are not just two different kinds of mathematical mind, but many” (p. 4), with an interpretative justification on the characteristics of some well-known mathematicians’ practices; including in his considerations those mathematicians’ epistemological beliefs related to mathematics.

Such a claim was empirically verified under the influence of the socio-cultural turn in mathematics education by Burton (2001), who argued that to the classical visual and the analytic styles of thinking mathematically, we must add another style, the conceptual style, which corresponds to “thinking in ideas, classifying” (Burton, 2001, p. 593). From the 70 professional mathematicians, with a position at universities as teachers and researches who participated in the research, 66% claimed they are visual, 37% considered themselves as analytic and 57% as conceptual. Numbers do not add up 70 because “the majority [of the participants],<sup>42</sup> used a combination of two out of the three [styles] and only 3 claimed to use all three styles” (Burton, 2001, p. 594).

For Burton, such models provide not only a framework for understanding mathematicians’ practices but also an epistemological perspective on students’ learning, as research mathematicians are, at the same time, university teachers.

### **REASONS TO ACCOUNT FOR LEARNING AND DOING MATHEMATICS FROM THE LEARNERS’ PERSPECTIVE**

Although the diversity of mathematicians’ minds or of mathematical practices seemed at first to be mainly interrelated to the different research areas in mathematics – such as to the analytic or to the applied context, Tall (1991) acknowledged “at a far deeper psychological level we all have subtly different ways of viewing a given mathematical concept, depending on our previous experiences.” (p. 6). Many other researchers agreed that any epistemological (or cognitive) perspective on students’

mathematics learning would necessarily include an investigation of students' different strategies when producing/reproducing mathematical knowledge. Pioneering this study, Krutetskii (1976) had already reported the results of a qualitative investigation of students' different styles of thinking, using mathematical problems in clinical interviews. He was, in fact, aiming to identify qualities that would distinguish gifted and capable students from average or below-average students, throughout the primary and secondary school. In his results, which were well-known to those interested in mathematics thinking and learning at the post-secondary level, he concluded that giftedness could be manifested in different ways, categorising them as *analytic*, *geometric* and *harmonic*, and exhibiting a spectrum of relative students' preferences for verbal-logical and visual thinking.

In the nineties, qualitative approaches oriented most research on undergraduate students' different modes of learning mathematics (see e.g., Alcock, 2001; Weber, 2001; Duffin & Simpson, 1993; Pinto, 1998). Even implicitly, the researchers' interest would include informing post-secondary classroom teaching.

Alcock (2001) interviewed first-year undergraduate mathematics students to investigate their reasoning in mathematical analysis. She identified two students' reasoning styles, named visual or non-visual, and acknowledged that students' reasoning styles combined with their "*sense of authority*, whether "internal" or "external" ", interferes in the "types of understanding a student develops" (p.10). Weber (2001) followed doctoral students and undergraduate students analysing their difficulties and their different use of forms of "strategic knowledge" – "knowledge of how to choose which facts and theorems to apply" (p. 101) when students were constructing proofs. He identified three distinct approaches that he named natural, formal and procedural. A decade earlier, Duffin and Simpson had focused on ways that learners build their own structures to respond to experiences, having the classroom as the research context. Based on evidence from classroom episodes, they firstly identified three basic types of *experiences* that an individual meets in a learning context. A new experience is taken as *natural* for the learner if the incident fits the internal structures already constructed by him/her. Otherwise, the experience is *alien*, when the new information is ignored or absorbed as a new isolated structure, or *conflicting*. Then they categorize the constructions that learners make to respond to a certain experience conjecturing two broad "types" of *learners*, other than strategies of learning, named *natural* and *alien* (see, for example, Duffin & Simpson, 1993).

Pinto (1998) reported on an inductive analysis of data collected from classroom observation and interviews with eleven first year undergraduate mathematics students, every two weeks, during two academic terms. From a cross-sectional analysis of three pairs of students, and with the aim of investigating students' sense-making of the formal axiomatic mathematics, two prototypical sense-making strategies were identified: extracting meaning from the concept definition through formal deduction, and giving meaning to the concept definition by building from earlier concept images. Those who *extract meaning* build new knowledge based on

routines (of actions on objects), constructing the formal mathematics if they routinize reflectively. Students who *give meaning* embed or interpret at first new knowledge in terms of the old, basing their arguments on thought experiments, sometimes retaining old images, sometimes adding some knowledge pieces to the concept image, or even reconstructing the concept image. These results are in dialogue with Tall's (2001) conceptions on the formal world of mathematics cognitive development, when he

... identified two different ways for the cognitive development of this formal world: natural thinking, which builds from the concept image towards the formalism on the one hand, and formal thinking, which builds from the concept definition, marginalizing imagery and focusing on logical deduction. (Artigue et al., 2007, p.1018)

As expressed in Artigue, Batanero and Kent (2007)

...the research initiated in the Advanced Mathematical Thinking working group of PME in this area, in spite of the diversity of its developments, has avoided a fractionalization of its perspectives and been able to integrate its previous achievements into complementary and coherent constructs ... (p. 1044)

Recent research on mathematical thinking and learning at the post-secondary level were "inspired by the increasing influence in the educational field of sociocultural and anthropological approaches" (Artigue et al., 2007, p.1025). Such perspectives revealed a transition from secondary to post-secondary levels that is not restricted to the difficulties related to the contrast of formal/informal approaches to mathematics. Researchers must consider the different institutions and practices where mathematics circulates. For instance, "the increasing importance taken in post-secondary mathematics education by service courses faces us with the necessity of taking a wider perspective" (ibid, p. 1044) which cannot be discussed "by considering only the practice of pure mathematicians working in traditional fields." (ibid, p. 1044).

## **RECENT RESEARCH INTERRELATED WITH SENSE-MAKING STRATEGIES IN MATHEMATICS**

The recent changes on the nature and possibilities of research developments on sense-making strategies in mathematics are organised according to internal and/or external factors to the post-secondary mathematics education that seem to be promoting them. Exemplary cases of the movements in other educational levels are included, and are represented by the following first two cases in this section.

A movement from informing the teaching of mathematics classroom to support broader educational policies seems to be a research interest instigated by a review of recent research related with students' sense-making strategies in mathematics. They indicate variations in focus, method, and context. Recent presentations given at ICMI-13 made evident a predominance of quantitative studies, even if considering mixed methods. Large quantitative studies were used to investigate strategies of learning organised under a cognitive and resource management strategies umbrella (see Khanal, 2016). Figueiredo and Guimarães (2016) based their quantitative research on Vermont's (1994) ILS (Inventory of Learning Styles), using a

questionnaire and confirming Vermont's learning styles named meaning-oriented and reproduction-oriented. Like Khanal (2016), both studies relate learning style and students' success in mathematics learning, suggesting an evaluation manner of the teaching and learning of mathematics in the researchers' respective countries.

As a second case, the qualitative research on inclusive education from Nardi, Healy, Biza and Fernandes (2016) investigates students' particular forms of sense-making of mathematics accounting for individuals who do not share the perceptions that shape the school mathematical programs and curricula. They reported on an evolving investigation project, focusing on aspects of mathematics that are typically associated with visual and auditory perception and that most of the time they have their related pedagogical practices shaped by non-ableist perspectives. Amongst their aims is to understand how the ableist perspectives of sense-making impact the teaching of these aspects. Such an attitude to education is at variance with earlier understandings of mathematical disabilities and empowers the learners. In those researchers' views, the perceptions are sensorial. However, they share an educational attitude with the researchers who embrace investigations on giftedness and creativity (Leikin et al., 2009). Although expressing that "prominent mathematicians are mathematically gifted and creative individuals" (Leikin et al., 2009, p. vii), which is a different interest in education from that of inclusion, the similar perspective assumed to understand the individuals may be of benefit for each others research.

The third case of a research movement is due to internal factors for mathematics education. Pinto (1998) observed that, for students who give meaning, the existent frameworks did not offer an explanation of their mathematical constructions and reasoning (see also Pinto; Scheiner, 2014). For Scheiner (2016), reasons seem to be related to the historical origins in Piaget's work on operational rather than figurative knowledge. A widespread conviction that mathematical knowledge can mostly be constructed through actions on objects led unattended other actions involved in the different sense-making strategies of mathematics. He reconsidered Pinto's (1998) notions of extracting meaning and giving meaning and presented a framework on structural abstraction capable of offering a systemic explanation of the processes required and involved in the constructions and reasoning of those who give meaning. In order to do that, mathematical meaning is not understood as an inherent quality of mathematical objects to be extracted, but something that is attributed to them by one's thinking.

An evolving investigation (Pinto, 1998; Pinto & Scheiner, 2015, 2016; Scheiner & Pinto, 2014, 2016, 2017, 2018-a, 2018-b; Scheiner, 2016, 2017, 2018-a, 2018-b) develops in this direction, of better understanding the interrelations of individuals' knowing and learning processes and their sense-making strategies. On the one hand, from an analysis of the qualities perceived in Chris' actions with the visual resources he created and had available, Pinto (1998) considered that the core of Chris's formal constructions could not be reduced to simple visualization. On the other hand, the alternative represented by Scheiner's framework and the discussion of the processes

of contextualizing, complementazing and complexifying ( Scheiner; Pinto (2017), submitted; 2018-a; 2018-b) allows an interpretation of Chris's learning phenomena (Pinto, 1998; Pinto & Tall, 2002) that pulls together aspects of a diversity of frameworks. Those must be considered as complementary rather than conflicting, in a direction of analyzing formal mathematics constructions; though for some of these frameworks formal mathematics was not in their research focus of attention.

## FINAL REMARKS

This talk was instigated by the rationales underlying the research on sense-making strategies in mathematics and by the long-term dynamics of the research groups' approaches to this theme.

Late nineteenth century initiatives reveal mathematicians and secondary school teachers starting a dialogue with psychology. The notions of creativity and invention, approached in a questionnaire responded by working mathematicians on their professional activities in mathematics, have their results analyzed by psychologists. Those were published in a journal edited by the mathematics community, to communicate and discuss results about teaching. A study on the German case shows that, at the same time, psychologists and mathematics teachers at elementary school investigated students' errors to design classroom activities, aiming at students' inclusion. In both cases, the main aim is improving teaching, and the initiative may be considered as related to the society industrialization effect.

As a matter of fact, genuine research developed by mathematics educators research groups was set up during the seventies – after the 'empirical turn' in mathematics education. Till the beginning of the XXI century, research on strategies of making sense in mathematics became mainly student centered. Creativity and invention are approached and inclusion is reflected on the common research method of considering a range of students' performance in the investigation. The main research aim is informing classroom activities.

The last decade's research on strategies of making sense in mathematics seem related to expansions of the school system and of the post-secondary education; and to the open questions left for the psychology of mathematics education. In respect, my own reasons are related to earlier research interests on the diversity of students' strategies of sense making of mathematics better explained by different perspectives on abstraction.

Undoubtedly, the methodological organisation touched the interests and some driving forces interfering in the movements within the groups embraced research on the teaching and learning of mathematics. The adopted methodology is promising for further research.

## REFERENCES

Alcock, L.J. (2001) *Categories, Definitions, and Mathematics: Students' reasoning about objects in Analysis*. Unpublished Ph.D. thesis. Coventry, UK: University of Warwick.

- Artigue, M., Batanero, C., & Kent, P. (2007) Mathematics Thinking and Learning at Post-secondary Level. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: a Project of the National Council of Teachers of Mathematics*, pp. 1011-1049. Charlotte, NC: Information Age Publishing, 2007.
- Burton, L. (2001) Research mathematicians as learners and what mathematics education can learn from them. *British Educational Research Journal*, 27(5), 589-599
- Cajori, F. (1894). *A history of mathematics*. New York: Macmillan.
- Duffin, J. & Simpson, A. (1993) Natural, conflicting and alien. *Journal of Mathematical Behaviour*, 12 (4), 313-328.
- Figueiredo, M. & Guimarães, H.M. (2016) *Learning styles in mathematics of the 10<sup>th</sup> grade Portuguese students*. Oral communication presented at 13<sup>th</sup> International Congress on Mathematics Education. Hamburg, Germany.
- Furinghetti, F. (2003) Mathematical instruction in an international perspective: the contribution of the journal L'Enseignement Mathématique. 21-46.
- Furinghetti, F. & Radford, L. (2002) Historical Conceptual Developments and the Teaching of Mathematics: from phylogenesis and ontogenesis theory to classroom practice. In 631-654.
- Hadamard, Jacques (1954) *The Psychology of Invention in the Mathematical Field*. New York, NY: Dover.
- Hardy, G.H. (1940) *A Mathematician's Apology*. Cambridge: Cambridge University Press.
- Khanal, B. (2016). Learning Strategies Used by Public and Private School Mathematics Students of Nepal. *IRA International Journal of Education and Multidisciplinary Studies (ISSN 2455-2526)*, 5(2), 66-75.
- Kilpatrick, J. (2008). The development of mathematics education as an academic field. In M. Menghini, F. Furinghette, L. Giacardi, & F. Azarello (Eds.), *The first century of the International Commission on Mathematical Instruction (1908–2008): Reflecting and shaping the world of mathematics education* (pp. 25–39). Rome, Italy: Istituto della enciclopedia Italiana.
- Klein, F. (1908) *Elementary Mathematics from a Higher Standpoint. Volume I: Arithmetic, Algebra, Analysis*. Translated by Gert Schubring (Berlin & Heidelberg: Springer, 2016).
- Krutetskii, V.A. (1976). *The psychology of mathematical abilities in school children* (Transl. by J. Teller; Ed. by. J. Kilpatrick & I. Wirszup). Chicago, Il: University of Chicago Press.
- Leikin, R., Berman, A., & Koichu, B. (Eds) (2009) *Creativity in Mathematics and the Education of Gifted Students*. Rotterdam, The Netherlands: Sense.
- Miguel, A.; Miorin, M. Â. (2011) *História na Educação Matemática: Propostas e desafios*. Coleção:Tendências em Educação Matemática. Belo Horizonte: Autêntica, 2 ed.
- Nardi, E.; Healy, L.; Biza, I.; Fernandes, S. H. (2016) Challenging ableist perspectives on the teaching of mathematics through situation-specific tasks. In C. Csíkos, A. Rausch, &

- J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (pp. 347-354). Szeged, Hungary: PME.
- Pinto, M. M. F. (1998). *Students' understanding of real analysis*. Coventry, UK: University of Warwick.
- Pinto, M., M. F. & Tall, D. O., (2002). Building formal mathematics on visual imagery: a case study and a theory. *For the Learning of Mathematics*.
- Pinto, M. M. F.; Scheiner, T. (2015) Visualização e ensino de análise matemática. *Educação Matemática Pesquisa*, v. 17, n. 3.
- Pinto, M. M. F., & Scheiner, T. (2016). Making sense of students' sense making through the lens of the structural abstraction framework. In E. Nardi, C. Winslow & T. Hausberger (Eds.), *Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics* (pp. 474-483). Montpellier, France: INDRUM.
- Poincaré, H. (1913) *The Foundations of Science*. New York, NY: The Science Press.
- Scheiner, T.; Pinto, M. M. F. (2014) Cognitive processes underlying mathematical concept construction: The missing process of structural abstraction. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, v. 5, pp. 105-112. Vancouver, Canada: PME.
- Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense-making strategies. *Educational Studies in Mathematics*, 91(2), 165-183.
- Scheiner, T.; Pinto, M. M. F. (2016) Images of abstraction in mathematics education: contradictions, controversies, and convergences. In: *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*. p. 155-162.
- Scheiner, T.; Pinto, M. M. F. (2017) The evolving framework of structural abstraction: a tool for and an object of research in knowing and learning mathematics. (submitted)
- Scheiner, T. (2017) Conception to concept or concept to conception? From being to becoming. 2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41<sup>st</sup> Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 145-152. Singapore: PME.
- Scheiner, T. (2018-a) Mathematics cognition reconsidered: on ascribing meaning. The 21<sup>st</sup> Annual Conference on Research in Undergraduate Mathematics Education. USA: San Diego, CA.
- Scheiner, T. (2018-b) Sense-making in mathematics: Towards a dialogical framing. In Hunter, J., Perger, P., Darragh, L. (Eds) *Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia*, pp. 669-676. Auckland, New Zealand.
- Scheiner, T.; Pinto, M.M.F. (2018-a) *Contextualizing, complementizing and complexifying in mathematical cognition*. Short Communication in International Congress of Mathematicians ICM 2018. Brazil: Rio de Janeiro.

- Scheiner, T.; Pinto, M.M.F. (2018-b) Theoretical Advances in Mathematical Cognition. To appear in *Proceedings of the First PME Regional Conference: South America*. Rancagua, Chile: PME.
- Schubring, G. (1989) Pure and Applied Mathematics in Divergent Institutional Settings in Germany: the Role and Impact of Felix Klein. In David Rowe, John McCleary (Eds) *The History of Modern Mathematics. Volume II: Institutions and Applications*, pp. 171-220. Boston: Academic Press.
- Schubring, G. (2012). 'Experimental pedagogy' in Germany, elaborated for mathematics: a case study in searching the roots of PME. *Research in Mathematics Education*, 14(3), 221-235.
- Shulman, L. S. (1986) Those who understand: knowledge growth in teaching. *Educational Researcher*, v.15, n. 2, p. 4-14.
- Skemp, R. R. (1986). *The psychology of learning mathematics*. (Second edition, first published 1971). London, UK: Penguin Group.
- Stone, L. (1971) Prosopography. *Daedalus*, 100(1), 46-79.
- Tall, D.O (1991). The Psychology of Advanced Mathematical Thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 3-21). Dordrecht, The Netherlands: Kluwer.
- Weber, K. (2001) Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101-119.
- Zeuthen, H.G. (1902). *Histoire des mathématiques dans l'antiquité et le Moyen Âge*. Paris: Gauthier-Villars. (Original work published in 1892)





# PME SESSIONS



# **PME SESSIONS: AN INTRODUCTION TO THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION**

Wim Van Dooren<sup>1</sup>, David M. Gómez<sup>2</sup>

(1) University of Leuven, Belgium, (2) Universidad de O'Higgins, Chile

One of the main goals of PME Regional Conferences is to strengthen bonds between the local community of researchers in Mathematics Education and the PME community. As such, this Regional Conference focused in South America includes three 40-minute sessions called “PME sessions”. These sessions aim to introduce PME to conference participants who are new to it, by focusing on PME as an international organization as well as on the PME conferences.

## **SESSION 1: WEDNESDAY, NOVEMBER 14**

This session will present PME as an organization: its goals and history, its governance and the role of the International Committee, how the organization is related to the PME conferences, as well as how to become a member of PME and how to get involved in the community.

## **SESSION 2: THURSDAY, NOVEMBER 15**

This session will present the different types of contributions that researchers can submit to a PME conference. This includes the personal presentation formats used in the Regional Conference (Research Reports, Oral Communications, Poster Presentations), but also includes additional group formats: Research Forums, Colloquia, Working Groups, and Seminars. When introducing these formats, we will highlight important points to focus when preparing your submission.

This session will also include information about the Skemp fund, a form of support for researchers from underrepresented countries, or countries in which no financial support is available, who have an accepted contribution in order to allow them to attend a PME conference.

## **SESSION 3: FRIDAY, NOVEMBER 16**

The last session will introduce the reviewing process used in PME conferences, explaining the reviewing criteria for Research Reports and how a final decision is reached based on reviews. This knowledge will surely come in handy when writing your own Research Report proposal.

Finally, we will present the PME criteria for becoming a reviewer yourself.





# RESEARCH REPORTS



# UNCERTAINTY AND 3D DYNAMIC GEOMETRY: A CATALYST FOR THE STEP FROM 2D TO 3D GEOMETRY

Armando Echeverry<sup>2</sup>, Leonor Camargo<sup>1</sup>, Ángel Gutierrez<sup>2</sup>, Carmen Samper<sup>1</sup>

<sup>1</sup> U. Pedagógica Nacional (Bogotá, Colombia) <sup>2</sup> U. de Valencia (Valencia, Spain)

*Preservice teachers' training in 3-dimensional geometry can be strengthened with technological resources currently available in mathematics education. In this paper, we report how a task proposed to preservice teachers in a 3D geometry course promoted learning due to the intellectual need generated by uncertainty that encouraged argumentation. The analysis is part of a design-based research whose goal is to propose a model of 3D geometry tasks, to be solved by using dynamic geometry, that generate uncertainty. The results of the analysis of the interaction that took place while discussing the solutions to the task suggest that a suitable use of dynamic geometry, together with a carefully designed task, can generate uncertainty that provokes an intellectual need which students express in their arguments.*

## INTRODUCTION

Preservice teachers' difficulties to visualize in 3-dimensional geometry have been documented in various studies (Moore-Russo & Schroeder, 2007; Sgreccia, Amaya, & Massa, 2012). This is a problem due to the role visualization plays in argumentation and, in general, in student performance in 3-dimensional geometry. Researchers such as Prusak, Hershkowitz, & Schwarz (2013) present the need to design classroom tasks that encourage argumentation to favor students' construction of mathematical meaning. A strategy that we believe promotes argumentation is to pose problems which provoke students' uncertainty that induces the intellectual need to argue. In this paper we present a study of student reactions, when uncertainty was provoked by two representations of a *folded quadrilateral*, that is, a 3-dimensional quadrilateral whose vertices are not coplanar (Figure 1,2), obtained by redefining some of them. The task design was inspired by the use that Ferrara & Mammana (2014) make of the tool *Redefinition* in Cabri 3D. Our research objective in this paper is to analyze students' activity while solving a task designed to generate uncertainty, to identify ways the uncertainty emerged and how it influenced students' need to argue.

## THEORETICAL FRAMEWORK

Students' motivation for learning arises from *uncertainty*, a concept that encompasses notions of conflict, doubt and perplexity resulting from social interaction in the classroom when solving a task that faces the students with a situation that is incompatible with their current knowledge or is not solvable with it (Zaslavsky, 2005). We agree with Stylianides & Stylianides (2009) that uncertainty acts as a mechanism that stimulates the emergence of the intellectual need to develop mathematical knowledge. *Intellectual need* is defined by Harel (2013) as the need to extend or reorganize knowledge to make it compatible with the situation that needs to be

understood.

In the task that we analyze, for instance, uncertainty is provoked by the effect produced in the students when the representation of a *quadrilateral* is changed. Initially it was constructed with vertices  $A$ ,  $B$ ,  $C$  and  $D$  in plane  $\alpha$  (determined by  $A$ ,  $C$  y  $D$  in figure 1). Later, the Cabri 3D *Redefinition* tool was used on vertex  $B$  to move it out of plane  $\alpha$  (Figure 1); afterwards, vertex  $D$  was also moved out of plane  $\alpha$ , using the same tool (Figure 2).

*Argumentation*, to convince oneself or others (Harel & Sowder, 1998), is the manifestation of intellectual need provoked by uncertainty. In the situation that we analyze, uncertainty led to argumentation as the students examined the fulfillment or not of the definition of a folded quadrilateral, that had been constructed collectively in the class. Specifically, in the above situation described, students argued whether the last represented figure continued being a folded quadrilateral or not, a situation that forced them to visualize different planes in space.

## METHOD

The task that gave way to the interaction that we analyze was planned, with other tasks, as part of a design-based research (Bakker & van Eerde, 2015), developed in a 3D geometry course of a pre-service secondary mathematics teachers' program, in a university in Bogotá, Colombia. The course consisted of 33 third semester students. They had studied two previous plane geometry courses, where a 2D DGS was frequently used. Cabri 3D was used for the first time in the experimental course. The students worked in groups of three.

Students were asked to *consider points  $A$ ,  $B$ ,  $C$  and  $D$  in space and to study the figure that is the union of the segments determined by these points, no two of which intersect in points different from their endpoints*. All the groups, except one, proposed as a solution a standard quadrilateral, with the four vertices in the plane ( $\alpha$ ). They did not imagine the situation that one group proposed, in which the four points were not coplanar. The teacher represented the situation suggested by the majority, with Cabri 3D, and then used *Redefinition* to move  $B$  out of plane  $\alpha$  (Figure 1). The teacher encouraged the students to define the resulting geometric figure, unknown to them until then; they collectively defined the geometric object, and, due to the teacher's suggestion, labelled it *folded quadrilateral: A folded quadrilateral is a four-sided figure with four non-coplanar vertices, for which every three vertices are not collinear, and every vertex is the endpoint of exactly two segments*. Later, some students questioned what would happen if vertex  $D$  did not also belong to plane  $\alpha$ . The teacher used the *Redefinition* tool again on  $D$  (Figure 2) and asked the students whether the resultant object was a folded quadrilateral or not.

The information used for the analysis in this paper corresponds to the discussion instigated by the teacher's question. It was obtained from two sources: the interaction between the teacher and the whole class, and the dialogue between one of the

researchers who was in the classroom, and some students. Using the Complementary Accounts Strategy proposal suggested by Clarke (1997), in the next class the researcher showed certain moments of the teacher-student interaction extracted from the video of the classroom events and asked questions about them. This revision favored the exposure of ideas by the students; it permitted us to carefully track uncertainty moments, manifested in the students' facial expressions and in their argumentation about the situation, and to describe how these developed.

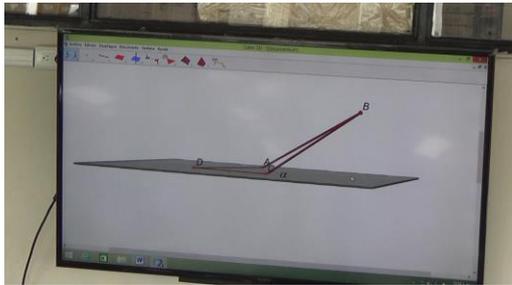


Figure 1. Solution with vertex  $B$  not belonging to plane  $\alpha$ .

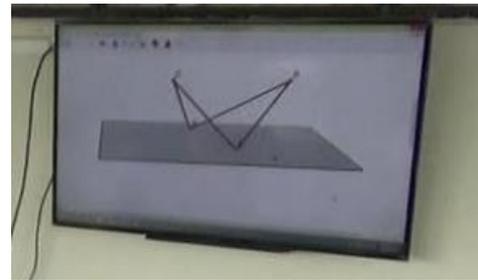


Figure 2. Folded quadrilateral with two vertices not in plane  $\alpha$ .

Using the transcriptions of the interactions that took place, the analysis commences with the identification of indicators of uncertainty during the interaction with the teacher. Once these are found, the emergence of uncertainty is corroborated in the transcriptions of the dialogues with the researcher. Then, traces of intellectual necessity, expressed as argumentation, are looked for; the teacher's role and the effect of the use of Cabri 3D in the development of the task are identified. The analysis leads to the establishment of a route to articulate elements identified in the task design, and thus advance in the construction of an answer to the research question we have formulated.

## **ANALYSIS**

In a preliminary discussion, before the implementation of the task, the research group considered that uncertainty could appear when the vertices of the quadrilateral are redefined to extract them from plane  $\alpha$  (Figure 2); therefore, the class was questioned whether in each case a folded quadrilateral was represented. We had anticipated that if no student promotes further exploration, by extracting the second vertex, the teacher would do it. We expected that intellectual necessity would be expressed with arguments in favor of and against accepting it as such, a product of uncertainty generated by the situation.

The first redefinition, when point  $B$  is extracted from the plane, caused uncertainty; some students were surprised, expressed by the look on their faces, that such a figure actually satisfied the established properties. They could only imagine coplanar figures. During the teacher-guided production of the definition of a folded quadrilateral, uncertainty, as an expression of doubt, arose. This becomes evident, with Adriana's (the names are pseudonyms) objection to the proposed definition when she suggested

that the possibility of redefining another vertex, as a point not in the plane, could modify the definition:

Teacher: Then, we are going to list the properties and, from it, arises the definition. Yes? Then, four non-coplanar points [writes] (...) non-coplanar. Do I need four non-coplanar points?

Juan: Every three not collinear.

Teacher: Every three [writes] not collinear.

Adriana: Teacher, if we redefine  $D$  and take it out of the plane (...)?

This possibility created the intellectual need for Adriana and Juan to question the established definition for a folded quadrilateral. They argued that the written list did not include the case of a representation in which two vertices were not in plane  $\alpha$ . There are two options that can give place to different arguments: when the representation with two points out of the plane  $\alpha$  is considered as an example of a folded quadrilateral and when it is not. The second was the case for Adriana and Juan, who felt that the definition lacked something: “We want [the definition to state that there is] exactly one point that is non-coplanar [in plane  $\alpha$ ]”. This would prevent accepting, as a folded quadrilateral, a representation with two vertices that are not points of plane  $\alpha$ .

Once the teacher redefined vertex  $D$ , as a point that does not belong to  $\alpha$  (Figure 2), she encouraged discussion by asking if the representation was an example of a geometric object, different from a folded quadrilateral. The intellectual need, which up to the moment was expressed with arguments against accepting the four-sided figure with two vertices not in  $\alpha$  as a folded quadrilateral, is now expressed in favor by John.

Teacher: Do I have another figure [different from a folded quadrilateral] that we may want to give another name to? (...) Doubly folded or something like that?

John: But it is... It is the same [figure]!

Teacher: It's the same? (...) Why?

John: If we make a plane that contains points  $B$ ,  $A$  and  $C$ , we are going to obtain the same thing.

The teacher illustrated John's idea with a representation of the plane that he mentions (Figure 3):

Teacher: [...] You say that it should be the plane that contains  $B$ ,  $A$  and  $C$ . And you say that one sees (a figure) like my folded quadrilateral (Figure 3). Does one see my same folded quadrilateral?

In chorus: Yes!

John's intervention, which is expressed with confidence, together with the group's unified answer, transmitted the sensation of having reached consensus. Yet, the facial expression of various students reflected doubt. Since our interest was to obtain more information from those expressions of doubt, which we consider as indications of

uncertainty, we decided to promote, during the next class, student interaction with the researcher, using the Complementary Accounts (Clarke, 1997) strategy. Presenting extracts of the previous class video, he asked questions about the students' comments and expressions.

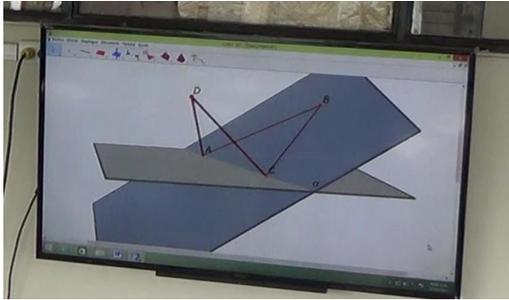


Figure 3. Quadrilateral with two vertices not in  $\alpha$



Figure 4. Lina illustrating her ideas

In their interaction with the researcher, Juan specified why he and Adriana thought that the definition should specify that exactly one vertex must not belong to the plane:

Juan: Well, I was thinking about the representation, but then [...] we could see that something is missing [in the definition], because we could find a counterexample. Yes? That is, the representation (...) when we take another point out [from  $\alpha$ ], would stop being [a folded quadrilateral]. Yes?

Laura, who did not agree with the given definition of a folded quadrilateral, though she did not say so previously, and expresses this to the researcher:

Laura: Up to this moment of the class, we had three points [in  $\alpha$ ] and a [point]  $B$  that was not in the plane. The question was: if another vertex is taken out (...) ? It seemed to me that it was not a folded quadrilateral.

Other students explained why they thought the written definition was correct:

Sergio: Nora said that it does not matter if we take [another] point [ $D$ ] out [of the plane, because]  $B$ ,  $A$  and  $C$  are going to determine a plane (...).

Santiago: Since there were two sides that intersect in a point, this already determines a plane.

Then Lina explains to the researcher how she had imagined the situation:

Lina: I imagined this visualization, but let's say in a drawing. It is that if we take first plane  $\alpha$  and redefine  $B$  and [re]define  $D$ , for example, it would be something like this [she modelled with her hands: she placed a hand in horizontal position and the forefinger of another hand over the palm, but without touching it, indicating the position of a point not in plane  $\alpha$ ] (Figure 4). Two points  $A$  and  $C$  and another point  $B$ . I see this [plane]  $B$ ,  $A$  and  $C$ , because [the points] are contained in a plane since they are not collinear; and we would have this plane (plane determined by the three points) and we would have the other one ( $\alpha$ ) [models this moving her finger from her

palm in vertical position]. For me it continued being a folded quadrilateral.

Laura's next reflection was different because she questioned whether the use of the tool *Redefinition* to convert point  $D$  in a point that is not in  $\alpha$ , could place it in the plane determined by points  $A$ ,  $B$  and  $C$ . Then the figure would simply be a quadrilateral.

Laura: I believe that the issue is, when another point is redefined [ $D$ ], is that moving it, the risk of moving it so that (...). In moving it, it could end up being in the same plane determined by [point  $B$ ] that is not [in  $\alpha$ ] and [points]  $A$  and  $C$  that remained fixed in plane [ $\alpha$ ]. [...] That  $D$  could be in the plane determined by  $C$ ,  $A$  and  $B$  [a plane different from plane  $\alpha$ ]. Then it would not be a folded quadrilateral. Then I was saying, no. (...) But also, there are many possibilities that it might not end up [in that plane], in which case, yes it would be a folded quadrilateral.

Laura's explanation shows an interesting extension of visualization, because she is considering a plausible result from the use of the *Redefinition* tool of Cabri 3D.

Some students changed their previous idea about the resulting figure, because initially they did not recognize it had the configuration of a folded quadrilateral. They expressed this to the researcher:

Juan: Because I had not seen the other visualization of another plane. Because let's say a point, because anyway another point is going to remain outside. Then I said: something is needed.

Arturo: For me also [I was interpreting the same], only until the other plane was constructed.

Researcher: Only until the plane was constructed you knew (...)?

Arturo: (...) that it was folded.

Evidently, Juan's and Arturo's visualization of folded quadrilaterals broadened as a result of their observation of the representation of the plane determined by points  $A$ ,  $B$  and  $C$ . Together with Santiago's argument: "two sides that intersect determine a plane", these became useful theoretical resources which the students could use to determine planes in 3D geometry.

Up to this point, the discussion illustrates two elements in the development of the task: the appearance of two divergent points of view regarding what is or not an acceptable representation of a folded quadrilateral, and the way the students try to resolve those differences. This development is what we now discuss in terms of the aspects incorporated in the task design: uncertainty and intellectual necessity.

## DISCUSSION

The use of the Cabri 3D *Redefinition* tool allowed teacher and students to represent a figure, with two vertices outside plane  $\alpha$ , that fulfilled the conditions stated in the definition of a folded quadrilateral. The modification of the initial representation of a folded quadrilateral, by taking another vertex out of plane  $\alpha$ , generated doubt in the

students, giving place to the desired uncertainty situation (is the new representation a folded quadrilateral?). This doubt was solved by means of arguments proposed by the students. The use of this tool, together with the construction of the plane determined by a new set of three non-collinear points, helped visualize the existence of planes in space, not represented in the initial configuration. In addition, *Redefinition* became a tool to solve 3D geometry problems. Having redefined two points, plane  $\alpha$  became a distraction in the representation, and generated doubt in some students, which was solved when they noticed that plane  $\alpha$  was not the only plane in space. The student's visual schema of space as a horizontal plane and some points not in that plane was modified.

The arguments exhibited by the students, especially in their interaction with the researchers, allowed us to identify that their concept image (Vinner, 1991) of a folded quadrilateral consisted of the specific position of three vertices lying in a horizontal plane, the other vertex above it. This motivated a class discussion to decide whether to accept or not other representations of the figure fitting the definition. Their arguments, as a result of the uncertainty provoked by the task, were a resource to analyze how to develop the intellectual need to clarify that only the properties given in the definition are needed to decide whether the representation is the defined figure. In addition, Laura's argument enriched our forecast of what can happen during the discussion of the solution of the task in future years.

As for the elements that must be articulated in the design and teacher management of a task, one is the transformation of familiar situations, without losing basic properties, to create doubt and to modify restricted images. Only if the students can visualize planes other than the one represented initially, will they be able to solve problems in 3D geometry. They must identify the geometric figures that are in each plane and the properties they have, so that their knowledge of plane geometry can become a resource to solve the problems.

## CONCLUSIONS

The first relevant issue we observe is that uncertainty is produced and expressed when the students voice different points of view regarding what is or not a representation of a folded quadrilateral. Doubt probably arises due to the difficulty to visualize other planes in a configuration in which a plane is already represented. The second relevant issue is that uncertainty gives way to intellectual necessity, for the students resort to the axiomatic system conformed in the course, specifically how to determine planes, to be able to find a different plane than the one represented. The third issue is that the ideas the students expressed to the researcher should be heard in a normal class setting. Thus, only if a teacher is aware of the moments when uncertainty arises during a task development, and, precisely at that moment, induces students to communicate their ideas and promotes argumentation, will intellectual necessity be generated, and, as a consequence, meaning-making favored.

## References

- Bakker, A., & van Eerde, D. (2015). An introduction to design-based research with an example from statistics education. En A. Bikner-Ashbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 429-466). Dordrecht, The Netherlands: Springer.
- Clarke, D. J. (1997). Studying the Classroom Negotiation of Meaning: Complementary Accounts Methodology. *Journal for Research in Mathematics Education. Monograph*, 9, 98-111.
- Ferrara, F., & Mammana, M. (2014). Seeing in space is difficult: an approach to 3D geometry through a DGE. En S. Oesterle, P. Liljedahl, C. Nicol, & A. Darien (Eds.), *Proc. 38th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 57-64). Vancouver, Canada: PME. Recuperado a partir de <http://www.pme38.com/>
- Harel, G. (2013). Intellectual need. En K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119-151). New York: Springer.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: results from exploratory studies. En A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (pp. 234-283). Providence, RI: American Mathematical Society.
- Moore-Russo, D., & Schroeder, T. L. (2007). Preservice and inservice secondary mathematics teachers' visualization of three-dimensional objects & their relationships. En T. de S. Lamberg & L. R. Wiest (Eds.), *Proc. 29th Conf. of the North-Amer. Chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 389-392). Stateline (Lake Tahoe), NV: PME.
- Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education*, 15(3), 266-285.
- Sgreccia, N., Amaya, T., & Massa, M. (2012). ¿Qué dicen los docentes, futuros docentes y formadores de docentes sobre su formación en didáctica de la geometría 3d? *Quaderni di Ricerca in Didattica /Mathematics (QRDM)*, (22), 1-20. Recuperado a partir de <http://math.unipa.it/~grim/quaderno22.htm>
- Stylianides, J., & Stylianides, J. (2009). The transition from facilitating to proof empirical arguments. *Journal for Research in Mathematics Education*, 40(3), 314-352.
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer
- Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, 60(3), 297-321.



in the fact that most PTs have problems working with basic operations, and difficulties relating different mathematical concepts and developing arguments.

Other studies highlight the deficit of the PTE system. According to Ávalos (2014), PTs study in a system with great diversity, which often does not meet minimum quality standards. Moreover, a study focused on characterizing primary school PTE showed that many MTE have precarious working conditions and do not have opportunities for professional development (Mineduc, 2016). The same study showed that the math courses in these programs do not meet the requirements of the new school curriculum, particularly regarding the development of mathematical skills.

In view of this need, the Laboratory of Education of the Center for Mathematical Modelling (CMM) of the Universidad de Chile is developing a R+D project that has two main goals: first, to increase MKT in PTs through activities that foster inquiry, the analysis of learning situations, and the development of mathematical skills; second, to support and guide MTEs' use of active learning methodologies. In the project, Learning Units for teacher training are being designed, which are sequences of lessons around a mathematical topic of high impact for PTE. The units include mathematical tasks for teaching and supporting resources for MTEs.

This study focuses on the pilot experiences of the Learning Units. We sustain that these units have a significant impact in the perceptions of the mathematical teaching of PTs participating, and that those perceptions acknowledge some of the principles that guided the design of the Learning Units. Considering this as an ongoing project, we address the following research question: How do PTs participating in the pilot recognize and value some of the principles that guided the model's design?

The article is structured as follows: The first part provides a description and justification of the Learning Units design, including an overview of the three principles analyzed in this work. This is followed by an explanation of the research methodology. Finally, there is an analysis of the study's results and its main conclusions.

