PME44 VIRTUAL

THE 44th CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

July 19-22, 2021

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Proceedings

of the 44th Conference of the International Group for the Psychology of Mathematics Education

VOLUME 3

Research Reports (H-R)

Editors:

Maitree Inprasitha, Narumon Changsri and Nisakorn Boonsena

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PREFACE

We are pleased to welcome you to PME 44. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. The PME 44 Virtual Conference is hosted by Khon Kaen University and technically assisted by Technion Israel Institute of Technology. The COVID-19 pandemic made massive changes in countries' economic, political, transport, communication, and education environment including the 44th PME Conference which was postponed from 2020. The PME International Committee / Board of Trustees decided against an on-site conference in 2021, in accordance with the Thailand team of PME 44 will therefore go completely online, hosted by the Technion - Israel Institute of Technology, Israel, and takes place by July 19-22, 2021. A national presentation of PME-related activities in Thailand is part of the conference program.

This is the first time such a conference is being held in Thailand together with CLMV (Cambodia, Laos, Myanmar, Vietnam) countries, where mathematics education is underrepresented in the community. Hence, this conference will provide chances to facilitate the activities and network associated with mathematics education in the region. Besides, we all know this pandemic has made significant impacts on every aspect of life and provides challenges for society, but the research production should not be stopped, and these studies needed an avenue for public presentation. In this line of reasoning, we have hosted the IGPME annual meetings for the consecutive year, July 21 to 22, 2020, and 19 to 22 July 2021, respectively by halting "on-site" activities and shift to a new paradigm that is fully online. Therefore, we would like to thank you for your support and opportunity were given to us twice.

"Mathematics Education in the 4th Industrial Revolution: Thinking Skills for the Future" has been chosen as the theme of the conference, which is very timely for this era. The theme offers opportunities to reflect on the importance of thinking skills using AI and Big Data as promoted by APEC to accelerate our movement for regional reform in education under the 4th industrial revolution. Computational Thinking and Statistical Thinking skills are the two essential competencies for Digital Society. For example, Computational Thinking is related to using AI and coding while Statistical Thinking is related to using Big Data. Therefore, Computational Thinking is mostly associated with computer science, and Statistical Thinking is mostly associated with statistics and probability on academic subjects. However, the way of thinking is not limited to be used in specific academic subjects such as informatics at the senior secondary school level but used in daily life.

For the PME 44 Thailand 2021, we have 661 participants from 55 different countries. We are particularly proud of broadening the base of participation in mathematics education research across the globe. The papers in the four proceedings are organized according to the type of presentation. Volume 1 contains the presentation of our Plenary Lectures, Plenary Panel, Working Group, the Seminar, National Presentation, the Oral Communication presentations, the Poster Presentations, the Colloquium. Volume 2 contains the Research Reports (A-G). Volume 3 contains Research Reports (H-R), and Volume 4 contains Research Reports (S-Z).

The organization of PME 44 is a collaborative effort involving staff of Center for Research in Mathematics Education (CRME), Centre of Excellence in Mathematics (CEM), Thailand

Society of Mathematics Education (TSMEd), Institute for Research and Development in Teaching Profession (IRDTP) for ASEAN Khon Kaen University, The Educational Foundation for Development of Thinking Skills (EDTS) and The Institute for the Promotion of Teaching Science and Technology (IPST). Moreover, all the members of the Local Organizing Committee are also supported by the International Program Committee. I acknowledge the support of all involved in making the conference possible. I thank each and every one of them for their efforts. Finally, I thank PME 44 participants for their contributions to this conference.

Thank you

Best regards

M. Inprasitha

Associate Professor Dr. Maitree Inprasitha
PME 44 the Year 2021
Conference Chair

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RESEARCH REPORTS (H-R)

ENGAGING STUDENTS WITH ONLINE ELABORATED FEEDBACK: TWO MEDIATING TOOLS

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Automatic feedback provided to students' work has the potential to improve their mathematical thinking and skills. A commonly used type of online feedback is elaborated feedback. Engaging students with elaborated feedback is a well-known challenge, which we addressed by examining two mediating tools that are part of a dynamic diagram environment: self-assessment, and interactive feedback. We focused on the students' process of constructing position-over-time graphs, while they were working on an online example-eliciting task on the Seeing the Entire Picture platform. We present and compare two case studies of students' use of the elaborated feedback, engaging both mediating tools, and analyze changes in the students' examples after processing the elaborated feedback.

INTRODUCTION

Elaborated feedback

Carless refers to educational feedback as "a dialogic process in which learners make sense of information from varied sources and use it to enhance the quality of their work or learning strategies" (2015, p. 192). Researchers distinguish between two main types of feedback: verification and elaborated. While verification feedback usually aims to ascertain the correctness of information, the elaborated feedback is provided to stimulate an explanatory dialog regarding the responsiveness of the answer. In the current research, we focused on "attribute isolation elaborated feedback," which prompts the dialog based on mathematical characteristics of the objects and processes involved in a task. Provided observations are important regardless of whether the referred attributes meet the requirements of the task or not. Elaborated feedback provides opportunities for higher-order learning processes in mathematics (Van der Kleij, Feskens, and Eggen, 2015). At the same time, the learning that involves elaborated feedback is considered to be more complex, and therefore more challenging, as to engaging students in the self-assessment process (Shute and Rahimi, 2017).

Interactive diagrams

We used interactive diagrams as a digital learning environment. Interactive diagrams include an initial example, its representations, and tools to manipulate it. Thus, interactive diagrams have the potential to engage students in the inquiry process (Mariotti, 2006). The illustrating interactive diagram is one of three types of interactive diagrams (Naftaliev and Yerushalmy, 2017). It usually consists of a single representation of a sketch example and it doesn't offer links between representations. Thus, an illustrating interactive diagram by itself offers fewer opportunities for engaging students into inquiry. There is a wide range of problems and learning situations, however, that benefit from the use of this type of interactive diagrams. For example, constructing graphs by direct manipulation of a given sketch. For the current research, we used

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the Seeing the Entire Picture (STEP) platform as an interactive environment that supports example-eliciting tasks and provides automatic elaborated feedback (Olsher, Yerushalmy, and Chazan, 2016). In example-eliciting tasks, students are asked to construct examples in a multiple linked representation (MLR) environment to support their answers. Eliciting examples is a vital element in the reasoning processes. Elicited examples or learner-generated example space may also be indicative of the students' mathematical reasoning (Zaslavsky and Zodik, 2014).

We address the challenge of engagement by creating a problem-solving digital environment that is based on illustrating interactive diagram construction tasks and involves inquiry with elaborated feedback. We observed the work of students by exploring their work with two mediating tools. The first is a self-assessment tool. The self-assessment process has the potential to enhance student learning (Stacey and Wiliam, 2013). We used self-assessment in which following the task submission, the students are asked to evaluate their work. The second tool dynamically provides interactive feedback, which, upon the construction of the required graph, verbally makes observations on the requirements of the task. By doing that, the interactive feedback acts as a parallel verbal representation. We studied the problem-solving process with each tool and sought to identify engagements with the automated elaborated feedback that grew out of the experience of the tools. We asked: how did the students use the elaborated feedback that follows engagement with each mediating tool? And were there any changes in the students' examples inspired by the elaborated feedback following the use of each mediating tool?

METHODOLOGICAL CONSIDERATIONS

The current study is part of a larger study in which we explore the use of online elaborated feedback by students and describe a small-scale experiment that enables qualitative analysis. We present two case studies involving 10th-grade math students at an advanced level. Each student carried out the same activity within the framework of a task-based interview. The activity contained two example-eliciting tasks in which the students were asked to construct position-over-time graphs. Each task required to construct and submit three different examples that meet specific given conditions. Asking the students to create three examples as different as possible aims to achieve a diverse personal example space. The submissions were automatically assessed and analyzed with STEP, to produce post-submission elaborated feedback for the students (hereafter, post-feedback) (Olsher et al., 2016). Once the students submitted their examples to the first task, they received the post-feedback from STEP. After processing the post-feedback, they proceeded to the second task. The post-feedback consisted of a list of task requirements and a list of mathematical characteristics of the task (see Figures 5 and 7). The lists were prepared in advance, as part of the task design. STEP can analyze the submitted work and mark the identified characteristic of the submitted example. By that STEP automatically produced a post-feedback. Figure 1 and 2 present the study procedure and a screenshot of task 1

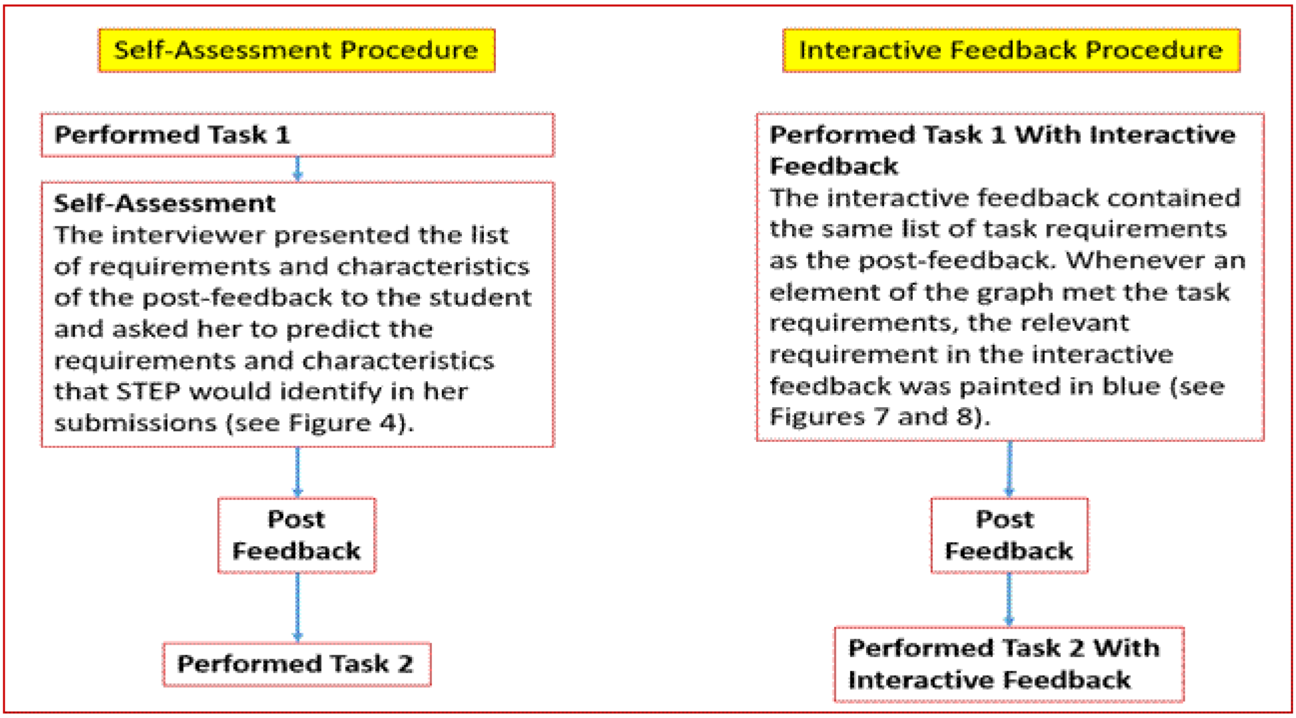

Figure 1: Study procedure

Task 2 contained the same instructions but instead of 4 segments of riding, task 2 included 2 segments. The use of two almost identical tasks allowed us to analyze the changes as a result of post-feedback. The data were collected through the STEP platform, and the students' work was also recorded, to check against the elements collected automatically. We analyzed the segments in the interviews where the students used the mediating tools and the post-feedback. To detect the effect of the post-feedback, we analyzed the changes in the personal example space and in meeting the task requirements in the first and the second task submissions.

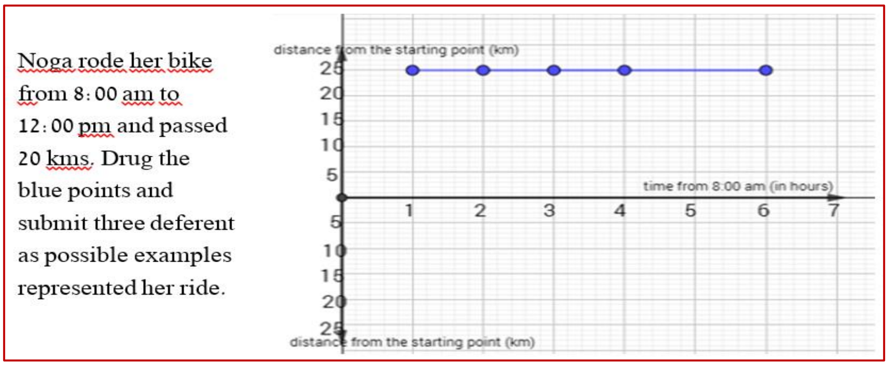

Figure 2: Task1

Task 2 contained the same instructions but instead of 4 segments of riding, task 2 included 2 segments. The use of two almost identical tasks allowed us to analyze the changes as a result of post-feedback. The data were collected through the STEP platform, and the students' work was also recorded, to check against the elements collected automatically. We analyzed the segments in the interviews where the students used the mediating tools and the post-feedback. To detect the effect of the post-feedback, we analyzed the changes in the personal example space and in meeting the task requirements in the first and the second task submissions.

FINDINGS

Self-assessment – Ela: After reading the instructions, Ela constructed three examples. While doing so, she explained that the left blue point on the graph represents the first location Noga reached. Later, she submitted three examples represented in Figure 3.

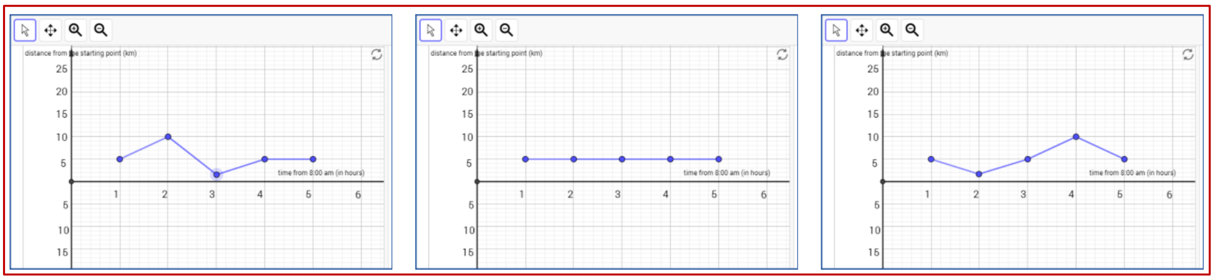

Figure 3: The three examples Ela constructed

Had the left point indeed represented Noga's first location, all the examples Ela submitted would have met the task requirements. Yet, according to her interpretation, this was not the case. After submission, the interviewer gave Ela a page represented in figure 4 and asked her to mark all the characteristics she predicts STEP would identify in her examples.

Task requirements	**Task requirements**	**Task requirements**
Overall time meet the task requirements	Overall time meet the task requirements ✓	Overall time meet the task requirements ✓
Overall distance meet the task requirements	Overall distance meet the task requirements ✓	Overall distance meet the task requirements ×
Starting point meet the task requirements	Starting point meet the task requirements ×	Starting point meet the task requirements ×
End point meet the task requirements	End point meet the task requirements ×	End point meet the task requirements ×
Example characteristics	**Example characteristics**	**Example characteristics**
Noga changed direction at least once	Noga changed direction at least once ×	Noga changed direction at least once ×
Noga stopped at least once	Noga stopped at least once ×	Noga stopped at least once ×
Noga finished riding in the city from which she left	Noga finished riding in the city from which she left ?	Noga finished riding in the city from which she left ?
Noga rode at different speeds	Noga rode at different speeds ×	Noga rode at different speeds ✓
Noga passed through the city from which she left	Noga passed through the city from which she left ×	Noga passed through the city from which she left ×

Figure 4: Ela's markings

Figure 4 shows that Ela marked correctly that the starting point and the end point of the right and the middle examples are not in the correct position. Namely, during the self-assessment, she realized that she had placed the first and last points in the graph incorrectly. Ela also identified whether her examples represent situations in which Noga stopped or passed through the city from which she started. Regarding other characteristics, Ela marked incorrectly whether or not those characteristics existed in her examples. Furthermore, regarding one characteristic Ela mentioned that she did not know whether it existed in her examples. Note that from the marking, we can obtain information about the knowledge constructs Ela created during the self-assessment process, and about those that she has not achieved yet. Figure 5 presents a screenshot of the post-feedback Ela received following her self-reflection.

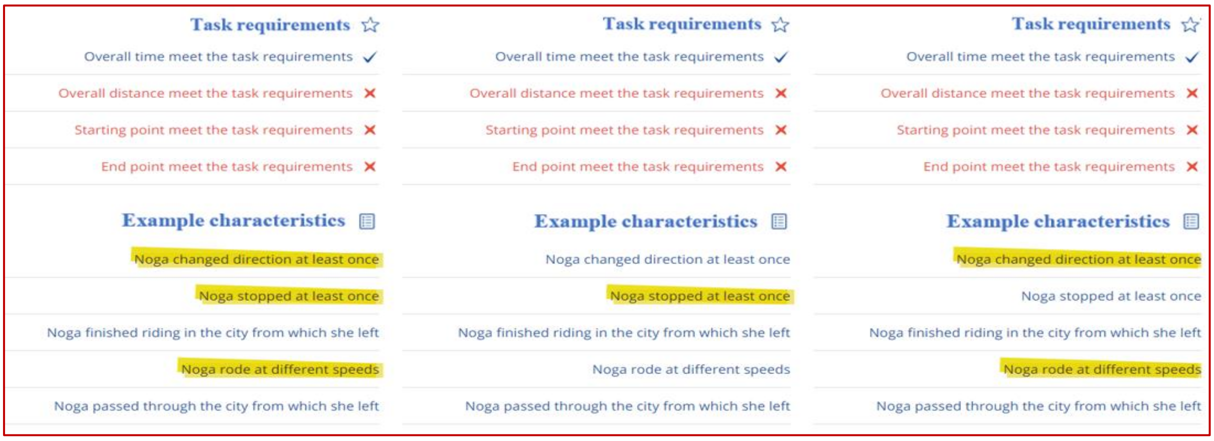

Figure 5: The post-feedback Ela received

Below are some of Ela's reactions to the feedback:

1	E:	(Focused on the middle drawing) Got it! I didn’t address the distance from the starting point.
2	I:	What made you understand this?
3	E:	That I saw I stayed at the same point (middle drawing). The computer told me it was a mistake. So I thought about it one more time.
4	E:	Now I also know that in the first hour she has reached a distance of five kilometres and then she can go back and reach a distance of zero.
5	I:	What does it mean that she came back to the city which she left?
6	E:	That she reached the point from which she left. In my drawing, she didn’t return. She drove five kilometres and stayed there.

In line 1 Ela realized that she addressed the starting point incorrectly. In line 3 Ela explained that the post-feedback helped her realize it. In line 6 Ela was able to identify the characteristic which she didn't know how to refer to during the self-assessment process (Noga ended up in the city from which she departed). Thus, Ela was assisted by the post-feedback in her learning process. Ela proceeded to the second task. Figure 6 presents the submission she made:

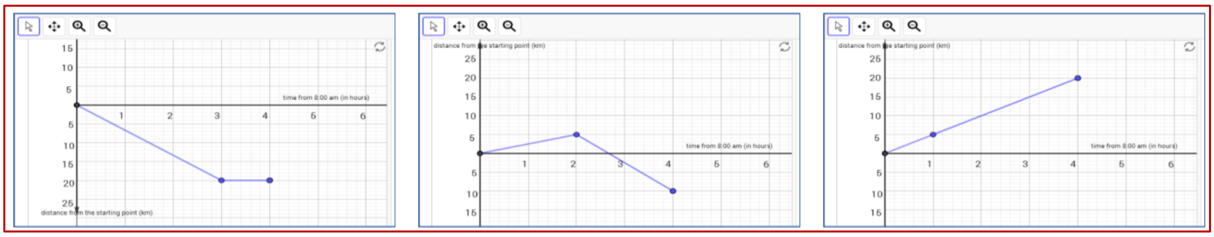

Figure 6: Ela’s submission for the second task

All the examples Ela submitted in the second task met the task requirements. We can also observe the great diversity of Ela's examples in which four of the five characteristics included in the post-feedback existed (Noga changed direction, stopped, progressed at different speeds, passed through the city from which she left).

Interactive feedback – Anna: After reading the task instructions, Anna addressed the interactive feedback and read the first mathematical characteristic in the list: "starting point." She reacted as follows: "Okay, so this should start from here (pointed at the first point of the graph)." Anna dragged the first point to the origin and noticed that the first characteristic was painted in blue. Then, she completed the construction of the example by placing the second point, the third point, and finally the fourth point. Doing so, Anna constructed a graph that meets the task requirements (see the example on the right in Figure 7) and all the characteristics in the interactive feedback turned blue. It seems that Anna used the interactive feedback to find a strategy for constructing the graph by placing the first point and to proceed according to the order of the points. Figure 7 shows a screenshot of Anna's submission and the online post-feedback:

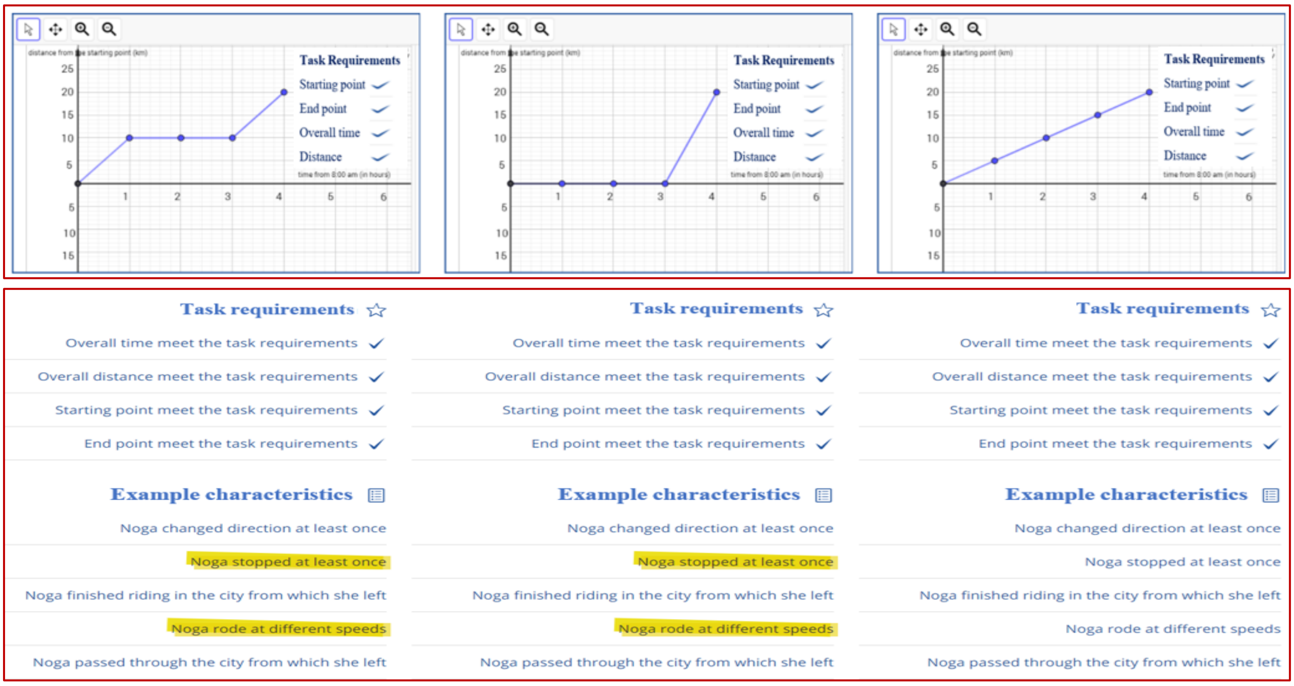

Figure 7: Anna's submission and elaborated feedback

Unlike Ela's first submission, Anna’s submitted examples met the task requirements. According to Figure 7, Anna's examples were not varied and included only one direction riding. Below are some of Anna's reactions to the feedback:

2 A: What does it mean that she passed through the city from which she left and she ended in the city from which she left? Did she drive in circles?

3 I: I cannot answer you. You can choose whether you want to crack it or move on.

4 A: It can be done in the negative direction.

5 I: What made you think of this?

6 A: I thought only of the positive direction and then in the feedback it’s written that Noga changed the direction. So that gave me the idea.

In line 1 Anna mentioned characteristics she didn't understand in the post-feedback. As a result of the interactive feedback, however, at this point, Anna had to deal only with the part of the post-feedback that relates to characteristics. In line 4 Anna expressed a new idea ("negative direction"). In line 6 she explained that she got the new idea from the post-feedback. Figure 8 shows a screenshot of her submission to the second task:

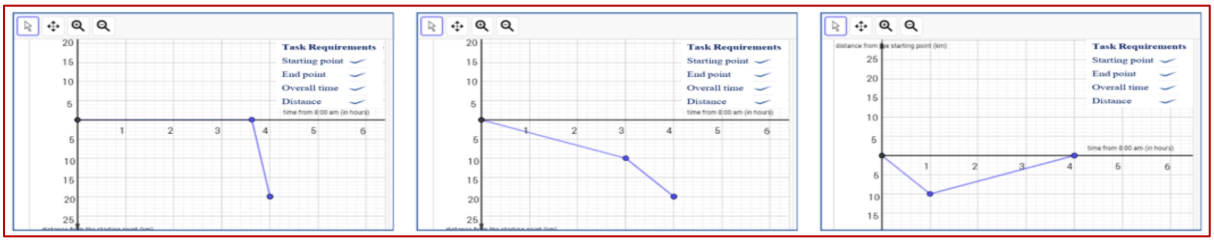

Figure 8: Anna's second task submission

Anna indeed constructed graphs that are below the X-axis as she had planned. Moreover, unlike her first submission, the second one was very varied and included four of the five characteristics from the post-feedback (Noga changed direction, stopped, progressed at different speeds, and finished in the city from which she left). Thus, Anna used the post-feedback to diversify her examples.

DISCUSSION AND CONCLUSIONS

The aim of the present research was to examine two mediating tools for engaging students through the post-feedback on their work in a dynamic diagram environment: self-assessment, and interactive feedback. To this end, we explored the students' use of the post-feedback with each mediating tool, and the changes in the students' examples as a result of the provided post-feedback. Ela’s self-assessment process helped her compare her predictions and the results of the post-feedback. As a consequence, she was able to identify the characteristics in the post-feedback she misunderstood. Namely, she used the post-feedback that followed the self-assessment to deepen the learning process. Following the post-feedback, Ela improved the correctness and diversity of her examples. The interactive feedback helped Anna construct examples that met the task requirements from the first submission. After receiving the post-feedback, Anna focused on a subset of the characteristics and mentioned that the list of characteristics gave her new ideas. We suggest that due to the interactive feedback Anna could use the post-feedback to diversify her examples. According to the students’ use of the post-feedback and according to the changes in the students' examples, we suggest that the mediating tools promoted the students' engagement with the post-feedback.

Shute and Rahimi (2017) noted that complexity is one of the reasons for students' difficulties in engaging with elaborated feedback. Both students indeed had difficulty in understanding the characteristics included in the post-feedback, but in the end, they were very much engaged. Ela’s self-assessment process led her to gradually address characteristics she did not understand. The self-assessment process may have helped her with the post-feedback complexity. The interactive feedback helped Anna focus on the part of the characteristics in the post-feedback, which may have helped her overcome the complexity of the post-feedback.

Studies have noted the benefit of using self-assessment for the active involvement of students (Stacey and Wiliam, 2013). According to our results, the interactive feedback was found to be effective as well. To compare the characteristics of the tools, we examined them separately. Combining these tools will likely make it possible to significantly promote the use of elaborated feedback by the students. This of course requires further research that may be designed on the basis of the current research.

Acknowledgment

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CONCEPTUALIZATION PROCESSES OF 6TH GRADERS FOR ROTATIONAL SYMMETRY

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The aim of our project is to explore students' conceptions of rotational symmetry at the beginning of secondary school. Thereby, we are interested in how children build up a concept of rotational symmetry working in a concrete learning environment. In the sense of subject didactic development research we performed designed experiments and explored the learner's conceptual ideas about rotational symmetry. This paper presents the concept of the learning environment and first reconstructive results based on student conversations using Steinbring's epistemological triangle as an interpretative method. We succeed in identifying central aspects of the concept of rotational symmetry in different reference contexts.

RATIONALE

Symmetry is an important concept in mathematics and a subject matter from elementary school to university (Bornstein & Stiles-Davis, 1984). Although it can be defined clearly, we don't know much about the evolvement of students' conceptual understanding. At the same time, learning environments are used in mathematics classes to initiate an idea of rotational symmetry. But are they helpful to build up a fundamental conception? We answer both questions with the help of a design-based research project (Nührenbörger et al., 2016), which is characterized by the fundamental aspects: (1) designing learning environments for classrooms and (2) investigating the initiated learning processes and contributing to local theories. So, we ask how learning environments for rotational symmetry can be designed and which concepts students develop. Prediger, Schnell and Rösike (2016) identified four intertwined working areas in a design-based project, which are connected and passed several times: a) specifying and structuring goals and contents of the learning environment, b) developing the design, c) conducting and analyzing design experiments and d) developing contributions to local theories. We structure our paper according to these working areas.

a) Specifying the Content: Rotational Symmetry

A figure is called symmetric, if apart from the identity, there exists at least one other mapping which maps the figure onto itself. Each of these mappings beside the identity is called "a symmetry" of the figure. In this sense a rotationally symmetric figure as a whole (the hexagon) can be mapped onto itself by a suitable rotation: mapping a figure onto itself by rotating it (figure 1). In a second perspective it is possible to identify an elementary figure (triangle) from which the whole figure can be built by suitable turns: using an elementary figure to build a symmetric figure. This elementary figure can then be identified in the overall figure as a repeating partial figure (Leuders, 2016).

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 9-15. Khon Kaen, Thailand: PME.

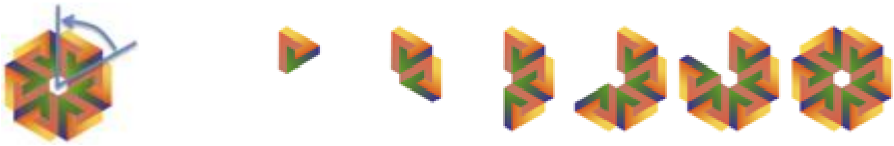


Figure SEQ Figure * ARABIC 1: Different perspectives of rotational symmetry (Leuders, 2016)

In mathematics education research we know a lot about students' difficulties with axial symmetry and their conceptualization processes in primary and partially in secondary school (Bornstein & Stiles-Davis, 1984; Ramful, Ho & Lowrie, 2015). In contrast, rotational symmetry is frequently only an additional part when researching students' difficulties and understanding of symmetry, but rarely the exclusive object of research. This also applies for the findings of the following studies and makes a research desideratum visible. Aktas & Ünlü (2017) identify students' problems in determining the rotation angle and the direction of rotation. This goes along with Küchemann's (1981) research who classifies children's understanding of reflections and rotations into a number of levels. More recent research findings in the area of (rotational) symmetry tend to examine the differences in the use of digital and traditional media (e.g. Chan, Leung, & Ong, 2017). Regarding research on the conceptualization processes of rotational symmetry Seah & Horne (2019) can be mentioned. They deduced that students' knowledge is fragmented as well as that their reasoning ability is poor and more research is needed to describe the underlying learning processes of rotational symmetry. It can be stated, that there is a need for research regarding the design of learning environments and especially students' conceptualization processes while learning the concept of rotational symmetry. This leads to the following research questions:

- How can a learning environment for rotational symmetry be designed, which allows students to develop an understanding of the concept?
- Which conceptualization processes regarding the concept of rotational symmetry can be reconstructed among learners of grade six?
- Which reference contexts (in the sense of Steinbring) are used by the learners to develop a concept of rotational symmetry?

b) Developing the Design

The learning environment is aimed at children of grades five and six in Germany (age 10 to 11) according to the German curriculum. In order to stimulate a comprehensive understanding of rotational symmetry, we have designed the learning environment according to the following principles:

Action-oriented approach: Due to the age of the children and the fact that they deal with rotational symmetry the first time, a constructive and action-oriented approach is chosen explicitly. Children are given lots of opportunities to explore figures, to move and to rotate them and to construct a figure out of elementary figures, which are repeatedly rotated.

Integrated conception: The environment takes both concepts of rotational symmetry into account: a) using an elementary figure to build a symmetric figure and b) mapping a figure onto itself by rotating it. Within the learning environment we worked at the beginning with rotationally symmetric figures, so-called "windmills" (see figure 2 for some examples, Götze & Spiegel, 2004).

Cooperative realization: Verbalizing, explaining, defending, asking and arguing are key activities for productive learning processes (Dekker & Elshout-Mohr, 1998). Cooperative learning situations seem to be suitable for such key activities to emerge. Regarding these design principles the following tasks are offered after a short introduction to groups of children.

Creating rotational symmetric figures

In this part of the learning environment the focus is on action as well as on the perspective of an "elementary figure" of rotational symmetry (figure 1). Students work in groups of four and everyone gets a geoboard and a workbook on their own. They receive wings of various windmills represented on transparencies which they have to share. So, the cooperation between the students is fostered on a natural way. They fix the wing at the centre of their geoboards, rotate the wing and stretch the figure with the elastics on the geoboard (see figure 2). By documenting the whole windmills in the workbook, the students change the perspective from an "elementary figure" to the entire rotational symmetric figure. According to their competences and speed the children stretch and draw a different number of figures.

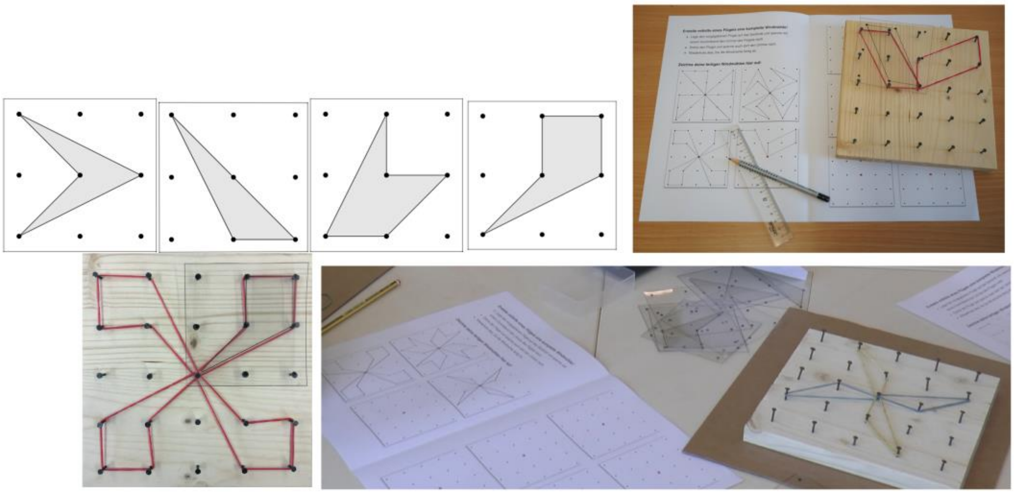


Figure 2: Examples of wings of windmills and stretched windmills on a geoboard

Distinguishing rotationally symmetric and non-rotationally symmetric figures

A larger number of figures (rotationally symmetric and not rotationally symmetric) are given to each group table. Students were asked to judge if a figure is rotationally symmetric or not (figure 3). All figures allow to identify an elementary figure, but they are arranged in a way, that some are axisymmetric, rotationally symmetric, or not symmetric at all. So, it is not enough to focus on the identification of the elementary figure, but the figure as a whole must be considered. The students are explicitly encouraged to work together, to discuss and to verbalize their judgements. In doing so, they can consider the figure as a whole, rotate it mentally or stretch it on the geoboard and rotate the board to solve the task.

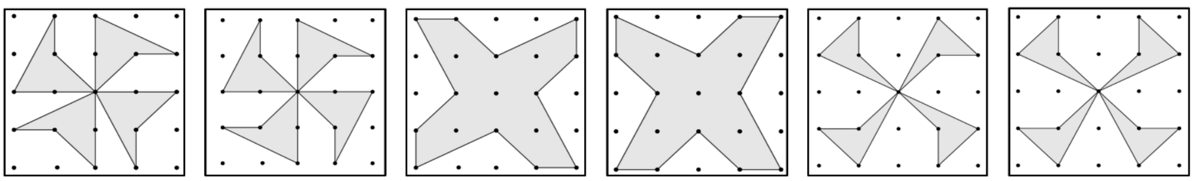


Figure 3: Examples of figures to sort

c) Conducting and Analyzing the Design Experiments

The design experiments were carried out in the teaching and learning laboratory at Paderborn University. The classes worked in the learning environment for 120 minutes. The groups were put together by the teacher, so that within the group children with different competencies work together. In this way, a productive discourse about different approaches and concepts may come up, which allows to reconstruct students' conceptual understanding.

In order to answer our research questions about the reconstructable concept of rotational symmetry, a qualitative approach is necessary to analyze students' actions and statements. To characterize the conceptual understanding, we use the epistemological theory of Steinbring (2005). Steinbring explains that mathematics, as the knowledge of abstract relations, is neither directly accessible by communication nor by the use of signs. Students have to interpret mathematical signs and develop a meaning on their own. This meaning has to be produced by means of establishing a mediation to appropriate reference contexts or objects. "The triangular connecting scheme between the mathematical signs, the reference contexts, and the mediation between signs and reference contexts, which is influenced by the epistemological conditions of mathematical knowledge, can be represented in the epistemological triangle" (Steinbring, 2005, p. 22).

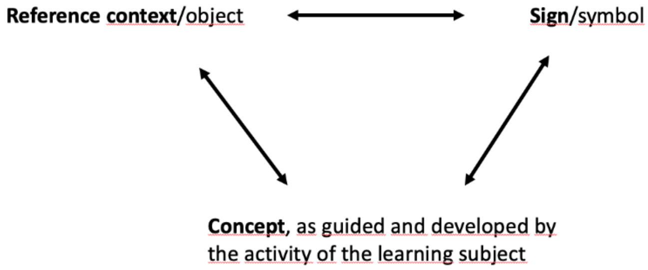


Figure 4: Steinbring's epistemological triangle

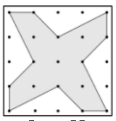
Thus, the epistemological triangle helps to reconstruct the connection between the concrete signs (e.g. symmetrical figures), the verbalized concept (of rotational symmetry) and the chosen reference contexts.

This mutual conceptualization process can be reconstructed by interpreting the statements and actions of the students (Steinbring, 2005). Therefore, the groups were videographed and corresponding transcripts were interpreted by a group of researchers (Krummheuer & Naujok, 1999). The aim of the interpretation process consists of reconstructing the ongoing development of the knowledge and characterizing typical (pre)concepts of rotational symmetry as local theories (Prediger, Schnell & Rösike, 2016).

Case: Sorting of symmetric and non-symmetric figures

The episode takes place during the second activity. The students have already sorted some of the windmills when the teacher joins the group.

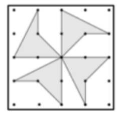
1 Teacher (T):	And how did you sort that out now? Now how did you know it was a windmill?
2 Oskar:	So, if that, so if this is exactly the same on the other side, because then it can be better (silenced) …
3 T:	Mhhh.
4 Noel:	But that is not a windmill (*pointing to windmill 1, quiet*).



windmill 1

5 T: You mean this isn’t a windmill. What do the others say?

6 Julian: Can I have a look?



“windmill” 2

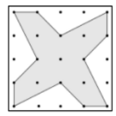
7 Oskar: gives the card with windmill 1 to Noel

8 Robert: What's the matter with this one (*pointing to “windmill” 2*)?

9 Oskar: That’s not a windmill, because on the opposite side, it isn’t the same.

10 Julian: (*speaks to Noel and stretches out his hand*) Show! Please! (*takes the card (windmill 1) in his hand, turns the card clockwise*) This is one.

11 T: Are we gonna say this is one or this isn’t one?



windmill 1

12 Julian: Because, if you hold in this position, then this is the same as when it rotates.

The discourse can be divided into two parts, where two reference contexts of rotational symmetry can be reconstructed: a) “parts are the same on the other side”, b) “windmills can be rotated”. Oskar chooses a figure sorted as a windmill and tries an explanation (line 2). He thus selects one concrete object, windmill 1, as his sign to explain the abstract concept of a windmill (rotational symmetry). It is not quite clear what he means with his reference to "exactly the same" and to "the other side". It could be that he possibly means a horizontal, vertical or diagonal axis of symmetry that divides the figure into two subfigures and probably refers to axial symmetry. These two partial figures would then be “on the other side” and congruent to each other. However, the assumed explanation referring to axial symmetry does not fit exactly to his example, because windmill 1 is a rotationally symmetric figure consisting of four partial figures and not an axial-symmetric figure, so that his formulations "on the other side" and "exactly the same" probably have to have different meanings. With the expression "the other side" Oskar could also mean the diagonally opposite partial wings. "Exactly the same" could then refer to the individual partial figures in the sense of congruence. Probably, Oskar’s strategy is to search for the “elementary figure” of the rotational symmetric figure. Answering Robert’s question in line 8, Oskar carries his idea forward, but now uses a new figure, a second sign, (“windmill” 2) as a counterexample. He seems to want to check the reference context he invokes on a second example, but even at this point it remains unclear what he exactly means by "on the other side". Similar to line 2, the wings could be meant diagonally opposite each other. "The same" could be an allusion to an even distance between the wings or, as assumed in line 2, it could point to congruent wings. His concept of rotational symmetry seems to be based on the idea of corresponding parts.

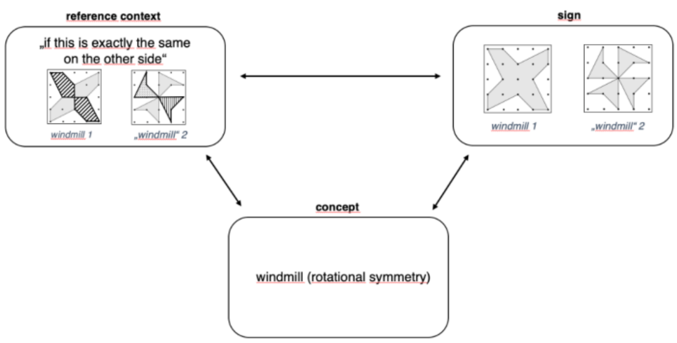


Figure 5: Epistemological triangle of Oskar’s concept

Julian comes up with a different interpretation and a different reference. He turns the windmill clockwise and decides "This is one" (line 10-12). Julian verbalizes his decision. If you "hold it in this position, this is the same as when it rotates" (line 12). He uses the same sign, but a different reference to develop his idea of rotational symmetry. In his statement, it is not entirely clear whether he is looking at the entire figure or focuses on individual parts as his classmates. But since the individual figure elements are shown in a different position when the figure is rotated, it can be assumed that it refers to the entire figure.

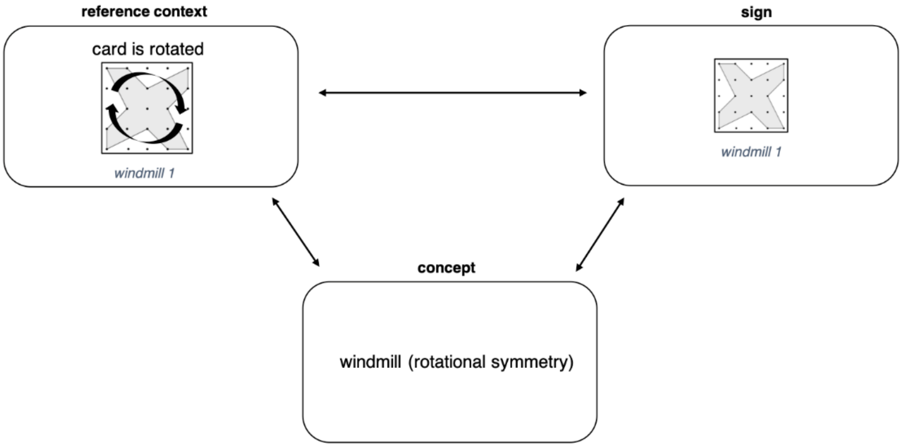


Figure 6: Epistemological triangle of Julian's concept

d) First Results contributing to Local Theories

The analysis of this and further episodes shows that students in guided discourse succeed in addressing central aspects of the concept of rotational symmetry using various reference contexts. The data analysis using Steinbring's epistemological triangle let us succeed in reconstructing two different reference contexts for the (pre)concept of rotational symmetry as local theories: a) "parts are the same on the other side", b) "windmills can be rotated", whereas b) represents the concept to be built of mapping a figure onto itself by rotating. The concept regarding the reference context a) is not sufficient from a mathematical point of view, because the idea behind holds for axis-symmetry as well. Regarding the examination of used materials and methods of the learning environment, it can be supposed that the activity of creating rotational symmetric figures encourages these concepts. The repeated depiction of a wing might strengthen the reference context of "the same on the other side" while the rotation seems to be less important. So, for the next design circle we change the order of the activities. Additionally, we give transparencies of the whole figures to the students to validate their decisions by rotating the transparency on the geoboard.

CONCLUSION AND PERSPECTIVES

In summary, it can be stated that students use the identification of an elementary figure to build a symmetric figure as well as mapping a figure onto itself to identify rotational symmetric figures in their conversations. Thus, both perspectives from part a) of rotational symmetry kicks in. But it remains still questionable in how far students are able to distinguish between the different concepts of axial and rotational symmetry, when they mainly focus on the elementary figure.

In further analyses, the development of selected group conversations over a longer period of time is examined in order to be able to describe decision-making processes over their entire length. We will try to describe conceptualization processes for the concept of rotational symmetry which go beyond individual group discussions. Ideally, these results can be abductively condensed into theories concerning the conceptualization processes of learners to the concept of rotational symmetry.

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A TEACHER'S CONCEPTUALIZATION OF THE DISTRIBUTIVE PROPERTY FOR CONTINUOUS (FRACTIONAL) UNITS

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In this case study of a grade-4 teacher, we address the question: How might a teacher conceptualize fractional units as multiplicative relations (measures) while operating on continuous quantities? We analyze two video-recorded coaching sessions in which Lily (pseudonym) solved tasks she also used for promoting her students' construction of the equi-partitioning scheme. Lily's case provides a glimpse into how knowledge from research on children's fraction learning can support and be extended to studying teachers' fractional reasoning. We infer and discuss the implications of changes in Lily's coordination of operations on concatenated (composed) sub-units and on those units separately, which led to her use of the distributive property for solving tasks she previously solved through guess-and-check.

INTRODUCTION

In this case study with Lily (pseudonym), a grade-4 teacher, we address the question: How might a teacher conceptualize fractional units as multiplicative relations (measures) while operating on continuous quantities? Our work with Lily drew on, and extended, research that articulated children's construction of the equi-partitioning scheme (Steffe & Olive, 2010; Tzur & Hunt, 2015). To this end, researchers engaged children in tasks of iterating units as a means for them to conceptualize unit fractions (1/n) as measures (Simon et al., 2018), that is, multiplicative relations to a given whole (Tzur, 2019). Given a continuous whole and the task to share it equally among a given number of people (e.g., 5), a learner would estimate the size of one person's share, iterate it 5 times, and compare the iterated whole with the one to be partitioned (see Figure 1a). If not equal – the learner would adjust the length of the estimated piece and repeat the process. Simon et al. (2004) called such a sequence of actions the *repeat strategy*. Two issues arise in such adjustment, namely, (a) its direction (making the next estimate shorter or longer) and (b) the amount of change, including how to improve the estimate by using the difference between the given and iterated whole (left-over or overage piece). Hunt, Tzur, and Westenskow (2016) showed that children progress through four stages, the last of which involving conceptualization of the amount of change as a unit fraction of the left-over (or overage; see next section). In Lily's case study we examined the extent to which her progression would be similar.

This study contributes to the line of work in which teachers' fractional reasoning is examined by using research with children (Lovin et al., 2018). It provides a glimpse into how a teacher may progress to a way of operating on fractional units that lays the foundation

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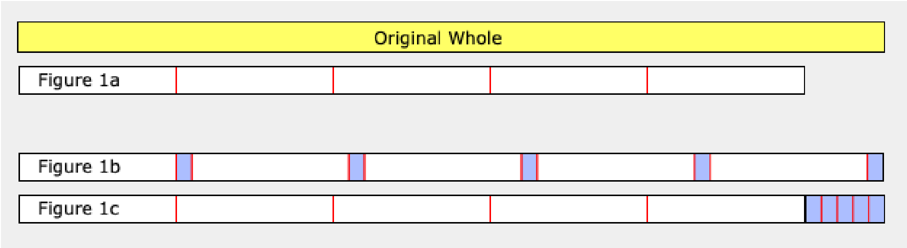
for constructing higher-level understandings. Such a way of operating seems a critical component of a teacher's mathematical knowledge for teaching (MKT, see Hill & Ball, 2004) so she can effectively teach fractions to her students. This study could thus provide mathematics educators with an empirically grounded model of what constitutes a teacher's reasoning with fractional units as well as tasks to promote it.

CONCEPTUAL FRAMEWORK

We draw on the constructivist research program that has been articulating conceptual progressions in children's schemes for reasoning about fractional units (Norton et al., 2015; Steffe & Olive, 2010; Tzur, 2019). The commencing scheme in this progression, called *equi-partitioning*, underlies two related understandings about unit fractions (1/n). First, the defining characteristic of such units is their multiplicative relation to another unit considered as a whole (a unit of 1). That is, a unit fraction (e.g., 1/5) is determined by the whole being 5 times as much of it, which is compatible with the number of times it needs to be iterated to fit within a given whole. Second, because iteration of larger units results in fitting fewer of them within the whole, an inverse relation is anticipated among unit fractions (e.g., $1/5 > 1/6$ precisely because $5 < 6$). The equi-partitioning scheme is postulated to be rooted in a person's ability to disembed pieces from a whole while keeping the whole intact.

To promote teachers' (e.g., Lily) construction of the equi-partitioning scheme through solving tasks with the repeat strategy, we have been using the *French Fry* activity (Tzur and Hunt's, 2015). In fact, Lily and her partner had been using this activity in their classrooms to foster their students' learning of unit fractions. Figure 1a shows a computer screen with a yellow strip (whole French fry) to be shared equally among 5 people. Below it we see that a piece, estimated to be one person's share, has been iterated 5 times to check if it accomplished the intended partitioning. Indeed – it did not, as the iterated, 5-piece whole is not equal to the given French fry. A learner would thus face two challenges: determine if the next estimate should be shorter or longer than the first estimate – and by how much. Hunt, Tzur, and Westenskow (2016) showed that children may progress through four stages in resolving each of those challenges. First, they have no conception of the nature (direction) of adjustment (shorter or longer). Second, an anticipation of the direction is evolving while the amount of adjustment is not yet conceptualized. Third, an anticipation of the amount of adjustment is evolving. Finally, both anticipations are integrated into a conception underlying both aspects of equi-partitioning (unit fractions as multiplicative relations to a given whole and the inverse relations among them). Figures 1b and 1c depict the fourth stage, including the two ways in which the left-over piece could be purposely operated on (Fig. 1b shows iteration of a unit concatenating the first estimate with 1/5 of the left-over piece; Fig. 1c shows separate iterations of those sub-units). When coordinated, these two ways provide a basis for using the distributive property while iterating (multiplying) continuous fractional units.



Figures 1a-1b-1c: Attempts at equally sharing a French fry among five people

METHODS

Participant and Settings

This was a case study with Lily, a grade-4 teacher working at a school in a large US city. We chose Lily as her work exemplifies the conceptual transition (similar to children) on which this study focused. She participated in a larger project (see Acknowledgements) to promote teachers' understandings of mathematics and mathematical pedagogy.

Lily's work took place during coaching sessions with her partner ("V"), conducted after they taught fraction lessons to students in one of their classes (with the partner observing that class). In those sessions, a researcher from our team led them to reflect on what they considered to be critical in the observed lesson. We focused on the teacher's own understanding of mathematical aspects of what students could (or not) do during the lessons. To this end, in the coaching sessions we used tasks that mostly followed tasks the teachers used, while further probing for their thinking.