## **DESCRIPTION OF THE LEARNING UNITS**

The Learning Units were developed by a multidisciplinary team composed of mathematics teachers, mathematicians and MTEs from several universities across the country. The development followed an elaboration-testing-adjusting design cycle. Four units were developed during 2017, whose topics were selected for their high impact on initial training and the feasibility of being tested by the MTEs. Two of the units deal with Numbers: *Addition and Subtraction Problems* (N1), that covers the classification of these problems according to the actions involved and the place of the unknown (Lewin, López, Martínez, Rojas & Zannoco, 2010); and *Representing addition and subtraction problems* (N2), which seeks to identify concrete and pictorial representations of these problems and discussing their pertinence (Veloo & Parmijt, 2017). In addition, two Geometry units were developed: *Definition of perimeter* (G1), which addresses the process of constructing a definition of a contour of a shape, and problem solving involving perimeters (Lu, Weng & Tuo, 2013); and *Variations of area*

and perimeter (G2), which deals with the relationship between area and perimeter when changing geometric shapes (Ma, 1999).

Each unit is designed to be used in two consecutive 90-minute lessons, and includes a sequence of 4 or 5 activities. The supporting material for the students consists of worksheets and a lesson plan for the MTE, which includes the purpose, instruction modality, possible student answers, teaching suggestions, pedagogical notes of each activity, and recommendations to carry out the transitions between activities.

### **Principles for the design of the Learning Units**

The Learning Units developed through this project correspond to a didactic device whose design considers a series of principles, such as:

- To offer opportunities to develop *Mathematical Knowledge for Teaching* (MKT) (Ball, 2008).
- To foster a *collaborative construction of mathematical knowledge* (Ball & Bass, 2000), which acknowledges the role of interactions between students mediated by the MTE.
- To promote a teaching process that recognizes *situated learning* (Brown, Collins & Duguid, 1989) as a way to favor learning to teach through activities that bring PTs closer to specific tasks in their future professional work.

According to the MKT model, the Learning Units focus mainly on Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT). In the Geometry units, the emphasis was on the SCK. The proposed activities seek that PTs construct a definition of the boundary and perimeter of a plane shape, solve problems related to perimeters using visual, inductive and deductive reasoning, and develop arguments to analyze statements about changes of area and perimeter. On the other hand, the Numbers units mainly address KCS and KCT. In them we find activities whose purpose is for PTs to observe different actions and recognize the various types of addition and subtraction problems, distinguish those that are easier or more difficult to solve for a child, identify concrete and pictorial representations for these problems and what motivates the transit from one to another.

With respect to the second principle, the methodology of Mathematical Discussion (Chapin, O'Connor and Anderson, 2003) was adopted as a strategy to encourage classroom interactions that can lead to the collaborative production of mathematical knowledge. Thus, the lesson plan includes whole-class discussions in most of the activities and provides teaching suggestions with questions to foster the discussion.

The situated learning approach was incorporated by using classroom contexts, such as activities based on the analysis of videos, student productions and case studies. These activities seek to bring the PTs closer to didactical problems inside school classrooms. An example is the use of video clips of a child solving different types of addition and subtraction problems to motivate reflection on the difficulty in their resolution.

## METHODOLOGY

This study builds on pilots for the Learning Units applied in the third week of classes in mathematics courses for primary-school PTE programs of two universities in Santiago (A and B). Prior to the implementation, the MTEs were able to review in detail the planning of the units, and agreed to adjust to the guidelines and suggested times. During the implementation, the students were aware that all the activities carried out, as well as the material used in classes, were done in the context of a project.

A qualitative research approach was used to understand the perspective of students regarding their experience with the applied Learning Units (Flick, 2002). Focus groups, applied immediately after the end of the implementation, were used as a data collection technique, as they facilitated dialogue and discussion among participants, contributing to the exchange of ideas, opinions and reflections (Kidd & Parshall, 2000). A sample of 35 students volunteered to participate, distributed as the following table shows:

Course	University	Learning Units	Participants
1	A	N1 and N2	6
2	A	G1 and G2	7
3	B	G1 and G2	9
4	B	G1 and G2	4
5	B	G1 and G2	9

Table 1: Summary of the sample of the study.

Each focus group was guided by two researchers, a math teacher and a researcher in education. They had an average duration of one hour and included a set of guided questions aimed at understanding the perceived main learnings achieved, the types of mathematical knowledge that were developed and the teaching practices adopted by the MTE. All the focus groups were recorded and subsequently transcribed.

The preliminary analysis was carried out by two members of the research team, who coded the responses in several emerging categories. These categories were defined through a Constant Comparative Method (Strauss & Corbin, 2007). Those segments were then cross-referenced with the three principles considered in this study and reviewed once again. To ensure the reliability of the results, they were also cross-checked by the members of the research team.

## RESULTS

We will now examine the main results obtained in the analysis, with quotes that were selected to faithfully convey the meaning intended in the evidence. Each quote is identified by the course number described in Table 1.

The first interesting result is that in all the focus groups, the PTs recognize that the lessons were different from the ones they were used to, considering their school and university experiences, as it is seen in the following quotes:

Something I want to say about the class is that in school, at least in my case, I was used to being given the formula, and I only applied it. But here, it is how I discover things (FG Course 2).

I noticed a lot the difference when these four classes started. It was very clear the difference with respect to how we had been working previously (FG Course 4).

What was also very noticeable was the participation that each one had. Because in previous classes we write and [only] the one who knows the content speaks. However, in these classes it was not like that, because one could say what one thought about the content and see if it really was that way or not. So that is what Peter said about constructing our own learning (FG Course 4).

As these quotes suggest, the PTs recognize the change in teaching methodology that the device promotes, which involves a more active participation of students.

The results that follow are organized according to the three principles that underlie the Learning Unites design:

### **Development of Mathematical Knowledge for Teaching**

The students pointed out that the Learning Units contributed positively to the development of different types of mathematical knowledge. Faced with the question "What have you learned throughout these lessons?", the PTs recognize that the units develop knowledge they consider relevant for their future teacher practice. In addition, their description suggest that they identify certain learnings related to the SCK, KCT and KCS, which coincide with the types of knowledge that promote the activities, as the following quotes show:

Because we have the idea of a perimeter, which apparently, we all thought was right, [...] and suddenly someone comes out and says no, [...], and you realize that you must agree. And it seems that an idea as basic as a perimeter, something that perhaps we should have very clear, is not so clear. And that [it is] interesting to agree to remake a definition. (FG Course 2, SCK).

When we were in school it was not like this, they give the problem and it was more intuitive, "ah, I only have to add" and ready, but I did not know "this is a comparison or transformation [problem]". So, in that sense, as Denisse said, being able to classify helps you as a teacher in the future to formulate problems and that was, for me, the most important thing we saw in these four classes. (FG Course 1, KCT).

Sure, analyze or deduce what the child might do. I think that was a very important discussion topic: how to think about what a child would do, and not about what I would do. (FG Course 1, KCS).

### **Collaborative construction of mathematical knowledge**

Another interesting result is that the PTs recognized that learning was built collaboratively through the sequence of activities, highlighting Mathematical Discussion as an element that significantly contributes to this process. The PTs distinguished various ways in which it can be articulated. First, they recognized the role of the MTE as a mediator, as can be seen in the following quote:

It was very interesting that when someone had a question and asked it, the teacher expected us to answer it ourselves, to try to explain it. And it was very cool, because it was very hard to explain. (FG Course 2).

In other cases, the Mathematical Discussion was perceived as a means for building knowledge among peers. This was positively valued, although sometimes it caused concern about the conclusions reached through these discussions:

S1: At one point, for example, if the majority said that it was such and such thing and another smaller group said that it was not, that it was the other one, then they had to defend why they thought it was so. Then, that generated arguments. That was, like, I do not know, it was cool. But on the other hand it was also confusing, because all the arguments were good.

S2: And there was a problem because we came to the supposed conclusion while the ideas were still up in the air and immediately another exercise sheet came, so [the conclusion] was like getting lost.

S1: That debate was cool, but if it had been left more settled, it would have been better (FG Course 1).

It is relevant to observe how the interactions proposed in the device, which sought to promote that the MTE involves the work of the groups and leads whole-class discussions, was highlighted by the PTs.

### **Situated learning**

One way in which the students identified that the units brought the school classroom closer to the PTE is the concrete use of classroom contexts in the activities:

When they showed the videos like this: the child would end up adding and they would ask him to subtract and it was like, "Oh, he was wrong for this reason. And then you had to analyze and think about how the child had made the mistake. (FG Course 1).

On the other hand, the data shows that students also approach the classroom symbolically, that is, when they reflect on their learning experiences in relation to the challenges they will face in their future teaching work:

What he is looking for is our knowledge about the topic, and then he takes our knowledge, which is going to be very diverse, and that is what happens in the classroom: one exposes a case in the classroom and all the children have different thoughts about that topic, and that is what the activities are looking for somehow. [...]

So, it is not so much about the discipline, but how we are going to, somehow, model how children will learn. (FG Course 3).

The interesting thing about this is that it shows how students begin to position themselves as future teachers because of their learning experience:

That I think [...] that we are just learning how to stop thinking of ourselves as students and [begin to] think of ourselves as teachers. Because before, when we are students, we are kind of focused on our way of developing problems. But [now with these activities] we have to put ourselves in the child's place and see the different approaches that children can use and be prepared to see the big picture. (FG Course 1).

## DISCUSSION AND FUTURE WORK

The aspects highlighted by the PTs suggest that the Learning Units support lessons consistent with the three principles that underlie their design, thus contributing to developments in the PTs mathematical and didactic knowledge relevant to their future work. This would confirm the feasibility of using these types of devices to support PTE in mathematics.

The evidence suggests that Mathematical Discussion contributes to making PTs aware of the idea that learning has a significant social component and that mathematical knowledge can be constructed collaboratively (Putnam & Borko, 1997). Although the PTs valued the discussions as instances to share their ideas and thinking, they made evident at times the lack of systematization at the close of the discussions.

On the other hand, it is observed that learning becomes more significant for the PTs when they are placed within a classroom context. It is striking that the reflection concerning the classroom not only takes place in the activities designed for that purpose, but also in the way they connect their learning experience with those of their future students when addressing other mathematical tasks. A possible explanation for this is that the activities that entail classroom situations trigger a permanent reflection regarding the teaching endeavor. This would also be a contributing factor to their perception as future teachers.

Finally, the study shows the need for an in-depth analysis of the data to answer questions such as:

- i) What improvements to the design of the device could help the discussions to better systematize closure ideas?
- ii) What characteristics of the didactic device trigger pedagogical reflection in PTs and contribute to making them envision themselves as future teachers?

**Funding:** This research was funded by FONDEF ID16I10119, FONDECYT #3170423 grant and Basal project CMM U. de Chile and UMI 2807 CNRS.

## References

Ávalos, B. (2014). La formación inicial docente en Chile: Tensiones entre políticas de apoyo y control. *Estudios pedagógicos*, 40, 11-28.

- Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the National Society for the Study of Education, Constructivism in Education*, 193–224. Chicago: University of Chicago Press.
- Ball, D.L., & Forzani, F. (2011). Building a common core for learning to teach and connecting professional learning to practice. *American Educator*, 35(2), 17-21, 38-39.
- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(3), 14-22, 43-46.
- Ball, D.L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Boyd, P. (2014). Using ‘Modelling’ to improve the coherence of initial teacher education. In P. Boyd, A. Szplit & Z. Zbróg (Eds.), *Teacher Educators and Teachers as Learners: International Perspectives* (pp.51-73). Kraków: Wydawnictwo Libron.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18 (1), 32-42.
- Chapin, S., O’Connor, C., & Anderson, N. (2003). Classroom discussions. Using math talk to help students learn. California, *Math Solutions Publications*. Sausalito, California, USA.
- Flick, U. (2002). *An introduction to qualitative research*. London: SAGE.
- Kidd, P., & Parshall, M. (2000). Getting the focus and the group: enhancing analytical rigor in focus group research. *Qualitative Health Research*, 10(3), 293–308.
- Lewin, R., López, A., Martínez, S., Rojas, D. & Zannoco, P. (2010). *Números para futuros profesores de educación básica, REFIP*. Santiago: Ediciones SM.
- Lu, P., Weng, K. & Tuo, Y. (2013). Perimeter in the primary school mathematics curriculum of Singapore. *Journal of the Association of Teachers of Mathematics*, 236, 27-30.
- Ma, L. (2010). *Knowing and Teaching Elementary Mathematics. Teachers’ Understanding of Fundamental Mathematics in China and United States*. New York: Routledge.
- Mineduc (2016). Final Report Identification of critical elements to strengthen the pre-service elementary teacher training of teachers in Mathematics.
- Putnam, R. T., & Borko, H. (1997). Teacher learning: Implications of new views of cognition. In B. J. Biddle, T. L. Good, & I. F. Goodson (Eds.), *International handbook of teachers and teaching*. (Vol. 2, pp. 1223-1296). Dordrecht, The Netherlands: Kluwer.
- Strauss, A., & Corbin, J. (2007). *Basics of qualitative research*. New York, NY: SAGE.
- Tatto, M. T., Schwille, J., Senk, S. L., Ingvarson, L., Rowley, G., Peck, R., Bankov, K., Rodríguez, M., & Reckase, M. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-M)*. IEA, Holanda: Amsterdam.
- Veloo, P. & Parmijt, S. (2017). Using Representations to Develop Mathematical Thinking. In B. Kaur & N. H. Lee, *Empowering Mathematics Learners. Yearbook 2017* (pp. 249-269). Singapore: Association of Mathematics Educators.

# TENSIONS AND DILEMMAS ABOUT WHAT IS “LEGITIMATE” IN A GEOMETRY COURSE

Oscar Molina, Luis Pino-Fan, Vicenç Font

U. Pedagógica Nacional (Colombia), U. de Los Lagos (Chile), U. de Barcelona  
(Spain)

*In this Research Report we present three tensions (about the legitimacy of the use of theoretical objects, the way to communicate productions, and the evaluation), and associated dilemmas, that students or teachers experience when faced with a 3D geometry task. We illustrate how certain norms, explicitly set by a teacher, can create tensions, and how the negotiation process of these norms brings momentary dissipation of such tensions, due to the students’ interpretation of informal argument. We highlight the teacher’s role in the negotiation process and how she legitimizes certain informal arguments used to include an object in the theoretical system consolidated in the course.*

## INTRODUCTION

The constitution of a microculture that enables interaction among participants is characterized by a system of knowledge, beliefs, behaviours, etc., shared by the members of the group. Each classroom establishes, maintains, negotiates or eliminates patterns (norms, standards, etc.) that distinguishes it (Cobb & Yackel, 1996). Argumentation is a key element for learning mathematics. From this sociological perspective, several authors have pointed out that teachers need to manage a system of norms that allows students, in inquiry-based classrooms, to engage in argumentation (Cobb & Yackel, 1996; Yackel, 2002) or in the classroom activity *doing proofs* (Dimmel & Herbst, 2014). In this context, an interest or need being, for example, clarifies what counts as a legitimate or valid argument in the math classroom.

To address the above-mentioned need, our research seeks to clarify how the dynamics of a system of norms that regulates discursive interaction in a 3D geometry course (of a preservice mathematics teacher program in Colombia) influences both argumentation and the inclusion of objects in the theoretical system being consolidated. A review of the data suggests that students and teachers experience tensions related to the legitimate use of theoretical objects and the production of informal arguments, generated by the system of norms that regulate their argumentative practices (Herbst, 2003; Gorgorió & Planas, 2005). We intend to provide answers to the following questions: (i) How do certain norms specified by the teacher influence argumentation? (ii) Does the system of norms create tensions for students and teachers? If so, (iii) how are such tensions managed by the teacher? With this, we contribute by illustrating how the tensions between norms and their management is another way of seeing the influence of norms on the production of particularly informal arguments, and how these are legitimized through their use to install objects to the theoretical system.

## THEORETICAL FRAMEWORK

The theoretical framework consists of the following constructs: norms, tension, dilemma and argument. Cobb and Yackel (1996) defined norms as characterizing “regularities in communal or collective classroom activity [which] are considered to be jointly established by the teacher and students as members of the classroom community” (p. 178). Given the varied conceptualization about norms in the literature (e.g., *social* and *sociomathematical* norms –Cobb & Yackel, 1996–, norms of *mathematical practice* –Gorgorió & Planas, 2005–, *mathematical norms* –Sánchez & García, 2014–), the Onto-Semiotic Approach (OSA) has made a theoretical proposal on the normative dimension that includes the previous perspectives (D’Amore, Font & Godino, 2007). They point out that the norms are not only regularities; they are also the language rules that are intended to regulate classroom practices. They allude to epistemic norms –labeled Ne– (regulate mathematical activities in any context), interactive –labeled Ni– (regulate interaction processes), cognitive –labeled Nc– (regulate teaching-learning processes), mediational –labeled Nm– (regulate the use of resources). With this, sociomathematical and mathematical norms can be considered into epistemic norms, and social norms into interactive or cognitive ones. A norms system serves as a reference with which to explain the meaning of actions that emerge from classroom practices (Herbst, 2003). We used the OSA’s proposal to typify the norms identified in class practices and, with this, to characterize what happened in them. The norms system is dynamic; the dynamism can create *tensions* between the responsibilities suggested by the norms; they must be managed by students and teachers to overcome the *dilemmas* that such tensions cause. A *tension* is the latent opposition between two equally important but potentially conflicting aspects in a certain context (Herbst, 2003). A *dilemma* is the reason that explains why, having various options, whichever one is chosen leaves a sensation of loss (Lampert, 1985).

Given that we intend to specify the influence of the norms in the argumentation, specifying a conceptualization in this respect is convenient. We use Toulmin’s basic Model (Pedemonte, 2007). In such basic model, an argument includes three elements: the speaker’s claim (C), data (D) supporting the claim C, and warrant (W) or the inference rule, which relates the data with the claim. *Argumentation* is the process by which an argument is produced. The model is quite useful for specifying different types of arguments (informal or not) according to the way the three elements are related (Pedemonte, 2007). *Deduction* is an inference allowing the construction of a C starting from some D and a W. *Abduction* is an inference of D from an observed fact C and the evocation or discovery of W. *Induction* is the inference of a W from some cases of D in which a pattern of regularity C is observed. *Analogy* (Juthe, 2005) starts from the comparison of two domains (S, O) by means of statement  $p' \text{ is to } q' \text{ as } p \text{ is to } q$  (W). Given  $p, q$  in S;  $p', q'$  in O (domain less known than S);  $p', q'$  respectively comparable with  $p, q$ , and a relationship between  $p, q$   $[R(p, q)]$  valid in S (D). Then  $[R(p', q')]$  can be inferred as possibly valid in O (C). Figure 1 shows the diagrams for each type of

argument using Toulmin's Model. The rectangles with a thicker line indicate what is inferred; those with dashed lines indicate that the inference is likely.

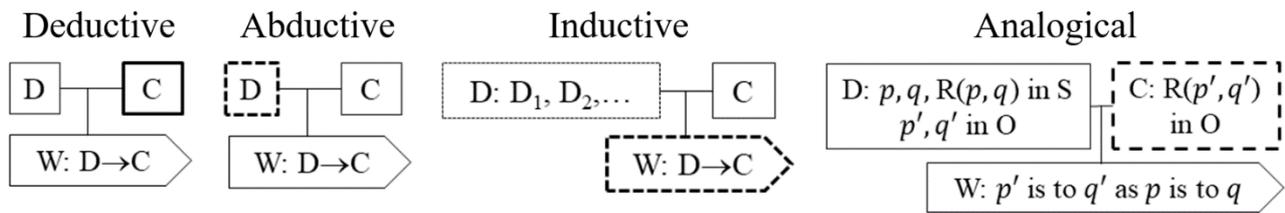


Figure 1. Diagrams for each type of argument

## METHOD

The study uses classroom-based research as a strategy (Kelly & Lesh, 2000). The sample consists of 33 preservice mathematics teachers of a 3D geometry course (Bogotá, Colombia) who had passed two 2D geometry courses. The data for this report is obtained from the activity that took place when the students addressed two problems. These were chosen because they were intended to allude to 3D objects for the first time in the course; in addition, the review of the data regarding its solution by a group of students suggested certain tensions. Its statements are as follows:

P1: Consider points  $A$ ,  $B$ ,  $C$  and  $D$ . What figure would the union of segments determined by these points be if the segments that intersect do so only at their endpoints and each of such points is the intersection of exactly two segments?

P2: Is space different from a point, a line and a plane?

Regarding the 3D geometry, the main aim of the problem was to generate for the student the theoretical need to establish the Space Postulate (given a plane, there exists a point that does not belong to it). In P1, because having four non-coplanar points is a possibility; in P2 because only that possibility makes the space and plane different.

The analysis was centered on the production of a group of three students (group I) for P1, and the production of thirteen students for P2 (which were chosen by the teacher to present in class), including the teacher modulated social communication of results. The group I was chosen from the beginning of the research to record their productions; the criteria for its selection were frequent attendance and disposition to talk to each other. Both situations were analyzed because the first scenario revealed certain norms (and associated tensions and dilemmas) that led to the need for negotiation, and the second revealed the students' interpretation of the adjusted norms after such negotiation. The data was obtained from the students' written productions, videotapes (of the students when they tackled the problem; of the whole-class when results were shared with the teacher's guidance) and from semi-structured interviews. The transcripts were studied according to the levels of didactic analysis proposed by the OSA (Font, Planas & Godino, 2010) with emphasis on *object and process analysis* and *normative analysis*. Through the former, primary objects (concepts, proposition, arguments, etc.) that emerge from practices were identified; specifically, Toulmin's basic model was used to structure the arguments produced by the students. Through

the latter, the norms were identified and typified according to OSA's proposal. These were identified in three ways: by clarification of what is expected, by reiteration of certain acts, or by violation of a norm (which implies the emergence of another one to replace it). The normative analysis permitted identifying, as an emergent process, tensions between norms that regulated practices when problems were addressed. The detected tensions were typified in relation to: the legitimacy of the use of objects (which objects can be used in a solution procedure or in an argument), communication (what should be reported in the solution), and evaluation (which criteria are most valuable to the teacher). The dilemmas were contextual and made explicit with the analysis.

## ANALYSIS AND RESULTS

At the beginning of the course, the teacher exposed several norms to regulate class activity; some of them were: (i) objects are introduced into the theoretical system through problem solving –Nc1–; (ii) solving a problem involves constructing and exploring in a Digital Geometry Software (DGS), formulating a conjecture-solution and justifying it –Nc2–; (iii) in the solution of a problem, their own ideas must be presented –Nc3–; every argument must use objects from the theoretical system available in the course –Ne4–. We present examples of how these norms regulated the activity of some students with respect to P1 and P2.

In relation to P1, we analyzed a brief fragment of the dialogue of group I (Adr, Bra, Jef) when they discussed how to approach the problem.

2. Bra: Well it's a square... A quadrilateral, I mean.
3. Jef: But... Wait. If it says [in the statement] that the segments only intersect at the endpoints ... Oh, yes, it can be any quadrilateral.
4. Adr: And are the points coplanar?
5. Jef: Well... that is not there [in the statement] ... Wait, let's look [reads the statement]. It only says four points.
6. Adr: So, it can be a pyramid or something like that
12. Bra: Well, we always have to put it [the conjecture], and justify it, and the... what is done in Cabri [...] So quadrilateral

They did not refer to non-coplanar points in their final report. We interviewed them to find out why they ruled out such a possibility (Res indicates Researcher):

5. Res: You only considered that the points [A, B, C, D] be coplanar ... But at the beginning, I believe Adr said, questioned whether the points were coplanar
6. Jef: It's that in the problem, being coplanar was not required.
7. Res: And, did you take into account that possibility?
8. All: No.
9. Res: And, why? At the beginning you said something about that...
10. Bra: Well... it's that we don't know ... what we would put there ...

11. Jef: It's that we could talk about quadrilaterals with everything we have [referring to the objects included in the theoretic system].
15. Bra: [...] It's that, if we were to justify something, well about that... the quadrilaterals, the types, we can say something.
16. Res: And what would happen if you had taken the points as non-coplanar? ...
17. Adr: It's that, there we are missing things ... To know that that can be done... Have points that are not in the same plane.
20. Res: Mmmm I see. You went towards what you know, surely, so you can justify, right?
21. Bra: Well yes...

Even though the students understood that not only coplanar points [4, 5, 6, group interaction] should be considered, they decided to approach the problem considering characteristics, so that the figure ended up being a convex quadrilateral, and therefore, the given points were coplanar. Bra highlighted the need to comply with the required characteristics of the written report of their solution to the problem [12]. We see that Bra exercised normative control by reminding them the need to comply with **Nc2**. In the interview, when asked why they did not study the possibility of non-coplanar points [7], Bra [10,15] and Jef [11] agreed that they have theoretical objects with which to justify something (**Nc2** y **Ne4**), when the points are coplanar. This idea was complemented by Adr, when they were asked what would happen if they had considered non-coplanar points [16]. He mentioned that, in that case, they would have needed objects not included in the theoretic system, such as, *the existence of non-coplanar points* (statement **s1**) [17]. Although they considered the possibility, they discarded it because it was not part of the theoretic system, and they had no way to justify it. This interpretation was confirmed by Bra's intervention [21] in response to Res's claim in [20]. We conceive that students experienced two tensions between norms (regarding question ii), and a dilemma caused by those tensions (Table 1).

Tensions	Dilemma
Regarding <i>the legitimacy of using objects</i> : they used new objects (e.g., <b>s1</b> ) to solve the problem ( <b>Nc1</b> ), or they presented arguments using previously included 2D geometry objects ( <b>Ne4</b> ). Regarding <i>communication</i> (product of the previous tension): they reported all their ideas, including considering <b>s1</b> ( <b>Nc3</b> ), or they reported arguments with known objects ( <b>Ne4</b> ).	Between issues of <i>authenticity</i> and <i>formality</i> : if they were authentic (they presented original ideas such as <b>s1</b> ), they could not provide formal arguments (a deductive argument using system objects); if they tried to argue formally, the original ideas could not be reported.

Table 1. Tensions and dilemmas

The teacher made two comments about the above when she addressed the students' productions for P1:

1. There were two groups that did not report all the ideas they had [...] It is important to report everything you do or think [...]; there might be interesting things.
2. Some groups gave an informal argument [for their conjecture] ... You didn't have the necessary objects to do it completely [the proof]. [...] But they gave ideas. I expect arguments like those [informal]; later we will see how to "prove" it.

Based on above, we see that the tensions probably had an effect on the teacher, causing her to feel a tension relative to the evaluation of the students' productions. The tension contrasts two criteria: She values more the report of authentic ideas (**Nc1**, **Nc3**), or she values more the production of formal arguments (**Ne4**). In this case, she gave priority to the former and considered a flexibilization of **Ne4**. That is, as part of the solution to the problem, *it is acceptable to provide informal arguments, understood as those that present ideas that use objects not included in the system (Ne4')*. This flexibilization is a result of the (implicit) negotiation (regarding question iii) between students and teacher that arose from the students' own practice (described by the teacher in 2). Since some groups considered the possibility that the given points in P1 be non-coplanar, the teacher proposed P2. In response, the students produced informal arguments according to their own interpretation of Ne4'. As an example, we present the productions of students E1 and E5, and the corresponding analysis in Table 2 (regarding question i).

E1: Since infinite points belong to a line, infinite lines are in a plane. If space is taken to be the plane, then the same thing happens. For space to be different, infinite planes must be contained in space.

E5: I think that if two points determine a line and three points determine a plane, perhaps at least four points are required to talk about space. If you have at least four non-coplanar points, the existence of space is somehow guaranteed. Since space contains everything, it is different.

The students' interpretation of Ne4' led to the production of arguments that the specialized literature has recognized as informal (Pedemonte, 2007). That is, of analogical and inductive arguments (to infer s3 and s8) or abductive arguments (to see the need for s8), where s3 and s8 would be new theoretic objects.

While sharing students' answers to P2, the teacher commented on the students' argumentative ideas refuting the objects (data, warrants) that make them up; she legitimized informal arguments (inductive and analogical) to install s1 as a postulate.

## CONCLUSION

The normative analysis allowed us to identify tensions between *epistemic* and *cognitive* norms. The tensions of *legitimacy* and *communication* brought to light four interesting issues: (i) A student dilemma that contrasts issues of idea authenticity and aspects of argument formality; (ii) an effect on the teacher, translated into a tension regarding her responsibility to evaluate confronting assessment criteria (Which has more value, original ideas or production of formal arguments?); (iii) an indicator of the negotiation of norms that corresponds to the teacher's management of this tension, a negotiation that was induced by the students' practices and that led to the flexibilization of a norm; (iv) the production of different types of empirical arguments (inductive, abductive and

analogical) to solve P2, product of students’ interpretations of informal argument (concept present in Ne4’). Although the teacher did not expect such variety of arguments, she used them to include a proposition (s1) as a postulate. Our main contribution is not only to illustrate tensions between specific types of norms – complementing works of Lampert (1985); Herbst and his colleagues (2003, 2014); and Liljedahl and his colleagues (2015)–, but also how their management by teachers can elicit informal arguments that are legitimized because, though them, (i) concepts or propositions are installed as postulates, or (ii) plans for constructing object or development proof can be proposed. So, we provide a concrete criterion of legitimacy for informal arguments in a formal geometry course –complementing works of Yackel and colleagues (1996, 2002)–. This is another matter, among the facets of professional knowledge about the norms, that a teacher must consider.

Student E1	Student E5
<p>The assertion is <i>Space <math>\varepsilon</math> is different from a plane <math>\alpha</math></i> (s2). Two arguments constitute the global argument, one analogical and one abductive. The former has as claim <i>infinite planes are subsets of <math>\varepsilon</math></i> (s3 -equivalent to s1-). The analogy is <i>infinite plane are in <math>\varepsilon</math> as infinite lines are in <math>\alpha</math></i> (s4), which associates the domains 3D Geometry and 2D Geometry, respectively. The latter states the need to have s3 as data to conclude s2. The warrant “s3 then s2” and s3 must be proven (Figure 2).</p>	<p>The assertion is <i>Space <math>\varepsilon</math> is different from a plane <math>\alpha</math>, a line <math>m</math> and a point <math>P</math></i> (s6). Two arguments constitute the global argument, one inductive and one abductive. The former has as cases: <i>Two points determine a line <math>m</math></i> (s7) and <i>three non-collinear points determine a plane <math>\alpha</math></i> (s8). The warrant <i><math>n</math> points can determine an object with <math>n-1</math> dimension</i> (s9 –equivalent to s1–) is induced. Hence, using this warrant to the 3D case implies that <i>four non-coplanar points determine <math>\varepsilon</math></i> (s10). The latter has as data <math>P, m, \alpha</math> and <math>\varepsilon</math> (s11) and warrant the <i>space definition</i> (<math>\varepsilon</math> contains everything –s12–) –Figure 3–.</p>

Table 2. Analysis of students’ arguments

References

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3-4), 175-190.

D'Amore, B., Font, V., & Godino, J. (2007). La Dimensión Metadidáctica de los Procesos de Enseñanza y Aprendizaje de la Matemática. *Paradigma, 28*(2), 49-70.

- Dimmel, J., & Herbst, P. (2014). What details do geometry teachers expect in students' proofs? A method for experimentally testing possible classroom norms. En C. Nicol, P. Liljedahl, S. Oesterle, & D. Allan, *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 2, págs. 393-400). Vancouver, Canada: PME.
- Font, V., Planas, N., & Godino, J. (2010). Modelo para el análisis didáctico en educación matemática. *Infancia y Aprendizaje*, 33(1), 89-105.
- Gorgorió, N., & Planas, N. (2005). Social representations as mediators of mathematics learning in multiethnic classrooms. *European Journal of Psychology of Education*, 20(1), 91-104. doi:<https://doi.org/10.1007/BF03173213>
- Herbst, P. (2003). Using Novel Tasks in Teaching Mathematics: Three Tensions Affecting the Work of the Teacher. *American Educational Research Journal*, 40(1), 197-238.
- Juthe, A. (2005). Argument by Analogy. *Argumentation*, 19(1), 1-27. doi:<https://doi.org/10.1007/s10503-005-2314-9>
- Kelly, A., & Lesh, R. (2000). *Handbook of Research Design in Mathematics and Science Education*. . N.J.: Lawrence Erlbaum Associates, Inc.
- Lampert, M. (1985). How Do Teachers Manage to Teach? Perspectives on Problems in Practice. *Harvard Educational Review*, 55(2), 178-194.
- Liljedahl, P., Andra, C., Di Martino, P., & Rouleau, A. (2015). Teacher tension: important considerations for understanding teachers' actions, intentions, and professional growth needs. En K. Beswick, T. Muir, & J. & Wells, *Proceedings of 39th Psychology of Mathematics Education conference* (Vol. 3, págs. 193-200). Hobart, Australia: PME.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23-41. doi:<https://doi.org/10.1007/s10649-006-9057-x>
- Sánchez, V., & García, M. (2014). Sociomathematical and mathematical norms related to definition in pre-service primary teachers' discourse. *Educational Studies in Mathematics*, 85(2), 305-320. doi:<https://doi.org/10.1007/s10649-013-9516-0>
- Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior*, 21(4), 423-440. doi:[https://doi.org/10.1016/S0732-3123\(02\)00143-8](https://doi.org/10.1016/S0732-3123(02)00143-8)

# CONCEPTIONS OF MODELING BY CHILEAN MATHEMATICS TEACHER EDUCATORS: A PHENOMENOGRAPHIC STUDY

Helena Montenegro

Facultad de Educación, Pontificia Universidad Católica de Chile

*This study reports phenomenographic research which aimed to explore the conceptions of modeling held by mathematics teacher educators. Data were collected through semi-structured interviews conducted face-to-face with fifteen mathematics teacher educators working in three Chilean primary initial teacher education programs. The analysis identified four categories of description, ranging from modeling as performing pedagogical activities to modeling as implementing congruent teaching with the theoretical model to which they adhere. Besides, three dimensions of variation were found, providing a more accurate picture of the outcome space. These findings contribute to developing a more comprehensive and relational approach to studying mathematics teacher education.*

## INTRODUCTION

In Chile, mathematics teacher education has been identified as a critical area where, despite all the initiatives implemented, results have been far from satisfactory. For example, the Teacher Education and Development Study in Mathematics (TEDS-M) indicates that the pedagogical and mathematical education of recently graduated primary school teachers is insufficient for high-quality teaching (Ávalos & Matus, 2010). Besides, Rojas (2017) notes that primary preservice teachers feel that they develop more theoretical than practical knowledge, regardless of the mathematical contents imparted. These findings are worrying because they reveal that future mathematics teachers will be unable to provide quality learning opportunities due to the preponderance of traditional models focused on seeking mastery and memorization of skills.

According to Russell (1999), if we desire substantial changes in schools, then those changes may have to happen first in teacher education. This author also points out that initiatives aimed at strengthening the education of future teachers would be more impactful if they take into account the crucial role of teacher educators. For mathematics teacher education, these changes involve strengthening the knowledge necessary for teaching in a way that meets the educational demands of the 21st century (Ball, Hill, & Bass, 2005). These new pressures pose a challenge not only to preservice teachers but also to teacher educators who collaborate in this process (Loughran, Keast, & Cooper, 2016).

The above is especially relevant considering how unique the work of teacher educators is: when they teach preservice teachers how to teach, they are also modeling how to do so through their teaching practices. That is, they adopt a role model for preservice teachers associated with the strategies they use and the professional values they apply (Boyd, 2014; Goizueta, Montenegro, Rojas, & González, 2017; Loughran, 2006;

Lunenberg, Korthagen, & Swennen, 2007). Several studies have demonstrated that teacher educators' conceptions of teaching and learning influence their approaches to teaching (Boyd, 2014; Lovin et al., 2012; Struyven, Dochy, & Janssens, 2010); and pedagogical innovations developed by teacher educators, made explicit through modeling, may be a powerful tool for changing the practice of their preservice teachers (Boyd, 2014; Lunenberg et al., 2007; Struyven et al., 2010). Hence, researching the underlying conceptions of the modeling approach adopted by mathematics teacher educators may contribute to evidence and empirical results that will strengthen mathematics teacher education, helping the changes that we hope to see in the school system.

The present study might help us take on this challenge by exploring the conceptions of modeling held by mathematics teacher educators in three Chilean primary initial teacher education programs. The results found are expected to provide new ideas and insights about how mathematics teacher educators could improve their teaching practices, as well as specific suggestions about how modeling research may become a powerful strategy for enhancing mathematics teacher education.

## **THEORETICAL FRAMEWORK**

Traditionally, modeling has been defined as a practice of intentionally displaying exemplary teaching practice with the aim of promoting professional learning in preservice teachers (Lunenberg et al., 2007; Swennen, Lunenberg, & Korthagen, 2008). Lunenberg et al. (2007) have identified four types of modeling: (1) implicit; (2) explicit; (3) explicit modeling with emphasis on facilitating the translation to the student teachers' own practices; and (4) connecting exemplary behavior with theory. These types of modeling differ in terms of explicitness, the relation between theory and practice, and the role of preservice teachers in the process.

Despite the critical role that teacher educators play in the professional learning of teaching, several authors show that good modeling is not enough for preservice teachers to become aware of the teaching practices that teacher educators request them to learn about teaching (Boyd, 2014; Loughran & Berry, 2005; Lunenberg et al., 2007). In this regard, Darling-Hammond (2006) suggests that this apprenticeship of observation constitutes a significant hurdle for preservice teachers because they are faced to understand teaching in a way that is radically different from that which they experienced as students in the school system. As for teacher educators, Loughran (2006) states that their primary challenge is to modify the naive belief of teaching as 'telling' and learning as 'listening.'

In this context, it seems reasonable to suggest that modeling becomes relevant for mathematics teacher education programs because it is always present, either intentionally or unintentionally, when mathematics teacher educators teach (Loughran, 2006). In other words, the mathematics teacher educator is continuously an example for preservice teachers. He or she may have a substantial impact on the preservice teachers' views on teaching mathematics; as a consequence, they must always be fully

aware of what and how they are teaching (Loughran & Berry, 2005). On the other hand, preservice teachers can only learn to teach if mathematics teacher educators make this (normally tacit) process explicit by highlighting the pedagogical reasoning that supports the teaching of a specific set of contents or a pedagogical strategy (Loughran, 2006; Lunenberg et al., 2007). Only then, preservice teachers will be able to become aware of the complexity that underlies the experience of teaching mathematics.

Concerning research on conceptions held by mathematics teacher educators, studies have focused on their beliefs about specific topics such as mathematics teacher education (Lovin et al., 2012) and the role of school teachers in teaching mathematics (Aydın, Baki, Yıldız, & Köğçe, 2010). With respect to research on conceptions of modeling, a study conducted by Boyd (2014) showed that teacher educators report modeling explicitly and in a way that is consistent with the theoretical models that they enact in class; however, their descriptions of practice do not clearly distinguish modeling from role taking. In other words, few studies have examined mathematics teacher educators' conceptions of their teaching practices in general or their conceptions of the modeling that they implement with their preservice teachers in particular. Exploring this topic in more detail would help expand a more comprehensive and relational approach to the study of the teaching-learning processes that take place in mathematics teacher education. Furthermore, research on this topic contributes to developing more complex teaching models with greater applicability and impact on the school system.

## **AIM AND RESEARCH QUESTION**

The purpose of this research is aimed at exploring the conceptions of modeling held by mathematics teacher educators. The following research questions guide this study: What are the conceptions of modeling held by mathematics teacher educators? How are these conceptions linked to the pedagogical reasoning that underlies their teaching practices?

## **METHODS**

This study adopted a phenomenographic research approach. Phenomenography seeks to explore the different conceptions or structures of awareness which people constitute from the world of their experience (Bowden & Walsh, 2000). The aim is to describe the full richness of variation in the experience of a phenomenon, providing insight into what would be required for individuals to move from less powerful to more powerful ways of understanding a phenomenon (Åkerlind, 2005).

Participants were selected using purposive sampling (Bowden & Green, 2005), and the final sample comprised fifteen mathematics teacher educators working in three Chilean primary initial teacher education programs. According to Trigwell (2000), this number of participants is adequate for the maximizing variation (samples of 15 to 20 participants). Participants were invited to take part voluntarily and signed a consent form. Data were collected through semi-structured interviews and this method allows participants to describe and reflect on their own experience of the phenomenon

investigated (Åkerlind, 2005). The interview questions focused on the mathematics teacher educators' experience of modeling, teaching practices, and a reflection on their approaches to teaching mathematics. Interviews lasted between 45 and 95 minutes. All of the interviews were audio recorded and fully transcribed in preparation for analysis.