Data Collection and Analysis

We focus on two coaching sessions with Lily and "V," involving two graduate students, the project's PI, and another researcher on the team (the first session was led by a graduate student, the second by the PI). One graduate student video recorded each session, focusing the camera on the teachers' actions (e.g., hand gestures) and written work. Analysis began with each team member observing (individually) the sessions and making detailed logs of the main events. Then, the four team members met to discuss what they noticed, and to identify themes (and data segments) that later informed the focus of this study. A graduate assistant transcribed the selected video segments and the team then discussed and co-authored the qualitative analysis presented in the next section (including tasks used to promote Lily's progression).

RESULTS

We present data excerpts that indicate the transition in Lily's reasoning about continuous, concatenated fractional units. We highlight how she moved from a guess-and-check method to purposely partitioning the piece left-over after iterating an estimate of one person's share. She realized that combining 1/n of the left-over with her initial estimate and re-iterating it would solve the task of equally sharing a whole through unit iteration. This realization indicated her construction of a for of the distributive property for concatenated (composed), continuous units.

Data in Excerpt 1 (the first, post-lesson session) illustrate the starting point in Lily's reasoning, and thus the conceptual challenge she faced, when having to operate on the left-over piece for a task of equally sharing the whole among four (4) people. Working on this task followed a previous task in which Lily and her partner shared the whole equally among 3 people. Both anticipated that one person's share would have to be shorter than the piece they iterated to equally share the same whole among 3 people. Thus, the researcher followed by asking how much shorter it needed to be, which both teachers did not know. (R stands for Researcher, L for Lily, and V for her partner.)

Excerpt 1

R: So, I'm just going to guess. Let's try there (makes a piece shorter than the 1/3 piece). So, I'm going to measure it (iterates the piece four times along the yellow strip; Fig. 2). So, I have this much left-over (points to it). I need to make this piece (points to the estimated-then-iterated piece) longer or shorter?

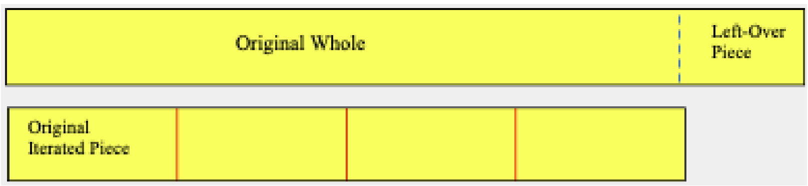


Figure 2: Estimated piece iterated 4 times; iterated whole is too short

L: You need to make them longer.

R: How much longer [with so] much left over? What would be the way to go about it?

L: *Guess and check.* [authors italicized words to highlight Lily's ideas.]

R: So how much longer? [Lily replies first, then her partner, V.]

L: *A little bit* (points to the left-over piece). *Like about there.*

V: So, taking this [leftover], splitting it into 4 equal parts, then take that little part [1/4 of the leftover] to add on this (the original white part), then remeasure.

R: Why would that work?

V: Because it is taking the same amount to split four times equally. So, each piece has ¼ of this (the leftover part).

R: (To V) Let's try Lily's way, then [yours]. (N iterates Lili's estimate). Lily, yours [now] has this much leftover. Would you make this longer or shorter?

L: A little bit longer.

R: How much longer?

L: *I guess a sliver.*

R: Lily, I want to push you. I want to make sure that this makes sense to you.

L: I don't understand why [V's way] is faster. Because *you can eye-ball it easier.*

R: I have these four pieces already and I have this much left-over. Rather than just guess and check, why would it make sense to have the precision to split this into four pieces and add one of those pieces to my pieces?

L: I don't know. You're still eyeballing, *you're still guessing the fourth here*.

Excerpt 1 illustrates Lily's initial idea, which is similar to Ana's work in the study by Hunt, Tzur, and Westenskow (2016). Like Ana, Lily understands the iterated piece needs to be longer, but when asked, "How much longer?" she simply says a little bit or by a sliver. Two pieces of the data support our inference she is yet to coordinate the left-over piece with both the iterated piece and the whole to be shared. First, Lily heard V's explicit suggestion to split the left-over into 4 pieces, but this did not seem to change her reasoning. Instead, she followed her own idea and faced yet another imprecise partitioning with some piece left-over. This leads to our second point. Lily realized the effect of this activity differed from the intended partition. Yet, she continued to suggest changes ("a sliver") that do not account for the number of people among whom the whole is to be shared. We infer that Lily's scheme (guess-and-check, eye balling) precluded assimilating V's and R's suggestions in a way that would yield her understanding and, hence, shift in operating. This could be seen particularly in her comment that what V has been doing is like her own eye balling method. For us, this brought forth a perturbation: How might we provoke a change in Lily's reasoning?

Our plan was to first orient Lily's attention to and possibly anticipation of the effect of adding a small piece (not related to a left-over) to the estimated-then-iterated piece once n-iterations would be performed. To this end, the project's PI (last author) joined the team for a second coaching session with Lily. For the first task of that coaching session he first created a piece and iterated it 7 times, then drew a longer piece considered as an adjusted one (Figure 3). Excerpt 2 presents what followed.

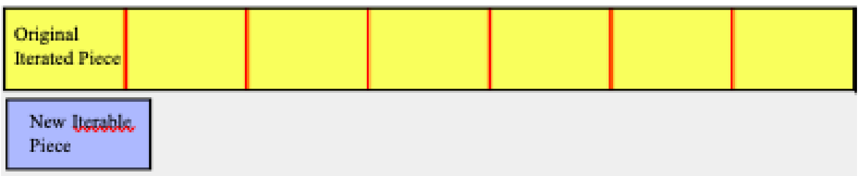


Figure 3: A task to bring forth Lily's anticipation of the effect of iteration

Excerpt 2

R: Can you estimate how far we are going to end up [if the blue piece is iterated 7 times]? Are we going to end up here? Here? (points to different places on the bar). Can you see where it would end up before we do it? Do you have a way of thinking about it?

L: (*Uses her fingers to measure how much longer (extra) the new piece is, then points to where it would end up*.) Like, around here.

R: And why would that be the place? What were you trying to do with this?

L: *Multiply this (extra only) by 7* [then add it to the original whole].

Excerpt 2 indicates the task promoted Lily's attention to the iteration of one piece as if it is composed of two sub-units, which she would need to set as a sub-goal for the partitioning of a left-over. Instead of asking her to determine how much of the left-over is to be added to the initial estimate (original yellow piece), she only had to compare that yellow piece with another, given one (blue piece). The task was based on our inference that Lily's ability to decompose whole numbers would enable such a way of operating also on concatenated, continuous units. As intended, Lily did not measure the entire (blue) piece and iterated it 7 times. Rather, she measured just the extra piece she knew to be the difference between the yellow and blue pieces – and iterated it 7 times. From her seemingly purposeful actions we inferred Lily anticipated that the total of iterating the entire blue piece 7 times would be equal to composing the 7-piece yellow stick with the effect of iterating (just) the extra 7 times. In a continuous context, she could reason about the concatenated aspect of the distributive property equation.

The researcher presented her with a follow-up task of determining the effect of iterating just one small (extra) piece. He drew another original (yellow) piece, iterated it 5 times, and then added an extra (blue) piece to its right end (Fig. 4). Lily, operating on the added (blue) piece separately from the initial (yellow) piece, copied the blue piece, iterated that copy 5 times, and pointed to the end of the 5-piece addition as the location of what she anticipated would have been produced if the original (yellow) and extra (blue) piece would have been iterated 5 times. That is, in a continuous context, she could reason about the aspect of the distributive property pertaining to each of the added units multiplied separately. To further her reflection on this equivalence, the researcher asked Lily to check – which she confirmed by combining a copy of the yellow and the blue piece and then iterating the combined piece 5 times.

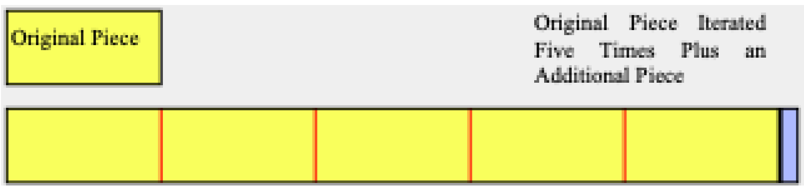


Figure 4: Lily's initial step in anticipating the 5-time iteration of an extra piece

The researcher believed Lily's reflection on and coordination of the effects of her two separate actions could support her transition to operating on a left-over. He inferred from her purposeful actions that, for the first time we could observe it, she began operating on a continuous piece composed of two sub-units and linking it with the effect of iterating each of the sub-pieces separately. He thus presented Lily with a follow-up task intended to further promote her coordination of this way of operating with the challenge of using a left-over in a task to equally partition a given whole. He thus drew a whole ("French fry") to be equally partitioned among 6 people, then a piece estimated to be one person's share (which he purposely made too short) and replicated it 6 times to produce a "too-short" iterated whole. Knowing Lily would think the next estimate would need to be longer than his first estimate, he continued as follows.

Excerpt 3

R: How much of this left-over should I add to this piece (points to the first estimate) so I get it right the next time?

L: I will take this piece and *eyeball six little parts in it* (points to the left-over piece).

R: How many little parts should I eyeball [and] where will you put the mark?

L: I'm thinking, *take this (points to the left-over piece) and splitting it into 6 parts*.

R: Six parts? Why six?

L: Because you have to *add the little part to each of the original ones you had*.

Excerpt 3 indicates the intended transition in Lily's reasoning. We infer from her explanation, prior to any action, that she anticipated the effect of adjustment to the initial estimate by purposely partitioning the left-over into the desired number of shares. Having worked on the previous tasks, she seemed to bring forth her anticipation of iterating a continuous piece composed of two sub-pieces to bear upon the task at hand. Specifically, she seemed to coordinate the impact of adding a piece and then iterating it 6 times (which she worked on in the previous tasks) with the size of piece relative to the left-over that would allow her to accomplish this. In contrast to Lily's lack of understanding of V's suggestion (Excerpt 1), it then made sense to her that 1/6 of the left-over would produce both (a) the entire left-over (if iterated 6 times) and (b) a 6-piece whole equal to the given French fry if the extra is first composed with the estimate and then iterated 6 times. She conceptualized how the two actions of iterating (multiplying) continuous fractional units—one for a unit composed of two sub-units and one iterating each of those sub-units separately—could lead to the goal.

DISCUSSION

Our case study with Lily illustrates how research on children's fractional reasoning (Steffe & Olive, 2010) can support, and be extended, to research on teacher's reasoning. We point out two key contributions. First, we further articulated the fourth stage in Hunt, Tzur, and Westenskow's (2016) study, in terms of conceptualizing the distributive property for continuous fractional units. Our starting point with Lily, and the conceptual change she went through, were similar to their study of Ana's case. By paying further attention to the coordination in her actions on continuous fractional units (concatenated or separated), we characterized this change as learning to apply the distributive property in a continuous context. Lily is a case of a person, adult like child, who uses iteration to construct unit fractions as multiplicative relations – measures consisting of one or more sub-units (Simon et al., 2018). That is, 1/n is a unit fraction in the sense of the whole being *n* times as much of it, regardless of the particular way the continuous unit is composed and/or iterated. We do not claim that Lily has conceptualized this at a stage she will not need prompting to engage in future tasks. However, her work on the tasks provides a glimpse into how coordinating operations on the left-over piece with the initial estimate may support conceptualization of unit fractions by bringing forth the distributive property in a continuous context.

Second, for teaching (and teacher development), our study reveals a crucial way of reasoning about fractional units needed to effectively promote students' understanding of fractions as measures. It thus adds to the growing body of knowledge on teachers' mathematical knowledge for teaching (Hill & Ball, 2004). Conceptualizing operations on fractional units presented in this study can become a basis on which teachers implement practices that build on students' available understandings. Our work with Lily and her partners showed increased abilities to identify where one's own students are in their reasoning and thus set goals and activities for moving them along a similar trajectory.

Acknowledgment

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FACILITATING LEARNERS' APPRECIATION OF THE AESTHETIC QUALITIES OF MATHEMATICAL OBJECTS: A CASES STUDY ON LEARNERS' PROBLEM SOLVING

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Although the importance of being able to appreciate the aesthetic qualities of mathematical objects during problem solving is now clear, there has been little research on facilitating students' appreciation of the aesthetic qualities valued by mathematicians. This paper presents a case study on problem solving by pairs of high-school students to describe and characterize the process by which learners gain such an appreciation under pedagogical interventions. The results show that participants could appreciate aesthetic qualities; and that to facilitate learners' appreciation more effectively, it is necessary to consider the curriculum and the order of the mathematical objects presented.

INTRODUCTION

By analyzing the mathematical problem-solving processes of experts, the importance of being able to appreciate the aesthetic qualities of mathematical objects during problem solving has become clear. Some studies have also shown empirically that being able to appreciate aesthetic qualities is a characteristic of experts (e.g., Dreyfus & Eisenberg, 1986; Silver & Metzger, 1989). Even gifted students tend to appreciate different aesthetic qualities than those valued by mathematicians (Tjoe, 2016).

On the other hand, the subjectivity and contextual dependence of the concept of "beauty" have been empirically noted (e.g., Wells, 1990); therefore, some studies have come to consider the role of aesthetic judgments, rather than the aesthetic qualities themselves. Sinclair (2004) identified three roles of aesthetic judgments in mathematical inquiry— the evaluative role, the generative role, and the motivational role. In addition, Sinclair (2006) suggested that a "learner's own aesthetic qualities" could play these roles. However, "learner's own aesthetic qualities" do not necessarily have the roles that Sinclair identified. Therefore, in order to enrich learners' problem-solving process—for example, to increase their motivation—it is useful to facilitate their appreciation of the aesthetic qualities valued by mathematicians. A method for doing so has not yet been clarified and remains as an important research question.

The purpose of this study is to elucidate a teaching method that helps learners to appreciate the aesthetic qualities of mathematical objects. For this purpose, the present study describes and characterizes the process whereby learners gain such an appreciation under pedagogical interventions (PIs).

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THEORETICAL FRAMEWORK

The aesthetic qualities of mathematical objects are often explained by attributes such as simplicity and generality (e.g., Hardy, 1956; Poincaré, 1908), but some studies have highlighted the subjectivity of these values, or even the concept of "beauty" itself (e.g., Inglis & Aberdein, 2014; Wells, 1990), and have tended to avoid clear definitions for aesthetic qualities (Davis & Hersh, 1981). However, for the present study, which is aimed at helping learners to appreciate such qualities, it is essential to be able to define them. In addition, this definition must satisfy the characteristics of the aesthetic qualities valued by mathematicians. Hence, the author defines the aesthetic qualities of mathematical objects by adopting the following four viewpoints based on the theory of Takeuchi Toshio, a Japanese aesthetician, which is in turn based on the principle of "unity in variety" (Takeuchi, 1979). The first viewpoint is the "form," which is the relationship among the components of a mathematical object. This viewpoint can be split into (i) "equivalence relations" (e.g., proportion) and (ii) "quasi-equivalence relations," which are not full equivalence relations but rather similarity relations (e.g., mappings). The former reasonably yields unity among the components and therefore gives rise to the aesthetic qualities of perceived objects in a wide area of fields, including mathematics, while the latter unifies components according to cultural and individual perception. In mathematics, similarity relations such as mappings are very important and give rise to attributes such as simplicity. The second viewpoint is the "essence" of a mathematical object, which is the property that is preserved during generalization and extension. The third is the "whole" of a mathematical object, which is the range in which its "essence" is established. The fourth is the "vastness" of a mathematical object, which is perceived through intuition about the "essence" and the "whole" of the mathematical object.

The above four viewpoints and the principle of "unity in variety" are also well suited to the below reference by Poincaré (1908):

> What is it that gives us the feeling of elegance …? It is the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts (pp. 30-31).

METHODOLOGY

This paper selects a multiple-cases study as its method based on the methodology of Yin (2014). The author observed learners' problem-solving processes under PIs. Three pairs of high-school students were chosen from a school that has among the highest standards in Japan, with the expectation that they would be able to use the mathematical knowledge and skills learned up to middle school.

Participants were asked to solve a problem collaboratively (Figure 1). This problem is widely used in junior high schools in Japan as a teaching material for the use of the symbolic expression $ax + bx = (a + b)x$. On the other hand, by interpreting this figure as "a set of three similar figures sharing a base", it is possible to understand at a glance that the two routes are equidistant. In other words, by identifying similarity or proportionality as a "form," and at the same time, by intuiting this "form" as an "essence," the aesthetic quality in this study can be appreciated.

A property of isosceles triangle [*]

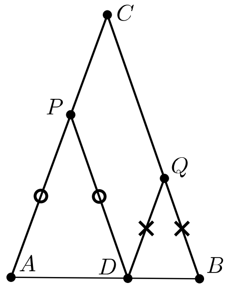

Fig. 1

In the Fig. 1, the triangle ABC is an isosceles triangle satisfying AC = BC. The three points D, P, and Q on the sides of the triangle ABC satisfy the following:

$$AP = PD, \; DQ = QB$$

In this case, the route (i) that departs from A, passes C, and reaches B, and the route (ii) that departs from A, passes through P→D→Q, and reaches B are equidistant.

<Question>Please prove this property.

Figure 1: The first form of problem presented to participants

The participants received the following PIs from the observer (the author) when their thought processes seemed to stop: (PI-i) presenting some derived figures in which the triangles are not similar to each other or in which the figures are rectangular (Figure 2); (PI-ii) asking the common properties among problems with different figures with two equidistant routes; (PI-iii) asking in what cases such common properties as are identified by (PI-ii) are established; and (PI-iv) asking to compare the primitive way of proving the statement with the means by intuiting the "whole" from the viewpoint of similarity or proportionality. These PIs are refinements of the method used in previous studies (Dreyfus & Eisenberg, 1986; Tjeo, 2016).

Figure 2: Derived figures for (PI-i)

The data obtained from the study are qualitatively analyzed as follows.

- Identify the point at which the "equivalence relation" was identified and intuited by the participants, as well as the point at which the "whole" was intuited by the participants.
- Describe the problem-solving process leading up to the identification and intuition of the "equivalence relation." If a part of this process can be considered to play an important role in the later appreciation process, it is also described in detail.

- Identify the "vastness" felt by the participants from the utterances in the series of problem-solving processes and the answers to interview questions.

RESULTS: AN APPRECIATION PROCESS BY A PAIR OF STUDENTS

Because a detailed description consumes a lot of space, the problem-solving process followed by only one pair of students (S1 and S2) is described here. The processes followed by other pairs were almost the same. This pair solved the problem by "detouring" more than the other pairs; therefore, they can be expected to obtain rich information. Their entire problem-solving process can be divided roughly into two parts.

Identifying and Intuiting a Parallelism as the "Form" and the "Essence"

In order to answer the <Question>, the pair constructed the proof as follows: based on the fact that the base angles of the isosceles triangles are equal, they claimed that $\angle CAB = \angle CBA$, $\angle PAD = \angle PDA$, and$\angle QDB = \angle QBD$. Next, since the isotope angles are equal, $PC \parallel DQ$, $CQ \parallel PD$, and the quadrilateral PDQC is assumed to be a parallelogram. Finally, they explained that the two routes are equal based on the fact that the opposite sides of any parallelogram are of equal length.

Because they stopped thinking after this proof, the author asked them to think about the figure on the left-hand side of Fig. 2 as (PI-i) and to clarify the differences between the two different figures composed of isosceles triangles as (PI-ii). In response to this question, they referred to one of the figures as being composed of similar triangles, and attributed the factor of whether the isometric property is established to the presence or absence of a parallelogram. In response to their reaction, the author asked them what they would consider to be a similar property that could be realized in the case of a rectangle, as (PI-i) and as (PI-iii). The author asked them this without giving them the image on the right-hand side of Fig. 2, so they drew an image like that presented in Fig. 3. Then, based on the attention paid to the parallelogram by S1, they finally saved parallelism as the "essence" and identified the set of figures inside the parallelogram as the "whole" by drawing Fig. 4.

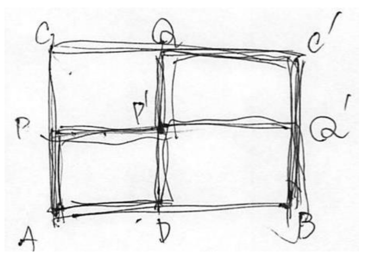

Figure 3: The figure drawn first

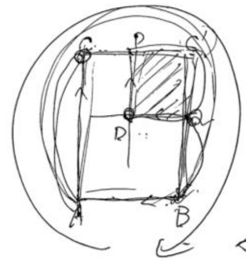

Figure 4: The figure drawn last

Identifying and Intuiting a Similarity as the "Form" and the "Essence"

While affirming the focus on parallel by the pair, the author asked them to think about the figure on the right-hand side of Fig. 2. They noticed that the route passed through SE twice, but could not be convinced that the two routes were of equal distance.

Since their thoughts seemed to be stuck, the author asked them to consider the case in which the figure was composed of squares as (PI-i). In response, they inductively confirmed the property by drawing many figures and explained that the property is satisfied by expressing each side length of the two inner squares by symbols.

In response to their reaction, the author asked them the range of the figure in which isometric routes exist as (PI-iii). Then, they considered parallelograms, pentagons, and diamonds. They found it difficult to draw a figure composed of pentagons, so they gave up drawing it. This confirms that they had not yet intuited similarity as the "essence" at this point. When considering diamonds (Figure 5), S1 said, "it will always hold if another shape is determined when the first internal shape is drawn."

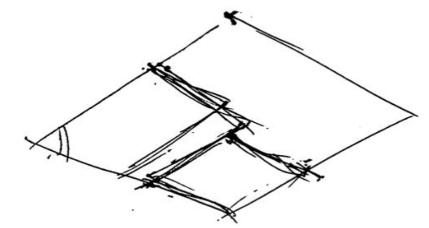

Figure 5: The figure composed of diamonds drawn by S1

On the other hand, S2 focused again upon similarity while referring to S1's remark and to the picture drawn by S1; however, S2 was not confident in focusing upon similarity without knowing the similarity condition of a square. However, S1 supplemented S2's idea, and they began to analyze figures composed of isosceles triangles or squares with a view to similarity. Hence, they determined that the similarity of the three figures was a sufficient condition to establish the isometric property. In addition, they specifically constructed figures containing isometric routes using various rectangles, pentagons, semi-circles, and circles (Figure 6).

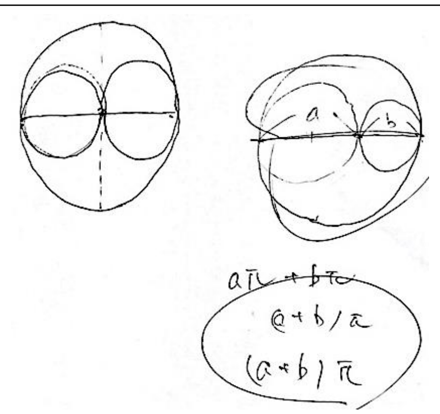

Figure 6: The figure composed of semi-circles drawn by S2

The "Vastness" Felt by the Participants

After asking participants to reflect upon the entire problem-solving process, the author asked them what they thought or felt. Their responses are shown in Table 1.

S1	S2
Until now, I have solved both this problem about the figure composed of isosceles triangles and this problem about the figure composed of semi-circles. However, there is a great sense of accomplishment that can be developed from the former problem to the latter problem. Even if it has a certain property, even if it is a straight line or a curve, it is like a circle with no corners, no matter how many corners there are, if we can find some common parts, if we can find common properties from among them, (Interviewer: What is it now?) What is similarity now, without being caught in concrete cases like isosceles triangles, we can apply them to various things. It was interesting to know this.	Until now, even if I thought, it was only about this problem [1]. Rarely have I developed a square or a circle from here. Well, by deeply exploring when it holds, I'm interested in the question about one case or one problem, "What about the other case?" I mean, it's good to know that.

Table 1: What was thought or felt (Translated by the author)

DISCUSSION

The Learners' Process of Appreciating the Aesthetic Qualities of Mathematical Objects

This study defined the process for appreciating the aesthetic qualities of mathematical object as four subprocesses: identifying the "form," intuiting the "essence," intuiting the "whole," and feeling the "vastness." Particularly in this paper, since similarity is regarded as the "form" and the "essence", identifying the "form" and intuiting the "essence" were regarded as the same process, without distinction.

In the case described in this paper, the pair of high-school students followed the above process successfully under PIs. They first identified and intuited the parallelism as the "equivalence relationship" and then identified and intuited similarity as the "equivalence relationship." It was only in this case that the learners identified and intuited an "equivalence relationship" as the "form" and as the "essence" that the author did not envision (cf. Hanazono, 2019).

The particularity of this case is thought to be due to the mathematical task employed in this study and the curriculum experienced by participants. In Japan, in the second grade of junior high school, formal proof is positioned as a learning objective in the national curriculum. When learning to prove things formally, the properties of the parallel lines (which are introduced immediately beforehand in the curriculum) are frequently used. As these properties are used from the beginning, they will naturally be widely used in subsequent proof problems. Therefore, it is natural for many Japanese learners to focus on the parallelism when asked to prove statements.

The curriculum can also explain why similarity is a viewpoint that is hardly noticed by Japanese learners when considering figures other than triangles. In mathematics education in Japan, enlargement and reduction are treated in elementary school subjects as being directly connected to similarity (Grade 6). Then, in junior high school, the fact that two figures are similar when straight lines connecting their corresponding points all intersect at one point is again treated (Grade 9). However, after the definition of similarity of triangles is handled following confirmation of the definition of similarity in general, many proof problems that require the identification of a set of triangles that satisfy the similarity condition will be presented. Therefore, as was the case with S2, the concept of similarity and the similarity conditions of triangles are strongly linked for many Japanese learners. Thus, it can be assumed that similarity will not function adequately as a viewpoint when considering a figure other than a triangle.

In the elementary school mathematics curriculum in Japan, concepts such as parallelism, symmetry, and enlargement/reduction are treated as viewpoints for considering figures in the context of concept formation for figures. When children learn these viewpoints, they recapture the figures which they have already learned from these viewpoints. On the other hand, in the junior high and high-school mathematics curricula, the context of concept formation for figures may be weakened, and the role of properties as viewpoints for considering mathematical objects becomes less noticeable. The curriculum needs to be reevaluated to develop educational methods aimed at facilitating learners' appreciation of the aesthetic qualities of mathematical objects.

Effects and Limitations of PIs

As mentioned in the previous section, in the case detailed in this paper, a pair of high-school students who underwent PIs derived from a theoretical framework were found to appreciate the aesthetic qualities of mathematical objects. The effects of such PIs have been confirmed in other case studies (cf. Hanazono, 2019). In other words, the case study in this paper supports the effectiveness of the theoretical hypothesis of the PIs.

As for the details of the appreciation process, in the case where participants consider figures composed of rectangles, it is particularly effective to use PIs that encourage learners to consider figures composed of squares or diamonds. By constructing the two inner diamonds such that they are inscribed in the outer figure, the three diamonds are similar to each other. By considering a figure composed of diamonds (for which similar cases are easy to handle) and squares (which are always similar to each other), the author believes that it is possible to help learners to pay attention to similarity as a common point for isosceles triangles with two equidistant routes.

On the other hand, in the case of an isosceles triangle in which the isometric property does not hold (the left-hand side of Fig. 2), or of a rectangle in which the isometric property holds but not the similarity (the right-hand side of Figure 2), learners' attention to similarity may be hindered. Although it is indispensable to consider the case in which the "essence" does not hold in order to intuit the "whole", there is room for reconsideration of the order in which figures are handled.

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A MODEL OF STUDENT MATHEMATICAL WELLBEING: AUSTRALIAN GRADE 8 STUDENTS' CONCEPTIONS

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To support academic performance, many schools are increasingly focussed on supporting student wellbeing. Yet the high incidence of mathematics anxiety and disengagement suggests that many students experience poor wellbeing in many mathematics classrooms. This paper proposes a seven-dimensional model of student mathematical wellbeing. To test this model, 488 eighth grade students responded to one open-ended question; responses were analysed using a combined deductive/inductive thematic analytic approach. Findings supported the model. The study illustrates the importance of focusing on wellbeing within specific subjects, provides a model for studying student wellbeing specific to mathematics education, and points to areas to target to improve and develop student mathematical wellbeing.

BACKGROUND

Complementing the traditional focus on developing cognitive skills such as numeracy and literacy, schools worldwide are increasingly considering strategies for supporting holistic student development, including the development of non-cognitive skills and student wellbeing. The United Nations includes wellbeing as one of 17 goals for sustainable development to achieve by 2030 (United Nations, n.d.), and student wellbeing is increasingly included as a key priority within policy and practice (NSWDET, 2015; VICDET, 2018).

Mathematics is a foundational subject within education, as the benefits of attaining competence within the subject accrue over the lifespan; enhance employment opportunities; inform choices about the environment, health, and wellbeing; and correlate with longer life expectancy (Plunk, Tate, Bierut, & Grucza, 2014). However, despite significant global financial investments to improve mathematics education, student engagement in mathematics generally has remained low, with negative emotions and attitudes towards the subject persistently reported by students (Clarkson, Seah, & Pang, 2019). Many students experience 'mathematics anxiety', and perceive mathematics to be boring and unenjoyable (Grootenboer & Marshman, 2015). In Australia, the proportion of secondary students taking advanced mathematics has steadily been declining over the past three decades (Kennedy, Lyons, & Quinn, 2014). These trends all point to poor student wellbeing in mathematics education.

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Despite the growing focus in education policy and practices on student wellbeing, there has been little attention exploring student wellbeing within individual subject disciplines, including mathematics. The current study addresses this gap, proposing and testing a model of mathematics wellbeing.

THEORETICAL FRAMEWORK

Wellbeing – also termed as ‘flourishing’ or ‘thriving’ – can be defined and operationalised in a number of ways. We focus on subjective wellbeing, in which wellbeing is defined based on a person’s subjective perception of the extent to which they are feeling and functioning well across a number of dimensions (e.g., physically, mentally, socially, cognitively). Drawing on research and theory from the field of positive psychology, a number of frameworks have been applied within schools over the past decade within Australia to operationalise student wellbeing (Slemp et al., 2017; White & Kern, 2018). For instance, the PERMA model includes five dimensions that together support wellbeing: positive emotions, engagement, relationships, meaning, and accomplishment (Seligman, 2011). The EPOCH model suggests five positive psychological characteristics that support positive development: engagement perseverance, optimism, connectedness, and happiness (Kern et al., 2016).

These various models and frameworks identify *general* student wellbeing, rather than wellbeing within specific subjects. To our knowledge only two studies have specifically explored ‘mathematical wellbeing’. Clarkson and colleagues (2010) proposed a three-dimensional model: cognitive, affective/values and emotions. Part (2011) includes an individual’s functioning’s and capabilities. Both frameworks ignore social aspects of wellbeing and mathematics learning and lack discrete measurable entities.

We propose an updated seven-dimensional model of mathematical wellbeing (Table 1), which includes seven dimensions that have been linked to positive mathematics learning outcomes (e.g., Grootenboer & Marshman, 2015). The model integrates the five aspects of Seligman’s (2011) PERMA model, four aspects of Kern and colleagues’ (2016) EPOCH model; and two dimensions of Clarkson and colleagues’ (2010) mathematical wellbeing model (MWB).

METHODS

Our research question was: To what extent do students’ conceptions of wellbeing in mathematics education align with our updated mathematical wellbeing model? To test this question, we compared our theoretical model with students’ experiences of mathematical wellbeing based on the open-ended question: *What makes you feel really good or function well in maths?*

Participants included 488 grade eight students (223 females) aged 13 - 14 years, from nine urban and regional secondary schools (3 private; 4 Government/public; 2 Catholic) located in Melbourne and surrounding cities in Australia. Students self-identified their ethnicities as Australian (71%), Asian (14%), European (6%), Indian/Pakistani (6%), Indigenous Australian (2%), Middle Eastern (1%), or North/South American (2%). Schools serviced socioeconomic neighbourhoods ranging from low to high.

Dimension	Description	Source
Positive Emotions	Positive emotions in mathematics e.g., fun	PERMA; EPOCH; MWB
Engagement	Concentration, absorption, deep intertest, or focus when learning/doing mathematics	PERMA; EPOCH
Relationships	Supportive relationships, feeling valued/cared for, connected with others, or supporting peers in mathematics	PERMA; EPOCH
Meaning	A sense of direction, feeling mathematics is valuable, worthwhile or has a purpose	PERMA
Accomplishment	A sense of achievement, reaching goals, or mastery completing mathematical tasks/tests	PERMA
Cognition	Having the skills, and understanding to undertake mathematics	MWB
Perseverance	Drive, grit, or working hard towards completing a mathematical task/goal	EPOCH

Table 1: An updated mathematical wellbeing model

Responses were imported into NVivo 11 and analysed using a combined deductive/inductive thematic analysis adapted from Braun and Clarke (2006). Initial nodes/themes were generated inductively. For example, "*having friends in my class to support me*" was coded as *peer support*. Nodes were then categorised deductively into common themes based on our theoretical model. For example, the nodes *peer support* and *teacher support* were categorised as *relationships.* As students often mentioned multiple themes, the response for each student could be categorised into multiple nodes and themes (for students mentioning the same theme multiple times, the theme was only counted once).

RESULTS

Table 2 summaries the final categorisation of nodes into themes and how student responses aligned with our theoretical model. Most students associated their mathematical wellbeing with positive classroom relationships followed by a sense of engagement, mathematical cognitions, accomplishments, positive emotions, perseverance, and meaningful mathematics learning. In addition, music was mentioned several times, with numerous students suggesting music facilitated engagement and positive mood. As such, music appeared to contribute to the other themes, rather than being a separate dimension of wellbeing.

Themes & nodes	Student examples	N (%)
Relationships		**200 (32%)**
Teacher support	*A supportive or good teacher*	94
Peer support	*Having friends to help me*	83
General support	*When I get help with my learning*	38
Engagement		**126 (20%)**
Interesting/hands on	*Learning interesting stuff*	55
Focused working	*Being absorbed in my work*	37
Independent/quietness	*When it is quiet and I'm by myself*	27
Music (engagement)	*Music helps me concentrate well*	15
Cognitive	*When I understand the material*	**96 (15%)**
Accomplishment		**86 (14%)**
Good marks	*When I do good in a test*	31
Accuracy	*When I get the answers right*	24
General mastery	*When successful at learning something*	17
Completing tasks	*When I complete my work*	13
Confidence	*When I'm really confident*	3
Positive emotions		**60 (9%)**
Enjoyment/fun/happy	*If the maths class is enjoyable*	47
Relaxed/no pressure	*When there is no pressure*	12
Music (emotions)	*Music to listen to, to enjoy it more*	4
Perseverance		**31 (5%)**
Challenge	*Having work I find challenging*	21
Working hard/practice	*When I work hard*	13
Music (no reasoning)	*Listening to music in class*	**24 (4%)**
Meaning		**10 (2%)**
Future skills	*Knowing these skills will help me in life*	5
Real world relevance	*I like when problems relate to real life*	5

Table 2: Results, with coded student responses by theme and node

DISCUSSION

Despite a growing focus in schools on student wellbeing, existing models focus on general student wellbeing rather than considering how wellbeing might depend on the context or specific subject. Extending existing theoretical models developed within positive psychology and mathematics education, we proposed a seven-dimensional mathematical wellbeing model. Based on qualitative data from 488 Australian Grade 8 students that asked students about what helps them feel and function well in mathematics classes, we found that students' responses generally aligned with the seven proposed dimensions. Student responses also pointed to the importance of music in promoting positive emotions and a sense of engagement. Thus, our model offers a useful starting point to explore the factors that might promote student wellbeing, specifically within mathematics.

Most students pointed to the importance of supportive classroom relationships with both teachers and peers. Previous mathematical wellbeing frameworks (Clarkson, Bishop, & Seah, 2010; Part, 2011) did not include a relationship dimension, pointing to the benefits of drawing on models developed through other disciplines. The impact of supportive social relationships on wellbeing and academic achievement is well recognised (e.g., Allen & Kern, 2017; Hattie, 2008). Within mathematics education, supportive teachers are associated with improved student mathematical achievement; positive emotions, academic enjoyment and effort; and mathematical engagement (OECD, 2019; Sakiz, Pape, & Hoy, 2012). Interestingly, a similar proportion of students referenced peer and teacher relationships. In many countries, mathematics classrooms are teacher led, with limited opportunities for peer-collaboration (Geist, 2010). Peer collaboration can also greatly impact on student mathematical learning outcomes, especially students from minority cultures (Hill, 2018). The prevalence that students noted the importance of positive relationships for helping them feel and function well suggests that greater attention to the social aspects of mathematics learning could be beneficial.

The second most common theme was engagement. Much research attention has focused on the impacts of engagement on student academic performance (e.g., Attard, 2013). Our findings illustrate that engagement also contributes to mathematical wellbeing. Notably, the comments by students pointed to factors that make the classroom more disengaging (e.g., distracting peers, repetitive pedagogy) or engaging (e.g., listening to music, quietness). Many students find mathematics boring, disengaging and repetitive (Grootenboer & Marshman, 2015), especially when teachers rely on textbook focused pedagogies (McPhan, Morony, Pegg, Cooksey, & Lynch, 2008). Our findings support the incorporation of "fun" pedagogies that have been supported in other countries to increase engagement and enhance wellbeing (Clarkson et al., 2019; Hill, 2018).

The progressive yet linear nature in which mathematics is often taught can contribute to fears or anxieties about being left behind in a fast paced mathematics curriculum, resulting in greater anxiety and poor learning outcomes (Geist, 2010). Poor performance can result in greater pressure to perform well, at the expense of student mental health. Yet recent studies suggest that strategies to develop student wellbeing and academic performance can be complementary, rather than competing (White & Kern, 2018). Our findings support this, pointing to reinforcing spirals in which wellbeing supports performance and performance supports wellbeing. Students indicated that aspects such as understanding the problems, successfully completing tasks, and correctly solving problems resulted in positive feelings and greater confidence, whilst pressure to not make mistakes and to perform in tests promoted negative emotions.

The low prevalence of perseverance was surprising, considering that perseverance is central to student academic achievement and mastery of goals (Duckworth & Gross, 2014). As mathematics involves reasoning and problem solving skills, perseverance is particularly important for mathematical accomplishments (Sullivan et al., 2013). The low prevalence could mean that perseverance is more relevant to achievement than to wellbeing, or as the data were based upon qualitative responses, other dimensions may have been more obvious. Future studies might use quantitative approaches to test the importance of this theme to wellbeing

Meaning similarly was only mentioned by a small number of students. Meaningful, real world or useful mathematical pedagogies are associated with greater student interest and motivation, improved effort and engagement, and improved mathematical performance (Dobie, 2019). In Western countries, studies point to students desiring meaningful learning experiences (Hill, 2018). The sample included a number of non-Western participants, who might not value meaningful learning in the same way. Alternatively, as a required subject, meaningful learning may be considered less relevant by students. Future studies might further explore the role that meaning plays in mathematics education for both academic and wellbeing outcomes.

Pointing to strategies for supporting student's wellbeing, some students indicated that listening to music promoted their mood or engagement. Thus, music appeared to be something that supports other themes of our mathematical wellbeing model, rather than as representing a separate dimension of wellbeing. Other studies similarly find that adolescents often strategically listen to music to enhance their wellbeing, motivation and concentration (Papinczak, Dingle, Stoyanov, Hides, & Zelenko, 2015). Future studies will benefit from identifying other strategies for promoting the seven wellbeing dimensions proposed by our model.

IMPLICATIONS AND CONCLUSION

While wellbeing is often considered as a global construct, a key message of this paper is that wellbeing is domain specific. Thus, student wellbeing should be explored in individual subjects. Using insights from positive psychology to inform mathematics education, this study provides a seven-dimensional model that aligns with aspects that help students feel and function well within mathematics. Necessary extensions of this research include the development of a measure to assess mathematical wellbeing, and understanding what factors enable students to thrive in mathematics education.

Our mathematical wellbeing model points to areas to target to improve students' experiences in mathematics. The model might not apply to students' experience of wellbeing when engaging in other subjects, but the methods proposed in this study can be adapted to explore student wellbeing in other academic subjects. Future studies might test the relative importance of each dimension, using quantitative methods and extending to other year levels and populations. Cultural, school and gender differences in students' conceptions of their mathematical wellbeing should also be explored.

The widespread negative reactions experienced by students in mathematics education are well publicised, pointing to a poor sense of wellbeing in many mathematics classrooms (e.g., Attard, 2013; Fielding-Wells & Makar, 2008). Providing a balance to the considerable research and media attention in mathematics education focused on the negative aspects, the 'gaps', and what is going wrong in the subject, our study offers a glimpse of the aspects of students mathematical learning that are working well and enable students to thrive in mathematics education. Considering the somewhat global preoccupation with student mathematics (under)achievement (e.g., OECD, 2019), this study makes a timely contribution offering a sense of hope that there is more to mathematics education than merely achieving academic benchmarks.

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WHAT DO TEACHERS LEARN ABOUT THE DISCIPLINE OF MATHEMATICS IN ACADEMIC MATHEMATICS COURSES?

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This study investigates the contribution of academic mathematics courses to teacher learning about the discipline of mathematics. Analysis of interviews with 14 secondary school mathematics teachers, who graduated from a master's program that included a strong emphasis on academic mathematics studies, identified references of contribution to nine topics that can be grouped into three aspects of knowledge about the discipline of mathematics: (1) essence, (2) doing, and (3) worth. Each of the three aspects was addressed by all or almost all the teachers. However, the number of teachers that addressed each topic varied considerably among and within aspects.

INTRODUCTION

Academic mathematics studies are, in many countries, an important component of the professional education of secondary school mathematics teachers. Theoretical contemplations and empirical studies suggest potential contribution in two dimensions of subject-matter knowledge that appear to be critical for teaching. One dimension is knowledge of specific topics, concepts, and procedures (Dreher et al., 2018; Weber et al., 2020), and the other – which is the focus of this study – is a more general epistemological knowledge about the discipline of mathematics (e.g., Even 2011; Zazkis & Leikin 2010; Ziegler & Loos 2014).

The existing empirical research literature reports on contribution of academic mathematical studies to teaching in relation to knowledge about the discipline of mathematics (e.g., Baldinger 2018; Even 2011; Zazkis and Leikin 2010). Yet, this literature has shortcomings. First, the existing research does not clearly link between academic mathematics studies and modifications in teachers' conceptions about mathematics; findings are commonly based on reports from teachers who studied in different academic programs, and, in addition, the course instructors' intentions regarding what to teach these teachers about mathematics has not been examined. Second, the contribution of academic mathematics studies, not only to knowledge, but to actual teaching, has rarely been examined. Our study addresses these shortcomings.

The starting point for our investigation are results from a study that examined what mathematicians who teach academic mathematic courses to secondary school mathematics teachers would like to teach teachers about the discipline of mathematics (Even, 2020; Hoffmann & Even, 2018). The findings revealed that enriching, expanding and deepening teachers' knowledge about the discipline of mathematics was a central goal of all the participating mathematicians. They referred to nine topics that can be grouped into three key aspects: (1) the essence of mathematics, which deals with the question: What is mathematics?

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(2) doing mathematics, which deals with the question: How is mathematics done? and (3) the worth of mathematics, which deals with the question: What good is it to engage in mathematics? (see Figure 1).

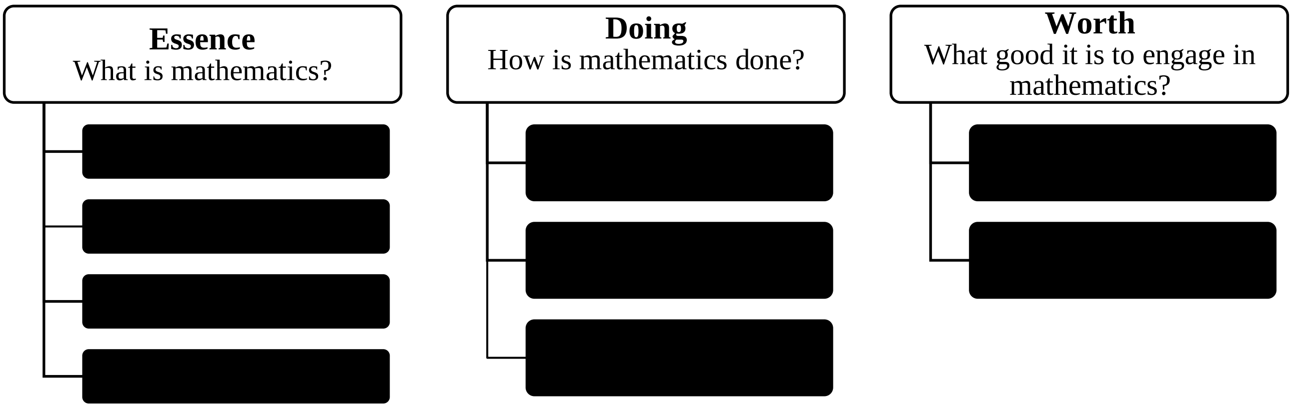

Figure 7: Framework for teacher learning about the discipline of mathematics.

In the current research, we focus on teachers who studied in these mathematicians' courses, using the conceptual framework in Figure 1 for data analysis, extending Hoffmann and Even's (2019) study, which addressed one aspect (the essence of mathematics) to all aspects and topics. The research question is: *What do secondary school mathematics teachers learn about the discipline of mathematics in academic mathematical courses, and how does this knowledge contributes to their teaching?*

METHODS

Setting and Participants

The study is situated in a two-year master's program for practicing secondary school mathematics teachers. A major component of the program comprised eight academic mathematics courses tailored for teachers, designed and taught by research mathematicians. Four of these courses dealt with topics in the school curriculum at an advanced level: algebra, analysis, geometry, and probability and statistics. Three courses were devoted to the use and application of mathematics in other domains: computer science, applied mathematics, and everyday technologies. One course dealt with the history and philosophy of mathematics. In addition, a final project was carried out under the guidance of a mathematician. Fourteen program graduates participated in the study; ten women and four men. Their teaching experience varied 3-23 years.

Data Collection and Analysis

The main data source included individual semi-structured in-depth interviews with the teachers. The interviews took place between 0.5-2.5 years after graduation, and lasted between 45-90 minutes. The aim was to learn how the mathematics courses contributed to the teachers' knowledge about the discipline of mathematics, and how that knowledge contributed to their teaching. The interview consisted of eight open-ended questions. The main questions were:

- Has there been any change in the teacher you were before the program and the teacher you are today?

- The program has two main components: courses in mathematics education and courses in mathematics. Have the mathematics courses contributed to you as a teacher?
- Have you learned anything new about what mathematics is from the mathematics studies in the program?

Following each question, the interviewees were asked to explain their responses and to give examples from their experiences in the program and their teaching. Additional data sources were participant observations in most courses and informal conversations with the teachers. The aim was to strengthen the internal validity of the study.

The interviews were transcribed and then analysed, employing the method of directed content analysis (Hsieh & Shannon 2005). First, we marked all excerpts in the transcripts that dealt with the contribution of academic mathematics courses to the teachers' knowledge about the discipline of mathematics and changes related to their teaching. Then we used the framework in Figure 1 for coding these excerpts, enabling additional categories to be created if needed. The analysis was iterative and comparative, and included peer validation.

FINDINGS

Analysis of the interviews revealed that reports on contribution of the academic mathematics courses to the teachers' knowledge about the discipline of mathematics and to related changes in their teaching were associated with all three aspects and with all nine topics of the coding scheme. Also, no additional aspects or topics were mentioned. Whereas each of the three aspects was addressed by most of the teachers, the number of teachers that addressed each topic varied considerably (1-10 out of 14), and the teachers varied considerably in the number of topics to which each of them referred (1-7 out of 9). Below we describe the main characteristics of the teachers' reports for each topic.

The essence of mathematics

Twelve teachers addressed this aspect; between four to nine of them addressed each of its four topics.

Wide and varied

Nine teachers reported on contribution associated with this topic. The teachers reported on learning that mathematics is much broader and more diverse than what they had thought; that the mathematics they were familiar with is just the tip of the iceberg of the existing mathematical knowledge, both in terms of the variety of fields and topics, and in terms of the amount of what is known or studied in mathematics. The following excerpt from J's interview exemplifies this.

Interviewer:	From the mathematical studies, was there anything you learned about what mathematics is? ...
Teacher J:	Mostly perhaps how many areas it covers. Like when I'm studying mathematics, I learn that I don't know... Actually, you see that mathematics is like a whole world. It is impossible to know it...You learn whatever you learn, but it is more than that. It gives you, like, the feeling of the size…

Eight of the teachers reported that their new understanding promoted change in their teaching. They try now to extend students' horizon regarding mathematical questions and mathematical domains beyond contents studied in school. For instance:

> …when we study vectors and we learn normals, then every plane has one normal. Today, after studying differential geometry, I know that surfaces have normals in different directions. I mean, you can talk now about, that a normal is not one normal to a plane. If the surface is a bit curved, then you have infinite number of normals. You can convey this to a certain extent, just open it, and curious girls fly with it further. (K)

Lively and developing

Eight teachers reported on contribution associated with this topic. The teachers reported that they learned how and why mathematics has evolved, and that mathematics is an intensely researched scientific field also nowadays. For example,

> …it is very significant how mathematics has evolved... Let's say Euclid, his book of the elements of geometry, and then over the years how things evolved, and so on, and projective geometry, and so on, and then like all the developments of recent years. There is something in this conception that straightens out my head. I can understand much more deeply. How it was created. (B)

Seven of the teachers reported on contribution to instruction related to this topic, saying that they began to present students with background on the development of the mathematical topics they study. For example,

> Interviewer: Is there anything else you teach differently that you can point out?
>
> Teacher A: …if I talk to them about the development of mathematics then it is to teach differently. Most definitely… anything new from the curriculum that I want to bring to class… I try not to dump it on them, but to prepare them for it…

Rich in connections

Six teachers reported on contribution associated with this topic. The teachers talked about contribution to their understanding about the existence, and importance, of connections among different mathematical fields, and among different concepts within a field. For instance, "I saw the connection between the different topics much more, how one domain uses another in order to prove, to make progress, to illustrate" (E).

All six teachers mentioned contribution to teaching, reporting that they started to look for connections among different topics in the curriculum, and that they strive to present mathematical concepts in a variety of contexts. For example, C said, "When I teach something, I try to connect it to previous and subsequent contents, and to other domains." She exemplified this by describing how she stresses for students that the straight line they sketch in geometry, the linear equation $f(x) = mx + n$ they see while learning functions, and the equation $Ax + By + C = 0$ that they meet in Cartesian geometry, are all the same mathematical concept in different representations.

Structured deductively

Four teachers reported on contribution associated with this topic. The teachers mainly reported on better understanding the role and importance of axioms, definitions and proofs. All four teachers reported on contribution to their teaching as well, describing that now they explicitly discuss in class the role and importance of these elements in mathematics. For example, K said:

> In any mathematical course. It suddenly pops up. You didn't look for it until you became aware of it. When you are aware that you look for definition of everything, I just look for it... I suddenly explain to the kids: That is a definition, you cannot argue about it, let us define it properly. I also explain what 'well defined' is... I think it made me a much more organized teacher, and it makes order in the students' heads... (K)

Doing mathematics

Ten teachers addressed the aspect doing mathematics; between three to eight of them addressed each of its four topics.

Asking questions and explaining why

Eight teachers reported on contribution associated with this topic. They reported on improved knowledge about the centrality of thinking and understanding in mathematics and the way questions and explainations advance understanding. The teachers reported also that they now pay more attention in their teaching to mathematical inquiry and understanding that involve explaining "why", in contrast to focusing merely on technical aspects and explaining "how". The focus on asking questions and explaining why included not only teachers' focus on explanations and answering "why" questions, but also a strong expectation that their students would also do so. For example, by asking students questions that require them to practice explaining their reasoning. The following excerpt from E's interview illustrates this:

> The class discourse is always a discourse of why and of what, and less of how. There is a lot of how, a lot, but if you say something, then: 'Why is this thing true? Explain, prove, in your own words, look for justifications, reasons'… I think that it is very very prominent in my teaching today, that first it is based on understanding. If I need not to explain something, but just to give a formula, I really have difficulty with it. And before I used to live very peacefully with it. (E)

Using intuition and formalism

Eight teachers reported on contribution associated with this topic. The teachers reported that the academic mathematics courses helped them understand the key roles of intuition and formalism in the process of doing mathematics. They developed their understanding regarding mathematicians' use of intuition. For example, G described how surprised she was to discover that mathematicians examine specific examples when they start working on a problem:

> First of all I learned that also mathematicians can start from examples… I thought they think in formulas all their lives… And suddenly I discovered... that when a mathematician wants to check something, and he checks that on numbers.... I thought that I do that because I am a human being with a limited head and mathematicians don't do that, they immediately think about the general case. (G)

The teachers reported learning also that formalism and precision is important in mathematics; yet not as a goal by itself, but because intuition could be misleading. Thus, both intuition and formalism are needed. For example, C said,

> Another thing that we didn't talk about… knowing that there is feeling the mathematics and then there is the tiring formalism. But there are two stages of the process, and to really distinguish between them. I think that it goes with me to the classroom, in explanations and also in thinking. (C)

All eight teachers reported also that their new understandings enabled them to better judge the extent of accuracy that is appropriate in different teaching situations.