Regarding data analysis, the initial pool of meanings was constructed by categorizing data based on shared ideas expressed by all participants. Through this process, different categories of description were defined and hierarchically organized to create an outcome space (Åkerlind, 2005). These categories of description were tested against the data and adjusted in an iterative process until the set of the emerging categories established was the best in describing the data. Besides, we explored several dimensions of variation to enrich the logical relationship between the categories of description. Finally, to maintain the research rigor and validity of the findings, the final categories of description were discussed and refined with one expert in phenomenography research.

## **FINDINGS**

Four interrelated categories of description were identified from the analysis. Below, each category of description is discussed and analyzed in part using illustrative quotations. At the end of each quote, a number was provided to identify them in the transcripts while keeping interviewees anonymous.

### **(A) Modeling as a medium to learning good teaching practices.**

In this category, modeling is viewed as a teaching practice where mathematics teacher educators model pedagogical activities that preservice teachers will be able to replicate when they become teachers. For instance, I1 describes an exercise that has two aims: first, to teach a specific set of contents practically; second, to share a pedagogical activity that students will be able to conduct in school classrooms.

Angles, parallel lines, you can show all that using your arms. I said to them “everyone, show me an obtuse angle with your arms, an acute angle,” that sort of thing... and I also told them explicitly that it is good for them to do that with their students (I1).

### **(B) Modeling as a medium to learning pedagogical interactions.**

In this category, mathematics teacher educators model pedagogical interactions that can facilitate learning in the classroom. For doing so, it is fundamental to establish an appropriate bond with students, as this extract shows:

There is also the emotional aspect... In my opinion, if there is no emotion, there is no learning. So I become emotionally involved with students. I mean, I tell them that they can do it (I9).

### **(C) Modeling as a medium to learning to teach connected to the school classroom.**

In the third category of description, mathematics teacher educators view modeling as a teaching practice linked to the school classroom. Here, mathematics teacher

educators model a kind of teaching related to teaching school mathematics, predicting the most frequent errors and difficulties observed in school students.

I try to model, with a theoretical basis, a way of thinking about designs and their objectives that is not unique... it is like thinking aloud about what I want to achieve in the classroom regarding a mathematical lesson plan (I6).

#### **(D) Modeling as a medium to learning a congruent teaching approach.**

In this category, mathematics teacher educators view modeling as the use of a consistent set of teaching practices that allow student teachers to experience mathematical learning and replicate it with students in the school system. The central idea guiding this view is that they enact a role model and, therefore, they should be consistent with the theoretical model that they ascribe to since they regard this as essential for learning how to teach mathematics.

Because otherwise there is no consistency, how can I... so if I am not a model, I can just babble about how I think students should learn mathematics. But if I am not [a model], students will not have a point of reference to observe how you can do those things that the teacher says you can do. So, I think discourse and practice must coincide (I2).

#### **Dimensions of variation between categories of description**

The above-defined categories of description show their complex relationship through the variation between each of them in accordance to three different aspects as shown in Table 1.

	The role of mathematics teacher educator	Potential impact	Situated context
A	Transmitting good teaching practices	Improving student learning	University classroom
B	Transmitting pedagogical interactions	Improving student learning	University classroom
C	Showing school teaching	Transforming school teaching	University and school classroom
D	Reflecting a mathematics congruent teaching	Transforming school teaching	University and school classroom

Table 1: Dimensions of variation and their relationship with the categories of description.

In the first dimension called ‘The role of mathematics teacher educators’, the variation represents an expanding focus on the different activities performed by mathematics teacher educators related to enacting modeling practices. Category A and B focus on strategies related to exposing good teaching practices and promoting pedagogical

interactions, respectively. By contrast, in Category C the approach is focused on showing school teaching experience by linking mathematics teacher education to school systems. Concerning Category D, the strategy is mainly focused on reflecting what means congruent teaching inside mathematics teacher education. The second dimension, 'Potential impact', represents an expanding focus on the expected impact of modeling practices: from improving student learning, in Category A and B, to transforming school teaching, in Category C and D. Finally, in the third dimension 'Situating Context,' the variation represents an expanding focus on the context where the modeling is oriented: from university classroom, in Category A and B, to university and school classroom, in Category C and D.

To sum up, these findings make it possible to infer that mathematics teacher educators have a range of conceptions of modeling in teaching prospective teachers. These notions involve multiple views of teaching, which differ concerning the kind of role they adopt, the type of strategies that they use, and the perception of the impact that they may have on transforming school teaching.

## **DISCUSSION**

The present study sought to explore the conceptions of modeling held by mathematics teacher educators. Four categories of description were identified: modeling as a medium to learning: (A) good teaching practices that can be replicated in the school classroom; (B) pedagogical interactions that should be established with school students; (C) to teach connected to the school classroom; and (D) a congruent teaching approach that is consistent with the theoretical model to which they adhere. Furthermore, three dimensions of variations were found between the descriptive categories: (1) the role of mathematics teacher educator; (2) potential impact; and (3) situated context. These findings, despite being in line with those reported in similar studies (Boyd, 2014; Swennen et al., 2008), provide a more comprehensive perspective of the phenomenon by revealing that these conceptions vary in terms of how the complexity of the school classroom is explained, which makes it possible to transform teaching inside the school system.

For instance, mathematics teacher educators who regard modeling as a practice with a focus on performing pedagogical activities and interactions with students visualize their teaching practices as being mainly connected with the university classroom. In consequence, they attempt to recreate the complexity of the school classroom hypothetically by teaching activities and kinds of interaction that can be replicated in the school system. In contrast, mathematics teacher educators who regard modeling as a teaching practice linked to the school classroom and supported by the theoretical model to which they adhere not only connect their teaching to the university classroom: they also invite students to think about and recreate the school classroom where they will work as teachers in the future. That is, learning to teach is viewed as a complex phenomenon that can be only understood if it is discussed and pondered, considering the context where it will take place (Boyd, 2014; Loughran, 2006). Interestingly, these findings also reveal that the potential impact of the modeling adopted by mathematics

teacher educators varies depending on the type of conception that supports it. For instance, mathematics teacher educators who hold A- or B- type conceptions of modeling tend to prioritize the learning of preservice teachers. In contrast, mathematics teacher educators with C- and D- type conceptions are more focused on transforming teaching in school classrooms. This advanced perspective involves not only paying attention to the learning of preservice teachers but also proposing new ways of teaching mathematics capable of modifying the traditional teaching patterns that remain present in school classrooms, thereby influencing student learning in the school system.

Regarding the limitations of the present study, its pioneering nature makes it necessary to collect more data in order to refine and consolidate the results obtained, even though the number of interviews conducted meets the level recommended for phenomenographic research. Nevertheless, the results reported are valuable for teacher education in general and mathematics teacher education in particular because they reveal that mathematics teacher educators –depending on the type of modeling adopted– not only play a key role in how preservice teachers learn to teach, but also have the chance to influence and change teaching in the school system. Finally, it is concluded that research on mathematics teacher educators' conceptions of modeling generates valuable knowledge and empirical evidence that can help preservice teachers tackle their future challenges in the school system.

### Funding

This research was funded by a grant from FONDECYT-CONICYT (#3170423).

### References

- Åkerlind, G. (2005). Learning about phenomenography: Interviewing, data analysis and the qualitative research paradigm In J. Bowden & P. Green (Eds.), *Doing developmental phenomenography* (pp. 63-73). Melbourne: RMIT University Press.
- Ávalos, B., & Matus, C. (2010). *La Formación Inicial Docente en Chile desde una Óptica Internacional. Informe Nacional del Estudio Internacional IEA TEDS-M*. Santiago de Chile: MINEDUC.
- Aydın, M., Baki, A., Yıldız, C., & Köğçe, D. (2010). Mathematics teacher educators' beliefs about teacher role. *Procedia-Social and Behavioral Sciences*, 2, 5468–5473.
- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(3), 14-22, 43-46.
- Bowden, J., & Green, P. (2005). *Doing developmental phenomenography*. Melbourne: RMIT University Press.
- Bowden, J., & Walsh, E. (2000). *Phenomenography*. Melbourne: RMIT University Press.
- Boyd, P. (2014). Using 'Modelling' to improve the coherence of initial teacher education. In P. Boyd, A. Szplit, & Z. Zbróg (Eds.), *Teacher Educators and Teachers as Learners: International Perspectives* (pp. 51-73). Kraków: Wydawnictwo Libron.
- Darling-Hammond, L. (2006). *Powerful Teacher Education: Lessons from Exemplary Programs*. San Francisco: Jossey-Bass.
- Goizueta, M., Montenegro, H., Rojas, F., & González, M. P. (2017). Rethinking modeling in pre-service teacher education: implications for teacher educators. In M. Juanjo, A.

- García-Valcárcel, F. García-Peñalvo, & M. Martín del Pozo (Eds.), *Search and research: teacher education for contemporary contexts* (pp. 139-147). Salamanca: Ediciones Universidad de Salamanca.
- Loughran, J. (2006). *Developing a Pedagogy of Teacher Education. Understanding teaching and learning about teaching*. New York: Routledge.
- Loughran, J., & Berry, A. (2005). Modelling by teacher educators. *Teaching and Teacher Education*, 21(2), 193-203.
- Loughran, J., Keast, S., & Cooper, R. (2016). Pedagogical reasoning in teacher education. In J. Loughran & M. L. Hamilton (Eds.), *International Handbook of Teacher Education* (Vol. 1, pp. 387-448). Singapore: Springer.
- Lovin, L., Sanchez, W., Leatham, K., Chauvot, J., Kastberg, S., & Norton, A. (2012). Examining beliefs and practices of self and others: Pivotal points for change and growth for mathematics teacher educators. *Studying Teacher Education: A journal of self-study of teacher education practices*, 8(1), 51–68.
- Lunenberg, M., Korthagen, F., & Swennen, A. (2007). The teacher educator as a role model. *Teaching and Teacher Education*, 23(5), 586-601.
- Rojas, F. (2017). Oportunidades para aprender a enseñar matemática en la formación inicial. In L. Pino-Fan & V. Díaz (Eds.), *Perspectivas de la investigación sobre la formación de profesores de matemáticas en Chile* (pp. 45- 84). Osorno, Chile: Cuadernos de Sofía.
- Russell, T. (1999). *The challenge of change in (teacher) education*. Keynote address to ‘The challenge of change in education.’ Conference. Sydney.
- Struyven, K., Dochy, F., & Janssens, S. (2010). Teach as you preach: the effects of student-centred versus lecture-based teaching on student teachers’ approaches to teaching. *European Journal of Teacher Education*, 33(1), 43-64.
- Swennen, A., Lunenberg, M., & Korthagen, F. (2008). Preach what you teach! Teacher educators and congruent teaching. *Teachers and Teaching: Theory and Practice*, 14(5-6), 531-542.
- Trigwell, K. (2000). A phenomenographic interview on phenomenography. In J. Bowden & E. Walsh (Eds.), *Phenomenography* (pp. 62-82). Melbourne: RMIT University Press.

# PROSPECTIVE PRIMARY TEACHERS' KNOWLEDGE OF PEDAGOGICAL ORCHESTRATION IN PROBLEM-SOLVING INSTRUCTION

Juan Luis Piñeiro, Elena Castro-Rodríguez, & Enrique Castro

University of Granada

*This study explored future primary teachers' knowledge of pedagogical orchestration in problem-solving instruction. The tool used was a questionnaire distributed to 77 future teachers who are finalizing their pre-service training, divided into two groups on the grounds of differences in their university training. Our findings showed that the respondents had a suitable theoretical understanding of the approaches to teaching problem solving and the associated practices. In light of the contradictions in some of their replies, however, that understanding may not necessarily be carried over into their classroom delivery.*

## TEACHERS' PROBLEM-SOLVING KNOWLEDGE

Teachers' ability to solve complex, cognitively demanding problems does not suffice to guarantee appropriate problem solving (PS) instruction. Elements in addition to problem-solving skills that mathematics teachers must master need to be elucidated (Lester, 2013).

In light of progress made in the field, research linking teachers' knowledge to PS has been identified as an area in need of attention (Cai & Lester, 2016). The studies conducted to date focus primarily on teachers as problem solvers, with a paucity of papers addressing PS from the perspective of their knowledge (Lester, 2013). Lin and Rowland (2016) spotlighted a need to enlarge on certain particulars or reinterpret existing teacher knowledge models, deemed to be overly general and to omit PS-related elements. Chapman (2015) authored one of the more prominent studies on the issue; however, research is still insufficient to determine the utility of the various elements comprising teachers' knowledge of PS.

These considerations led us to pose the following research question: what knowledge do prospective primary teachers have about pedagogical orchestration in PS instruction upon completion of their pre-service university training? To achieve this, we analyse and describe the knowledge of two groups of prospective primary teachers who have differences in their university training about mathematics education, given the impact that might have on their replies.

## THEORETICAL PERSPECTIVE

Identifying professional knowledge about PS teaching calls, firstly, for addressing teachers' knowledge of processes rather than their mathematical content knowledge, the perspective adopted in traditional teachers' knowledge models. For this reason,

we resorted to the teaching triangle, which we adapted to the PS process to develop and reinterpret the pedagogical dimension of teachers' PS knowledge. We deem the triangle to afford a holistic understanding of PS teaching without distorting the nature of the process. The interactions among the vertices of the triangle disentangle elements of teachers' pedagogical knowledge of PS omitted in the literature characterising that notion. The pedagogical triangle and its relations are depicted in Figure 1, along with our interpretation of the elements of teachers' PS knowledge stemming from it.

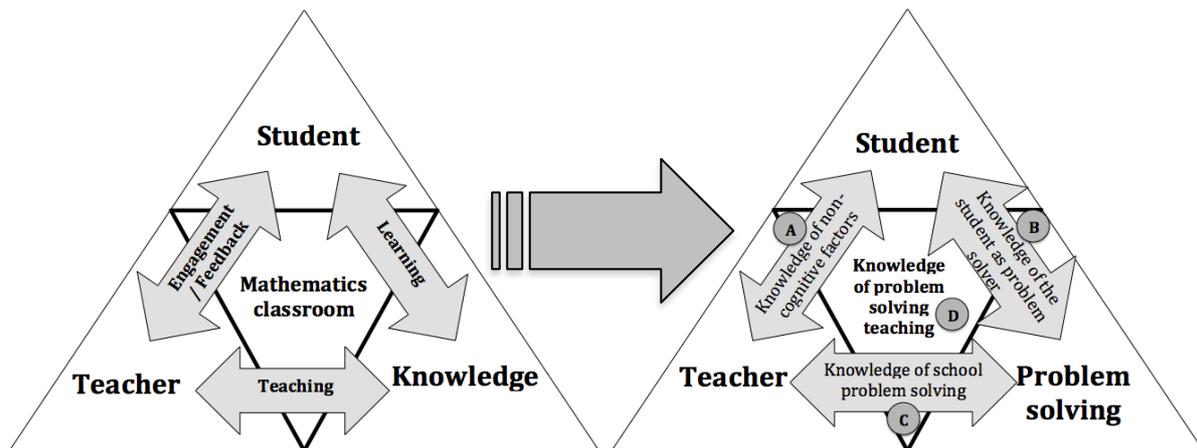


Fig. 1. Pedagogical triangle and pedagogical knowledge of PS proficiency

Those interactions underlie important elements of teachers' pedagogical knowledge of PS such as: a) non-cognitive factors that affect PS (Charles et al., 1987); b) viewing students as problem solvers (Chapman, 2015); b) PS knowledge of school problem solving (Lester & Cai, 2016); and d) identifying and establishing knowledge of problem solving teaching (Lester, 2013). The first three components are related to learning and the fourth to teaching PS. This study addressed the fourth, related to instructional practice.

### Knowledge of problem-solving instructional practice

Schroeder and Lester's (1989) perspective on teaching problem solving acknowledges the teaching approaches. In this framework, it can be grouped under four areas of practice or orchestration: discourse, blockage, assessment and resources. These four orchestration practices and approaches constitute five elements of professional knowledge addressed here.

*Teaching approaches* to PS (Schroeder & Lester, 1989) are informed by models describing classroom actions that further PS proficiency. Each approach (for, about, and through) gives rise to certain actions.

*Discourse* is understood to mean all the actions (verbal or otherwise) that converge in lessons conducted in a way that encourages students to participate, cooperate and genuinely engage in PS. Lester and Cai (2016) described it as the manner in which 'teachers orchestrate pedagogically sound, active problem solving in the classroom' (p. 124). It includes actions such as furthering the use of multiple representations or

explaining that problems can have more than one solution. Lester (2013) contended that the most successful classroom approaches for training good problem solvers entails discourse tailored to PS lessons.

Dealing with obstacles that obstruct successful PS calls for an understanding of specific teaching strategies with which to mediate an overcoming *blockage*. Such tools for confronting students' possible difficulties constitute part of teachers' instructional knowledge (Chapman, 2015). The practice involved may consist, for instance, in suggesting an alternative PS strategy.

PS *assessment* is yet another factor to be considered. Chapman (2015) noted the importance of having a command of the elements that are consistent with, i.e., focus on, genuine PS, as collected by Charles et al. (1987). With such knowledge, educators can set goals that further learning and translate into suitable and varied tools.

The fourth practice relates to the manipulative and intangible *resources* used in PS. Teachers should know how to use both PS manipulatives (Kelly, 2006) and representations (Smith, 2003). The latter merits particular attention, for all students need to master a variety of notations in the various stages of PS.

These four orchestration practices and teaching approaches constitute five elements of professional knowledge considered in this study.

## **METHOD**

With the purpose of determining future primary school teachers' professional knowledge of PS pedagogical orchestration, we designed and applied a questionnaire to 77 last-year Education students at the University of Granada (Spain). They were divided into two groups (Group A=54; Group B=23) on the grounds of differences in their training, given the impact that might have on their replies.

### **Context**

Both groups had taken three mathematics courses as part of their university training. One focused on classroom mathematics content, the second on teaching and learning the mathematical contents from a cognitive and pedagogical standpoint, and the third on the study of the mathematical curriculum of primary school and lesson design and implementation. PS was treated as a cross-curricular topic in all three courses.

Group B had also taken an elective in which PS, and more specifically PS strategies and heuristics, problem posing and teaching strategies, were explicitly addressed in the syllabus.

### **Instrument**

The questionnaire, which contained five sections and 66 questions, was formulated by stages as described in Piñeiro, Castro-Rodríguez and Castro (2018). We used a closed binary design because we sought answers that would denote the presence or

absence of certain types of knowledge (Fink, 2003). Figure 2 lists some of the items in the section on blockage, by way of example.

- What action should a teacher take if students gets blocked when solving a problem?
23. If they made a calculation mistake, ask them to re-read the problem until they understand it.
  24. Determine whether the mistake has to do with understanding the problem as worded or implementing the strategy.
  25. Suggest alternative representations to contribute to understanding the elements of the problem.
  26. Suggest alternative strategies to overcome blockage in implementing the plan.
  27. Tell students the answer to avoid frustration.

Figure 2. Examples of questions

## Procedure

The participants were asked to complete the questionnaire at the end of academic year 2017/2018, which they did individually in a session that lasted around 20 minutes. One of the researchers was present throughout.

## ANALYSIS AND RESULTS

Two methods were deployed to analyse the data. First, dimensional scaling (ALSCAL, SPSS) was used to group the replies for multivariate analysis. The dimensions defined were agreement and doubt, i.e., replies in which most respondents concurred in their affirmative or negative answers, and those where no clear majority opinion was observed. The stress values for the dimensions were 0.10 in the group A, and 0.11 in group B. Despite group B's stress, we have decided to maintain 2 dimensions, because understanding the data should prevail (Bisquerra, 1989). Secondly, we proceeded to a descriptive analysis of the replies. This article discusses the most prominent findings of the second analysis, arranged in keeping with the theoretical pre-analysis conducted and the two dimensions defined.

### Teaching approaches

Group A exhibited agreement in questions on the existence of classroom environments that favour problem exploration (100%), discussion of the strategies used (87.5%), the process deployed (96.4%), the teacher's role in exemplifying PS (82.5%) and beginning lessons with problems to subsequently explore their implicit mathematics (85.7%). When asked about what type of classroom organisation is appropriate, the future teachers agreed that it should focus *on* (92.9%) and *through* (82.1%) PS. Polarisation, with around 50% agreement and 50% disagreement, was observed in questions about whether a concept or procedure should first be taught and then applied to solve a problem, whether stages and strategies should be taught directly and whether educators should teach *for* PS.

In their replies to the characteristics of teaching approaches, Group B agreed to the need for an environment favouring problem exploration (100%), that the discussion should focus on the process (100%) and that lessons should begin with a problem with a view to exploring and discovering the underlying mathematics (100%). They

also agreed that concepts or procedures should be first learnt and then applied to PS (76%) and that students should be taught how to solve a problem before it is posed (76%). This group of future teachers agreed that all the types of classroom organisation cited are suitable for PS lessons, although the percentages differed. The approach *for* PS scored 64%, *about* PS 100% and *through* PS 96%. Polarisation in their replies was found, for instance, in the question on whether a concept or procedure should be taught first and subsequently applied to solve a problem, with 56% answering yes and 44% no.

### **Discourse**

Group A replied affirmatively to questions on the utility of furthering different solving strategies (100%), discussing student-used or -posed strategies (100%), asking students to justify their replies and explain the mathematics implicit in problems (100%) and guiding discussion to address such issues. They also agreed, at a rate of 98.2%, that teachers should explain and exemplify strategies for solving the problems posed. This dimension included negative replies on questions about furnishing students with a list of the answers to the problems in the current lesson (60.6%), deeming the exercise to be over when a solution is found (94.6%), urging students to find solutions quickly (100%), posing easily solvable tasks (78.6%) or showing students how to solve problems (80.4%). Respondent polarisation was observed for the question on validating students' results and whether this should be a classroom exercise before obtaining the teacher's conformity.

Group B agreed to most of the proposals around discourse. Their replies were essentially similar [to those of Group A], except that 96% agreement was recorded for the question on validating results. This group's replies were polarised around the question on the need to show students how to solve problems.

### **Blockage**

Group A agreed that teachers should determine whether students' mistakes stem from understanding or implementing a strategy (98.2%), that alternative representations of the elements of the problem should be suggested if blockage is due to understanding (96.4%) and that alternative strategies should be suggested if blockage is attributable to strategic error (92.9%). The doubt dimension replies referred to the question on the suitability of asking students making a calculation error to re-read the problem to understand its elements. More specifically, the disparities arose around what a teacher should do if students understand and correctly choose the solving strategy but make a calculation mistake. Future teachers were unsure about the utility of suggesting a different strategy, with 53.6% agreeing and 46.4% disagreeing.

Most of Group B's replies lay in the agreement dimension. The same item as identified in the preceding paragraph was classified in the doubt dimension. Here, future teachers' answers were polarised at 50%-60% for each option when asked to determine whether representing the problem in another manner would favour greater understanding. Similar results were recorded for the option around suggesting an

alternative strategy. This group agreed (100%) that students should be asked how they performed their calculations.

### **Assessment**

The questions in this section broached assessment from two perspectives: what should be assessed and what tools should be used.

In the questions on what should be assessed, Group A respondents agreed on the importance of assessing students' command of the process (94.6%), their perseverance throughout (98.2%) and their ability to choose and use strategies (100%), explain what they did (94.6%), find the correct answer (98.2%) and give the answer meaning in the context of the problem (94.6%). They also deemed that neither finding the answer quickly (85.7%) nor using only abstract mathematical symbols to represent ideas and replies (75%) should be assessed. They agreed that the tools to be used, among others, should include direct observation (98.2%), problem posing (82.1%) and written tests (92.9%). No agreement (doubt dimension) was observed about the use of tools such as multiple choice and fill-in-the-blank tests.

Group B agreed that assessment should cover understanding the problem (100%), data arrangement and representation (96%), process control (96%), explanation of progress and the answer (100%), ability to find the answer quickly (76%) and perseverance (100%). Doubts arose in their replies around some tools. This group was polarised at around 60%-40% in the use of personal interviews, self-reporting, multiple choice and fill-in-the-blank tests to assess PS.

### **Resources**

The questions in this section were also grouped around two areas: the role of materials and the role of representation.

Group A agreed that materials enable students to visualise and manipulate the relationships and ideas present in problems (98.2%) and disagreed that only abstract mathematic symbols should be used (89.3%). The use of representation in solving processes elicited agreement around the non-restriction of the type of representation (89.3%), furtherance of its use as a way of ascertaining students' understanding of the problem (98.2%) and encouraging students to use their own representation system (92.9%) to favour transition to other more formal systems (100%). Doubts arose around focusing the use of representation on the understanding stage, with around 60%-40% polarisation in the replies to the various items.

All of Group B's replies were positioned in the agreement dimension. They agreed, for instance, that abstract symbols should not be used exclusively (100%) or used only with students exhibiting difficulties (72%). They also agreed that the use of more than one type of representation should be encouraged when solving a given problem (100%).

## **DISCUSSION AND CONCLUSION**

The gaps in teachers' knowledge of PS and how they conceive PS are among the deterrents to its introduction in classroom environments (Lester, 2013). This study contributes to that complex scenario by identifying and analysing primary education teachers' pedagogical orchestration of PS instruction when they are finalizing their university training. We identified five critical types of knowledge associated to PS orchestration practice: teaching approaches, discourse, blockage, assessment, and resources.

We found aspects relating to all three approaches to PS teaching (for, about, and through). As Schroeder and Lester (1989) noted, arguing in favour of one or the other is futile, for in the classroom they overlap. Nonetheless, more doubts were identified among group A, specifically in questions alluding to elements of the approaches for and about SP. That may be attributable to the group's lower level of training in this respect.

In these types of knowledge—approaches, discourse, blockage, assessment, and resources—we observed replies agreeing to what has been identified as good PS classroom practice (Lester and Cai, 2016). Participants' replies nonetheless co-existed with ideas that may prevent the translation of such knowledge into teaching practice. The two findings discussed below illustrate that concern.

In the area of classroom discourse, both groups agreed that different solving strategies should be encouraged and that PS is not over when an answer is found. Some of the replies in both groups call for reflection, however. Group A deemed, for instance, that students should be explicitly shown the various strategies to solve a problem before confronting PS. Group B, in turn, believed that students should be asked to solve problems quickly. Both those ideas run counter to successful learning experiences: Schoenfeld (2013) noted that a truly problematic task is one where the pathway to the solution is unknown. Lester and Cai (2016), in turn, contended that genuine PS calls for more time than traditionally devoted to this task.

The second example relates to dealing with blockage. Both groups' replies exhibited uncertainty around whether students may become blocked in different stages of the process and the sort of action teachers should take, depending on the nature of the difficulty, to guide their learning process. According to Chapman (2015), one of teachers' tasks is to identify students' difficulties, interpreted, however, from the student's perspective. In other words, mistakes should be viewed from the vantage point of the person in error to be able to furnish truly effective learning assistance.

The two groups' replies were essentially similar and indicative of a sufficient theoretical knowledge of approaches and practice among respondents, a finding consistent with the results of an earlier survey on future teachers' understanding of problems and the solving process (Piñeiro et al., 2018). In light of the contradictions in some of their replies, however, that understanding may not necessarily be deployed in the classroom. We therefore believe that in addition to addressing theory,

education curricula should foster skills that enable future teachers to transfer that knowledge to classroom practice. The future teachers surveyed exhibited suitable theoretical training, but some of their replies revealed a divide between theory and classroom delivery.

### Acknowledgements

This study was funded under Spanish R&D Project EDU2015-70565-P (MICINN) and a PhD. grant awarded by Chile's CONICYT (folio 72170314).

### References

- Bisquerra, R. (1989). *Introducción conceptual al análisis multivariable: un enfoque informático con los paquetes SPSS-X, BMDP, LISTEL y SPAD*. Barcelona, Spain: PPU.
- Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT*, 3(1), 19-36.
- Charles, R. I., Lester, F. K., & O'Daffer, P. G. (1987). *How to evaluate progress in problem solving*. Reston, VA: NCTM.
- Fink, A. (2003). *How to ask survey questions*. Thousand Oaks, CA: Sage Publications.
- Kelly, C. A. (2006). Using manipulatives in mathematical problem solving: A performance-based analysis. *The Mathematics Enthusiast*, 3(2), 184-193.
- Lester, F. K. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, 10(1&2), 245-278.
- Lester, F. K., & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 117-135). New York, NY: Springer.
- Lin, F.-L., & Rowland, T. (2016). Pre-Service and in-service mathematics teachers' knowledge and professional development. In Á. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of Mathematics Education* (pp. 483-520). Rotterdam, The Netherlands: SensePublishers.
- Piñeiro, J. L., Castro-Rodríguez, E., & Castro, E. (2018). Prospective primary teachers' conceptual understanding of mathematical problems and problem solving. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd PME International Conference* (Vol. 3, pp. 465-472). Umeå, Sweden: PME.
- Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, 10(1&2), 9-34.
- Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics. 1989 yearbook* (pp. 31-42). Reston, VA: NCTM.
- Smith, R. P. (2003). Representation in school mathematics: Children's representations of problems. In J. Kilpatrick (Ed.), *A research companion to principles and standards for school mathematics* (pp. 263-274). Reston, VA: NCTM.

# STRUCTURES AND GENERALISATION IN A FUNCTIONAL APPROACH: THE INVERSE FUNCTION BY FIFTH GRADERS

Eder Pinto and María C. Cañadas

University of Granada

*Against the backdrop of functional thinking to early algebra, this paper discusses an initial study of how 24 fifth year elementary school students (10 to 11 years old) perceived the inverse function. This notion has been scarcely tackled in the context of early algebra and, particularly, within the functional thinking approach. Based on structure and generalisation notions, we analysed the responses of a group of 24 students when solving a problem, which contains questions involving the direct and inverse forms of a problem involving a linear function. Ten of the 24 students were observed to establish structures involving inverse function and five to generalise that form of the function.*

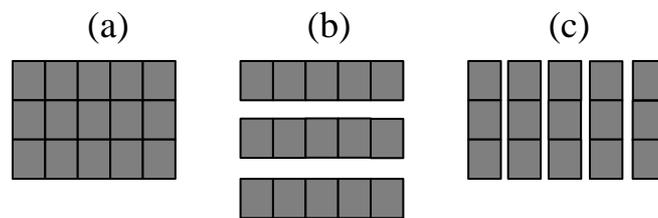
## INTRODUCTION

Generalisation is a crucial element in research on algebraic thinking in early schooling (hereafter, early algebra). Some authors (Schifter, Monk, Russell, & Bastable, 2008) contend that children are naturally inclined to perceive and discuss regularity, which is the key to generalisation, even when they lack the resources needed to represent general relationships. The present study was conducted in the context of early algebra, to which generalisation and the way it is expressed are core aspects (Kaput, 2008).

Functional thinking is a vehicle for introducing algebra in the early years of schooling. This type of algebraic thinking focuses on the relationship between two or more covarying quantities. Relationships maybe identified for specific cases or in general (generalisation) (Smith, 2008). Given that functions constitute the prime mathematical content in functional thinking, some researchers recommend focusing on how students perceive both direct and inverse forms of functional relationships (Oehrtman, Carlson, & Thompson, 2008). Whilst several researchers are interested in generalisation at elementary school (e.g., Carraher & Schliemann, 2016; Pinto & Cañadas, 2017), very few studies have been published on how such students perceive and generalise inverse functions, the issue addressed in this article.

Regularities from a functional approach to early algebra have to do with the relationships between the dependent and independent variables involved in a given situation. Specifically, the notion of structure is associated with how regularity between variables is organised, as perceived by students through different representations when working with specific cases as well as when generalising. Some researchers note that before being able to generalise, students must ‘see’ the structure in a mathematical situation (Mason, Stephens, & Watson, 2009). Structure, although seldom dealt with in the context of functional thinking, provides a way to describe generalisation as

engaged in by lower year students. Authors like Mulligan and Mitchelmore (2009) exemplify the notion of structure through rectangular grids represented in Figure 1.



*Figure 1:* Rectangular grid perceived as (a) 3x5, (b) 3 rows of 5, (c) 5 columns of 3 (Mulligan, & Mitchelmore, 2009, p. 34)

Often students at elementary grades have difficulties identifying the pattern 3x5 in the three grids of Figure 1 because they are not able to recognize the implicit structure of three rows of five squares each. In this case, the way in which the squares are organized is one important characteristic of the structure.

In the present study about generalisation with fifth graders within a functional thinking approach, in Spain, we focus on the inverse function. According to MacGregor and Stacey (1995) inverse function poses greater difficulty than direct function to students, and is barely addressed in the literature (Carraher & Schliemann, 2016). Fifth year elementary school was chosen because whilst research has been conducted on the inverse function with older ages, it has not been studied in elementary school.

The general aim of the paper is to describe how fifth year students (10- to 11-year-olds) perceive the inverse function when working with a problem involving a linear function in the context of early algebra. The two specific aims pursued are: (a) to identify the structures detected by students; and (b) to describe the students' generalisation based on notion of structures.

## FUNCTIONS

This study focuses on linear functions, a type recommended for elementary school students, for instance,  $f(x) = mx + b$ , in which constants  $m$  and  $b$ , as well as  $x$  and  $y$  are natural numbers (Carraher & Schliemann, 2016). A function is a rule that establishes a relationship between two variables, with the emphasis on how the changes in one are related to changes in the other (Thompson, 1994). The direct and inverse forms of a function are consequently related to the roles played by each variable involved. The independent variable in the direct form of the function is the dependent variable in the inverse form, and vice-versa.

Prior research on inverse functions has been conducted with post-elementary school students. MacGregor and Stacey (1995), for instance, explored how 143 14 to 15 years old perceived functional relationships in exercises involving direct and inverse functions. They reported 63 % of correct answers for the direct, but only 43 % for the indirect, function. The authors gave no information on how students perceived regularity when the inverse function was involved.

## GENERALISATION AND STRUCTURES

Generalisation “involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly and exposing commonality across cases” (Kaput, 1999, p. 136). From the functional approach to early algebra, generalisation is related to the various ways students expressed a general functional relationship involving two variables. According to Radford (2002), we assume that generalisation at elementary grades can be expressed through different representations (natural language, numerical, tabular, for instance).

A pattern can be defined as a spatial or numerical regularity and its structure as the relationships among its components (Mulligan, Mitchelmore, & Prescott, 2006). The distinction between pattern and structure is considered to be pertinent here, for the former is associated more with recurrence than with the establishment of a functional relationship such as covariation between two quantities. From the functional thinking approach and further to the literature (such as Kieran, 1989), we assume here the notion of structure as the numbers and numerical variables (expressed via different representations), operations and their properties present when students identified a regularity. Previous authors have argued that to generalise, students must previously identify the structure of the relationships observed (Mason et al., 2009). Some studies have shown that when they identify structures in mathematical tasks, students experience mathematics more deeply (Mason et al., 2009).

Structure and generalisation were exemplified here in a problem involving the linear function  $y=2x+6$ . To find the number of grey tiles ( $g$ ) that can be laid around a given row of white tiles ( $w$ ) (see Küchemann, 1981), students might identify the relationship between variables as: (a) “three tiles on the right, three on the left, eight at the top and eight at the bottom” (specific case, eight white tiles); or (b) “double the number of tiles plus six” (general case). In the first, non-generalised description (a), the structure would be  $3+3+x+x$  (if applied to more than one specific value) and in the second generalised response (b) it would be  $2x+6$ . These two structures are equivalent with the function given in the problem.

## METHOD

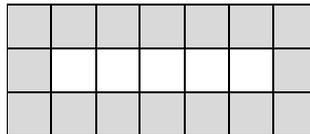
This study forms part of a broader teaching experiment on functional thinking in different years of elementary education, focused on fifth year students (10 to 11 years old).

### Students and tools

A group of 24 Spanish 10 to 11 year olds participated. They had previously studied the four arithmetic operations with natural numbers, integer and rational number sets. Their only experience with the type of problems proposed here in the session at issue was during three prior sessions, in which they were introduced to functions involving addition, multiplication and both.

The fourth session was divided into three parts. In the first part, the students were shown the tiles problem (see Figure 2) and asked questions to ensure they understood it. In the second part, each student worked individually on the worksheets. In the third part, students shared orally their answers to some of the questions. This paper deals only with students' written answers in the worksheets.

A school wants to re-floor its corridors because they are in poor condition. Its administration decides to use a combination of white and grey tiles, all square and all the same size, to be laid as in the drawing.



The school contracts a company to re-floor the corridors. We want you to help the workers answer some questions before they get started.

- Q1. How many grey tiles do they need for a corridor with 5 white tiles?
- Q2. As some corridors are longer than others, the workers need a different number of tiles for each. How many grey tiles do they need for a corridor with 8 white tiles?
- Q3. How many grey tiles do they need for a corridor with 10 white tiles?
- Q4. How many grey tiles do they need for a corridor with 100 white tiles?
- Q5. The workers always lay the white tiles first and then the grey tiles. How can they calculate how many grey tiles they need in a corridor where they've already laid the white ones?
- Q6. In some corridors, the workers mistakenly laid the grey tiles before the white tiles. They laid 20 grey tiles. How many white tiles do they need?
- Q7. In another corridor where they laid the grey tiles before the white, they laid 56 grey tiles. How many white tiles do they need?

Figure 2: The tiles problem

Questions Q1 to Q5 involved the direct form of the function (particular and general cases) and questions Q6 and Q7 the inverse form of the function (only particular cases).

### Data analysis

Taking all the students' answers, structures were observed in the answers to all the questions (involving either specific cases or the general case). A structure was regarded to have been identified when the same student answered two or more questions with the same regularity or generalisation. In other words, identification of the structure consisted in the use of the same regularity in at least two questions involving the direct function (Q1-Q5) or of generalisation in one of those questions, and analogously for inverse function questions Q6 and Q7. Students' perception of the inverse function (Q6 and Q7) was explored in depth in this study.

The structures detected are represented here using algebraic symbolism, although some students used other representations. For instance, in Table 1 we show different possible students' answers and structures inferred from the answers.

Questions	Students' responses	Underlying structures
Q2 (8 white tiles)	$(2 \cdot 8) + 3 + 3$	$2x + 3 + 3$
Q3 (10 white tiles)	$(2 \cdot 10) + 3 + 3$	$2x + 3 + 3$
Q4 (100 white tiles)	$(2 \cdot 100) + 3 + 3$	$2x + 3 + 3$
Q6 (20 grey tiles)	$(20 - 6) / 2$	$(x - 6) / 2$
Q7 (56 grey tiles)	$(56 - 6) / 2$	$(x - 6) / 2$

Table 1: Examples of possible students' responses

In Table 1 we observe that in Q2, Q3, and Q4 the structure was identified as the double of the number of tiles plus three on the right and three on the left, it is inferred here as  $2x + 3 + 3$ . However, in Q6 and Q7, students subtracted the tiles from both sides to the total number of grey tiles and divided this quantity by two, which can be inferred as  $(x - 6) / 2$ .

## RESULTS

A total of 14 students showed no sign of detecting structures as defined in the preceding section. These students: (a) answered the question without giving any information about their used procedure; (b) merely copied the problem wording; (c) drew or referred to illustrations of the problem; (d) performed inconsistent operations; or (e) did not answer the question.

Structures were detected in the remaining 10 students' replies to the questions for both the direct and inverse forms of the function. In their answers to Q5, which involved the direct function, all 10 students generalised the function, while one student's answers to Q1 and Q2 were also a generalisation. The three structures identified in these students' answers, which appeared in the specific and the general case, were:  $2x + 6$ ,  $2x + 3 + 3$ , and  $2x + 2$ . The first two structures matched the situation set out in the problem, but the third did not. The structure  $2x + 6$ , the most frequent, was identified by eight students.

Five of these 10 students generalised the inverse function structure when we asked for particular instances (Q6-Q7). Broadly speaking, four structures were inferred in their replies to questions Q6 and Q7:  $(x - 6) / 2$ ;  $(x/2) - 3 - 3$ ;  $(x/2) - 6$  and  $(x/2) - 2$ . A discussion of these inverse function structures and generalisation follows.