Experiencing struggles and insights

Three teachers reported on contribution associated with this topic while referring mainly to their improved understandings about the prolonged mental effort that doing mathematics requires and the inevitable mistakes that are done along the way. For example:

> You have to be diligent... it requires effort. Do not give up ... because mathematicians… if until today I thought that everything comes to them easily, then I think that, today I understand that mathematicians work hard to make discoveries. Work very hard. (G)

The teachers added that following their new understanding they linger longer on tasks discussed in class, and are more open to expose their own struggles to students as a model for doing mathematics. For example, after teacher K described her struggels with a piece of mathematics during her studies, she was asked how such experiences contributed to her teaching. She then said:

> …if I do not know [how to solve a problem in class] ... I'm not ashamed. I will tell them [the students] that we will think and figure it out, and we will rack our brains during the break... (K)

The worth of mathematics

Ten teachers addressed this aspect; all of them addressed the first topic and one teacher addressed the second topic as well.

Practical worth of mathematics

Ten teachers reported on contribution associated with this topic. The teachers said that their academic mathematical studies made them aware of the practical worth of mathematics and of the central role that mathematics has in everyday life. For example:

> So what he [the course instructor] actually said, that mathematics, it's role is, actually, there is a problem in life, and one needs to construct a model, a mathematical model, in order to solve that problem. That's basically the role of mathematics. Like to create, like for life, for real life. And it was something that was really new and interesting for me to see mathematics like this, as a field that solves problems. Beforehand, I viewed it as an intellectual field, which is fun to deal with, it is interesting in itself. I did not look at the, at what it actually gives in practice. (I)

The teachers said that becoming aware of the practical worth of mathematics helped them better address students' questions regarding why they should learn mathematics. For instance, F said: "It was interesting to see how it fits into life" and added: "They ask me: 'Why do I need that?' Then I do tell them… You tell them what it does to cancer patients… It's of great interest to them."

Worth per se

Only one teacher elaborated on the contribution of her mathematical studies to her knowledge about the worth of mathematics per se. She said that now she better understands what beautiful mathematics means and that she tries to present such mathematics to her students. Several more teachers mentioned this topic but they either did not elaborate on it or rather contrasted it (as the excerpt's from I's interview above illustrates) with the practical worth of mathematics.

CONCLUSION

As shown, the conceptual framework we developed, based on interviews with mathematicians about what they wished to teach secondary school teachers about the discipline of mathematics (Hoffmann & Even 2018), was useful for examining the contribution of academic mathematical studies to secondary school teachers' knowledge about the discipline of mathematics, and in turn to their practice. Analysis of interviews with teachers, who graduated from a program in which academic-level mathematics that focus on specific fields (geometry, analysis, etc.) comprised a main component, generated the same three aspects and nine topics that the mathematicians who taught in that program mentioned. Furthermore, no additional aspects or topics were mentioned by the teachers.

Although each of the three aspects was addressed by all or almost all the teachers, the number of teachers that addressed each topic of the coding scheme varied considerably among and within aspects. For example, most teachers (8-10) mentioned the contribution of the mathematics courses they studied to the following five topics: *wide and varied*, *lively and developing*, *asking questions and explaining why*, *using intuition and formalism*, and *practical worth*. However, only few teachers (1-4) mentioned the following three topics: *deductively structured*, *experiencing struggles and insights*, and *worth per se*. As expected, analysis of the teachers' interviews indicated that whereas the topics that were adressed by most teachers included characteristics that were new and exciting for the teachers, the topics that were adressed by a small number of teachers included characteristics with which the teachers were already familiar. Yet, the topic *experiencing struggles and insights* was an exception. It appears that for quite a few of the teachers, having to struggle themselves with the mathematics during their studies, impeded their ability to view struggles as a general characteristic of *doing* mathematics. Instead, they perceived it as characterising *learning* mathematics only.

Our findings suggest that all participating teachers learned something new about the discipline of mathematics in the academic mathematics studies. Yet, even though all the teachers studied the same academic mathematics courses with the same instructors, different teachers attended to different topics. More research is needed to better understand these variances among teachers, and how they relate to teacher characteristics and to course instruction.

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IMPACT OF DIFFERENT PROBLEM CONTEXTS ON STUDENTS' TASK PERFORMANCE IN THE HOSPITAL PROBLEM

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This paper aims to clarify whether differences in problem contexts affect students' task performance in the hospital problem. Our survey showed that 29/79 students in a Japanese junior high school applied different thinking approaches to different problem contexts. Some students made decisions based on personal life experience, and some made decisions based on their experience of school-based probability education. These results confirm the relationship between contextual knowledge and probabilistic knowledge. Our results and consequent discussion underline the necessity, when teaching probability, of raising awareness of the relationship between probabilistic knowledge and contextual knowledge, and of emphasising that examples of "equal likelihood" presented in probability problems represent subjective assumptions.

INTRODUCTION

In research of misconceptions regarding probability, a commonly highlighted issue is that many people neglect to consider the effect of sample size (Fischbein & Schnarch, 1997). According to the law of large numbers, the larger the sample size, the smaller the fluctuation of the statistic (Fischbein & Schnarch, 1997); however, many people make decisions without considering the size of the corresponding fluctuation, meaning they do not factor in the effects of the sample size (Fischbein & Schnarch, 1997). This issue has been examined across various fields, including mathematics education and psychology; notably, it is often investigated through consideration of the "hospital problem" (Figure 1) developed by Kahneman and Tversky (1972).

> A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50% of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than 50%, sometimes lower.
>
> For a period of 1 year, each hospital recorded the days on which (more/less) than 60% of the babies born were boys. Which hospital do you think recorded more such days?
>
> (The larger hospital / The smaller hospital / About the same)

Figure 1: Hospital problem (Kahneman & Tversky, 1972, p. 443)

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Previous studies on the reasons people do not consider sample sizes in probability problems have suggested that both the character of the problems and the respondents' characteristics are influencing factors (Weixler, Sommerhoff, & Ufer, 2019). One such factor is the "problem context"; for the hospital problem, the context is the presentation of births in a hospital. Weixler et al. (2019) presented subjects with the hospital problem and a similar problem featuring a coin-toss situation. They found no difference between the correct answer rates for the hospital context problem and the coin context problem. This indicated that the problem context has little influence on respondents' task performance in the hospital problem.

However, Weixler et al.'s (2019) conclusion was only derived from comparing the correct answer rate for the hospital context with that for the coin context. It is not clear whether, for example, the respondents applied the same task-performance approach in both problem contexts or different task-performance approaches across the two problem contexts. The latter would indicate that the problem context influences respondents' task performance in the hospital problem.

Considering the above, the present paper aims to clarify whether differences in problem contexts affect respondents' task performances in the hospital problem. To achieve this, we conducted a survey and analysed and discussed the results. From these results, we derive implications for the teaching of probability in schools.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

The hospital problem and its context

Several previous studies, such as Fischbein and Schnarch (1997) and Watson and Callingham (2013), have considered problem contexts. However, no study has adequately investigated the relationship between problem contexts and correct answer rates because the problem structure is significantly different from the original hospital problem, or the problem structure has a different numerical setting. Weixler et al. (2019) conducted a survey of 242 mathematics teacher education students at a German university, using two problems that featured the same problem structure and numerical data, but that concerned births in hospitals and numbers of coin tosses, respectively (Figure 2). They consequently found no difference between the respective correct answer rates for the problem featuring the hospital context and the problem featuring the coin context. From this, Weixler et al. (2019) concluded that, as the two problems featured the same problem structure, problem context has little effect on respondents' task performance in the hospital problem.

However, as mentioned above, Weixler et al.'s (2019) results do not comprehensively clarify whether problem context influences a change in task-performance approach. Respondents may or may not apply the same approach in each context. Individuals who select "is less likely than" in the hospital context and "is as likely as" in the coin context (both of which are incorrect responses and would consequently, when only correct answer rate is considered, be grouped together) may not be applying the same task-performance approach; this would represent problem context influencing respondents' task performance in the hospital problem.

[Hospital context]	[Coin context]
Children are born again and again in the University-Hospital. That in 10 births at least 7 times a boy is born (is more likely than / is as likely as / is less likely than) that in 100 births at least 70 times a boy is born.	Coins are thrown again and again into the Trevi-Fountain. That in 10 throws at least 7 times tails are face up (is more likely than / is as likely as / is less likely than) that in 100 throws at least 70 times tails are face up.

Figure 2: Part of survey items by Weixler, Sommerhoff, & Ufer (2019)

Research questions

Fischbein and Schnarch (1997) and Matsuura (2006; who translated Fischbein and Schnarch's [1997] survey items into Japanese and surveyed a pool of Japanese students ranging from 5th grade to university) both found that, in the hospital problem, the more knowledge students have of probability, the more likely they are to choose the typical incorrect answer "about the same". This suggests that learning probability alters associated thinking approaches. Consequently, along with problem contexts, we sought to examine the influence of learning probability on task-performance approaches in the hospital problem. Two research questions were developed:

(RQ 1) How do problem contexts affect students' task performance in the hospital problem?

(RQ 2) How does students' knowledge of probability affect their task performance in the hospital problem across different problem contexts?

METHOD

Survey problems

The survey problems used in this paper are shown in Figure 3. These problems were created with reference to Weixler et al. (2019) and Kahneman and Tversky (1972). The difference between the problems used in this study and those of Weixler et al. (2019) is that the former state that the birth of a boy or a girl is generally equally likely and that receiving heads or tails in the coin toss is also generally equally likely. In classical probability, it is commonly implicitly assumed that individual results (for example, 1 to 6 when we throw a die) are equally likely (Borovcnik & Kapadia, 2014); thus, Weixler et al. (2019) did not mention events that are equally likely. However, to ensure a uniform interpretation of the survey items across the sample, it is necessary to state which events are equally likely. In fact, in Weixler et al.'s (2019) coin problem, respondents may consider that the coin is biased. Meanwhile, in the real world the respective likelihoods of giving birth to a boy or girl are not exactly equal; in Japan, more boys are born annually (see: Portal Site of Official Statistics of Japan, 2020). Thus, we decided to explicitly state which events are equally likely. Further, the numerical settings in this study's problems differ from those in Weixler et al. (2019). First, the frequency value was set to 60% instead of 70%. This was because Weixler et al.'s is the only version of the hospital problem to feature a frequency value of 70%; in other studies (e.g., Fischbein & Schnarch, 1997; Watson & Callingham, 2013), the frequency matches that used by Kahneman and Tversky (1972): 60%. Next, we used sample sizes of 45 and 15 rather than 100 and 10, respectively. Again, this decision was based on the approaches used in various previous studies, such as Kahneman and Tversky (1972) and Fischbein and Schnarch (1997).

[Hospital context]	[Coin context]
In a certain town there are two hospitals, a small one in which there are, on the average, about 15 births a day and a big one in which there are, on the average, about 45 births a day. The likelihood of giving birth to a boy is approximately 50% (nevertheless, there are days on which more than 50% of the babies born were boys, and there are days on which fewer than 50% of the babies born were boys.) In the small hospital a record has been kept during the year of the days in which the total number of boys born was greater than 9, which represents more than 60% of the total births in the small hospital. In the big hospital, they have kept a record during the year of the days in which there were more than 27 boys born, which represents more than 60% of the births. In which of the two hospitals were there more such days? a. The big hospital b. The small hospital (Correct) c. About the same	Kazu and Manabu are both tossing a fair coin. Kazu tosses his coin, on average, about 15 times a day, and Manabu tosses his coin, on average, about 45 times a day. For both coins, the likelihood of heads occurring is approximately 50% (nevertheless, there are days on which heads occurs for more than 50% of the tosses, and there are days on which heads occurs for fewer than 50% of the tosses). Kazu has kept a record during the year of the days on which heads occurred more than nine times, which would represent over 60% of his daily tosses of the fair coin. Meanwhile, Manabu has kept a record during the year of the days on which heads occurred more than 27 times, which would also represent over 60% of his daily tosses of the fair coin. Which of the two boys recorded the highest number of days? a. Kazu (Correct) b. Manabu c. About the same

Figure 3: Our survey items

If, for the problems in Figure 3, respondents select (a, b), (b, a), or (c, c) ([hospital context], [coin context], respectively), this would indicate that the problem contexts did not impact their task performance. However, selection of any other combination would suggest that the problem contexts influenced their task performance.

Sample and procedure

The survey was conducted in July 2020 on 39 students from grade 7 (aged 12–13 years) and 40 students from grade 9 (aged 14–15 years) of a junior high school attached to a national university in Japan. The reason that 40 students were selected from each grade is that there are 40 students per class in a Japanese junior high school. The survey was to be conducted on 40 students from grade 7 too, but one of the students was absent on the day. In Japan, frequentist probability is taught in the first half of grade 8, while classical probability is taught in the latter half of grade 8 and in high school (grades 10–12; Otaki, 2019). Thus, we felt that junior high school (grades 7–9) students would have a greater understanding of the effect of sample size than would students in other grades; notably, in Matsuura (2006) the correct answer rate was higher among 8th-grade students (44%) when compared to university students (19%). The reason we targeted students in grades 7 and 9 was that in grade 8, probability is taught using coins and dice as teaching materials (Otaki, 2019). Therefore, it is possible that 9th-grade students who have learned probability have strong familiarity with the problem context of coin tossing, but not with the problem context of births in a hospital; alternatively, it is possible that learning probability in grade 8 using coins helps students correctly discern solutions even in the context of births in a hospital. On the other hand, 7th-grade students, who have not learned probability, may take a different approach to the problem context of coin tossing, which they are likely to have experienced in their daily lives, when compared to the problem context of births in a hospital, with which they would generally be unfamiliar. Pfannkuch et al. (2016) highlighted that contextual knowledge and probabilistic knowledge are interrelated.

In this survey, the [hospital context] and [coin context] problems shown in Figure 3 were distributed to the students separately. Students were asked to choose one of the options ("a" to "c"), and to provide the reason for their choice. They were informed of the correct answers after they had provided answers to both problems.

RESULTS AND ANALYSIS

The results are shown in Table 1 and 2. One 9th-grade student did not answer (N) either problem, saying that "'a,' 'b,' and 'c' are all incorrect". This student was considered to have applied the same task-performance approach in both contexts.

First, we examined the relationship between problem context and students' task performance (RQ 1). Overall, 50 of the 79 students were considered to adopt similar task-performance approaches across the two contexts: those who answered (a, b), (b, a), (c, c), or (N, N); ([hospital context], [coin context], respectively). Meanwhile, 29 of the 79 students were considered to have adopted different task-performance approaches for the two contexts. Thus, approximately 37% of the students applied different approaches to the two problem contexts.

Next, we examined the relationship between the students' existing knowledge and their task performance (RQ 2). Overall, 22 of the 39 7th-grade students were considered to have adopted the same task-performance approach across the two problem contexts, and 17 were considered to have adopted different task-performance approaches. Meanwhile, 28 of the 40 9th-grade students were considered to have adopted the same task-performance approach across the different contexts, and 12 were considered to have adopted different task-performance approaches. Thus, the percentage of students who adopted the same task-performance approach across the two contexts was higher among the 9th-grade students, who had learned probability, than the 7th-grade students.

		Hospital context			
		a	b	c	N
Coin context	a	4	5	2	0
	b	9	2	2	0
	c	3	4	8	0
	N	0	0	0	0

Table 1: Results for the 7th-grade students ($n = 39$)

		Hospital context			
		a	b	c	N
Coin context	a	2	9	0	0
	b	5	3	2	0
	c	2	3	13	0
	N	0	0	0	1

Table 2: Results for the 9th-grade students ($n = 40$)

DISCUSSION AND IMPLICATIONS

Discussion of the survey results

The survey results indicate that problem context frequently influence students' task performance in the hospital problem. Some 7th-grade students who selected "b" ("Manabu") for the [coin context] problem and "c" ("about the same") for the [hospital context] problem explained their decisions as follows: Their life experiences caused them to believe that people can improve their coin-tossing ability through repetitive practice, and can consequently improve their ability to receive heads; thus, they chose Manabu, who performed more tosses than Kazu. On the other hand, for births in a hospital, they chose "about the same" because they could not judge which was more likely to occur. The factor underlying these students' judgment is their life experiences. According to Pfannkuch et al. (2016), contextual knowledge and probabilistic knowledge are interrelated, and different probabilistic knowledge is used in different contexts. Therefore, it is considered that students who utilise their own experiences as contextual knowledge change their task performance depending on the problem context.

Some 9th-grade students who selected "c" ("about the same") for the [coin context] and "a" or "b" ("the big hospital" or "the small hospital") for the [hospital context] reported selecting "c" ("about the same") for the [coin context] because they knew from learning probability in junior high school that there is an equal likelihood of heads or tails occurring in a coin toss. On the other hand, they knew from probability studies that the likelihood of giving birth to a boy or girl is actually not equal. Students make such judgments because they think the probabilities they learn in school apply to real-world events (e.g., coin tosses and baby gender), even though probabilities apply to our information about that world at any given moment in time (Devlin, 2014). Thus, students feel that the presentation in education of an event as having an equal likelihood reflects the nature of that event in the real world, rather than a human-manufactured assumption. Therefore, for them, the likelihood of heads or tails in a coin toss, which is presented in probability education as equally likely, is equal in the real world, but the likelihood of giving birth to a boy or girl, which is presented in probability studies as being unequally likely, is not equal. This thinking approach is natural among students who have learned probability because their probability education is based on performing measures of the frequencies by which various kinds of outcomes occur (Devlin, 2014). From the above, it can be considered that students educated in such probability recognition change their task-performance approach depending on the problem context, even if the problem structure remains the same.

Implications for teaching probability

The above discussion underlines two elements that are essential for teaching of probability. First, teachers and students should be aware that probabilistic knowledge is dependent on contextual knowledge. The present results suggest that students may not always answer correctly across different contexts, even if the problem structure remains the same. Teachers should take this into consideration, and should present in probability class diverse problems in diverse contexts. It is possible that exposure to a range of problems and contexts can help students acquire sufficient probabilistic knowledge to negotiate different contexts; however, it is preferable to present a single problem across various contexts, rather than presenting each problem in a unique context. Such an approach may help students learn to think critically and avoid erroneous judgments, as it can encourage them to self-reflect and change their thinking approaches depending on the context of the problem.

Second, teachers should emphasise which events are assumed to be “equally likely” in that problem. Experts in classical probability may be able to apply the correct understanding that, even if the assumption is implicit, probabilities relate to our information about events in a specific situation. However, this is not easy for students. Our results indicate that students think “equally likely” is not an assumption but the nature of the event in the problem. Thus, teachers, when presenting probability problems, should state which events are equally likely and that this likelihood is not linked to the corresponding real-world event but is an assumption.

CONCLUSION

The results of the present survey suggest that differences in problem contexts affect students’ task performance regarding the hospital problem. This differs from the findings of Weixler et al. (2019). Contextual knowledge and probabilistic knowledge have been found to be interrelated, regardless of the problem structure (Pfannkuch et al., 2016); this supports the validity of this study's finding that students' task-performance approaches can differ depending on the problem context.

A future task is to develop and administer lessons based on the teaching suggestions presented in this paper.

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NOTICING ENHANCEMENT THROUGH THE RECONSTRUCTION OF PRACTICAL ARGUMENTS

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Professional noticing of children's mathematical thinking implies teachers' ability to use their knowledge to attend to students' strategies, interpret their understanding and decide how to respond. Developing this competence implies a shift from general descriptions to descriptions that include teachers' reasoning based on students' understanding. Practical arguments are post hoc examinations of actions that serve to explain or justify what was done. This study focuses on examining to which extent the elicitation and reconstruction of practical arguments help pre-service teachers enhance the noticing skill during their internship period at school. Participants were 17 pre-service teachers. Results show evidence that pre-service teachers were able to develop progressively more complete practical arguments during this period.

INTRODUCTION AND THEORETICAL FRAMEWORK

Jacobs, Lamb and Philipp (2010) conceptualised professional noticing of children's mathematical thinking as a progression through three interrelated skills: (i) attending to children's strategies (ii) interpreting children's understanding, and (iii) deciding how to respond based on children's understanding. Professional noticing of children's mathematical thinking implies teachers' ability to use their knowledge (mathematical content knowledge and pedagogical content knowledge) to attend to, interpret and decide what to do next (Brown, Fernández, Helliwell, & Llinares, 2020; Thomas, Jong, Fisher, & Schack, 2017). Therefore, developing this skill in teacher education programs can prepare pre-service teachers for classroom practice.

Previous research has shown that developing this skill in teacher education programs is a challenging task, examining different contexts for its development (Amador, 2020; Fernández & Choy, 2020; Schack, Fisher, & Wilhelm, 2017). A crucial common assumption underlying them all is that changes in pre-service teachers' discourse on students' mathematical thinking indicate changes in their noticing expertise. In other words, the development of noticing can be inferred from their written professional discourse and perceived as a shift from general strategy descriptions, to descriptions that included teachers' reasoning based on mathematically relevant details of students' mathematical thinking (Ivars, Fernández, Llinares, & Choy, 2018). The internship period at primary schools is a proper context to develop pre-service teachers' noticing. However, as far as we know, little is known about its development during this period (Fernández, Llinares, & Rojas, 2020; Stockero, 2020).

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The notion of practical argument has been used from general educational perspectives to study how pre-service teachers learn to reason about a teaching situation (Fenstermacher & Richardson, 1993; Vesterinen, Toom & Krokfors, 2014). Practical arguments are *post hoc* examinations of actions that serve to explain or justify what was done. A complete practical argument comprises four types of premises and is concluded with an action or intention to act (conclusion of the practical argument; Fenstermacher & Richardson, 1993). These premises are:

- Situational premises: statements that describe the context of the situation.
- Stipulative premises: statements arising from the theory that interpret what happened in the situation.
- Empirical premises: statements that provide evidence or empirical support for future action.
- Value premises: statements that include information regarding the benefit derived from performing a future action.

In developing reasoning about teaching situations, Fenstermacher and Richardson (1993) consider two different processes: eliciting and reconstructing a practical argument. Eliciting is the process by which pre-service teachers provide reasoned descriptions of the teaching-learning situation and the actions taken. Reconstructing is the process by which pre-service teachers assess the practical argument elicited to improve it. From this perspective, engaging pre-service teachers in the elicitation and reconstruction of practical arguments could help them learn to reason about students' understanding and decide how to respond considering students' understanding. Therefore, if pre-service teachers provide progressively more complete practical arguments (seen as a growth in their professional discourse), from the point of view of the premises indicated above, this could reflect the development of noticing children's mathematical thinking skill.

Thus, in this study, we aimed to contribute to the field of noticing examining to which extent the elicitation and reconstruction of practical arguments help pre-service teachers enhance the noticing skill during their internship period at school.

METHOD

Participants and instrument

Participants were 17 pre-service primary school teachers (PTs) enrolled in the last year of their four-year-degree to become a primary school teacher. They were in the internship period at primary schools (8 weeks). In the first two weeks, they had to observe the school tutor's teaching and the remaining six weeks they had to plan and implement a lesson. PTs had completed two mathematics education courses related to numerical and geometrical sense and a mathematics method course.

Writing narratives can be considered a facilitator for eliciting practical arguments. A narrative is a story described sequentially in which the author recounts some events that, according to his/her internal logic makes sense to him or her (Chapman, 2008). Moreover, writing narratives can be used as a tool that allows PTs to generate explanatory schemes and relationships between theory and practice (Ivars & Fernández, 2018; Schultz & Ravitch, 2013).

Writing narratives provides a context in which: (i) PTs can express and describe what they consider relevant in a teaching-learning situation related to students' thinking (providing situational premises); (ii) PTs can interpret students' understanding using their knowledge about mathematics and the teaching and learning of mathematics (providing stipulative and empirical premises), and (iii) PTs can propose a learning objective and new actions to continue the instruction (providing value premises to conclude the argument).

PTs had to write two narratives in which they considered that students were developing mathematical competence. PTs were provided with the following prompts as a guide to write their narratives:

- *Describe in detail* a mathematics teaching-learning situation. The task (curricula contents, materials, resources…). What did the primary school students do? For example, you can indicate some students' answers to the task, difficulties…What did the teacher do? For instance, you can describe the methodology and some aspects of the interactions.
- *Interpret the situation*. Indicate the mathematical objectives of the task and how the implementation of the task pursued the objective. Indicate evidence of students' answers that show how they achieved the objectives (students' understanding of the mathematical content) and/or the difficulties they had.
- *Complete the situation.* Complete the situation indicating what to do next to support students in their conceptual understanding.

After PTs had written the first narrative during the observation period, the narrative was shared with their university tutor who provided them with feedback. Feedback provided are tutor prompts to promote pre-service teachers' learning, providing information that guides them towards the learning objectives (Wang, Gong, Xu, & Hu 2019). In our study, the role of feedback is relevant since it can help pre-service teachers to reconstruct their practical arguments. The feedback focused on the premises: asking for more detailed descriptions of students' mathematical strategies, interpretation of students' understanding, more evidence to support their actions, and information regarding the benefit derived from performing a future action. Afterwards, PTs wrote a second narrative during their practice (reconstruction of the practical argument after feedback). Data of this research are the two narratives written by PTs.

Analysis

In the analysis, we focused on whether the narratives included:

- Situational premises: whether PTs provided detailed descriptions of the teaching-learning situation and students' strategies.

- Stipulative premises: whether PTs interpreted students' understanding using their mathematical knowledge and knowledge of teaching and learning mathematics.
- Empirical premises: whether PTs provided evidence from students' understanding that can support their actions.
- Value premises: whether PTs provided statements that include information regarding the benefit derived from performing a future action.

Three researchers analysed the two narratives individually, then, compared, and discussed their agreements and discrepancies (triangulation process) until a consensus was reached. Afterwards, changes in PTs practical arguments regarding the premises were identified.

RESULTS

Table 1 shows the changes in the premises included by pre-service teachers in their narratives. In the first narrative, nine out of the 17 PTs provided only situational premises. These PTs only described students' strategies identifying some mathematical details. Eight out of the 17 PTs also provided stipulative premises interpreting students' understanding. These PTs linked the mathematical details identified with their mathematical knowledge and knowledge of teaching and learning mathematics. However, only three PTs provided evidence from students' understanding supporting their future teaching decisions (empirical premises) and included information regarding the benefit derived from performing this future action (value premises).

	Situational premises	Situational premises Stipulative premises	Situational premises Stipulative premises Empirical premises Value premises
1st narrative	9	5	3
2nd narrative	2	6	9

Table 1. Premises included in the narratives of pre-service teachers.

In the second narrative, 15 out of the 17 PTs provided situational and stipulative premises. Moreover, nine out of these 15 PTs provided empirical and value premises.

These data highlight that PTs were able to develop progressively more complete practical arguments showing evidence of noticing enhancement. This finding suggests that reconstructing practical arguments during the period of practices at schools and the feedback provided seem to be powerful tools to develop noticing. We are going to show extracts of the narratives written by PT03 and the feedback provided as an instance of this result.

The first narrative of PT03 -Eliciting-

In his first narrative, PT03 described a 2nd-grade classroom teaching-learning situation with 21 primary school students focused on problem-solving. This PT only provided situational premises describing the problem solved and some students' difficulties. Following, some excerpts from his narrative are shown:

> Maria is saving money to buy a dollhouse in two different piggy banks. There are 325 euros in the first piggy bank and, 172 euros in the second piggy bank. How many more euros does the first piggy bank have than the second one?
>
> […] instead of performing a subtraction operation to find out the difference between both piggy banks, they performed an addition operation. When the teacher asked one of these students: "What should we do to solve the problem?" he said "We have to add" "Why? "Because it appears the word *more* in the problem".
>
> When students read the problem, they realised that they had to perform a mathematical operation to solve the problem. However, some of them had difficulties since they did not identify the correct operation to solve it.

PT03 focused his attention on the difficulty of performing "the correct operation" since students were linking the word *more* from the problem with the use of addition. Therefore, this PT identified and described an important mathematical detail of students' strategies (situational premises). Nevertheless, PT03 associated students' difficulties with the incorrect identification of the operation without providing additional information. He did not interpret students' difficulties; for example, using knowledge of the type of problem solved (stipulative premises). For instance, the missing-value is the difference in this comparison word problem. These word problems are more difficult than the change, and combination word problems (Fuson, 1992) and students try to associate the words that appear in the problem with a type of operation (*more* with addition and *less* with subtraction).

This PT concluded his narrative providing a future action focused on the use of smaller numbers in the problem:

> […] presenting the problem, first, with smaller numbers to make the situation easier and introducing gradually bigger natural numbers […].

PT03 focused his future action on using smaller numbers instead of focusing on students' understanding that would imply working on the relationships between quantities involved in the problem (identifying the structure of the problem). This part of the narrative shows that PTs' difficulties providing stipulative, empirical and value premises did not let him thoroughly interpret the situation to make a teaching decision considering students' understanding.

Tutor's feedback

The university tutor's feedback focused this PT attention on providing stipulative premises to interpret students' understanding: "[…] you should analyse students' difficulties deeply considering, for instance, the characteristics of the problem […]". Besides, the feedback asked for more concrete future actions based on students' understanding (empirical and value premises): "[…] it would be better to suggest a concrete action focused on students' understanding […]. How would you help them overcome their difficulties with the problem?"

The second narrative of PT03 -Eliciting and Reconstructing-

In the second narrative, PT03 described a teaching-learning situation that involves the subtraction algorithm. He provided descriptions of the teaching-learning situation using situational premises:

> I am going to describe a situation that happened in a 2nd-grade classroom of a primary school. Students were solving individually different activities involving subtractions.

PT03 described some students' difficulties:

> Initially, it seemed that the great majority of students understood what to do. Nevertheless, I corrected the subtractions, individually, and I identified that some students had difficulties with the subtraction algorithm since they answered "365" as a solution for the subtraction 418-173.

In the following excerpt, PT03 used his knowledge about the characteristics of the decimal numbering system (place value and the idea of grouping, Battista, 2012) to interpret students' difficulties with the subtraction algorithm (providing stipulative premises):

> […] These students seemed to understand that they always have to subtract a smaller digit from a bigger one. So, they seemed to have acquired a limited knowledge of subtraction; especially they seemed not to understand the place value of the digits and how to subtract by regrouping […].

Moreover, he provided evidence from the situation to support his interpretation through empirical premises when he wrote:

> […] I confirmed my thoughts when I asked one student to explain how she solved the subtraction 418-173 and she answered: From 3 to 8, 5, from 1 to 7, 6 and, from 1 to 4, 3.

Continuing with his narrative PT03 wrote a value premise when he provided a learning objective and included information regarding the benefit derived from performing a future teaching action using didactical resources that help students progress in their learning. He used his knowledge of teaching and learning mathematics to deal with the situation considering students' difficulties:

> […] we should work the decomposition of the numbers from the beginning. First, the canonical decomposition of the numbers using base ten blocks to help students understand that 1 ten is 10 units or that 1 hundred is 100 units or 10 tens. Following, we should work with multiple decompositions of the numbers to help students understand how to group and regroup numbers and, understand the value represented by a digit in a number (place value). For instance, a decomposition of 117 is one hundred, one ten and seven units and other decompositions are 11 tens and 7 units or 9 tens and 27 units.

This value premise (and implicitly an empirical premise) was followed by an intended action linked with the interpretation of the situation considering students' understanding of the subtraction algorithm.

> […] Working with the different decompositions of the number will help students understand how the subtraction by regrouping works. In this case, I would provide students with base ten blocks and ask them to represent the quantity of the minuend (418) and subtract the quantity of the subtrahend (173). Then, using the base-ten blocks, students would realise that to subtract 7 tens to 1 ten they should break a hundred into 10 tens ($418 = 3 \cdot 100 + 11 \cdot 10 + 8$).

The more complete practical argument provided in his second narrative (evidenced by the premises given) can be considered evidence of noticing enhancement.

DISCUSSION AND CONCLUSIONS

Results indicate changes in the PTs' practical arguments regarding the premises considered. These changes derived in providing complete practical arguments progressively. Therefore, the reconstruction of practical arguments and the university tutor's feedback allowed PTs to explicitly establish the connections between what is observed (data, evidence) and theoretical knowledge that can help them to interpret the situation and justify future teaching actions (Ivars et al., 2018). This finding suggests that reconstructing practical arguments during the internship period at primary schools and the tutor's feedback seem to be powerful tools for the enhancement of noticing.

The reported changes allow us to identify the potential of the prompts provided to pre-service teachers to write the narratives and the nature of the tutor's feedback addressed to give more details of students' understanding and make more explicit the reasons behind their interpretations and actions. It seems that the tutor's feedback pushes PTs to generate complete practical arguments providing reasons (premises) for their actions (Vesterinen et al., 2014). Therefore, our results contribute to the field of noticing examining other contexts for the enhancement of noticing in teacher training programs while pre-service teachers are in the period of apprenticeship at schools (Stockero, 2020).

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COORDINATING CO-OCCURENCE AND SAME EXTENT WHEN GENERATING LINEAR EQUATIONS

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We conducted 1-on-1 interviews to investigate how 10 future middle grades mathematics teachers in the United States generated equations to express proportional relationships between two co-varying quantities. Our analysis revealed the importance of coordinating 2 perspectives on quantitative relationships we term the co-occurrence and same extent perspectives. We argue that past research on school algebra has overlooked as important the interplay between the 2 perspectives and demonstrate that the future teachers were successful generating normatively correct equations when they accounted for and coordinated both. The results provide new insight into proficient reasoning when generating equations that model problem situations.

INTRODUCTION

We conducted one-on-one interviews to investigate how 10 future middle grades mathematics teachers generated equations to model proportional relationships between two co-varying quantities. When analysing where these future teachers were more and less successful generating normatively correct equations, we came to see that a critical aspect of their proficient reasoning has been overlooked by past research. In particular, the future teachers succeeded when they coordinated two perspectives on relationships between quantities that we call the co-occurrence and same extent perspectives.

To illustrate the two perspectives, consider a situation in which gold and copper are mixed in a 7-to-5 ratio. When generating an equation, teachers or students might intend to express weights of gold and copper that co-occur—for instance, 7 ounces of gold co-occur with 5 ounces of copper. Alternatively, they might intend to express weights of gold and copper that are the same—for instance, 5 ounces of gold weigh the same as 5 ounces of copper. More generally, a person might want to convey two quantities that co-occur or go together or two quantities that have the same extent, such as the same weight or volume. We found that when future teachers generated and explained normatively correct equations that modelled proportional relationships, they successfully coordinated these two perspectives.

BACKGROUND

Past research has delineated different strands of school algebra, and the two strands pursued most often have emphasized algebra as an extension or generalization of arithmetic and as the study of functions (e.g., Kaput, 2008; Kieran, 1992, 2007). Much of the research on

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generalized arithmetic has relied on numeric tasks with no reference to quantities in problem situations, while much of the research on functions has emphasized expressing co-occurrence, not same extent, of two quantities. The separate emphasis on these strands is one explanation for why the field has overlooked the importance of coordinating the co-occurrence and same extent perspectives.

Equations in the context of generalized arithmetic

Earlier research on school algebra as generalized arithmetic reported difficulties that students have with the use of letters (e.g., Küchemann, 1981), rules for transforming expressions (e.g., Matz, 1982), and interpretations of the equal sign (e.g., Kieran 1981). For purposes of locating the contribution of the present study, an important finding has been students' tendency to use an *operator* perspective in which the equal sign separates an arithmetic expression on the left-hand side from an evaluated answer expressed as a single number on the right-hand side. This perspective contributes to students' tendencies to use the equal sign in a chain of computations, as in 2.3 + 3.2 = 5.5 – 1.5 = 4 (e.g., Kieran, 1992), and difficulties making sense of equations like _ = 3 + 4, 4 + 5 = 3 + 6, and 2 x 6 = 10 + 2 (e.g., Kieran, 1981). Researchers have contrasted the operator perspective with a *relational* perspective in which the equal sign indicates two equal values or equivalent expressions, which is consistent with same extent. More recent research has continued to rely on numerical tasks when assessing students' interpretations of the equal sign (e.g., Blanton et al., 2015, 2019; Carpenter, Franke, & Levi, 2003; Carraher & Schliemann, 2007; Jones & Pratt, 2012). Further recent studies of teachers' attention to the operator and relational interpretations have also relied on numerical tasks (e.g., Asquith, Stephens, Knuth, & Alibali, 2007; Hohensee, 2017; Prediger, 2010). One finding is that teachers can overestimate the frequency with which students use the relational interpretation.

Equations in the context of functions

Research on school algebra as the study of functions (e.g., Blanton et al., 2015, 2019; Kaput, 2008; Kieran, 1992, 2007) has often focused on connections among tables, graphs, equations, and language. We simply point out that tables and graphs are based on ordered pairs and, thus, provide students experience using algebraic notation to express co-occurrence more directly than same extent.

Equations that model proportional relationships

One of the best known results about equations that model proportional relationships is the so-called Students-and-Professors reversal error (e.g., Clement, 1982). A sample of 150 college engineering students were asked two write an equation for the following statement: "There are six times as many students as professors at this university." Thirty-seven percent answered incorrectly, and two-thirds of the erroneous answers were of the form $6S = P$. One explanation for the error was that students generated equations left-to-right as they read the problem statement phrase-by-phrase. A second explanation was that students intentionally compared a larger group to a smaller group. The first error has been termed "word order matching" and the second "static comparison." Various researchers have continued to investigate similar reversal errors.

THEORETICAL FRAME

In previous reports (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019), we have explicated an overlooked, yet promising, perspective on proportional relationships we term *variable-parts*. This perspective views quantities in terms of fixed numbers of equal-sized parts that can vary in size. Consider again a situation in which gold and copper are mixed in a 7-to-5 ratio. From a *variable-parts perspective*, one imagines fixing seven parts for gold and five parts for copper and allowing the units in each part to increase or decrease in such a way that the number of units in each of the 12 parts is the same (Figure 1b). We emphasized this perspective when designing interview tasks.

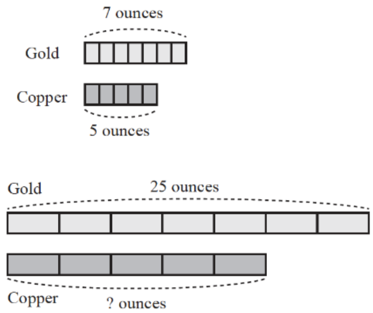


Figure 1. Gold and copper in a 7-to-5 ratio from the variable-parts perspective.

We have found aspects of the knowledge-in-pieces epistemological perspective (e.g., diSessa, 1993, 2006) useful for making sense of how future mathematics teachers reason about topics related to multiplication, including proportional relationships. The most important features of this perspective for the present report are that (a) future teachers can hold multiple ideas about a given problem situation that do not necessarily fit together into a coherent whole, (b) teachers' activation of ideas are often sensitive to particular features of problem situations, and (c) learning does not proceed through a series of hierarchically organized stages but rather through complex processes in which knowledge is refined and reorganized.

METHODS

Data for the present report come from a larger, multi-year study in which we investigated future middle grades mathematics teachers' reasoning about topics related to multiplication. We recruited future teachers enrolled in preparation programs at two large, public universities in the United States and used a survey targeting multiplication and division with fractions to select a mathematically diverse sample for interviews.

During the 80-to-90 minute, semi-structured interviews (Bernard, 2006), the future teachers worked tasks we designed to engender reasoning about variable parts. Typical follow-up questions asked future teachers to explain their thinking in more detail, to discuss any additional ways they might have for thinking about a task, and to consider using a drawing.

Such questions drew out diverse ways in which the future teachers could think about the tasks.

We recorded each interview using two cameras, one to capture the future teacher and the interviewer and one to capture the future teacher's written work. At the end of each interview, we collected all written work for later analysis.

First, we reviewed the videos and verbatim transcripts independently of each other. We generated summaries of each future teacher's reasoning, then met to compare summaries, and resolved discrepancies by reviewing the data in light of points each of us raised. Through several cycles of this process, we improved the fit between verbatim statements, gestures, and inscriptions we observed and our accounts of the concomitant reasoning. Finally, as we identified similarities and differences across cases, we began to suspect that ways future teachers did or did not combine the co-occurrence and same extent perspectives contributed to important differences in their performance.

As we sharpened our analysis, we converged on the following research question: *How did the future teachers' attention to and coordination of the co-occurrence and same extent perspectives contribute to their success when generating equations to model proportional relationships?*

RESULTS

All 10 future teachers attended to both the co-occurrence and the same extent perspectives at one point or another during their interviews. Of these, five coordinated the two perspectives to a much greater extent than the others and, when so doing, were more successful in generating normatively correct equations that modelled proportional relationships between two co-varying quantities. We illustrate this result with two cases, the first illustrates coordinating the two perspectives to a lesser extent and the second illustrates coordinating the two perspectives to a greater extent.

Nina

Nina attended to the co-occurrence and same extent perspectives at different times over six approaches to the *Jewelry Gold* task. The initial version of the task read as follows:

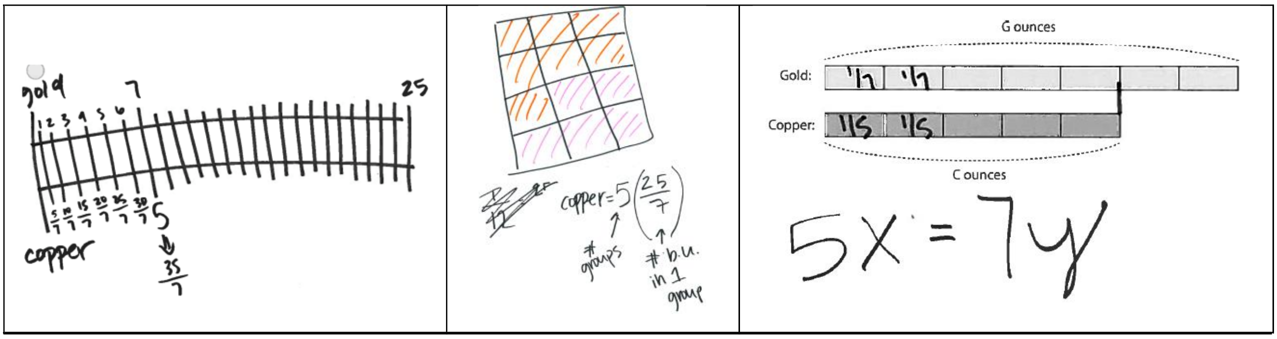
> A company makes jewelry using gold and copper. The company uses different weights of gold and copper on different days, but always consisting of a total of 7 parts gold and a total of 5 parts copper, where all parts weigh the same amount.
>
> One day the company uses 25 ounces of gold. Please explain a relationship among the number of ounces of copper the company uses that day and the 25, 7, and 5.

For her first approach, Nina quickly set up the following proportion: "7/5 = 25/*X*." The interviewer did not follow-up to determine how Nina made sense of her equation.

When the interviewer asked for a solution using a drawing, Nina initiated her second approach for which she drew a double number line. One number line showed ounces of gold and one showed ounces of copper. She used vertically aligned tick marks to coordinate each 1 ounce of gold with 5/7 ounces of copper (Figure 2a). Thus, at least tacitly, she focused on co-occurrence.

When the interviewer asked for other drawings, Nina's third approach was to draw a rectangular array with 12 parts (Figure 2b) and to write "copper = 5 (25/7)." She expressed the ounces in each part correctly, but her explanation of equality was vague:

> You said to write an expression. This isn't really mathematical [gestured across "copper = 5(25/7)"], but because you said that, I put an equal sign, just like the copper, colon, copper is…like it could have been anything. I just put equal.



(a) (b) (c)

Figure 2. Three of Nina's approaches to the *Jewelry Gold* task

When the interviewer asked for "other kinds of expressions that you could generate based off this drawing that you've produced," Nina generated her fourth approach. She quickly wrote "gold = x," "copper = y," and "$7x = 5y$." The last equation was consistent with those characterized as reversal errors. Nina then attended to co-occurrence and same extent but expressed uncertainty:

> What I meant was…like for every seven groups of gold there would always be five groups of copper, but if I look at it in the drawing, then like these seven group…like it's not actually equal. There's more [pointed to the seven gold parts] than there is [pointed to the five parts copper]…there's more gold than there is copper. So I feel like that is confusing.

She commented that equations should "balance," but she was unclear what would balance in the mixture situation. She knew to look for, but did not see, same extent.

Next the interviewer presented a version of the *Jewelry Gold* task that introduced the letter G to stand for the unspecified number of ounces of gold, C to stand for the corresponding number of ounces of copper, and a strip diagram showing seven parts for gold and five parts for copper (Figure 2c). For her fifth approach, Nina commented:

> I feel like if there were five of these whole things [pointed to the seven part strip for gold], it'd be equal to seven of these whole things [pointed to the five parts strip for copper], which would change it.

She then wrote "$5x = 7y$," using x and y instead of G and C, and explained that this was balanced because both sides indicated 35 total parts. This expressed same extent.

When the interviewer asked for any other equations, Nina generated her sixth approach by writing $G = 7/5C$ and $C = 5/7G$. She explained:

> C is five sevenths the size of G makes sense because these are all one seventh of G [wrote "1/7" in two parts of the gold strip, Figure 2c] and it's the size of five of them….one seventh, two seventh, three seventh, four seventh, five seventh. That's the size of C.

She then gave a similar explanation for $G = 7/5C$ and wrote "1/5" in two parts of the copper strip (Figure 2c). Finally, she explained that both sides of her equation "would give the same number of ounces." Nina now articulated same extent explicitly.

Leann

In contrast to Nina, who initially found coordinating the co-occurrence and same-extent perspectives "confusing," Leann coordinated the two perspectives quickly when using both subtraction and multiplication. For her first approach to the *Jewelry Gold* task (the version with the variables C and G), she generated "$C = G – 2$" and explained:

> The way I was thinking about it was if you have seven gold and you're using five copper, every time you get to seven gold you've used five copper, which is two less than the number of gold that you used, which would give you the total number of copper that you used.

Leann's statement that "every time you get to seven gold you've used five copper" was consistent with co-occurrence, while her statement that the number of copper was "two less than the number of gold that you used" was consistent with same extent.

For her second approach, Leann computed $7 \div 5 = 1.4$ and explained:

> So for every one bar—is it bar?—one part of copper, they used 1.4 parts of gold. So if I used, so I guess I want to say the gold equals the copper multiplied by 1.4 [wrote "$G = C$ x 1.4"], since there's 1.4 more gold than copper every time.

Leann's statement that for every one part of copper there would be 1.4 parts of gold was further evidence of attention to co-occurrence. At the same time, Leann's "$G = C$ x 1.4" equation combined with her comment that "gold equals the copper multiplied by 1.4" evidenced attention to same extent. In subsequent work, Leann produced drawings, made co-occurrence statements like "every seven gold is five copper," and made same extent statements such as: "The total gold would equal the total copper plus another two fifths of the copper." Thus, her explanations for correct equations combined the co-occurrence and same extent perspectives fluidly.

DISCUSSION AND CONCLUSION

Our results suggest that coordination of the co-occurrence and same extent perspectives is a subtle, yet important, aspect of school algebra that has been overlooked by past research on students and teachers. In contrast to past research on Students-and-Professors and similar problems, which has characterized static comparison as an impediment to generating normatively correct equations, in our data future teachers were successful when they found ways to account for both perspectives. Nina had to find ways of viewing the jewelry gold mixture that fit with the same extent perspective, while Leann moved fluidly between the co-occurrence and same extent perspectives.

Our results have straightforward implications for teacher education programs. We imagine discussions in courses that identify the co-occurrence and same extent perspectives, highlight where each is useful, and examine how they are not in opposition but can complement one another. Such discussions could help prepare future teachers to recognize when students are similarly challenged by the two perspectives.

Finally, the present study is limited both by the small number of participants and the narrow focus on a small number of tasks that included the *Jewelry Gold* task. We imagine future research that builds on insights reported here by investigating how future teachers and students generate and explain equations using a wider variety of tasks about proportional relationships and tasks in which other relationships are central.

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CHANGES IN MATHEMATICAL REASONING – A LONGITUDINAL STUDY WITH MATHEMATICALLY GIFTED CHILDREN

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Mathematical reasoning shows relevance for the support of mathematical giftedness. In order to combine both aspects from elementary school age, knowledge on two aspects is required: firstly, on changes in mathematical reasoning and secondly on possible connections between mathematical reasoning and the mathematical giftedness profile. This article takes up already known indicators for changes and connections and presents the methodology and results of a longitudinal study with potentially gifted children between 9 and 12 years. Firstly, a scheme for the analysis of the orally produced arguments is developed. Afterwards, a detailed examination of the changes in the categories of Content and Validity results in six types that characterize differences in the temporal changes.

INTRODUCTION

With emphasis on the goal of training the later ability to prove, the importance of argumentation becomes apparent in the support of mathematical giftedness at a young age. Nevertheless, the PISA results, for example, show that even mathematically gifted students in lower secondary schools still find it difficult to argue correctly, completely and conclusively (OECD, 2016). However, it is consensus that the development of the giftedness potential is a process that takes several years and that the promotion of mathematical giftedness should be designed for the long term (Fritzlar & Nolte, 2019). This leads to the question how the long-term training of mathematical reasoning can be reconciled with the support of mathematical giftedness. The article takes up this research interest in a longitudinal study. After the derivation of the research questions, it presents the methodological development of a suitable analysis scheme. This takes into account the age of the children, orality as a form of the survey, the product character of the arguments and the longitudinal character of the study. Furthermore, the results on typical courses of change and their evaluations, as well as on a possible connection between the argumentation characteristics and the giftedness profile is emphasized.

THEORETICAL BACKGROUND

Analysis of mathematical arguments

The relevance of mathematical argumentations opens up the need for suitable analysis possibilities of arguments. In addition to competence models, e.g. in PISA and TIMSS, this is also done with the help of a structural analysis based on the Toulmin scheme. By definition, the core of the Toulmin scheme contains the *Conclusion*, i.e. a statement that asks for legitimation, the *Data*, i.e. the unquestioned statements to which the *Conclusion* is

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attributed and the *Warrant*, i.e. the statement which substantiates the step from *Data* to *Conclusion* (Toulmin, 2003).

The occurrence or non-occurrence of these elements does not allow any statement about the content or the conclusiveness of an argument (Koleza, Metaxa & Poli, 2017). Especially the warrant can be the basis for a distinction whether an argument contains all necessary information for the step from data to conclusion and thus is *complete*, or if the argument leaves questions open and has to be evaluated as *incomplete*. Furthermore, different types of arguments give reason to analyze and categorize arguments in detail. Children, in particular, do not usually argue on a general level or use mathematical rules increasingly as evidence for their arguments (Koleza, Metaxa & Poli, 2017). Therefore, the argument's method can be distinguished, e.g. through being based on an example or on a general mathematical statement. In addition, especially oral arguments might be promoted by questions, e.g. questions that request an evaluation or questions that request a warrant (Cervantes-Barraza & Cabañas-Sánchez, 2020). Therefore, it is possible to analyze the independency of the argument and their creation context.

Mathematical giftedness

Giftedness at a young age is often defined by means of competence definitions, i.e. a potential for an outstanding performance (Lucito, 1964). Such a definition raises the question of appropriate diagnostic methods for giftedness. Especially in the area of mathematical giftedness, there is a tendency to describe the construct with the help of catalogues and systems that contain several characteristics of mathematically gifted people. The system for third- and fourth-graders with a potential mathematical giftedness by Käpnick (1998) and the counterpart for potentially gifted fifth- and sixth-graders by Sjuts (2017) reconcile specific mathematical traits with personality traits of potentially mathematically gifted children, e.g. structuring and creativity. So-called indicator tasks are based on the idea of operationalizing mathematical giftedness with the help of these area-specific characteristics.