### Particular cases of the inverse function

Among the 10 students who described structures in both direct and inverse function questions, five did not generalise in their answers to Q6 and Q7. Three inverse function structures were inferred in this group:  $(x - 6) / 2$ ;  $(x/2) - 6$  and  $(x/2) - 2$ . The first is correct with the problem, whereas the second and third are not.

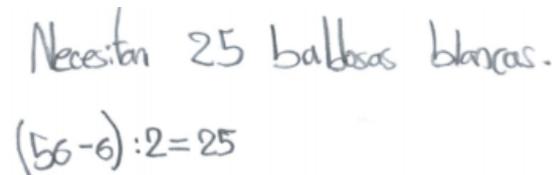
Four students defined the structure  $(x-6)/2$ . Mario, is one of them, he answered Q6 (number of white tiles needed for 20 grey tiles) as follows: “they need seven tiles. You have to calculate backwards, which means  $(20-6):2=7$ ”. In this answer, the student took the total number of grey tiles, subtracted the ones on the sides (6) and then divided the remainder by two, to get seven. Like the other three students, Mario started from an identified structure of the direct function  $(2x+6)$  to describe the structure for the inverse function. In his response to Q7 (number of white tiles needed when 56 grey tiles are laid) Mario answered: “25 white tiles.  $(56-6):2=25$ ”.

Lara was the only student who described the structure  $(x/2)-2$ , which is not equivalent with the function describing the problem. She replied to Q6: “18 white tiles. You divide 10 by 2 and -2”. She used the same structure for Q7, failing to consider the constant part of the function (6).

### Generalisation of the inverse function

Five students generalised the inverse functional relationship when calculating the number of white tiles from the 20 grey tiles cited in Q6 and the 56 in Q7. The three generalised structures were:  $(x-6)/2$ ,  $(x/2)-6$  and  $(x/2)-3-3$ . A number of examples of students’ answers follow.

Juan generalised in his reply to Q6: “(...) You have to subtract six tiles (from the sides) [and divide] by two”. This student generalised the structure  $(x-6)/2$ . His reply to the next question (Q7) is shown in Figure 3.



Necesitan 25 baldosas blancas.  
 $(56-6):2=25$

Figure 3: Juan’s reply to Q7 (English translation of the first line: ‘They need 25 white tiles’).

In his reply (Figure 2), Juan applied the structure identified in Q6 and wrote the answer to the problem in natural language. He used symbolic-numerical representation for the specific case (56 grey tiles).

Two students generalised in both Q6 and Q7. In her answer to Q6, for instance, Ana reasoned: “(...) Dividing the grey tiles by two and subtracting the three at the beginning and the three at the end (...)”. Here, irrespective of the specific case, the students divided the number of tiles in half and then subtracted the number of tiles on the right (3) and left (3). In this case generalisation was identified as  $(x/2)-3-3$ .

### DISCUSSION AND CONCLUSION

The present findings contribute to the description of how elementary education students perceive the inverse function, an issue barely researched in the context of functional thinking (Carragher & Schliemann, 2016). By focusing on structure and

generalisation, the study provides insight into how students interpret the relationships between variables.

Stacey and MacGregor (1995) explored secondary education students' perception of relationships in direct and inverse functions. Complementing that research by studying elementary education students, this study distinguishes between students who identified structures when working with specific cases and those who generalised and described the elements comprising those structures.

The 10 students who generalised the direct form of the function 'saw' the structure before answering Q5, confirming the ideas put forward by Mason et al. (2009). Only five students also generalised the inverse form of the function, inferring that generalisation is more difficult in this type of question than in the direct form, as contended by Stacey and MacGregor (1995).

In a study on direct functions, Pinto and Cañadas (2017) distinguished between fifth year elementary school students who generalised in questions referring to specific cases (spontaneous generalisation) and those who did so when confronting the general situation (prompted generalisation). Further to that distinction, generalisation observed here in the inverse function was systematically spontaneous, for both Q6 and Q7 refer to specific cases. The most common strategy for generalising the inverse function, implemented by four of the five students who did so, was to 'reverse' the structure generalised for the direct function ( $2x+6$ ). In all five cases, the generalisation was equivalent with the inverse function set out in the problem. These results suggest that a new line of research might address the generalisation strategies used by students in problems involving inverse functions after working with the direct form.

### Acknowledgements

This study was funded by the Spanish Ministry of Economy and Competitiveness under National R&D Projects EDU2013-41632-P and EDU2016-75771-P; the first author benefited from a PhD. grant awarded the by Chilean Government through the CONICYT, folio 72160307-2015.

### References

- Carraher, D. W. & Schliemann, A. (2016). Powerful ideas in elementary school mathematics. In L. English & D. Kirshner (Eds.), *Handbook of international research in Mathematics Education. Third edition* (pp. 191-218). New York, NY: Routledge.
- Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.
- Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5-17). New York, NY: Lawrence Erlbaum Associates.

- Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Vol. 4, pp. 33-56). Reston, VA: NCTM.
- Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London, United Kingdom: Murray.
- MacGregor, M. & Stacey, K. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal*, 7(1), 69-85.
- Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, 21(2), 10-32.
- Mulligan, J., Mitchelmore, M., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: the Australian pattern and structure mathematics awareness Project (PASMAPP). In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings 30<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 209-216). Prague, Czech Republic: PME.
- Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Cambridge, United Kingdom: Mathematical Association of America.
- Pinto, E. & Cañadas, M. C. (2017). Generalization in fifth graders within a functional approach. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 49-56). Singapore: PME.
- Radford, L. (2002). The seen, the spoken and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, 22(2), 14-23.
- Schifter, D., Monk, S., Russell, S. J., & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413-448). New York, NY: Lawrence Erlbaum Associates.
- Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133-163). New York, NY: Lawrence Erlbaum Associates.
- Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate Mathematics Education*, (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

# THEORETICAL ADVANCES IN MATHEMATICAL COGNITION

Thorsten Scheiner<sup>1</sup> & Marcia M. F. Pinto<sup>2</sup>

<sup>1</sup>The University of Auckland, New Zealand

<sup>2</sup>Federal University of Rio de Janeiro, Brazil

*This paper articulates and explicates theoretical perspectives that emerged in accounting for the complex dynamic processes involved when individuals ascribe meaning to the mathematical objects of their thinking. Here the focus is on the following processes that are convoluted in the complex dynamics in mathematical concept formation: contextualizing, complementizing, and complexifying. The paper elaborates these three processes in detail, recognizing their epistemological, conceptual, and cognitive significance in mathematical knowing and learning.*

## INTRODUCTION

Theoretical advancement is key to driving progress in mathematics education research and practice, and the deep understanding it can foster is essential when confronting fundamental problems. However, as diSessa (1991) asserted, in the learning sciences “theory is in a poor state” (p. 221), and the mathematics education community has “not reached deep theoretical understanding of knowledge or the learning process” (p. 221). For diSessa (1991), this is problematic particularly as “intuitive frames are not powerful enough to constitute theories of the mind in general and learning in particular” (p. 225). Reaching deep theoretical understanding of knowing and learning mathematics is challenging not only due to the complexity of phenomena under consideration but also because these phenomena are studied from a diversity of viewpoints both socially and culturally situated (Sierpinska & Kilpatrick, 1998) and relying on different philosophies and paradigms (Cobb, 2007).

Over the past two decades, various theoretical frameworks have arisen to account for cognitive development in mathematical knowing and learning. Here the focus is explicitly on local theories of knowing and learning in mathematics education to explain a specific set of phenomena, instead of global theories that are often tools to produce knowledge of or about mathematics education. Such local theories “are constructions in a state of flux” (Bikner-Ahsbahs & Prediger, 2010, p. 488) that shape, and are shaped by, research practices. This paper outlines some of the theoretical advances gained in our recent research that has been dedicated to better accounting for the complexity of mathematical knowing and learning on a fine-grained level.

Over the past five years, we explored critical processes in mathematical cognition and searched for dialogical possibilities to both move the discussion beyond simple comparison and offer new insights into complex phenomena in mathematical knowing and learning. In Scheiner (2016), two seemingly opposing forms of abstraction (i.e., abstraction from actions and abstraction from objects; Piaget, 1977/2001) and sense-

making strategies when learning formal mathematics (i.e., extracting meaning and giving meaning; Pinto, 1998, 2018) were put in dialogue. This dialogue contributed to reconsidering the notion of abstraction – as ascribing meaning to the objects of an individual’s thinking from a perspective an individual has taken rather than as recognizing a previously unnoticed meaning of a concept (for a discussion of different images of abstraction, see Scheiner & Pinto, 2016). Within this reinterpretation, meaning is construed not as an inherent quality of objects to be extracted, but something that is attributed to objects of one’s thinking. To this end, Scheiner’s (2016) theoretical discussion acknowledged three processes as central to mathematical concept formation that are the substance of this paper, namely contextualizing, complementizing, and complexifying.

This paper reports theoretical perspectives and insights gained over the past few years that advance our understanding of contextualizing, complementizing, and complexifying, particularly concerning their epistemological, conceptual, and cognitive significance in mathematical knowing and learning. These new perspectives and insights inform research on mathematical cognition and enable one to see not only new phenomena in mathematical concept formation, but to think about these phenomena differently.

## **THEORETICAL ORIENTATIONS AND ORIENTING ASSERTIONS**

The theoretical perspectives put forth here emerged as elaborations of a diversity of points of view on mathematical knowing and learning, organized around critical insights provided by the German mathematician and philosopher Gottlob F. L. Frege (1848-1925). Here we cultivate these theoretical insights as means of advancing our understanding of at least two critical issues involved in mathematical cognition. First, we share Frege’s (1892a) assertion that a mathematical concept is not directly accessible through the concept itself but only through objects that act as proxies for it. Second, mathematical objects (unlike objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, ‘see’ the object), but only via some ‘mode of presentation’ (Frege, 1892b) – that is, objects need to be expressed by using signs or other semiotic means such as a gestures, pictures, or linguistic expression (Radford, 2002). The ‘mode of presentation’ of an object is to be distinguished from the object that is represented, as individuals often confuse a sense<sub>F</sub> (‘Sinn’) of an expression (or representation) with the reference<sub>F</sub> (‘Bedeutung’) of an expression (or representation) (the subscript F indicates that these terms refer to Frege, 1892b). The reference<sub>F</sub> of an expression is the object it refers to, whereas the sense<sub>F</sub> is the way in which the object is given to the mind (Frege, 1892b), or in other words, it is the thought (‘Gedanke’) expressed by the expression (or representation).

Consider, for instance, the two expressions ‘ $4=4$ ’ and ‘ $2+2=2\cdot 2$ ’. The expression ‘ $2+2=2\cdot 2$ ’ is informative, in contrast to the expression ‘ $4=4$ ’. The two expressions ‘ $2+2$ ’ and ‘ $2\cdot 2$ ’ express different thoughts but have the same reference<sub>F</sub>, the natural number 4. The upshot of this; senses<sub>F</sub> capture the epistemological significance of expressions. Indeed, the algebraic structure consisting of the set of natural numbers

equipped with the arithmetic operation of addition could be a possible context for both the expression ‘ $2+2$ ’ and for the expression ‘ $2\cdot 2$ ’, where multiplication would be understood as repeated addition. Notice that in this case, expressions such as ‘ $3\cdot 5$ ’ and ‘ $5\cdot 3$ ’ may be understood as different operations, because the former means ‘adding five three times’ while the latter is ‘adding three five times’. However, there is another possible context for the expression ‘ $2\cdot 2$ ’: the algebraic structure of the set of natural numbers equipped with the arithmetic operation of multiplication. In this case, the epistemological significance of the same expression ‘ $2\cdot 2$ ’ would be different, as it would represent an operation per se, which is commutative. Thus, expressions express different thoughts concerning the different contexts where they are used. Similarly, Arzarello, Bazzini, and Chiappini (2001) called this the ‘contextualized sense of an expression’ that is, “a sense which depends on the knowledge domain in which it lives” (p. 63). These ideas are used as a way of recovering one of Frege’s decisive insights: what  $\text{sense}_F$  comes into being is itself dependent on the context in which an object actualizes. That is, context is constitutive for  $\text{sense}_F$ .

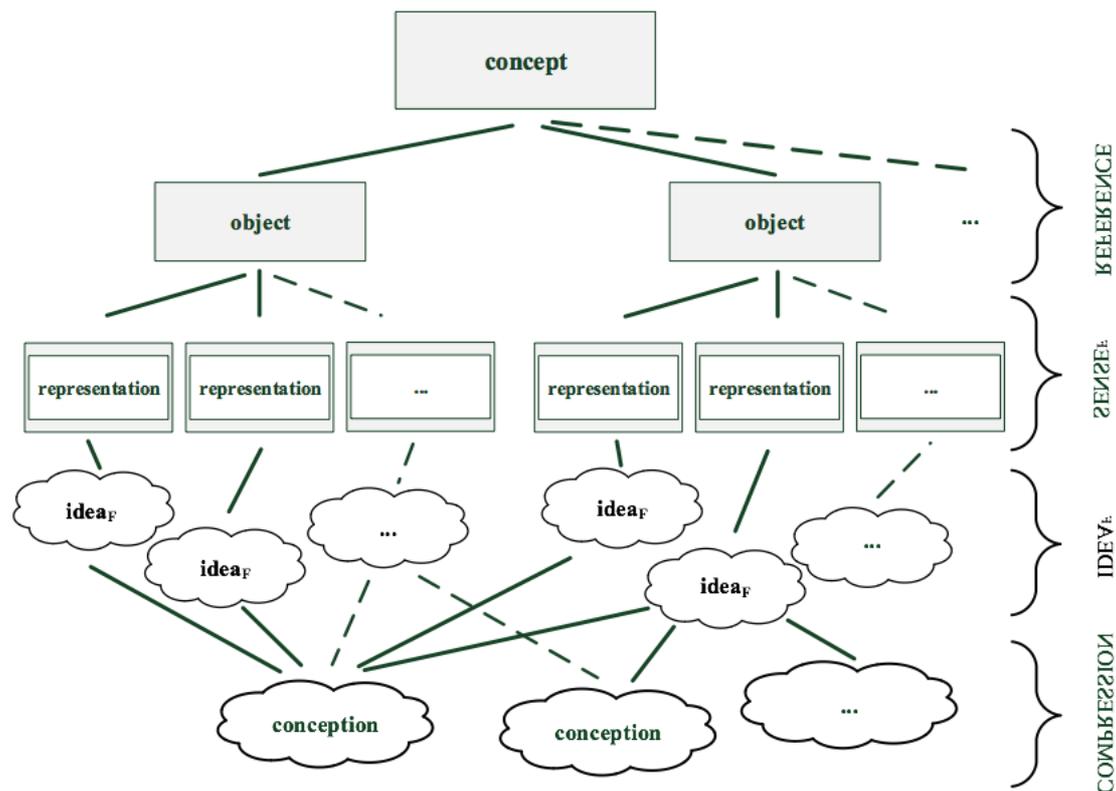


Figure 1: On  $\text{reference}_F$ ,  $\text{sense}_F$ , and  $\text{idea}_F$ , (reproduced from Scheiner, 2016, p. 179)

From this position, it seems to follow that we may understand Frege’s notion of an  $\text{idea}_F$  the manner in which we make  $\text{sense}_F$  of the world. For instance, one might attach the  $\text{idea}_F$  of repeated addition to the notion of multiplication. Ideas<sub>F</sub> can interact with each other and form more compressed knowledge structures, called conceptions. For instance, one might construe ‘ $2+2$  being equal to  $2\cdot 2$ ’ as ‘adding twice a number is the same as multiplying this by two’, whereas one might construe ‘ $2\cdot 2$  being equal to  $2+2$ ’ as ‘multiplication is repeated addition’. Alternatively, focusing on the sum and product,

instead of the addition or multiplication, the sum ‘2+2’ is equal to the product ‘2·2’. A general outline of the relations between concept, objects (the references<sub>F</sub> of representations), representations (expressing senses<sub>F</sub>), ideas<sub>F</sub>, and conceptions is provided in Figure 1.

### ON CONTEXTUALIZING, COMPLEMENTIZING, AND COMPLEXIFYING

In acknowledging Frege’s (1892a, 1892b) assertions, Scheiner (2016) argued that a concept does not have a fixed meaning. Rather, the meaning of a concept is relative (a) to the senses<sub>F</sub> that are expressed by representations that refer to objects falling under a concept and (b) to an individual’s system of ideas<sub>F</sub>. In the following, three processes are outlined that are considered to be critical in mathematical concept formation: contextualizing, complementizing, and complexifying.

#### Contextualizing: the epistemological function of particularizing senses<sub>F</sub>

In Frege’s view, a sense<sub>F</sub> can be construed as a certain state of affairs in the world and an idea<sub>F</sub> in which we make sense<sub>F</sub> of the world. Here, we started from an understanding of sense<sub>F</sub> as not primarily dependent on a mathematical object, but as emerging from the interaction of an individual with an object in the immediate context. That is, a sense<sub>F</sub> of an object at one moment in time can only be established in a more or less definite way when the process of sense<sub>F</sub>-making is supported by what van Oers (1998) called *contextualizing*. Van Oers (1998) argued for a dynamic approach to context that provides the “particularization of meaning” (p. 475), or more precisely, the particularization of a sense<sub>F</sub> that comes into being in a context in which an object actualizes.

Consider, for instance, the object  $\frac{3}{4}$ . There are many different ways of bringing to mind  $\frac{3}{4}$ , even within a particular representation system (e.g., as an iconic representation as illustrated in Figure 2a and Figure 2b). Different thoughts can be expressed in different contexts: Figure 2a expresses the thought ‘part of a whole’ (via dividing a whole into four equal parts and directing mind to three of these four parts), whereas Figure 2b expresses the thought ‘part of several wholes’ (via taking three wholes, each divided into four equal parts, and directing mind to one part of each whole).



Figure 2a: Part of a whole



Figure 2b: Part of several wholes

Recent research suggests that individuals seem to reason and make sense<sub>F</sub> from a specific perspective (see Scheiner & Pinto, 2018). It might be suggested that individuals take a specific perspective that orients their sense<sub>F</sub>-making, or more accurately: in taking a particular perspective, individuals direct their attention to particular senses<sub>F</sub>. Contextualizing, in this view, means taking a certain perspective that calls attention to particular senses<sub>F</sub>. Attention in such cases, however, may not involve an attempt to ‘sense’ or ‘see’ anything, but it seems to be attentive thinking: attention as the direction of thinking (see Mole, 2011). As such, calling attention to particular senses<sub>F</sub>, then, means directing mind to sense<sub>F</sub>. In this respect, contextualizing is intentional: it directs one’s thinking to particular senses<sub>F</sub>.

### **Complementizing: the conceptual function of creating conceptual unity**

Frege (1892b) underlined that a particular sense<sub>F</sub> “illuminates the reference<sub>F</sub> [...] in a very one-sided fashion. A complete knowledge of the reference<sub>F</sub> would require that we could say immediately whether any given sense<sub>F</sub> belongs to the reference<sub>F</sub>. To such knowledge we never attain.” (p. 27). This is to say, that just from sense<sub>F</sub>-making of one representation that refers to an object, we are typically not in a position to know what the object is (see Duval, 2006). As contextualizing serves to particularize only single senses<sub>F</sub> of a represented object, the same object can be ‘re-contextualized’ (see van Oers, 1998) in other ways that support the particularization of different senses<sub>F</sub> of the same object. Notice that senses<sub>F</sub> can differ despite sameness of reference<sub>F</sub>, and it is this difference of senses<sub>F</sub> that accounts for the ‘epistemological value’ of different representations. It is the diversity of senses<sub>F</sub> that has ‘epistemological significance’ and forms conceptual unity (see structuralist approach, Scheiner, 2016), not the similarity (or sameness) of senses<sub>F</sub> (as might be advocated in an empiricist view). This means, what matters is to coordinate diverse senses<sub>F</sub> to form a unity, a process called complementizing. However, the notion of ‘complementizing’ might be misunderstood as accumulating various senses<sub>F</sub> (until an individual has all of them); this is not the case. Complementizing means to coordinate different senses<sub>F</sub> to create conceptual unity.

Consider, once again, the object  $\frac{3}{4}$ . The two different thoughts of ‘part of a whole’ and ‘part of several wholes’ as expressed by the two different ways the object can be brought to mind are coordinated into a single unified way of presentation (see Figure 3).

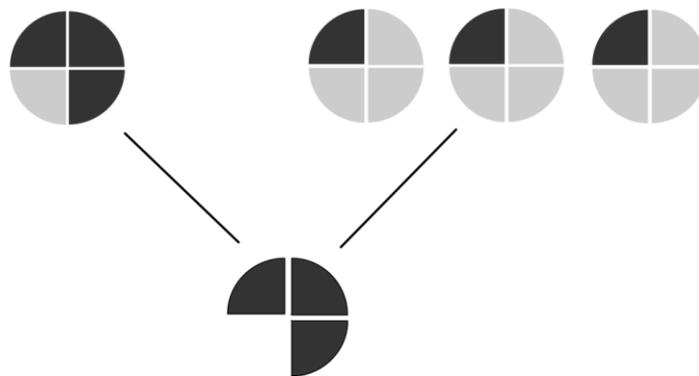


Figure 3: A conceptual unity of ‘part of a whole’ and ‘part of several wholes’

As each  $idea_F$  is partial in the sense of being restricted (in space and time) and biased (from a particular perspective), it needs to be put in dialogue with other  $ideas_F$  that offers an epistemological extension. The function of complementizing, then, is extending the epistemological space of possible  $ideas_F$ . Complementizing as extending the epistemological space of possible  $ideas_F$  brings a positive stance, indicating that seemingly conflicting  $ideas_F$  can be productively coordinated in a way such that these  $ideas_F$  are cooperative rather than conflicting. Hence complementizing is the ongoing expansion of one’s epistemological space, the ever-unfolding process of becoming capable of new, perhaps as-yet unimaginable possibilities.

### Complexifying: the cognitive function of creating a complex knowledge system

It is not only creating a unity of diverse  $senses_F$ , but creating an entity in its own right that forms a ‘whole’ from which emerges new qualities of the entity. That is, rather than treating the unity as a collection of different  $senses_F$  that can be assigned to objects that actualize in the immediate context, it is the forming of the unity that emerges new  $senses_F$  that might be assigned to potential objects.

For instance, with respect to the object  $\frac{3}{4}$ , the two different thoughts of ‘part of a whole’ and ‘part of several wholes’ cannot only be coordinated into a single unified way of presentation (see Figure 3), but also be blended so that it might promote the emergence of a new  $idea_F$  such as, for a given sequence of entities (e.g. balls), three entities are marked and one is left out respectively (see Figure 4). Put differently; every fourth entity is not in the focus of one’s attention.



Figure 4: Sequence of three colored balls and one non-colored ball

In forming a unity,  $senses_F$  are not merely considered as the parts of the unity, but “they are viewed as forming a whole with distinct properties and relations” (Dörfler, 2002, p. 342). It is, therefore, not an unachievable totality of  $senses_F$  (or  $ideas_F$ ) that matters, but how  $senses_F$  (or  $ideas_F$ ) are coordinated that develop emergent structure. This brings to the foreground a critical function of complexifying that has not been attested

yet: blending previously unrelated ideas<sub>F</sub> that emerge new dynamics and structure (for a detailed account of conceptual blending, see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called a cross-space mapping, between frames from established domains (known as inputs), in order to project selectively from those inputs into a novel hybrid frame (a blend), comprised of a structure from each of its inputs, as well as a unique structure of its own (emergent structure). This strengthens Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

## CONCLUSION

The emerging interpretive possibilities in thinking about contextualizing, complementizing, and complexifying have implications for theoretical, conceptual, and philosophical considerations in cognitive psychology in mathematics education. On the one hand, these perspectives call attention to a new understanding of mathematical concept formation: mathematical concept formation does not so much involve the attempt to recognize a previously unnoticed meaning of a concept (or the structure common to various objects), but rather a process of ascribing meaning to the objects of an individual's thinking from the perspective an individual has taken. That is, meaning is not so much an inherent quality of objects that is to be extracted, but something that is given to objects of one's thinking. On the other hand, in contrast to Frege (1892b), who construed a sense<sub>F</sub> in a disembodied fashion as a way an object is given to an individual, it might be suggested that individuals assign sense<sub>F</sub> to object. One is now in a position to interpret that what sense<sub>F</sub> is assigned to an object is related to what ideas<sub>F</sub> is activated in the immediate context. Recall the previous construal of Frege's notion of idea<sub>F</sub> as a manner in which an individual makes sense<sub>F</sub> of the world: ideas<sub>F</sub>, it can be asserted then, orient forming the modes of presentation under which an individual refers to an object.

## References

- Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analysing algebraic processes of thinking. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 61-81). Dordrecht, The Netherlands: Kluwer.
- Bikner-Ahsbabs, A., & Prediger, S. (2010). Networking of theories—an approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 483-506). New York, NY: Springer.
- Cobb, P. (2007). Putting philosophy to work. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3-38). Greenwich, CT: Information Age Publishing.
- diSessa, A. A. (1991). If we want to get ahead, we should get some theories. In R. G. Underhill (Ed.), *Proceedings of the 13th annual meeting of the North American Chapter of the*

- International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 220-239). Blacksburg, VA: PME-NA.
- Dörfler, W. (2002). Formation of mathematical objects as decision making. *Mathematical Thinking and Learning*, 4(4), 337-350.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1-2), 103-131.
- Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York, NY: Basic Books.
- Frege, G. (1892a). Über Begriff und Gegenstand (On concept and object). *Vierteljahresschrift für wissenschaftliche Philosophie*, 16, 192-205.
- Frege, G. (1892b). Über Sinn und Bedeutung (On sense and reference). *Zeitschrift für Philosophie und philosophische Kritik*, 100, 25-50.
- Mole, C. (2011). *Attention is cognitive unison: An essay in philosophical psychology*. Oxford, UK: Oxford University Press.
- Piaget, J. [and collaborators] (1977/2001). *Studies in reflecting abstraction* (Recherches sur l' abstraction réfléchissante) (translated by R. Campbell). Philadelphia, PA: Psychology Press.
- Pinto, M. M. F. (1998). *Students' understanding of real analysis*. Coventry, UK: University of Warwick.
- Pinto, M. M. F. (2018). Making sense of students' sense making of mathematics. In Proceedings of the First PME Regional Conference: South America. Rancagua, Chile: PME.
- Radford, L. (2002). The seen, the spoken and the written: a semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, 22(2), 14-23.
- Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, 91(2), 165-183.
- Scheiner, T., & Pinto, M. M. F. (2016). Images of abstraction in mathematics education: contradictions, controversies, and convergences. In C. Csíkos, A. Rausch, & J. Sztányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155-162). Szeged, Hungary: PME.
- Scheiner, T., & Pinto, M. M. F. (2018). *Emerging perspectives in mathematical cognition: Contextualizing, complementizing, and complexifying*. Manuscript submitted.
- Sierpinska, A., & Kilpatrick, J. (Eds.). (1998). *Mathematics education as a research domain: A search for identity*. Dordrecht, The Netherlands: Kluwer.
- Tall, D. O. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics*. Cambridge, UK: Cambridge University Press.
- van Oers, B. (1998). From context to contextualizing. *Learning and Instruction*, 8(6), 473-488.

# **WHY WOMEN TEND NOT TO CHOOSE MATHEMATICALLY DEMANDING CAREERS: A SYSTEMATIC REVIEW OF ALL TIME LITERATURE**

Darinka Radovic

Center for Mathematical Modeling, Universidad de Chile

*More than 30 years of intensive research on women's participation in sciences, technology, engineering and mathematics (STEM careers) have shown how women tend to be severely underrepresented, particularly in mathematically demanding careers. This research has explored reasons of underrepresentation from different perspectives and reached contrasting results. Existing summary reviews have offered useful but non-systematic approaches, presenting difficulties with the identification of emphases and gaps in the literature. This paper presents preliminary results of a comprehensive systematic literature review. By analysing purposes, methods and perspectives from a broad sample of papers, this study aims to explore gaps in the literature and advance a model that allows for integration of dissimilar approaches.*

## **UNDERREPRESENTATION OF WOMEN IN MATHEMATICALLY DEMANDING CAREERS**

The problem of the representation of women in mathematically demanding careers has been described as both progressive and persistent: it remains despite several interventions and treatments and becomes more acute in more advanced levels of the academic career (Cronin & Roger, 1999). For example, in Chile, while women in life and social sciences account for over 50% of students in undergraduate programs, in mathematics, engineering and computing they are 34%, 24% and 10% respectively (SIES, 2016). Women in the academic careers are even less represented. For example, in three of the biggest engineering programs in Santiago de Chile women account for only 16-17% of lecturers and professors with more than half-time contracts (number derived from public information in websites of Universidad de Chile, Universidad Católica y Universidad de Santiago).

Promoting female participation in careers with intense mathematical content is important for several reasons. Firstly, more participation of women can strengthen sciences through diversification of the labour force. Several studies have proven that diverse groups of people work better, are more creative and provide better solutions to complex problems (e.g. Smith & Schonfeld, 2000; Woolley et al., 2010). In addition, increasing women's participation in STEM can reduce social inequities produced by their low participation in highly valued careers and the corresponding gender pay gap (Petersen & Morgan, 1995).

Following the persistence and relevance of the problem, many researchers have attempted to understand the reasons for women's underrepresentation on mathematically demanding careers. 30 years of intensive research have come from different paradigms and theories to explore why women tend not to choose STEM, yielding dissimilar results and emphasizing different facets of the problem. Many researchers have tried to summarize this evidence following their own reviews of the literature. For example, in the process of reviewing the literature I have found at least 20 reviews. Even though these reviews provide valuable information, as a group there are important weaknesses.

Available reviews have either provided atheoretical summaries of reasons why women remain underrepresented in mathematically demanding careers or presented systematic focused reviews on the evidence of one theory as an explanation of this issue. On one hand, the main weakness of the summaries reviewed is that they have not presented a systematic approach in their methodologies. Some examples are Roger and Duffield (2000), Blickenstaff (2005) and Wang and Degol (2013). These reviews provide lists of factors with and without evidence of the impact on women's STEM career choices, but their narrative approach makes it difficult to get a sense of how the research has emphasized different explanations and what areas remain unexplored. On the other hand, systematic reviews have been focused on understanding the literature that has used one particular theoretical model or approach. For example, recently Su and Rounds (2015) and Boucher and colleagues (2017) explored how stereotypic perceptions of STEM as not affording communal goals that influence who enters, stays and excels in engineering, mathematics and computing. These focused reviews allow a more accurate account of the amount of evidence for this particular theory, but do not allow the integration of this evidence with competing theories in the field.

Systematic and integrative analyses of the literature in the understanding of women's mathematically demanding career choices are needed for two main reasons: 1) it can provide a clear sense of what the literature has been focused on and therefore which areas remain unexplored; and 2) it can advance the development of models that could bring together literature coming from diverse paradigms and allowing better integration of evidence. This research addresses these needs by systematically analysing the literature on women's STEM career choice using an identity lens.

### **USING IDENTITY AS A LENS FOR UNDERSTANDING LITERATURE ON WOMEN'S CHOICE OF MATHEMATICALLY DEMANDING CAREERS**

By and large, the literature that has tried to explain women's choice of mathematically demanding careers has presented complex relationships between individual choices and social influences. One relevant concept that can be used to understand these complex processes is the concept of identity. On one hand, identity considers different individual dispositions that have been found related to individual academic decisions, like interest (what I like or enjoy), aspirations and goals (who I would like to become) and perceptions of one's own ability and expectations of

success (what I am capable of), etc. (Eccles, 2009). On the other hand, many authors have suggested that this concept allows for broadening the focus of the interrelated understanding of subjective experiences in mathematical social contexts, including learning activities (a sense of who I am in a particular learning context), belonging to social groups (who I am in relationship with others), and social categories and influences of culture and society (who I am in a group and who we are in society) (e.g. Lerman, 2001).

The complex interrelation of aspects that the identity concept has tried to account for has given rise to multiple identity definitions that differ theoretically and methodologically (for reviews on mathematical identities see Darragh, 2015 and Radovic, Black, Williams & Salas, 2018). In our review, my colleagues and I suggested that these different definitions at least differ in three dimensions: a change/stability, a representational/enacted and a subjective/social dimension (see Radovic et al., 2018). For example, if we apply these dimensions to the study of the underrepresentation of women in mathematically demanding careers, the process can be understood as a development in time or as a choice (change/stability), focusing on the act of choosing or on representations that may influence this act (representational/enacted), and considering individual or/and social circumstances (see examples in Radovic et al., 2018). These different emphases and focuses will have implications on the general understanding of the problem. For example, if the focus is placed on individual characteristics, it can be assumed that there is only one way of doing mathematics (in which I am good or bad) or that there is only one type of mathematician (with which I identify/or not) independent of context. In contrast, if the social construction of individual choices (relationships and identities) is considered, mathematics and gender can become constructed phenomenon, where local and cultural negotiation of meanings and discourses happen.

Following the usefulness of identity as a concept for the understanding of choice and the previous analysis of its different uses for the understanding of individuals' relationships with educational subjects, this review uses this concept for the analysis of literature on women's mathematically demanding career choice. The analysis is focused on answering how this literature has approached the choice process from an identity perspective, considering how individual and subjective aspects of the decision, disciplinary context, relationships with others, gender, and social discourses are considered. This analysis will allow identification of emphases in the literature and a critical analysis of the available evidence.

## **METHODOLOGY**

A systematic search of concepts related to gender (gender OR women OR girl), career choice (career choice OR career interest OR career aspiration) and STEM (stem OR mathematics OR computing OR engineering OR physics OR technology) was carried out in two of the most important databases [Web of Science and Scopus]. These searches yielded 1125 hits after duplicate deletion. As this is a working project, this report will be focused on 400 randomly chosen articles.

Titles and abstracts of the 400 articles were screened to see if they fit the purpose of exploring reasons for the underrepresentation of women in STEM, focusing specifically on women or gendered influenced choices of STEM careers. I excluded 116 articles that were not focused on the process of choosing a STEM career (these were mainly articles that were focused on persistence in STEM academic careers or into work after the process of choosing) and 41 articles that did not present data (reviews, commentaries and projects). After this process 243 articles were selected to be included in this report.

The process of analysis of the 243 papers followed two steps. First, general characteristics of the articles were coded including types of article (study or intervention), area of STEM covered, how categories of women and gender were considered (differences between sexes, focus on women, intersectional, other), in which stage of the career trajectory the sample of the article was focused and methodology. All of this information serves as context for the purpose of the articles and as categories for comparisons. After that I followed a thematic synthesis approach (Thomas & Harden, 2008) in analysing the articles' purposes and focus as described in the abstract and method sections, going from open coding to more abstract categories. I was interested in understanding how each article engaged with the problem, how authors operationalized the decision process and variables that were considered in the analysis. Using theoretical tools derived from the concept of identity, I went from codes highly attached to data to more general descriptions of the approach of the article, considering how the main focus of the article was defined (e.g. interests and aspirations, choice, decision and participation, motivations, identity, etc.) and what intervening variables were considered (individual variables, local contexts, and socio-cultural constructions). Implications from this conceptual approach were explored in relation to how mathematics (and STEM) and how gender (or being a women) was approached.

## **RESULTS**

The first article included in this review was written in 1989. Since then, research on women's decisions to study mathematically demanding careers has grown significantly, with most articles being published in the last 5 years. In relation to specific areas 95 articles focused on STEM careers in general (39%), 49 in computing and information technologies (20%), 34 in engineering (14%), 29 in science (12%), 21 in mathematics and physics (9%) and 15 in other STEM careers or did not specified (6%). Initially, articles were categorized into two main groups: articles that presented interventions (including or not its evaluation) (38%, n= 92) and articles that presented studies (62%, n= 151).

### **Articles focused on Interventions**

The big number of articles that present interventions shows how increasing representation of women (and other minorities) in STEM careers has been a matter of concern and a focus of initiatives aimed at solving this problem. Most interventions

(55) where explicitly designed and implemented with a focus on women or girls. 15 were open for boy/men and girls/women, but had an explicit focus on girls/women, and 18 were focused on students in general (9 of them on minority students), with mentions of gender and other diversities represented but not with an explicit focus on them. Only 4 intervention programs had an intersectional approach, focusing their design and implementation on specific problems of girls from ethnic minorities (e.g. appalachian girls, hawaiian girls). Most interventions were focused on school students, with 7 focused in elementary school, 32 in middle school, 29 in high school and 8 in general K-12.

The vast majority of articles that presented interventions aimed at increasing girls (and other minorities) interests and motivation to pursue a mathematically demanding career by offering different programs of “engaging” activities. In this sense, these interventions used an understanding of the individual choice and the commitment with a STEM identity as determined by how the social activity of doing STEM is presented and experienced by students. Examples were after school activities, summer programs, elective courses and workshops with hands-on experiences, focused on problem solving, students’ (and girls) interests and some considered mentoring and communication with role models.

An interesting finding regarding interventions was that most of the interventions were designed as out of school activities, where students visited universities, colleges or private companies (65 articles, 71%). Only 20 articles (22%) described interventions to school science and mathematics. These interventions offered elective courses and workshops, usually designed by academics or experts in the field and implemented by school teachers or embedded new approaches to school science and mathematics by offering professional development to school teachers in curricular or teaching innovations.

### **Research Studies**

In relation to research studies, a huge emphasis on quantitative data and methodologies was found (n= 115, 76%). Only 24 articles used in-depth exploration and qualitative data (16%) and 12 attempted to mix different sources, mainly by using interview data (qualitative) to explore more detailed survey data (quantitative) (8%).

When analysing how the problem of underrepresentation was approached by research studies, it was found that **all of them** considered subjective experiences and/or individual characteristics as explaining factors. Students’ interests and motivations, beliefs about their abilities (self-efficacy and self-concept), confidence, and even some personality traits (competitiveness, perfectionism and orientation towards people or things) were considered. An interesting result regarding how studies explored women’s underrepresentation was that a third of the reviewed articles (50) only considered these individual level variables, leaving any social influence unexplored.

In the majority of studies the individual experience of choosing a STEM career was analysed in relation to social variables and considered social contexts. On the one hand, 71 studies (47%) considered proximal contexts influencing individual choices, including background characteristics and the beliefs of parents and teachers (20, 13%), perceived support (29, 19%), how mathematics, science and general STEM is presented in local practice (including teaching and controlled manipulation of environment) (24, 16%), experiences of advanced science/mathematics courses (9, 6%), school characteristics (type, socioeconomic status of school) (6, 4%), and in a smaller number, relationships and interactions with peers (3, 2%).

On the other hand, 46 studies (31%) considered more distal, macro social variables in their inquiries, including institutional distribution of privilege (through for example access to courses), cultural differences in equity indexes between countries, and stereotypes, public images and cultural discourses about mathematics, STEM and gender roles in society. Only a very few (15, 10%) explored macro variables and local variables at the same time, with about half of them exploring how social discourses and institutional constraints are expressed in support and characteristics of socializers, and half on how STEM and mathematics is presented in the classroom/school.

## **CONCLUSION AND DISCUSSION**

This preliminary analysis of articles aimed at understanding women's choice of mathematically demanding careers offers interesting insights regarding the emphasis of the literature in the field. Interventions worked with the idea that individual attitudes and dispositions were intrinsically related with the local experience of doing mathematics and STEM activities. By working with meanings and practices related with STEM, interventions were aimed at changing these individual dispositions. An interesting result regarding these interventions was how most of them were designed as out of school activities. This suggests that activities were offered as independent from regular students' learning activities, contributing to the disconnection between school mathematics/sciences and university/professional STEM careers.

In contrast, research studies showed a big emphasis on individual attitudes, with many of them exploring these individual dispositions in relation to social variables. One interesting result in this regard is how the construction of general discourses in local practices is a neglected topic of research. There is an established literature about mathematics and STEM gendered stereotypes, and this study surveyed some of them (e.g. Cheryan et al., 2013; Leslie et al., 2015). Only very few articles explored how local practices constructed mathematics and STEM as masculine/feminine, for example, through lack of support for women and through particular practices (e.g. non communal).

Although these preliminary results are encouraging about the value of this analysis in advancing the understanding of women's (lack of) participation in mathematically demanding careers, this is only the first step. More detailed explorations are needed.

For example, in interventions there is a clear emphasis on STEM constructions and local activities and how these may affect individual dispositions and identities, but the current analysis does not allow for the exploration of how these local practices relate to gender. Are activities designed as more engaging by, for example, being more similar to “real” STEM (what STEM professionals do in their professional lives or what STEM students do in their careers in HE) or by being closer to “women’s/girls” needs (making STEM more female friendly)?

A more detailed analysis of how gender is considered in research studies can also strengthen the results that were presented in this report. For example, Risman (2004) has argued that gender can also act simultaneously on different levels, influencing individual choices, social practices (expectations, bias and interactions), and institutions and institutional constraints. Again, on these different levels gender (and gendered identities) can be conceptualized as a sex category that is stable and given or can be understood as a social construction, where individuals are required to perform in certain ways that are culturally determined. Although in this preliminary analysis, a strong emphasis on gender as differential and therefore as sex category was observed, further analyses are needed to explore in which cases more detailed constructions of gender are explored.

In summary, this article suggests that there are concerning disconnections between literature and between approaches for the understanding and intervention of the problem of women’s underrepresentation in STEM and mathematically demanding careers. Developing different analysis from the entire literature and testing different models of understanding, their approaches and evidences are needed for advancing in this integration. This study will be an attempt towards this direction.

## ACKNOWLEDGEMENTS

This work was funded by CONICYT Proyecto Fondecyt Post-doctorado # 3180291 and by the Center for Mathematical Modeling through CONICYT+PIA Basal funding for scientific and technological excellence centers AFB170001 (CMM) and Unité Mixte Internationale UMI CNRS 2807.

## REFERENCES

- Blickenstaff, C.J. (2005). Women and science careers: leaky pipeline or gender filter? *Gender and education*, 17(4), 369-386.
- Boucher, K. L., Fuesting, M. A., Diekman, A. B., & Murphy, M. C. (2017). Can I work with and help others in this field? How communal goals influence interest and participation in STEM Fields. *Frontiers in psychology*, 8, 901.
- Cheryan, S., Plaut, V. C., Handron, C., & Hudson, L. (2013). The stereotypical computer scientist: Gendered media representations as a barrier to inclusion for women. *Sex roles*, 69(1-2), 58-71.

- Cronin, C., & Roger, A. (1999). Theorizing progress: Women in science, engineering, and technology in higher education. *Journal of Research in Science Teaching*, 36(6), 637-661.
- Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, 93(1), 19-33.
- Eccles, J. (2009). Who am I and what am I going to do with my life? Personal and collective identities as motivators of action. *Educational Psychologist*, 44(2), 78-89.
- Franklin, D. (2013). A Practical Guide to Gender Diversity for Computer Science Faculty. *Synthesis Lectures on Professionalism and Career Advancement for Scientists and Engineers*, 1(2), 1-81.
- Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational studies in mathematics*, 46(1-3), 87-113.
- Leslie, S. J., Cimpian, A., Meyer, M., & Freeland, E. (2015). Expectations of brilliance underlie gender distributions across academic disciplines. *Science*, 347(6219), 262-265.
- Petersen, T. & Morgan, L.A. (1995) Separate and Unequal: Occupation-Establishment Sex Segregation and the Gender Wage Gap. *American Journal of Sociology* 101(September), 329-65.
- Radovic, D., Black, L., Williams, J. & Salas, C. (2018). Towards conceptual coherence in the research on mathematics learner identity: a systematic review of the literature. *Educational Studies in Mathematics*, 99(1), 21-42.
- Risman, B. J. (2004). Gender as a social structure: Theory wrestling with activism. *Gender & society*, 18(4), 429-450.
- Roger, A., & Duffield, J. (2000). Factors underlying persistent gendered option choices in school science and technology in Scotland. *Gender and education*, 12(3), 367-383.
- SIES (Servicio de Información de Educación Superior) (2016) *Bases de datos matriculados*. Disponible en: <http://www.mifuturo.cl/index.php/bases-de-datos/matriculados>
- Smith, D. G., & Schonfeld, N. B. (2000). The Benefits of Diversity what the Research Tells Us. *About campus*, 5(5), 16-23.
- Su, R., & Rounds, J. (2015). All STEM fields are not created equal: People and things interests explain gender disparities across STEM fields. *Frontiers in psychology*, 6(189), 1-20.
- Thomas, J., & Harden, A. (2008). Methods for the thematic synthesis of qualitative research in systematic reviews. *BMC Medical Research Methodology*, 8(1), 45.
- Wang, M. T., & Degol, J. (2013). Motivational pathways to STEM career choices: Using expectancy-value perspective to understand individual and gender differences in STEM fields. *Developmental Review*, 33(4), 304-340.
- Woolley, A.W., Chabris, C.F., Pentland A., Hashmi, N., Malone, T.W. (2010) Evidence for a Collective Intelligence Factor in the Performance of Human Groups. *Science*, 330(6004), 686-688.

# THE BASIC IDEAS (BIS) OF PROSPECTIVE TEACHERS FOR THE FIRST CLASS OF TEACHING ADDITION

Pamela Reyes Santander & Rudolf vom Hofe

Universität Bielefeld

**Abstract:** *Focusing on Primary Education Initial Teaching Training, the objective of this study is to determine the BIs of addition that students would use to teach this concept in primary school. A cross-sectional comparison was made of Chilean students in teacher training, from a first year cohort and a group of higher years. The BIs of addition presented by students were observed from their proposals about the first class they would develop to teach addition. The results show that first-year students present BIs of addition similar to those of final year students, although there is an important difference in the inclusion and use of materials, as also in the relationship with the real world.*

**Resumen:** *Centrándonos en la formación inicial de profesores de primaria, el objetivo de este estudio es determinar las BIs de la adición que utilizarían los estudiantes para enseñar este concepto en la escuela primaria. Se realizó una comparación transversal de estudiantes chilenos en formación docente, una cohorte de primer año y un grupo de curso superior. Las BIs de la adición que presentan los estudiantes se observaron a partir de sus propuestas de una primera clase para enseñar la adición. Los resultados muestran que los estudiantes de primer año presentan BIs de la suma similares a las de los estudiantes de cursos superiores, aunque hay una diferencia importante en la inclusión del uso de materiales y la relación con lo concreto.*

## Introduction

Arithmetic skills are essential for the development of mathematical thinking as well as for the effective development of society (Butterworth, 2005). Arithmetic is associated with verbal and spatial skills (Dowker, 2005; Lehmann & Juling, 2002) and it has a closed relationship with algebraic thinking (Carpenter, Levi, Loef & Koehler, 2005). Despite the importance of arithmetic, we have not yet managed to ensure the learning of this and other topics that requires a good arithmetical basis such as algebra (Carpenter & Franke, 2001; Warren 2004).

According to Dowker (2005), most of the difficulties in arithmetic are due to a mismatch between the cognitive strengths of an individual and the way he is taught in school. For this reason, it is necessary to characterize the way in which future elementary teachers learn and attend to the tools they acquire to teach. (Llinares & Krainer, 2006). So this work has a focus on primary education Initial Teacher Training (ITT) and in the arithmetical concept of addition.

According to Ernest (1989), beliefs play an important role in ITT. Ernest identified three belief components: (i) conception of nature of mathematics, (ii) model of the

nature of mathematics teaching, and (iii) model of the process of learning mathematics. This indicates, among other things, that what the teachers teach are their own beliefs about the mathematical concept and beliefs about models for teaching and learning. On the other hand, the Basic Ideas (BIs), which today are known as category of subject matter didactics (vom Hofe & Blum, 2016) reappear as an alternative to unify teacher beliefs and mathematical knowledge.

The BIs of mathematical concepts started with Pestalozzi around the years 1800, continued with other investigators such as Herbart, Kühnel, Oehl and resurfaced with the three-dimensional characterization of vom Hofe (1995). This characterization indicates that the BIs (1) come from the subject's experience (in the same way that beliefs) that allows the construction of significance for the mathematical concept, (2) the generation of a visual representation appropriate for such a concept (in the same way as mathematical knowledge) and (3) the ability to apply the BIs to reality, recognizing the corresponding structure in factual connections or via modelling.

We consider also the BIs and the three-dimensional characterization as an analysis method for the data (Reyes-Santander, 2012; Reyes-Santander & vom Hofe, 2018; Wartha & vom Hofe, 2005; Vohns, 2005; Wartha & Schulz, 2012). In particular, they are considered the BI for addition: adding, combining and completing (vom Hofe, 1995; Reyes-Santander & vom Hofe, 2018; Wartha & Schulz, 2012).

Although there are other studies on addition that consider the additive structure problem (combine, change and compare) proposed by Nesher, Greeno & Riley (1984) or the main additive relationships proposed by Vergnaud (1982), these focus on the written problem and are appreciated by some differences with the BIs of addition. BIs consider also the static and dynamic concept of the situations, but the focus is on the real actions that children can do to generate the appropriate visual representations of the mathematical concept and the relationship with modelling.

This study considered a group of Chilean students in ITT and how they would run a first class to teach addition. Considering the first dimension related to the significance for the mathematical concept from real actions, we can say that an environment with concrete material or active situation for the children is very important. We would expect that all ITT programs promote different BIs of addition and change the vision on the use of the material for teaching and learning. The above raises the following research questions:

- Are there any differences in the BIs of addition in first year and senior students?
- Is there use of concrete material in the first class about the teaching of addition?

### **Basic Ideas for addition**

BIs are given substantive interpretations of mathematical objects, definition, operations, and mathematical relations, they give appropriate constructs to locate the

abilities of translation from the real world to the mathematical world (vom Hofe, 2003; vom Hofe & Blum, 2016). To give significance to addition it is necessary to involve experiences such as adding, combining or completing with concrete materials (Reyes-Santander & vom Hofe, 2018).

As we already mentioned, the BIs for addition are: adding S-C-S; combining S-C-S; combining S-S-S; completing S-S-C. In figure 1, the action of adding stones in a bowl can be observed as the basic notion of adding in state (S), change (C), state (S). The image can be mathematically represented by the expression “there are five stones in the bowl (S), three are added (C) so eight are left in the bowl (S)”, which as written equals  $5 + 3 = 8$ .



Figure 1: Basic notion of adding S-C-S.

In Figure 2, it can be observed that there are 3 stones in a group and 5 in another group, the change is done when these two groups get together and the final state is achieved, when these two groups are mixed, resulting in a set of 8 stones. The image is associated to the mathematical representation of combining, in which if 3 objects and 5 objects get together, there are 8 objects. This corresponds to the addition  $3 + 5 = 8$ . In the case of the BI combining S-S-S there is a mental union of two existing sets, and there is no need to concretely get both groups together in order to indicate the total amount of existing objects.

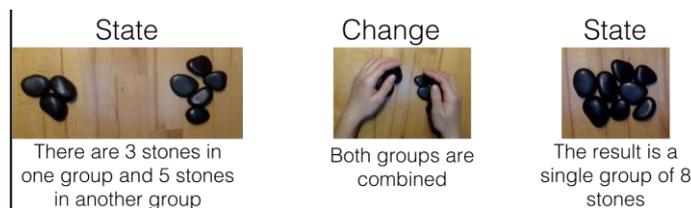


Figure 2: Basic Notion of combining S-C-S.

The BI completing S-S-C (see figure 3) gives a concrete solution to the question: how many squares are left to get 8? The strategy used in this case is the comparison between sets, the initial set 5 squares and the final set 8 squares. In figure 3, the initial state (S) is to have the first 5 squares; the following state (S) is having 8 squares, and the change is given by the comparison and observation of wanting 3 squares (C). The answer given in this case, are 3 wanting squares to obtain 8, and the addition associated to this action is  $5 + 3 = 8$ .



Figure 3: Basic notion of completing S-S-C.

The backwards work that some students can perform to answer this question could be the subtraction  $8 - 3 = 5$ . Although the action that is related to the sum is completing, the idea is "how much is wanting" where the student "wants to reach a certain amount by completing" (the student needs more).

### **Methodology**

This is a cross-sectional comparison of two different cohorts of teacher education students and non-experimental methods. The groups of comparison are a first-year and a higher-year cohort of students in ITT from a Chilean university. 48 subjects are first year students who have never participated in a mathematics didactic class; and 47 students who have had at least one mathematics didactic class.

In 2013 the department began a program change and this study was carried out in 2016, thus 11 higher-year students belong to the old program and the other 37 students were in the third year before going on to the final practice. The major issues that all students could have accessed in the new program were: mathematics in the primary school system; development of mathematical thinking in primary school; elementary notions of mathematics didactics; decimal system, its operations and resolution of problems; development of mathematical language; the representations in the initial mathematization; development of mathematical skills in primary school.

The data considered in the study corresponded to the student's proposal for the first lesson that they would run to teach addition. The categories of this study are:

- Category 1 with four subcategories: 1.1 adding S-C-S; 1.2 combining S-C-S; 1.3 combining S-S-S and 1.4 completing S-S-C.
- Category 2, with three subcategories: 2.1 actions using concrete material, where the meaning of addition is given; 2.2 verbal explanations with material (only visually) about what is addition and 2.3 no use of concrete material.

### **Results for the category 1**

BI was found in 22 proposals (.46) out of the total of 48 first-year students. In the case of the 47 students of higher years, 7 of them did not answer the question and in 30 proposals (.75) we found BI. Table 1 shows the results for each BI, for the subcategory 1.1 the students used the words adding or included directly an example. In the subcategory 1.2 and 1.3 the students used directly words like grouping, joining or union.

In Table 1 it can be observed that the subcategory 1.1 is the most considered BI. An example of the answers given by senior students is the one presented by the student H31, where he mentions that he would use the term adding because it was the one used in primary years. This indicates that the student has incorporated knowledge to his belief; hence, this belief is closer to being sophisticated.

H31: I would start the teaching using concrete materials, asking the children to use bottle caps. During the first exercises I wouldn't use the term addition, but adding, which is the one used in preschool. After those exercises using concrete materials, I would express the

same on the whiteboard as addition and I'd transfer their knowledge of adding to the formal concept of addition "+".

	1.1 Adding S-C-S	1.2 Combining S-C-S	1.3 Combining S-S-S	1.4 Completing S-S-C
First year	F1; F2; F5; F8; F12; F13; F15; F16; F22; F25; F26; F34; F37; F45	F13; F20; F21; F38; F39	F23; F35; F46	--
Higher years	H1; H2; H3; H8; H43; H7; H12; H13; H16; H17; H20; H29; H31; H34; H35; H37; H41; H42; H45; H46	H1; H4; H14; H18; H23; H24; H26	H19; H21	H15

Table 1: Grouping of students according to the four subcategories of the Basic Ideas of Addition.

Within the same subcategory, first year student F34 did not give explanations related to the usage of the word adding. The alternative strategy to use was counting using the fingers. Then the use of drawing is mentioned. Here it could be said that the student's belief (first year) was also closer to being sophisticated. In contrast, the rest of the first year students, within the same category were closer to naive beliefs.

F34: I would teach them first the idea of "adding" before "addition", first through counting using their fingers, then through the use of drawings and then, once they have that clear, I'd teach the concept of addition.

In the case of the subcategory 1.2 it could be said that there is a minor amount of students who would start with the action of combining to give meaning to the addition in the first class.

The three first year students that are in subcategory 1.3, mentioned examples and from this starting point it is inferred that there is a mental union of objects. An example F23 of this subcategory is when the student uses a visual presentation with apples to explain addition.

The only senior student H15 that is within subcategory 1.4, presented a problem related to completing according to amount. The ITT student would use concrete material, which is meant to be handled by students, and he would give a starting number that is meant to be reached by children.

## Results for the category 2

Regarding category 2, related to the use of concrete material to give meaning to addition, the amount of senior students who base their methodologies on the use of concrete material in their first class about addition is shown in Table 2. It can be observed that to first year students the use of this type of material has a role more visual than active; and there is a great amount of first year students who do not consider the use of concrete materials in their first class, this probably happens because of their own experiences regarding addition in primary years.

	2.1 Concrete actions	2.2 Explanations with material	2.3 Without material
First Year	F1; F3; F5; F12; F13; F20; F21; F25; F29; F32; F34; F37; F47	F6; F8; F10; F11; F17; F19; F22; F23; F24; F28; F30; F35; F36; F38; F39; F45; F40; F43; F44; F46	F2; F4; F7; F9; F14; F15; F16; F18; F26; F27; F31; F33; F41; F42; F48
Higher Years	H1; H2; H4; H6; H7; H8; H14; H15; H16; H17; H18; H19; H20; H21; H23; H24; H25; H26; H29; H30; H31; H32; H35; H40; H43; H46	H3; H12; H13; H33; H34; H35; H36; H37; H38; H39; H41; H42	H5; H45

Table 2: Grouping of students according to the three subcategories related to the use of material to give meaning to addition.

An example for the subcategory 2.1 is giving by F12 when they mentioned the word “grouping” things, for instance, apples, corn or beans, which counts as working with concrete material.

F12: In my first class to teach addition I’d start with an easy example about grouping things, like apples or something like that, clearly knowing that kids know how to count or know what a number represents. Then I’d do an example using corn or beans in each desk, to add in every exercise.

Senior students are more or less precise in relation to the type of material they would use. And the moment in which they would use the material corresponds to the right one in the development of the class. Additionally, the way of working includes theoretical elements, such as in the case of student H4, where he mentions how colored marbles can be used first counting and combining.

H4: I’d use a series of known objects, for example, colored marbles. First, we would do counting. Then I’d gather a certain amount of marbles in bags, five in one bag and five in another bag, to later ask what would happen if we mix (combining) both.

It is important to highlight that senior students consider the use of recyclable materials in concrete work and the use of dices to start the addition class. Two students mentioned the use of the number line, drawn in the playground, where students can jump over it.

In subcategory 2.2, explanations and use of concrete material, the answers were based mainly on initial verbal explanations and there is use of material in relation to the visual area, as was shown in the transcripts of answers in the case of first year students F8.

F8: I'd start explaining the definition of addition, then I'd give examples using objects and finally I'd teach the method using numbers.

In comparing the answers of first year students and senior students, it can be observed that first year students mention concepts without going in depth. For example, the student F8 would teach in a first class the numerical method to add, which would be completely inappropriate for a first, or even a second class.

In the subcategory 2.3, the classic teaching methodologies can be observed. The student F27 is an example of the first year student answers, in which they based their lessons on the theoretical and frontal teaching of addition.

F27: I'd start giving a short definition of addition and then I'd explain it with my words so they understand the concept. Then I'd do some easy addition examples so they can do some exercises ( $5 + 4 = 9$ ,  $2 + 1 = 3$ ).

## **Discussion**

The results show that less than half of the first year students would not use any of the BIs to run a first class to teach addition. This is because they are just entering the studies. On the other hand, it indicates that their beliefs, based on their learning experiences, do not include BI and that ITT must accurately include this topic in their programs.

It can also be said that most of the higher year students would include NB adding or combining (S-C-S) in their first class. This is a more or less expected result, in the sense that these students already include knowledge at this stage. It is not as expected, because all students should have responded and because there are many productions where there is no mathematical significance of this concept from the real actions of children. Unlike the completed BI, it can be said that first year students and seniors handle similar IB.

Regarding the use of material, we can say that in the case of first year students, there is a use of concrete materials, but in general this usage is not related to the mathematical concept. It is surprising that senior students consider the use of materials only by the teacher and not for all students in the class.

## **References**

Butterworth, B. (2005). The development of arithmetical abilities. *Journal of Child Psychology and Psychiatry*, 46(1), 3 – 18, doi: 10.1111/j.1469-7610.2005.00374.x

- Carpenter, T., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalisation and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent. (Eds.), *The future of the teaching and learning of algebra, Proceedings of the 12th International Commission on Mathematics Instruction study conference*, Vol. 1, pp. 155-162). Melbourne: University of Melbourne.
- Carpenter, T., Levi, L., Loef, M., & Koehler, J. (2005). Algebra in Elementary School: Developing Relational Thinking. *Zentralblatt für Didaktik der Mathematik*, 37(1), 53 - 59.
- Dowker, A. (2005). *Individual Differences in Arithmetic*. NY: Psychology Press.
- Ernest, P. (1989). The knowledge, beliefs, and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, 15, 13-34.
- vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*. Heidelberg: Spektrum.
- vom Hofe, R. (2003). Grundbildung durch Grundvorstellungen. *Mathematik Lehren*, 118, 4-8.
- vom Hofe, R. & Blum, W. (2016). "Grundvorstellungen" as a Category of Subject-Matter Didactics. *Journal für Mathematik-Didaktik*, 37 (1), 225-254, <https://doi.org/10.1007/s13138-016-0107-3>
- Lehmann, W. & Juling, I. (2002). Spatial reasoning and mathematical abilities: Independent constructs or two sides of the same coin? *Psychologie in Erziehung und Unterricht*, 49, 31-43.
- Llinares, S., & K. Krainer (2006). Mathematics (student) teachers and teachers educators as learners. In A. Gutierrez & P. Boero (eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, present and future*, Rotterdam/Taipei, Sense Publishers, 429-459.
- Reyes-Santander, P. (2012). *Charakterisierung des mathematischen Denkens – Szenarien mit Gymnasiasten und Studenten unter Verwendung von Themen der Gruppentheorie*. Doctoral Thesis, University Augsburg, Germany. OPUS: <http://opus.bibliothek.uni-augsburg.de/opus4/frontdoor/index/index/docId/2627>
- Reyes-Santander, P. & vom Hofe, R. (2018). Basic Ideas for addition in primary education trainees. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds). *Proceedings of the 42nd Conference International Group for the Psychology of Mathematics Education* (Vol. 5, p. 146). Umeå, Sweden: PME.
- Vohns, A. (2005). Fundamentale Ideen und Grundvorstellungen: Versuch einer konstruktiven Zusammenführung am Beispiel der Addition von Brüchen. *Journal für Mathematik-Didaktik*, 26 (1), 52 – 79.
- Warren, E. (2004). Generalizing arithmetic: supporting the process in the early years. In M. Johnsen y A. Berit (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 417-424). Bergen, Noruega: Bergen University College.
- Wartha, S. & vom Hofe, R. (2005). Probleme bei Anwendungsaufgaben in der Bruchrechnung. *Mathematik lehren*, 128, 10 – 17.
- Wartha & Schulz (2012). *Rechenproblemen vorbeugen*. Berlin: Cornelsen.

# THE TRANSITION FROM MATHEMATICS TEACHER TO FACILITATOR: CHALLENGES IN SCALING UP PROFESSIONAL DEVELOPMENT PROGRAMS

Luz Valoyes-Chávez, Natalia Ruiz, Carmen Espinoza, Carmen Sepúlveda, & Nicole Fuenzalida

Centre for Advanced Research in Education – University of Chile

*Activating Problem Solving in the Classroom or ARPA is a professional development program aimed to enhance Chilean teachers' abilities to implement mathematics problem solving. As ARPA shows its impact on mathematics teachers' practices, an educational program for facilitators able to scale it up has been designed and implemented. Drawing upon the Situated Learning theoretical perspective, in this paper we focus on the learning process of Luis, a novice facilitator. In particular, we analyze his learning to develop teachers' abilities to set up and sustain meaningful mathematics and pedagogical discussions, a key feature of ARPA. The paper contributes to the incipient body of research aimed at building capacities to scale up PDs to help teachers implement reform-based mathematics teaching.*

## INTRODUCTION

During the last decades, the education of mathematics teachers able to enact the core of mathematics education reforms has received special attention. From different theoretical perspectives, researchers have designed, put into practice, and evaluated professional development programs (PDs) aimed at producing a new mathematics teacher (Even, 2008). However, scaling up PDs that positively impact mathematics teaching and learning has proved difficult. For instance, an important challenge in expanding a PD is the education of facilitators who are able to faithfully and flexibly replicate its core aspects (Borko, Knoellner & Jacobs, 2014). Little is known about the knowledge and abilities required for facilitators to recreate the fundamental principles of PDs in new and unfamiliar school contexts (Roesken-Winter, Hoyles & Blömeke, 2015) nor those necessary to fully respond to mathematics teachers' needs. Furthermore, Even (2008) underlines the lack of scientific knowledge about facilitators' practices and education, indicating a significant gap in the literature.

To address this issue, several studies from different theoretical perspectives have focused on designing and implementing educational programs for facilitators (e.g., Jackson et al., 2015). These studies have shed light on the critical role of learning opportunities that allow facilitators to experience the PDs' fundamentals as well as to participate in their main activities. Features such as collective engagement, ongoing individual and collective reflection over practice, and close interactions with expert facilitators seem to contribute to the enhancement of participants' knowledge and abilities to scale up PDs. Yet, results indicate the limited and ephemeral impact of these educational programs on facilitators' knowledge, abilities and leadership (Jackson et al., 2015).

In this paper, we discuss the preliminary results of an ongoing study aimed to comprehend the learning processes of novice facilitators. The study is being conducted as part of an effort to scale up the PD *Activating Problem Solving in the Classroom* or *ARPA* (its acronym in Spanish) in Chile. As part of a national movement toward mathematics curricular reform, ARPA was set up to enhance teachers' knowledge and abilities to implement problem solving in their classrooms. Research on ARPA has shown its positive impact on teachers' conceptions of mathematics teaching and learning (Cerda et al., 2017) and on their own abilities to solve non-routine mathematics problems (Felmer & Perdomo-Díaz, 2016), resulting in the challenge to replicate it across Chile.

In particular, we consider the learning process of Luis, a novice facilitator. We focus on the process he undergoes in learning to set up and sustain meaningful pedagogical and mathematical discussions. This is a critical aspect of ARPA that seems to be difficult for facilitators to perform as it involves building upon the participating teachers' thinking and previous experiences. By drawing upon the theoretical perspective of Situated Learning (Lave & Wenger, 1991), we focus on Luis' participation and engagement in the first stage of the ARPA Educational Program for Facilitators (ARPA-EPF). The research question guiding this study is:

To what extent does a novice facilitator's engagement and participation in the learning experiences proposed in the ARPA-EPF contribute to enhancing his knowledge and abilities in setting up and sustaining meaningful pedagogical and mathematics discussions?

## **THE ARPA PD INITIATIVE**

The goal of ARPA is twofold. First, the program intends to provide Chilean teachers with opportunities to experience non-routine mathematics problem solving. Second, it aims to help teachers learn to develop this ability among their students. ARPA introduces the Problem Solving Activities in Classroom or PSAC (Felmer & Perdomo-Díaz, 2016) to model the mathematics activity in class. PSAC is structured into four stages: Delivery, Activation, Consolidation and Discussion. In randomly organized groups, the students engage in solving problems. The teacher's main role is to pose questions that would help the students move forward in the solving process. If a group has difficulty solving the problem, the teacher provides a simplification. Otherwise, the group is given an extension. The PSAC ends with a plenary, bringing students together to think about and discuss the process of solving the problem. During the ARPA PD, the participants have multiple opportunities to experience PSAC in order to learn how to implement it in their own classrooms.

ARPA is comprised of 3 practical-oriented workshops that build upon the principles of doing and reflecting (Felmer & Perdomo-Díaz, 2016), including Problem Solving Action (PSAction), Problem Solving Content (PSContent) and Problem Solving Classroom (PSClassroom). PSClassroom is the most important workshop, consisting of 8 sessions distributed throughout the course of the school year. The workshop

gradually moves from engaging the participants in an intensive problem-solving activity towards planning their own PSAC. The first two sessions are devoted to solving non-routine mathematics problems to strengthen teachers' identities and abilities as problem solvers. During the next sessions, the participants learn to set up the four stages of a PSAC. The main facilitator's role in the workshop is to model the enactment of the PSAC for the participating teachers and to provide opportunities for them to reflect about the process.

### **THE ARPA EDUCATIONAL PROGRAM FOR FACILITATORS**

The ARPA Educational Program for Facilitators, ARPA-EPF, draws upon the theoretical developments of Situated Learning (Lave & Wenger, 1991). In this paper, Situated Learning is utilized as “a conceptual framework from which to derive a consistent set of general principles and recommendations for understanding and enabling learning” (Wenger, 2009, p. 201). It is conceived of as an analytical and theoretical tool to comprehend the novice facilitators' process of learning to implement ARPA. In contrast to cognitive perspectives on learning, this theoretical approach underlines its social and collective nature. Learning is defined as “an aspect of participation in socially situated practices” (Wenger, 2009, p. 211). It implies not only engaging in and building meaning of social practices but also, and mainly, *becoming*. As Wenger (2009) posits, participation shapes “not only what we do, but also who we are and how we interpret what we do” (p. 211). Learning involves making sense of the collective practices in a community while also being an active participant in such practices. Thus, identity, meaning and participation emerge as key conceptual tools in understanding novice facilitators' learning processes.

Accordingly, ARPA-EPF defines learning as increasing participation in the shared practices of the ARPA facilitators' community. Such practices include, but are not limited to, designing and implementing ARPA workshops, elaborating non-routine mathematics problems, providing feedback to teachers, and engaging in professional enrichment. By understanding learning in this way, both professional collaboration and engagement in collective practices emerge as critical features for the development of ARPA facilitators. Thus, drawing upon Wenger's (2009) model of a social theory of learning, we introduce a preliminary framework to understand and guide our analyses of the process of learning to become an ARPA facilitator (see Figure 1).

The ARPA-EPF model is comprised of 4 stages: Initial Education Program for Facilitators, Guided Practice, Autonomous Practice, and Professional Development for Facilitators. Engaging and participating in different learning experiences are the underlining principles that support the ARPA-EPF model. These learning experiences are practically oriented and advance the main practices of ARPA.

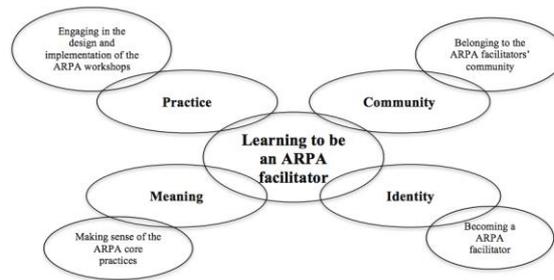


Figure 1: ARPA-EPF Theoretical Framework

The **Initial Education Program for Facilitators** lasts two months. During this stage the participants engage in solving problems and carrying out the key activities of ARPA workshops. The initial program has an important practical component, which involves giving the participants multiple opportunities to experience these activities first-hand being modelled by an expert facilitator. Reflection on the activities and their own role as facilitators is a critical component of this stage. Upon successful completion of the Initial Program, novice facilitators join the second ARPA-EPF stage, **Guided Practice**. Over the course of a school year, novice facilitators implement their own PSClassroom workshops under the guidance of an expert facilitator. They gather once per month to discuss, evaluate, and analyse their previous workshops. They also collectively design their upcoming sessions. Expert facilitators observe the participants' sessions and provide collective and individual feedback to enhance learning. This is an important moment of the program in which novice and expert facilitators engage as a community of learning to foster their own abilities as ARPA facilitators. After the **Guided Practice** ends, the novice facilitators engage in the **Autonomous Practice**. It aims at allowing the participants to engage in designing and implementing the ARPA workshops. Novice and expert facilitators meet twice per year. In the first meeting, they set up annual goals and plan the work ahead. In the second meeting, the novice and the expert facilitators follow up the implementation of the workshops. The expert facilitator also observes the PSClassroom sessions and provides feedback. As the novice facilitators move towards expertise, the fourth and final stage of the program, the **Professional Development** stage allows them to engage in seminars and other academic activities aimed at enhancing their knowledge about the main issues of mathematics problem solving and teacher education.

## METHODOLOGY

During the second semester of 2017, 9 elementary and secondary teachers joined the ARPA-EPF to become facilitators in one of the regions of southern Chilean. They participated in 9 sessions during the Initial Program, which were delivered by Pedro, an expert facilitator. We use a qualitative approach to explore and comprehend the learning process of Luis, one of the participating teachers in the program.

### Luis, the Participant

Luis has 8 years of mathematics teaching experience. After he successfully finished the Initial Program, Luis and 4 other participants were selected to become ARPA

facilitators. We selected Luis for this study because he was asked to conduct a PSClassroom during the entire 2018 school year, so his transition from mathematics teacher to ARPA facilitator could be closely observed. Besides, Luis did not have previous experiences with ARPA, unlike other participants. His lack of familiarity with ARPA would allow us to link his learning process to the learning experiences as proposed in the Initial Program.

### **Data Collection and Analysis**

The 9 sessions of the initial program were filmed and constituted the main body of data. A camera positioned at the back of the classroom was used to capture the entire group activity. When working in small groups, the camera followed Pedro and zoomed in. This allowed us to capture discussions and interactions as they occurred. We use video-analysis (Knoblauch & Schnettler, 2012), an interpretative approach to recorded social interactions in natural settings, to understand Luis' learning process. Within this approach, it is assumed that the meanings of such interactions are to be constructed by way of the actions recorded audio-visually. However, making sense of these social interactions depends on the contextual knowledge that both the participants and the observers bring to the analysis (Knoblauch & Schnettler, 2012).

The 5 authors met once per week to collectively watch the videos. We first focused on the participants' engagement in the different learning experiences proposed in the program. We watched each session and recorded both, the learning experiences and the interactions between Pedro and the participants. Second, we selected episodes in which the main goal of the learning experiences was to foster the participants' abilities to set up and maintain pedagogical and mathematical discussions. In this phase of the analysis, we maintained close contact with Pedro in order to obtain his feedback about the context of the episodes and the interactions (Knoblauch & Schnettler, 2012). Third, within these episodes, we examined how the learning experiences proposed in the Initial Program allowed Luis to re-examine and redefine his own understanding of meaningful discussions. We finally characterized how Luis' participation and engagement in the program activities boosted his own learning process. In this sense, our analysis is located in the interaction between the community and meaning dimensions of our theoretical framework (See Figure 1).

### **RESULTS AND DISCUSSION**

Our analysis reveals the ways in which Luis builds upon his teaching and personal experiences to make sense of "meaningful pedagogical and mathematics discussions". For instance, during the second session of the initial program, the participants were asked to design and implement a PSAC, emphasizing on the plenary session. Pedro first randomly selected Lina, a participant, to conduct the implementation of her PSAC. After Lina finished, Pedro brought the group together to reflect on the experience while he himself modelled a plenary discussion. The learning experience ended with a second process of reflection focused on Pedro's activity leading the discussion. Through participation in the activity, Luis' experiences as a teacher emerged in the questions he

posed and the interpretations he made. These experiences constitute critical aspects of his identity as a teacher and provide a baseline to understand his process of becoming an ARPA facilitator. For instance, based on his teaching experience, Luis casts doubts on the possibilities to set up meaningful discussions with students because of class time and size groups:

Luis: We have to consider the context. In a normal class, we have 40 students. So, a teacher could argue: “I have 40 students; I decide to make 5 or 8 groups”. Let’s say, we have 10 groups, 4 students in each one. So, as a teacher, I realize that although not each group solved the problem, some made great progress. He (pointing to a teacher in the group) is motivated to explain his solution on the whiteboard. And each group wants to explain its own solution. I planned a PSAC to last 45 minutes. It is impossible.

Pedro: What would you do?

Luis: I don’t know. It is impossible. I’m saying this because it is what really occurs in class; that is my experience. Each group wants to get help, the teacher moves around, answers some questions. But I do not know what to do. I would give every group the opportunity until the class time ends. I could also choose a group with the right answer.

In this exchange, Luis positioned himself as a “normal” teacher to express an important challenge in the current efforts to reform mathematics teaching. Rather than assuming the facilitator’s role, he switched his position to point out that within the Chilean educational system, class time is highly controlled to cover the proposed mathematics curriculum. Limitation on time is a critical obstacle to the implementation of reform-based instruction. As a result of this experience, Luis still needs to be convinced that it is possible to set up meaningful discussions in the actual context of the Chilean schools before he can do it with other teachers.

During the second part of the experience, Pedro invited the participants to reflect on his modelling. In this discussion, they highlighted several techniques he employed in the modelling process, such as asking for explanations rather than looking for and giving right answers, engaging everyone, asking for clarifications, and building upon others’ ideas. Although quite general, the participants agreed these are important aspects to take into account to have meaningful discussions. The participation in the collective reflection on the activity allowed the negotiation and renegotiation of meanings (Lave & Wenger, 1991) about meaningful discussions as a relevant practice in ARPA workshops.

The second episode corresponds to a learning experience during the fifth session. After responding to a questionnaire in small groups, one of the participants had to lead a plenary session. Luis was selected to conduct the discussion. The questionnaire focused on the main aspects of implementing a PSAC. Luis opened the plenary and allowed the participants to express their ideas. Throughout the discussion, Luis systematically asked for explanations rather than just evaluating the answers:

Luis: Would somebody like to answer the third question? (Carlos answers). Carlos, could you say a little bit more? (After he finishes, Luis addresses Luisa). Luisa, do you agree with Carlos?

In addition, Luis also recognized and tried to build upon the experiences the participants bring to the discussion:

Luis: Ok. What Luisa says is very important for helping teachers to set up a PSAC. She is telling us her own experience. Has anybody else had that experience?

However, the participants brought to the discussion the difficulties they have had designing and implementing their own PSAC, while Luis again positioned himself as a teacher and lost control over the discussion. At the end, Pedro brought the group together to reflect over the experience.

Pedro: Luis, how did you feel?

Luis: It was simple, because here I have an erudite audience. They know what they are talking about. So, it is easy.

Pedro: What would you tell to Luis to improve?

Luisa: He knew how to control us. However, I think he addressed only a few of us.

While the analysis needs to be expanded to include the relationships between participation in the proposed learning experiences and Luis' learning processes, our preliminary findings point to the critical role of professional collaboration between novice and expert facilitators in building meaning for the ARPA practices. Engagement in the practical-oriented learning experiences seems to contribute to the development of a collaborative relationship from which both novice and expert facilitators benefit. As stated by Lave and Wenger (1991), co-participation and social engagement in community practices are critical aspects of learning. These aspects of the ARPA-EPF learning experiences allow Luis to negotiate new meanings about pedagogical and mathematics discussions and to foster his knowledge and abilities to deliver this important practice. This process of making sense of practices seems to be mediated by Luis' teaching experiences. He entered the ARPA-EPF program with the same needs of the teachers he will eventually work with. They share similar teaching experiences as they belong to the same school culture. Thus, to some extent, becoming an ARPA facilitator for Luis means to unpack, confront and reconstruct these shared experiences through increased participation in the learning experiences. The process of transitioning from being a teacher to being a facilitator seems to be a critical feature of their education that needs to be taken into account in our ongoing analysis.

**Acknowledgement.** The authors gratefully acknowledge funding from FONDEF ID14I20338 and PIA-CONICYT Basal Funds for Centers of Excellence Projects FB003.

## References

- Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *Journal of Mathematical Behavior*, 33, 149-167
- Cerda, G., Pérez, C., Giaconi, V., Perdomo-Díaz, J., Reyes, C., & Felmer, P. (2017). The effect of a professional development program workshop about problem solving on mathematics teachers' ideas about the nature of mathematics achievement in mathematics, and learning in mathematics. *Psychology, Society & Education*, 9(1), 11-26.
- Even, R. (2008). Facing the challenge of educating educators to work with practicing mathematics teachers. In B. Jaworski & T. Woods (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional (Vol. 4)* (pp. 57-74). The Netherlands: Sense Publishers.
- Felmer, P., & Perdomo-Díaz, J. (2016). Novice Chilean secondary mathematics teachers as problem solvers. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving problems. Advances and new perspectives* (pp. 287-308). Springer: Switzerland.
- Jackson, K., Cobb, P., Wilson, J., Webster, M., Dunlap, C., & Appelgate, M. (2015). Investigating the development of mathematics leaders' capacity to support teachers' learning on a large scale. *ZDM Mathematics Education Journal*, 47, 93-104.
- Knoblauch, H., & Schnettler, B. (2012). Videography: analyzing video data as a "focused" ethnographic and hermeneutical exercise. *Qualitative Research*, 12(3), 334-356.
- Lave, J. & Wenger, E. (1991). *Situated Learning. Legitimate peripheral participation*. Cambridge, MA: Cambridge University Press.
- Roesken-Winter, B., Schuler, S., Stahnke, R., & Blomeke, S. (2015). Effective CPD on a large scale: examining the development of multipliers. *ZDM Mathematics Education Journal*, 47, 13-25.
- Wenger, E. (2009). A social theory of learning. In K. Illeris (Ed.), *Contemporary theories of learning* (pp. 209-218), New York, NY: Routledge.