RESEARCH QUESTION

Mathematical argumentation and its analysis play a role in international school achievement studies (e.g. PISA and TIMSS), but also in numerous empirical research projects (e.g. Reid & Knipping, 2010). The analysis is done either by classification in complex models or by recourse to schemata such as the Toulmin scheme. The hereby identified characteristics of reasoning are mainly cross-sectionally, i.e. describing mathematical reasoning in one specific age. "Real" descriptions of changes in argumentation derive from the area of developmental psychology. Piaget's Step Theory (Piaget, 2002) and the LOGIK study (Ahnert, Boes & Schneider, 2003) describe cognitive changes in logical thinking and provide indicators of changes in children's general reasoning without reference to mathematical reasoning.

Besides the lack of longitudinal studies on mathematical reasoning, the relation between mathematical giftedness and mathematical argumentation is still uncertain (Durak & Tutak, 2019). On the one hand, it is assumed that there are no differences between mathematically gifted and normally gifted primary school students with regard to the need for proof or argumentative reasoning (Fritzlar, 2011). Instead, the mathematical intuition is taken into account here, which could hypothetically also have a negative impact on the explicitness of argumentation. Contrary to this assumption, the hypothesis that "[m]athematically gifted students [...] tend to show unusual paths of reasoning" (Gutierrez et al. 2018, p. 170) seems to be internationally widespread. This potential for increased reasoning skills is based on a possible cognitive advantage and on characteristics of mathematical giftedness, e.g. a creative approach to problem solving, special abilities in structuring mathematical facts and special abilities in reversing thought processes (Fritzlar & Nolte, 2019).

With reference to the comparison of the systems of characteristics for potentially mathematically gifted third- and fourth-graders by Käpnick (1998) and for potentially mathematically gifted fifth- and sixth-graders by Sjuts (2017), a possible temporal change of the mathematical reasoning in mathematically gifted children becomes obvious. While logical reasoning is not listed in the system for children in the third and fourth grades, it is taken into consideration as a gifted-specific characteristic for children in grades five and six. This addition is based on empirical studies and a theoretical justification with reference to the transition to the formal-operational phase defined by Piaget (2002). This leads to the following research question:

How can typical temporal courses (types) be characterized, evaluated and connected to mathematical giftedness?

METHODOLOGY

In an explorative approach, the indicators for changes in argumentation are investigated. With the emphasis on changes, a longitudinal study with an identical sample and survey method is designed. For a detailed overview on the study design see Jablonski & Ludwig (2019).

The sample comprises 37 participating children from the enrichment program "Junge Mathe-Adler Frankfurt". It is an out-of-school program for the long-term, regular support of mathematically interested and potentially gifted students from Frankfurt, Germany. The children are selected by nomination of their mathematics teachers. The teachers receive indicator tasks in advance that operationalize special mathematical giftedness characteristics.

The children's argumentation products and possible changes are recorded with the help of problem-oriented interviews with arithmetic tasks. The task formats of the interviews are the number pyramid and the numerical lattice as well as the number relations contained therein (see Figure 1 and 2; de Moor, 1980).

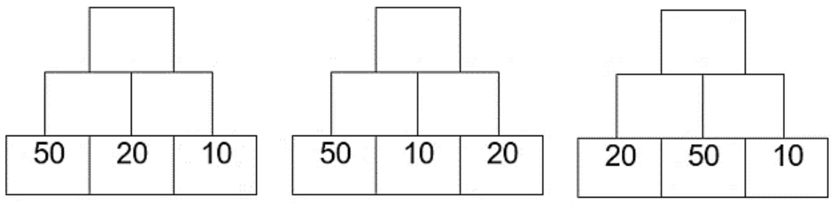

Figure 1: Task format number pyramid

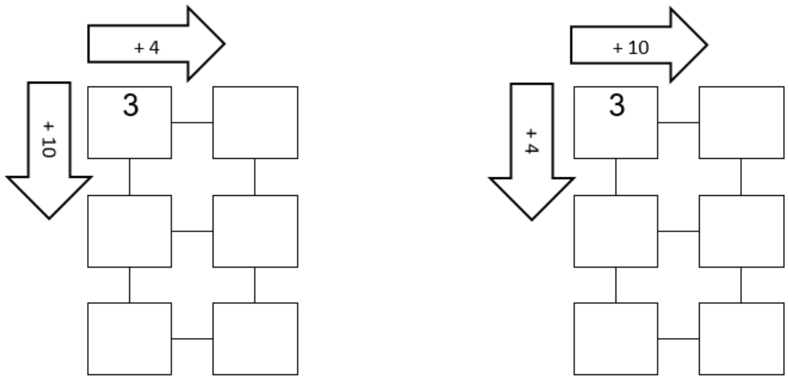

Figure 2: Task formats numerical lattice

First, a statement is presented in which it is claimed that the order of the basic elements (basic stones or arrow numbers) has no influence on the result of the respective format. The children are asked to comment on this (false) statement. Afterwards, the children are asked to investigate the relationship between the basic elements and the result in more detail, for example by asking how they have to be arranged so that the result is maximal. A guideline includes that the children firstly formulate discoveries and that – if not independently formulated – justifications and generalizations are initiated by the interviewer. In the last part of the interview, the children are asked to give reasons for their own "special" number pyramid or a "special" numerical lattice. The data collection of the interviews takes place every six months. There are four interview surveys in total, so that an observation period of 18 months is taken into account.

The resulting argumentation products are coded with the help of an analysis scheme for oral individual interviews in the categories *Structure*, *Content*, *Validity* and *Independence* (see Figure 3).

On the one hand theory-guided, on the other hand empirically tested on the basis of the interviews of the pilot study, the analysis scheme is delimited from already existing analysis schemes. Firstly, the involved Toulmin elements are coded. Afterward, the content and validity of the conclusion and warrant are analysed. Both happens with regards to the context, i.e. whether the element is formulated independently. The scheme is empirically confirmed in the context of the study with good values (Kappa between 0.66 and 0.74) of intercoder reliability.

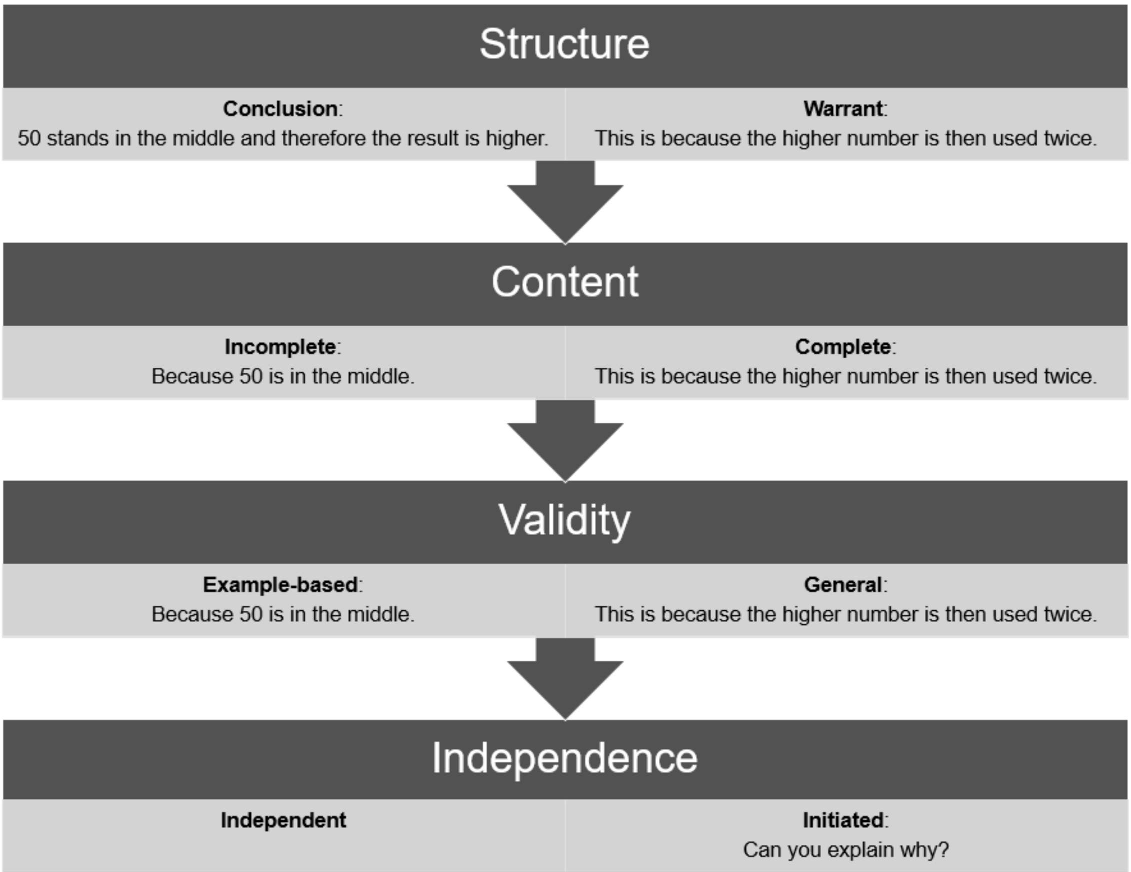

Figure 3: Chosen Categories of the Analysis Scheme

For the typology, two comparative dimensions are selected. These are the categories *Content* and *Validity*, which result in the corresponding feature space:

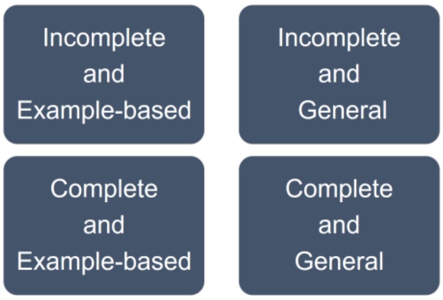

Figure 4: Feature Space as Basis for the Typification

The 37 children are ranked for each of the four data collections by means of empirical boundaries. Finally, the children are grouped according to their longitudinal classification.

RESULTS

Characteristics of the Types

The children can be assigned to the following six types:

- Type 1.1 argues stably generally
- Type 1.2 argues stably generally and completely

- Type 2.1 argues stably incompletely
- Type 2.2 argues stably incompletely and example-based
- Type 3 argues increasingly generally
- Type 4 argues increasingly completely

These types characterize the interindividual differences in the progressions over the study period. Two of these types - type 3 and type 4 - also show intraindividual differences over the period of examination.

Their characterization results from the following properties

(1) concrete numerical examples/direct reference to the material

(2) beyond concrete numerical examples/general facts

(3) missing necessary information to complete the argument

(4) all relevant information for a complete warrant

In Table 1, these characteristics are assigned to the types, whereby a differentiation is made regarding their stability and variability. Additionally, their frequency among the children is quantified by N. Especially type 3 is strongly represented empirically. Therefore, a hypothetical statement on a group-independent increase of generalizing elements can be assumed. All in all, 34 children are allocated to the types, whereby three children show irregular courses of change.

Type	Characteristics	N
1.1	Stably: (2)	6
1.2	Stably: (2) and (4)	5
2.1	Stably: (1)	4
2.2	Stably: (1) and (3)	6
3	Increasing: (2)	10
4	Increasing: (4)	3

Table 1: Chosen Categories of the Analysis Scheme

Problems, Prognoses and Connections to Mathematical Giftedness

Despite the initial descriptive orientation of the analysis, at this point three evaluative classifications of the argumentation characteristics should help in the normative evaluation of the types:

1. The children should be able to formulate a complete warrant.
2. The children should be able to formulate a general, valid conclusion.
3. The children should be able to formulate valid warrants according to the task and be able to generalize them when asked.

The combination of the three normative statements allows a one- or two-dimensional positive classification of the types 1.1, 1.2, 3 and 4. In contrast, there are no positive classifications regarding the types 2.1 and 2.2. On the basis of this evaluative classification,

it is possible to formulate problems and prognoses for those types that do not develop positively. With the help of a qualitative content analysis of all interviews, there are indicators for problems in arguing, especially with regard to types 2.1 and 2.2, e.g. an empirical reasoning (a generalizing conclusion is drawn on the basis of an example-based warrant) and the lack of need for justification (no independently formulated warrants).
These problems occur not exclusively, but predominantly in types 2.1 and 2.2. This raises the question of the extent to which early diagnosis and support are possible. This is done retrospectively with the help of prognoses for a classification in the first data collection to a type. According to this, a low rate of generalizing elements on the one hand and a low rate of complete elements on the other hand serve as indicators for a later assignment to a non-positive type. Conversely, especially the generalizing argumentation can be seen as an indication for an assignment to a positive running type.

Also in the context of generalizing argumentation, there are possible connections to increased abilities in structuring and mathematical creativity. In the evaluation of the indicator tasks, the generalizing types show, in comparison to the other types, a clearer connection to the generalized reasoning of structures. In contrast, the incomplete types show difficulties with the complete specification or justification of a structure. They primarily formulate a detailed description of the overall situation without explicitly presenting what has been discovered.

CONCLUSION AND OUTLOOK

The results of the study show that the changes in mathematical reasoning at the age of investigation cannot be described one-dimensionally, despite the restriction to a potential mathematical giftedness. Nevertheless, the type formation in particular has shown that it is possible to structure, group and simplify the course of change. Together with the identified problems, prognoses and characteristics of giftedness, it is thus a decisive basis for the practical teaching goal of the study, which is to reconcile the long-term support of mathematical giftedness with the training of mathematical argumentation. In particular, initiation processes and the teaching of methodical argumentation knowledge seem to be promising. The formulated results are a first, explorative approach to describe the longitudinal change in reasoning of potentially gifted children. This explorative approach provides starting points for further questions that may validate the hypotheses. The types can be the basis for complex case studies. These could be characterized in detail by further categories from the analysis scheme, for example by extending the range of characteristics using the category *Independence*. Also in this context, the inclusion of further mathematic specific giftedness characteristics seems promising in order to promote mathematical argumentation as a learning goal and basis of understanding in the context of mathematical giftedness.

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PLACE VALUE AND REGROUPING AS SEPARATE CONSTRUCTS OF PLACE VALUE UNDERSTANDING

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Place value understanding plays an important role in children's mathematical learning. Various studies show a need for further support of children regarding their development of place value understanding. What form such support might take has yet to be clarified. While we know from a theoretical point of view that place value understanding is based on two key ideas, that can be called "place value principle" and "regrouping principle", empirical results on place value understanding differ considerably concerning underlying constructs. Our own study indicates that the two constructs place value principle and regrouping principle are two separate constructs. These results can contribute to the provision of individually tailored support for children's place value understanding.

INTRODUCTION

The development of place value understanding in elementary school is amongst other things important for computational competence, e.g., in dealing with crossing the tens boundary, and the use of computational strategies (Verschaffel et al., 2007). However, place value understanding seems also to be a predictor of mathematics achievement in general (Moeller et al., 2011). Various studies show a need for action regarding support for children's development of place value understanding (e.g., Herzog et al. 2019). But it is unclear what such support might look like, also because in several studies, albeit the authors conceptualize place value understanding similarly on a theoretical basis, they use different methods and tasks to examine it empirically. In the following, we take a comparative look at some empirical approaches, locating the core theoretical ideas of place value understanding. An own empirical study is presented afterwards, in which the two core ideas "place value" and "regrouping" were investigated in a comparative way. Thereby, our study aims at clarifying the possible contribution of these two core ideas in place value understanding in order to provide starting points for developing support options.

THEORETICAL BACKGROUND

Two concepts, named the "place value principle" and the "regrouping principle" by Padberg and Büchter (2019), are consistently described as fundamental properties of

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place value systems. The place value principle states that each digit gives two pieces of information: First, the position indicates which bundling unit the digit stands for, and second, the value of the digit indicates the number of represented bundling units. The regrouping principle describes that continuously the same number of units is bundled. In the decimal system the number of units is ten. Models for developing place value understanding have been proposed by Fuson et al. (1997), Ross (1989), Herzog et al. (2019), and Fromme (2017), among others. In their empirical studies, the researchers capture place value understanding in different ways. In the following, the two principles presented are used to analyze the empirical approaches.

Some authors try to get insight into place value understanding via observation of counting processes and solving behaviour in multi-digit addition and subtraction (Fuson, 1990; Fuson et al., 1997; Cobb & Wheatley, 1988). In this context the use of grouping and regrouping in crossing the tens boundary is considered a crucial indicator for place value understanding. Hence, the focus of these studies lies on the regrouping principle. However, it should be noted that by observing computational processes with regard to the use of tens structures the place value notation plays a subordinate role. Another option to analyse place value understanding is to focus on translation processes between different kinds of number representations used for example by Ross (1989) and Fromme (2017). Both authors draw attention to the differences between the procedures the children use. Ross (1989) asked children about the meaning of the digits in a two-digit numeral. She distinguishes levels regarding the awareness of the quantity of objects corresponding to each digit. Thus, she examines primarily the understanding of the place value principle. Ross (1989) notes that some children refer to each digit by the names “units” and “tens”, but without associating the tens digit with the corresponding number of bundles of tens. Fuson et al. (1997) observe a similar kind of approach, when children perform digit-by-digit arithmetic while treating all digits in isolation as ones (“concenated single digit conception”). Although Fromme (2017) examines translation processes, she works out the problem of bundling as an important element. She differentiates two kinds of assigning numerals/number words to number representations on the abacus in children’s answers: She highlights whether the children use the tens structure compared to counting the objects individually. Here, the focus is on using the tens structure analogous to the conceptualization in Fuson (1990), Fuson et al. (1997) and Cobb and Wheatley (1988).

Herzog et al. (2019) develop a competency model based on the models of Fuson et al. (1997), Cobb and Wheatley (1988), and Ross (1989), among others, which they validated empirically conducting a Rasch analysis. The model consists of four levels and the authors differentiate procedural place value understanding and conceptual place value understanding. Here, the place value principle as described in the model of Ross (1989) is included as procedural place value understanding, which consists of being able to name positions by e.g., “tens”, “hundreds” in a multi-digit number or to show the right positions in a multi-digit-number according to the given names (already developed at level 1). The regrouping principle is included in the conceptual place value understanding, which develops from level II to IV. It consists of understanding the relationships between different bundle units, e.g. the relation between tens and ones. It is initially at level II still dependent on visual support (e.g., base ten blocks). According to Herzog et al. (2019), the ability to translate non-canonical representations such as 3T 14U into a place-appropriate notation by regrouping ten ones into one ten is crucial for the determination of place value understanding. This is accomplished without visualization at level III and in larger number spaces at level IV.

Looking at the presented empirical approaches, Herzog et al. (2019) take up the place value principle only as part of the procedural place value understanding at level 1, while the regrouping principle seems to be mainly relevant for all higher levels. Fuson et al. (1997) and Ross (1989) both point out that the activity of naming positions, that Herzog et al. (2019) include in procedural place value understanding, is not yet a sign of a full place value understanding. Because of that, the question arises whether place value understanding could also be differentiated at higher levels, or at least play a further role for place value understanding next to regrouping.

Insofar it is still an open question, whether the understanding of the meaning of place values as described by the place value principle is included in the overall construct of conceptual place value understanding and whether it is inseparable of the regrouping principle, or if the both principles emerge as separate constructs. To address this question, we conducted a study and used tasks in which the children had to consider the two principles. Our study aims at examining to which extent the two principles contribute to place value understanding. The results may help to diagnose specific difficulties in place value understanding and to address difficulties more precisely to support children's development.

Our research questions are:

- Is place value understanding a unidimensional construct that combines the two principles “place value” and “regrouping” or a two-dimensional construct with “place value” and “regrouping” as two separate constructs?
- If place value understanding proof to be two-dimensional: To what extent contribute the understanding of “place value” and “regrouping” each to children's errors?

METHODS

To answer our research questions, we conducted a cross-sectional study: 100 third graders (8 to 10 years old) from 7 classes in three elementary schools were presented with twelve tasks. In the tasks the place value principle (PVP) and/or the regrouping principle (RP) had to be considered. In order to test understanding of the place value principle on a higher level than only naming, we chose tasks, in which the correct order of bundling units had to be considered.

- Two tasks addressed translating a representation with base ten blocks into numerals. One representation was non-canonical, but sorted by bundling units according to the place value representation (RP, see figure 1), the other presented unsorted material (PVP).

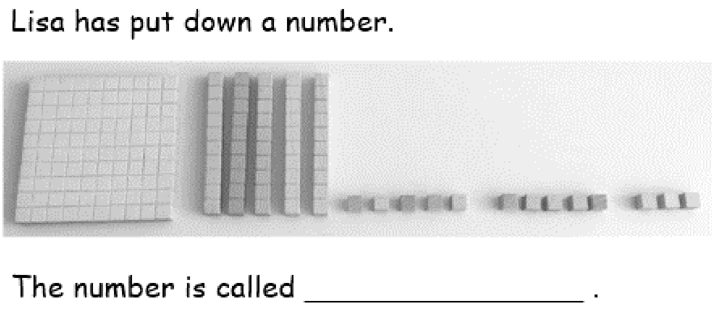

Fig. 1: Task in which regrouping is needed.

- Seven tasks request translations between representation with named units into multi-digit numbers e.g., 7U 1H 4T = ______ (PVP), 4H 15T 6U = ______ (RP), 7H 3U 19T = ______ (PVP and RP).
- One task calls for describing material handling with base ten blocks at 92 - 8 (RP).
- Two tasks gave prompts to take away tens: a) 1 ten from 305 and b) 14 tens from 168 (both PVP and RP).

For the analysis, we coded first dichotomously true/false and determined the solution rates of the tasks. Based on the solution rates, we excluded two tasks from further analysis:

(1) The task for the transition from the representation with base ten blocks with unsorted material into a numeral was solved correctly by 91% of the children, the nine incorrect answers suggested counting errors instead of comprehension problems.

(2) The task for the description of the usage of base ten blocks for solving 92 – 8 was not worked on by the majority of the children in the intended sense. Most children described taking away a ten and giving back two ones instead of a regrouping process (take ten ones for one ten and then take away eight ones) thus offers no gain in knowledge for the children's understanding of the regrouping principle.

This leaves ten tasks, three of which require only the observation of the place value principle and three of which require only the observation of the regrouping principle.

In the remaining four tasks both principles have to be applied (see table 1). To examine the extent to which the solution frequencies confirm the two principles as independent constructs, a confirmatory factor analysis was performed on the six tasks, that each cover only one of the principles (three tasks address the place value principle, the other three tasks the regrouping principle). For the one-dimensional model, the latent variable is place value understanding, while for the two-dimensional model, the two principles are assumed to be two latent variables.

In order to be able to include the tasks for both principles in the analysis, a reliability analysis is carried out over all seven tasks in each of which the regrouping and/or place value principle must be considered. The four tasks in which both principles are addressed are included in both scales. To examine the causes of the children's errors in these tasks with regard to the both principles, the errors are examined qualitatively. For this purpose, for each error we analysed, which of the two principles (or both) was violated. For example, in 7H 3U 19T = ______ the solution 7193 indicates that the place value principle was considered, but the regrouping principle was not considered (regrouping error). Solution 749 shows a disregard of the place value principle (place value error), and in 7319 both principles are violated (regrouping and place value error).

To understand whether children are more likely to make errors regarding one of the principles we compared the numbers of the two types of errors per child.

FINDINGS

Findings of the confirmatory factor analysis

The mean value of the solution rate is 6 tasks out of ten (standard deviation 2.78). The confirmatory factor analysis shows that one-dimensional data does not fit our data well as indicated in the comparison of our model and the data as well as in the Fit-indices RMSEA, CFI and SRMR. The two-dimensional model however, shows no significant difference to our data and fits our data comparably well. If both models are compared, the AIC and the sample-size-adjusted BIC also indicate that the two-dimensional model represents our data better than the one-dimensional model (see table 1).

	df	χ^2	*p*	*RMSEA*	*CFI*	*SRMR*	*AIC*	*adjBIC*
1-dim model	9	53.067	.000	.221 [0.166, 0.281]	.645	.104	706.388	696.433
2-dim model	8	11.766	.162	.069 [0.000, 0.146]	.970	.040	667.087	656.579

Table 1: Results of the confirmatory factor analysis

Note: RMSEA (Root Mean Square Of Approximation); CFI (Comparative Fit Index); SRMR (Standardized Root Mean Square Residual); AIC (Akaike Information Criterion); adjBIC (Sample Size Adjusted BIC Information Criterion)

Reliability analysis across the "place value principle" and "regrouping principle" scales (seven tasks each, with four tasks included in both scales) yields a Cronbach's α of .749 for the place value principle and Cronbach's alpha of .783 for the regrouping principle, so that a further analysis of the different kind of errors according to the two principles seems to be reasonable.

Findings of the error analysis

The solution rate in the three tasks in which the regrouping principle have to be considered is 68.33%, the solution rate in the three tasks for the place value is exactly the same. In the four tasks in which both principles have to be considered, the solution rate is 47.5%. The error numbers in the four tasks regarding both principles are shown in table 2.

Table 2: Numbers of both kinds of errors in the tasks regarding both principles

	7H 3U 19T	*3H 15T*	*Take away 1 ten of 305*	*Take away 14 tens of 168*
Regrouping errors	41	45	16	30
Place value errors	38	21	15	45

The comparison of the numbers of the two kinds of errors per child shows that 33 children made more regrouping errors than place value errors (in the mean 2,67 more regrouping errors than place value errors) and 44 children the other way round (in the mean 2.05 more place value errors than regrouping errors).

DISCUSSION

The first research question was, whether the two principles "place value" and "regrouping" are empirically confirmed as two separate constructs. The results of the confirmatory factor analysis express a preference for a two-dimensional model with the two principles as separate latent variables over a one-dimensional model.

The second research question was, to what extent the two principles each contribute to children's errors. The comparison of the solution rate of tasks regarding only the place value principle with that of the tasks regarding only the regrouping principle shows, that neither principle seems to be more error-prone than the other, because the solution rates are exactly the same (68.33%). The solution rate of the four tasks regarding both principles is lower (47.5%). We therefore conclude that these tasks seem to be more error-prone. Counting and relating the regrouping errors and the place value errors made in all tasks show that there is not a great difference in problems with one of the two principles in these tasks either. Interestingly, one of the four tasks (3H 15T = ____) seems to be more error-prone for the regrouping error. The common error "315" could be a result of the fact that the number of digits matches the number of digits actually required at the end. At the same time another task (Take away 14 tens of 168) seems to be more error-prone for the place value error. The common error “154” might be due to no hundreds beeing left over. Maybe a task like “Take away 14 tens of 268” would lead to less errors.

Our study emphasizes that problems in place value understanding are a highly important issue: Only of 60% of the tasks were solved correctly. This comparably low solution rate confirms results of other studies that show insufficient place value understanding (e.g. Herzog et al., 2019; Fuson et al. 1997). Our results can provide suggestions for individual support options.

Our results indicate, that an understanding of both principles can be more or less present in children’s solutions. That is why an understanding for each principle has to be built up through mathematics education.

While overall both principles present equal difficulty, some children show significantly greater difficulty with one of the two principles. Insofar, it seems highly important to consider children’s individual problems in place value understanding, and in doing so to include the two principles both specifically when diagnosing problems in place value understanding.

Finally, our results can serve as an empirical foundation to further differentiate the existing models for place value understanding. For this, it might be necessary to describe the more or less separate development of both principles at the lower levels. A highest level, on which the complete place value understanding is described, would have to include the integration of the understanding of both principles.

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DOES SUCCESSFUL PREPARING AND REFLECTING ON LESSONS SUPPORT PRE-SERVICE TEACHERS' ACTIONS? A MEDIATION STUDY

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Pre-service teachers acquire mathematical teacher knowledge at university, but often fail to apply that knowledge to spontaneous teaching situations in the classroom. In this study, we investigate whether an ability to prepare and reflect mathematics instruction may facilitate the application of mathematical knowledge in instructional situations under time pressure. For that, we assessed pre-service teachers' mathematical teacher knowledge, their ability to prepare and reflect mathematics instruction and their ability to act in time pressuring teaching situations with partly video-based instruments. The results show that an ability to prepare and reflect instruction mediates between mathematical teacher knowledge and the ability to act in teaching situations. Our results give implications for teacher training at university.

INTRODUCTION

Teacher education programs at university aim at providing pre-service mathematics teachers with mathematical teacher knowledge such as *Content Knowledge* (CK) and *Pedagogical Content Knowledge* (PCK) (Kaiser et al., 2014; Shulman, 1986). After graduating from university, early career teachers have to apply that knowledge to deal with the real-life demands of mathematics instruction. However, many of those teachers struggle with applying their knowledge in teaching situations under time pressure (Rowland et al., 2001; Stender et al., 2017). Regarding this issue, research has not yet conclusively explained what enables teachers to apply their knowledge in a teaching situation. Prior research suggests that an ability to prepare and reflect on instruction may facilitate an ability to apply knowledge for teaching (Stender et al., 2017). So far, quantitative empirical support for this assumption has not been provided. The present study investigates this assumption by examining relationships between pre-service and in-service mathematics teachers' mathematical knowledge, their ability to prepare and reflect instruction and their ability to act in teaching situations under time pressure.

THEORETICAL BACKGROUND

Modelling Teacher Competence

To investigate mathematical teacher knowledge, their ability to prepare and reflect mathematics instruction and their ability to act in teaching situations, we use the model

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 88-95. Khon Kaen, Thailand: PME.

of subject-specific teacher competence by Lindmeier (2011). In this model, three aspects of teacher competence are differentiated:

1. *Mathematical Teacher Knowledge*: This contains a teacher's decontextualized (i.e., independent from specific contexts such as an instructional situation) mathematical knowledge. It encompasses content knowledge and pedagogical content knowledge.
2. *Reflective Competence* (RC): This contains a teacher's cognitive, motivational and affective characteristics that contribute to mastering the typical demands of preparing and reflecting instruction. Teachers need RC, for example, to plan an upcoming lesson (e.g., to choose content or teaching materials) or evaluate past teaching episodes.
3. *Action-related Competence* (AC): This contains a teacher's cognitive, motivational and affective characteristics that contribute to mastering the typical demands of teaching a subject under time pressure. Teachers need AC, for example, to react to a conceptual misconception displayed via a student's statement during classroom discourse or to give immediate feedback to a student's mathematical question.

In particular, RC and AC contain mathematical teacher knowledge that is applicable for mastering demands of preparing/reflecting instruction and teaching situations, respectively (Kersting, 2008; Stürmer et al., 2013). To assess RC and AC of mathematics teachers, computer-based instruments were developed to approximate the typical demands that mathematics teachers may face (Lindmeier, 2011; see *Instruments*). Prior studies using those instruments showed that mathematical teacher knowledge, RC and AC are related but empirically separable from each other (Jeschke et al., 2019; Knievel et al., 2015).

How Mathematical Teacher Knowledge Becomes Part of AC

The theoretical model by Stender et al. (2017) describes how mathematical teacher knowledge becomes applicable for teaching situations under time pressure (i.e., part of AC). The model refers to the ACT-R theory of cognition (Anderson, 1983). In ACT-R, factual, decontextualized knowledge is called *declarative* knowledge. Declarative knowledge can only be applied to a real-life situation, if (1) its *level of activation i*s high enough to be triggered in the demanding situations and (2) it underwent *proceduralization* so that the activated knowledge can be executed towards a specific goal in a demanding situation. Especially proceduralization is described as a time-consuming process that is unlikely to be completed in situations under time pressure (Anderson, 1983).

Based on the model of Stender et al. (2017), preparation and reflection of instruction can be assumed to facilitate both the activation and the proceduralization of teacher knowledge: (1) Processing teacher knowledge in contexts of instructional preparation and reflection increases its level of activation. Thus, it is more likely to be activated in teaching situations under time pressure. (2) Anticipating future teaching actions, such as reactions to possible student's misunderstandings, as well as considering past teaching experience, such as previous (reactions to) student misunderstandings, facilitate the proceduralization of teacher knowledge (details see Stender et al., 2017). Thus, by using knowledge for preparation and reflection of teaching, it is structured to a form that is easier for a teacher to apply in teaching situations.

Based on the model described above, it can be hypothesized that RC (as the ability to prepare and reflect instruction) might facilitate a teacher's ability to apply knowledge for mastering the demands of teaching (AC) in pre-service teachers.

Research Questions

So far, no quantitative study has investigated whether RC facilitates the acquisition of AC due to teacher knowledge in pre-service mathematics teachers. Accordingly, we approached the following research questions:

(RQ1) *What relationships can be found between mathematical teacher knowledge (CK, PCK), RC and AC in pre-service mathematics teachers with little teaching experience?*

(RQ2) *Is the relationship between mathematical teacher knowledge and AC mediated by RC?*

METHOD

Sample

In order to investigate our research questions, we examined a total sample of $N = 140$ mathematics pre-service teachers (ca. 53% female, mean age ca. 26.3 years) from 10 federal states in northern and southern Germany. The participants were enrolled in university teacher training programs ($n = 116$, mean semester: ca. 6.4) or in their first semester of in-service training ($n = 24$, on average since ca. 2.7 month in in-service training). As the first months of in-service teacher training in Germany is mostly limited to observing experienced teachers, it can be assumed that the pre-service teachers in our sample had only few opportunities to teach on their own. Participation in the study was on a voluntary basis and each participant received a financial compensation.

Instruments

In this sample, we administered well-established instruments for AC, RC, and mathematical teacher knowledge (Lindmeier, 2011; Loch et al., 2015). The selected instruments aim at similar topics within secondary-level algebra, calculus, stochastics, and geometry. However, to avoid priming or repetition effects, each task contained in the instruments focusses a different mathematical issue (e.g., all instrument address misconceptions of fractions, but no particular misconception is featured more than once). The tests were administered in small groups (max. 20 test-takers) and under supervision of trained personnel.

For AC, we used 9 computer-based items (Jeschke et al., 2019; see Figure 1 for an example item). Each item contains a short video-clip of a classroom situation typical for secondary mathematics instruction. Depending on the item type, the task is to provide, for example, an explanation that solves a student's question or a hint that helps students with a mathematical problem without telling them the correct solution. Since AC is characterized by its spontaneous and immediate demands, the items had to be answered verbally (again, recorded via headset) and under time pressure.

For RC, we used 9 computer-based items (Lindmeier, 2011, see Figure 2 for an example item). The items address demands of evaluating teaching material (4 items; e.g., mathematical representations), reviewing students' homework (2 items; e.g., giving feedback to typical mathematical mistakes) and planning mathematics instruction. Whereas items for material evaluation and homework review contain a picture (of the material/homework in question) and required a written response, items for planning instruction contained a video clip of the previous lesson's ending and required a verbal description (recorded via headset) of how the instruction would be continued in the following lesson.

For mathematical teacher knowledge, we used paper-pencil items of mathematics PCK and (school-related) CK (for sample items see Jeschke et al., 2019, or Loch, Lindmeier, & Heinze, 2015). The scale contained in total 24 items (3 open-ended and 10 closed-ended items for PCK; 5 open-ended and 6 closed-ended items for CK).

All responses were scored to 0, 1 (partial credit), or 2 (full credit). Open responses (including the audio recordings for AC and RC) were coded and scored independently by two trained persons under usage of a detailed coding/scoring scheme. The scheme includes item-specific criteria for each item and code. For the scoring, interrater agreement (Cohen's Kappa) was $\kappa = .77–.90$ for AC, $\kappa = .80–1.00$ for RC, and $\kappa = .70–1.00$ for mathematical teacher knowledge. The three scales showed (marginally) acceptable internal consistency (Cronbach's Alpha) of $\alpha = .63$ for AC, $\alpha = .69$ for RC, and $\alpha = .72$ for mathematical teacher knowledge in this study.

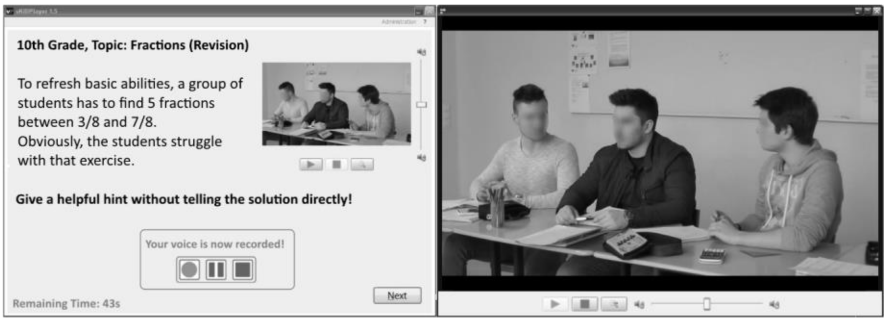

Figure 1: Example item for AC (translated and edited for publication).

In the video, the students state that they only found 4/8, 5/8 and 6/8. Whilst one student claims that there are no more fractions, another thinks that there should be more. The test-taker is asked to give the students a helpful hint without telling the correct solution.

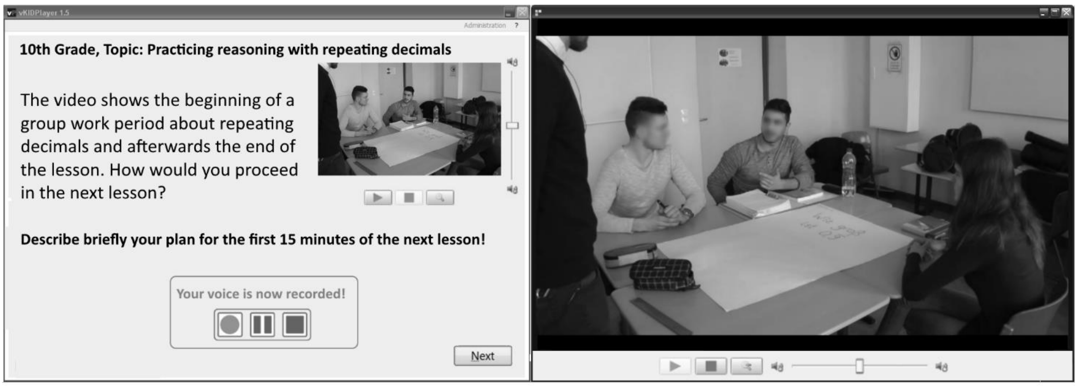

Figure 2: Example item for RC (translated and edited for publication)

The vignette features a lesson in which the students of a class work on different tasks in groups. The video shows the end of the lesson and centers one group with the task to answer the question *'What is 0.999… ?'* on a poster. After a short dialogue with the teacher about what they learned about 0.999…, one student says '*0.9… is very close to 1, almost 1 itself*'. The other students of this group agree. The lesson ends in that moment. The test-taker is asked to describe how he/she would start the first 15 minutes of the subsequent lesson.

Data Analysis

Using sum scores for each scale, we conducted correlational analyses and computed path models to investigate mediation effects. In those path models, standard errors were estimated using bootstrapping procedures (10 000 draws). Missing values (Mathematical Teacher Knowledge: 0; RC: 1; AC: 12) were estimated using full information maximum likelihood (FIML) methods. The computations were performed using *R* (version 3.6.0) and the package *lavaan* (Rosseel, 2012).

RESULTS

	AC (1)	RC (2)	Mathematical Teacher Knowledge (3)
(1)	–		
(2)	0.57***	–	
(3)	0.42***	0.61***	–
M	7.28	7.21	18.58
SD	3.29	3.41	6.51
Max (theoretical)	18	18	47

Table 1: Descriptive statistics for mathematical teacher knowledge, RC and AC including Pearson correlations between those variables. *** $p < .001$.

Table 1 contains the descriptive statistics and Pearson correlations for mathematical teacher knowledge, RC, and AC. It shows strong correlations between AC and RC ($r = .57$, $p < .001$) and between RC and mathematical teacher knowledge ($r = .61$, $p < .001$). It also shows a moderate to strong correlation between AC and mathematical teacher knowledge ($r = .42$, $p < .001$).

However, as Pearson correlations do not control for possible mediation effects, we computed a corresponding mediation model with mathematical teacher knowledge as the independent variable, AC as the dependent variable and RC as the mediator (Figure 3). The results show that the relationship between mathematical teacher knowledge and AC is fully mediated by RC (direct effect: .11, $p = .20$; indirect effect: .31, $p < .001$; total effect: .42, $p < .001$).

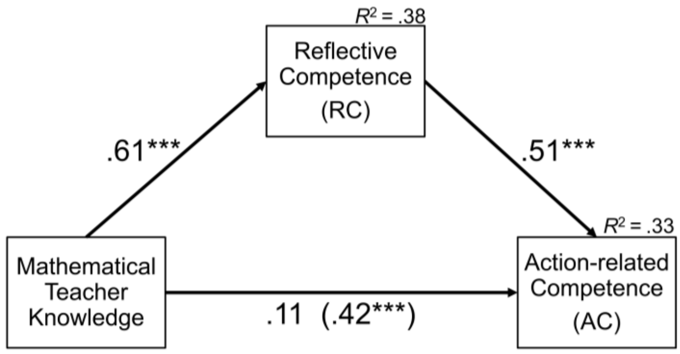

Figure 3: Mediation model with standardized regression coefficients for the relationship between mathematical teacher knowledge and AC mediated by RC. Total effect in parentheses. *** $p < .001$.

DISCUSSION

Aim of the present study was to quantitatively investigate relationships between mathematical teacher knowledge, RC and AC (RQ1), including a possible mediation of the relationship between mathematical teacher knowledge and AC by RC (RQ2) for pre-service teachers.

The results show moderate to strong correlations between mathematical teacher knowledge and both RC and AC. This is in line with the assumption that mathematical teacher knowledge is highly relevant for pre-service teachers' preparation and reflection of mathematical instruction as well as for their actions during instruction. Furthermore, the mediation analysis indicates that the relationship between mathematical teacher knowledge and AC is fully mediated by RC. This fits the theoretical assumption that an ability to apply knowledge to master the demands of instructional preparation and reflection (RC) facilitates an ability to apply teacher knowledge in teaching situations under time pressure (AC). The present study thus provides quantitative empirical evidence for the model by Stender et al. (2017), including first evidence on how teacher knowledge and RC relate to teaching actions.

Having said this, some limitations of our study should be considered. First, the cross-sectional design of our study only allows correlational interpretation of our data. Furthermore, our results should be interpreted carefully as possible selection effects, as well as the limited sample size might have affected the findings. Finally, we might have underestimated the strength of correlations due to rather low scale reliabilities, especially for AC.

IMPLICATIONS

Considering those limitations, future research should reproduce our results with larger samples. In particular, as this study is limited to correlational interpretation, future studies may investigate how RC affects the acquisition of AC by mathematical teacher knowledge with an experimental or longitudinal design.

If our results find corroboration in future studies, some implications on how to foster AC in teacher training programs at university can be derived. First, following the assumption that RC facilitates the acquisition of AC, specifically designed learning opportunities for RC with focus on anticipating future teaching actions and reflecting past teaching experience—explicitly encouraging the inclusion of the own (mathematical) teacher knowledge—may foster AC in the university training phase. Second, our assessment approach for AC presenting teaching situations via video-vignettes could be used as a template to develop video-based learning environments for AC. Pre-service teachers could use those learning tools to reflect and apply the knowledge shortly after its acquisition. In this way, pre-service teachers may be able to transform their acquired knowledge into AC while still at university.

This study thus paves the way for future strategies on how pre-service teachers can use their mathematical teacher knowledge more effectively for teaching mathematics.

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TEACHERS’ AND STUDENTS’ PERCEPTION OF RATIONAL NUMBERS

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It is known from several studies, that students have problems when treating rational numbers. In this paper, there is a focus on rational numbers as equivalent classes, in teaching expressed as extending of fractions. The current research shows how three teachers mediate extending of fractions and how their students understood this concept. The main outcome of the research is that most of the misconceptions were caused by shortcomings in teachers’ subject matter knowledge. For that reason, they were not able to find crucial aspects of the object of learning. Consequently, their students tried to find other, however irrelevant, patterns.

BACKGROUND

International research shows students’ problems and misconceptions when treating rational numbers (van Dooren, De Bock & Verschaffel, 2010). According to Adler and Sfard (2017) this is a problem that can be decomposed into three separate issues

> *First, how can one explain he is lingering pervasive failure in mathematics experienced by so many students around the world [...]. How can one transform the resulting understanding into educative action? [...] How do we enable the impact of such interventions, and how do we make sure that local improvements or reforms are scaled-up. (p.1)*

To give an answer to these questions we must understand reasons for the student’s misconceptions. So, during some years researchers have studied the teaching process in 2nd to 8th Grade, in different schools and communities, in order to understand how students are initially introduced to fractions and how this is followed up during the following school-years. As a part of these studies, a special attention was paid to teachers’ content knowledge (Shulman, 1986) and how this influenced the object of learning (Marton, 2015). In this paper there is a focus on teaching and learning fractions in 4th and 5th Grade.

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THEORETICAL FRAMEWORK

Variation and Learning

Variation theory (Marton, 2015) is a theory of learning that originates from phenomenography. A central concept in variation theory is the object of learning.

> *The somewhat curious implication of our argument here is that the object of learning is constituted while learning. Or to sharpen the claim a touch further, learning is the constitution of the object of learning. (p.161)*

This means that the purpose of teaching is to plan and carry through activities that make it possible for the student to “constitute the object of learning”. For learning to take place, some crucial aspects of the object of learning must vary while others must remain constant. From a teacher’s point of view, it requires a good survey and insight of the actual content as well as possibilities to identify students’ multiple conceptions of the actual phenomenon. If not, it is neither possible for the teacher to present a content that makes it possible for the student to find the core of awareness, nor to offer a relevant variation.

Against this background researchers analyzed how teaching of fractions was carried though and how learning took place in mathematics classrooms. One crucial aspect was to study to what extent teachers were capable to arrange learning conditions that gave their students possibilities to experience relevant objects of learning and then deepen the knowledge by suitable variation? This question is extended in the next section.

Teachers’ Content Knowledge

For the teachers to find the crucial aspects of the learning objects and a suitable variation, they have to master the current content as well as students’ pre-knowledge of it and how to teach it. Shulman (1986) describes the concept of pedagogical content as an intersection of subject knowledge and pedagogical knowledge. However, a crucial question is which type of subject knowledge teachers require. According to Ma (1999) teachers need a profound comprehension of mathematics to understand the content to teach. Ball, Thames and Phelps (2008) call this a Mathematical Knowledge for Teaching (MKT). MKT contains two groups of knowledge, namely Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). According to Ball et. al (2008), SMK contains three subsections, namely Common Content Knowledge (CCK), Knowledge at the Mathematical Horizon (KMH), and Specialized Content Knowledge (SCK).

The importance of a developed subject matter knowledge is claimed by Even (1993). On the same subject, Stein, Baxter and Leinhardt (1990) wrote:

> *[…] limited subject matter knowledge led to the narrowing of instruction in three ways: (a) the lack of provision of groundwork for future learning in this area; (b) overemphasis of a limited truth, and (c) missed opportunities for fostering meaningful connections between key concepts and representations. (p.659)*

At the same time, it is important to remember that not even elementary mathematics is superficial and must be studied hard to be understood in a comprehensive way (Ma, 1999).

Rational Numbers and Subject Matter Knowledge

In his section researchers explain our understanding of CCK as a background for our research. According to van der Waerden (1971), rational numbers can be defined as a number field Q constructed from the ring of whole numbers Z as a Cartesian product, Z x Z = $\{ \frac{a}{b} : a, b \epsilon Z, b \neq 0\}$. The rules for calculation are (b, $d \neq 0$):

$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. $\quad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

From this definition we can learn two important properties of rational numbers.

- As Q is a field, the laws for addition and multiplication are the same as for natural numbers. It is just a matter of using them in a new number range.
- In contrast to the whole numbers, the rational numbers consist of equivalence classes like $\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \frac{18}{24} = \dots$

These two properties are paid too little attention to define teachers' subject matter knowledge. In our opinion, CCK deals with understanding of definitions like the one presented above. However, the complexity of CCK requires translations into a more comprehensible language, into a theory for practice (Subramaniam, 2019), to make it a complement to SCK.

Rational Numbers in Teaching and Learning

In a research agenda for mathematics education NCTM (1988) there is a comprehensive survey of the actual research on multiplication and rational numbers. Most of it is still valid and make up an important background to our research. In the survey, Vergnaud (1988) deals with multiplicative structures, an essential background for handling fractions. He emphasizes that

> *The concept field of multiplicative structures consists of all situation that can be analyzed as simple and multiple proportion problems, and for which one usually needs to multiply or divide. […]. A single concept usually develops not in isolation but in relationship with other concepts …. (p.142)*

To this, researchers want to add the results from a study in 3rd and 5th Grade (Karlsson & Kilborn, 2018) showing that a one-sided focus on multiplication as repeated addition caused serious problems in understanding, and carrying through, operations with fractions. In the survey, Ohlsson (1988) calls attention to the fact that the concept of fraction has several mathematical meanings as well as several applicational meanings. He refers to Kieren (1975) who mentions seven different interpretations of rational numbers. Three of them are:

- Rational numbers are fractions which can be compared, added, subtracted, etc.
- Rational numbers are equivalent classes of fractions. This {1/2, 2/4, 3/6, …}and {2/3,4/6, 6/9, …} are rational numbers. […]
- Rational numbers are numbers in the form p/q where p, q are integers and q ≠ 0. In this form, rational numbers are “ratio” numbers. (Kieren, 1975, pp102–103)

METHOD

The overall design refers to variation theory, where the object of learning can be understood at three different levels:

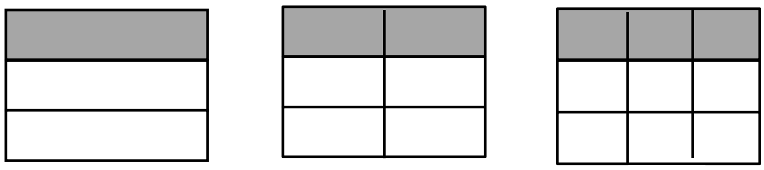
- The *intended object*: What the teacher intended to teach.
- The *manifest object*: How the object of content in fact was mediated.
- The *experienced object*: What the students in fact experienced.

The correspondence between the intended object and the experienced object is a criterion of the quality of the teaching/learning. As the object of learning and the crucial aspects are central parts of variation theory, researchers paid certain attention to the teachers' subject matter knowledge and their ability to find key components in a current content

Material

In order to study teaching and learning of fractions in 4th and 5th Grade, researchers constructed a teaching material inspired by Vergnaud (1988), Ohlsson (1988), and van der Waerden (1971) where the object of learning was expanding fractions or more formally expressed, rational numbers as equivalent classes. In a first step researchers wanted to study the teachers' ability to concretize the idea of expanding the fractions $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{3}$.

The following type of pictures was presented one by one to the students:



$\frac{\square}{3}$ $\qquad\frac{\square}{6}$ $\qquad\frac{\square}{9}$

By dividing the left figure into two equal parts by a vertical line, and then into three equal parts by two vertical lines, the students were supposed to understand that $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$. When the number of rectangles becomes two (three) times as many, also the number of painted rectangles becomes two (three) times as many. To create variation, this was done for three different fractions, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{3}$. As the object of concretization is abstraction (verbalisation) this activity was followed up by the following questions as a basis for discussion and reasoning.

> *Which number is bigger,* $\frac{1}{3}$, $\frac{2}{6}$ *or* $\frac{3}{9}$? *Explain why.*
> *Can you find a pattern here? Describe the pattern.*
> *Is it possible to write* $\frac{1}{3}$ *in more ways than* $\frac{2}{6}$ *and* $\frac{3}{9}$. *Give two more examples.*

After an abstraction, it is time to apply and generalise what is learnt (verbalised). For that purpose, teacher and students were supposed to discuss how to reason when solving the following types of tasks.

a) $\frac{1}{4} = \frac{}{8} = \frac{}{12} = \frac{}{20} = \frac{6}{}$ $\qquad$ b) $\frac{3}{4} = \frac{}{8} = \frac{}{12} = \frac{}{20} = \frac{18}{}$

Participants and Realization

The participants of the current research were three so called “first” teachers from different communities outside Stockholm and their classes, in all 60 students. During five lessons the teachers used the teaching material. During three of the five lessons, the teaching process was documented by video and an extra microphone on the teacher. At the end of the fifth lesson the students got a test and after still one week, a sample of the students were interviewed about their understanding of fractions. Finally, the teachers were interviewed with focus on their subject matter knowledge and how they apprehended their teaching and the results.

RESULTS AND ANALYSIS

To follow up the *experienced object* the student was tested and interviewed a week after the last lesson. It was quite easy to establish that the students in two of the classes had experienced the wrong object of learning. One example of this is presented in Table 1.

Task	$\frac{1}{3}=\frac{2}{6}$	$\frac{2}{3}=\frac{6}{9}$	$\frac{1}{4}=\frac{2}{8}$	$\frac{1}{4}=\frac{2}{8}=\frac{4}{20}$	$\frac{3}{4}=\frac{6}{8}$	$\frac{3}{4}=\frac{6}{8}=\frac{15}{20}$
T1. n = 21	19	12	19	18	18	17
T2. n = 20	8	4	10	3	1	0
T3. n = 19	12	3	14	8	14	1

Table 1. Number of students, taught by teachers T1, T2 and T3, who gave a correct answer to a certain task.