# PEDAGOGICAL PROBLEM SOLVING KNOWLEDGE OF CHILEAN MATHEMATICS TEACHERS AND INSTRUCTIONAL REFLECTION

Farzaneh Saadati, Eugenio Chandía, Natalia Ruiz

Centro de Investigación Avanzada en Educación, Universidad de Chile

*This study focuses on the pedagogical problem solving knowledge (PPSK) of mathematics teachers and their ability to reflect, make decisions, and generate a plan to guide their students. 47 Chilean mathematics teachers participated in this study. They have been asked to reflect on students' solution to a non-routine problem. The results showed 85% of teachers could identify one or more key elements in the procedure while 51% recognized at least one key element in the conceptual understanding. The latent class analysis also revealed that only 38% of teachers planned to help the student based on their explanation about the student's misunderstanding. These results question the adequacy of mathematics teachers' PPSK and the potential instructional planning to help students in problem solving.*

## INTRODUCTION

The results of a recent international exam PISA 2012 showed the majority of Chilean students (51.5%) failed to reach an adequate level of proficiency (level 1 or below) in mathematics and problem solving (OECD, 2012). The Chilean students' deficiency in mathematics problem solving (MPS) has created a concern among educators and policymakers. Several studies have revealed the reasons behind this phenomenon (e.g. Felmer & Perdomo-Díaz, 2016) in which the findings indicated that problem-solving activities are virtually absent in the Chilean classrooms. It's important to mention that while the Chilean curriculum has highlighted problem solving as a core of mathematics, there is a lack of an appropriate interpretation of it to school practices.

The main question that arises here is what factors are causing teachers' resistance against changes and moving toward more active and student-centered practices. The quality of an educational system is, indeed, basically dependent on teachers and the quality of their teaching. A previous study in a Chilean context highlighted that the lack of self-efficacy in doing and teaching problem solving can be an inhibiting factor for teachers to shift from traditional practice to active student-centred practices (Saadati, Cerda, Giaconi, Reyes, & Felmer, 2018).

Besides this factor, teachers' mathematical and pedagogical problem solving knowledge (PPSK) should be considered as another factor that affects the sustainability of changes in practices (Ball, 2000; Cady, Meier, & Lubinski, 2006). In order to have an active mathematics classroom, teachers are being required to explore students' thinking and make use of those thinking styles in their classroom practices (Franke et al., 2009). Therefore, to move beyond those traditional teaching practices, mathematics

teachers' PPSK and their ability to reflect, make decisions, and generate a plan to guide the students are also important. This study is an initial report of an attempt to extend our understanding of Chilean mathematics teachers' PPSK and their instructional planning in teaching problem solving in the classroom. This study purposely aims to answer the following research questions:

1. Is the procedural and conceptual knowledge of Chilean teachers adequate for analysing students' MPS thinking?
2. Is there any relationship between teachers' decisions for an instructional plan and their knowledge of students' MPS thinking?

## **LITERATURE REVIEW**

According to Chapman, teachers' knowledge of and for teaching MPS proficiency must be broader than their general ability in MPS, which requires teachers to know more than only how to solve problems. In general, there are different perspectives to interpret teachers' knowledge (e.g. Ball, Thames, & Phelps, 2008; Shulman, 1986). Chapman (2015) has divided teachers' mathematics knowledge into three components which are:

- Mathematical problem solving content knowledge or knowledge of problems, problem solving, and problem posing,
- Pedagogical problem solving knowledge (PPSK) or knowledge about what students know, think, and do as problem solvers, and having a plan or knowing how to help them,
- Affective factors and beliefs or understanding the nature of affective factors and beliefs on learning and teaching problem solving.

Among these three components, we sought to focus on PPSK as knowledge which teachers ought to hold in regard to support their students' development of MPS proficiency. We believe that PPSK is one of the important ones for mathematics teachers due to its importance for comprehending what a student knows and is disposed to do (e.g. his/her MPS thinking) and understanding how to help a student to become a better problem solver (e.g. decision making and instructional planning for MPS after or during a student act). Therefore, this type of knowledge is about teachers' ability to find a student's errors in solutions, misunderstanding in concepts and procedures during problem solving, and more importantly having a right and sufficient plan to help the student to overcome these misunderstandings. Accordingly, PPSK can be measured based on teachers' adequacy of conceptual knowledge and procedural knowledge to analyse a student's MPS thinking, as well as their ability in decision making and follow up planning.

In a study conducted by Lui and Bonner (2016) on pre-service and in-service elementary teachers' PPSK, which is considered as their ability to analyse a student's work, it showed that these teachers have more competence in procedural than

conceptual knowledge. The study also revealed that teachers' conceptual knowledge of a student's MPS thinking plays a significant role in predicting their instructional planning. According to Darling-Hammond (1994), the information that teachers obtain from their students' knowledge and their way of thinking gives them opportunities to be more efficient. This consideration recently has caused great attention, particularly with regards to the knowledge that teachers have to have to teach mathematics (Chapman, 2015). Planning as a mental process can support teachers' actual intention in instructional implementation and classroom practices.

## METHODOLOGY

### Study Overview and Participants

47 in-service mathematics teachers were considered in this study who participated and completed the survey. There were 25 teachers from elementary schools (grade 6<sup>th</sup> to 8<sup>th</sup>) and 22 secondary teachers. These teachers were volunteer participants from various elementary and secondary schools, public, charter, and private schools with the only requirement that they had taught mathematics before. Among the participants, 24 were males and 23 women, while the majority (58%) of them had more than 5 years of experience in teaching mathematics at schools.

### Instruments

The instrument is a survey called "thinking about math problems" designed by Lui and Bonner (2016). The survey involves a scenario and 4 open ended questions designed to examine teachers' PPSK. We designed two different scenarios; one for elementary and another for secondary teachers. Each scenario was about a non-routine mathematical problem and a sample of the solution presented by a student. The intention of choosing specific samples was to show a solution done by an ordinary student which includes some common and general procedural and conceptual misunderstanding that typically occurs in students' works for both elementary and secondary grades. The scenario was, indeed, a reflection to challenge mathematics teachers' knowledge and experience in teaching MPS.

Each scenario is followed by four open ended questions based on the given scenario and the solution. First, the participants were asked to interpret the student's procedural and conceptual understandings or misunderstanding that she had while solving the problem. Second, they were required to reflect on the student's solution as a teacher and describe their possible instructional plans to help the student based on her interpretations in the first part. Table 1 showed the questions, descriptions, and a sample of responses made by the teachers for the following scenario:

*Students were asked to create a word problem to represent this expression, then to solve the problem:  $1\frac{3}{4} : \frac{1}{2}$ . One student wrote:*

*Word Problem: Tommy has  $1\frac{3}{4}$  apple pies, and wants to share them with his friend Ben. How can he make sure that both he and Ben get an equal share?*

*Answer: Both Timmy and Ben get  $\frac{2}{7}$  apple pies each.*

N°	Question	Variables	Example from an elementary teacher
1	Is the student's solution correct?	General Understanding	No, if he wants to share the apple pie with his friend, then he must divide the total amount into 2, which is not the same as the expression proposed. A situation for the expression could be: If you have 1 and 3/4 portions of the apple pie, how many pieces of 1/2 reach the previous amount?
2	Describe the mathematical concepts (correct or incorrect) that you think the student used to reach it.	Conceptual Understanding	The student started well by turning the mixed number to the improper fraction, and it was good to want to do a multiplication, the error occurred when placing the result of this multiplication since he invested the values. Another mistake was in the interpretation of 1/2.
3	Describe the mathematical steps or procedures that you think the student performed to get it.	Procedural Understanding	First: converted the mixed number to an improper fraction. Second: crossed multiplied 7/4: 1/2 Third: misplaced the values, multiplied denominator by opposite numerator but the result placed it in the numerator and it should have been in the denominator (4/14).
4	As a teacher, how would you support this student to clarify and/ or expand his skills and understanding?	Instructional Planning	- It may be a common mistake to invert the values at the time of cross multiplication to solve a division of fractions, therefore it would be good to express the formula: $a / b : c / d = a / b \times d / c = ad / bc$ . In this way, and observing the algebraic expression, it can be determined that the division is transformed into multiplication by inverting the second fraction and then its resolution is horizontal. - Explain that 1/2 corresponds to half of an integer.

Table 1: PPSK Variables and a Sample of Responses

### Data Analysis

To analyse the data, we used two different methods. Descriptive statistics were used to answer the first research question. Then, latent class analysis (LCA) was used to answer the second question. LCA allowed us to investigate the relationships among observed variables in order to find the possible latent categorical variables. In fact, we fit a latent class model to identify clusters of teachers based on their responses to the question number 4 that measured their instructional planning. The decision that they made in question 4 (as their instructional plan) was compared with their previous responses to questions 1, 2, and 3 to generate three conditions correspondingly. In each condition, we have two codes: *related* (1) or *unrelated* (0). Here, the term of *relation*

in each condition means the teacher has considered his/her previous answer in the corresponding question to make the decision on the last question. For example, the value of 0 in condition 2 for a teacher revealed that his/her instructional planning is unrelated to her/his response in question number 2.

## RESULTS

**General Understanding.** The first question was about if teachers believe the student showed a mistake in the solution or not. Among the participants, just 2 teachers did not find the mistake in the solution and they responded that the solution was correct. 4 teachers did not respond to this question and the rest mentioned that the solution was not correct. The sample in Table 1 is one of the complete and most relevant answers made by an elementary school teacher.

Responses		Conceptual		Procedural	
		N	%	N	%
Key Elements Discussed	Not relevance	23	49	7	15
	Partially relevance	4	8.5	5	11
	Relevance to one element	12	25.5	14	30
	Relevance to two elements	8	17	12	25
	Relevance to more than two elements	0	0	9	19

Table 2: Teachers' conceptual and procedural knowledge in PPSK

**Conceptual Understanding.** When teachers were asked to describe student' conceptual understandings (correct or incorrect) to get that solution, 23 (49%) teachers explained it in a way that was not relevant at all. 24 (51%) teachers explained partially relevant or relevant. There were 20 out of 47 who pointed to at least one key element in the solution (refer to Table 2). Among them, some responses were partially correct or included some mistakes. For example, the teacher in our example explained 2 relevant conceptual elements without mistakes. However, he could mention some more relevant elements like a student's sense of fractions (to justify that  $2/7$  is too small to function as a fraction and as an answer).

**Procedural Understanding.** Chilean teachers have better results in the procedural understanding (Table 2). A group of 7 or about 15% of teachers could not clarify any key element in students work. 40 (85%) teachers mentioned at least one key element clearly relevant to the procedural process. However, some of them made a mistake in their explanation. In our specific example (Table 1), the teacher showed a clear expression by highlighting 3 relevant factors.

## Instructional Planning

The responses of all teachers to the reflective question were analysed based on three conditions. A 2-class model shows a satisfactory goodness of fit (AIC=166.99, BIC=179.94,  $G^2 = 0.58$ ,  $\chi^2 = 0.32$ , Entropy=1.62), which means that it fits with two latent groups in the participants based on their decision plan. The outcomes (Table 3) show the probability of belonging to each condition for each teacher (considering their responses while making a decision plan or not). In fact, every teacher belongs to one and only one class. An interpretation of the results identifies the following patterns:

- 38% of the teachers belong to class 1 as teachers who consider the students' mistakes (that they mentioned for questions 1, 2, and 3) while making a decision to help him/her.
- 62% of the teachers belong to class 2 as teachers who do not consider the students' mistakes (that they mentioned for questions 1, 2, and 3) while making a decision to help him/her.

Latent Classes	Class 1: Considering	Class 2: Not Considering
Probability of belonging	0.38	0.62
Condition 1	0.40	0.13
Condition 2	<b>1</b>	0.19
Condition 3	<b>0.95</b>	0.09

Table 2: Two-Latent-Class Model of Teachers' Instructional Plan in PPSK. Item-response probabilities  $>.5$  in bold to facilitate interpretation.

The table shows that teachers of latent class 2 (62% or 29 teachers) were very unlikely to choose a plan based on their knowledge about student mistakes, but their most prevalent plan is based on conceptual knowledge (19%), and general understanding (13%). Members of latent class 1 (in total 38% or 18 teachers) were most likely to have a plan based on student conceptual understanding (100%). They were also significantly more likely to choose their plan based on procedural understanding as well (95%). In the sample (Table 1), the teacher belongs to class 1. His instructional plan (or response in question 4) shows he clearly used his procedural and conceptual knowledge about the student's MPS thinking. In question 3, he explains the students' mistake as "*Third: misplaced the values multiplied denominator by opposite numerator*", then he uses this consideration for making a plan as "*it would be good to express the formula:  $a / b : c / d = a / b \times d / c = ad / bc$ .*"

## DISCUSSION AND CONCLUSION

This study focused on the PPSK of Chilean teachers as the knowledge that teachers must have for teaching mathematics problem solving efficiently and competently. The

examination of participants' answers displays that some mathematics teachers have difficulties in understanding and analysing students' work. Analysing students' solutions and explanations are considered important knowledge that mathematics teachers need to have (Hill, Rowan, & Ball, 2005). The results of descriptive statistics showed that teachers' procedural knowledge was better than their conceptual knowledge based on their ability to analyse student's work. The findings for Chilean teachers are similar and consistent with research conducted by Lui and Bonner (2016), which declared many teachers in the United States have stronger mathematical procedural knowledge than conceptual knowledge.

The results of the LCA revealed that most of the teachers (62%) reflect on students' work immediately without analysing and considering their mistakes. This group of teachers, in fact, showed less tendency or caution for considering the students' knowledge of mathematical concepts or procedures while planning to help them. The decisions could be influenced by their belief system related to teaching or learning mathematics (Blömeke, Gustafsson, & Shavelson, 2015). Moreover, Chilean teachers normally teach mathematics in a traditional way (Saadati et al, 2018), so it might be because there is no space for planning even if they have adequate knowledge about their students' understanding.

Teachers with differentiated and integrated knowledge will have a greater ability to plan and enact lessons that help students develop deep and consistent understandings than others whose knowledge is limited and fragmented. A Teachers' lack of knowledge of analysis for students' work can cause the lack of effective instructional planning (Lui & Bonner, 2016). This group of participants is more open to considering this idea, particularly in light of the answer to the first research question which highlighted teachers' insufficient procedural and conceptual knowledge. This is also an obstacle for being an effective teacher who supposed to have a plan to help students. Therefore, there is a gap in the adequacy of mathematics teachers' PPSK and their potential instructional reflection for helping students sufficiently in problem solving.

In this challenging and demanding situation for teachers, where they are responsible for the students' outcome, this is surely the wisest way to grow and maintain teachers' pedagogical knowledge. There are several professional development (PD) programs, especially in the Chilean educational system, to help teachers to improve their quality of teaching. This body of work can serve as the foundation and preliminary effort for the policymakers and local PD organizers while designing a content-based program for mathematics teachers. Future studies need to extend results on teacher expertise and transform it into effective localized models for PDs. Finally, the results presented here are based on only 2 scenarios, that might be unclear for concluding teachers' PPSK, therefore we suggest to repeat this study with more complex scenarios.

## References

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of teacher education*, 51(3), 241-247.

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher education*, 59(5), 389-407.
- Blömeke, S., Gustafsson, J.-E., & Shavelson, R. J. (2015). Beyond dichotomies: Competence viewed as a continuum. *Zeitschrift für Psychologie*, 223(1), 3–13. doi:10.1027/2151-2604/a000194.
- Cady, J., Meier, S. L., & Lubinski, C. A. (2006). The mathematical tale of two teachers: A longitudinal study relating mathematics instructional practices to level of intellectual development. *Mathematics Education Research Journal*, 18(1), 3-26.
- Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT (2013-2015 Issues)*, 3(1), 19-36.
- Felmer, P., & Perdomo-Díaz, J. (2016). Novice Chilean secondary mathematics teachers as problem solvers. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds), *Posing and solving mathematical problems* (pp. 287–308). Cham: Springer International Publishing.
- Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education*, 60(4), 380-392.
- Darling-Hammond, L. (1994). Performance-based assessment and educational equity. *Harvard Educational Review*, 64(1), 5-31.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, 42(2), 371-406.
- Lui, A. M., & Bonner, S. M. (2016). Preservice and inservice teachers' knowledge, beliefs, and instructional planning in primary school mathematics. *Teaching and Teacher Education*, 56, 1-13.
- OECD: Organisation for Economic Co-operation and Development (2012). Development co-operation report 2012. Paris, France: OECD Publishing. <https://doi.org/10.1787/dcr-2012-en>.
- Saadati, F., Cerda, G., Giaconi, V., Reyes, C., & Felmer, P. (2018). Modeling Chilean Mathematics Teachers' Instructional Beliefs on Problem Solving Practices. *International Journal of Science and Mathematics Education*, 1-21.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher*, 15(2), 4-14.

# TO ADD OR TO MULTIPLY? AN IN-DEPTH STUDY ON PRIMARY SCHOOL CHILDREN'S PREFERENCE FOR ADDITIVE OR MULTIPLICATIVE RELATIONS

Tine Degrande, Lieven Verschaffel, and Wim Van Dooren

KU Leuven, Centre for Instructional Psychology and Technology

*Previous research shows that children erroneously solve multiplicative word problems additively, and others solve additive word problems multiplicatively. Recently, a preference for additive or multiplicative relations has been raised as an explanation for these errors, besides lacking abilities. The present study aimed to characterise this preference by means of semi-structured interviews in which open problems were solved by both fifth and sixth grade children who did and did not prefer additive or multiplicative relations in a pre-test. Results characterised children's preference – and especially multiplicative preference – as something strong that is resistant to change. Moreover, children experience difficulties in justifying their preferred answer, which suggests that a children's preferred answer arises rather in a non-deliberate way.*

## INTRODUCTION

Learning to reason multiplicatively is an important goal in primary math education. One major way in which it is practiced is through missing-value word problems, which involve three given values and a fourth one that needs to be found (Kaput & West, 1994). For example, the missing value in the problem “A car of the future will be able to travel 8 miles in 2 minutes. How far will it travel in 6 minutes?” can be obtained by noticing that the time is tripled, and apply this to the distance ( $2 \times 3 = 6$ , so  $8 \times 3 = 24$  miles). Despite the omnipresence of multiplicative missing-value problems in primary education, children experience many difficulties. Children, especially in lower grades, often solve multiplicative problems additively (e.g.,  $2 + 4 = 6$ , so  $8 + 4 = 12$  miles in the word problem above) (e.g., Kaput & West, 1994; Vergnaud, 1988). Numerous other studies reported the inverse error as well: Children in upper primary education massively respond multiplicatively (e.g.,  $4 \times 3 = 12$ , so  $8 \times 3 = 24$  laps) to additive missing-value problems such as “Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?” (e.g., Van Dooren et al., 2010).

While children's errors in multiplicative or additive problems were frequently attributed to either their inability to reason multiplicatively or additively, the completeness of this explanation has been recently questioned (Degrande, Verschaffel, & Van Dooren, 2017a, 2017b). Kindergartners have been shown to correctly model a problem situation as multiplicative and to correctly execute multiplicative operations (e.g., Nunes & Bryant, 2010). Likewise, older children still solved additive problems multiplicatively, despite having acquired the ability to execute additive operations and

the ability to distinguish additive from multiplicative word problems (e.g., Degrande et al., 2017b). It thus seems that children's errors do not merely depend on their *ability* to reason additively and multiplicatively. An additional explanatory element has been raised by Resnick and Singer (1993), who interpreted children's additive answers to multiplicative word problems as indicating a *preference* for additive relations. Likewise, children's multiplicative answers to additive word problems may be due to a *preference* for multiplicative relations. The distinction between preference and ability is not new in mathematics education literature, where preference refers to the way of reasoning that "has precedence over" the other (Pellegrino & Glaser, 1982, p. 310).

Previous studies documented the existence and development of a preference for additive or multiplicative relations in primary school children (Degrande et al., 2017a, 2017b). In this research line, problems open to both additive and multiplicative reasoning were used (i.e., both were equally correct and valuable). These problems were best suited to measure children's preference, as children's ability to reason additively or multiplicatively in line with the underlying mathematical model was ruled out. Mainly younger children preferred additive relations and older children preferred multiplicative relations, but inter-individual differences in the relations that children preferred were still found *within* each grade (Degrande et al., 2017a, 2017b). Moreover, it has been shown that preference cannot be fully explained by children's calculation skill (Degrande et al., 2017a), and that preference accounts for children's errors in word problems beyond children's abilities (Degrande et al., 2017b).

### **AIM OF THE PRESENT STUDY**

Despite this evidence base, information on the thinking processes behind children's answers in open problems is lacking. So far, collective tests were administered to large groups of children. In the present study, we conducted an in-depth investigation by means of individual semi-structured interviews in which children who did and did not prefer additive or multiplicative relations were stimulated to consider an alternative answer in open problems. Contrasting those groups' responses (i.e., their answer, how convinced they were of their answer, and how they articulated their reasoning) throughout the interview allowed us to characterise the nature of this preference.

### **METHOD**

**Participants:** The participants of the pre-test were 145 fifth or sixth graders from three Flemish primary schools. The 27 with the most pronounced profiles were selected to participate in individual interviews. Their ages ranged from 10- to 12-years old.

**Materials:** The pre-test, measuring children's preference for additive or multiplicative relations, consisted of open problems in which "several relations" (Pellegrino & Glaser, 1982, p. 302), including additive and multiplicative ones, were equally valuable

and correct. Such open problems (for an example, see Figure 1) have been shown to validly measure children’s preference (Degrande et al., 2017a, 2017b). The pre-test had 45 items, 32 of which had integer ratios and 13 had non-integer ratios.

The interview consisted of four similar open problems, two containing integer (see Figure 1) and two non-integer ratios. Due to space restrictions, this paper exclusively focuses on integer problems. In each problem, three numbers were given, a fourth one was missing, and arrows pointed out the relational structure.

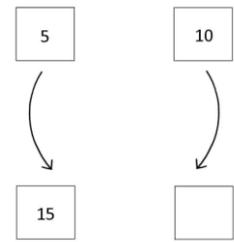


Figure 1:  
Integer open problem.

**Procedure:** The pre-test was collectively administered. Based on the results, we identified children with the most distinct profiles in terms of preferences, and invited them to participate in individual semi-structured interviews. The first two groups will later be denoted as the “preference groups”, and the latter two as the “non-preference groups”.

- Additive preference (A-group,  $n=11$ ): mainly gave additive answers
- Multiplicative preference (M-group,  $n=8$ ): mainly gave multiplicative answers
- Additive-and-multiplicative (A&M-group,  $n=4$ ): gave both the additive and multiplicative answers in integer problems, and sometimes in non-integer ones too
- Mix-group (Mix-group,  $n=7$ ): switched between additive and multiplicative answers, both in integer and in non-integer items

The semi-structured interviews took place about three weeks after the pre-test. All interviews were audiotaped. Four open problems (see “Materials”) were subsequently given. The interview around each problem consisted of four phases in which children were increasingly stimulated to consider the second and thus alternative answer too. The interview stopped in the phase where both the additive and the multiplicative answer were accepted. In *Phase 1*, each child was asked to answer the open problem, to explain how his answer was found, and to indicate and explain how convinced (s)he was of that answer by choosing a position on a four-point scale (from “not at all convinced” to “very convinced”). If the child only gave one answer, in *Phase 2* (s)he was asked whether any other answer could be possibly correct in the problem at hand. If (s)he thought this was not the case, (s)he proceeded to *Phase 3*, where (s)he was shown the alternative answer from a fictitious pupil and was asked about his/her thoughts about that answer. Whenever a child still could not explain the fictitious child’s answer by the end of Phase 3, the alternative answer was explained by the interviewer in *Phase 4*, and the child could again react to it. Whenever the alternative answer entered the picture – whether in Phase 1, 2, or 3 – the child was asked to indicate and explain on another four-point scale how convinced (s)he was of that alternative answer, and this conviction score was explicitly compared with the one of the child’s original answers. Children were, moreover, told that they could change both conviction scores at any time during the interview.

## RESULTS

In this section, we report children’s responses in each phase of the two interview trials consecutively. In line with the goals of this study, this report will focus on similarities and differences between (1) children who clearly preferred additive or multiplicative relations in the pre-test (preference groups) versus the other two groups (non-preference groups), as well as (2) children who preferred additive relations (A-group) versus children who preferred multiplicative relations (M-group).

### First trial

When initially confronted with the open problem in **Phase 1**, most children who preferred additive relations solved the problems solely additively (6 out of 8), and likewise, most children who preferred multiplicative relations solved the problems solely multiplicatively (7 out of 8), as expected based on the pre-test. This implies that only a few children who preferred additive or multiplicative relations immediately gave both the additive and multiplicative answer (2 out of 8 and 1 out of 8). This was in contrast with the A&M-group where 3 out of 4 children gave both answers, and with the Mix-group where children’s answers were spread over the answer categories (i.e., additive, multiplicative and additive-and-multiplicative).

	Trial 1				Trial 2			
	Phase 1	2	3	4	1	2	3	4
A ( $n=8$ )	2	2	1	3	4	3	1	0
M ( $n=8$ )	1	1	2	4	3	5	0	0
A&M ( $n=4$ )	3	1	0	0	4	0	0	0
Mix ( $n=7$ )	2	3	0	2	5	2	0	0

Note: Three children of the A-group who only gave the multiplicative answer in Phase 1 of the interview were not further included in this paper.

Table 1: Number of children who gave both answers per phase per trial.

Children in the two preference groups were strongly convinced of their answers. 3 out of 8 children of the A-group and 4 out of 8 children of the M-group indicated the highest score on the conviction scale, which was more frequent than in the non-preference groups (2 out of 4 and 2 out of 7 in the A&M- and Mix-groups).

Despite their high conviction scores, the preference groups had great difficulties explaining *why* they gave that answer, except for some rather superficial explanations such as implicit task-

related expectations, “*It is plus 6 here, and here you maybe have to do plus 6 too*” (A3) or “*I look at what happened from 5 to 15, that was times 3. Then I also did 10 times 3*” (M119), or the difficulty level of the task, “*It is actually pretty easy*” (M17). In addition, children who preferred multiplicative relations frequently referred to the correctness of their answers, “*Because I am sure that it is correct*” (M115). Children belonging to one of both preference groups did not seem to have clear arguments justifying their answers, nor did they realise that there were competing answers. They gave their answers in a rather impulsive and non-deliberate way, which was in contrast

with the children of the other groups, who often made conscious considerations like the two equally possible solutions *“other operations may be possible as well”* (MIX5).

In **Phase 2**, children who preferred additive or multiplicative relations rarely gave another answer when explicitly asked to do so. Most children responded *“No”* or *“I don’t think so”* when asked whether there was any other answer. The few children who did come up with the alternative answer had a hard time in justifying this answer. Just like in Phase 1, they referred to the difficulty level of the task *“because again, it was an easy question”* (M17) or the correctness of their answer *“because I am almost sure that it is correct”* (A11). This was in contrast with the non-preference groups, where children had less difficulty in giving both answers and indicating that they were equally valuable, *“I think both are possible”* (MIX80) or *“Because there are actually two possibilities”* (MIX123). In addition to that, some of them expressed why they initially only gave one of both answers; one answer was more logical *“First, this one logically made sense to me, and only later this one...”* (A&M29) or easier *“because I think that + is much easier than  $\times$ ”* (MIX85) than the other. As a result, most children belonging to the non-preference groups had given both answers by the end of Phase 2, while this was only the case for half of the additive and one fourth of the M-group.

Only very few children who were confronted with the alternative answer of a fictitious pupil in **Phase 3** accepted it. This held for all groups, including the A-group (1 out of 4) and M-group (2 out of 6). However, the underlying reason for this behavior clearly differed between both preference groups. 2 of the 3 children in the A-group did not know how the fictitious pupil obtained this solution, *“I think this [answer] is strange, because I don’t know what he [the fictitious person] does here”* (A3), whereas all 4 remaining children of the M-group explicitly discarded the (additive) solution as being erroneous, *“A little bit wrong [...]. Because you need to do the same with this [arrow]. And she [fictitious pupil] has not done that. She did times 2 and here you should have done times 4”* (M115). This also resulted in lower scores on the conviction scale of this alternative answer, compared to the preferred one. A similar discrepancy in conviction scores was found in children of the M-group who did accept the alternative answer. They were still more convinced of the multiplicative than the additive answer, which clearly corresponded with their preference, *“Yes, this [the additive answer] is good but I tend to work with  $\times$  more quickly than with +. [...] Because I rather think that  $\times$  is correct than +”* (M130). However, we did not find a discrepancy in the conviction scores in the A-group who accepted the alternative answer.

At the end of Phase 3, about half of all children of the A-group and M-group and only a minority of the children in the non-preference groups had not given both answers yet (see Table 1). Those children proceeded to **Phase 4**, where the alternative answer was explained by the interviewer. This led to an increased conviction in almost all children in the A-group (2 out of 3) and M-group (4 out of 4), as well as in all children belonging to the Mix-group (2 out of 2). For many children belonging to one of both preference groups – and especially the M-group – the explanation of the fictitious answer functioned as an *“Aha-Erlebnis”* (Wertheimer, 1945), *“Aaaaah! Now I know it”*

(M115). They named this alternative solution smart “*Also smart that she has seen well that you multiply by 3 here*” (A93) and admitted that they had not thought about it yet, “*I was not thinking of plus*” (M114) or “*because only now, I see that this goes like this [plus] as well*” (M115). Despite this increase in conviction, still half of the remaining children in the M-group in Phase 4 were more convinced of their own answer, “*Yes, + is possible too, but I think it is  $\times$* ” (M83). This was not the case in the A-group, which often referred to the equality of the two answers “*I just think both are correct answers, fine*” (A74).

## Second trial

All children in the first trial at some point were brought to understand and get somewhat convinced of the alternative answer as well. We were then interested in the reaction of the children when they were involved in a second, very similar trial. We will report on the second trial, but not as systematically as the first one due to length restrictions. We focus on the new information that was obtained, compared to trial 1.

In **Phase 1** of the second trial, 5 out of 8 children of the M-group answered solely multiplicatively, while only 2 out of 8 children of the A-group solely gave the additive answer. In addition, none of the children of the M-group gave the additive answer, while 2 out of 8 children belonging to the A-group gave the multiplicative answer. This resulted in a larger number of children who gave both the additive and multiplicative answer in Phase 1 in the A-group than in the M-group (see Table 1). These relative differences suggest that the M-group stuck more strongly to their preferred answer than the A-group. One-sided additive or multiplicative answers were also scarce in the Mix-group (2 out of 7) and were totally absent in the A&M-group, since all other children of non-preference groups gave both the additive and multiplicative answer.

Moreover, the children who one-sidedly answered additively or multiplicatively in the preference groups were much more convinced of their answers (2 out of 2 and 3 out of 5 gave the highest conviction score) than the children who did so in the Mix-group (only 1 out of 2). However, children belonging to the preference groups had difficulties in explaining *why*, indicating that they just “saw” what was expected based on the given numbers, such as “*Here they have done 2 times 4, and then here I have done 6 times 4*” (M119) or “*Because 6 plus 6 is 12. Here you do plus 6, so here you need to do plus 6 too*” (A93). Children who preferred multiplicative relations again stressed the correctness of their own answer, but now implicitly considering the other answer too: “*Mine is correct, but I am still doubting*” (M83) or “*Because my answer is correct too, that’s why I am very convinced*” (M115).

Children belonging to one of the two preference groups who gave both answers in Phase 1 reported that they were equally sure of both answers, “*This solution is correct too, it is just another way of thinking, but it is simply all correct because it is not written here whether you need to multiply or add*” (A78) or “*And actually there is still another way...*” (M17). However, especially amongst the M-group, many children still seemed to verbally privilege their preferred answer, “*There are different ways to solve the*

*exercise, so you don't know for sure whether this is the correct one, but you do already know that this [multiplicative] is a good solution for this exercise" (M26) or "Simply this one times 3 and the other one times 3. Because I do think that this one is good, but I don't know for sure whether it is completely correct" (M114).*

Compared to the non-preference groups, more A- and M-group children proceeded to **Phase 2** (see Table 1). Here, almost all children came up with the alternative answer, including those belonging to the preference groups. While most children of the A-group (2 out of 3) and M-group (3 out of 5) did not report a discrepancy in the conviction scores of both answers, still many of their verbalisations somehow expressed that they attached different values to both answers. This occurred in the A-group, "*Uhm, because the first exercise [solution], I find it somewhat better. And I find the second one a little weirder. There you need to think more about it. Well, at least in my case...*" (A3) and in the M-group, "*This is the first time that I used plus. I do think that it is correct, but I am not very convinced*" (M95), or "*Because I am a little less convinced, I am not quickly inclined to work with plus*" (M130). Those children literally expressed their preference but seemed unable to explain *why* this was the case. Only one child of the A-group proceeded to **Phase 3**, but not to Phase 4 (see Table 1).

## CONCLUSION AND DISCUSSION

Previous research has repeatedly shown that children erroneously solve multiplicative word problems additively, while others solve additive word problems multiplicatively. Recently, a preference for additive or multiplicative relations has been raised as an explanation for these errors, besides lacking abilities. The present study aimed to characterise this preference by means of semi-structured interviews in which open problems were solved by both children who did and did not prefer additive or multiplicative relations in a pre-test. By comparing and contrasting the responses (i.e., actual answer, the degree of conviction of that answer and the self-reported reasoning underlying it) of children belonging to different pre-test profiles, we characterised the typical behavior of children who preferred additive or multiplicative relations.

Children who preferred additive or multiplicative relations initially only gave an additive or multiplicative answer, and were very convinced of it, but did not seem to have a clear justification. This suggests that they gave their preferred answers in a rather impulsive and non-deliberate way, without realising at all that there was an alternative answer. Most of them did not come up with another answer when explicitly asked, and did not immediately accept the alternative answer when shown. Even in a second trial, many children fell back to their preferred answer at first. When they finally accepted the alternative answer, their verbalisations still suggested a preference.

Children's preference for multiplicative relations seemed to be stronger than that for additive relations. Compared to the additive preference group, the multiplicative preference group more often called their preferred answer "correct" and the alternative additive answer "incorrect". Once acknowledging the alternative answer, they stayed more convinced of their own answer. Even in the second trial, more of them only gave

the multiplicative answer, and they kept on privileging this answer more strongly.

The results of this in-depth investigation confirm the existence of a preference for additive or multiplicative relations in some children and characterise it as something that is strong and resistant to change. Moreover, these children seemed to give their preferred answer in a rather non-deliberate way. They did not realise at first that there were competing answers, and they experienced difficulties in justifying their preferred answer. Hence, instruction might aim at (1) making children aware of their preferred ways of reasoning, (2) remedying them and (3) preventing the development of such preferences in the future. In this light, it seems useful to use educational approaches wherein children are stimulated to explicitly discuss the considerations they make when deciding on the appropriateness of a solution in both classical and open problems, as well as to avoid a stereotyped offer of problems in mathematics curricula (e.g., only multiplicative missing-value problems in upper primary education, never being interchanged with additive ones) that may shape children's preference.

## References

- Degrande, T., Verschaffel, L., & Van Dooren, W. (2017a). Beyond additive and multiplicative reasoning abilities: How preference enters the picture. *European Journal of Psychology of Education*. Advance online publication. doi:10.1007/s10212-017-0352-y
- Degrande, T., Verschaffel, L., & Van Dooren, W. (2017b). Erroneous additive or multiplicative reasoning: the role of preference besides ability. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of PME41*, Vol. 2, pp. 273-280. Singapore: PME.
- Kaput, J. J., & West, M. M. (1994). Missing- value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). New York, NY: New York Press.
- Nunes, T., & Bryant, P. (2010). Paper 4: Understanding relations and their graphical representation, in: T. Nunes, P. Bryant, & A. Watson (Eds.), *Key understanding in mathematics learning*.
- Pellegrino, J. W., & Glaser, R. (1982). Analyzing aptitudes for learning: Inductive reasoning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 2, pp. 269-345). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107- 130). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication... and back. The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction*, 28, 360-381. doi:10.1080/07370008.2010.488306f
- Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.
- Wertheimer, M. (1945). *Productive thinking*. New York, NY: Harper and Brothers.



# ORAL COMMUNICATIONS



# SITUATED LEARNING IN AN INTERDISCIPLINARY PROSPECTIVE TEACHER COURSE

Jenny Patricia Acevedo-Rincón

University of Campinas (Brazil)

The prospective teacher Course at the Education Institute at the University of Campinas (FE / Unicamp) is interdisciplinary, bringing together prospective teachers from different institutes in the same scenario. This study aims to understand the teaching professional practices of prospective mathematics teachers from their participation in interdisciplinary experiences situated in the FE/Unicamp. This is qualitative research, with a Prospective Teacher Course as research context, which contained eighteen students from eight different undergraduate Institutes. However, four Prospective Teacher of Mathematics Education programs were selected to analyze narratively their trajectories of professional teaching. The research has a theoretical reference to Situated Learning (Lave & Wenger, 1991; Wenger, 1998). The empirical materials that compose the corpus of analysis and interpretation were obtained from the researcher's field diary, records in trainee diaries, intervention plans, questionnaires, final report on the problematization of the practices, individual interviews, and the interdisciplinary Communities of Practice (CoP). In order to analyze this information, two enterprises of CoP were found, which were characterized as actions. Therefore, for the enterprise “Construction of interdisciplinary intervention project in the prospective teacher Course”, we analyzed the actions: (1st) identification of the school context, (2nd) observation and registration of teaching mathematics in school; and, (3rd) identification of interdisciplinary experiences in the school. For the “construction of the final of the prospective teacher course report”, the following actions were identified: (i) being a teacher in the school and (ii) reflections about the teaching profession. The participation in the different interdisciplinary scenarios of teaching and learning in the school helped the trainee to problematize the professional practices of teachers, with a re-meaning of the contexts of school practices, knowledge and specific processes of school mathematics, interdisciplinary situations, teaching methodologies, and interdisciplinary intervention actions in school.

## References

- Acevedo Rincón, J. P. (2018). *Professional teacher learning of the mathematics prospective teacher situated in an interdisciplinary prospective teacher course*. 282 p. Thesis (PhD in Education) – State University of Campinas (Unicamp), Campinas.
- Lave, J.; Wenger, E. (1991). *Situated Learning: legitimate peripheral participation*. Cambridge, Cambridge University Press, 135-197.
- Wenger, E. (1998). *Communities of Practice: Learning, Meaning and identity*. New York, Cambridge University Press.

## BRIDGING MATHEMATICAL MODELING AND ARGUMENTATION IN THE CLASSROOM

María Aravena<sup>a</sup>, Horacio Solar<sup>b</sup>, Manuel Goizueta<sup>c</sup>,  
Andrés Ortiz<sup>d</sup>, Rodrigo Ulloa<sup>d</sup>, Leonardo Cárdenas<sup>a</sup>

<sup>a</sup>Universidad Católica del Maule,

<sup>b</sup>Pontificia Universidad Católica de Chile,

<sup>c</sup>Pontificia Universidad Católica de Valparaíso,

<sup>d</sup>Universidad Católica de la Santísima Concepción

Mathematical modeling and argumentation are two competencies that are widely present in curriculums of many countries. Different authors conceive of modeling as a cornerstone of mathematical activity, mainly because modeling tasks emphasize students' processes over produced models (Blomhøj, 2004). Argumentation, on the other hand, is considered in many curricular proposals to be a key feature of dialogic, inquiry-oriented classrooms where students engage in and take responsibility for the collaborative construction of mathematical knowledge (Krummheuer, 1995).

Although both competencies have been investigated from a variety of theoretical lenses and methodological perspectives, it seems to us that the arguably fertile ground where both competencies meet remains under researched. Therefore, our project aims at studying students' mathematical learning when mathematical activity in the classroom is organized around the articulation of modeling and argumentation.

The first stage of the project consists of the design, application and assessment of a 42 hours course focused on modeling, argumentation and their articulation in the classroom. The course will be delivered to a group of 25 primary teachers (grades 1-8) in Santiago and Concepción (Chile) during the second semester of 2018. During 2019, we will select and follow 9 teachers in their classrooms to study how students learn mathematics and develop argumentation and modeling competencies in a classroom where mathematical activity is organized around the promotion and articulation of modeling and argumentation.

During the conference we will present the principles behind the design of the course and the related criteria for the articulation of argumentation and modeling in the mathematics classroom. We aim at characterizing the type of learning that takes place when the development of both competencies is fostered.

Blomhøj, M. (2004). Mathematical modelling—a theory for practice. En B. Clarke et al. (Eds.), *International perspectives on learning and teaching mathematics* (pp. 145-160). Göteborg University: National center for mathematics education.

Krummheuer, G. (1995). The ethnography of argumentation. En P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum.

# THE HYPOTHETICAL LEARNING TRAJECTORY AS A TEACHING TOOL TO PROMOTE MATHEMATICAL THINKING OF STUDENTS

Andrea Cárcamo

Facultad de Ciencias de la Ingeniería, Universidad Austral de Chile, Chile.

The results on hypothetical learning trajectory (HLT) show that it is a tool that can help the teacher to have an overview of the class before starting it (Daro Mosher & Cocoran, 2011). In particular, Cárcamo and Fortuny (2017) indicate that the intention of the design of a HLT, at the university level, is to support the teacher to create models of thinking in order to try to interpret the mathematical thinking of the students.

In this research, I present an exploratory study of a HLT that aims to support the construction of the possible types of solution sets that can be obtained by solving systems of linear equations using matrices. The HLT was designed in terms of the instructional design heuristic of emergent models (Gravemeijer, 1999) and the mechanism of reflection on the activity-effect relationship (Simon, Tzur, Heinz & Kinzel, 2004). The methodology of this study is a design-based research. During the teaching experiment phase, the HLT was used to guide 70 students in the construction of new knowledge. Audio recordings and written protocols of the tasks developed by the students were analysed. The results give indications that this HLT helps the students to identify the relationship that exists between the type of solution set that has a system of linear equations, and the ranges of the augmented matrix and the coefficient matrix. Likewise, the results provide evidence that some students confuse a system that has an infinite solution set with a system that has an empty solution set.

**Acknowledgments:** I thank the Facultad de Ciencias de la Ingeniería of the Universidad Austral de Chile that allowed the development of this study.

## References

- Cárcamo, A., & Fortuny J. (2017). A hypothetical learning trajectory for spanning set and span. 2017. In Kaur, B., Ho, W.K., Toh, T.L., y Choy, B.H. (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 209-216). Singapore, Singapore: PME.
- Daro, P., Mosher, F., & Cocoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. Philadelphia, PA: Consortium for Policy Research in Education.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1, 155-177.
- Simon, M., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, 305-329.

# THE GAMIFICATION, AND ITS USE TO TEACH NUMBERS IN THE FIRST SCHOOL YEARS

Josefa Castillo Funes, Elisabeth Ramos Rodríguez

Pontifical Catholic University of Valparaíso

We study the role that gamification can play in 8 and 9 year old students in developing strategies for mental calculation and the identification of odd and even numbers. We situate ourselves in the quantitative paradigm with experimental design with pre and post tests in control and pilot groups, both randomly selected from a Chilean school. The control group worked with written material related to exercises and problems of mental calculation similar to the pilot group. With this last group, we worked with a software that considers the following aspects:

1. The game (gamification), García and Llull (2009) point out that children, through their experience and context, acquire new knowledge and skills through the practice of ludic activities.
2. Interdisciplinarity. To provide knowledge about the regions of Chile, working in an integrated way with the subject of history and geography.
3. The use of error. In each stage of the game, feedback and suggestions are given on how to solve problems, considering error as a learning opportunity, "good management of an error can generate significant learning" (Guerrero, Castillo and Chamorro, p. 367).

Analyzing the data, better results are obtained for solving mental calculation problems and identifying odd and even numbers in the pilot group (55% achievement to 78%), presumably thanks to the use of gamification.

It is expected that we will apply the game with other students, increasing the sample, in addition to forming a platform with various types of games that point to different objectives and varied mathematical objects.

## **Acknowledgments**

Funding from project "Intelligent educational platform" code SI.02, Seminar call for Interdisciplinary Research, project CORFO 14ENI2-26905 "New Engineering for 2030".

## **References**

- García, A., & Llull, J. (2009). *El juego infantil y su metodología*. Madrid: Editex.
- Guerrero, J. I., Castillo, E. J. S., & Chamorro, H. G. (2013). El error como oportunidad de aprendizaje desde la diversidad en las prácticas evaluativas. *Plumilla educativa*, 12, 361-381.

# LINKING MATHEMATICS AND ENGINEERING THROUGH MODELLING: THE CASE OF CIVIL ENGINEERING

Saúl Ernesto Cosmes Aragón

Pontificia Universidad Católica de Valparaíso

We present here an advance of research that deals with mathematical modelling in engineering education. Training in modelling competence is a competence required both by the educational programmes themselves, and by accreditation agencies and agencies that investigate their academic training, for example in SEFI (2013).

Mathematical modelling is a line of research that has been present in mathematics education for more than forty years; nonetheless there are still many challenges that need to be addressed when working with it. Specifically, within the area of the training of engineering students, which has yet to be addressed in terms of modelling (Romo, 2014). Our research question is posed: What is the link that could be generated to relate mathematics subjects to subjects in the engineering specialty, and what didactic implications is it possible to find in this link, so as to establish modelling tasks for mathematical engineering training?

In order to study and analyze the above, we use as theoretical tools the theory of Mathematical Workspaces (Kuzniak & Richard, 2014) in interaction with the Blum-Borromeo (2009) modelling cycle. As a methodological design, we used an instrumental case study which is made up of civil engineering students from a Chilean university. With what was done, we are able to begin to establish routes that link mathematics and engineering through the modelling tasks that we have studied in the course of structural analysis. The next step is to analyze projects and design tasks that allow the teaching of mathematical modelling in engineering.

## References

- Alpers, B.A., Demlova, M., Fant, C.H., Gustafsson, T., Lawson, D., Mustoe, L. et al. (2013). A framework for mathematics curricula in engineering education. A report of the mathematics working group. Brussels: *European Society for Engineering Education* (SEFI).
- Borromeo, R. (2013). Mathematical modelling in European education. *Journal for Mathematics Education at Teachers College*, 4, 18-24.
- Kuzniak, A. & Richard, P. (2014). Espacios de Trabajo Matemático. Puntos de vista y perspectivas. *Revista Latinoamericana de Matemática Educativa*, 17, 29-39. Cd. de México, México.
- Romo-Vázquez, A. (2014). La modelización matemática en la formación de ingenieros. *Educación Matemática 25 años*, 314-338.

# STATISTICS AS A TOOL FOR DECISION-MAKING: TWO COUNTRIES AND ONE PATTERN

Raimundo Elicer<sup>a</sup>, Vanessa Neto<sup>b</sup>

<sup>a</sup>Roskilde University, <sup>b</sup>Federal University of Mato Grosso do Sul

From a sociopolitical perspective of mathematics education, we have analyzed school mathematics as an important element for building current notions about citizenship in western countries (Andrade-Molina, 2018). This idea has been mobilized as a partnership between researchers from both Brazil and Chile. This study focuses on the ways in which statistics have been announced in the official mathematics high school curricula in both countries as a fundamental tool for students to critically evaluate quantitative information. In the same sense, decision-making has become a relevant skill, and an ubiquitous statement in research in mathematics education, policy making, and official curricula in general. This movement has had consequences for daily life in the two countries analyzed. Our goal is to describe and problematize statements about statistics being a tool to improve notions about citizenship by means of decision-making as a skill (which can be) built in the school mathematics context. In an ongoing investigation, we pursue this goal by analyzing excerpts from official curricula and electoral campaigns from these two countries, using elements of discourse analysis. Our analysis strategy considers the statistical as the “knowledge of State” (Traversini & Bello, 2009), which has operated as a way to conduct subjects’ behaviour and, at the same time, it has been announced as a way to enforce individual and rational decision-making, supposedly improving society as a whole. In order to inquire about this last widespread notion, we unpack uses (by politicians) and repercussions about statistical arguments used in recent electoral campaigns in both countries. Preliminary results show manipulated statistical results and graphical representations to appeal to voters. These campaign strategies have provoked many discussions (in these countries) but they do not seem to affect the elections’ results (at least in our examples). Therefore, the question that will be conducted by this study is: In what ways and to what extent has school mathematics (policy makers, educators, etc.) been producing results and statements in societal decision-making? In our interpretation, ideas about mathematical knowledge to ensure social justice, for example, are powerful nowadays, but need to be investigated.

## References

- Andrade-Molina, M. (2018) "MINDNIAC: The reasonable citizen of schooling (Chilean Edition)," *The Mathematics Enthusiast*: Vol. 15: No. 1, Article 4.
- Traversini, C. S.; Bello, S. E. L. (2009). O Numerável, o Mensurável e o Auditável: estatística como tecnologia para governar. *Educação & Realidade*. 34 (2): 135 – 152.

# RELATIONSHIPS BETWEEN MATHEMATICAL CREATIVITY AND MOTIVATION: MATHEMATICAL CREATIVITY WORKSHOPS IN HIGH SCHOOL STUDENT TRAINING

Mateus Gianni Fonseca<sup>1,2</sup>, Cleyton Hércules Gontijo<sup>1</sup>,  
Juliana Campos Sabino de Souza<sup>2</sup>, Matheus Delaine Teixeira Zanetti<sup>1</sup>

1 University of Brasilia (UnB), 2 Federal Institute of Education of Brasilia (IFB)

Mathematics learning is a topic that has raised concerns in different countries, specifically in the development of basic skills for 21st century citizens, after all, many do not even want to devote the least amount of time to this area of study. Motivating students to study math is one of the greatest challenges. Several studies suggest that students are more interested in math when creativity is involved (Fonseca, 2015; Haavold, 2018). The purpose of this research is to analyze if a group of 8 students in the last year of basic education in Brazil (Average age of 16.75 years, SD = .6) in a public school in the administrative region of the Federal District feel more motivated in mathematics from of a series of seven creativity workshops in mathematics at a time contrary to the regular school time. The Mathematical Motivation Scale of Gontijo (2007) was used as a data collection instrument, which was applied before and after the previously mentioned workshops. According to the results, we found a significant increase in the level of motivation in mathematics in the group of students, demonstrating that the stimulus to creativity in mathematics can increase their motivation.

## References

- Fonseca, M. G. (2015). *Construção e Validação de Instrumento de Medida de Criatividade no Campo da Matemática*. Dissertation of Master Degree. University of Brasilia (UnB), Brasilia, Federal District, Brazil.
- Gontijo, C. H. (2007). *Relações entre Criatividade, Criatividade em Matemática e Motivação em Matemática dos Alunos do Ensino Médio* Thesis of PhD. University of Brasilia (UnB), Brasilia, Federal District, Brazil.
- Haavold, P. (2018). An empirical investigation of a theoretical model for mathematical creativity. *The Journal of Creative Behavior*, 52(3), 226-239.

# CONCEPTIONS FROM FRESHMEN ENGINEERING STUDENTS ON THE CONCEPTS OF ABSOLUTE VALUE AND INEQUALITY

Claudio Fuentealba

Facultad de Ciencias de la Ingeniería, Universidad Austral de Chile, Chile.

The concepts of absolute value and inequality are fundamental tools in the development of university mathematics. Proof of this is that the limit of a function at a point, which corresponds to one of the fundamental and structuring concepts of calculus courses, is based on the coordination of these concepts. Also, these two concepts allow for the analysis of the functions, their approximation and optimization. However, the understanding and coordination of both concepts turns out to be a cognitive challenge that very few students can face successfully (Bazzini & Tsamir, 2004). That is why knowledge of the correct and incorrect conceptions from students about these and other mathematical concepts acquires great importance, since this information allows us to build a better instructional design for their learning (Karsenty, Arcavi & Hadas, 2007).

The participants of this study were 60 university students from a course of precalculus at the undergraduate program of Industrial Civil Engineering. The data of this work was collected during the first semester of the year 2018 and corresponds to the students' productions obtained through the application of a questionnaire. This questionnaire was composed of nine sections that addressed different aspects associated to the concepts of absolute value and inequality. The application of the questionnaire took approximately 90 minutes.

The analysis and classification of the information provided by the questionnaire allowed for the identification of the correct and erroneous conceptions from students about the concepts of absolute value and inequality. In addition, from this analysis it was possible to infer the conceptions that students have about the possible set solution of an equation and of an inequality.

**Acknowledgments:** I thank the Facultad de Ciencias de la Ingeniería of the Universidad Austral de Chile that allowed the development of this study.

## References

- Bazzini, L., & Tsamir, P. (2004). Algebraic equations and inequalities: Issues for research and teaching. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group of Psychology of Mathematics Education*, Vol. 1 (p. 137-139), Bergen, Norway: PME.
- Karsenty, R., Arcavi, A., & Hadas, N. (2007). Exploring informal mathematical products of low achievers at the secondary school level. *Journal of Mathematical Behavior*, 26(2), 156–177.

# MIND THE GAP(S): SELF-CONCEPT AS A MEDIATOR OF MATH ACHIEVEMENT DIFFERENCES IN CHILE

Denisse Gelber, Lorena Ortega, Ernesto Treviño, Horacio Solar, Alonso González,  
Ignacio Wyman

Pontificia Universidad Católica de Chile

In the context of the implementation of the Educational Inclusive Law, Chile faces a challenge: to reduce achievement gaps by gender, social origin, nationality and ethnicity. Considering that, compared to OECD countries, Chilean students underperform in math and that achievement gaps are more accentuated than in other disciplines, this study focuses on the mediating effect of a set of important psychological aspects, namely general self-concept and math self-concept, on students' math achievement (Marsh, 1986). PISA studies conclude that female students have lower self-concept in math, are less interested in the subject and assume that they are less competent in it than male students, even after controlling for achievement level (OECD, 2015). However, there is no information regarding how these psychological factors affect achievement gaps in math for other groups, such as indigenous and immigrant students. This research aims to shed light on the literature, by analysing whether psychological aspects affect differently math achievement by interest group (indigenous, foreign and female students).

Data was collected from 1,400 7<sup>th</sup> grade students (14 years old) from 50 public diverse schools in Santiago, Chile. Student outcomes were obtained from a math standardized test (SEPA) and measures of general and math self-concept were collected via a self-administered survey.

Using Hierarchical Linear Models (students nested within classrooms), the study identifies significant differences in math achievement by group (foreign students, indigenous students, and female students). These gaps among groups are partially mediated by student's general and math self-concept. This is particularly the case for the gender gap, where female students are found to be at disadvantage, mainly because of their lower levels of math self-concept.

The study provides recommendations for future research, such as, incorporating a qualitative approach to explore the roots of low self-concept within each student group, shedding light on pedagogical practices that may overcome these differences and promote higher effectiveness and equity in Chilean schools.

## References

- Marsh, H.W. (1986). Verbal and math self-concepts: An internal-external frame of reference model. *American Educational Research Journal*, (23), 129-149.
- OECD. (2015). *The ABC of gender equality in education. Aptitude, behaviour, confidence*. OECD. Retrieved from: <http://dx.doi.org/10.1787/9789264229945-en>

# DEVELOPMENT OF A QUESTIONNAIRE FOR TEACHERS ABOUT TEACHING PRACTICES AND MOTIVATIONAL BELIEFS RELATED TO PROBLEM SOLVING IN MATHEMATICS

Valentina Giaconi<sup>1,2</sup>, Josefa Perdomo-Díaz<sup>3</sup>, Gamal Cerda<sup>4</sup> and Farzaneh Saadati<sup>2</sup>

<sup>1</sup>Universidad de O'Higgins, <sup>2</sup>Universidad de Chile, <sup>3</sup>Universidad de La Laguna and <sup>4</sup>Universidad de Concepción.

*Problem solving is a fundamental aspect of mathematics education and teachers are key actors in its implementation. This study presents the design of a self-reported questionnaire for teachers about teaching practices and motivational beliefs related to problem solving (Giaconi, Perdomo-Díaz, Cerda and Saadati, 2018). We present evidence of validity based on the internal structure of the questionnaire. The results give support to the theoretical design of the instrument and show that motivational beliefs are related to student-centered practices and not to teacher-centered practices.*

The design of the questionnaire was made considering previous measures of teaching practices (Swan, 2006) and variables that impact teachers practices in problem solving (Chapman, 2015). The dimensions measured by the questionnaire are student-centered practices during problem solving, teacher-centered practices during problem solving, self-efficacy in doing problem solving, self-efficacy in teaching problem solving, and the value of problem solving.

In total, 579 mathematics teachers answered the questionnaire and exploratory and confirmatory factor analysis was applied to the data. The results distinguished theoretical dimensions, with the exception of student-centered practices that were divided in two sub-dimensions. The correlations between dimensions showed that motivational beliefs are positively correlated with student-centered teaching practices, but not with teacher-centered teaching practices. These results support the usefulness of the questionnaire for doing quantitative research and show that teachers who feel more capable and value problem-solving more report using student-centered practices more frequently.

## References

- Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT*, 3(1), 19-36.
- Giaconi, V., Perdomo-Díaz J., Cerda, G. and Saadati, F. (2018). Prácticas docentes, autoeficacia y valor en relación con la resolución de problemas de matemáticas: diseño y validación de un cuestionario. *Enseñanza de las Ciencias*. 36(3), 99-120. doi: 10.5565/rev/ensciencias.2351
- Swan, M. (2006). Designing and using research instruments to describe the beliefs and practices of mathematics teachers. *Research in Education*, 75(1), 58-70. doi: 10.7227/RIE.75.5

# THE CONSTRUCTION OF KNOWLEDGE ON THE TEACHING OF MATHEMATICS OF PRESERVICE MATHEMATICS TEACHERS ON ONLINE DEBATES

Oscar Guerrero C., Salvador Llinares

Universidad Católica del Maule, Chile; Universidad de Alicante, Spain

This investigation aims to answer the following questions: What do preservice mathematics teachers learn, and how do they learn through interaction in the analysis of segments of mathematics teaching provided through videos in a virtual environment?

From a sociocultural perspective, the emphasis is on the role of participation and discourse in the construction of knowledge, about the teaching of mathematics by mathematics teachers (Goos, 2013; Wenger, 2001) as well as on the importance of learning to see “professional noticing” in the teaching of mathematics (Llinares, 2013; Mason, 2017). Twenty-three mathematics teacher students (9 women and 14 men) participated in a blended-learning environment, which integrated virtual debates, conceptual tools and video. They focused on the analysis of the teaching of mathematics, in particular to see the development of mathematical competence in secondary school students. The results indicate that the analysis of the participation in the virtual debates and the resolution of the tasks have allowed us to characterize the learning of knowledge on the teaching of the mathematics as a change in the speech of the mathematics teacher students. This change was evidenced by a gradual integration of the knowledge of didactics of mathematics for the interpretation of the teaching that was presented in the video. In addition, the contributions of the type "agrees", "agrees and amplifies", "disagrees", or "disagrees and amplifies", privileged an instrumental use of conceptual tools come from the research on didactics of mathematics. Therefore, these spaces of social interaction allow the construction and mediation of meanings related to the teaching of mathematics.

## References

- Goos, M. (2013). Sociocultural perspectives in research on and with mathematics teachers: A zone theory approach. *ZDM Mathematics Education*, 45(4), 521–533.
- Llinares, S. (2013). El desarrollo de la competencia docente “mirar profesionalmente” la enseñanza aprendizaje de las matemáticas. *Educar em Revista*, Curitiba, Brasil, 50, 117-133.
- Mason, J. (2017). Probing Beneath the Surface of Experience. In E. O. Schack, M. H. Fisher and J. A. Wilhelm. (Eds.), *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks* (pp. 1–20). USA: Springer.
- Wenger, E. (2001). *Comunidades de práctica. Aprendizaje, significado e identidad*. Barcelona: Paidós.

# COMPARATIVE STUDY ON THE CHANGES AND CONSISTENCY OF THE MATHEMATICS CURRICULUM IN MEXICO THROUGH THE CURRICULAR REFORMS OF 1993-2009/2011

V. Hoyos, M. Navarro, V. Raggi, & S. Rojas

Universidad Pedagógica Nacional, México

There are several researchers (see the different chapters by Remillard & Reinke, Van Zanten & Van Den Heuvel-Panhuizen; Hemmi et al.; Lee et al.; in Thompson et al., 2018) who recently documented the characteristics of the curriculum reforms carried out in recent years in the USA, Holland, Finland, and Korea, through the study of the official documents in their countries of origin. These authors have agreed on conceptualizing the curriculum at least three levels: the intended curriculum, the implemented curriculum and the attained curriculum. These authors (see Thompson et al., 2018) also report that research around the mathematics curriculum was a relatively recent phenomenon.

Following principles posed by these authors we have posed the subsequent questions: (i) What is the vision of mathematics and mathematics education that is portrayed in the Mexican curriculum, and what does this curriculum seem to value? (ii) What is the role in Mexico of evaluation in the intended and in the enacted curriculum? Or even, (iii) What is the role of textbooks and other resources in the Mexican enacted curriculum? We then carried out a review, analysis and discussion of the official documents endorsed by the Ministry of Public Education in Mexico City, the results of Mexico in PISA 2009 and 2015, and several of the documents prepared by the Mexican National Institute for the Evaluation of Education (INEE). This paper is part of an ongoing research project that is looking at recovering and classifying the information and knowledge that primary or elementary school teachers in Mexico put into practice on the understanding of concepts that are key to the development of mathematical thinking at this educational level.

While PISA's purpose has been to impact public education policies in the different participant countries, it appears that this has not been the case for Mexico. In practice, there is still no registered progress on the promotion of the changes that public schools need, or that PISA results had influenced decision making or the implementation of public policies that had sought a sustained development of mathematics public education, especially for the most marginalized sectors of the Mexican population.

## References

INEE (2010/2016). México en PISA, 2009/2015. México: INEE.

Thompson, D. R., Huntley, M. A., & Suurtamm, C. (Eds.) (2018). *International Perspectives on Mathematics Curriculum*. Charlotte, NC (USA): IAP.

# INTERACTIONS OF IMMIGRANT SECONDARY SCHOOL STUDENTS DURING THE RESOLUTION OF ALGEBRAIC PROBLEMS

Atif Lodhi

Beatriz Cantero

Universidad Católica del Maule

Universidad de Barcelona

In this research we have researched the types of interactions that occur among immigrant students during the resolution of mathematical activities in a workshop. We proposed a series problem solving activities, where they had to use mathematical reasoning. The mathematical relationships among numbers, objects and geometric forms play a fundamental role in mathematical thinking (Mason, 2001). The translation of mathematical problems into equations implies a fundamental question about the transition from arithmetic to algebra, in terms of symbolism, such as reasoning.

This study focuses on a group of Pakistani students attending school in Cataluña. In this context we asked: What mathematical algebraic and linguistic learning difficulties do Pakistani immigrant students show when they are solving mathematical problems in a language that is not their first language? We observed the language switching during the resolution of mathematical activities and we asked - What kind of interactions are promoted in a workshop regarding the resolution of mathematical activities? What strategies do they use to overcome the difficulties?

In order to organize the workshop, we had four 15-year-old Pakistani students from a public school in Cataluña. We presented a contextualized problem. The methodology used is a case study, and we analyzed the interactions between a pair of students, using the criteria proposed by (Cobo, 1998). We also analyzed their language switching and difficulties during the resolution of a problem. The analysis of students' interactions showed that during their resolution of mathematical activities, they interact mainly in a cooperative way and that it was in their first language, depending on the type of mathematical knowledge they had previously learned. These results show that the algebraic mathematical knowledge in the classroom grows through personal interactions.

## References

- Cobo, P. (1998). *Análisis de los procesos y de las interacciones sociales en la resolución de problemas matemáticos: Doctoral Tesis*. Universidad Autónoma de Barcelona.
- Mason, J. (2001). Aspects of generalisation and algebra: exploiting children's power. In: Haggarty, Linda ed. *Aspects of teaching secondary mathematics: perspectives on practice*. London, UK: RoutledgeFalmer, pp. 105–120.

# PROBLEM POSING AS A TOOL FOR FORMATIVE ASSESSMENT: AN EXPERIENCE WITH PROSPECTIVE TEACHERS

Uldarico Malaspina, Carlos Torres, Magaly Campos

Pontificia Universidad Católica del Perú

Problem posing (PP) in mathematics teacher education has recently become more interested in empirical research (Felmer, et al., 2016). One of the important aspects of research is the formative assessment of learnings through PP (Kweck, 2015), since this assessment must go beyond the understanding and passive application of knowledge when solving problems. PP provides opportunities to ask questions, establish mathematical connections and identify problems (Lavy & Shriki 2007).

In this context, we are developing a research that seeks to answer the following question: How does PP contribute to the formative assessment of prospective teachers? To this aim, we gave a course on PP, using a sample of 60 freshmen, future elementary teachers. For this reason, we designed a PP test including a rubric to evaluate posed problems. In the PP test, we presented a realistic situation and asked students: (a) to pose and solve problems A and B, where B should be more complex than A; (b) to identify the information, requirement, context and mathematical environment in the posed problems; (c) to self-assess the problems and their solutions using the designed rubric; and (d) to answer a short questionnaire with open and closed-ended questions. With the rubric, expert judgment and content analysis, we analyzed the students' products.

The results obtained in our empirical research allow us to state that PP helps assess fundamental aspects of teacher training, such as consistency in the proactive management of mathematical contents and the criteria they use to determine mathematical complexity when posing a problem. The most notorious deficiency found is the lack of clarity in the problem wording. Finally, we believe these results will contribute to expanding research related to the use of PP as a formative assessment tool in mathematics teacher education.

## References

- Felmer, P. L., Pehkonen, E., & Kilpatrick, J. (Eds.). (2016). *Posing and Solving Mathematical Problems*. Springer International Publishing.
- Kweck M.L. (2015) Using Problem Posing as a Formative Assessment Tool. In Singer, F., Ellerton, N., & Cai, J. (Eds). *Mathematical Problem Posing* (273-292). New York: Springer.
- Lavy, I. & Shriki, A. (2007). Problem posing as a means for developing mathematical knowledge of prospective teachers. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of PME 31*, 3, pp. 129-136. Seoul: PME.

# REPRESENTATIONS OF STUDENTS' CONCEPT IMAGES ABOUT CONTINUITY OF A FUNCTION

Maria Alice de Vasconcelos Feio Messias, João Cláudio Brandemberg  
Federal University of Pará - UFPA (Brazil)

We present, in this summary, part of the results obtained in a research where we aimed to investigate students' concept images about continuity of a function. Our goal was to answer the following question: *In what way(s) do mathematics students understand the concept of continuity of a function?* With the aim of verifying the way students comprehended the concept of continuity of a function, we conducted a test-based survey with five mathematics students who had just finished a Calculus Course. We based the analysis of these individuals' responses on Vinner's (1991) descriptions about the possible ways the concept image and concept definition cells are activated at an individual's cognitive system when they are solving a task. The five individuals who participated in our investigation had to answer a 5 question-test that was elaborated with the aim of having them reflect upon specific situations related to continuity and its relation with the limit concept. In the obtained results, the students' understanding about the continuity concept was mainly based on the Evoked Concept Images (ECI) that follow: [ECI 1] –  $f$  is continuous at  $p$  if  $p$  belongs to the function's domain (such as highlighted by Tall & Vinner, 1991; Vinner, 1987; Amatangelo, 2013, among others); [ECI 2] – Continuity at a point depends on the limit's existence (such as pointed by Tall & Vinner, 1981; Vinner, 1987; Amatangelo, 2013, Jayakody, 2015; Jayakody & Zazkis, 2015) ; [ECI 3] – Discontinuity is represented by 'gaps', 'jumps', 'holes' or 'breaks' in the function's graph (as observed in the results of Tall & Vinner, 1981; Cornu, 1991; Jayakody, 2015, among others); [ECI 4] – Continuity means 'connectivity' (the same way observed by Cornu, 1991; Jayakody, 2015).

## References

- Amatangelo, M.L. (2013). Student understanding of limit and continuity at a point: a look for potentially problematic conceptions. Brigham Young University (Utah/USA).
- Cornu, B. (1991). Limits. In D. Tall (Ed), *Advanced Mathematical Thinking* (pp. 153 – 166). Dordrecht, The Netherlands.
- Jayakody, G. (2015). University first year students' discourse on continuous functions: a commognitive interpretation. Simon Fraser University.
- Jayakody, G. & Zazkis, R. (2015). Continuous problem of function continuity. In *For the learning of mathematics* (vol. 35, pp. 9 – 14). New Brunswick (Canada).
- Tall, D. & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. In *Educational Studies in Mathematics* (n. 12, pp. 151 – 169).
- Vinner, S. (1987). Continuous functions and reasoning in college students. In Bergeron, J. (ED) *Proceedings of the international conference on the psychology of mathematics education (PME)* (vol. 1, pp. 177 – 183). Montreal, Canada.
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed), *Advanced Mathematical Thinking* (pp. 65 – 81). Dordrecht, The Netherlands.

# THE ROLE OF ANALOGY IN THE FORMATION OF EARLY NUMERICAL REPRESENTATIONS

Navarrete Jairo, Gómez David, Dartnell Pablo

Universidad del Bío-Bío, Universidad de O'Higgins, Universidad de Chile

Susan Carey (2009, p. 340) posed questions about the cognitive mechanisms that allow children to build mappings between numerals and internal representations of magnitude. Corre and Carey (2007) proposed a mechanism whereby children draw on the order information contained in the count list to infer a rule “later in the list implies a greater number”. We draw on research about the development of early numerical representations to hypothesize that the formation of these mappings is supported by analogy—the human faculty to contrast and compare two entities by aligning their structure (Gentner, 1983). To establish whether analogy plays a role in this process of learning, we devised a classroom game based on an analysis of a number-line analogy that leverages children's spatial intuitions for helping them understand the numbers (Navarrete & Dartnell, 2017). To investigate the influence of spatial positioning along the number line in learning numbers, we conducted a pre-test/post-test scholar intervention where seventy-seven preschoolers were randomly assigned to control tasks (control condition) or to embodied number line tasks in two possible spatial locations: watching numbers increasing either from left to right (space-number alignment condition) or from right to left (space-number misalignment condition). In contrast with children in the control group, children performing learning activities improved their proficiency in four tasks of numerical knowledge. However, only children under spatial and numerical alignment improved in a task of number line estimation as shown by a repeated-measures ANCOVA displaying a significant interaction  $\text{Session} \times \text{Condition}$ ,  $F(2,70)=3.25$ ,  $p=.044$ ,  $\eta_p^2=.07$ . This finding shows that numerical representations generated by children during training were integrated with spatial information that was implicit in the learning activities. Hence, children's incipient representations of numbers were aligned to the spatial structure that was cued during training in the learning activities, thus leading to the formation of numerical representations of magnitude with links to spatial structure. This suggests that the mechanism proposed by Corre and Carey is implemented through analogies that align children's number representations (count list) and everyday experiences of magnitude.

## References

- Corre, M. L., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles.
- Carey (2009). The origin of concepts.
- Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy.
- Navarrete, J. A., & Dartnell, P. (2017). Towards a category theory approach to analogy: Analyzing re-representation and acquisition of numerical knowledge. *PLoS Computational Biology*, 13(8), e1005683.

# CITIZENSHIP NOTIONS IN MATHEMATICS TEXTBOOKS FOR RURAL PRIMARY SCHOOLS IN BRAZIL

Déborá Reis Pacheco, Vanessa Neto

Federal University of Mato Grosso do Sul – Brazil

As part of a research analyzing mathematics textbooks for rural populations in Brazil, we employ discourse analyses (see Foucault, 1972) to describe and to problematize notions about citizenship in these textbooks. We analyzed ten mathematics textbooks that were sent to rural primary schools in 2013 and 2018 (the textbooks are renewed every three years). Ideas about citizenship have strong links to school mathematics as Valero (2017) has shown us. These textbooks were made as a result of the political struggle of rural populations and the different social movements that recognize rural life forms and conditions. For this reason, we assume that these materials are a potential resource for understanding what are the features, values and practices that represent citizenship in rural communities in Brazil by means of mathematics textbooks. For us, the starting point is to admit that notions about social justice were the ground that guaranteed the appearance of these materials as government educational policy.

Our preliminary results show, for example, notions about environmental problems (especially water use, rural practices with pesticides, trash destination, etc.) in the mathematics tasks, exercises, and images in order to contextualize the activities in the textbooks. However, these approaches have hidden specific rural problems (e.g. the absence of selective garbage collection in rural areas) and they have assigned individual responsibilities. These practices are aligned with the neoliberal rationality in society. Therefore, between math tasks and under the tutelage of the notions of social justice, practices have been constructed to ratify both values and morality, erasing specificities and difficulties as a part of countryside lifestyle and to attribute individual responsibility to the subjects in a movement that ratifies neoliberal practices.

## References

- Foucault, M. (1972) *The archaeology of knowledge*. (World of man). Pantheon Books, New York.
- Valero, P. (2017) Mathematics for all, economic growth, and the making of the citizen-worker. In T. S. Popkewitz, J. Diaz, & C. Kirchgasler (Eds.), *A political sociology of educational knowledge: Studies of exclusions and difference* (pp. 117- 132). New York: Routledge.

# THE EFFECT OF BENCHMARKS ON STRATEGY USE IN FRACTION COMPARISON PROBLEMS

Andreas Obersteiner<sup>1,2</sup>, Vijay Marupudi<sup>1</sup>, & Martha W. Alibali<sup>1</sup>

<sup>1</sup>University of Wisconsin-Madison, U.S. <sup>2</sup>Freiburg University of Education, Germany

When people compare the numerical values of two fractions, the fractions' natural number components may interfere with reasoning about magnitudes, yielding a "natural number bias" (Ni & Zhou, 2005). However, not all studies reveal the bias, and some studies have revealed a reverse bias (e.g., DeWolf & Vosniadou, 2015). In this study, we investigated whether the strategies people use to compare fractions depend on features of the problems. We were particularly interested in the role of benchmarks (reference numbers, e.g.,  $\frac{1}{2}$ ), which people may use to compare fractions. Moreover, we investigated whether strategy use affects the occurrence and strength of a natural number bias.

Adults solved complex fraction comparison problems and reported their strategies on a trial-by-trial basis. Half of the pairs were congruent (i.e., the larger fraction had the larger components) and half were incongruent (i.e., the larger fraction had the smaller components). The congruent and incongruent sets were balanced in terms of the fractions' magnitudes relative to common "benchmarks" (i.e., reference points, specifically,  $\frac{1}{4}$ ,  $\frac{1}{2}$ , or  $\frac{3}{4}$ ). In "straddling" problems, one fraction was smaller and the other larger than one of these benchmarks. In "in-between" problems, both fractions were in between two adjacent benchmarks. In a special subcategory of "in-between" problems, both fractions were either smaller than  $\frac{1}{4}$  or larger than  $\frac{3}{4}$ ; in these problems, one fraction was close to 0 or 1, which may be especially salient benchmarks. Some participants also received a tip that benchmarks could be useful.

Overall, we found a reverse "smaller components—larger fraction" bias. Participants varied in their strategy use across problem types, indicating that they used strategies adaptively. On problems in which one fraction was close to 0 or 1, they used generally incorrect, component-based strategies more often than on other problems. For the other two problem types, participants used component-based strategies less often, and used benchmark strategies somewhat more often. The tip about using benchmarks had little effect. Participants seemed to adapt their strategies to the affordances of different problems, including the fractions' relative positions to benchmarks. Thus, patterns of strategy use may at least partially explain the occurrence and the direction of the natural number bias in fraction comparison.

## References

- DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, 37, 39-49.
- Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educational Psychologist*, 40, 27-52.

# KINDERGARTEN AND PRIMARY TEACHERS' SPECIALISED KNOWLEDGE IN THE CONTEXT OF CLASSIFICATION

<sup>1</sup>Alessandra Rodrigues de Almeida, <sup>2</sup>Diana Zakaryan, <sup>3</sup>Milena Policastro,  
<sup>3</sup>Marcos Paulo de Oliveira, <sup>3</sup>Miguel Ribeiro

<sup>1</sup> Pontifical Catholic University of Campinas (Brasil); <sup>2</sup>Pontifical Catholic University of Valparaíso (Chile); <sup>3</sup>State University of Campinas – UNICAMP (Brasil)

Teaching Mathematics at kindergarten and primary school, with the aim that pupils understand what they do and why they do it, is a challenging endeavour. Such difficulties are grounded in the nature of teachers' knowledge of mathematical topics (e.g., Policastro, Almeida & Ribeiro, 2017) as well as on the fact that, frequently, official documents are too broad and don't serve as guidelines for teachers' practices. For kindergarten, one of the topics that needs to be dealt with is classification. Fostering situations that entail objects classification enables students to develop a line of thinking which, later, would allow them to build definitions. Thus, it is essential to obtain a broader understanding of teachers' content knowledge, in order to conceptualize ways for its improvement.

On the work we have been developing, the Mathematics Teachers' Specialised Knowledge (Carrillo et al, 2018) is perceived as a theoretical and analytical tool for analyzing practice and conceptualizing tasks for teacher education. Here we focus on the knowledge revealed by a group of kindergarten and primary teachers when discussing a task in the context of classification. The results reveal teachers' knowledge about the concept of classification associated to disjoint classification, and even if they recognize an element as being part of the intersection of two groups, they do not associate the kinds of classification to the mathematical knowledge to be learned by pupils. Such work led to designing tasks for teacher education which focused on developing teachers' knowledge on the connections involving classification and different mathematical and non-mathematical topics.

This research has been partially supported by the grant 2016/22557-5, São Paulo Research Foundation (FAPESP) and is part of the activities of the project CONICYT PCI/Atracción de capital humano avanzado del extranjero, n° 80170101 (Chile).

## References

- Carrillo, J., Climent, N., Montes, M., Contreras, L.C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The Mathematics Teacher's Specialised Knowledge (MTSK) model. *Research in Mathematics Education*, to appear.
- Policastro, M.S., Almeida, A.R. & Ribeiro, M. (2018). Conhecimento especializado revelado por professores da educação infantil dos anos iniciais no tema de medida de comprimento e sua estimativa. *Espaço Plural*, 18, 123-154.

# TEACHING REFLECTION ON STUDENT'S PERFORMANCE IN THEIR MATHEMATICAL LEARNING ASSESSMENT

Grecia Gálvez Pérez, Daniela Rojas Bastías

Facultad de Educación, Pontificia Universidad Católica de Chile

Centro de Modelamiento Matemático, Universidad de Chile

## **Theoretical Framework**

This research articulates a theoretical approach on reflexivity (Shön, 1998) with an approach that considers classroom assessment as an essential part of teaching and learning processes. In this context, tests are a contribution to classroom assessment when teachers reflect on their results and use them to improve student learning (Stobart, 2010).

## **Objectives**

To describe teachers' reflective practices when analyzing the mathematical learning performance of their students.

## **Methods**

The thinking aloud technique was used to record the reflections of six teachers while analyzing the performance of their students in learning assessments. Subsequently, a content analysis was used to raise categories that would characterize the reflections of these teachers.

## **Results**

Teachers' reflection can be characterized in three ways: reflection that is limited to corroborating if the students' answers are correct; reflection on the difficulty of the proposed task, analyzing the performance as a student's learning achievement; and reflection that questions the teaching practice within the classroom and analyzes its implications for student performance.

## **Discussion**

The first two forms are observed in most moments, while the third one was observed on very few occasions. The use of learning evidences in an autonomous way as a means of reflection, and feedback of teaching practice within the classroom, requires that teachers have mathematical and didactic tools to analyze students' performances regarding the knowledge that is brought into play.

## **References**

- Schön, D. (1998). *El profesional reflexivo. Cómo piensan los profesionales cuando actúan*. Barcelona: Paidós.
- Stobart, G. (2010). *Tiempos de prueba: los usos y abusos de la evaluación*. Madrid: Ediciones Morata.

# PROFESSIONAL NOTICING OF ARGUMENTATION IN MATHEMATICS CLASSROOMS

Horacio Solar<sup>a</sup>, Jonathan Cervantes<sup>b</sup>, Andrés Ortiz<sup>c</sup>, Rodrigo Ulloa<sup>c</sup>

<sup>a</sup>Pontificia Universidad Católica de Chile, <sup>b</sup>Universidad de Guerrero,

<sup>c</sup>Universidad Católica de la Santísima Concepción

Argumentation is a complex and structured process whose purpose is to convince others that a given statement is valid (Krummheuer, 1995). In general, analyses of argumentation in the classroom are based on Toulmin's model, which follows a linear process that comprises six elements: data, warrant, backing, rebuttal, modal qualifier and claim. While there are some researches on the interpretations that teachers make of the different elements of Toulmin's argumentative structure, they haven't reported how these interpretations change over time.