According to variation theory, the quality of teaching can be apprehended by comparing the *intended object* and the *experienced object.* For teacher T1, there is a clear relation, however for teachers T2 and T3 the relation is very weak. One explanation for the result is that their students usually looked for additive patterns like “2-jumps” and “3-jumps”, instead of multiplicative patterns.

To understand the weak relation between the *intended object* and the *experienced object*, researchers must study the *manifest object*, that is what happened during the lessons. To begin with, only one of the three teachers apprehended what was the crucial aspects of the object of learning, that is to understand the concept of extending a fraction. Moreover, only that teacher was able to concretize, that is, to transfer a given illustration into abstraction. In the Table 2, there is an overview of what the teachers focused on during their lessons.

During interviews with the teachers it became evident that just one of the teachers had sufficient subject matter knowledge. As the two other teachers lacked durable subject matter knowledge, they had problems in finding the object of teaching. Consequently, they had also problems to transfer their subject matter knowledge into pedagogical content knowledge.

Teachers focus on	crucial aspects	number sequences	students pre-knowl.	concretising into abstraction	language and reasoning
T1	mostly	no	yes	yes	mostly
T2	no	mostly	not enough	no	not enough
T3	no	mostly	not enough	no	not enough

Table 2. The teachers focus during the lessons

DISCUSSION AND CONCLUSIONS

From this research, researchers find that variation theory is a suitable tool to analyze the quality of teaching, that is, the correspondence between the *intended object* and the *experienced object.* If the correspondence is insufficient it is important to study the *manifest object*, not least how the teacher mediated the *object of learning* and its *crucial aspects*. However, to do so, one must leave the variation theory and interpret the results from another angle, from teachers' *subject matter knowledge*.

Teaching and learning fractions are internationally recognized as a difficult domain (van Dooren et al., 2010). A crucial question is 'why'. The outcome of the research shows that only one of the teachers was able to find and mediate the object of learning and its critical aspects, while the two other teachers were not. The main reason for these differences was the teachers' subject matter knowledge. Without such a knowledge it is not possible to know which pre-knowledge is required to find the crucial aspects of the object of learning, or to decide if an attempt to concretize really ended up in abstraction.

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DEVELOPING STRUCTURAL THINKING FOR EQUIVALENCE OF NUMERICAL EXPRESSIONS AND EQUALITIES WITH 10- TO 12-YEAR-OLDS

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In the numerical-algebraic world of school mathematics, equivalence has two faces: computational and structural. This paper offers firstly a brief review of the theoretical and empirical literature related to equivalence and structure, with attention to the properties associated with the structural dimension of equivalence of numerical equalities. It then provides a description of the six phases that the participating students passed through in shifting their thinking from the computational to the structural dimension of equivalence. The discussion focuses on the role of the interviewer's interventions, as well as the evolving usage of the decomposing property that characterized students' structuring activity. It also highlights the support offered by their sense of numerical structure and relationships.

RATIONALE AND RESEARCH QUESTION

In the numerical-algebraic world of school mathematics, equivalence has two faces: one, computational, and the other structural. For students in primary school, a great deal of research has been carried out on expanding their views of the equals sign from that of an operator symbol to one that encompasses a more relational interpretation involving multiple operations on both sides of the numerical equality. While advances have clearly been made in developing their notions of the meaning of the equals sign, the multi-facetted aspects of equivalence within a numerical setting remain largely unstudied – despite their relevance for later work in school algebra and the importance of beginning to develop such algebraic thinking at the primary levels of schooling. The present study aimed at filling part of this gap by fostering the growth of 10- to 12-year-old Mexican students' structure sense regarding equivalence of numerical expressions and equalities. The study included tasks and group interviews that focused on indicating the truth value of numerical equalities within the activity of generating equivalent equalities – where equivalence was grounded in properties related to the order and addition structures, as well as in its reflexive, symmetric, and transitive properties. The research question addressed by the study was the following: *What are the ways in which a structural form of thinking unfolds during the implementation of tasks related to the generation of equivalent numerical equalities by students who are nearing the end of their primary schooling?*

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THEORETICAL FRAMEWORK AND RELATED LITERATURE

This three-part section on the theoretical framework and related literature deals firstly with mathematical equivalence, secondly with structure, structuring, and structural thinking, and thirdly with prior research in this area.

While the actual term *equivalence* came into its own as late as the 20th century, Asghari (2018) recounts the history of this notion as a rather winding path that goes back to much earlier times. Eventually, the following definition was arrived at: An equivalence relation is a binary relation that is reflexive, symmetric, and transitive. The relation "is equal to" is the canonical example of an equivalence relation, where for any objects a, b, and c: $a=a$ (reflexive property); if $a=b$, then $b = a$ (symmetric property); and if $a=b$ and $b=c$, then $a=c$ (transitive property). Equality is a relationship between two quantities, or more generally two mathematical expressions, asserting either that the quantities have the same value, or that the expressions represent the same mathematical object, or that an object is being defined.

With respect to the numerical world, equivalence clearly has a computational dimension. This dimension arises from the above definition of the equivalence relation "is equal to" whereby the two quantities "have the same value" – as in 5+9+3=10+7 is true because both sides evaluate to 17. But, equivalence also has a structural dimension. This dimension arises from the same definition of equivalence, but capitalizes on the reflexive property where $a=a$ with a being a mathematical expression – as in 5+9+3=5+9+3 where the structure of the non-computed sum of each side of the initial equality (5+9+3=10+7) is reflected in the same decomposition on the two sides of the second equality and whereby the truth value of the equality statement is visibly obvious without computing.

As with equivalence, structure is another of the fundamental ideas of mathematics. However, unlike *equivalence*, which has been well defined, *structure* is often treated within the mathematics education community as if it were an undefined term (Kieran, 2018). While many researchers use the term *structure*, it is just assumed that there is universal agreement on its meaning. In fact, Venkat et al. (2019) argue that the community seems to have difficulty in defining *structure* in a coherent way.

For Mason et al. (2009), the structural is closely intertwined both with the general and with attending to properties. We would argue, however, that the intertwining of the structural with the general has resulted in giving more emphasis to generalizing within arithmetic and algebra, and much less to the structural. To open up the notion of structure and the structural, we turn to Freudenthal (1991) who points out that the system of whole numbers constitutes an *order structure* where addition can be derived from the order in the structure, such that for each pair of numbers a third, its sum, can be assigned. The relations of this system are of the form $a+b=c$, which he refers to as an *addition structure*. In his related discussions on the properties of both the addition and multiplicative structures, Freudenthal (1983) emphasizes multiple means of structuring and properties that can be characterized in a manner not restricted to their axiomatic formulation within the basic properties of arithmetic.

Linchevski and Livneh (1999) have drawn our attention to young students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. They suggest that instruction in arithmetic be designed to foster the development of structural thinking by providing experience with equivalent structures of expressions and with their decomposition and recomposition. Our broadening of perspective on structure and structuring leads us to consider decomposing, composing, and recomposing also as properties – properties that are tied to the order, addition, and multiplicative structures. The use of these properties is anchored in the symmetric property of equality; that is, symmetry allows for the rewriting of the addition fact of, say, 5+7=12 as 12=5+7. In other words, 5+7 can be composed into 12, and 12 decomposed into 5+7, or any of its other combinations.

During the past several decades, research on the ways that students use the equals sign (e.g., Carpenter et al., 2003; Kieran, 1981) has characterized students' thinking primarily in terms of "operator" (i.e., as a "do something signal") versus "relational" views (i.e., as a "sign denoting the relation between two equal quantities"). The kinds of tasks that have generally been used in these studies have included: (i) true-false equalities where students are asked to state their truth-value and (ii) open sentences requiring them to determine which number will make the sentence true. By means of such tasks, Rittle-Johnson et al. (2010) found that by about the 5th grade most students can compare both sides of an equation and thus hold a basic relational view of equivalence (e.g., they can accept equations with operations on both sides). However, very little of the existing research literature on the equals sign and equivalence has included explicit attention to decomposing, composing, and recomposing. One exception is a study by Warren (2003) who found that only 5% of the 672 7th and 8th graders she tested responded that there was an unlimited number of possibilities for answering the question: "Write other sums that add to 23; how many can you write?"; and fully one-third of the students failed to respond to the question in any way. In sum, little research has focused on the kinds of structural thinking involved in the generation of equivalent numerical equalities by primary students.

METHODOLOGY

This section presents both general aspects of the methodology, as well as brief information related to the design of the tasks. Six students from the 6th grade of a public community school in Mexico participated in the first part of the study, which was conducted when the students were halfway through their last year of primary school. The second part of the study took place when the students were finishing their last year. None of them had had any prior experience in structural activity with equalities or, in particular, with generating equivalent numerical equalities, but they had been exposed to equalities with numerical terms on both sides of the equals sign.

The data collection technique was that of the Group Interview, a method that involved the students first working individually on a given task question or set of questions. This was followed by an interviewer-orchestrated, discussion segment where the students would share their responses with the rest of the group. During this group sharing, the interviewer (i.e., 2nd author) might probe their thinking by asking for clarification or might also pose additional questions.

Data from Part 1 of the study were obtained during three sessions, one session per task-set, with sessions lasting about 60 minutes each in one of the rooms of the school. All six students participated in each of the three sessions. Data from Part 2 of the study were obtained during one session that lasted about 60 minutes and involved three of the original six students. The data sources for both parts of the study include the individual students' worksheets, videotaped footage of the interviewer interacting with the group of students and the recording of all their verbalizations, and the researcher's field notes. All interactions and task-sets were in the Spanish language.

For Part 1 of the study, three task-sets were designed to explore students' strategies related to structure within the equivalence of expressions and equalities. The first two task-sets did not include the equals sign. Task-set 3 involved the equals sign in numeric equality statements and included two or more numerical terms on each side of the equals sign (i.e., 4+5=4+3+2, 480+6+123=486+123, 172+10+75=182+50+25, 150-70=125+25-70, and the non-equality 2+8=1+1+5). The task-set consisted of two types of questions: one asking students whether a given equality statement was true and to explain why, and the other to rewrite the true equalities in another way so as to show they were true.

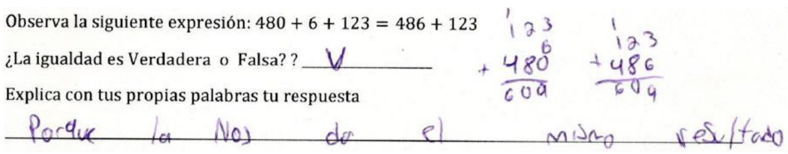
Due to the computational nature of the results obtained for Task-set 3 in Part 1, a new Task-set (Task-set 4) was designed for Part 2. This task-set involved true equalities of equivalent expressions (10+7=5+12, 530+200=300+430, and 8+2+16=10+12+4). However, the task instructions specifically requested that the students <u>not</u> calculate the total of each side in order to show that the equality was true, and then to explain their reasoning. The last question asked for a generalization of their main strategy. The structural properties that would be at play in students' generation of equivalent numerical expressions and equalities would include: composition and decomposition, as well as the common-form property of equivalence (i.e., converting pairs of expressions into a common form to indicate equivalence) and the transitive property.

RESULTS

The development of students' structural thinking related to the generation of equivalent numerical equalities evolved through six specific phases. The students featured in the extracts below are two girls (S1, S2) and one boy (S3).

Phase 1: Computation Without Decomposition

The first two task-sets evoked computational views. For the first task, *Can the number 7 be written from the numbers 6 and 1? If so, how?,* the students answered affirmatively and explained their thinking by means of a computation involving the property of composing the addends. Similar thinking was evidenced throughout the tasks of the second set. For Task-set 3, where all of the questions included the equals sign and involved, for some equalities, smaller numbers and for others, larger numbers, the students could accept without hesitation equalities in the form of $a+b=c+d$ and justified their truth (or falsity) by calculating the result on each side and stating that an equality was true "*because we get the same result*" (see Fig. 1).


Observa la siguiente expresión: 480 + 6 + 123 = 486 + 123

¿La igualdad es Verdadera o Falsa? ? V

Explica con tus propias palabras tu respuesta

Porque la Nos de el mismo resultado

Observe the following expression: 480 + 6 + 123 = 486 + 123

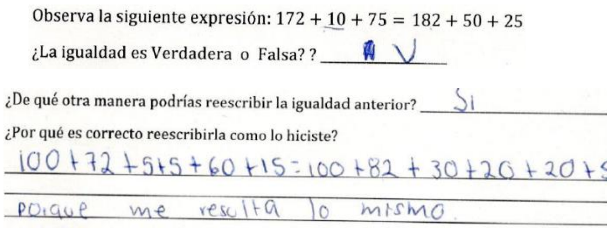
Is the equality True or False? _T

Explain with your own words

Figure 1: S3's computational strategy (English translation also provided)

Phase 2: Computation with Unrelated Decomposition of Each Side

Each of the tasks in Task-set 3 also included the question: "In what other way could you rewrite the given equality?" While the totals for each side of the given equalities had already been computed, the students were still able to generate alternative forms by independently decomposing each side (see Figs. 2 & 3). There was no inclination, however, to re-express the equalities so that both sides of the equalities looked alike.


Observa la siguiente expresión: 172 + 10 + 75 = 182 + 50 + 25

¿La igualdad es Verdadera o Falsa? ? V

¿De qué otra manera podrías reescribir la igualdad anterior? Si

¿Por qué es correcto reescribirla como lo hiciste?

100+72+5+5+60+15=100+82+30+20+20+5

Porque me resulta lo mismo

Observe the following expression: 172 + 10 + 75 = 182 + 50 +25

Is the equality True or False? _*T*

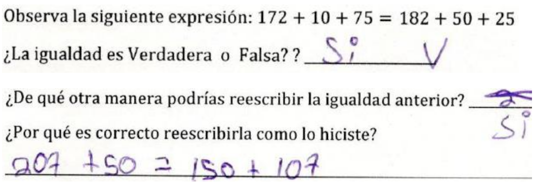
In which other way could you re-write the previous equality? _*.Yes.*

Why it is correct re-write the expression in such a way?

100 + 72 + 5 + 5 + 60 + 15 = 100 + 82 + 30 + 20 +20 + 5

Because I get the same result

Figure 2: S1's equality rewriting


Observa la siguiente expresión: 172 + 10 + 75 = 182 + 50 + 25

¿La igualdad es Verdadera o Falsa? ? Sí V

¿De qué otra manera podrías reescribir la igualdad anterior? Sí

¿Por qué es correcto reescribirla como lo hiciste?

207 + 50 = 150 + 107

Observe the following expression: 172 + 10 + 75 = 182 + 50 +25

Is the equality True or False? *Yes*

In which other way could you re-write the previous equality? _*.Yes.*

Why it is correct re-write the expression in such a way?

207 + 50 = 150 + 107

Figure 3: S3's equality rewriting

Phase 3: Ad Hoc Decomposition of One Side and Copying to the Other Side

When the students came together for Part 2 of the study six months later, Task-set 4 requested that they show the truth-value of each of the given equalities, but this time without first calculating the total of each side. There were initially looks of puzzlement among the students, as if to say, "What other way is there?" Some prompting on the part of the Interviewer was necessary (the English transcriptions below are verbatim translations of the original Spanish version; I is the Interviewer):

I: Would there be a way to write also [referring to rewriting the given equality] but using, say, these same numbers? [points to the board to the equality 10+7=5+12]

S3: Yes. 5+5+5+2 [verbalizes the expression].

I: Ok, S3 says that this [referring to the left side of 10+7=5+12] could be written as 5+5+5+2 [writes on the blackboard the expression stated by S3]. Is this OK?

S1 & S2: Yes [both at once].

I: […] This [referring to the right side of 10+7=5+12], in which other way? Look, this [referring to 5+5+5+2] already has a form of, I mean, it [10+7] can be re-expressed in this way [referring to 5+5+5+2]?

S3: Yes.

S2 & S1: It gives the same [see Fig. 4].

However, it was not yet certain whether or not the right side had actually been decomposed or simply recopied by S3 on his sheet. With the next equality involving larger numbers, S2 proposed transforming the left side of 530+200=300+430 to 200+300+30+200. When she subsequently and clearly copied this to the other side and was asked how the new right side related to the initial right side, she was unable to do so. Eventually, she rewrote the left side differently; but this time she could reconcile it with the right side of the initial equality when she recopied it to the other side. Then the interviewer returned to the earlier equivalent equality that had been generated (see Fig. 4) and inquired into the need (or not) to calculate the total:

I: Is it necessary to add?

S2: No, but if it is the same [in the same form], obviously it will give the same [the total will be the same]. If the expression is the same, it will be equal, it will give the same.

Phase 4: Decomposition of Both Sides into a Third Common Form

As the Group Interview progressed, it became obvious that the ad hoc decomposition strategy was evolving into a genuine decomposition of both sides of the initial equality so as to obtain an equivalent second equality (see Fig. 5).

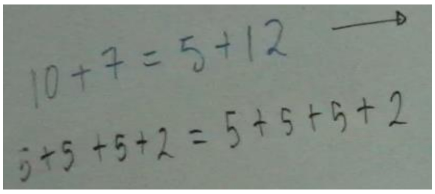


Figure 4: Ad hoc decomposition

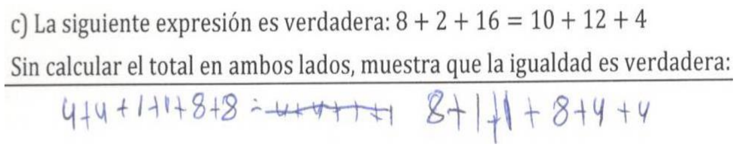


Figure 5: S3's finding a third common form

The work illustrated in Fig. 5 suggests a structural strategy that involves the search for a *third common form*: Rewrite by decomposing both sides of the equality so as to show that both sides are the same – even if the order may be different. This strategy based on decomposition differs from the *ad hoc* decomposition strategy in that both sides of the equality are genuinely decomposed and the decomposition of the second side is guided by the first decomposition so as to obtain the same common form.

Phase 5: Decomposition/Composition of One Side to Match the Other Side

The last step in the consolidation of the common-form strategy was seen when the interviewer asked if it would be possible to rewrite the equality 8+2+16=10+12+4 in such a way that the common form would be the same as one of the two sides of the initial equality. S3 simultaneously related both sides of the initial equality by composing 8+2 into 10 and decomposing 16 into 12+4 (see Fig. 6a & 6b).

a) $\underbrace{8+2}+\underbrace{16}_{4}=10+12+4$ b) $10+12+4$

Figure 6: Composing and decomposing the left side (a) into (b)

Phase 6: Expressing Structural Approaches in a General Way

The last question was of a more general nature: "What should be done, regardless of the numbers involved in the equality and without calculating the total of each side, to show that the equality is true?" With generic examples, the students conveyed – even with somewhat imprecise language (e.g., S1's initial response: "We must simplify the numbers, or convert them in a different way, but they should be the same") – the dual aspect of both *matching both sides of the* equality and doing this matching in such a way as to *safeguard the uncalculated values of each side*.

DISCUSSION AND CONCLUSIONS

This study has investigated how a group of 6th graders moved from the computational to the structural dimension of equivalence with respect to numerical expressions and equalities. The excerpts presented above highlight how the interviewer's responses and prompts were crucial to enabling this evolution, especially during the third phase where five key interviewer interventions occurred:

- The first intervention was the request to show that the given equalities were either true or false, but without calculating the total for each side.
- This was supplemented by the suggestion, implicit in the interviewer's follow-up question, that asked if the students could find a way to rewrite the given equality, but based on the numbers that were in the equality.
- When students then decomposed one of the two expressions of the given equality, they were asked where the decomposed numbers came from.
- Since the students then simply recopied the decomposed expression over to the other side of the equals sign in order to complete the transformed equality, they were asked to justify their recopied expression, that is, to relate explicitly the recopied expression to the corresponding expression of the initial equality.
- The last prompt involved eliciting the awareness that the students no longer needed to total each side of an equality in order to determine its truth value.

From this third phase onward, the realization that equivalence could be shown by converting either/both expressions of an equality into a common form was actualized and came to be expressed in a generalized, albeit generic, manner.

In the movement of the students to the structural dimension of equivalence, two features of a cognitive nature are noteworthy: (i) the role played by the decomposing property of the addition structure in the students' evolution, and ii) the way in which their use of properties, especially that of decomposition, was grounded in their sense of number and computational knowledge. Regarding the first feature, the way in which the decompositions of the various numerical terms of the expressions were written showed that these students had been able to develop a rather strong number sense in the growth of their computational knowledge throughout primary school. Not only could they, as expected, compute forward (as was seen in their initial approaches to determining the truth-value of equalities), but they could also work backwards by breaking up numbers into their principal structural parts.

The grounding of the students' use of decomposition in their sense of number and computational knowledge and the role this played in their building of structure sense for equivalent equalities relates to the operational-structural theory of Sfard (1994, p. 53), who argues that "from the developmental point of view, operational conceptions precede structural." While we do not, in any way, minimize the crucial importance of the structural dimension of equivalence that we fostered within our study, we would be remiss if we did not raise the point about the significance of students' number sense and computational knowledge in making their transition to the structural.

Note

I dedicate this paper to my young co-researcher, Cesar Martínez-Hernández, whose life was sadly taken away from him by COVID-19 on December 15, 2020.

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EXCELLENCE IN MATHEMATICS IN HIGH SCHOOL AND THE CHOICE OF STEM PROFESSIONS OVER SIGNIFICANT PERIODS OF LIFE

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The current study investigates the relation between excellence in mathematics in high school and the choice of STEM professions over significant periods of life. The study employs a big data analysis based on a total of about 550K records obtained from the central bureau of statistics over the last decade and a half. Main results suggest the importance of studying the advanced level in mathematics in high school, as well as the importance of excelling in this field. The study suggests theoretical, practical, and methodological contributions in relation to the factors that affect and predict STEM choice for study and employment in significant periods in human's life: high school, higher education, and employment.

INTRODUCTION AND THEORETICAL BACKGROUND

Science, Technology, Engineering, and Mathematics (STEM) professions are in constant demand (Caprile, Palmén, Sanz, & Dente, 2015). The foundations for pursuing a STEM career are laid early in a student's life. Studies (e.g., Addi-Raccah & Ayalon, 2008) revealed a correlation between students' experiences in high school and their post-secondary pursuit of STEM professions. Another study which was reported by Kohen, Nitzan, & Gafni (2019) revealed a similar trend, by which students who had studied AP STEM subjects in high school, were 1.6 more likely to pursue academic studies as junior STEM students, to graduate STEM academic studies and subsequently, to pursue STEM careers. Mathematics has always been considered an invaluable and imperative component for STEM study and for many cases also for career in STEM (Maaß, O'Meara, Johnson, & O'Donoghue, 2018). Yet, studies that deal with the relation between excellence in high school mathematics and future employment in STEM professions for study and career, are not common, specifically studies that combine national cohort data in several time points with wide graduates and employee's data.

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This study aims to investigate the relation between excellence in mathematics in high school and actual STEM choice for study and employment, based on records retrieved from the central bureau of statistics (CBS).

The theoretical framework that guided this study is based on Reinhold et al. (2018) integrative model. This model is based on the Social Cognitive Career Theory (SCCT: Lent, Brown, & Hackett, 1994), but refers to four main phases in a person's life which represent different life periods: 1) the pre-decisional phase that indicates an interest in some field of study, 2) the pre-action phase that indicates an interest with commitment in some field of study, (3) the action phase that indicates actual choice of some field for study or career, and finally – (4) the post-action phase that indicates that a person pursued a chosen career. In this study we focus on the three last phases, which represent a person's actual choices in study and/or career and refer specifically to the choice in STEM professions in the following life periods: high school, higher education, and employment.

RESEARCH QUESTIONS

(1) What are the trends over the past decade and a half in actual STEM choice for study and employment in different life periods, by a person's excellence in mathematics in high school?

(2) Can a person's excellence in mathematics in high school affect and predict actual STEM choice for study and employment in different life periods?

(3) Can a person's excellence in mathematics in high school predict future success in STEM studies in higher education?

METHODOLOGY

The study is based on a big-data analysis, using a systematic sampling data obtained from the CBS in our country. The study population for this analysis is approximately 350K high school students over the last one-and-a-half decades (2001 - 2017), and about 200K bachelor's degree graduates, of whom about 70K employees in the industry in our country.

The codebook that guided the analysis was comprised of the following data: (a) students' excellence in mathematics in high school that was defined by study level in mathematics: elementary, standard or advanced; (b) field of study in high school: STEM (physics, chemistry, biology, computer science, or electronics) or Non-STEM; (c) field of study in institutions of higher education: STEM (such as computer sciences, physics, or engineering) and Non-STEM; (d) grade in graduation from higher education; and (e) employment in the industry: STEM (such as information and communications, or technological services) and Non-STEM.

Descriptive statistics and Chi-Square tests were applied to examine trends in choosing and persisting in STEM fields throughout students' lifespan (from high school to employment in the industry), and their correlation with students' study levels in mathematics in high school. Also, Logistic regression analyses were performed to investigate the predictability of students' excellence in mathematics in high school, on their likelihood to pursue subsequent STEM academic studies and careers. Finally, Regression analyses were performed to predict future success in STEM studies in higher education, by excellence in mathematics in high school.

RESULTS

Responding to the first research question, our findings revealed that on average, students who study mathematics at the advanced level in high school are about 3.7, 5, and 3 times more likely to choose STEM in high school, to graduate STEM in higher education, and to being employed in the STEM industry (Respectively) [compared to students who study in the elementary level] and about 1.4, 3, and 1.5 (Respectively) [compared to students who study in the standard level]. See Graph 1 (a, b, & c) for illustration of these findings in relation to STEM graduation in high school, in higher education, and in employment.

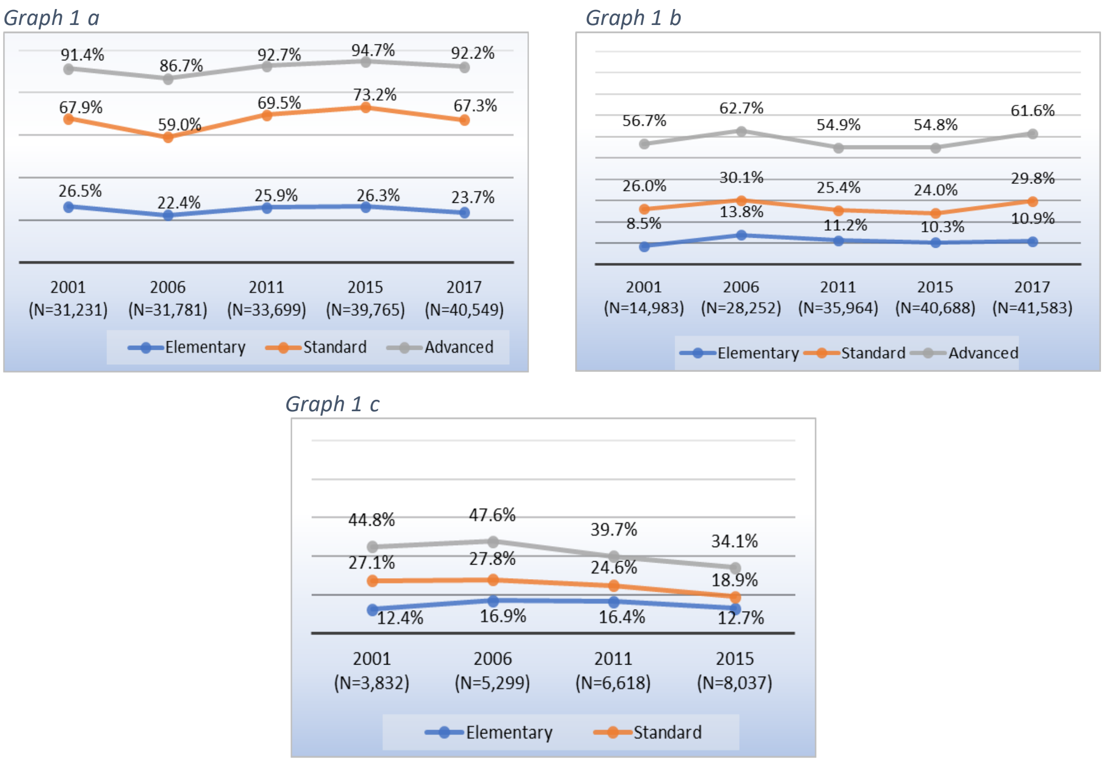

Figure 1: Graph 1 (a, b, & c) - STEM graduation in high school, in higher education, and in employment (respectively), by students' level in mathematics in high school

Due to the relatively similar trend that is observed over the years in regard to all life periods: high school, higher education, and employment, for responding the second research question, we elaborated the definition of excellence in mathematics to also inducing students' level of success in the matriculation exam in mathematics in high school: excellence; success; moderate; or weak. Our findings revealed a rank by which student's STEM choice in different life periods, is linearly affected by the combination between students' study level and level of success, i.e. the more advanced level you study and the more you succeed, then you are more likely to choose STEM for study and for career. Since similar findings were found in respect to predicting STEM graduation and employment, we illustrate the findings in Graph 2 in relation to STEM graduation in higher education.

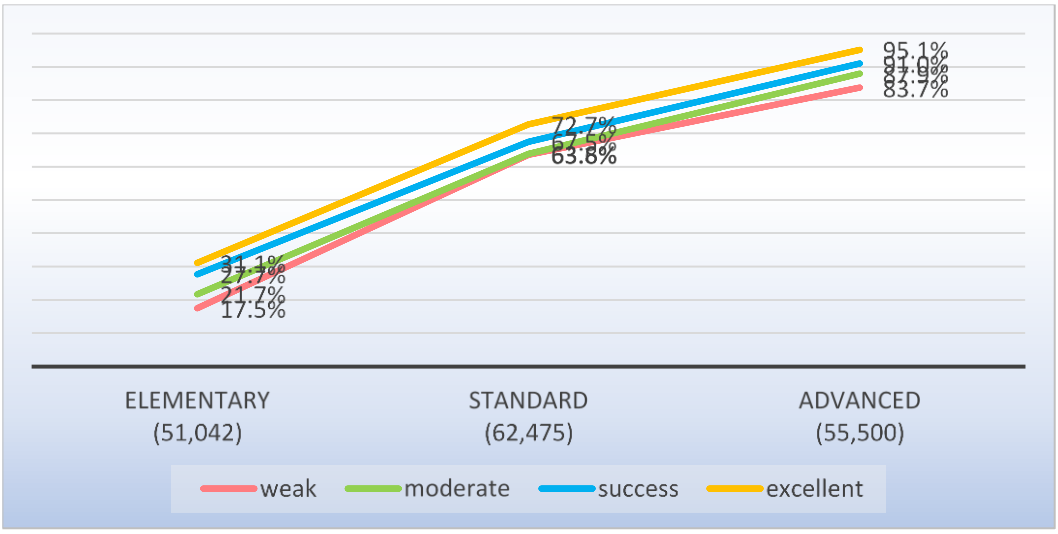

Figure 2: Graph 2 - STEM graduation in higher education by students' level & success in mathematics in high school

Focusing on the advanced level for predicting STEM graduation in high education, the regression model for the comparison between weak learner and moderate learner was found to be significant, $^2(1,15290) = 19.01, p < .001$, while the chances of a moderate learner to graduate STEM track is about 1.2 times more than a waek learner (B= .21, p<.001). Also, the regression model for the comparison between moderate learner and successful learner at the advanced level was found to be significant, $^2(1,28745) = 248.86, p < .001$, while the chances of a successful learner to graduate STEM track is about 1.5 times more than a moderate learner (B= .37, p<.001). Finally, the regression model for the comparison between successful learner and excellent learner was also found to be significant, $^2(1,38276) = 733.30, p < .001$, while the chances of an excellent learner to graduate STEM track is about 1.8 times more than a successful learner (B= .58, p<.001).

Finally, responding to the third research question, our findings revealed that a students' future success in STEM studies in higher education is predicted by his/her excellence in mathematics in high school. The regression model was found to be significant, $F(2{,}26850) = 1611.86$; $p < .001$; $R^2 = 0.11$., and indicates that students' level of success in high school mathematics is more contributing to future success in STEM studies in higher education, than the level of study in mathematics.

DISCUSSION

The current study demonstrates a positive relation between students' excellence in mathematics in high school and future STEM choice for study and employment. The linear rank that was observed in this study suggests the importance of succeeding in mathematics, and not just choosing an advanced level in mathematics. The most important insight of these findings is that in order to choose STEM as a major for study or employment, it is better to choose a more advanced level of mathematics in high school, even if you less succeed.

The big data analysis that was obtained over the last decade and a half enabled us to have an objective data about students' actual choice, and to observe the trends in choice over the years, as well as in significant life periods. This observation enables us to link between global changes and the trend of choosing STEM for study and career. For example, the decrease in choosing STEM as a major in high school at 2006, might be a consequence of the "dot-com crash" in the STEM industry which occurred in 2001.

The study adds on the literature on career choice, particularly the SCCT, by focusing on significant steps in a person's life toward STEM career choice. With that the study also contributes practically to the STEM field, as excellence in mathematics was found to be significantly contributing to choice of STEM for study and career. Methodologically, the study is based on a big-data analysis, which is not common in the literature that investigates the relation between high school studies and future study and career. Studies in this area are often based on prospective or retrospective data, rather that data that is based on a person's actual choices, specifically over the years in which these choices might be changed.

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GESTURES AND CODE-SWITCHING IN MATHEMATICS INSTRUCTION – AN EXPLORATORY CASE STUDY

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While gestures are considered an important resource in the mathematics classroom, gesture use in bilingual instruction has been largely neglected in research so far. This paper presents a case analysis of two teachers' bimodal bilingual instruction in the context of isosceles triangles to show how they coordinate meaning making of content and terminology through code-switching and gestures. Based on the analysis, we will introduce the concept of 'bimodal mnemonics' as means to support learning mathematical terminology together with the respective concept and conclude with a brief discussion of our preliminary findings and an outlook on future research.

INTRODUCTION

Mathematics teachers of English Language Learners (ELL) – and generally of students that learn in a second language – face additional challenges as they need to coordinate the students' varying levels of both mathematical proficiency and language proficiency (Moschkovich, 2013). However, language should not be seen as a barrier to learning that needs to be addressed first before it can be engaged with content. While learning mathematical terminology is important, "mathematics instruction should address more than vocabulary and support ELL participation in mathematical discussions as they learn English" (p. 50). For this, teachers are asked to acknowledge the students' home language as an important resource for meaning making rather than seeing it as a deficiency. For example, using two or more languages within a single communicative event – a phenomenon referred to as *code-switching* – can become a powerful tool for the teacher to support bilingual learners of mathematics by providing a scaffold for accessing mathematical content, and by facilitating links between verbal and visual representations of this content (Prediger et al., 2019). Furthermore, Moschkovich (2002, 2013) advocates for acknowledging the multiple resources beyond words in the mathematics classroom, like objects, drawings, and gestures. In fact, Church and colleagues (2004) showed that bilingual students benefit from gesture-rich instruction even more than their monolingual peers. It is hence surprising that in the steadily growing research on gestures in mathematics instruction (e.g. Alibali & Nathan, 2007; Arzarello et al., 2009), only very few scholars directed their attention to the bilingual classroom to understand better how teachers' gesture use can support these students' learning (e.g. Shein, 2012), and even less is known about how gestures interact with the practice of code-switching in bimodal bilingual mathematics instruction.

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This paper presents an early exploration of teachers' gestures in relation to their use of two languages in bilingual mathematics classrooms, aiming at getting a better grasp of how gestures can contribute to instruction to support meaning in mathematics across languages. We will present two cases from Farsi-English bilingual classrooms in a complementary school in the UK, dealing with the context of isosceles triangles. The cases are analysed for how teachers use gestures coordinated with language to navigate the tension between concept and terminology. In particular, we focus on the questions:

(1) How do the teachers use gestures and the two languages Farsi and English in their bilingual mathematics instruction of language learners?

(2) How might bimodal bilingual instruction support learning mathematical content as well as the second language?

Our work is embedded within the larger body of research on bilingual mathematics education that adopts a situated-sociocultural perspective (Moschkovich, 2002), emphasizing on the social dimension in mathematics teaching and learning, especially on the role of mathematics communication in which meaning develops by drawing on social, linguistic, cultural, and material resources (ibid.). Learning mathematics can then be understood as increasing participation in mathematical discourse practices with the teacher facilitating the students' active engagement in these practices by acknowledging their resources for meaning making.

Gestures are considered those "idiosyncratic spontaneous movement[s] of the hands and arms accompanying speech" (McNeill, 1992, p. 37) that do not serve any practical or manipulative purpose. As semiotically different components of a single linguistic unit, speech and gesture are coordinated in thinking and expression as embodying "different sides of a single underlying mental process" (p.1), collaborating in forming an utterance as well as in its interpretation. With respect to the development of mathematical meaning, a special focus is set on representational gestures (Alibali & Nathan, 2007): the relationship between the content of the verbal utterance and the gestural reference is established through *pointing*, through perceived similarity to a physical object or action (*iconic*), or by representing an abstract idea mediated through concrete reference (*metaphoric*). The different categories are not mutually exclusive.

METHODOLOGY

The data has been collected in the context of a study on multimodal communication in multilingual mathematics classrooms (Farsani, 2015), conducted in the UK in a bilingual British-Iranian complementary school. In this school, instruction was bilingual in Farsi and English and learners were encouraged to value both languages equally. What seemed to be at the heart of the school was creating multilingual spaces by using languages flexibly and integrating a full range of learners' linguistic repertoires. This bilingual school welcomed teaching strategies that supported learning both content and language simultaneously. The students in the recorded lessons were between 14 and 16 years old and of varying proficiency of the English language.

The selected excerpts are taken from longer episodes analysed for the relationship between gestures and code-switching in mathematics instruction. They present instances of teachers' use of gesture co-expressive to English and Farsi, in which they provide an additional visual component that enriches meaning making. The audio-visual data was transcribed, transliterated, and translated for analysis, with transliteration and translation being carried out by the second author (a native speaker of Farsi) and kept as literal as possible while overall adopting the language structure of the goal language English. The transcripts as presented in this paper have been prepared to visualize the use of two different registers by using a different colour for the Farsi register in the original transcript and its translation in the English transcript. Simultaneity of performance of gesture (start and end of the main movement) and speech is indicated in the transcript using squared brackets. For spatial reasons only selected gestures discussed more in detail in the analysis can be represented in pictures.

A speech-based analysis focuses on the teachers' code-switching practices as they can be identified in the discourse, followed by an analysis of the gestures in context as related to speech and inscription. The former was carried out based on the transcripts, the latter by additionally reviewing the video.

CASE ANALYSES

The two excerpts are taken from different classrooms and display the gesture use of the two teachers Ebi – native speaker of Farsi, fluent L2-speaker of English – and Mamad – native speaker of Azeri and Kurdish, fluent L2-speaker of Farsi and English.

In both excerpts, Ebi and Mamad are dealing with the topic of isosceles triangles in the contexts of regular polygons. Isosceles triangles are characterized by having two sides of equal lengths. While English adopts the Greek terminology of 'isosceles', many other languages (such as German and Farsi) use a quite literal translation referring to variations of 'legs of equal length'. The Farsi translation for isosceles triangle is *motasaavi-al saaghain* – literally 'equal shins' – which will be used in both episodes.

Excerpt 1

In the first excerpt, the teacher (Ebi) explains to the students how to find an inner angle of a regular pentagon. For this, he divided the regular pentagon into five equal isosceles triangles, asking for the angle *x* located at the centre (see Figure 1a, in the transcript).

		Original Farsi-English dialogue	English version
1	Ebi	That's a regular pentagon obviously, and each side is four, ok.	That's a regular pentagon obviously, and each side is four, ok.
2	Ebi	Chon regular pentagon-e *(points at the centre)* centre-esh *(points briefly towards the diagram)* [age maa be behesh] *(traces lines, Fig. 1b)* vasl bekonim mitoonim hamash *(short pause)* [isosceles] (*points to his eyes, Fig. 1c)* triangles [peida bekonim, dorost bekonim, khob]? *(traces all five inner lines similar to the three lines in Fig. 1b)*	Because this is a regular pentagon *(points at the centre)*, [if we connect] *(traces lines, Fig. 1b)* to the centre, *(points briefly towards the diagram)* [it will all become *(short pause)* [isosceles] (*points to his eyes, Fig. 1c)* triangles, okay?] *(traces all five inner lines similar to the three lines in Fig. 1b)*

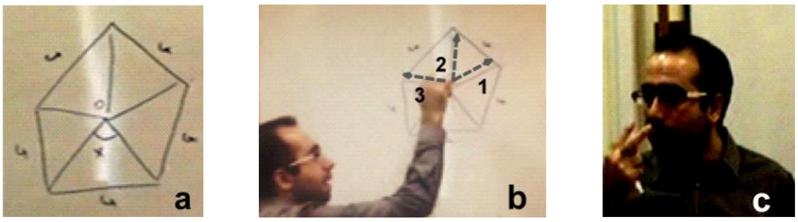

Figure 1: Ebi’s inscription on the white board; (a), Ebi’s gestures co-timed to “age maa be behesh” (b) and to “isosceles” (c)

3	S1	motasaavi-al saaghain	Isosceles triangles
4	Ebi	motasaavi-al saaghain	Isosceles triangles
5	Ebi	khob, [iino *(points at “x” in the pentagon)* ke peida mikonim] angle of x is equal to angle of x equal to angle of x and so on.	Okay, [when we find this] *(points at “x” in the pentagon)* angle of x is equal to angle of x equal to angle of x and so on.

Ebi starts with establishing a common ground of reference by clarifying the figure drawn on the white board being a “regular pentagon” (1). While this first introduction into the situation is carried out in English, he switches back and forth between English and Farsi in his following utterance when explicating how isosceles triangles emerge through construction, using English exclusively for the technical terms – “regular pentagon”, “centre”, and “isosceles triangles” (2). He ends with ‘okay?’ inviting the students to ask for clarification. To this, one of the students reacts by repeating ‘isosceles triangles’ in Farsi – “motasaavi-al saaghain” (3) – potentially asking for reassurance that he can draw on his prior knowledge on a concept he already encountered in his mother tongue. After Ebi confirms by repeating the Farsi terminology (4), he returns to the actual task of finding the angle *x*, starting with Farsi but switching back to English to complete his statement about the central angles all being the same (5). His use of Farsi and English seems to emphasize mathematical terminology in the English language while using Farsi for the wider explanation that links the mathematical components. This reflects in praxis his approach to using English and Farsi in instruction as mentioned by him in an interview carried out prior to this study: There he stated that he emphasizes the use of English in instruction since “At the end of the day they go to an English school and they learn everything in English” but that he uses Farsi to support his students when he notices problems in understanding grounded in language use (see Krause & Farsani, *under review*).

The gestures Ebi uses during his explanation are reminiscent of gesture use of teachers described by Alibali and Nathan (2007) during monolingual instruction in an algebra lesson: There, they found gestures to ground instructional language by linking it to physical referents, including inscriptions, potentially making “the information conveyed in the verbal channel more accessible to students” (p. 350). In line 2, we see two forms of grounding through gestures: pointing and tracing. Remarkably, the pointing gestures are all co-timed to the English mathematical terminology, while the tracing re-enacting the drawing of the lines that result in isosceles triangles (e.g., Fig. 1b) are co-timed with Farsi. Moreover, the gestures accomplished co-expressively to English and Farsi differ not only qualitatively, they also seem to offer different functions in grounding: The tracing gestures provide additional semantic information to clarify the incomplete/imprecise verbal expression; the pointing gestures can be considered semantically redundant to speech but grounding the terminology in the second language through providing a visual frame of reference. However, Ebi's gesture co-timed to "isosceles" remarkably appears to be different than the other two as it does not point to the diagram, but – seemingly unrelated – to his eyes (Fig. 1c). The pointing can be interpreted as a phonological rather than a semantic reference: emphasizing the "i" in "isosceles" by referencing the phonologically similar 'eye' by pointing to it, the gesture grounds the terminology and offers potential to provide a mnemonic device at service for the students with various degrees of English proficiencies to recall the technical term. In addition to the deictic reference, the concrete shaping of the gesture with two adjacent fingers of the same hand carries a reference to the semantic content of speech in the gesture's iconic dimension, reminding of the two sides with equal length an isosceles triangle itself. The gestures' deictic and iconic reference together offer a twofold semantic-phonemic link to both mathematical content and terminology.

Ebi's use of gestures coordinated with the two languages offers representational and phonemic support to potentially provide a scaffold for engaging in mathematical discourse about the respective topic through language and gesture, thereby helping the students learn and remember both the new mathematical concept and mathematical terminology. However, given that the gestural reference is rather implicit, further analyses of the classroom interaction would need to confirm this hypothesis.

The second excerpt will illustrate a variant of the "isosceles gesture" (Fig. 1c), contrasted with a gesture accompanying the teacher's explanation of the Farsi terminology. For reasons of space we will set the focus of our analysis of excerpt 2 on those gestures that are linked to the teacher's code-switching to English.

Excerpt 2

The second excerpt is taken from a lesson in which another teacher (Mamad) and some bilingual students are reviewing angles in the context of a regular octagon projected on the whiteboard. When they encounter isosceles triangles, Mamad (M) uses the occasion to discuss the terminology in both Farsi and English, using gestures as visual support.

		Original Farsi-English dialogue	English version
1	M.	aha, Farsi sho baladin bacheha? saagh yanni chi?	Aha/yeah. Do you know what's the Farsi for it guys? What does 'saagh' mean?
2	S1	paa	leg
3	S2	saagh yanni paaiinesh	saagh means the lower part
4	S1	chii bood dobare?	What was it again?
5	M.	motasaavi-al saaghain. [motasaavi yanni chi?] *(moves both extended index finger together and apart, Fig. 2a)* yanni equal. Motasaavi-al saaghain yanni do ta saghash	Isosceles. [What does 'motasaavi' mean?] *(moves both extended index finger together and apart, Fig. 2a)* It means equal. Hence 'Motasaavi-al saaghain' means both

		chi-an, [do ta paahash chi-an?] *(points down his legs, Fig. 2b)*	shins are what, [both legs are what?] *(points down his legs, Fig. 2b)*

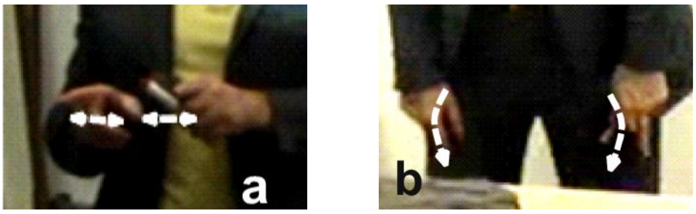

Figure 2: Gestures co-timed to "motasaavi yanni chi?" (a) and "do ta paahash chi-an?" (b)

6	S-s	mosavi	Is equal
7	M.	mosavi. Englisi besh migan [isosceles] *(Fig. 3a)* yanni dota chesha shabihe haman, Irania [migan motasaavi-al saaghain.]	Is equal. In English they call it [isosceles], *(Fig. 3a)* meaning just like our two eyes. In Iranian [it is referred to as 'equal shins']

Figure 3: Gestures co-timed to "isosceles" (a),"migan motasaavi-al saaghain" (b)

Together with the students, Mamad recalls the Farsi term for the concept under investigation – motasaavi-al saaghain – and the meaning of its single components (1-6): First, they identify the reference to the shin (saagh) – the lower part of the leg (2, 3) – then Mamad asks 'What does motasaavi mean'. He does not wait for a response but provides it himself right away as "yanni equal" (5). His following prompt to put both together ('Hence (…) their legs are what?') is then responded by students with 'is equal'. Mamad then contrasts the terminologies in English and Farsi (which he mistakenly refers to as Iranian), explicating 'how they call it (in English)' and what the Farsi word refers to as just established, but now showing in the diagram (7, Fig. 3b).

Mamad switches to English twice in this short excerpt – first in line 5 ("equal") then in line 7 ("isosceles"). Similar to the use of English in excerpt 1, these switches to English concern mathematical terminology: While "equal" (5) is a rather general translation of the Farsi motasaavi, the teacher asks for its meaning in the context of the mathematical terminology motasaavi-al saaghain. Code-switching here serves the function of expressing meaning in the second language, using vocabulary supposedly known by the students. In the second instance (7), Mamad explicitly uses "isosceles" as naming the English terminology to refer to the same concept, establishing new mathematical vocabulary through code-switching.

Both instances of using English can be linked to gesture use, even though the first one is carried out just before, co-timed to the Farsi equivalent of 'equal', carrying the same semantic meaning: While asking 'What does 'motasaavi' mean?' (5), he moves his extended index fingers horizontally and symmetrically together and apart in front of his body (Fig. 3a), adding a visual dimension to his question that might be seen as a metaphoric reference to equality, but also as iconic indicator of two sides (represented by the index fingers) with equal length. Co-timed to the English "isosceles" (7), Mamad points to his eyes, similar to what we have seen in Ebi's gesture in excerpt 1, but – different to Ebi – verbally explicating the connection to 'our two eyes' right after. Again, the gesture grounds the accompanying language phonologically and anticipates the verbal reference in Farsi. In its performance, it however shows another difference to Ebi's gesture: Ebi points to his eyes with <u>two</u> fingers of <u>one</u> hand, Mamad's gesture is bimanual, pointing with the index fingers he used earlier in his reference to equality (line 5). This might indicate a *catchment* in the sense of McNeill (2005) – a recurrent discourse theme whose link can be identified in the recurrence of gesture features, here the use of the two index finger in the gestures for 'equal' and "isosceles". Even though this contextual link is made only implicitly, one can argue that gesture and speech are processed and interpreted as a unit such that the link carried in gesture can serve as potential additional resource for meaning making for both students and the teacher.

SUMMARIZING DISCUSSION AND OUTLOOK

The bimodal analyses of the two teachers' bilingual instruction provided some first insights on how bilingual teachers coordinate the use of both languages and gestures and how this might support meaning making in mathematical discourse. Similar to what has been observed in monolingual classrooms (Alibali & Nathan, 2007), the teachers grounded their instructional language in the physical world around them to support the students' understanding of the verbal explanation. However, the bilingual setting also directs our attention to some peculiarities in the teachers' gesture use.

A striking commonality in both cases is the teachers' use of gestures as linked to the English terminology "isosceles": Importantly, this does not only concern the phonemic support in the reference to the phonologically similar 'eye', but the concrete shaping of this reference that aligns with representational features of the concept. We see two variants – one-handed with two adjacent fingers (Ebi) and two-handed with the two index fingers (Mamad) – that both carry a reference to the two sides of equal length, a defining feature of isosceles triangles. We do not claim that the teachers established this link to the mathematics consciously – they certainly did not make it explicit. However, keeping in mind the multimodal nature of communication and speech and gesture being perceived as a unit in interpreting an utterance, the combination of gesture and speech can provide assistance for remembering the mathematical idea together with its English terminology – a combination we call *bimodal mnemonic*.

A difference in both cases concerns the focus of instruction, causing differences in the coordination of gestures with language. Ebi's explanation concerns the mathematics towards solving the problem with his gestures providing a referential frame to ground the English terminology by identifying the respective concepts in the diagram, and specifying the imprecise Farsi expression. The gestural reference to the English mathematical terminology is integrated implicitly. Mamad foregrounds the terminology in both languages, reflected in his gestures as they are largely related to making meaning of this terminology. Here, the gestural reference to the mathematics with respect to isosceles triangles in a regular polygon is rather implicit. In both cases, the use of gestures and two languages played an integral role in helping the students being engaged with the ongoing flow of the lesson content.

The analyses presented here are only a first step in investigating the role of gestures for coordinating bilingual instruction with mathematical content to support language learners. They serve us as a starting point by pointing at different forms and functions of bimodal bilingual instruction along the three axes of code-switching, gesture, and focus of instruction. Our further research will concern the development of a methodological tool that integrates these three axes for systematic analysis. Furthermore, we plan a wider exploration of mathematical contexts, including also other language backgrounds in bilingual classrooms.

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A NEGATIVE EFFECT OF THE DRAWING STRATEGY ON PROBLEM SOLVING: A QUESTION OF QUALITY?