Our hypothesis is that change in the teachers' interpretation of the argumentative structure occurs to the extent that they develop their Noticing of argumentation, analysing the element's content structure. This process can be encouraged by strengthening the professional noticing of teachers, by means of evaluating teachers' written accounts and the use of video recordings of classroom activities (Sherin, Jacobs & Phillip, 2011). Both are employed as resources during in-service teacher education, to foster the reflection and understanding of mathematics teachers. The objective of this study is to evaluate how the strengthening of Noticing of mathematics teachers favors changes in the interpretation of the argumentation.

Within the frame of a larger research project, 10 Chilean teachers participated in a 20-session course which aimed at developing mathematical argumentation within the classroom. We analysed teachers' responses during two selected sessions, focused on evaluating the teachers' appropriation of argumentation structure, through indicators from the literature that were refined empirically through a constant comparison method. The findings show changes in three indicators of teachers' interpretation: data, warrant and rebuttal. In the oral presentation, we will show how these changes can be analysed from a Noticing perspective in which the professional development experience has favoured teachers' identification, interpretation and decisions-making about the development of argumentation.

## References

- Krummheuer. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Ed.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Lawrence Erlbaum.
- Sherin, M. G., Jacobs, V. R., & Phillip, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–13). New York: Routledge. <https://doi.org/10.1080/13632434.2014.992777>

# UNDERSTANDING AND PROMOTING CHILDREN'S PROPORTIONAL REASONING

Alina Galvão Spinillo

Federal University of Pernambuco

Children's difficulties with the concept of proportion have long been reported in the literature. However, there is evidence that children can successfully solve proportional tasks presented with continuous amounts (Boyer & Levine, 2015), by using ratio representation (Howe, Nunes & Bryant, 2010), by using the 'half' boundary to estimate and by plotting out the first-order relations (Spinillo & Bryant, 1991). Based on the possibilities children have when dealing with this concept, a teaching programme was carried out in a classroom setting with third-graders aged 8 to 9 years old. None of them had been formally instructed about proportion. The programme comprised 18 two-hour sessions, two sessions per week over nine weeks. The sessions involved working with small groups and, also, whole class activities focusing on proportionality tasks. The approach to proportions was based on (i) children's initial understanding of this concept; (ii) the invariant principles governing proportional reasoning; and (iii) a metacognitive orientation in which children were systematically asked to think about their own thought processes when solving proportional problems and encouraged to explain their solution procedures. The instruction focused mainly on tasks where children had to estimate rather than to perform precise calculations, to make proportional judgments on the basis of part-part terms, and to discriminate between absolute and relative quantities. Data analysis allowed us to follow up on the children's way of reasoning and the progress they made with regards to the concept of proportion. To ask children to explain how they solve problems has proven to be a key teaching strategy, not only for the teacher, who had the opportunity to understand the student's reasoning process, but also for the student him/herself who became aware of his/her own ways of thinking, being therefore able to monitor them. These results support the call to teach the concept of proportion in elementary school.

## References

- Boyer, T.W., & Levine, S. C. (2015). Prompting children to reason proportionally: processing discrete units as continuous amounts. *Developmental Psychology*, 51(5), 615-620.
- Howe, C., Nunes, T., & Bryant, P. (2010). Rational number and proportional reasoning: using intensive quantities to promote achievement in mathematics and science. *International Journal of Science and Mathematics Education*, 9, 39-417.
- Spinillo, A.G., & Bryant, P.E. (1991). Children's proportional judgements: the importance of 'half'. *Child Development*, 62, 427-440.

# FEATURES OF MATHEMATICS TEACHER ARGUMENTATION

Jorge A. Toro, Walter F. Castro

Universidad de Antioquia

Research in Mathematical Education recognizes the importance of favoring argumentation in class, as it is essential in the construction of knowledge (Metaxas, 2015). It also recognizes the teachers' role, the tasks' design, opportunities for participation, or the analysis of arguments. However, there are few studies that inquire about argumentative features of the mathematics teacher while teaching the class. In this communication we show features of a teacher's argumentation in the mathematics class, which is part of a doctoral research in progress.

We consider argumentation, a collective activity, as a complex, communicative and interactive speech act that consists in a constellation of statements aimed to justify or to refute a statement. Instead of being only a structural identity, argumentation is a verbal activity that happens through the use of language; it is a social activity addressed to other people; it is a rational activity based on intellectual considerations (Eemeren & Grootendorst, 2004); and it is a didactical activity intended to educate in mathematics. With this approach, we try to gather elements to carry out our object of study.

We have taken data from a ninth-grade class (15-year-old students) in a public school in Medellin (Colombia), where the teacher and students discuss probability. In the analyzed segments we have recognized different types of intentions immersed in the teacher's verbalization while teaching. We can identify argumentative features that respond to social, rational and didactic intentions which are all together expressed spontaneously by the mathematics teacher in the natural environment of the classroom when teaching mathematics topics to the students. Argumentation in a mathematics class is a complex speech act that responds to several intertwined factors and whose presence derives from changing conditions in the classroom environment. Even though the criteria used by the teacher to regulate her speech acts are beyond the scope of this study, we acknowledge that the speech acts suit intentions that the teacher, based on her experience, identifies as important to reach both, her instructional objectives and to keep students' interest. We have also been able to identify possible links between the teacher's argument and a teacher's knowledge model.

## References

- Eemeren, F., & Grootendorst, R. (2004). *A Systematic Theory of Argumentation. The Pragma-Dialectical Approach*. Cambridge: Cambridge University Press.
- Metaxas, N. (2015). Mathematical argumentation of students participating in a mathematics–information technology project. *International Research in Education*, 3(1), 82–92. doi: <https://doi.org/10.5296/ire.v3i1>.

# HOW DO CHILEAN MATH TEACHER EDUCATORS PREPARE PRESERVICE TEACHERS TO ACHIEVE AND DEVELOP ARGUMENTATION SKILLS?

Rodrigo Ulloa<sup>1</sup>, Sara Rivera<sup>1</sup>, Horacio Solar<sup>2</sup>, Andrés Ortiz<sup>1</sup>

<sup>1</sup>Universidad Católica de la Santísima Concepción, <sup>2</sup>Pontificia Universidad Católica de Chile

The study of the teacher educator and his knowledge and practices is relatively new, despite its central role in teaching education programs (Jaworski & Huang, 2014). Here, mathematics courses are given by educators who do not respond to a single profile, and literature has distinguished them between mathematics educators or didacticians, and mathematicians (Schoenfeld, 2004; Fried & Dreyfus, 2014).

In order to characterize teacher educators in a Chilean context, we focused on the description of the conception of teacher educators in the province of Concepción with regards to argumentation, considering the recent adding of this competence to the Chilean curriculum. We interviewed almost every teacher educator of the province (n=8), through a semi-structured interview designed to keep certain comparison criteria, with a focus on what is argumentation, how to develop it in children and preservice teachers, and the reflections about their own practice in that context. For the analysis, we engaged a bottom-up coding following the methods of grounded theory.

We identified and described different conceptions of argumentation: as exposition, as explanation, as mathematical justification or proof, and as the content of a mathematical debate. These conceptions seem to be related to how to develop argumentation in preservice teachers, and in the perspective of teacher education for teaching mathematics. So, we found closed and semi-open discourses, linked to their closed and semi-open declared practices. Closed practices are selective, for they aim to identify those students capable of doing math; semi-open practices are more inclusive, but still search for the adherence of certain ideas. In all, both discourses consider the act of understanding as an act of convincing. In the communication, we'll present this result in detail.

In our findings, in its final stages of development, teacher educators show a range of discourses tensioned around whether mathematics is a filter for selective excellence, or a means for equity and social justice.

## References

- Jaworski, B. & Huang, R. (2014). Teachers and didacticians: key stakeholders in the processes of developing mathematics teaching. *ZDM*, 46(2), 173-188.
- Fried, M. & Dreyfus, T. (Eds.) (2014). *Mathematics & Mathematics Education: Searching for Common Ground*. Springer.
- Schoenfeld, A. (2004). The Math Wars. *Educational Policy*, 18(1), 253-286.

# THE EDUCATION OF URBAN MATHEMATICS TEACHERS

Luz Valoyes-Chávez

Centre for Advanced Educational Research – University of Chile

The mathematics education reform seems to be failing in schools mainly attended by marginalized student populations. Despite the implementation of multiple models of professional development programs (PD), the education of teachers able to materialize the “mathematics for all” promise has proved hard to achieve. Some researchers (e.g., Gutierrez, 2013) have questioned the alleged existence of a unique type of teacher knowledge needed to teach mathematics regardless of the social, political and cultural school contexts. Gutiérrez (2013) introduces the construct of “political knowledge” to designate a particular type of understandings for teachers to challenge and deconstruct educational practices that keep urban students away from learning meaningful mathematics. Thus, a critical issue within the field relates to the distinctive knowledge needed for teaching mathematics in urban contexts.

In this paper, I explore the nature of learning experiences for urban teachers that would allow them to enhance the participation, mathematics experiences and identities of marginalized student populations. I present the results of a long-term PD intervention aimed at supporting urban teachers in implementing reform-based mathematics teaching in Chile. Besides attending 8 workshops along the school year, one participating teacher engaged in a learning experience aimed to provide opportunities for unpacking, confronting and challenging representations and stereotypes about his students’ abilities to learn mathematics. The teacher and I met once per month to analyse the mathematics activity of a student he positioned as a low achiever and lacking motivation. We also met to discuss excerpts of the student’s journal in which she told her expectations, dreams, and quotidian school experiences. We contrasted such narratives with the ones the teacher constructed about the student at the beginning of the school year.

Preliminary results evidence the positive impact of the teacher’s participation in the learning experience. By following the student and unravelling her school and home experiences, the teacher had the opportunity to reconstruct his discourse and expectations and to engage in a new pedagogical relationship with the student, allowing her to participate in the class. In this sense, the results seem to confirm that a different type of knowledge is needed for urban teachers to successfully implement the school mathematics reform.

**Acknowledgement.** Funding from Fondecyt #3180238 is gratefully acknowledged.

## References

Gutierrez, R. (2013). Why (urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education*, 6(2), 7–19.

# PROBLEM SOLVING, AN ABILITY TO BE DEVELOPED IN PRIMARY TEACHERS

Patricia Vásquez-Saldías, Mayra Cerda-Montecinos

Pontificia Universidad Católica de Valparaíso

In Chile, the ability to solve problems must be developed in math classes, from the first years of scholarship until the end of secondary school (MINEDUC, 2015). But, how can we do it? What is a problem? How can I manage the class if the students are solving a problem? These are some of the questions that the teachers have to face and that lead to them participating in a professional development program, the RPAula courses, in the inner area of the Valparaiso region. This communication will talk about the evolution of a group of teachers, that is manifested in a way in which, according to Kuzniak and Richard (2014), the ideal Mathematical Working Space (MWS) is organized by leading the students to commit themselves to the resolution of the problem. To carry out this analysis, the three levels of MWS will be described: the reference MWS, the ideal MWS and the personal MWS, analyzing the tasks proposed by these teachers in a problem solving activity, and the way in which they carry out the circulation between the different poles of the epistemological and cognitive planes (Vásquez, Mena-Lorca, & Mena-Lorca, 2016).

## References

- Kuzniak, A. y Richard, P. (2014). Espacios de Trabajo Matemático. Puntos de vista y perspectivas. *Revista Latinoamericana de Matemática Educativa*, 17(4), 181-197.
- MINEDUC (2015). Curriculum Nacional. Reviewed on July 11th 2018 from <http://www.curriculumnacional.cl/>
- Vásquez, P., Mena-Lorca, A. y Mena-Lorca, J. (2016). Construcción de un espacio de trabajo matemático idóneo en álgebra lineal: episteme versus currículo. In F. Rodríguez, R. Rodríguez y L. Sosa (Eds.), *Investigación e Innovación en Matemática Educativa*. Oaxaca, pp. 204-209. México: Red de Centros de Investigación en Matemática Educativa A. C



# POSTER PRESENTATIONS



# THE DEVELOPMENT OF VISUALIZATION SKILLS AND PROCESSES

Daniela Camila Ojeda Guerrero, Jenny Patricia Acevedo-Rincón

Chiloe's Adult Vocational Technical School (Chile),  
Industrial University of Santander (Colombia)

This research aims to identify the development of visualization skills and processes (Del Grande, 1990; Acevedo, 2010) in one group of second year, high school students in Puerto Montt City (Chile) by participating in the unit *similarity of flat figures*, through the realization of a didactic sequence from the VAK model (Visual-Auditory-Kinesthetic). The research methodology is framed within the qualitative design, with development and application of a didactic sequence under the perspective of Teaching Experiment (Steffe & Thompson, 2000). The proposal was developed in three moments: (i) similarity and scale figures; (ii) triangles similarity criterion; and, (iii) Homothety and similarity. The data analysis focused mainly on the recognition of visualization skills and processes and on the identification of Visual, Auditory and Kinesthetic skills from the VAK model (Bedoya & Botero, 2006). For this investigation, it was concluded that in the development of activities characterized by the use of figures and graphic images, the visualization skills and processes are implicit, especially in the area of geometry. Throughout the development of the activities, students used visual skills to a greater extent. However, the other auditory and kinesthetic skills were used for the execution of activities. Based on the results, we conclude the need to incorporate activities that enhance the Visual, Kinesthetic and, as usual, the Auditory channel for the approach to new mathematical concepts in the area of Geometry, and thus create different scenarios and learning possibilities. from the VAK model.

## References

- Acevedo-Rincon, J. (2010). *Cognitive structural modifiability vs. Visualization: an exercise in the analysis of the use of tetris in tasks of rotation and translation*. Master's Thesis, National Pedagogical University, Sciences and Tecnology Faculty, Bogota, Colombia.
- Bedoya, C; Botero, C. (2006). Learning styles from model VAK and its incidences in achievement academic in children of degree 5 of primary. *Electronic Journal of Education and Psychology*, 2(4), pp. 4-12.
- Del Grande, J. (1990). Spatial Sense. *Aritmethic teacher*, 14-20.
- Steffe, L.; Thompson, P.W. (2000). Teaching experiment methodology: underlying principles and essential elements. In: Kelly, A.E. y Lesh, R.A. (Eds.). *Handbook of research design in mathematics and science education*, pp. 267-306. Mahwah: NJ: LAE.

# MIDDLE AND HIGH SCHOOL STUDENTS IN BRAZIL AND THEIR BELIEFS ABOUT MATHEMATICS

Felipe Augusto de Mesquita Comelli, Wanusa Rodrigues Silva, Ana Lúcia Manrique

Postgraduate Studies Program in Mathematics Education, Pontifical Catholic University of São Paulo (PUC-SP, Brazil)

This work provides an analysis of data on beliefs about mathematics and mathematics learning in middle and high school students from a private school located in the city of Santos, State of São Paulo, Brazil.

Beliefs about mathematics have been investigated with different purposes and approaches, sometimes seeking to understand them by focusing on teachers, and other times by focusing on students. This construct is part of the different components of the Affective Domain, such as emotion, attitudes, motivation. The debate around its difficulties with regards to its definition is constant in the literature (e.g. Pajares, 1992; Fives & Buehl, 2012). However, Philipp (2007) says that “Beliefs might be thought of as lenses that affect one’s view of some aspect of the world... Beliefs, unlike knowledge, may be held with varying degrees of conviction...” (p.259).

To examine their beliefs, we applied an instrument adapted from Grootenboer and Marshman (2016) to 469 students, between 10 and 18 years of age, containing three parts: 1) sociodemographic issues (school year, age and sex); 2) a five-point Likert scale questionnaire with 25 statements; and 3) open questions; ‘What is math for you? List things you think Mathematics is.’

Among other results, the perception of a strong presence of the belief that mathematics is something useful emerges from the study, despite negative attitudes towards them as observed in the students' responses: ‘Mathematics is boring, but it's still necessary’, ‘Boring, irritating, makes me fall asleep, but necessary’, ‘A very useful discipline’.

## References

- Fives, H., & Buehl, M. M. (2012). Spring cleaning for the “messy” construct of teachers’ beliefs: What are they? Which have been examined? What can they tell us. *APA Educational Psychology Handbook*, 2, 471-499.
- Grootenboer, P., & Marshman, M. (2016). *Mathematics, affect and learning*. Singapore: Springer.
- Pajares, M. F. (1992). Teachers’ beliefs and educational research: Cleaning up a messy construct. *Review of educational research*, 62(3), 307-332.
- Philipp, R. A. (2007). Mathematics teachers’ beliefs and affect. *Second handbook of research on mathematics teaching and learning*, 1, 257-315.

# A GLOCAL LESSON STUDY WITH PROSPECTIVE BRAZILIAN MATHEMATICS TEACHERS

Jenny Patricia Acevedo-Rincón, Campo Elías Flórez Pabón

University of Santander (UIS-Colombia),  
University of Pamplona (UniPamplona-Colombia)

Since 2006, the research group Pedagogical Practices in Mathematics (PraPeM) from the Education Faculty in the University of Campinas (Unicamp), proposed the inclusion of the Pedagogical Practices in Mathematics (PPM) Course in the Mathematics Education Course. The PPM course proposal for the second half of the year 2016 sought to adopt the Japanese teaching methodology Lesson Study (LS), which has many advantages in teacher education and research. From the LS perspective (Lewis, 2002; Fernández & Yoshida, 2004; Acevedo-Rincón & Fiorentini, 2017), there are multiple surveys conducted in several countries, mainly as part of the teacher continuing education, not only in mathematics, alluding to its slogan "Teachers learning together". This article describes and discusses the education process and the professional learning for prospective teacher education in the Pedagogical Practices in Mathematics course. The course aimed to know and problematize the teaching and learning practices in the school. The activities of the prospective teachers were developed under the Lesson Study methodology. Prospective teachers have developed a 'glocal' Lesson Study from a choice of topics - relevant to the school curriculum - along with lesson planning, sharing and discussing the lesson proposals, lesson implementation, lesson analysis, and presentation/discussion of results, culminating in the writing of articles. The process, and some results from the implementation of the Lesson Study in a pedagogical discipline of a degree course in mathematics will be highlighted. Finally, the continuous opportunities for teacher learning that graduates had in this formative experience, within the context of reflective and investigative participation in the practices of teaching and learning mathematics in the school, will also be a focus. The prospective teachers learned from the moment they chose a theme, through to the socialization and joint discussion of the planning and the execution of the class, finally culminating in the systematization of the lived experiences.

## References

- Acevedo-Rincón, J.; Fiorentini, D. (2017). 'Glocal' Lesson Study: the pedagogical practices in mathematics case. *International Journal of Mathematics Education*, 7 (2)2, 24-44.
- Fernandez, C.; Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah, NJ: Lawrence Erlbaum.
- Fujii, T. (2016). *Designing and adapting tasks in lesson planning: A critical process of Lesson Study*. ZDM, p. 411-423.
- Lewis, C. (2002). *Lesson Study: A Handbook of Teacher-Led Instructional Change*. Philadelphia, PA: Research for Better Schools, Inc.

# COMPARING FRACTIONS VS. COMPARING DIVISIONS: DIFFERENT UNDERLYING PROCESSES IN YOUNG ADULTS

David M. Gómez<sup>1</sup>, Pablo Dartnell<sup>2</sup>

<sup>1</sup>Universidad de O'Higgins, <sup>2</sup>Universidad de Chile

Cognitive investigations of mathematical thought are aimed at understanding its underlying mental processes. However, just as a mathematical task can usually be solved by means of a variety of strategies, the cognitive processes involved in mathematical tasks can vary widely, depending on subtle aspects of the task, the instructions, or the task items.

In this presentation, we focus on the processes that allow young adults to select the larger element from either a pair of fractions or a pair of natural number divisions. Previous research on fraction comparisons has revealed that adults tend to answer more correctly and quickly, for instance, when the fractions to be compared share a common denominator than when they share a common numerator (e.g. Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; see also Gómez & Dartnell, 2015). However, it is unknown if division comparison, a very similar mathematical task, will lead to similar results. We contrasted the accuracy and response time of answers from young adults to fraction comparisons (n=150) with those to division comparisons (n=50). Fraction items were constructed using proper fractions and a similar classification to previous studies (Obersteiner et al., 2013; Gómez & Dartnell, in press), whereas division items were classified in similar categories but used only items where the dividend was larger than the divisor.

Response accuracy was higher for divisions, reaching ceiling effects that did not allow us to compare effectively both tasks. However, contrary to our expectations, response times in the division task showed the opposite pattern to fractions: items with the same divisor were answered more slowly than items with the same dividend. This outcome suggests that these two tasks are approached cognitively in very different manners.

This research has been supported by projects Fondecyt Regular 1160188 and PIA/Basal FB0003.

## References

- Gómez, D. M., & Dartnell, P. (2015). *Is there a natural number bias when comparing fractions without common components? A meta-analysis*. In Beswick, Muir, & Wells (Eds.), *Proceedings of the 39th Meeting of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 1-8. Hobart, Australia: PME.
- Gómez, D. M., & Dartnell, P. (in press). Middle schoolers' biases and strategies in a fraction comparison task. *International Journal of Science and Mathematics Education*.
- Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, 28, 64-72.

# MECHANISMS UNDERLYING THE GENDER GAP IN MATH PERFORMANCE AT UNIVERSITY ADMISSION IN CHILE

Navarrete Jairo, Sandoval José, Espinoza Juan, Moreno Luis

Universidad del Bío-Bío

Women are underrepresented in STEM careers. In Chile, this phenomenon is tied to a wide gender gap in math performance at examinations that determine university admission. In this study we argue that this gender gap results from gender differences in a cognitive ability called cognitive reflection. Prior research has found gender differences in this cognitive ability (Frederick, 2005; Primi, Donati, Chiesi, & Morsanyi, 2018) and has shown that this ability is strongly related to mathematical reasoning and math anxiety (Primi et al., 2018). We tested whether the gender gap in math performance at university admission is better accounted for by differences in cognitive reflection, than by gender itself. A cross-sectional study collected data from 259 participants (60.6% females) regarding their math performance at university admission (PSU math scores) and their levels of cognitive reflection. A mediation analysis controlling for participants' linguistic abilities showed that the direct effect of gender on PSU math scores was small and not statistically significant ( $c = .06$ ,  $p = .57$ ,  $CI_{95} = [-.16, .30]$ ) whereas indirect effects of gender on math scores mediated by cognitive reflection were statistically significant ( $ab = .22$ ,  $p < .0001$ ,  $CI_{95} = [.13, .34]$ ). Significant relationships between gender and cognitive reflection ( $a = .73$ ,  $p = .0001$ ,  $CI_{95} = [.50, .96]$ ) and between cognitive reflection and PSU math scores ( $b = .30$ ,  $p = .0001$ ,  $CI_{95} = [.18, .42]$ ) were observed. In light of these results, our discussion draws connections from a wider context in order to outline how psychosocial mechanisms such as math anxiety and stereotype threat affect differently men and women in their development of cognitive and numerical abilities. Our discussion also identifies systematic negative self-reinforcing loops that affect women's math performance at university admission. For example, when a girl receives a poor grade, it increases gender stereotype in her environment and diminishes her sense of self-efficacy. This triggers math anxiety, impairing her normal development of cognitive reflection and numeracy abilities. Her math performance is affected, meaning that future grades will likely be poor, thus starting the loop again. Some recommendations are given that might help in breaking out of these vicious cycles (e.g. Maloney & Beilock, 2012).

## References

- Frederick, S. (2005). Cognitive reflection and decision making. *The Journal of Economic Perspectives*, 19(4), 25–42.
- Maloney, E. A., & Beilock, S. L. (2012). Math anxiety: who has it, why it develops, and how to guard against it. *Trends in Cognitive Sciences*, 16(8), 404–406.
- Primi, C., Donati, M. A., Chiesi, F., & Morsanyi, K. (2018). Are there gender differences in cognitive reflection? *Thinking & Reasoning*, 24(2), 258–279

# INCLUSIVE PRACTICES IN THE MATH CLASS: THE USE OF ARGUMENTATION AS A MEANS TO PROMOTE EQUITY

Pilar Peña-Rincón, Horacio Solar

Pontificia Universidad Católica de Chile

A good math class should give learning opportunities to all of your students. In particular, it is thought that an inclusive math class includes fair and equitable teaching practices, high expectations for all students, access to rich, rigorous and relevant mathematics, and considers the family and community context to promote learning and the positive achievement of mathematics (NCSM & TODOS, 2016). However, there is not enough empirical research about which are the teaching practices that promote equity in the math class. We consider that an inclusive practice is argumentation understood as a key tool for the development of dialogical classes, oriented to research, where students engage in and take responsibility for the collaborative construction of mathematical knowledge (Krummheuer, 1995).

The objective of the research is to contribute to the characterization and promotion of inclusive pedagogical practices in the math class, and to analyze the effectiveness of an intervention for teachers of mathematics and differential education: i) the promotion of performance in mathematics, ii) the socio-emotional development of students, and iii) the promotion of the equity of learning opportunities. About this last point, we want to answer the following research question: In what way do the differential educator and teacher collaborate in the use of argumentation in the math class to develop inclusive pedagogical practices? The intervention consists of a training course for these teachers, focused on the analysis of videos of their own and other practices, whose purpose is to study and put into practice the use of argumentation in the math class. The sample considers 57 public schools in the Metropolitan Region (with a high percentage of immigrant, indigenous and high social vulnerability students), 61 mathematics teachers of 7th grade (12 and 13 years old), 50 teachers of differential education and 2048 students. We will analyze the recordings in the classroom to characterize the collaboration between the differential teacher and the math teacher by promoting argumentation in the math class. We will do interviews and questionnaires to identify which elements make collaboration difficult and easier for promoting argumentation and perceptions of the roles of the teacher and differential teacher in the math class. We hope to discover the collaborative practices that promote equality in the math class.

## References

- NCSM & TODOS. (2016). Mathematics Education Through the Lens of Social Justice: Acknowledgment, Actions, and Accountability. Retrieved of <http://www.todos-math.org/>.
- Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum.

# ELEMENTS TO OBSERVE AND DISCERN GOOD EDUCATIONAL PRACTICES IN MATHEMATICS

Elisabeth Ramos-Rodríguez<sup>1</sup>, Pamela Reyes-Santander<sup>2</sup>, Jorge Soto-Andrade<sup>3</sup> and Rudolf vom Hofe<sup>2</sup>

<sup>1</sup>Pontificia Universidad Católica de Valparaíso, <sup>2</sup>Universität Bielefeld, <sup>3</sup>U. de Chile

Delving into the good educational practice construct (Ladel, & Kortenkamp, 2013) we intend to contribute with indicators to discern good educational practices in mathematics. We conducted a case study of an expert primary teacher analyzing the planning, implementation and evaluation of one of her lessons, using as categories the indicators of good practices in the classroom (Preiss et al., 2014). The results show the need to complement the initials indicators with at least the characteristics of the reflective teacher (Ramos-Rodríguez et al., 2017), since they allow us to observe outside the classroom (Schön, 1983), elements of good educational practices.

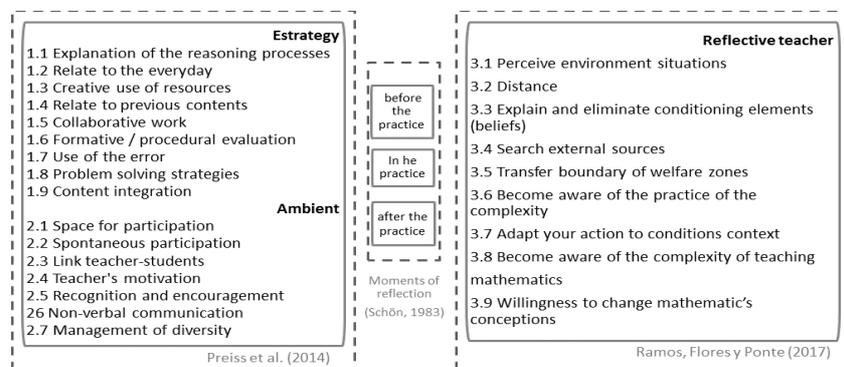


Figure 1: Indicators of good educational practices in mathematics

This model can serve as a basis for creating an instrument for observing good practices, in order to learn what to foster in teachers so as to improve their students' learning.

This study was funded by PUCV Project 039.315/2018; DAAD Project 57335022 and PIA-CONICYT Basal Funds for Centers of Excellence Project FB0003.

## References

- Ladel, S., & Kortenkamp, U. (2013). Designing a technology based learning environment for place value using artifact-centric Activity Theory. In A.M. Lindmaier, & Heinze, A. (Eds.), *Proceedings of PME 37*, Vol. 1 (pp. 188-192). Kiel, Germany: PME.
- Preiss, D. D., Calcagni, E., Espinoza, A. M., ..., & Volante, P. (2014). Buenas prácticas pedagógicas observadas en el aula de segundo ciclo básico en Chile. *Psykhé*, 23(2), 1-12.
- Ramos-Rodríguez, E., Flores, P., & Ponte, J.P. (2017). An approach to the notion of reflective teacher and its exemplification on mathematics education. *Systemic practice and action research*, 30(1), 85-102.
- Schön, D. (1983). *The Reflective Practitioner*. New York: Basic Books.

# DIVIDING EQUALLY – RESULTS OF A PILOT STUDY ON THE PRECONCEPTIONS OF GRADE 1 TO 2 CHILDREN IN CHILE

Pamela Reyes Santander & Andrea Peter-Koop

University of Bielefeld, Germany

Preconceptions, that is to say ideas held before formal instruction, have been the focus of research in science and mathematics education ever since Piaget's (1929) work. Current approaches to teaching and learning primary school mathematics all over the world frequently acknowledge that children bring important out-of-school knowledge to the classroom that influences their school-based learning.

Dividing equally is a basic concept for the understanding of division and fractions. Kieran (1993) found that being able to divide equally is central for the development of fractional number knowledge. International research in this area also suggests that children can solve sharing problems before the related mathematical operation and procedures are taught in school (e.g. see Pepper & Hunting, 1998).

The international research project underlying this poster presentation aims to explore young children's preconceptions of sharing /dividing equally on all five continents, analyzing data from Australia, Chile, China, Germany and Tanzania. Four tasks were presented to groups of children from these countries asking them to equally divide in situations that imply actions of quotitioning and partitioning with and without remainders. The problems were presented in reading and writing and were also supported by a drawing that the children could choose to elaborate on for their solution.

The sample presented here comprises 48 children from three classes (two Grade 1, one Grade 2) from three different primary schools in two Chilean cities. An initial data analysis shows that the majority of children could solve the two tasks without remainder (56 % for quotition, 66 % for partition) when using a drawing. Only 12 % (quotition) and 4 % (partition) of the children found the right answer without a correct drawing. When having to deal with remainders the success rate dropped (as expected) to 16 % (quotition) and 8 % (partition). In the poster, the four problems as well as typical solutions will be presented and further results will be discussed in detail.

## References

- Kieran, T. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of the research* (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Pepper, K., & Hunting, R. (1998). Preschoolers' counting and sharing. *Journal for Research in Mathematics Education*, 29(2), 164-183).
- Piaget, J. (1929). *The child's conception of the world*. New York: Harcourt Brace.

# CAN WE OBSERVE STUDENTS' MATHEMATICAL THINKING?

Jorge Soto-Andrade, Alexandra Yáñez-Aburto, Andres Beltrán-Gabrie

University of Chile, Mathematics Department, CIAE and PAB

Of course we can! This is the quick answer. However, metaphorically, what we do is like trying to study people's motricity by throwing balls at their heads and watching how they react. Then, only a narrow band of their motricity spectrum will be observed. Indeed, as researchers and teachers, unwittingly or unwillingly, we usually throw maths tasks at students' heads and we end up just watching their "defensive thinking". Most students – especially those not mathematically inclined – feel this is "cognitive aggression" (turning easily into "cognitive abuse"), although they do not usually voice it, as dramatically shown in our (secondary) student interviews.

According to our enactivistic approach however (Proulx & Maheux, 2017; Soto-Andrade, 2018), problems are not standing "out there" waiting to be solved, but we bring them forth, because of our historical structural coupling with the environment. Thus, we avoid giving the students tasks to be solved, and we focus instead on observing what kind of questions and problems emerge from them when engaged with just "situational seeds". Such a seed may be a frog jumping randomly on a row of stones in a pond. Students eventually focus on "impossible questions" like: Where will the frog be after a number of jumps, or in the long run? Depending on their previous history, idiosyncratic metaphorising emerges from them during group work: they see the frog splitting into pieces, or a fluid that drains on a network, a pack of deterministically jumping frogs, Doppelgängers of the frog popping up at each jump, etc. So in our view, mathematical thinking (Mason et al., 2010) is neither choosing the right tool in the box to solve a problem sitting "out there" nor just a defensive reaction to a challenge. It is a much more complex cognitive enactive process, hard to fathom, which unfolds over time with sky as the limit...

We gratefully acknowledge funding from: PIA-CONICYT Basal Funds for Centres of Excellence FB0003, FIDOP 2016-60PAB & DAAD Project 573 35022, D.

## References

- Mason, J., Burton, L., & Stacey, K. (2010). *Thinking Mathematically*. London: Pearson
- Proulx J., & Maheux J.-F. (2017). From problem solving to problem posing, and from strategies to laying down a path in solving: taking Varela's ideas to Mathematics Education Research, *Constructivistic Foundations*, 13(1), 161-167.
- Soto-Andrade, J. (2018). Enactive metaphorising in the learning of mathematics. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.). *Invited Lectures from the 13th Int. Congress on Mathematical Education* (pp. 619-638). Berlin: Springer.

# THE GAME AS A TEACHING RESOURCE IN MATHEMATICS FOR YOUTH AND ADULT EDUCATION

Stephanie Machado da Silva

Brenner Gomes Alvim

Matheus Procópio de Quadros

Juliana Campos Sabino de Souza

Instituto Federal de Brasília – IFB *Campus* Estrutural

"Adult education differs in many ways from the principles of teaching children and adolescents" (PAZIN, 2007, p.7). In this way, it is essential to know the specificities of this public to adapt the teaching methodology.

Games can be attractive to people of all ages. Cordeiro and Barcellos (2015, p.23) point out that "in any educational modality, and not unlike in the EJA (education for youth and adults), games are able to provide moments of relaxation without letting the principal goal, which is learning, get lost. "

In this way, the following research aims to analyze the relevance of using games as a didactic resource to the learning in math for young and adult students. To obtain such results, qualitative research was carried out, with an interview and a form application with two math professors from the EJA of a public school from Distrito Federal. A focal group was also carried with 6 students from the 6th and 7th grade of the EJA from the same school as the teachers that were interviewed.

It was possible to perceive that both, teachers and students, recognize the relevance of the use of games in the process of learning mathematics, demonstrating a great interest in the use of different teaching methodologies that seek to stimulate learning in mathematics. They pointed out that most students work and/or take care of the home and children, so they usually arrive at school on the night already tired from their daily journey. Using games in the classroom could help students get more excited and take away tiredness and sleep and provide some dynamism to school activities.

It is also worth mentioning the fact that the games are not infantilized. When working with games it is essential that the game is appropriate to the profile of the students that will play them, because depending on the game, they may not be stimulating or attractive to the students who will participate.

## References

- Pazin, A. (2007). Características do aprendizado do adulto. *Medicina, Ribeirão Preto, Simpósio: Didática: a aula teórica formal*.
- Cordeiro, K. M. S.; Barcellos, W. S. (2005). O uso de jogos pedagógicos na educação de jovens e adultos. *Revista Científica Interdisciplinar*, Nº 4, volume 2, artigo nº 20.

# PROBLEM-POSING STRATEGY TO STIMULATE THE DEVELOPMENT OF TEACHERS' DIDACTIC ANALYSIS COMPETENCE

Carlos Torres, Uldarico Malaspina

Pontificia Universidad Católica del Perú

Problem posing has long been recognized as a critically important intellectual activity in scientific investigation (Cai, Hwang, Jiang, & Silber, 2015). This importance has been reflected in the development of many empirical investigations. Among these, a prominent place is held by investigations where the focus of study is the teacher's didactic analysis competence in mathematical problem posing (PP) tasks. Also, PP tasks demand a person to expose his mathematical knowledge. However, if the posed problem is aimed at contributing to the student's knowledge – or more specifically, to understanding and solving other more complex problems – then the didactic-mathematical knowledge of the teacher must also intervene. This aspect is closely related to the teachers' didactic analysis competence, which has been studied within the onto-semiotic approach of cognition and mathematics instruction (OSA) (Breda, Pino-Fan, & Font, 2016).

In this research, we implement a problem-posing strategy to improve the teachers' didactic analysis competence. This strategy involves a phase to recognize the basic elements of a mathematical problem (e.g. information, requirement, context and mathematical environment). Moreover, we adopt the proposal related to problem posing, according to which problem posing is a process through which a new problem is obtained. If the new problem is obtained by modifying a given problem, it is said that the new problem was obtained by variation. For our research objective, we used a multiple case study with 16 in-service high school mathematics teachers who participated in a problem-posing workshop. Our study is exploratory, descriptive and analytical, taking as unit of analysis the problems posed by the teachers participating in the workshop. We analyze these problems using OSA tools. As a consequence of this implementation, we have evidence to state that our strategy could help to improve in-service teachers' didactic analysis competence

## References

- Breda, A., Pino-Fan, L., & Font, V. (2016). Establishing criteria for teachers' reflection on their own practices. In Csíkós, C., Rausch, A., & Sztányi, J. (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 283). Szeged, Hungary: PME.
- Cai, J., Hwang, S., Jiang, C, & Silber, S. (2015). Problem-posing research in mathematics education: some answered and unanswered questions. In F. M. Singer, N. F. Ellerton, & J. Cai (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 141-174). New York, NY: Springer.

# EVOLUTION IN THE DESIGN OF SCHOOL MATHEMATICAL TASKS FOR TRAINING TEACHERS

Macarena Valenzuela Molina<sup>1, 2</sup> and Elisabeth Ramos Rodríguez<sup>1</sup>

Pontificia Universidad Católica de Valparaíso<sup>1</sup>, Universidad Alberto Hurtado<sup>2</sup>

Investigations (Blömeke, Suhl, & Kaiser, 2011) show shortcomings in math and didactic knowledge from initial teacher training in Chile. This knowledge considers the design of school mathematics tasks (SMT). In this context, our objective is to study the evolution in the design of SMTs, and their cognitive demand for training teachers. Our theoretical support starts from the concept of tasks, understood as teaching proposals, that the student responds to with an action to achieve learning. We consider cognitive demand according to Stein and Smith (1989), see Table 1.

Cognitive demand levels	Type
Low cognitive demand	Memorization or Process without connections
High cognitive demand	Process with connections or Doing mathematics

Table 1: Cognitive demand levels (Stein, & Smith, 1998)

We consider two case studies, students from the 7<sup>th</sup> semester in university in the Basic Education program, and analyzed their writings in different moments, where they designed and reformulated SMTs. The study used content analysis, whose categories corresponded to the levels of cognitive demand observed in the SMTs. The results reveal that both cases show changes, from the initial presentation of a low demand SMT to a high demand SMT. Table 2 illustrates an example.

Initial TME statement	Final TME statement
Problem presentation: each student will be asked to represent the following fractions in their notebook: $\frac{1}{4}$ , $\frac{1}{3}$ , $\frac{1}{2}$ , $\frac{2}{3}$ .	Let's know the fractions: They will be given the fractional circles, through questions like: What do you think happens if we take all the $\frac{1}{3}$ and put them together? What did you observe? What happens if we take the means and put them together? What are we forming? Why?

Table 2: Example of TME statement proposed by one of the groups.

This helps us to understand the way in which they select and design appropriate SMT according to the context and the factors that interfere in its design. From the above we must make decisions regarding the initial training of teachers.

## References

- Blömeke, S., Suhl, U., & Kaiser, G. (2011). Teacher education effectiveness: Quality and equity of future primary teachers' mathematics and mathematics pedagogical content knowledge. *Journal of Teacher Education*, 62, 154-171.
- Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3(4), 268-275.