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Make a drawing is known to be a powerful strategy for solving mathematical problems. But surprisingly, the drawing strategy was found to negatively affect the ability to solve non-linear geometry problems. Our study replicates and extends this finding by addressing the quality of the drawing strategy, which might explain the negative effect. In a randomized controlled trial with 180 students (ninth- to eleventh-graders), we enhanced drawing quality by prompting the students to highlight important elements in their drawings. Our results replicated the negative effect of the drawing strategy on performance and confirmed the quality of the drawing strategy as an important factor that affected the number of linear overgeneralizations. The roles of drawing quality and other factors that might influence the ability to solve such problems are discussed.

INTRODUCTION

The drawing strategy is a heuristic method that is claimed to have strong positive effects on problem solving. However, studies that have investigated the effect of applying the drawing strategy have arrived at divergent findings. Some studies showed that the drawing strategy facilitates problem solving (e.g. Hembree, 1992), whereas others did not find any effects (e.g. De Bock, Verschaffel, & Janssens, 1998), and one study even provided surprising evidence for a negative effect of the drawing strategy on problem solving performance (De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003). The present study is aimed at replicating the negative effect of applying the drawing strategy and elaborating on potential explanations for why applying the drawing strategy can hinder problem solving.

THEORETICAL BACKROUND AND RESEARCH QUESTIONS

Drawing Strategy

Applying the drawing strategy involves constructing an external visual representation that corresponds to the structure of the mathematical problem. By drawing, the learner externalizes his or her mental model of the problem situation. This involves re-organizing the given information in such a way that important elements and relations become visible and can be processed more easily after the drawing is constructed (Larkin & Simon, 1987). Hence, drawing makes the key information from the problem explicit and facilitates the process of problem solving (Cox, 1999).

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Empirical evidence for the positive effect of drawing strategy was found in a number of studies (Rellensmann, Schukajlow, & Leopold, 2016; Van Essen & Hamaker, 1990; Zahner & Corter, 2010). Teaching the drawing strategy was even identified as the most effective treatment for improving mathematical problem solving in a meta-analysis conducted by Hembree (1992), in which drawing strategy was compared with other strategies such as verbalizing concepts. However, drawing strategy does not help all students solve the problem. Theoretical models of self-generated drawings emphasize that the benefits of applying the drawing strategy are strongly related to the quality of the use of the strategy (Cox, 1999).

The quality of the use of the drawing strategy is reflected in two properties of the drawing as the final product of the drawing process: the correctness and completeness of the drawing. High-quality use of the drawing strategy implies that students construct a correct drawing (correctness) that explicitly represents the key information from the problem (completeness). The first evidence for the importance of the quality of the use of the drawing strategy comes from research on text-based learning. Supporting students' drawing activities positively affected performance on items that required comprehensive elaboration activities (Van Meter, 2001). Moreover, empirical studies in science and mathematics confirmed theoretical considerations and revealed that the quality of the drawing strategy is positively related to demanding problem solving (Rellensmann et al., 2016; Schwamborn, Mayer, Thillmann, Leopold, & Leutner, 2010; Uesaka, Manalo, & Ichikawa, 2007). Students who constructed drawings of higher quality solved geometrical modelling problems better than other students (Rellensmann et al., 2016). The quality of the drawing strategy is expected to be particularly important when students are required to build connections and draw conclusions from the given information (Van Meter, 2001), as is the case for solving non-routine mathematical problems.

The Drawing Strategy for Solving Non-Linear Problems

An important type of non-routine mathematical problems is the non-linear geometry problem, in which the area or volume of similar figures or solids has to be determined by a given scaling factor. For example: "You need approximately 400 grams of flower seed to lay out a circular flower bed with a diameter of 10 m. How many grams of flower seed would you need to lay out a circular flower bed with a diameter of 20 m?" (De Bock et al., 1998, p. 68). This type of problem is important because it addresses students' strong tendency to engage in linear overgeneralizations – the application of linear models to non-linear situations – which is known to be a common error in problem solving (Van Dooren, De Bock, Janssens, & Verschaffel, 2008). A series of studies conducted by De Bock and colleges (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock et al., 1998; De Bock et al., 2003) showed that this type of problem is very difficult for students, who often seem to use the linear model in an intuitive manner without being aware of the model they chose (De Bock et al., 2002).

The drawing strategy can be helpful for solving non-linear geometry problems because it provides the opportunity to recognize the non-linear property of the area, and thus, it might facilitate the use of appropriate mathematical procedures. A drawing for a non-linear geometry problem should include the original and scaled figure, which enables the use of visual solution strategies aimed at estimating the relation of the areas (e.g. paving strategies). Contrary to these theoretical considerations, De Bock et al. (2003) showed that applying the drawing strategy did not facilitate the solving of non-linear geometry problems and even affected problem solving performance negatively. What can explain this unexpected finding? In the drawing condition, students between the ages of 13 and 16 were given a drawing that referred to the geometrical object from the problem (e.g. a square). They were then instructed to complete the drawing by using the given scaling factor to add a scaled geometrical object. Students in the drawing condition performed worse than students in the control group, who worked on the same problems without receiving any instructions (23% vs. 44%). An in-depth analysis of students' solutions indicated that the drawing strategy did not elicit visual solution strategies for determining and comparing the sizes of the areas. This argument provides a good explanation for why applying the drawing strategy was not beneficial, but it remains unclear why using a drawing negatively affected problem solving in this study. Another explanation might be that students use the drawing strategy inappropriately, which in turn decreases their performance in solving non-linear geometric problems.

Because of the surprising nature of the negative effects of the drawing strategy, we aimed to replicate De Bock et al.'s (2003) study in order to validate its findings. We expected a negative effect of using the drawing strategy on problem solving performance for non-linear geometry problems. Further, we expected the use of the drawing strategy to increase students' tendency to engage in linear overgeneralizations. As geometrical figures are typically depicted by their circumferences, students' attention is guided toward the linear property of the circumference while drawing instead of toward the non-linear property of the area.

Further, we considered the quality of the drawing strategy as a potential reason for the negative effect of using the drawing strategy on problem solving performance. In particular, we expected that key information such as the area and its non-linear relationship would not be made salient in the drawings so that the quality of drawing strategy would be insufficient with respect to the completeness of the drawings. Therefore, we expected that increasing the quality by highlighting the key information would diminish the negative effect of the drawing strategy on performance because it would prevent the linear overgeneralizations that usually result from drawing.

RESEARCH QUESTIONS

These considerations led us to pose the following research questions:

RQ 1: Does the use of the drawing strategy decrease problem solving performance and increase linear overgeneralizations?

RQ 2: Does increasing the quality by highlighting important information in the drawing diminish the negative effects of the drawing strategy on problem solving performance and on the number of linear overgeneralizations?

METHOD

Participants and Design

The sample involved 123 students (58.5% female, mean age = 16.19 years) from nine classes, including ninth-graders (11.4%), tenth-graders (48.8%), and eleventh-graders (39.8%). Students came from four high-track schools (German Gymnasium) and one comprehensive school (German Gesamtschule). Students in each class were randomly assigned to one of three groups: Students in the experimental conditions received either drawing (D) or drawing with highlighting (DQ) instructions, aimed at increasing the quality of the drawing strategy. Students of the control group (CG) received no drawing instructions. The instructions were embedded in the tasks given on a paper-and-pencil test. Figure 1 shows the drawing with highlighting instructions (DQ condition) embedded in one of the tasks.

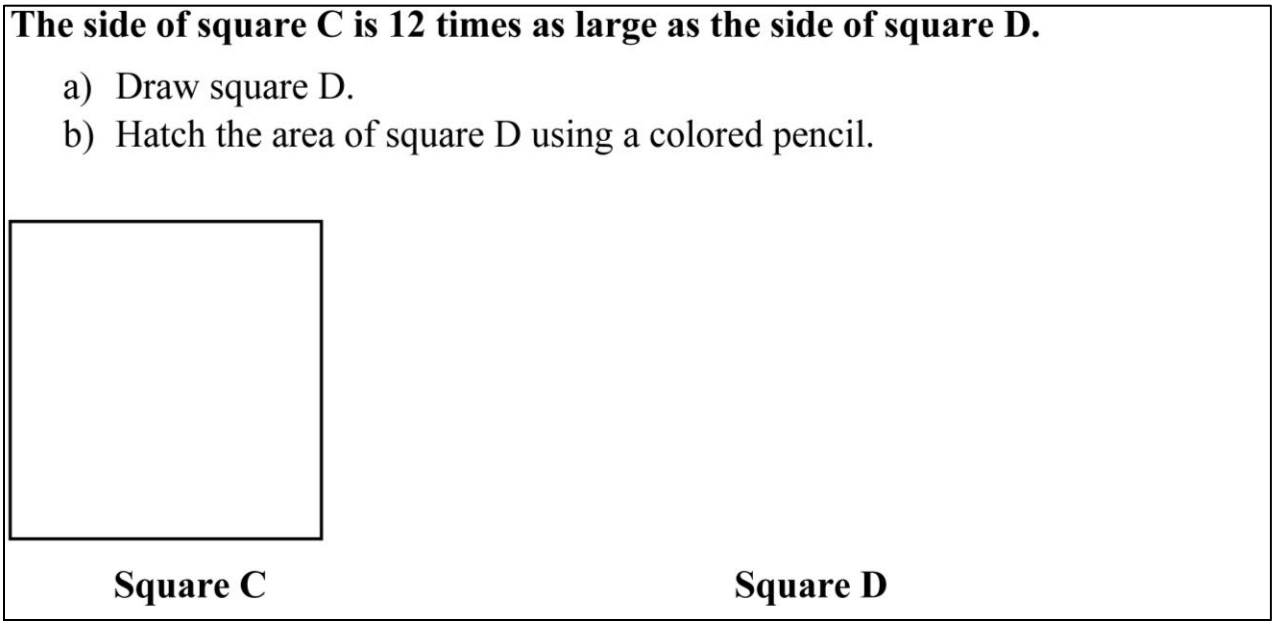

Figure 1: Sample item with drawing with highlighting instructions. Tasks were adopted from De Bock et al. (2003, p. 449)

To check the implementation of the treatment, we examined whether students in the experimental and control groups followed the instructions by analyzing the numbers of papers with no drawings, drawings (without highlighting), and highlighted drawings in the different conditions (CG: 63% no drawings, 35% drawings, 2% highlighted drawings; D: 7% no drawings, 92% drawings, 1% highlighted drawings; DQ: 8% no drawings, 22% drawings, 70% highlighted drawings). Significantly more drawings and highlighted drawings were made in the respective conditions, indicating that the majority of students followed the instructions as intended for the non-drawing, drawing, and drawing with highlighting groups.

Measures and Data Analysis

Students' performance and the number of linear overgeneralizations were assessed via a problem solving test, which included four experimental items and three additional buffer items. The experimental items were non-linear geometry problems in which the area or volume of a figure (square, circle) or a solid (cube, sphere), respectively, and a scaling factor were given with the question to find the size of the area or the volume of a similar figure. For example: "The side of square C is 12 times as large as the side of square D. If the area of square C is 1440 cm^2, what's the area of square D?" All items were taken from the study by De Bock et al. (2003).

To measure students' performance, we analyzed whether the solutions were correct (coded 1) or incorrect (coded 0). The number of linear overgeneralizations was assessed by analyzing if they were based on a linear model (coded 1) or not (coded 0). Two independent raters rated 20% of the answers to each problem with sufficient inter-rater agreement (Cohen's $\kappa \geq .827$). Scale reliability was satisfactory (Cronbach's $\alpha = .787$ for performance and .715 for linear overgeneralizations). To address the research questions, we compared the mean scores for students' performance and linear overgeneralizations between the CG and D groups (research question 1) and the CG and DQ groups (research question 2) by using *t*-tests. All alpha values we report are one-tailed due to our directional expectations. For reasons of comparability, we followed De Bock et al.'s (2003) procedure and conducted our analysis with only two of the four experimental items. The results remained nearly the same when all items were included in the analysis.

RESULTS

Our first research question was aimed at replicating the negative effect of the drawing strategy on problem solving performance. We found that students in the drawing condition had significantly lower solution scores than their peers in the control group ($M_D = 0.268$, $SD_D = 0.389$; $M_{CG} = 0.476$, $SD_{CG} = 0.460$; $t(80) = 2.203$; $p < .05$; $d_{Cohen} = 0.488$). In line with our expectations, applying the drawing strategy negatively affected students' problem solving performance for non-linear geometric problems.

The first research question further referred to the number of linear overgeneralizations. We found that students in the drawing condition made in tendency significant more linear overgeneralizations than students in the non-drawing condition ($M_D = 0.390$, $SD_D = 0.426$; $M_{CG} = 0.244$, $SD_{CG} = 0.389$; $t(80) = -1.624$; $p = .054$; $d_{Cohen} = -0.358$). As expected, applying the drawing strategy appeared to increase the number of linear overgeneralizations.

The second research question referred to the quality of the use of the drawing strategy and was aimed at investigating whether the negative effect of the drawing strategy could be diminished by increasing the quality. We found that students who used the drawing strategy in a high-quality manner (DQ condition) had significantly lower solution scores than students who did not use this strategy (CG) ($M_{DQ} = 0.220$, $SD_{DQ} = 0.388$; $M_{CG} = 0.476$, $SD_{CG} = 0.460$; $t(77.78) = 2.724$; $p < .01$; $d_{Cohen} = 0.602$). Increasing the quality apparently could not diminish the negative effect of the drawing strategy on performance.

However, a high-quality use of the drawing strategy was found to diminish the negative effect for linear overgeneralizations. Students who used the drawing strategy in a high-quality manner made a similar number of linear overgeneralizations as students in the control group ($M_{DQ} = 0.342$, $SD_{DQ} = 0.425$; $M_{CG} = 0.244$, $SD_{CG} = 0.389$; $t(80) = -1.08$; $p = .141$; $d_{Cohen} = -0.241$). Hence, increasing the quality helped prevent students from making linear overgeneralizations, but it did not help them solve the problems.

DISCUSSION

One of the goals of the present study was to replicate and extend the findings from De Bock et al.'s (2003) study. In line with the previous findings, we found a negative effect of the drawing strategy on students' problem solving performance for non-linear geometry problems. Even the solution scores in our study were very similar to the ones reported by De Bock et al. (2003), indicating that the negative effect is stable across time and different samples. This replication increases the validity of the surprising finding that drawing can hinder students' ability to solve mathematical problems.

Further, our study was aimed at elaborating on potential reasons that might explain the negative effect of applying the drawing strategy. The results confirmed the previous assumption that lower performance is caused by linear overgeneralizations (De Bock et al., 2003). Applying the drawing strategy without supporting students in using it in a high-quality manner increases the number of linear overgeneralizations. The process of drawing seems to guide learners' attention to the linear property of the circumference, which they mistakenly transfer to the area or volume of the figure. Moreover, in both conditions (D and CG), we found that linear overgeneralizations appeared frequently, which, in line with prior research (Van Dooren et al., 2008), highlights the pervasive role of students' tendency to apply linear models.

With the second research question, we investigated the role of the quality of the use of the drawing strategy. We expected that the negative effect of the drawing strategy on performance and on the number of linear overgeneralizations in students' solutions could be diminished by increasing the quality of their strategy use. Quality was increased by addressing the important feature of the drawing strategy to represent key information (completeness of drawings), which was done by instructing students to highlight the area or volume in their drawings. The results partly confirmed the expectations derived from the theoretical considerations.

Contrary to our expectations, improving the quality did not diminish the negative effect of the drawing strategy on students' performance. Hence, even the use of the drawing strategy with an increase in its quality had a negative effect on the ability to solve non-linear geometry problems. A possible explanation is that applying the drawing strategy when drawing the geometrical figures might hinder a covariation view (area as an alterable value that depends on the length of the side), by leading to a static view of a specific figure's lengths and area. Following this consideration, future studies should investigate how the drawing strategy affects different concept images (Vinner, 1997) for linear and non-linear functions. A promising approach for fostering the covariation view might be to construct a drawing by using dynamic geometry software.

Regarding the number of linear overgeneralizations, the results confirmed our expectation that the quality of the use of the drawing strategy is a crucial factor that determines whether the negative effect occurs or not. This finding is in line with previous research demonstrating the important role of the quality with which strategies are applied. A high-quality use of the drawing strategy helped to prevent at least some of the students from falling into the linearity trap, but it did not help the students find the correct mathematical procedure. This result indicates that apart from linear overgeneralizations, students also encounter other difficulties in solving non-linear geometry problems. This highlights the need for qualitative studies to investigate the process of solving non-linear geometry problems with the help of the drawing strategy in order to get a more complete picture of students' difficulties.

Taken together, our findings show that applying the drawing strategy is not a one-size-fits-all solution. Besides increasing the quality of the drawing strategy, teachers should consider that different conceptual images of linear and non-linear functions are essential for problem solving. Reflecting on the advantages and disadvantages of various representations is an important prerequisite for the beneficial use of this strategy. This stresses the need for further investigations on the drawing strategy.

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TEACHER STUDENTS' AND IN-SERVICE TEACHERS' AWARENESS OF STATISTICAL VARIATION AND RELATED LEARNING POTENTIALS IN DIFFERENT PROFESSION-RELATED REQUIREMENT CONTEXTS

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Dealing with statistical variation is frequently emphasised as a key component of statistical literacy – related abilities should, therefore, be fostered already in the primary mathematics classroom. For being able to create corresponding learning opportunities, teachers need awareness of statistical variation and related learning potentials, which should be fostered during university teacher education. As relatively little is known so far to what extent teacher students already have such an awareness, a study was conducted in which teacher students' awareness was compared with the awareness found in a sample of in-service teachers. The results imply a need for building up specific criterion knowledge with a particular emphasis on a flexible use across different profession-related requirement contexts.

INTRODUCTION

In the last years, an extensive amount of research has been conducted focusing on how to build up foundations for students' statistical literacy in the primary mathematics classroom (e.g. English, 2012). In the related literature, there is a broad consensus on the importance of being able to deal with statistical variation for students' development of statistical literacy (e.g. Watson & Callingham, 2003), and there are several classroom studies that have shown that engaging students to deal with statistical variation is a feasible endeavour (e.g. Ben-Zvi, 2005). Creating appropriate learning environments, however, requires teachers to be aware of the relevance of statistical variation and related learning potentials. Teachers should be aware of these learning potentials in different *profession-related requirement contexts*, such as task analysis, the analysis of data sets for being used in the classroom, or the analysis of classroom situations in which learning opportunities related to statistical variation could come into play, for instance. In a prior study with in-service primary school teachers (Krummenauer & Kuntze, in press) it has been found that only a minority of the participants showed such an awareness when analysing in profession-related requirement contexts, which points not only to a need of corresponding professional development, but also emphasises the importance of building up related professional knowledge from university teacher education on. Relatively little is known so far about teacher students' awareness of the learning potentials related to statistical variation. Addressing this research need, an empirical study has been conducted in which written

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analysis products from 57 teacher students have been collected and compared with data from 44 in-service teachers. The results indicate that only a relatively low percentage of the teacher students was aware of learning opportunities related to statistical variation, as it was also the case for the in-service teachers, and that the evidence of their awareness largely depended on the requirement context. We conclude that teachers should be supported in building up awareness and in its flexible use across different profession-related requirement contexts.

THEORETICAL BACKGROUND

As argued by Cobb and Moore (1997), variation is an omnipresent phenomenon when dealing with statistical data. A focus on dealing with statistical variation is thus frequently emphasised as a key element of the statistics classroom from primary school on (e.g. Franklin et al., 2005). Due to the omnipresence of statistical variation, data provide various opportunities for students to engage with statistical variation (e.g. English & Charles, 2000; Makar, 2018), e.g. when students interpret distributions (Ben-Zvi, 2005) or identify general trends in data, e.g. for making predictions based on patterns of data (e.g. Oslington, 2018). Whether or not such learning potentials are used for students' learning in mathematics classroom, however, can be expected to depend highly on whether teachers have an awareness of statistical variation and related learning potentials. Awareness has been described as *a part of professional knowledge which influences the readiness and ability of teachers to use related professional knowledge elements in instruction-related contexts* (Kuntze & Dreher, 2015, p. 298). The awareness of certain criteria – in this case, statistical variation and related learning potentials – can hence enable teachers to use relevant professional knowledge for analysing in profession-related requirement contexts and, therefore, can be expected to influence what learning opportunities teachers are able to provide to their students (e.g. Kuntze & Friesen, 2018).

To what extent teachers are aware of certain criteria becomes apparent when they analyse in profession-related requirement contexts (Kuntze & Friesen, 2018). *Analysing* is considered as an "awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation" (Kuntze, Dreher, & Friesen, 2015, p. 3214). As published in Kuntze & Friesen (2018), the process of analysing can be described by means of a model with a circular structure, similar to the modelling cycle described by Blum and Leiss (2005). The model primarily addresses teachers' analysing of classroom situations, but it can also be applied to teachers' analysing in other profession-related contexts, such as analysing students' answers to tasks or analysing textbook material. In prior research (e.g. Kuntze & Friesen, 2018; Krummenauer & Kuntze, in press), the model had already been applied successfully. The process of analysing is structured into four phases, which are, however, meant to be not necessarily followed by each other in a fixed order, but also can be skipped, interrupted, or repeated. In the first phase of the model, teachers generate a situation model ("real model") of the profession-related context they analyse, which is interpreted in the subsequent phase based on criteria, which are part of their professional knowledge. These criteria are the basis of models for explaning teachers' observations and developing implications, which subsequently can be validated against the situation model of the classroom situation in a further step. Teachers' criterion awareness plays a key role in this cycle, as it activates relevant professional knowledge for an analysis, when a criterion appears to be relevant; without the awareness of the relevance of a specific criterion in a certain situation, it cannot be used for analysis, even if a teacher in principle has relevant professional knowledge.

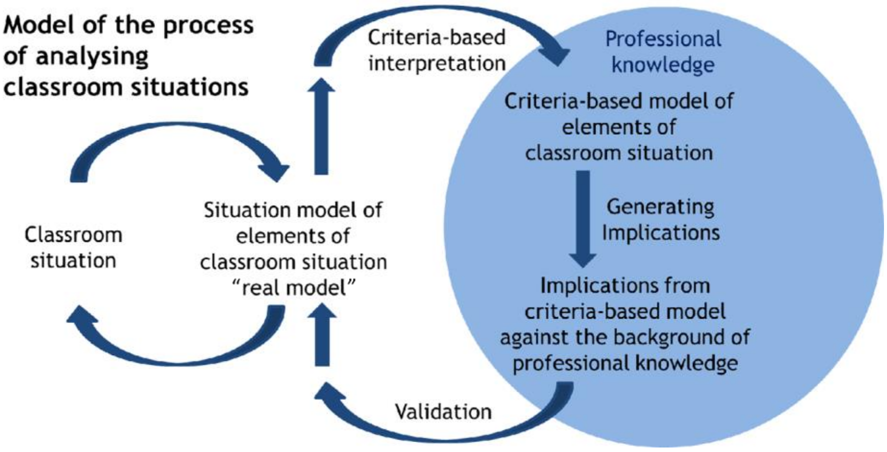

Figure 1: Model of the process of analysing (Kuntze & Friesen, 2018, p. 277)

As already introduced before, teachers' analysing occurs in different profession-related requirement contexts, such as classroom situations, choosing material from textbooks, or interpreting written student answers. In order to make these different situations accessible to teachers' analysis, different vignettes can be used representing these profession-related requirement contexts (Friesen, 2017; Skilling & Stylianides, 2020). In an earlier study with $N = 44$ in-service primary school teachers (Krummenauer & Kuntze, in press), such vignettes had been used in order to investigate whether the teachers showed an awareness for learning potentials related to statistical variation when analysing the vignettes representing several profession-related requirement contexts. Based on the teachers' written answers, a set of categories had been identified representing different types of answers. First of all, there were answers indicating an awareness of statistical variation and related learning potentials; in such answers, reference to the statistical variation implemented in the vignettes was made, and a learning potential related to the statistical variation was mentioned, e.g. a corresponding question focussed on engaging students with statistical variation or a suggestion on how to deal (better) with statistical variation in a classroom situation represented by a vignette. Besides such answers, there were further categories of answers which did not indicate an awareness of statistical variation. Some answers, for instance, only focused on didactical or pedagogical aspects of the represented profession-related context. Further, there were answers, in which teachers mentioned statistical variation (and therefore apparently identified it as a relevant criterion for their analysis), but it was considered as hindering for students' learning and suggested to be removed from the analysed textbook material. Overall, only a minority of the in-service teachers showed an awareness for learning potentials related to statistical variation. Although all vignettes had been designed so that statistical variation played a predominant role for students' learning in the respective contexts, differences in the rates of successful answers among the different vignette categories were found, which may indicate that there may be a relatively strong interrelation of teachers' awareness with particular profession-related contexts.

These first findings point not only to a need for specific professional development, but also appear as highly relevant for university teacher education. However, specific research on teacher students' awareness of learning potentials related to statistical variation is scarce. In particular, relatively little is known so far to what extent teacher students' awareness differs from that found in in-service teachers and whether they show that awareness when analysing in different profession-related requirement contexts.

AIM OF THE STUDY AND RESEARCH QUESTIONS

Consequently, this study addresses this research need; in particular, the following research questions have been investigated:

- Do teacher students show an awareness of the learning potentials related to statistical variation when analysing vignettes representing profession-related requirement contexts, and to what extent does their awareness differ from that found in a study with in-service teachers?
- Are there differences in the teacher students' awareness among different professional requirement contexts implemented in different types of vignettes?

SAMPLE AND DESIGN OF THE STUDY

The data analysed in this study comprises of N=101 data sets from n=57 primary teacher students and n=44 in-service primary school teachers (95 % female, 5 % male) from southern Germany. About 32 % of the in-service teachers studied mathematics as a major subject during their initial teacher education at university, 29 % studied mathematics as a minor subject and about 40 % of the teachers had not studied mathematics/mathematics education during their university education. The teachers' classroom experience ranged from 1 to 36 years with a mean experience of 8 years (SD = 7). About 70 % of the teacher students were in their third semester (M = 4 semesters); about half of the sample studied mathematics as a major subject, the other half as a minor subject.

For collecting data from the teacher students, the same vignette-based paper-pencil questionnaire was used as in the prior study with the 44 in-service teachers (Krummenauer & Kuntze, in press). The questionnaire consists of several vignettes representing three different types of profession-related requirement contexts: analysis of classroom situation (requirement context represented by means of a cartoon vignette; one item), analysis of learning tasks for students (4 items), and analysis of data sets intended for use in the mathematics classroom (2 items). Although the vignettes represent different types of profession-related requirement contexts, they are designed in a way so that awareness of learning potentials related to statistical variation is needed for creating optimal learning opportunities for students' learning in the respective requirement context. The sample vignette-based items presented in the following will provide insight into how this approach was implemented in the questionnaire instrument.

In Figure 1, a data set is displayed as it was part of a vignette. It shows the fictive development of the weight of a newborn including a (possible) drop of the weight in the fourth week. These data can function as a starting point for several learning opportunities related to statistical data; students, for instance, could be asked what in week four might have happened or how the data could continue, how other kids with different birthweight may develop. Initiating only basic activities, on the contrary, would indicate that teachers are not aware of statistical variation and its learning potentials.

Data set 2:

A baby was born. Every Sunday, the weight of the baby is measured.

Weeks	***Weight***
1. Week	*3600g*
2. Week	*3900g*
3. Week	*4200g*
4. Week	*3500g*
5. Week	*4700g*
6. Week	*4900g*

Figure 2: Example of a data set, which is part of a vignette used in the questionnaire.

In the learning tasks, the second vignette type of the questionnaire instrument, a data-set as shown in Figure 2 is combined with a question to students (e.g. "Draw a bar chart."). The third vignette type requires the participants to analyse a classroom situation represented as a comic. A key feature of the classroom situation used in the instrument is that statistical variation is embedded in a classroom scenario, in which a teacher discusses a task with students, in which statistical variation has to be taken into account, but the teacher does not take notice of the high relevance of statistical variation. In the questionnaire, all vignettes are combined with open questions, which are formulated in a way that they initiate the participants' analysing (e.g. in case of the data set in Figure 2: "Would you use this data set in the mathematics classroom? If yes: What questions would you ask the students? If no: What would you change?" or in the case of the classroom situation: "How could the teacher in the comic sequence do better?"), but without pointing to statistical variation in any way, in order to be able to distinguish whether the participants' identified statistical variation as a relevant criterion and were aware of the related learning potentials.

Data coding

Based on the coding developed in Krummenauer and Kuntze (in press), the teacher students' answers were subjected to a criteria-based top-down rating on whether the answers indicate an awareness of the learning potentials related to statistical variation. For a positive rating, the answer had to fulfil two criteria: they had to contain at least one reference to the statistical variation in the respective data set/task/classroom situation, and a possible learning potential related to statistical variation had to be mentioned. In the case of the data set displayed in Figure 2, for instance, the sudden drop of the weight or the development of the weight could be mentioned for fulfilling the first criterion. The second criterium (mentioning a related learning potential) is fulfilled, when it is, for instance, stated that teachers could engage students to reflect on the rapid decrease in the data set or ask students for several scenarios on how the data could develop further. In the case of the vignette representing the classroom situation, it is required to identify that the teacher in the represented classroom situation ignored the predominant relevance of statistical variation in the task and the related learning potential. The coding had been validated during the prior study by means of a second rating with a measured overall inter-rater-reliability of κ=.84 (Krummenauer & Kuntze, in press).

RESULTS

Figure 2 shows the joint results of the coding for both samples of teacher students and in-service teachers. It can be seen, that the teacher students' frequencies of answers, which indicate an awareness of statistical variation and the related learning potentials, are in a similar range compared with the frequencies of the in-service teachers; t-tests did not indicate any significant differences of the frequencies of teacher students and in-service teachers (p>.20). Further, the results show that the teacher students' frequencies vary among the profession-related requirement contexts almost to the same extent as it has been found for the in-service teachers (see Krummenauer & Kuntze, in press). The mean rate of answers indicating an awareness is 15.6 % for student teachers ans 18.3 % for in-service teachers.

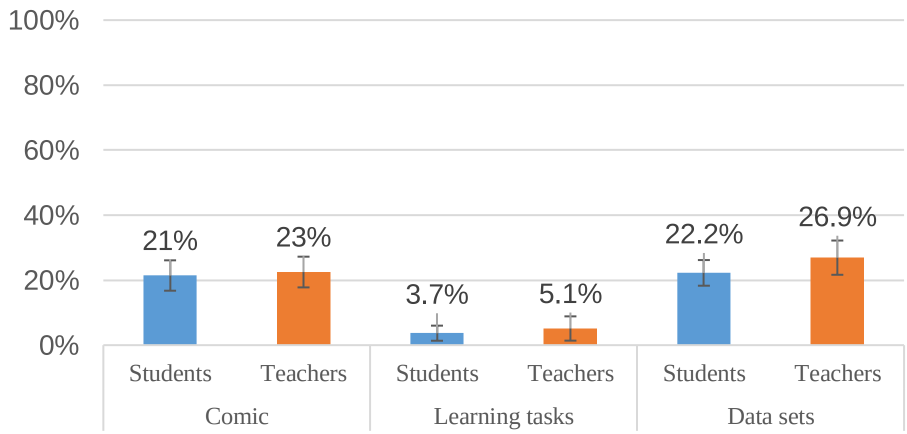

Figure 2: Frequencies of answers indicating an awareness of statistical variation and related learning potentials.

In order to further explore whether the teachers' and teacher students' awareness is dependent or independent from the different profession-related requirement contexts, bi-variate correlations between the different categories were computed. In the joint sample of teacher stiudents and in-service teachers, a correlation of r=.29 (p<.01) was found between the participants' responses to the data set vignettes and to the comic vignette, indicating a joint variance of about 8 %. Between the participants' answers showing awareness to tasks and data sets, we found a weak correlation of r=.22 (R^2=0.048, p<.05), while there is no correlation (r=.04) between answers to the vignettes containing learning tasks and answers to the comic vignette.

DISCUSSION AND OUTLOOK

Addressing the first research question, the results imply that the teachers students' awareness of learning potentials related to statistical variation was very low on average, and the evidence resembles the earlier findings from the sample of in-service primary school teachers. Although statistical variation has high relevance for students' learning in all vignettes, showing awareness of this criterion in one profession-related requirement context appears not to be highly predictive of whether teachers and teacher students show this awareness also in other profession-related requirement contexts. Against the background that the learning task vignettes mainly differ from the data set vignettes in that the given data set had been combined with a question to students, it appears that already relatively little changes in the professional requirements can affect whether or not statistical variation is used as a relevant criterion in the teachers' analysis.

Overall, the rate of answers indicating an awareness of statistical variation and related learning potentials is relatively low (mean frequencies below 20 %). As it has to be assumed that there is a strong relation of teachers' awareness and learning opportunities teachers are able to provide to their students, there is a clear need both for in-service teacher professional development and for building up professional knowledge in university teacher education. For strengthening the teacher's awareness across different professional requirement contexts, a special focus should be given to vignette-based professional learning opportunities in a variation of methodological settings: Different types of vignettes should provide opportunities for analysis of tasks, classroom situations, data material, or written student responses, which show their thinking related to statistical variation. In the European project "coReflect@maths", corresponding vignette-based material for further research and for teachers' professional learning is under development.

Acknowledgements

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MULTI-CRITERION NOTICING: PRE-SERVICE TEACHERS' DIFFICULTIES IN ANALYSING CLASSROOM VIGNETTES

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When confronted with the complexity of classroom situations, mathematics teachers are often required to notice aspects of a situation with respect to multiple criteria. Such multi-criterion noticing requirements can be assumed to come with high cognitive load for the teacher and teachers need multi-criterion awareness as well as corresponding professional knowledge in the relevant domains. Despite its importance for mathematics teacher expertise, studies approaching multi-criterion noticing in a systematic way are still scarce. Consequently, this study examines pre-service teachers' analyses of a classroom vignette designed to require noticing related to several criteria relevant for the students' learning. The findings provide insight into pre-service teachers' difficulties and point to the complexity of multi-criterion noticing.

INTRODUCTION

Mastering complex professional tasks in the mathematics classroom often requires that teachers analyse classroom situations with respect to multiple criteria. Dealing with students' heterogeneous learning prerequisites is one of such complex professional tasks (e.g., Hardy, Decristan & Klieme, 2019): It requires mathematics teachers to draw on a bundle of criteria within their professional knowledge, since they have to keep an eye on tasks, on the students' individual thinking, on multiple solution pathways and accesses for learning, while at the same time they have to avoid unnecessary obstacles when dealing with representations of mathematical objects, amongst others. Therefore, to provide optimal support to students, teacher noticing has to focus on multiple criteria. Such multi-criterion noticing is expected to be demanding, especially for pre-service teachers. Consequently, research into difficulties related to multi-criterion noticing is highly relevant in order to identify possibilities of supporting pre-service teachers in developing corresponding noticing competences.

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THEORETICAL BACKGROUND

Teachers' noticing has increasingly been considered as a key aspect of expertise (e.g. Berliner, 1991; Sherin, Jacobs & Philipp, 2011; Fernández & Choy, 2020; Amador et al., 2021). There is a growing body of research about noticing regarding specific criteria, e.g. regarding mathematical contents (e.g. Choy, 2014) or regarding how representations are dealt with in the classroom (e.g. Dreher & Kuntze, 2015; Friesen & Kuntze, 2016). Noticing in this sense has been characterised as a combination of selective attention and knowledge-based reasoning (Sherin et al., 2011): Different components of professional knowledge (Kuntze, 2012; Shulman, 1986) can be used as criteria which can be connected with observations in classroom situations (Sherin et al., 2011; Kuntze & Friesen, 2016). The complexity of classroom situations (e.g. Petko, Waldis, Pauli, & Reusser, 2003) implies that multiple criteria are relevant for successful noticing in such multifaceted situations, i.e. noticing that enables teachers to support their students' learning optimally (see below for specific examples). By the notion of *multi-criterion noticing*, we emphasise knowledge-based reasoning which draws on multiple knowledge-based criteria. As different noticing criteria can be expected to be in a competing relationship with each other (see findings of the prior study by Kuntze & Friesen, 2018), multi-criterion noticing is likely to bring high cognitive load (Sweller, 1994) for teachers who analyse classroom situations, particularly for pre-service teachers who are less experienced in noticing.

Further, and beyond only possessing (potentially inactive) corresponding professional knowledge, teachers need *criterion awareness* (Kuntze & Friesen, 2018) related to the different noticing criteria: this criterion awareness supports analysis cycles, which afford activating criterion knowledge and connecting it with observations in the classroom situation (see the process model published in Kuntze & Friesen, 2018). Such connections of professional knowledge elements with situation observations lead to successful knowledge-based reasoning (Sherin et al., 2011), hence successful noticing. In a somewhat similar way, Kersting and colleagues (2012) emphasise the aspect of activating professional knowledge elements and using them for interpreting observations, namely in their notion of *usable knowledge*.

A set of criteria based on mathematics teachers' professional knowledge which can be seen as an example in this context, is needed when teachers have to deal with heterogeneous learning prerequisites of students in their classrooms: For analysing classroom situations in which heterogeneous learning prerequisites have to be dealt with, noticing criteria often have to include several criterion-related foci, among which the following play key roles:

- a focus on possible obstacles for learning and understanding in tasks (cf. Choy, 2014; Vondrová & Žalská, 2013),
- a focus on the students' individual thinking, as it can reveal their individual learning prerequisites and individual difficulties (e.g. Fernández et al., 2018),
- a focus on how individual, multiple solution pathways can be supported (Hardy et al., 2019), and
- a focus on avoiding unnecessary obstacles when dealing with representations of mathematical objects (Ainsworth, 2006; Duval, 2006; Friesen & Kuntze, 2016).

For eliciting teachers' analysis of classroom situations in systematically designed research settings, vignettes offer various possibilities (Buchbinder & Kuntze, 2018). Vignettes are based on representations of practice, e.g. classroom situations represented in video, text or cartoon format, on which teachers can be asked to reflect and/or to report their observations or suggestions. The noticing criteria introduced above can thus be implemented in specifically designed vignettes so as to create an analysis situation for mathematics teachers in which they need multi-criterion noticing in order to detect non-optimal aspects of the classroom situation through corresponding knowledge-based reasoning (cf. also Buchbinder & Kuntze, 2018; Skilling & Stylianides, 2019). Although vignette-based empirical research offers a broad spectrum of possibilities, research about multi-criterion noticing is still relatively scarce. There is hence a need of studies exploring multi-criterion noticing, in particular whether and to which extent already pre-service teachers are able to show multi-criterion noticing in their analysis of classroom vignettes.

RESEARCH QUESTIONS

As outlined above, multi-criterion noticing is likely to be associated with high cognitive load and to depend on the availability of professional knowledge related to the noticing criteria, as well as on criterion awareness supporting the active use of this professional knowledge. In all three areas, pre-service teachers can be expected to have lower prerequisites than experienced in-service mathematics teachers. As empirical evidence is still scarce despite the high relevance for professional growth, this study examines whether multi-criterion noticing is possible for pre-service teachers. Moreover, as a second research aim, we explore the role of possibly non-available professional knowledge and criterion awareness for multi-criterion noticing, by offering the pre-service teachers a seminar in which the noticing criteria introduced above are focused, and then asking the pre-service teachers to analyse the vignette again. Consequently, the research questions of this study are the following:

(1) Are the pre-service teachers able to analyse situation aspects with respect to several criteria, which are relevant for helping the vignette students in their learning?

(2) To what extent does a university seminar offering learning opportunities related to multiple noticing criteria lead to improved multi-criterion noticing?

DESIGN AND METHODS

As mentioned above, this study uses a vignette-based approach to address multi-criterion noticing in a systematic way. The four noticing criteria introduced in the theoretical background section were implemented in the vignette, so that, from a normative point of view, an expert mathematics teacher could be expected to notice situation aspects related to these four noticing criteria.

The cartoon-based vignette (shown together with its key design features in Fig. 1) presents a classroom situation (material, student-teacher dialogues, students' notes, etc.) from the perspective of a pre-service teacher who reports on his teaching experien-ce in a lesson with 7th-graders during his school internship. The task, the vignette stud-ents are asked to work on, is presented, and the teacher gives some information about his intentions. In the vignette situation, two students ask the teacher for help related to the given task. The teacher mainly reacts by writing a "hint" on the blackboard, which is however disconnected with the students' thinking and the students' representations. As the students ask the teacher again by showing him their notes, he does not connect with the students' ideas and suggests them to use the formula he had presented as "hint" on the blackboard. The vignette ends with an analysis question: "Give your peer teacher student feedback on the classroom situation shown above. Write down everything that comes to your mind when looking at the classroom situation and refer as specifically as possible to what has been said in the situation, to materials, etc."

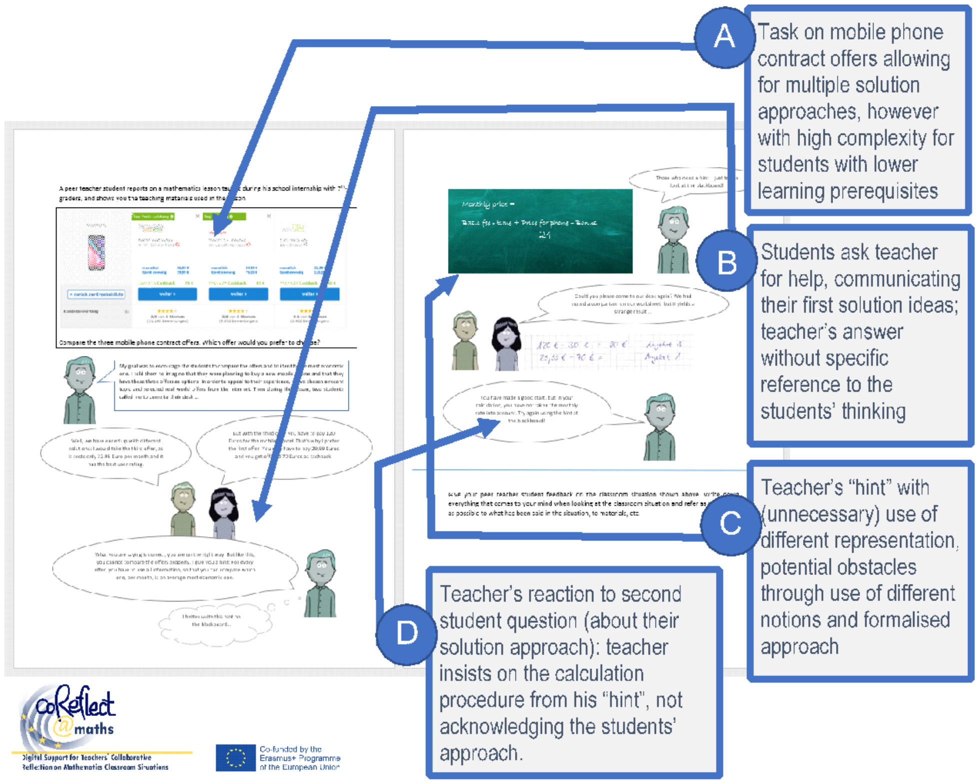

Figure 1: The vignette and key design features

Related to this analysis question, an expert teacher can be expected to notice possible

- learning/understanding obstacles in the vignette task (possibility of entering the work on the task at different levels, but need of providing facilitated access for students with lower learning prerequisites) (criterion A)
- learning/understanding obstacles arising from the vignette teacher's reaction as the teacher does not connect with students' thinking in his answers (criterion B)
- learning/understanding obstacles in teacher "hint" through an unexplained and unnecessary change of representation (including new notions in formalised representation) (Ainsworth, 2006; Duval, 2006) (criterion C)
- learning/understanding obstacles as the vignette teacher insists on his way of solving the task instead of supporting the students in further developing their own solution ideas (criterion D).

Based on this study's theoretical background, we consider these four noticing criteria as necessary for a successful analysis of this vignette, hence for successful multi-criterion noticing. Of course, further noticing criteria can be used as well for answering the analysis question in addition to the criteria given above. As mentioned before, we expect that the noticing criteria are in a competing relationship with each other, making successful multi-criterion noticing relatively complex.

The vignette was administered as a paper-and-pencil questionnaire to 20 pre-service master students (i.e. bachelor secondary teacher studies completed), at the beginning and end of a seminar. In the seminar, opportunities for professional learning related to the criteria A-D corresponded to core content goals. Throughout the seminar, several vignettes were used to stimulate the participants' analysis, however, mostly with a focus only on one of the criteria A-D at a time. Like this, we expect that professional knowledge and corresponding criterion awareness could be built up by the participants.

For analysing the pre-service teachers' answers, we used a top-down coding according to the four criterion categories (A-D) given above. This criterion-based content analysis (Mayring, 2015) led to a dichotomous coding whether the pre-service teachers' answers showed evidence that noticing based on the respective noticing criterion had taken place (see Fig. 3 and 4 in the results section for coding examples).

RESULTS

The first research question concentrates on whether pre-service teachers are able to notice situation aspects with respect to several criteria, which are relevant for helping the vignette students in their learning. The relative frequencies in Figure 2 on the left, show how many of the four noticing criteria were visible in the pre-service teachers' answers. Before the beginning of the seminar intervention, none of the participants used all of the four noticing criteria implemented in the vignette, more than 60% of the pre-service teachers used none or only one of these noticing criteria.

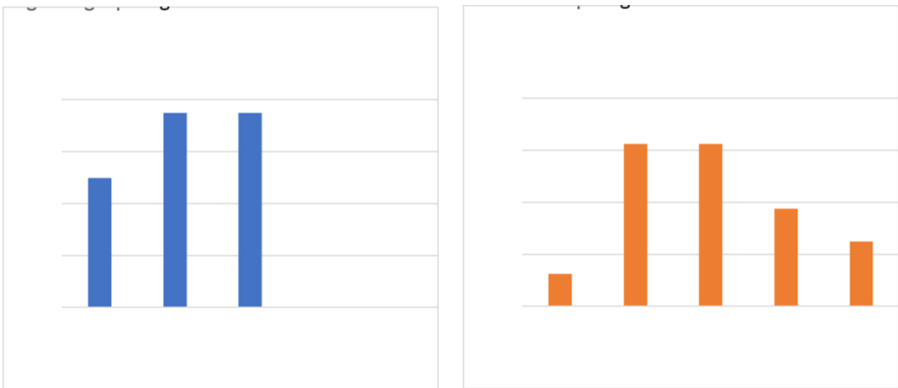

Figure 2: Number of (coded) noticing criteria used in the pre-service teachers' analysis (beginning and end of the seminar)

An initial approach explored the role of professional knowledge and criterion awareness by focusing on the pre-service teachers' answers at the end of the seminar (second research question). The corresponding relative frequencies are shown in Figure 2 on the right. In this context, we recall that these findings are not primarily intended to evaluate the seminar, but to explore the complexity of multi-criterion noticing for pre-service teachers. At the end of the seminar, there is evidence of successful multi-criterion noticing, even if more than 35% of the pre-service teachers used none or only one of the four implemented noticing criteria.

Figure 3 allows a closer look at the pre-service teachers' multi-criterion noticing as visible through their written analysis results, i.e. their answers to the analysis question. In this sample answer from the end of the seminar, the pre-service teacher first turns to the task and the contract offers displayed in it (see Fig. 1). Even if the answer does not give clear suggestions how to "change the representation", we have evidence that the pre-service teacher noticed that the representation of the offers is "complex" and "confusing", which constitutes a potential obstacle for the vignette students' understanding and work on the task. Further, the vignette teachers' reactions are analysed: As far as criterion B is concerned (*code B*), the pre-service teacher appears to have noticed that the vignette teacher did not connect with the vignette students' thinking ("hardly connects with the students' problems"), which is considered as non-sufficient (underpinned by the words "just" and "only" later in the sentence).

* Representation of [contract] offers is very complex and confusing. Perhaps it would have been more reasonable here to change the representation a little. * Even if mobile phones cover a big part of students' everyday life, I do not think that the majority of the students already had to do with such a webpage.	Code A
* The teacher hardly connects with the students' problems, but he just notes a formula on the blackboard, in which the students only are supposed to enter the values.	Code B Code C
* After the students did not work with the formula, but chose a different representation, the teacher hardly connects with it, but refers directly to the formula again. He does hence not let the students calculate in their representation.	Code D

Figure 3: Sample answer (end of the seminar)

Mentioning/describing the non-optimal use of representations in the teachers' hint ("formula", which is incongruent with the representation relevant for the "students' problems" and later the "different representation") indicates that noticing criterion C has been used in the pre-service teacher's analysis. Finally, the vignette teacher's non-optimal reaction related to multiple solution pathways ("He does hence not let the students calculate in their representation") is also noticed, corresponding to Code D.

Figure 4 shows a sample answer from the end of the seminar, for which none of the codes A-D was given, whereas different criteria were focused on. The answer appears to focus on the role of the vignette teacher's "hint" for the *other* students in the classroom and for their working process. This noticing result draws on related, rather pedagogical criteria and mainly stays with this criterion domain, possibly together with a suggested different intervention of the vignette teacher.

- I find it inappropriate to give hints for everybody in the middle of a working phase → students could be confused and led away from their own solution pathways.
- As a hint, the [following] stimulus could be given, for example: Do you find a way for each [contract] offer to find out about the monthly price/rate? Like this, you can compare better…

Figure 4: Sample answer (end of the seminar)

DISCUSSION AND CONCLUSIONS

Given the small sample and the explorative nature of this study, the results should be interpreted carefully. The results indicate that most pre-service teachers focused on a small number of noticing criteria in their vignette analysis. Even at the end of a one-semester university seminar in which the pre-service teachers had the opportunity to learn about the criteria implemented in the vignette, multi-criterion noticing remained difficult for a majority of the pre-service teachers. Therefore, providing professional knowledge learning opportunities is not sufficient for succeeding in related multi-criterion noticing. This result is in line with previous research focused on providing different theoretical lenses to support professional noticing in teacher education programs (a review in Fernández & Choy, 2020).

The findings also show that multi-criterion noticing is possible already for pre-service teachers (there are cases with four noticing criteria), in this case after they have been successfully supported in building up professional knowledge and criterion awareness for the corresponding noticing criteria.

Nevertheless, multi-criterion noticing needs focused support in mathematics teacher education. Such support could focus on a meta-level and target explicit strategies of monitoring one's own noticing: Pre-service teachers could be encouraged to explicitly monitor whether and how they draw on specific noticing criteria within their professional knowledge. Such strategy support can be combined with the analysis of vignettes: As vignette-based work is in the focus of the European project coReflect@maths (Digital Support for Teachers' Collaborative Reflection on Mathematics Classroom Situations, www.coreflect.eu), we expect more insight into multi-criterion noticing and its support in follow-up studies.

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NATURAL NUMBERS BIAS IN UNDERSTANDING VARIABLES – THE INTEGRITY AND THE PHENOMENAL SIGN EFFECT

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This report aims to present quantitative results from a study where students were asked to evaluate a series of statements about the numerical value of algebraic expressions. The participants were 138 8th and 9th graders and they were given a questionnaire with 48 statements about numbers that can or cannot be assigned to six algebraic expressions that contained literal symbols (e.g. a, -b, k+3). The results showed that the students tended to agree with the statements which were in line with their belief that variables stand only for natural numbers and not for rational or real numbers (integrity effect), and that algebraic expressions stand for numbers of the same sign as the expressions' phenomenal sign (phenomenal sign effect). It also appeared that the integrity effect was stronger than the phenomenal sign effect.

THEORETICAL AND EMPIRICAL BACKGROUND

Acquiring the concept of variable is widely recognized as a major challenge for students in the transition from arithmetic to algebra (Kieran, 2006). Prior research has shown that students initially think that literal symbols stand for abbreviated names of people or objects, or as coded numbers, with values corresponding to their positions in the alphabet (Booth, 1984; Kuchemann, 1981; Wagner, 1981). While such mistakes are usually abandoned by 10^{th} grade, some other difficulties are more persistent even when students receive detailed instructions to correct them. Such is the tendency to think that literal symbols can stand for only one, specific, unknown number as opposed to the generalized number which may take more than one value (Asquith et al., 2007; Knuth et al., 2005; Kuchemann, 1981).

A question that is only recently addressed in the literature is what happens when students understand that a variable stands for more than one number (Booth, 1984; MacGregor & Stacey, 1997): Are they ready to accept that it may represent any real number? Christou and Vosniadou embrace the framework theory approach to conceptual change, which assumes that, before they are introduced to non-natural numbers, students have already consolidated a complex system of interrelated concepts and beliefs, tied around their knowledge and experiences with natural numbers. This initial framework theory of number underlies students' expectations about what a number is and how it is supposed to behave, ending up in a bias towards natural numbers (Vosniadou et al., 2008). As *natural number bias* is characterized the phenomenon of students using their prior knowledge of numbers – which is organized in a coherent body of knowledge with numbers having the properties of natural numbers – in situations where this knowledge does not apply (Ni & Zhou, 2005).

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In a series of studies, Christou and his colleagues (Christou et al., 2007; Christou & Vosniadou, 2009, 2012) investigated the hypothesis that this bias would affect the kind of numbers that may be associated with algebraic expressions that contain literal symbols. This hypothesis was supported by the results of those studies with Greek students and were further affirmed from implication studies with Flemish students (Van Dooren et al., 2010).

Specifically, using open ended questions, students were asked to give numbers that they thought could be assigned to specific algebraic expressions that contained literal symbols. The majority of the students tended to respond by assigning natural numbers to the literal symbols of the expressions, saying for example that *a* stand only for natural numbers, *-b* stand for negative integers, and *k+3* for natural numbers larger than 3. Of course, these responses were not incorrect, however they showed a specific tendency on the part of the students to think of literal symbols as natural numbers only. When the students were asked about numbers that *could not* be represented by the same algebraic expressions, the majority of them replaced literal symbols with negative integers, saying, for example, that 1, 2, 3,… cannot be associated with *-b*, and -1, -2, -3,… cannot be associated with *a* (Christou & Vosniadou, 2012).

Quite the same were the results in the multiple-choice tasks of the Christou et al (2007) study where the students were offered a set of possible values and were asked to decide whether some of them should be excluded from the range of the values that the algebraic expressions could represent. Less than 20% of the students' responses to the combined questions corresponded to the correct answer, which was one of the given alternatives. Another 22% corresponded to the values that could not come out of the substitution of natural numbers. Finally, around 25% of the choices pointed to students' reluctance to accept that an algebraic expression that appeared to be negative (such as –b) could take positive values, and vice versa. This was observed for students who were prepared to accept non-natural numbers as possible substitutes for the literal symbols, provided that the phenomenal sign of the algebraic expression would not be violated.

The above results provided some empirical evidence that the natural number bias has a dual effect on students' understanding the use of literal symbols to stand for variables in algebra. First, it affects them to perceive literal symbols them as symbols that represent natural numbers and not any rational or real number, as they are specifically taught in school. Students who may think that literal symbols should be substituted with natural numbers only, they would conclude that the phenomenal sign of the algebraic expression is the actual sign of the numbers it can only represent. As a result, a second effect of the natural number bias would be to think that an algebraic expression can only stand for numbers with the same sign as the sign the expression appears to have as a superficial characteristic of its form, (e.g. negative in the case of *-b* and positive in the case of *k+3)*.

The purpose of the present study was to further test the hypothesis that there is a dual effect of the natural number bias which affects students to tend to misinterpret variables to stand for natural numbers only (integrity effect) and algebraic expressions to only represent numbers of the same sign as their phenomenal sign (phenomenal sign effect). A new methodology was used in this study instead of asking students to assign numbers to the given expressions, or exclude specific numbers as numbers that cannot be assigned to them, as the previous studies did. In this study students were asked to evaluate a series of statements about the numbers that can or cannot be represented by the given expressions. This way it could also be tested which effect of the natural number bias is stronger: the integrity or the phenomenal sign effect. Additionally, it could also be tested the effect of the expression's form (i.e., whether an expression has an integer-like versus a rational-like form, or a positive versus a negative-like form) in the numbers associations that may provoke.

METHOD

Participants

The participants were 138 students from two Greek public high schools. Sixty-eight attended 8th and 70 attended 9th grade; 61 boys and 77 girls. We chose 8th and 9th graders as an interesting age group, because they have in principle all knowledge necessary to deal with the research tasks correctly. The participants to our study have been introduced to the concept of variable since grade 7, and they had also extensive experience with non-natural numbers since they have been introduced to rational numbers already from grade 4.

Materials

A questionnaire was administered to the students including 48 statements about the numerical value that can or cannot be associated to an algebraic expression. Six algebraic expressions (*a, -b, k+3, -d-4, 4/5y, -4/5z*) were assigned to a different number in each task and students were asked to *agree* or *disagree* with the given statement. Half of the statements were given in an affirmative form (e.g. it is possible for -b to stand for number 2), in order for the answer "I agree" to be correct, and the other half were given in a negative form (e.g. it is not possible for -b to stand for number -8), in order for the answer "I disagree" to be correct.

The above resulted in a design of tasks that were either congruent or incongruent with the beliefs about the integrity and the sign of the number that are intuitively associated with the algebraic expressions due to the natural number bias. More specifically, congruent with integrity (CongInt) were the statements in which natural numbers were assigned to literal symbols and thus integer numbers were assigned to integer-like expressions (e.g., it is possible for *a* to stand for 6). Incongruent with integrity (IncongInt) were the tasks where fractions or decimals were assigned to integer-like expressions (e.g. it is not possible for *a* to stand for 1/6). Congruent with the phenomenal sign (CongSign) were the tasks where the numbers assigned to the algebraic expression were of the same sign as the expression's phenomenal sign (e.g. it is possible for *a* to stand for ¾). Statements incongruent with the sign (IncongSign) were those in which the numbers had a different sign from the phenomenal sign of the expression (i.e. it is possible for *a* to stand for -2). Thus, four task categories were formed depending on whether each task was congruent or incongruent with the integrity and with the phenomenal sign at the same time (See Table 1). The questionnaires were administered in the students' classrooms, in the presence of their mathematics teacher. The students had approximately 45 minutes to complete the questionnaires.

Task Category	Example
CongInt / CongSign	it is possible for -b to stand for -4
CongInt / IncongSign	it is possible for -b to stand for 2
IncongInt / CongSign	it is possible for -b to stand for -2.8
IncongInt / IncongSign	it is possible for -b to stand for 4/3

Table 1: Examples of tasks by category of congruency/incongruency with integrity/phenomenal sign

RESULTS

Students' responses in the tasks were scored on a right/wrong basis. Overall the questionnaire showed high reliability (Cronbach's Alpha = 0.738). The analysis of students' overall performance showed no main effect for gender [$F(1, 134) = 2.36$, $p = 0.127$] or grade [$F(1, 134) = 0.088$, $p = 0.768$]. The mean scores were calculated for each category of tasks and are presented in Table 2.

	Minimum	Maximum	Mean	S.E.
CongInt / CongSign	3	12	8,38	0,19
CongInt / IncongSign	0	12	6,01	0,21
IncongInt / CongSign	1	12	6,11	0,19
IncongInt / IncongSign	0	12	4,89	0,23

Table 2: Mean scores for each category of congruent/incongruent tasks with integrity and phenomenal sign

As expected, students' highest performance appeared in the tasks that were congruent with both integrity and phenomenal sign (i.e., CongInt/CongSign tasks) and the lowest performance appeared in the tasks that were incongruent with both integrity and phenomenal sign (i.e., the IncongInt/IncongSign tasks), which challenged students' belief than only natural numbers should be associated with literal symbols and only numbers of the same sign with algebraic expressions.

Students' performance was systematically higher for the tasks that were in-line with their belief that variables stand for natural numbers than when the tasks were challenging this belief. Specifically, paired-samples t-test showed that the students performed statistically significantly higher in the CongInt/CongSign tasks than in the IncongInt/CongSign tasks [$t(137)=11.064$, $p<.001$, $d=0.94$], when both kinds of tasks were in-line with their belief about the sign of the numbers that can be represented by the specific expressions (CongSign tasks). The same was the case for the pair of tasks that were both against students' beliefs about the sign of the numbers that can be represented by the expressions (IncongSign tasks); there was significantly higher performance in the CongInt/IncongSign tasks than in the IncongInt/IncongSign tasks [$t(137)=5.332$, $p<.001$, $d=0.45$], showing again that students tended to misinterpret variables to stand for natural numbers rather than rational numbers.

Considering the phenomenal sign effect, students' performance was systematically higher for the tasks that were aligned with their belief that algebraic expressions represent numbers of the same sign as their phenomenal sign than when tasks were presented sentences that were against this belief. Specifically, there appeared statistically significant higher scores in the CongInt/CongSign than in the CongInt/IncongSign tasks [$t(137)=9.267$, $p<.001$, $d=0.79$] when both kinds of tasks were in-line with their belief that only natural numbers can be associated with the literal symbols of the expressions (CongInt tasks). The same was the case when both tasks were challenging the belief that variable should stand for natural numbers only (IncongInt tasks); in those kind of tasks the students scored again significantly higher in the IncongInt/CongSign tasks than in the IncongInt/IncongSign tasks [$t(137)=4.275$, $p<.001$, $d=0.36$]. These results show that the students tended to misinterpret algebraic expressions to represent numbers of the same sign as the phenomenal sign of the expression in both categories of tasks.

In order to test which effect of the natural number bias was stronger between the integrity and the phenomenal sign effects, Cohen's D effect sizes for each of the above mention effects was compared. The size of the integrity effect for the pair of CongSign tasks ($d=0.94$) was larger than the size of the phenomenal sign effect for the pair of CongInt tasks ($d=0.79$). That was also the case for the integrity effect which size was larger for the pair of IncongSign tasks ($d=0.45$) than the size of the phenomenal sign effect for the pair of IncongInt tasks ($d=0.36$). Thus, the integrity effect appeared to be overall stronger than the phenomenal sign effect.

In an additional analysis it was tested the way the form of an algebraic expression affected students' anticipation about the numbers that can be associated with the given expression. Specifically, it was tested whether expressions that have a form that resembles integer values would provoke interpreting them as standing primarily for integer number rather than rational numbers, and accordingly, whether fraction-like expressions would provoke more rational number associations. The results showed statistically significantly higher performance in the CongSign tasks for the expression that had an integer-like form (e.g., a, -b, -d-4) (M= 10.31, S.E.= 0.23) than for the expressions that had a rational-like form (e.g., 4/5y, -4/5z) (M= 4.181, S.E.= 0.147), [t(137) = 28.683, p <.001].

Considering the effect of the phenomenal sign to positive-like versus negative-like expressions the results showed statistically significantly higher performance in the CognInt tasks for the expressions that had a positive-like from (e.g., a, k+3, 4/5y) (M = 13.29, S.E.= 0.331), than for those with a negative-like form (e.g., -b, -d-4, -4/5z) (M= 11.92, S.E.= 0.30), [t(137) = 4.127, p <.001].

DISCUSSION

This study aimed to further test the hypothesis that there is a dual effect of the natural number bias in students' number associations with the algebraic expressions that contain literal symbols, using a new methodology. As natural number bias is characterized the tendency on the part of the students to use their initial conception of the number concept- which is grounded on the representation and the properties of the mathematical concept of natural number- to situations where reasoning with rational or real number should be applied. Previous studies (Christou et al., 2007; Christou & Vosniadou, 2009, 2012; Van Dooren et al., 2010) have shown that due to this bias, students tend to misinterpret literal symbols that appear as variables in algebraic expressions to stand primarily for natural numbers rather than for rational or real numbers, as they are explicitly taught in school; this was named as integrity effect. Due to this tendency, students also appear to misinterpret the phenomenal sign of the algebraic expressions, which is the sign an expression appears to have as a superficial characteristic of its form, to be the sign of the numbers it can only represent; this was named as phenomenal sign effect. The results from a sample of 8th and 9th grade students supported the main hypothesis of the study.

Specifically, considering the integrity effect, the students tended to perform significantly better in the tasks that presented statements that were aligned to their beliefs that only natural numbers can be associated with literal symbols, e.g. *k+3 can stand for 9*, than in statements that challenged these beliefs, e.g. *k+3 can stand for 4/5*. These results provide further empirical support to previous findings in the field (Christou et al., 2007; Christou & Vosniadou, 2009, 2012; Van Dooren et al., 2010). Additionally, the students appeared reluctant to accept integer values for fraction-like algebraic expressions (e.g. *4/5y)* than to accept rational number values for integer-like expressions (e.g. *-4/5y).*

Considering the phenomenal sign effect, the students performed significantly better in the tasks that were congruent with the belief that numbers associated to an algebraic expression should be of the same sign as the phenomenal sign of the algebraic expression (e.g. *k+3 cannot not stand for -1*) than in the tasks that challenged this belief (e.g., *k+3 cannot not stand for 5)*. Additionally, the results showed that the students were more willing to accept negative number values associated with positive-like algebraic expressions (e.g. *k+3)*, than to accept positive numbers associated with negative-like expressions (e.g. *-d-4)*. This supports previous findings that students are less likely to be affected by the positive phenomenal sign of the given expressions, than by the negative phenomenal one which they tend to interpret as the symbol of negativity (see also Vlassis, 2004).

Overall the results showed that the two aspects of the natural number bias, i.e., the integrity and the phenomenal sign effects both affected students' responses in the given tasks. However, effect size comparison showed that the integrity effect appeared to be stronger than the phenomenal sign effect.

The above-mentioned misconceptions do not come without consequences. The findings of Christou & Vosniadou (2009) study suggest that students' limited interpretations of the use of literal symbols in algebra caused them great difficulty in drawing graphs of functions, in evaluating inequalities, and evaluating equalities that included square root functions. Teaching which emphasizes the connection between variables and numbers may enhance understanding in all these domains.

However, addressing the above-mentioned misconceptions is not expected to be an easy process, because it requires that the students accept positive and negative rational and real number substitutions for the literal symbols that appear in algebraic expressions, and thus that they have fully acquired the mathematical concept of variable as a symbol that can stand for any real number. For students to expand their conceptual fields beyond natural numbers they should address their natural numbers bias which is deeply ingrained in their knowledge base (Ni & Zhou, 2005). For students to do that they must reorganize their initial knowledge of number which is organized around natural numbers, to an understanding of the number concept that is closer to a number concept that resembles the characteristics of rational and real numbers. We would expect this to be a gradual and long lasting process that would require systematic support. For this reason, it is important for mathematics teachers to be familiar with the natural number bias in learning mathematical concepts in order to better understand students' mistakes and the possible reasons why they may appear, and reconcile their teaching approach to deal with these mistakes.

Some promising results comes from a teaching intervention that used a refutational argumentation to help students address the phenomenal sign misconception. Refutational argumentation consisted of directly stating students' erroneous conceptions and then refuting them by presenting students with alternative viewpoints. The results showed that students who attended the intervention did significantly better immediately after the intervention and those benefits were maintained in the retention test one month later (Christou, 2012). Plausible alternative learning environments that target the natural number bias phenomenon need to be designed and applied to school, as long as they, of course, are empirically validated.

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PUBLIC MUSINGS ABOUT MATHEMATICS TEACHING: SOCIAL RESPONSIBILITY AND IDEATIONAL ALIGNMENT

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The rise of social media has afforded new opportunities for professional activity around mathematics teaching. Thousands of users are posting publicly about their experiences with mathematics teaching on an ongoing basis at an unprecedented scale in an unprompted, unfunded, and unmandated setting. Given the challenges around engendering sustainable professional development, informal professional activity, such as that found within a social media setting, is worthy of investigation. This study explores features of one such setting, with a specific focus on the underlying social structure that supports ongoing engagement. To this end, various social locations in this structure are defined and characterized, and modes of engagement in locations are found to vary according to social responsibility and ideational alignment.

INTRODUCTION

Professional activity around mathematics teaching is considered vital in the improvement of mathematics education at all levels (Borko, 2004). Research in mathematics education has identified various aspects of teacher professional development settings that make it effective at stimulating rich professional activity. For instance, activities should reflect and be driven by teacher needs and interests, and community building and networking should be at the core (Lerman & Zehetmeier, 2008). While many initiatives prove successful in engaging teachers in meaningful activity around mathematics teaching, there is growing attention on the sustainability of such activity. As such, informal settings where such activity occurs naturally have become of growing interest (e.g., Horn & Kane, 2015).

With the rise of social media technology in recent years, education professionals are turning to resources that are becoming increasingly available beyond the confines of institutional boundaries. In turn, many of the constraints of traditional forms of professional activity are being bypassed, allowing for informal professional activity to flourish. In some cases, collectives of professionals have formed. One such collective, referred to as the Math Twitter Blogosphere (MTBoS), has remained resilient for almost ten years, with ongoing activity around mathematics teaching occurring daily via Twitter. MTBoS participants also have very promising statements about the possibilities for professional growth they experience and are often found suggesting that MTBoS is an effective space to share and develop ideas for teaching.

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 158-165. Khon Kaen, Thailand: PME.

> Following this weird #MTBOS hashtag on twitter has changed my teaching practice in so many ways. (@MrOrr_geek, 6 Feb 2018)

Although communication through Twitter is generally random and unprompted, MTBoS is treated as an established space, determined by participation from members who contribute and continue to use the MTBoS hashtag. For instance, contributors often refer to it as a place rather than a hashtag, and grow to expect like-minded peers to be available there (Larsen & Parrish, 2019). Studies have explored various facets of MTBoS, such as around content quality (Parrish, 2016), instances of negotiation (Larsen & Liljedahl, 2017), participant perspectives (Larsen & Parrish, 2019), and its capacity to support in-person conference events (Waddell, 2019). Taken together, these studies reveal strong potentialities of MTBoS as a tool for professional growth.

While most of the investigations of MTBoS to date have taken interest in the ideational opportunities in the space, none have focused on the unique nature of the social structures that support its ongoing activity. Interestingly, Twitter offers its users unreciprocated 'following' relationships. That is, when a user 'follows' (or subscribes to the 'tweet' activity of) another user, that user need not reciprocate the relationship. This means that tweets published by users who are highly 'followed' are more likely to be seen and interacted with, and those who follow more users who publish 'tweets' in their domain of interest can build a better view on the activity in that domain of interest. Since all tweets made by users one follows are organized in chronological order in one's newsfeed, this means each user has a unique view on the ideational space of MTBoS depending on the social relations they choose to create and maintain through both following other users and being followed by others. Therefore, following relationships can affect the nature of the ideational activity, and in turn, the social structure in MTBoS can influence the production and consumption of content. As such, the social location of a contributor could stimulate different kinds of engagement.

Given the interest in engendering communities of practice around mathematics teaching and the overall agreeance on the necessity of collaboration within professional activity around mathematics teaching, uncovering the implications of the unique social structure in MTBoS on professional activity around mathematics teaching is worthy of attention. As such, the study presented in this paper is driven by the global question – how and why does MTBoS invoke a sustainable form of professional activity around mathematics teaching? And more specifically in this paper – what is the social structure in MTBoS and how does it drive ongoing activity?

THEORETICAL FRAMEWORK

Although a variety of theoretical frameworks were taken into consideration when approaching MTBoS as a phenomenon of interest, the primary theoretical perspective in this study is drawn from *complexity thinking* (Davis & Sumara, 2006). Complexity thinking provides the tools to describe a system of individual agents who seem to generate emergent macro-behaviours from individual autonomous actions. It is best suited for studying decentralized and bottom-up emergent learning contexts, where learning is treated as expanding the space of the possible and the "emergence of the as-yet unimagined" (Davis & Sumara, p. 135). Since MTBoS has no central organization and is driven by individual professionals engaging in activity autonomously while simultaneously contributing to a collective that is often treated as a single entity, complexity thinking was the most well-aligned theoretical perspective for pursuing investigation into this context and served as the primary theoretical lens in this study. As such, we highlight some of its key theoretical notions.

Davis and Sumara suggest six interdependent conditions necessary for complex emergence, which they organize into three complementary pairs (or tensions): specialization (*diversity* and *redundancy*), trans-level learning (*neighbour interactions* and *decentralized control*) and enabling constraints (*randomness* and *coherence*). Specialization has to do with the diversities and redundancies among agents, trans-level learning has to do with the possibilities for individual ideas to 'bump' or interact with each other towards a greater whole, and enabling constraints consider the points of cohesion in the collective that maintain its structure while allowing for enough randomness for adaptation and learning. Although the focus in complexity thinking is primarily on 'collective-knowing', Davis and Sumara state, "the ideational network rides atop the social network" (p. 143). That is, through neighbour interactions among ideas, which are driven by agents with enough diversity and redundancy with each other, ideational emergence can occur. The strength and resilience of a collective is in some sense dependent on the *far-from-equilibrium* behaviour of these conditions.

Although the goal of complexity theory is not to identify interpersonal collectively, as do other social theories of learning, 'collective-knowing' cannot be considered without the social interactions that bring it to fruition. However, Davis and Sumara indicate these aspects should not be collapsed. Since the ideational network of MTBoS has been explored elsewhere (Larsen, 2019), the focus of the work presented in this paper is on the social network that supports the emergent ideational network in MTBoS. To this end, some terminology is also borrowed from *communities of practice* (Wenger, 1998), such as that of the existence of a *periphery* and a *core* in terms of participation in a community, and that there are *trajectories* along which participants move as they participate over time. While the terminology from communities of practice is used as appropriate, complexity thinking, and the conditions for complex emergence serve as the primary theoretical worldview underpinning the methods and analysis in this study.

METHODS

Towards the construction and examination of the social network underlying the ideational network of MTBoS, tweets published on Twitter that included the MTBoS hashtag were gathered for analysis. However, the sheer mass of data available on Twitter (and in MTBoS) made it impossible to investigate in its entirety. MTBoS has grown over the past ten years to include over 6308 users (the number of unique users tweeting with the hashtag at least twice between September and December of 2018), with an average of one tweet every two minutes (from this same timeframe). As such, a specific selection of data was chosen for analysis. Guided by the insider perspective of the presenting author, who gradually because a participant of MTBoS over a five-year period, the selection of data was made based on a 7-day period in late September of 2018. This timeframe was chosen strategically to avoid major influencing factors such as ‘back-to school’ or ‘midterm grading’ and the length of time was chosen to avoid influencing factors of certain days of the week while keeping data size manageable. As such, the data set was constructed by collecting all tweets from September 21-28, 2018 into a spreadsheet along with meta-information such as timestamps and usernames. This initial collection included 6146 tweets made by 2948 unique user accounts, 4653 of which were direct retweets (exact replicas of original tweets). Given the aims of this study, direct retweets were removed. This left 1493 unique tweets made by 694 unique users. However, to make the dataset feasible for analysis, 30% of these tweets were randomly selected, resulting in a final data set of 444 tweets made by 322 unique users.

While this data set was analysed in various ways, the study presented in this paper pertains to the analysis of the follower relations among the 322 users and the tweet contents they produced as relative to their social locations in the social network. To this end, the followers of each of the 322 users were identified, and then only those relationships that existed within the set of 322 users were determined. The unreciprocated nature of these relationships allowed for each user to be granted two values: an in-degree and an out-degree, representing the number of users in the set they followed or were followed by, respectively. After plotting these values on a scatterplot, it became evident there were four possible social locations that corresponded to low and high values in each of these two dimensions (of in-degree and out-degree). As such, cut values were determined by taking the top 20% of values in each dimension to define what counted as high for that dimension. In-degrees of 56 or more and out-degrees of 49 or more were considered as high, which came to serve as boundaries for constructing the four regions of social location, as determined through this process.

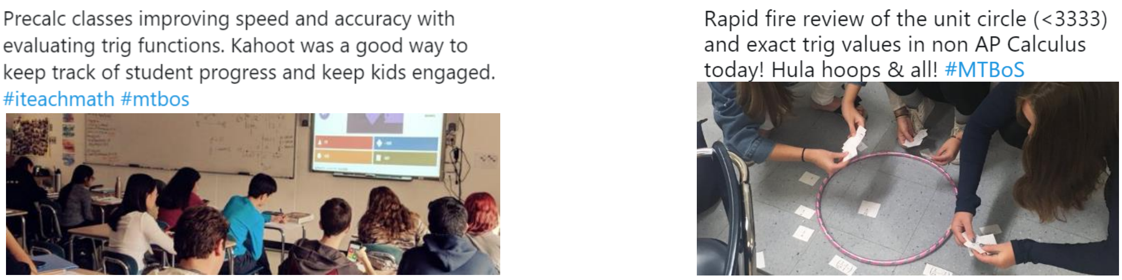
Tweet data was then organized by social location, and each set of tweet data was examined for redundancies and diversities among contributions. To achieve this, various aspects of contributions in each set were examined, such as key topics, presence of media in tweets, tone, and how content was presented by users. The most fruitful of these investigations was in looking at how content was presented. To this end, we drew on Remillard's (2012) distinction between *forms* and *modes* of address: where *forms of address* involve “particular ‘looks’ or formats that reflect and reinforce the mode of address” (p. 106). As such, each region of social location was examined in terms of the *forms* and *modes of engagement* used in publicizing the content. This was conducted via an open coding process that involved iterative coding with attention to redundancies and diversities within and across regions, beginning with tweets in the region with high values in each dimension. New codes were generated when necessary until a saturation point was reached in that every tweet fit into at least one coding. Overlap was permitted, and some tweets were coded with several codes if the tweet revealed multiple forms of engagement. The coding was then re-checked with a second pass. Once no new codes were found and each tweet in each of the categories was tagged with at least one of the codes, a final set of codes was determined. To explore how these forms of engagement compared across regions of social location, the proportions of each code within each region were compared across regions. A final pass was then made to interpret the findings within each region of social location to build characterizations of the nature of contributions made in each region. These characterizations are revealed and interpreted in the results and discussion that follows.

RESULTS

After pursuing analysis of all selected tweets from each of the four social locations, the forms of engagement identified fell into eight categories: *soliciting advice or resources*, *contributing resources or teaching advice*, *revealing practice*, *sharing accomplishments*, *endorsing*, *signalling an identity*, *advocating an opinion or stance*, and *building community*. These forms occurred to varying degrees in each region and served to characterize the kinds of engagement prominent in each region.

Those with relatively low numbers of followers and followings were referred to as *newcomers*. While this title may not be reflective of their longevity of participation, their social position and the forms of engagement they evidenced led us to consider them as newcomers in this setting. When examining the tweets made by this group, it because evident that they typically engaged in primarily practice-oriented ways, with a focus on pragmatism and utility towards classroom practice. Although their tweets were generally diverse in terms of topics, there was a strong redundancy among their tweets around being focused on either specific mathematical content or pedagogy, without making significant connections between them. Their tweets were relatively generic and could be considered as representations of the broader populous of mathematics teachers in the sense that they lacked reference to popular pedagogical approaches of MTBoS such as ‘noticewonder’ or ‘groupwork’.

In terms of tone, the tweets made by newcomers were often celebratory in nature, with share-outs of pride and excitement for accomplishments that were most often practice-oriented. For instance, in Figure 8a below, the contributor reveals excitement for using technology towards improving student ability with evaluating functions.



(a) (b)

Figure 8: Examples of practice-oriented contributions

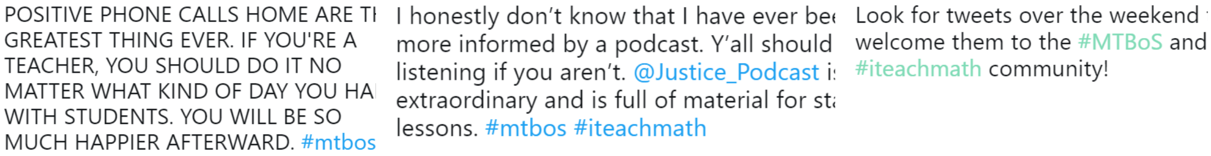
Although many newcomers offered indication of contributing resources, the resources they contributed were often partial in nature without complete enough detail to be used in someone else’s classroom. As such, newcomers primarily seemed to use MTBoS to build their identities as teachers of mathematics in general rather than specifically as MTBoSers. Their concerns were often personally oriented rather than concerned with others and tended to involve mathematical content rather than pedagogy.

On the other hand, slightly different forms of engagement were evident among users who followed more of the others in this dataset while still being not very highly followed. These users were referred to as *observers* due to their heightened capacity to view activity of others in MTBoS. Unlike newcomers, observers engaged by tweeting about practice in ways that revealed they were implementing pedagogies commonly found within MTBoS activity, such as 'instructional routines', 'desmos activities', and 'noticewonder', and not only mathematical ones such as 'geometry'. In such ways, they were projecting their identities as those who belonged to MTBoS. The focus in these tweets was around sharing personal accomplishments aligned with popular trends in MTBoS and not just in the more generic domain of mathematics teaching. For example, in Figure 8b above, one contributor shows excitement for doing a hands-on activity that involves group learning and meaning making, as is common in MTBoS activity, and reveals an image of students using a hula hoop to explore the unit circle.

Observers were also overall quite attention-grabbing in how they presented their contributions and came across as willing to be vulnerable to ask questions. Although many of their questions went unanswered, they revealed sensitive issues such as forgetting how to approach a mathematical topic or seeking opinions on an undesirable student response they encountered. In one case, an observer stated, "Can't seem to help students to see variables as possibilities. They just want numbers. Any ideas? #mtbos". In such a query, they suggested their desire for helping students understand concepts more deeply while simultaneously indicating their inability to achieve this. This aligns with commonplace themes in MTBoS such as helping students make meaning and evidencing growth mindset. Overall, observers seemed to tweet in ways that positioned them as MTBoSers while maintaining a focus on personal aims and interests.

In contrast to the personally focused newcomers and observers, those with more significant followings were more aptly focused on serving their community, such as by providing resources or sharing about initiatives. However, those who followed fewer MTBoS contributors seemed less aligned with the most central topics of interest than those who followed more. As such, highly followed contributors who did not follow as many MTBoS contributors were referred to as *influencers*, while those who also followed many MTBoS contributors were referred to as *leaders*. This distinction had to do with their capacity to view ideational trends in MTBoS based on who they followed.

Interestingly, a similar pattern of misalignment continued for influencers as for newcomers, the two social locations that did not highly follow others in the set. While influencers were often contributing resources that were specific enough to be used directly in classrooms, their resources were often misaligned and sometimes even overly polished or marketed. However, unlike their newcomers, influencers used their high social visibility to promote new materials, expressions of gratitude, and questions of those who were less followed. In one case, an influencer even developed a 'broken calculator' tool and used their following to solicit feedback on it, with significant response. Leaders, on the other hand, were similarly attentive in terms of serving the community and pushing boundaries, but did so in ways that maintained alignment with prominent topics and values in MTBoS. For instance, they advocated for building relationships with students (as in Figure 9a), for learning about social justice issues (as in Figure 9b), and for welcoming newcomers to MTBoS (as in Figure 9c).



(a) (b) (c)

Figure 9: Examples of advocacy in contributions of leaders

And when sharing resources, their contributions provided enough specificity that they could be used directly by others while remaining within the ideational space commonly found in MTBoS. For example, they included hyperlinks to 'desmos' activities or to student self-reflection sheets that reveal student progress. Moreover, they linked pragmatic resources with pedagogical descriptions and rationale while also signaling a MTBoS identity. Their capacity for visibility and awareness of others was evident, which translated to being community-oriented, boundary-pushing, but also, aligned.

DISCUSSION AND CONCLUSIONS

Taken together, when looking across regions of social location, two key factors emerged as pertinent to identifying the modes of engagement evident within each social location in MTBoS: *social responsibility* and *ideational alignment*. Those who were more highly followed seemed to engage with more concern for others, and therefore, with social responsibility, than those who were less highly followed. Conversely, those who were following more of the other MTBoS contributors in the data set seemed to include prominent topics and ideas commonly shared in MTBoS (as determined in previous studies), and therefore, had more ideational alignment to prevailing discourse in MTBoS. These findings are summarized in Figure 10 below.

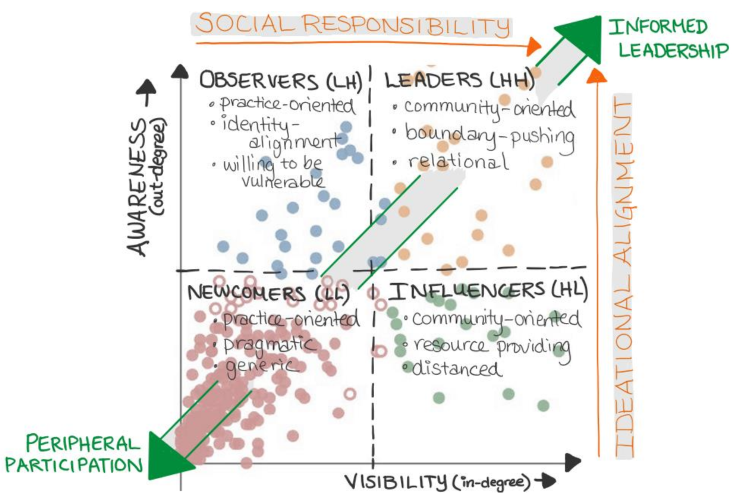


Figure 10: Modes of engagement in social location – as two dimensions

This implies that neighbour interactions among agents in MTBoS are not completely random, as suggested by *complexity thinking*, and the social structure is more nuanced than having a single core and periphery, as suggested by *communities of practice*. Rather, neighbour interactions are produced and influenced by a sort of social capital that privileges certain ideas and modes of engagement over others. There are also multiple social locations, determined by visibility by others and awareness of others, that can be characterized by two related dimensions to modes of engagement: social responsibility and ideational alignment. When both are invoked, a sort of informed leadership emerges and seems to equip leaders to advocate for change, push boundaries, and build community. Combinations of these dimensions offer other social locations which may contribute to the robustness of MTBoS by offering sources of diversity and redundancy. Various trajectories for engaging in MTBoS over time may also be possible to determine with further study. Overall, the features presented in this paper illuminate the unique social structure of MTBoS, and in turn, challenge perspectives on social structures in professional communities as being centered around a single core. It also suggests further attention to the visibility of emergent ideas and opportunities for advocacy in professional learning environments.

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ANALYSIS OF A PRESERVICE TEACHER'S REFLECTION ON THE ROLE OF MATHEMATICAL MODELLING IN HIS MASTER'S THESIS

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The aim of this study is to analyse the reflection made by a preservice mathematics teacher on the role of mathematical modelling for improving an instructional process. The interest focuses on the use of didactic suitability criteria as a tool for reflection in his master's thesis. The methodology consists of carrying out a content analysis of his written report. In terms of its extension, this is a case study. Based on the analysis of the teacher's reflections, the main results are: on the one hand, the redesign of the implemented lesson plan adds more modelling projects to complement the teaching of functions; on the other hand, the inclusion of mathematical modelling is mainly related to the improvement of the epistemic and ecological suitability of the implemented instructional process.

INTRODUCTION

A key aspect in teachers training is teacher reflection on their own practice, in particular, the assessment of didactic suitability of instructional processes (Pino-Fan, Font, & Breda, 2017). Once reflection becomes habitual for a teacher, it can become the main mechanism to improve his/her own practice (Schoenfeld & Kilpatrick, 2008; Schön, 1987).

Various studies have addressed teacher reflection in the training of mathematics teachers, specifically, analysing the preservice teachers' reflection in their Master's Theses, using the didactic suitability criteria (DSC) proposed by the Onto-Semiotic Approach (OSA) (e.g., Breda, 2020; Breda, Pino-Fan, & Font, 2016, 2017; among others). In contrast to these studies, which analyse the whole teacher reflection, our study focuses on preservice teacher's reflection on the implementation of mathematical modelling for the improvement of an instructional process.

Based on the latter, the following question arises: how do preservice teachers use the DSC in their master's theses to reflect on the implementation of mathematical modelling? In order to answer it, we analyse the reflection made by a preservice teacher (Justicia, 2020) on the role of mathematical modelling for the improvement of an instructional process, in which he used the DSC. Overall, the chosen master's thesis is a reflection and reformulation of the design and implementation of a lesson plan (LP), which contemplates the work with mathematical modelling to teach functions in the first grade of baccalaureate (students aged 16 to 17 years-old).

THEORETICAL FRAMEWORK

This section presents the theoretical references considered in this study.

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 166-173. Khon Kaen, Thailand: PME.

Mathematical modelling

The mathematical modelling process is understood, in general terms, as a transition between the real world and mathematics, to solve a problem-situation taken from reality. Although different cycles designed to explain this process coexist (Borromeo Ferri, 2006), and diverse perspectives on its implementation have emerged (Abassian, Safi, Bush, & Bostic, 2020), there is a clear consensus that its inclusion in curricula is necessary to improve the learning of mathematics (Kaiser, 2020).

This study does not adopt any particular modelling cycle or perspective, but rather, we consider the following consensual attributes that characterise the work with this process through projects. In this approach, a real-world problem (question or task) is posed to students, which must be solved in groups over a longer period of time (Blomhøj & Kjeldsen, 2006). This problem must be *open* (not limited to a specific answer or procedures), *complex* (useful information must be distinguished from the rest of the wording of the task), *realistic* (adding elements taken from reality), and *authentic* (a situation consistent with a fact from reality) (Borromeo Ferri, 2018). This kind of projects presuppose greater degrees of autonomy and responsibility on the part of students, who assume the discussion and resolution of the problem through a constant reflection on the situation (Blomhøj & Kjeldsen, 2006).

Didactic suitability criteria

In the OSA, the didactic suitability of a teaching-learning process is understood as the degree to which it (or a part of it) meets certain characteristics that allow it to be qualified as suitable (optimal or adequate) in order to achieve an adaptation between the *personal meanings* achieved by the students (learning) and the *institutional meanings* intended or implemented (teaching), taking into account the circumstances and available resources (environment).

This multidimensional construct consists of six partial suitability criteria: *epistemic criterion*, to assess whether the mathematics that is taught is 'good mathematics'; *cognitive criterion*, to assess, before starting the instructional process, whether what is intended to teach is at a reasonable distance from what the students know; *interactional criterion*, to assess whether the interaction solves doubts and difficulties of the students; *mediational criterion*, to assess the adequacy of resources and time used in the instructional process; *affective* (or *emotional*) *criterion*, to assess the involvement (interest, motivation) of the students in the instructional process; *ecological criterion*, to assess the adaptation of the instructional process to the educational project of the school, the curricular guidelines, the conditions of the social and professional environment, etc. (Morales-López & Font, 2019). Each of these criteria has its respective components and indicators (see Breda et al., 2017).

In the OSA, mathematical modelling is considered as a *hyper* or *mega process* (Godino, Batanero, & Font, 2007), since it involves other more elementary *processes* (representation, argumentation, idealisation, etc.). Furthermore, within this framework, enhancing mathematical modelling is an aspect that improves the suitability of the instructional process (Ledezma, Font, & Sala, 2021; Sala, Font, & Ledezma, in press).

METHODOLOGY

In this work, we followed a qualitative research methodology (Cohen, Manion, & Morrison, 2018), which consisted of conducting a content analysis of the reflection of a preservice teacher in his master's thesis; and, from the point of view of its extension, it is a case study (Stake, 2005).

Research context

This research was developed in the context of the Master's Program in Teacher Training for Secondary Education and Baccalaureate (mathematics speciality), taught by public universities of Catalonia (Spain), during the academic year 2019-2020. To obtain the Master's Degree, preservice teachers must design and implement a teaching-learning sequence, which is determined by the educational institution, the level of students, and the time of the academic year in which preservice teachers do their teaching practice. Thus, a preservice teacher has many restrictions to work exclusively on mathematical modelling in his/her LP, although this is not the case in the redesign that he/she can propose in his/her master's thesis.

Description of the master's thesis

After their teaching practice in the training centre, the DSC proposed by the OSA are introduced to preservice teachers, along with a guideline of components and indicators for the application of such criteria. Then, in their master's theses, the teachers are suggested to use these tools to assess the teaching-learning sequence they implemented and propose changes to improve the suitability of the instructional process.

The master's thesis of Justicia (2020) is a proposal to teach functions in the first grade of baccalaureate. It consisted of 15 sessions: 4 of them were held in person and the remaining 11 online, because of the lockdown due to the COVID-19 pandemic. In his master's thesis, the preservice teacher includes a modelling project, which consists of the profitability study of a small company. The project develops during two sessions (the fifth and the sixth) of his LP. This project is detailed in the next section.

Content analysis

The qualitative analysis of the data was developed in four steps, following a methodology similar to that used by Sánchez, Font, and Breda (2021): first, 122 master's theses (from the academic year 2019-2020) were classified according to the educational level and the mathematical content of the LP; second, a search for keywords related to mathematical modelling was carried out (context*, model*, problem*, real*, etc.); third, the master's theses were categorised according to the importance they gave to mathematical modelling (used in the implementation, in the redesign proposal, or both); fourth, the assessments of DSC were analysed to identify on which ones they focused their reflections on mathematical modelling.

The master's thesis of Justicia (2020) was classified in the group of master's theses where mathematical modelling is used both in the implemented LP and in its redesign (step 3), that is why we analysed this one. In his master's thesis, the preservice teacher writes evaluative comments related to the different components and indicators of the DSC. In this research, comments about working with mathematical modelling are analysed. The starting point of the content analysis are a priori categories, namely the criteria, components and indicators of didactic suitability to identify which criteria are related to mathematical modelling in the preservice teacher's reflections (step 4).

ANALYSIS OF THE MASTER'S THESIS

This section presents the analysis of the assessments made by the preservice teacher on the implementation of his LP. To this end, we describe this assessment and, for each DSC that is assessed, we point out the comments that justify his thought and that –directly or indirectly– allude to mathematical modelling. We also consider the components of each criterion to which these comments relate. The analyses focus on the assessment of the project *Profitability study of a small company* (see Figure 1), where the preservice teacher makes his reflections explicit. The aim of this activity is "to know, through multidisciplinary tools, whether a small business is profitable or not, setting economic reference values" (Justicia, 2020, p. 39, authors' translation).

Statement:
A person wants to enter the world of work with a simple business model. There would be more complex ways, such as implementing a *start-up* initiative that could be a very good research work. It is a multinational mobile phone operator parent company, that sells voice and data service provision contracts with franchised establishments.

Quarterly income:
- The first 100 contracts with € 20 each.
- From 101 to 150 contracts with € 30 each.
- From contract 151, € 35 each.

Costs paid by the small business owner:
- € 800 quarterly rent for the store.
- € 4 for each contract to a company that advertises the franchised store.

Taxation, the business activity tax applied by the Tax Agency has two sections per billing:
- From € 0 to € 1000 billing, 5%
- From € 1,001, 8%

Figure 1: Extract from the *Profitability study* statement
(translated from Justicia, 2020, pp. 39-40)

Epistemic suitability

In the assessment of the component 'richness of processes', the preservice teacher highlights that his *Profitability study* allows the students to develop the following processes: 'representation', 'communication', and 'argumentation', but above all, 'problem solving' and 'mathematical modelling'. With this last process, he intends to:

> interpret the properties of functions, checking the results with technological tools [software GeoGebra] in contextualised problems, and to extract and identify information derived from the study and analysis of functions in real contexts, such as the initial study to start a small business. (Justicia, 2020, p. 6, authors' translation)

In the assessment of the component 'representativeness', he alludes to the fact that this project emphasises the *partial meanings* related to the concept of function (domain, codomain, cartesian representation, limit, continuity, etc.). In addition, he considers functions as a unifying concept and a useful tool for mathematical modelling.

Interactional, affective, cognitive, and mediational suitability

These four DSC are presented together because the preservice teacher does not make very detailed assessments of these criteria. Regarding the component 'student's interaction' of the *interactional suitability*, the preservice teacher highlights the heterogeneous arrangement of student groups, the socialisation and collaboration between the students, and the minimum intervention by himself –as a teacher– during the tasks related to the *Profitability study* project. In the assessment of the component 'interests and needs' of the *affective suitability*, he mentions that this project was used to awaken curiosity and interest in his students, and it is related to their own environment. Regarding the *cognitive* and *mediational suitability*, the preservice teacher does not mention mathematical modelling to these criteria in his assessments.

Ecological suitability

In the assessment of the component 'adaptation to curriculum', the preservice teacher considers that the content addressed, the objectives set, and the tasks proposed (*Profitability study*), are in line with what is stipulated by the curriculum for the first grade of baccalaureate (Departament d'Ensenyament, 2008). In the assessment of the component 'intra and interdisciplinary connections', the preservice teacher points out that the *Profitability study* establishes connections with economy, addressing issues such as the identification and calculation of costs and incomes of a business, the minimum interprofessional salary, economic profitability, etc. In the assessment of the component 'social and labour utility', the role of the *Profitability study* stands out, since both the context of the task and the transversal use of functions in working life, represent a strong point of this proposal.

Global assessment of didactic suitability

Figure 2 presents the global assessment made by the preservice teacher of the didactic suitability of the implemented instructional process. This consists of a hexagonal radial graph, in which the outer regular hexagon represents an ideal instructional process, and the inner irregular hexagon represents the suitability of the implemented instructional process.

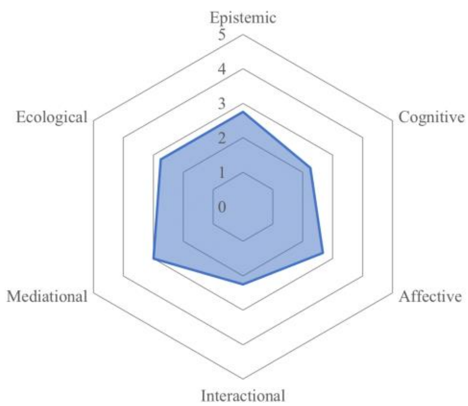

Figure 2: Hexagon for the assessment of didactic suitability
(adapted from Justicia, 2020, pp. 12-13)

Lesson plan improvement proposal

Based on the assessments presented in Figure 2, and in order to redesign his proposal, the preservice teacher focuses on improving the following components: ‘representativeness’ (*epistemic suitability*), ‘high cognitive demand’ (*cognitive suitability*), ‘teacher-student interaction’ (*interactional suitability*), and ‘didactic innovation’ (*ecological suitability*). To this end, he considers that posing more mathematical modelling projects and contextualised problems would contribute to the improvement of such components, based on Font’s (2011) proposal for the work with functions. Therefore, the preservice teacher adds the following new projects: The *Changes of phase of water* project, whose objective is to discover the equations of a line using data from the experiment of boiling water; the *Maximisation of the volume of a prism* project, related to cubic functions, their derivatives, and the maxima and minima of these functions; and the *Final speed of a car* project, which considers the problem of maximum speed posed by a car company.

DISCUSSION AND CONCLUSIONS

From the analysis of the explanation of the preservice teacher Justicia (2020), the results show that his appraisals about the incorporation of mathematical modelling in his LP mainly focus on the *epistemic* and *ecological suitability* criteria; and to a lesser extent on the *interactional* and *affective suitability* criteria.

The preservice teacher mentions mathematical modelling when he assesses the component ‘richness of processes’ of the *epistemic suitability*. However, he does not specify whether he adopted or not any modelling cycle, although the master’s training includes a mathematical modelling sub-module, which introduces, in particular, the proposal of Blum and Leiß (2007). Moreover, the preservice teacher did not use this cycle –or any other– to describe or analyse his *Profitability study* project, which could indicate that he did not have enough conceptual clarity about theories on mathematical modelling. It should be noted that, although the statement posed in Figure 1 meets some of the characteristics of a mathematical modelling problem (it is *open*, *complex*, *realistic* and *authentic*), the description of its implementation shows that it was not carried out according to the methodology of a project. This is due to the structure of the educational system, where the curricular times are limited and mathematical modelling is subject to its treatment, mainly, as a competence and not as a content.

Resuming the research question of this study, we can conclude that the DSC not only allowed the preservice teacher to reflect on his teaching practice, but also revealed both his conceptual and procedural notions of mathematical modelling, as well as the importance he attributes to it in the implemented instructional process. The latter conclusion is evidenced by the addition of more activities to develop mathematical modelling in his redesign proposal.

Acknowledgements

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STUDYING ADULTS' BELIEFS RELATED TO ENGAGING YOUNG CHILDREN WITH NUMBER CONCEPTS

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It is widely acknowledged that engaging young children with mathematics is both possible and beneficial. Being that young children spend a great deal of time with adults outside of school, this research investigates the beliefs of adults regarding engaging young children with number concepts. Questionnaires were handed out to 91 participants, none of whom were preschool teachers. In general, participants had positive beliefs regarding supporting children's engagement with various numerical activities. Some differences between participants with different backgrounds (e.g., connections with young children, professions) were found.

INTRODUCTION

Researchers agree that promoting numerical skills, such as counting, comparing sets, number composition and decomposition, and recognition of number symbols, is important during early childhood (Nguyen et al., 2016). Several studies have investigated preschool teachers' knowledge and beliefs related to teaching early number competencies (e.g., Vlassis & Poncelet, 2016). However, young children often spend much of their day in the care of other responsible adults who are not necessarily trained preschool teachers. Studies suggest that for children to take advantage of the academic opportunities provided at preschool, some level of support from the home environment, such as toys that stimulate learning number and shapes, is necessary (Anders et al., 2012). Furthermore, adults' beliefs regarding the importance of doing mathematical activities at home was found to be significantly related to the frequency with which children reportedly did mathematics at home (Sonnenschein et al., 2012). Thus, it is relevant to investigate adults' knowledge and beliefs related to numerical learning during the preschool years.

Based on our previous work with preschool teachers (e.g., Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015), this paper introduces a framework for investigating adults' knowledge and beliefs related to playfully engaging with mathematics in the early years. Researchers use the term "playful" in recognition of the dilemma that early childhood educators face, both teachers and researchers, in how to balance instruction and play. Playfully engaging with mathematics infers that the child is active, the activities are flexible, and an adult is present to guide the child toward a specific knowledge (Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009). After introducing the framework, the paper focuses on beliefs, and describes how this framework was used to investigate adults' beliefs

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concerning children's playful engagement with numerical concepts. Finally, the paper presents results regarding those beliefs.

THEORETICAL FRAMEWORK

For the past several years, researchers have used the Cognitive Affective Mathematics Teacher Education (CAMTE) framework, when investigating and promoting teachers' knowledge and self-efficacy for teaching number, geometry, and pattern concepts (e.g., Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015(. This framework differentiated between two components of subject-matter knowledge (SMK): being able to produce solutions, strategies and explanations and being able to evaluate given solutions, strategies and explanations. In line with Ball, Thames, and Phelps (2008) the framework differentiated between two aspects of pedagogical content knowledge (PCK): knowledge of content and students (KCS) and knowledge of content and teaching (KCT). In adapting the framework for adults, researchers referred to the same aspects of SMK as our previous framework. Researchers consider the mathematics knowledge researchers wish to promote among young children, and the mathematics knowledge adults need in order to promote children's knowledge (see Table 1, Cells 1 and 2). Thus an example of Cell 1 would be to request participants to count a set of objects, and then count them again in a different way (e.g., first by counting 1, 2, 3,… and then by skip counting 2, 4, 6,…).

	Mathematics for adults and children		Mathematics Engagement	
	Solving	Evaluating	Children	Playful learning
Knowledge	Cell 1: solving tasks. e.g., count the number of elements in a set using a variety of strategies	Cell 2: evaluating tasks. e.g., evaluate the efficiency of a counting strategy	Cell 3: knowledge of children's conceptions. e.g., which number symbols do children confuse.	Cell 4: knowledge of content and playful learning. e.g., which activities can foster children's acceptance of the one-to-one principle.
Beliefs	Cell 5: mathematics beliefs related to solving tasks. e.g., is it important to know several ways to count the number of items in a set.	Cell 6: mathematics beliefs related to evaluating tasks. e.g., is it important to know which solution methods are efficient.	Cell 7: beliefs regarding children and mathematics. e.g., believing that young children enjoy learning number concepts.	Cell 8: beliefs regarding ways of engaging children with playful mathematics. e.g., believing that adult guidance can foster the learning of early number concepts.

Table 1: The Cognitive Affective Mathematics Adult Education (CAMAE) Framework

Whereas for teachers, researchers referred to PCK, for adults, researchers refer to knowledge needed for engaging children with playful mathematics (henceforth, Mathematics Engagement Knowledge) (Cells 3 and 4). Thus, instead of KCS, researchers refer to knowledge of content and children, such as knowing that children aged three may not yet have acquired the cardinality principle of counting; instead of KCT, researchers refer to knowledge of content and playful learning, that is, knowledge of activities that can promote numerical thinking. Researchers call the adapted framework for adults who are not preschool teachers, the Cognitive Affective Mathematics Adult Education (CAMAE) framework.

Each knowledge cell has a corresponding belief cell. Whereas for teachers, researchers were interested in their self-efficacy for teaching mathematics, in our research with adults, researchers were interested in their beliefs regarding what mathematics children (and the adults who interact with them) should know, and how children can engage with mathematics. That is, researchers consider beliefs related to mathematics (Table 1, Cells 5 and 6) as well as engagement beliefs, i.e., beliefs related to engaging children with mathematics (Table 1, Cells 7 and 8).

BELIEFS REGARDING PRESCHOOL MATHEMATICS

Several studies investigated beliefs regarding the importance of learning mathematics in preschool. Most studies found that both preschool teachers and parents agree that mathematics should be and can be promoted in the years before Grade 1 (e.g., Missall, Hojnoski, Caskie, & Repasky, 2015). That being said, when comparing the importance of learning mathematics to other subjects, Vlassis and Poncelet (2016) found that first-year prospective preschool teachers rated engaging with mathematics in preschool as less important than engaging with language, arts, and psychomotricity. Similarly, some home day care providers believe it is less important for young children to acquire mathematics skills before entering kindergarten, than other social and academic skills (Blevins-Knabe, Austin, Musun, Eddy, & Jones, 2000). Furthermore, parents and caregivers reported that mathematics activities at home occurred less frequently than reading or other play. More specifically, it was found that parents help their children learn language skills more than mathematics in both everyday contexts, such as carrying out household chores, as well as during more structured contexts, such as direct teaching (Cannon & Ginsburg, 2008). Some participants claimed that teaching mathematics in preschool can hinder social and emotional development. These beliefs are in contradiction with educators' recommendations for supporting early mathematics (Nguyen et al., 2016).

Focusing on mathematical activities at home, Missall, Hojnoski, Caskie, and Repasky (2015) listed 19 activities related to number and operations and asked parents to rate how often they engaged their children with those activities. Among the most frequent activities were counting aloud, counting out several items from a larger group, and reading numbers. Among the least frequent activities were skip counting, counting backwards, and comparing the number of objects in two sets. Similar results were found by Skwarchuk (2009), who also found that many parents incorporated numerical concepts during natural settings at home.

Parents' backgrounds might also be related to their beliefs and home mathematical activities. For example, Chinese parents thought it was less important to do mathematics at home, than American parents (Sonnenschein et al., 2012). A different study found that middle socio-economic status (SES) parents were more likely than lower SES parents to endorse embedding mathematics in the children's home routine (DeFlorio & Beliakoff, 2015). Parents' educational backgrounds were found to be positively correlated with their attitudes towards mathematics, which in turn affected their home numeracy practice (LeFevre, Polyzoi, Skwarchuk, Fast, & Sowinski, 2010). Some studies found a positive relationship between parents' beliefs of their own mathematical ability and the types of mathematical activities they provide their preschoolers (Blevins-Knabe et al., 2000).

The aim of the present research is to investigate adults' beliefs regarding engaging young children in mathematical activities, in accordance with the CAMAE framework described above. Previous studies investigated parents' and home-care providers' beliefs. Taking into consideration that grandparents, aunts, uncles, and other adults may also engage children with mathematics activities, this research includes adults who are not necessarily parents, as well as adults who have no specific current relationship with young children. Specifically, researchers ask: (i) Is there a difference in beliefs regarding engaging young children in mathematical activities between adults who have a relationship with children between the ages of 3 and 6, and those who do not? Considering that adult's active engagement in mathematics during their daily life might also be related to their beliefs, the second research question is: (ii) Is there a difference in beliefs regarding engaging young children in mathematical activities between adults that work in a mathematics related profession and those who do not?

METHODOLOGY

Participants were 91 adults, between the ages of 20 and 60. Of the 91 adults who participated, 30 reported working in a mathematics related profession, such as mathematics teachers and engineers. The rest were teachers (not mathematics teachers and not preschool teachers), psychologists, occupational therapists, and municipal workers. A total of 58 participants stated that they have some relationship with children between the ages of three and six years. Researchers defined relationship as parent, uncle, aunt, grandparent, sibling, or other connection deemed close to the child. Questionnaires were handed out individually to the adults in the presence of the researcher.

A nine-Likert-scale questions were designed for this research. The range of the scale was from 1 (I do not agree) to 6 (I fully agree). Table 2 presents the questions and their relationship to the CAMAE framework. The first two questions relate to general beliefs regarding children's ability to learn mathematics at an early age and their enjoyment in doing mathematics. While previous studies investigated adults' beliefs regarding the importance of promoting mathematics at an early age, they did not address beliefs regarding children's enjoyment in doing so. Questions 3 and 4 are similar but have a subtle difference. In Question 3, researchers ask participants if it is worthwhile to engage children with number activities, but do not specify why it would be worthwhile. In other words, participants may believe that it is worthwhile because it can promote a positive attitude towards numbers. In Question 4, the statement is more direct, specifically asking if playing with such games can enhance children's number knowledge. Question 5 addresses participants' beliefs regarding general activities, and not necessarily those that deal with number aspects. One may think of eating dinner as a general activity that does not specifically deal with number concepts and ask themselves if such an activity can invite engagement with numbers. Question 6 is again a rather general question dealing with the importance of solving mathematical tasks. On the other hand, Questions 7 and Question 8 deal with the importance of evaluating solutions. Preschool mathematics educators have suggested that evaluating solutions and comparing strategies is beneficial for preschool children in learning number sense and examining relationships between numbers (e.g., Linder, Powers-Costello, & Stegelin, 2011). Finally, Question 9 focuses on the benefit of an adult's interaction. As can be seen from the questions, this part of the research mainly focused on participants' beliefs regarding engaging children with mathematics. Researchers also avoided terms such as instruction, and instead related to activities and games, suggesting a playful approach.

Question	Framework
1. Children enjoy activities/games that deal with number aspects.	Cell 7
2. Children's number knowledge can be promoted.	Cell 7
3. It is worthwhile to engage children with activities/games that deal with number aspects	Cell 8
4. Activities/games that deal with number aspects can increase children's knowledge of number concepts.	Cell 8
5. Almost every activity/game can invite children to engage with aspects of number.	Cell 8
6. It is important for children to be able to solve number tasks in various ways.	Cell 5
7. It is important for children to be able to identify if a suggested method for solving a number activity/task is correct.	Cell 6
8. It is important for children to be able to choose appropriate ways for solving number activities/tasks.	Cell 6
9. Interaction between a child and an adult while engaging in an activity/game can increase the child's knowledge of number.	Cell 8

Table 2: Relationship between beliefs questions and the CAMAE framework

RESULTS

In general, participants had positive beliefs regarding supporting children's engagement with various numerical activities (see Table 3). Participants were less sure if every activity/game can invite children to engage with aspects of number (Item 5) and less positive about the need for children to be able to identify if a suggested method for solving a number activity/task is correct (Item 7). Researchers also note that Item 5 was the only item where responses ranged from 1 to 6. It also had the highest standard deviation, inferring that participants did not agree on this item.

	Relationship with children		Mathematics related profession	
Question	Yes (N=58) M (SD)	No (N=33) M (SD)	Yes (N=30) M (SD)	No (N=61) M (SD)
1	5.21 (.93)	5.27 (.98)	5.27 (.91)	5.20 (.98)
2	5.79 (.61)	5.73 (.57)	5.50 (.90)	5.90 (.30)
3	5.55 (.84)	5.70 (.59)	5.63 (.72)	5.59 (.78)
4	5.78 (.56)	5.76 (.56)	5.57 (.73)	5.87 (.43)
5	4.48 (1.27)	3.70 (1.48)	3.90 (1.37)	4.36 (1.38)
6	5.24 (1.01)	4.91 (1.01)	4.93 (1.02)	5.23 (1.01)
7	4.83 (1.14)	5.00 (1.00)	4.93 (1.02)	4.87 (1.13)
8	5.28 (.89)	5.15 (.80)	5.07 (.91)	5.25 (.93)
9	5.74 (.61)	5.67 (.78)	5.47 (.97)	5.77 (.64)

Table 3: General results of the beliefs questionnaire

An analysis of variance showed that the effect of working in a mathematics related profession was significant to two items. Adults who were working in mathematics related professions believed less in the promotion of early mathematics (Item 2) than did adults in other professions, $F(1,93) = 5.07$, $p = .008$, $\eta^2_p = .102$. Mathematics related professionals believed less that activities/games dealing with number aspects can increase children's knowledge of number concepts (Item 4) than other adults (M=5.87, SD=.43), $F(1,93) = 3.194$, $p = .046$, $\eta^2_p = .067$. Regarding adults' relationship with children, an analysis of variance showed that adults with a relationship to young children had a stronger belief that almost every activity/game can invite children to engage with aspects of number (Item 5) than adults who did not acknowledge a relationship with children, $F(1,93)=7.249$, p=.008, $\eta^2_p = .075$.

DISCUSSION

In this paper researchers introduced a framework for investigating adults' beliefs regarding young children's engagement with mathematics. Our framework grew out of our work with preschool teachers. In answer to our first research question, it was interesting to find that the beliefs of adults who have a relationship with children hardly differ from those that do not have a relationship with children. The one exception was that adults with a connection to children had a stronger belief that almost every activity/game can invite children to engage with aspects of number than those with no connection to children. This may be due to having more experiences with children in their natural environment and the realization that even mealtime may give rise to engagements with number concepts. Educators of adults, including preschool teachers, may consider this result, and demonstrate how number concepts can arise in almost every activity.

In answer to our second research question, researchers found that adults working in mathematics related professions believed less in the promotion of early mathematics and believed less that activities/games dealing with number aspects can increase children's knowledge of number concepts. Perhaps those adults who have studied mathematics at the university level believe that mathematics is a subject best learned formally in school. Perhaps this result is related to those adults' beliefs of mathematics as a domain. Yet, previous studies that found parents' who have a positive attitude towards mathematics frequently engage their young children with number activities (LeFevre et al., 2010). This is a question for further studies.

Previous studies recognized that parents ask their children questions such as, how many? Or where is there more? However, it is also important for children to know different ways for solving a problem and to evaluate given solutions (Linder, Powers-Costello, & Stegelin, 2011). In fact, studies have shown that some young children can evaluate other's solutions, and can point out mistakes, for example, in one-to-one correspondence (Tirosh, Tsamir, Levenson, & Barkai, in press). A notable result of this research is that adults do believe that these issues are somewhat important, but believe it is more important for children to choose appropriate strategies than to be able to identify if a suggested solution is correct. It could be that adults believe that evaluating strategies is a task for a teacher, and not for children. Thus, it is important to introduce adults to these types of activities and demonstrate how they can be implemented in a playful way with children. The present research and studies such as these can help plan appropriate interventions for adults, that may motivate them to engage young children with number activities.

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ADULTS' INTERACTIONS WITH YOUNG CHILDREN AND MATHEMATICS: ADULTS' BELIEFS

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Taking into consideration that the home environment can impact on young children's mathematical knowledge, this study investigates adults' beliefs regarding the importance of adult interactions with children and mathematics. It also investigates beliefs regarding the importance of receiving guidance for these interactions. Results indicated that in general, adults, regardless of the types of relationships they had with children, believed in the importance of their interactions with children and mathematics, but felt it was less important to receive guidance. Participants' reasons for both positive and negative beliefs are explored.

INTRODUCTION

The importance of fostering mathematical development during the early years is supported by studies that found early mathematics competencies to be a predictor of later school success (e.g., Duncan et al., 2007). Acknowledging the importance of these studies, mathematics educators (e.g., Ginsburg, Lee, & Boyd, 2008) have become increasingly interested in how to foster mathematics knowledge during the preschool years. Ginsburg (2016) stated that while researchers may now agree that young children engage with powerful ideas of number and shape, the same cannot be said of many teachers and parents.

Regarding teachers, several countries have instated mandatory mathematics curricula for preschools. However, many young children spend a considerable amount of time at home. Furthermore, studies suggest that for children to take advantage of the academic opportunities provided at preschool, some level of support from the home environment, such as toys that stimulate learning number and shapes, is necessary (Anders et al., 2012). Thus, if we aim to promote young children's mathematical knowledge, the home environment should also be considered. This study is part of a larger project that focuses on adults' knowledge and beliefs regarding the teaching of number and geometry concepts during the early years, and ways of supporting adults' interactions with children and mathematics. In this paper we focus specifically on adults' beliefs regarding their intervention in children's learning number and geometry concepts and receiving guidance for this purpose.

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THEORETICAL BACKGROUND

Several studies investigated teachers' and parents' beliefs regarding the importance of learning mathematics in preschool. Most studies found that both preschool teachers and parents agree that mathematics should be and can be promoted in the years before first grade (e.g., Missall, Hojnoski, Caskie, & Repasky, 2015). Yet, some home day care providers believe it is less important for young children to acquire mathematics skills before entering kindergarten, than other social and academic skills (Blevins-Knabe, Austin, Musun, Eddy, & Jones, 2000). In one study, a few participants claimed that teaching mathematics in preschool can hinder social and emotional development (Cannon & Ginsburg, 2008). Several studies found that parents believe it is more important to enhance reading skills than mathematical skills (Sonnenschein, Stites, & Dowling, 2020).

Beliefs can impact on the amount of time and the types of activities adults engage with children at home. Several studies reported that parents help their children learn language skills more than mathematics in both everyday contexts, such as carrying out household chores, as well as during more structured contexts, such as direct teaching (e.g., Cannon & Ginsburg, 2008). In a more recent study (Sonnenschein, Stites, & Dowling, 2020), most parents reported engaging children with reading activities every day, while the frequency of engaging with mathematical activities varied and occurred sometimes as few as two days a week. Looking at specific types of mathematical activities, in one study parents kept a diary of mathematical activities carried out at home and were also observed during parent child laboratory interactions (Skwarchuk, 2009). Findings showed more activities related to number and operations, than geometry and shapes. However, in a different study (Missall, Hojnoski, Caskie, & Repasky, 2015), parents reported counting aloud and naming simple shapes as among the most frequent activities carried out at home.

Adult's knowledge may also impact on parents' engagement with their children and mathematics. In one study (Cannon & Ginsburg, 2008), parents stated that they lacked knowledge about early mathematics and were unaware of the goals for learning mathematics at a young age. Some parents may be unaware of activities that can be carried out home. Sonnenschein, Stites, and Dowling (2020) found that 64% of parents in their study wanted to receive information from their children's preschool teachers regarding how to support their children's mathematics. Most parents wished to receive ideas for carrying out fun mathematics activities with their children at home, about a third were interested in progress reports from the children's teachers, and a quarter were interested in worksheets.

Another factor that might influence adults' involvement with children and mathematics is anxiety. Parents with higher levels of math anxiety engaged less frequently in mathematical activities at home than parents with lower levels of math anxiety (Elliott, Bachman, & Henry, 2020). Another factor might be parental goals. Elliot, Bachman, and Henry (2020) identified two goals that parents held for their children's learning mathematics: they wanted their children to succeed in mathematics at school and they wanted their children to like mathematics. These goals led to different ways of interacting with children.

While previous studies have focused on parents' beliefs (e.g., Skwarchuk, 2009), children often spend time with a grandparent, aunt, or neighbour (e.g., Pilarz, 2018). Our research questions are: (1) Do adults believe that their involvement is important to children's number and geometry development, and what are the underlying reasons for these beliefs? (2) Do adults believe they need guidance in order to foster this development, and what are the reasons for these beliefs? (3) Is there a difference between parents' beliefs and other adults' beliefs? (4) Is there a difference between beliefs regarding number concepts and beliefs regarding geometry concepts?

METHODOLOGY

Fifty-one adults (labelled A1-A51), between the ages of 20 and 60, participated in the study. None of the participants were early childhood educators. Of the 51 adults, 22 were parents of children between the ages of three and six, and 18 had some other relationship with children of this age, either as a grandparent, aunt, or uncle. The rest, 11, claimed to have no connection with young children. As this was part of a larger study, participants were handed two separate questionnaires, at least a week apart. The first questionnaire related to beliefs and knowledge regarding promoting children's numerical knowledge and the second, beliefs and knowledge for promoting children's geometrical knowledge. In this study we analyse responses to two questions from each questionnaire: (1) In your opinion, is it important for an adult to be involved in developing preschool (ages 3-6) children's quantitative reasoning? Explain. (2) In your opinion, is it important for an adult to receive guidance so that he/she can help foster quantitative reasoning among young children (ages 3-6)? Explain. (3) In your opinion, is it important for an adult to be involved in developing preschool (ages 3-6) children's geometric reasoning? Explain. (4) In your opinion, is it important for an adult to receive guidance so that he/she can help foster geometric reasoning among young children (ages 3-6)? Explain.

A first step in data analysis was to assess the frequency of positive and negative responses to the questions. Using grounded theory, a qualitative analysis was then conducted to assess reasons participants offered for their responses. Two researchers categorized all reasons. A third researcher validated categorization. Initial agreement was 91% for the intervention categories, and 88% for the guidance categories. After discussion between the three researchers, 100% agreement was reached.

FINDINGS

Quantitative results

Table 1 presents the number of adults that reported positive beliefs regarding adult intervention for promoting young children's number and geometry knowledge, according to their relationships with children. It also presents beliefs regarding the importance of receiving guidance. As can be seen, nearly all participants believed that adult intervention is important for enhancing children's mathematics knowledge. Furthermore, using a McNemar test to compare marginal distributions of the two variables (number and geometry intervention), no significant differences were found between adults' beliefs in the importance of their intervention in number concepts and geometry concepts. In addition, no differences were found between parents, adults who have some other connection with young children, and those who claimed to have no connection with young children.

	Number concepts		Geometry concept	
	Intervention	Guidance	Intervention	Guidance
Parents (N=22)	21(95)	12(55)	20(91)	14(64)
Other relation (N=18)	18(100)	15(83)	15(83)	12(67)
No connection(N=11)	10(91)	8(73)	10(91)	8(73)
Total (N=51)	49(96)	35(69)	45(88)	33(65)

Table 1: Agreement (%) that adult intervention and receiving guidance is important

When it came to receiving guidance, beliefs were not as positive. Using a McNemar test to compare marginal distributions of the two variables (intervention and guidance), for each mathematics domain, significantly less participants agreed that guidance was important both in supporting number concepts, as well as geometry concepts ($p<.01$). Although no significant differences were found between beliefs in guidance in the number and geometry domains, nine adults agreed to the importance of receiving guidance in the number domain, but not in geometry. For example, A38 stated for number concepts, "Children's minds and their ways of thinking are different and so you must explain to them in ways they will understand." However, for guidance in geometry A38 stated, "No. Children are too young to learn complex ideas in geometry." Four adults did not believe that guidance was important in the number domain, but agreed to its importance in geometry. For example, referring to guidance in the number domain, A28 stated, "It's not necessary. In general, we are talking about simple numerical concepts, and in my opinion, any adult with a little imagination can explain it to children." However, in geometry, A28 wrote, "Yes, in order to know how to explain concepts correctly and lay the foundation for what comes next."

Intervention: Reasons for beliefs

Although nearly all adults agreed that adult intervention is important, the reasons for these beliefs differed. From the data arose seven categories of reasons, quite similar for both number and geometry domains. Table 2 presents those reasons along with their frequencies of occurrence and examples of participants' written responses. From Table 2, noting the statements given by A21, we can also see that each adult did not necessary give the same reason for intervening with numbers as with geometry.

Among the few adults who were against intervening (two for number concepts and six for geometry), two reasoned that children will learn on their own, "by watching an adult or other children" (A36). The rest believed that it was the job of the preschool teacher to teach mathematics, and not the job of the parents or other non-teachers.

Category	Frequencies Number (N=49) / Geometry (N=45)	Example
Children need adults to teach them	23 / 25	An adult should talk about shapes and their attributes, such as the difference between a square and rectangle, and that not everything round is a circle. (A21, geometry)
An adult can expose children to new concepts	9 / 8	The more a child is exposed to concepts and possibilities, the more his thinking can improve. (A19, geometry)
Preparation for school	4 / 5	You have to make sure that the child tries to learn concepts, so he will not enter first grade without having some idea (of numbers). (A45, numbers)
Mathematics is all around us	6 / 10	To see what things are made from, everything in nature, from which shapes they are composed. (A40, geometry)
An adult can promote practice	7 / 5	You need to talk to children… because they need more practice than what they get in kindergarten. (A1, number)
To promote a positive attitude	2 / -	To promote counting, a positive attitude, and connect them to numbers. (A21, numbers)
Age-related issues	4 / -	It's easy at this age to learn and observe new ideas. (A6, numbers)

Table 2: Reasons for believing in the importance of adult intervention (N=51)

Beliefs regarding guidance

As seen from Table 1, most adults agreed that guidance for adults is important. Reasons for agreeing with this statement are given in Table 3. As can be seen, most adults simply wished to acquire tools that may help enhance children's mathematics. As with reasons for interventions, when someone agreed that guidance in both domains was important, the reasons for wanting guidance were not necessarily the same. For example, regarding number concepts, A21 agreed to guidance, stating, "So many people are afraid of numbers." For geometry, she agreed to guidance stating, "Adults sometimes use incorrect terminology." Thus, for number concepts, A21 was interested in deflecting adult's mathematics anxiety (an affective issue), while for geometry, she was concerned in avoiding mistakes.

Category	Frequencies Number (N=35) / Geometry (N=33)	Example
To learn how to promote early mathematics	17 / 19	Because I don't have the tools to teach on my own. (A40, geometry)
To avoid making mistakes	4 / 5	For those who do not know geometry, so they won't make mistakes. (A23, geometry)
To raise adult's awareness	4 / 4	To raise parents' awareness that children's thinking can be developed informally. (A13, numbers)
To understand how children think	7 / 4	You need to know how to deal with children who are slower and those who are more advanced. (A17, numbers)
Affective issues	5 / 1	You have to know how to challenge children without frustrating them. (A3, numbers)

Table 3: Reasons why guidance for adults is important

Reasons for negating the necessity for guidance are given in Table 4. Recall that a few adults believed that interacting with children and mathematics was the job of the preschool teacher. So too, a few believed that it is teachers who should receive guidance. However, some simply believed that promoting early number and geometry learning comes naturally to adults and that the mathematics involved at this age is basic and simple, thus guidance is not necessary.

Category	Frequencies Number (N=16) / Geometry (N=18)	Example
Parents intuitively know what to do	8 / 5	It comes naturally to most people that encounter children in situations that allow for numerical thinking, and so they automatically teach, and children learn. (A2, numbers)
Preschool mathematics is basic knowledge	2 / 1	In general, we are talking about simple numerical concepts, and in my opinion, any adult with a little imagination can explain it to children. (A28, numbers)
It is the job of the preschool teacher	4 / 6	It's important to guide the preschool staff, because that is where most of the learning occurs (A2, geometry)
Age-related issues	- / 3	Children are anyway too young to learn complex geometry (A38, geometry)

Table 4: Reasons why guidance for adults is *not* important

DISCUSSION

The first focus of this study was beliefs regarding adults' involvement with children's number and geometry development, and the underlying reasons for those beliefs. We found that nearly all adults believed adult involvement is important. Furthermore, while some studies found that parents engage more with number activities than geometry activities (Skwarchuk, 2009), this study found no differences between beliefs in the importance of adults' involvement within these two domains. Perhaps this signifies a gap that needs to be filled. If adults believe that adult intervention is important in both number and geometry, but report being involved in fewer geometric activities, they might need more support in geometry. Significantly, while most previous studies focused solely on parents' beliefs (Missall et al., 2015), this study found that parents, as well as grandparents, aunts, and even adults who have no current relationship with young children, believe in the importance of adult interactions with children and mathematics. On a practical level, this means that those offering workshops for parents might consider opening their workshops to additional adults.

The reasons for adults' beliefs in the importance of adult interaction with children varied. These reasons may impact on the types of activities adults implement at home. A future study might investigate if those who believe that mathematics may be found all around us tend to engage children with mathematics during daily activities, such as setting the table. Might those who believe in the need to prepare children for first grade, engage children with more formal teaching activities, such as using flash cards? Interestingly, while previous studies found that parents' mathematics anxiety can impact on their mathematical interactions with children (Elliott, Bachman, & Henry, 2020), only a few adults in this study related to affective issues.

The second focus of this study concerned beliefs regarding the necessity for guidance. Finding in this respect were less positive. Among those that agreed that guidance is important, only a few expressed the need to learn more about children's ways of thinking. While knowledge of students is recognized as an important element of teachers' knowledge (Ball, Thames, & Phelps, 2008), it might also be that this type of knowledge can significantly affect the types of mathematical activities adults offer children to engage with at home. Understanding why adults may or may not be interested in guidance, might help educators attract adults to participate in workshops that encourage adults', not only parents', interactions with children and mathematics. It is not only the amount of time parents spend with children engaging in number concepts, but also the nature of that learning experience that is important (Pan, Gauvain, Liu, & Cheng, 2006).

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IN-SERVICE ELEMENTARY TEACHERS' KNOWLEDGE IN MATHEMATICS AND PEDAGOGY FOR TEACHING – THE CASE OF FRACTION DIVISION

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We focused on both in-service elementary teachers' (ETs) confidence about their knowledge and the extent of their knowledge of the specific topic of fraction division. The results revealed how these ETs' confidence may or may not be supported by their knowledge for teaching fraction division, a concept they are expected to teach as part of the elementary school curriculum in China. The results also illustrated the importance of specifying knowledge components in mathematics in order to help further or support ETs' confidence for classroom instruction.

INTRODUCTION

Worldwide efforts to facilitate teachers' learning of mathematics for teaching have led to the increased emphasis not only on the mathematics training provided through teacher preparation programs (e.g., CBMS, 2012), but also on in-service teachers' learning through teaching (e.g., Li & Huang, 2018). It is generally perceived that Chinese mathematics teachers had superior understanding of the school mathematics they teach (Ma, 1999). With the inclusion of both novice and experienced elementary teachers in her study, Ma indicated that "Chinese teachers begin their teaching careers with a better understanding of elementary mathematics than that of most USA elementary teachers'' (p. xvii). However, the results from recent studies that involved prospective elementary teachers in China did not seem to support the hypothesis that prospective elementary teachers in China may have strong preparation in pedagogical content knowledge (Li, Ma & Pang, 2008; Li et al., 2020). The results obtained with Chinese prospective elementary teachers prompted us to wonder what may happen to Chinese in-service elementary teachers, especially with the dramatic changes in school mathematics and instruction happening in China in recent years (Liu & Li, 2010). As part of a large research study of elementary school teachers' mathematical training, this paper focused on a group of ETs' confidence and knowledge of mathematics and pedagogy on the topic of fraction division in China.

The topic of fraction division is difficult in school mathematics not only for students (Li, 2008), but also for teachers (Li & Kulm, 2008; Simon, 1993). Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as "invert and multiply." However, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, or real world contexts (Greer, 1992; Li, 2008). The selection of the topic of fraction division,

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as a special case, can provide a rich context for exploring possible depth and limitations in in-service teachers' knowledge in mathematics and pedagogy. Specifically, this study focused on the following two research questions:

(1) What is the confidence of in-service elementary school teachers regarding their knowledge training for teaching?

(2) What is the extent of in-service elementary school teachers' knowledge in mathematics and pedagogy for teaching fraction division?

CONCEPTUAL FRAMEWORK

To be able to help students learn mathematics with understanding, teachers need to have mathematics conceptual knowledge for teaching (MCKT; Li et al, 2020). By MCKT we mean *topic-based conceptual knowledge packages that are needed for understanding, explaining, as well as teaching specific mathematics content topics with connections*. It can be specified as containing the following three topic-based knowledge components that can and should be acquired by mathematics teachers:

(a) Having knowledge and skills directly associated with a specific content topic;

(b) Being able to connect and justify the main points of a content topic, and to place it in wider contexts;

(c) Knowing and being able to use various representations for teaching the content topic, and being able to teach the relations between them.

Clearly, specific MCKT varies from one content topic to another. The task of specifying MCKT is needed but enormous for different content topics. Nevertheless, teachers' acquisition of MCKT would enable them to develop a profound understanding of mathematics content topics they teach as termed by Ma (1999). Given the dramatic variations across mathematical content topics, we focus on the MCKT that teachers would need to have for teaching fraction division.

The conceptual complexity of the topic of fraction division is evidenced in a number of studies that documented relevant difficulties pre-service and in-service teachers have experienced (e.g., Borko et al., 1992; Simon, 1993; Tirosh, 2000). Although both pre-service and in-service teachers can perform the computation for fraction division, it is difficult for teachers, at least in the United States, to explain the computation of fraction division conceptually with appropriate representations or connections with other mathematical knowledge (Ma, 1999; Simon, 1993). Teachers' knowledge of fraction division is often limited to the invert-and-multiply procedure, which restricts teachers' ability to provide a conceptual explanation of the procedure in classrooms (e.g., Borko et al., 1992). Because the meaning of division alone is not easy for pre-service teachers (e.g., Simon, 1993), fraction division is even more difficult (Li & Kulm, 2008; Ma, 1999). The findings from previous studies help provide specifics of these three components of MCKT as follows:

(a) Having knowledge and skills about fraction division, including conceptual and procedural knowledge (e.g., Borko et al., 1992), and solving problems involving fraction division (e.g., Greer, 1992)

(b) Mathematical connections and justifications of main points related to fraction division, including fraction concept; addition, subtraction, and multiplication of fractions (e.g., Ma, 1999; Tirosh, 2000)

(c) Representational variations and connections for teaching fraction division such as explaining the computational procedure for "division of fraction" with different representations (e.g., Li & Huang, 2008; Li & Kulm, 2008)

The specifications of these three components of knowledge provided a framework for the current study and served as a guideline for selecting items to examine the extent of ETs' knowledge and specific difficulties with fraction division.

METHODOLOGY

Subjects

The participants were in-service elementary school teachers sampled from three provinces and one major city in China. All of these provinces and city are traditionally classified as developing areas located mainly in southwest of China. 190 surveys were distributed, and 180 responses (returning rate: 94.7%) were collected. All 180 responses (130 females, 44 males) are used for data reporting, with 117 (65%) of responses self-indicated from schools located in inner city, 16 (9%) responses from sub-urban, 40 (22%) responses from rural areas, and 7 (4%) with no indication.

Instruments and Data Collection

A survey was developed for this study, containing two main parts with three items for Part 1 and seven items for Part 2. Part 1 contains items on elementary teachers' knowledge of mathematics curriculum and their confidence in their readiness for teaching. Part 2 has seven main items that assess elementary teachers' three knowledge components of MCKT on the topic of fraction division. Most items were taken from previous studies (Li, Ma, & Pang, 2008; Li et al., 2019), with some items adapted from school mathematics textbooks and others' studies (e.g., Tirosh, 2000). Given the limited page space, only three items (note: each item containing two questions) from Part 2 and ETs' responses to these items are included for analyses to provide a glimpse of sampled ETs' confidence and MCKT.

It was impossible to conduct the survey with in-service teachers with specified time and location. Thus, the survey was distributed and then collected the next day from in-service teachers in many schools, and was given during teachers' professional development session in a few other schools.

Data Analysis

Both quantitative and qualitative methods were used in analysing and reporting the participants' responses. Specifically, responses to the items in Part 1 were directly recorded and summarized to calculate the frequencies and percentages of participants' choices for each category. To analyse participants' solutions to the items in Part 2, specific rubrics were first developed for coding each item, and subsequently, the participants' responses were coded and analysed to examine their solutions/answers.

RESULTS AND DISCUSSION

In general, the results showed interesting relationships between ETs' confidence and their mathematical knowledge for teaching fraction division, which illustrates the importance of specifying knowledge components in mathematical training in order to help further or support ETs' confidence for classroom instruction.

For ET's confidence, the results from the survey indicated that (1) participating ETs in China did not know well about their national curriculum standards in general; (2) the majority of these ETs were confident in the knowledge needed for teaching; and (3) they knew very well about selected topic placement in mathematics curriculum. The results suggested that these ETs tend not to feel over confident.

For specific knowledge components of MCKT, these ETs' performance revealed that their mathematical knowledge was sound in the content topic itself, especially in the procedural aspect, and relatively weak conceptually in connecting the content topic with other topics both mathematically and pedagogically. The seemingly mixed results in their responses actually suggest that these ETs' confidence was built upon or supported by what they know that can and should be specified in concrete terms or knowledge components. The following sections are organized to present more detailed findings corresponding to the two research questions.

In-service Teachers' Confidence in Elementary School Mathematics

The following items are from Part 1 of the survey to illustrate ETs' confidence of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the National Mathematics Standards? On a scale of four choices (High; Proficient; Limited; Low), 63% and 1% of the participants chose "Limited" and "Low", respectively. Relatively smaller percentages of the in-service teachers felt to have high (3%) or proficient (33%) understanding of their national mathematics standards.

For item 2-(2): Choose the response that best describes whether elementary school students have been taught the topic – *Multiplication and division of fractions.* On a scale of five choices (Mostly taught before grade 5; Mostly taught during grades 5-6; Not yet taught or just introduced during grades 5-6; Not included in the National Mathematics Standards; Not sure), 92% participants indicated that the topic is "mostly taught during grades 5-6" (a correct choice), and most of the remaining (3%) chose the first response ("Mostly taught before grade 5", a partially correct choice if only fraction multiplication is considered). The results, in contrast to the participants' response to item 1, suggested that these ETs know very well about the content topic placement in mathematics curriculum, although the majority did not feel confident in knowing about their national mathematics standards.

For item 3-(2): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of "Number – Representing and explaining computations with fractions using words, numbers, or models?" On a scale of three (Very ready; Ready; Not ready), 69% of the participants thought they were "ready", while 21% chose "very ready," and 7% "not ready." The results indicated that the majority of these ETs were confident in their knowledge for teaching fraction computations, including fraction division.

There was also a small percentage of in-service teachers who are not confident. The diversity in responses suggested the need of learning more about their confidence and possible connections with their knowledge preparation.

Taking together, in-service elementary school teachers' responses to the Part 1 suggested that these ETs in China tend not to feel over confident, although they actually knew very well about some specifics. In fact, the results are consistent with what has been reported about in-service mathematics teachers in East Asian countries (Mullis et al., 2004) and China in specific (Li & Huang, 2008). The consistency in the general response pattern between ETs in the current study and in-service teachers in other studies suggested that culture likely plays an important role in expressing confidence by teachers in East Asia including China.

The Extent of i=In-service Elementary School Teachers' Training in MCKT for Teaching Fraction Division

These ETs' responses to Part 2 allowed a closer look at the participants' three knowledge components of MCKT, especially on the topic of fraction division. Results indicated that these ETs do very well on items related to fraction division computation and problem solving (MCKT knowledge component 1). For example, for the problem "Say whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without solving. Explain your reasoning.", 98% of these ETs answered the problem correctly (i.e., the first numerical expression is greater than the second one). Among those who provided the correct answer, 78% did not use fraction division computations and explained, "If the divisor is the smaller (and the dividend is the same), the result of the division is bigger; 2/3 (8/12) is smaller than ¾ (9/12)". The other 17.2% used the computation rule for fraction division (i.e., converting division into multiplication, then followed by comparing 3/2 and 4/3) to reach the correct answer. 4 out of 180 respondents (2.2%) used both methods. There were about 2.3% of those sampled in-service teachers who either answered incorrectly or did not answer at all. They did not infer further on what would be the result of division if the divisor was the smaller.

Moreover, these ETs also had great performance in solving multi-step word problems that involve fraction division. For example, 95% participants solved the following problem correctly.

> Johnny's Pizza Express sells several different flavour large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was 3/4 of the number of pepperoni pizzas sold, and 2/3 of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?

Specifically, 42% used a multi-step computation method to get the answer (e.g., 24 x ¾ =18, 18÷2/3 =27), about 49% used a combined computation method (e.g., 24 x ¾ ÷ 2/3 =27), 2.2% adopted an algebraic approach to set up and solve an equation for solution, and a few (about 1.8%) provided more than one solution approach. About 5% of these respondents did incorrectly, resulting from either computation errors (e.g., providing a computation as 24 x ¾

÷ 2/3 = 36) or misunderstanding of the problem (e.g., providing a computation as 24 ÷ 2/3 = 36).

For the knowledge component 2 of MCKT, ETs were asked to explain "the meaning of fraction division, and how fraction division relates to other content topics" that aims to assess their knowledge of fraction division and ability of connecting and justifying possible association between fraction division and other content topics. The results suggested that 82.3% provided one or more correct explanations to the first sub-question. The common explanation provided by 53.9% sampled in-service teachers is that "the meaning of fraction division is the same as the meaning of the division of whole numbers, and if knowing the product of two factors and the value of one factor, it is an operation to find the value of the other factor". 7.2% explained that "the meaning of fraction division is the same as the division of whole numbers", and 9.4% provided their answers as "if knowing the product of two factors and the value of one factor, it is an operation to find the value of the other factor". The rest (17.8%) either indicated "don't know" (6.1%) or did not answer this question (11.7%). For the second sub-question, about 40% of them provided correct answers, with 11.1% indicating "fraction division is an inverse operation of fraction multiplication", 10% explaining that "fraction division relates to inverse number, e.g., if divided by a number equals to multiplying its inverse number", 5.6% indicating that fraction division relates to ratio, e.g., $a \div b = a : b = \frac{a}{b}$ (b≠0), 0.6% with other explanation, and 12.6% providing two different explanations. There are about 60% sampled in-service teachers who either provided incorrect explanation (33.8%), or no answer at all (26.1%).

There were several items used to assess ETs' knowledge component 3 of MCKT. As an example, ETs were asked to explain how to explain/teach given computations of fraction division. In particular, the problem of "How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$?; Why $\frac{2}{3} \div \frac{1}{6} = 4$?" (adapted from Tirosh, 2000) was included in the survey. For the first fraction division (i.e., explaining why 2/3 ÷ 2 = 1/3?), 98% provided valid explanations and the majority (53%) relied on the meaning of fraction to provide their explanation as "dividing a whole into three equal parts, each part should be 1/3, so 2/3 mean to have two such parts. Dividing 2/3 into two equal pieces, so each piece should be 1/3." 19.3% used drawings or a number line to explain, and 16% explained using the fraction division algorithm, i.e., flip and multiply to get the answer. There are about 9% sampled in-service teachers provided two or more different explanations. For the second fraction division (i.e., explaining why 2/3 ÷ 1/6 =4?), 88% provided valid explanations but the dominant explanation (39.8%) was based directly on the fraction division algorithm, that is, flip and multiply to compute. The other 18.8% provided their explanations mainly as "changing fractions so that they have the same denominator first, and then using the meaning of fraction for solution". 19.3% used drawings or a number line to provide their explanations. A few (0.6%) used the first fraction division to help explain the second fraction division as "based on the first fraction division 2/3 ÷ 2 = 1/3, we can deduct why 2/3 ÷ 1/6 = 4 is correct. That is, keeping the dividend 2/3 unchanged, when the divisor 2 is decreased 12 times to become 1/6, the original quotient 1/3 should be increased 12 times to become 4." There are about 4.6% respondents provided two or more different explanations. 6.1% failed to provide correct explanations, with either incomplete explanation or simply copying the question without explanation. Taking together, these ETs did very well in explaining these two fraction divisions (98% and 88%, respectively). Their explanations were dominant with an approach that relies on either the meaning of fraction or the fraction division algorithm. Moreover, they performed better in explaining a fraction divided by a whole number than a fraction divided by a fraction.

The results from these ETs' responses on MCKT items revealed their strengths in many aspects of MCKT, as specified in the framework. However, ETs' strengths across these aspects varied to a certain degree. It appeared that these ETs have solid performance on items related to fraction division computation itself, especially in the procedural aspect and problem solving, but relatively weak conceptually in connecting the content topic with others both mathematically and pedagogically.

CONCLUSION

The findings from this study helped shed a light on the relationships between these ETs' confidence and their mathematical knowledge for teaching fraction division. Specifically, these ETs didn't feel over-confident about their understanding of national mathematics standards, but they knew very well about the curriculum placement of selected content topics. They also had better confidence in terms of their readiness to teach elementary school mathematics. Such confidence was likely supported by their solid knowledge and skill directly associated with fraction division, a knowledge component that is also typically required for school students. At the same time, their relatively weak performance on items that are conceptually demanding in mathematics or pedagogy likely failed to support their confidence in readiness for teaching. Such knowledge differentiations, as specified in the MCKT framework, help provide an important and feasible lens for us to know the strength and weakness of teachers' knowledge. For the case of China in this paper, the results suggested that ETs are likely strong on mathematics, somehow less on mathematical pedagogy, and limited on connections of mathematical ideas through their teaching practices and professional development. In turn, such results helped illustrate what teacher professional development needs to do more in mathematical and pedagogical training in order to help further or improve ETs' confidence and expertise.

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THE MATHEMATICAL BELIEFS AND INTEREST DEVELOPMENT OF PRE-SERVICE PRIMARY TEACHERS

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We investigated the development of interest in mathematics of pre-service primary teachers (N=62) during the transition from school to university using longitudinal data and examined whether their beliefs about the nature of mathematics explained their future interest. One main result is that although high correlations between dynamic beliefs (the process and utility aspects of mathematics) and interest were found in each of three surveys, dynamic beliefs did not predict future interest (in addition to prior interest). Instead of dynamic beliefs, formalism beliefs formed an additional significant predictor of future interest.

THE SECONDARY-TERTIARY TRANSITION

The transition to university brings many changes and is often perceived as a stressful endeavour (Gueudet, 2008). Many students feel that mathematics has changed without being able to handle this new form of mathematics. A fundamental change refers to what Tall (2008) calls the formal world. Definitions, logic, and proofs are new to most students, whereas calculations now play a minor role. In particular, dealing with proofs is difficult for students and may negatively impact their interest in mathematics. Consequently, many students lose interest during the transition (Rach & Heinze, 2017). Interest and beliefs may be helpful concepts to understand the psychological side of this transition. In particular, analysing their relationship may help understanding why some students struggle more than others.

We focus on primary teacher education that has some commonalities with secondary teacher education like the new role of formalism and proof. Mathematics in primary teacher education is less formal than in secondary teacher education. Unlike in school, however, even the primary teachers' mathematics courses emphasise argumentation and reasoning not only in the lectures but also in the homework. This work completes earlier analyses presented at PME (Liebendörfer et al., 2014).

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INTEREST AND BELIEFS

We use Krapp's (2005) interest concept, in which interest is defined as a motivational person-object relationship, which is rather stable over time. Interest is specific to a person, but, unlike other motivational concepts, it is also specific to a (mental) object which, in our case, is mathematics. Interest has a cognitive component, which refers to a high personal value, and an emotional component related to positive affect.

Interest has gained importance as a predictor of good learning processes, such as the use of deep learning strategies, effort, and good learning outcomes, as can be shown across various disciplines and settings (Krapp et al., 1992; Köller et al., 2001). Pre-service primary teachers have reported low interest in mathematics. In a study by Abel (1996), pre-service primary teachers' interest scores (N=171) were about one standard deviation (SD) below the theoretical mean of the scale and considerably lower than the interest scores of pre-service secondary teachers (N=36) who had opted for mathematics as the subject of their future teaching careers. Whereas in higher secondary teacher education, interest declines during the transition (Rach & Heinze, 2017), this is not the case in lower secondary teacher courses that focus less on formalism (Liebendörfer & Schukajlow, 2017).

We use Grigutsch et al.'s (1998) concept of beliefs on the nature of mathematics that distinguishes four dimensions: The process aspect describes mathematics as a vivid field of trial and discovery. The utility aspect emphasises the usefulness of mathematics in everyday life. The formalism aspect characterises mathematics by logic, proof, and abstraction. Finally, the toolbox aspect describes mathematics as the application of routine skills, formulae, and standard procedures (see also Table 1). The first two aspects are rather dynamic whereas the last two aspects reflect a rather static view on mathematics.

Dynamic beliefs are often favoured over static beliefs because they emphasise opportunities for learning and improvement. They are further positively correlated with students' interest, whereas toolbox beliefs are negatively correlated (Baumert et al., 2000). Beliefs generally affect the way we experience and deal with new mathematics. In particular, improper beliefs may be seen as one reason for the decline of interest during the transition (Daskalogianni & Simpson, 2001) and beliefs that fit to the new mathematics students are presented may help them taking interest (Liebendörfer & Schukajlow, 2017). Thus, the static formalism beliefs may also be important for students' interest development during the transition as they may help them understanding new elements like the role of axioms and definitions.

RESEARCH QUESTIONS AND DESIGN

Our main aim is to describe how pre-service primary teachers' interest in mathematics develops in the first semesters at university and whether beliefs about the nature of mathematics may explain this development.

RQ1: How is pre-service primary teachers' interest connected to their beliefs?

RQ2: How do interest and beliefs change during the first year at university?

RQ3: Do beliefs serve as a predictor of future interest?

Design of the Study

We used data from the KLIMAGS-project (Blum, Biehler, & Hochmuth, 2014) collected at Kassel University. There, mathematics courses are compulsory for all pre-service primary teachers. The data were collected in the first (T1) and last lectures (T2) of a course on arithmetic during the students' first semester. The third survey was collected at the end of a course on geometry (T3) during the students' second semester in which they also took a course on the didactics of arithmetic. These paper-and-pencil surveys were collected over two consecutive years to gain a reasonable sample size. The sample consisted of N=62 pre-service primary teachers who participated at all three time points, 57 of whom were female. They were on average 21.75 years old (SD 5.06) and all but two were first-year students.

Scale	Items	Example	α (T1-T3)
Interest in Mathematics	6	I am not interested in mathematics. (reverse scoring)	.74 / .81 /.78
Utility Beliefs	4	Mathematics is helpful for solving everyday tasks and problems.	.79 / .74 /.71
Process Beliefs	4	Mathematics thrives on inspiration and new ideas.	.80/ .80 /.74
Formalism Beliefs	7	Clarity, accuracy, and uniqueness are features of mathematics.	.63 / .68 /.76
Toolbox Beliefs	5	Mathematics is a collection of procedures and rules that specify exactly how to solve tasks.	.47 /.46 /.50

Table 1: Scales and their reliabilities

To measure interest and beliefs, well-tested Likert scales were taken from other projects and were modified slightly if needed (words were adjusted; e.g. "university" instead of "school"). To measure interest, we used Rheinberg & Wendland's (2000) scale; to measure beliefs, we took Grigutsch et al.'s (1998) scales from the COACTIV (Baumert et al., 2009) version with a 6-point format (1=not at all, 6=exactly). Reliabilities (Cronbachs □) ranged from poor to good, see Table 1. In particular, the toolbox scale had a low reliability. For the sake of completeness, we included this scale; however, results concerning toolbox beliefs should be handled cautiously.

RESULTS

We analysed data from a subgroup of the first-year pre-service primary teachers; namely, those who answered all three surveys. Using Levene's tests and t-tests to compare the variances and means of the reported constructs, we found no differences between this subgroup and students who missed one of the three tests and had thus been excluded from further analyses ($p>.10$ in each case). The means (SDs in parentheses) of the different constructs are displayed in Table 2. We found that the interest values were below the theoretical mean of the scale (3.5).

	T1	T2	T3
Interest in Mathematics	3.36 (0.83)	3.07 (0.92)	3.15 (0.80)
Beliefs: Utility Aspect	4.59 (0.79)	4.03 (0.88)	4.35 (0.75)
Beliefs: Process Aspect	4.27 (0.91)	3.99 (0.95)	4.13 (0.88)
Beliefs: Formalism Aspect	4.21 (0.65)	4.29 (0.68)	4.26 (0.66)
Beliefs: Toolbox Aspect	4.12 (0.66)	4.13 (0.64)	3.93 (0.61)

Table 2: Means and standard deviations of interest and beliefs

For RQ1, we found significant correlations between interest and both dynamic beliefs and toolbox beliefs on each survey. However, there were no statistically significant correlations between interest and formalism beliefs. The correlations between interest and the different aspects of belief are displayed in Table 3 for each time point.

Correlation between interest and …	T1		T2		T3	
	r	p	r	p	r	p
… Beliefs: Utility Aspect	**.46**	**<.001**	**.50**	**<.001**	**.47**	**<.001**
… Beliefs: Process Aspect	**.52**	**<.001**	**.52**	**<.001**	**.50**	**<.001**
… Beliefs: Formalism Aspect	-.10	.444	-.11	.393	.04	.758
… Beliefs: Toolbox Aspect	-.13	.308	**-.28**	**.026**	**-.31**	**.013**

RQ2 asks for the development of interest and beliefs. Interest was a stable construct in our study. Correlations were .66 (both T1-T2 and T2-T3) and .60 (T1-T3). The beliefs were also rather stable; correlations ranged from .44 to .59 (T1-T2) and .37 to .43 (T1-

T3). The changes in mean scores of both interest and beliefs can be derived from Table 2.

	Between T1 and T2			Between T2 and T3		
	p	t(df=61)	d	p	t(df=61)	d
Interest in Mathematics	**.003**	**3.14**	**.33**	.423	-0.81	.09
Beliefs: Utility Aspect	**<.001**	**5.95**	**.67**	**.003**	**-3.08**	**.39**
Beliefs: Process Aspect	**.006**	**2.85**	**.30**	.227	-1.22	.15
Beliefs: Formalism Aspect	.249	-1.17	.12	.706	0.38	.04
Beliefs: Toolbox Aspect	.744	-0.33	.12	**.017**	**2.45**	**.32**

Table 4: Significance values, t-values, and effect sizes

Results of paired samples t-test for these differences and the effect sizes (Cohen's d) are shown in Table 4. There was a considerable decline in interest as well as in dynamic beliefs during the first semester, followed by a slight recovery in the second semester. Static beliefs were less affected; only toolbox beliefs decreased in the second semester.

	Effect on interest at T2			Effect on interest at T3		
	β	p	R^2	β	p	R^2
Pre-Interest	**.615**	**<.001**		**.666**	**<.001**	
Beliefs: Utility Aspect	.149	.297		-.088	.444	
Beliefs: Process Aspect	.093	.462	.47	-.089	.410	.50
Beliefs: Formalism Aspect	-.168	.326		**.292**	**.013**	
Beliefs: Toolbox Aspect	.168	.310		-.094	.464	

Table 5: Results of linear regressions.

RQ3 was to investigate, whether beliefs could predict students' future interest. We calculated a linear regression and took interest and beliefs at T1 and T2 as independent variables to predict interest values at T2 and T3 respectively. In a simple linear regression using interest values only, the explained variance of the dependent variable (R^2) was .44 at T2 and .43 at T3. Including beliefs increased the R^2 to .47 and .50 for T2 and T3, respectively (cf. Table 5). The additional variance in interest explained by beliefs was rather low and not significant in the first semester. In the second semester, formalism beliefs explained an additional 7% of the variance in future interest.

DISCUSSION

Answers to the Research Questions

In terms of their general level, the students in our study had little interest in mathematics. This result compares to other findings and fits the idea that primary teachers often have a stronger pedagogical than content-specific (e.g. mathematical) interest (Abel, 1996). In addition, at Kassel University, mathematics courses were compulsory for pre-service primary teachers.

The answer to RQ1 is that the correlations between beliefs and interest during secondary school were positive for the two dynamic aspects (utility, process) and negative for formalism beliefs. The correlations of interest and beliefs even appeared to be slightly higher than those reported by Baumert et al. (2000). The answer to RQ2 for the development of beliefs and the interest in mathematics of pre-service primary teachers over the first year at university is threefold. For interest and dynamic beliefs, a strong decline was followed by a weak recovery. Toolbox beliefs decreased in the second semester, whereas formalism beliefs were constant. To answer RQ3, modelling the influence of interest and beliefs on future interest surprisingly revealed no effect of dynamic beliefs but a significant positive influence of formalism beliefs.

How can this development and the predictive power of beliefs be explained? Students' loss of interest is similar to the loss of interest reported for future higher secondary teachers (Liebendörfer, 2018; Rach & Heinze, 2017). An important reason for their loss of interest lies in the restrictions in students' self-determination. Formal mathematics requires competencies in handling symbols and working with definitions, that cause students problems in solving their tasks and it may even become difficult to understand the task itself. In such situations, competence and autonomy are hard to perceive (Daskalogianni & Simpson, 2001; Liebendörfer, 2018); however, they are necessary for a positive interest development (Krapp, 2005). Students who share a more formal view on mathematics may better handle the "formal world" (e.g. proving theorems) at this point and see its elegance and use. The changes in the second semester might then be an adaption of the students to the new situation. Our data thus underline the idea that beliefs that fit the mathematics addressed in future may help taking interest (Schukajlow & Liebendörfer, 2017).

Strengths, Limitations and Practical Implications

One strength of our study is the longitudinal sample that revealed differences between correlations and predictors. One limitation is that it is possible that more interested students have a greater willingness to participate in the testing thus affecting the results. We should further mention that our study could not cover students' prior knowledge, performance, and other motivational factors, which most likely interact with interest and its development.

Our results shed some new light on interventions, which mainly focus on dynamic beliefs (e.g., Grootenboer, 2008). Formalism beliefs should not be seen as something obstructive but can also help students take an interest in mathematics.

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Rethinking Authentic Assessment in Mathematics Education: A Holistic Review

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Authentic assessment is developed for overcoming the problems caused by standardized tests. In mathematics education, many economies regard students' problem-solving ability as the essential goal of mathematics education and foster students' mathematical literacy by applying mathematics teaching standards related to authenticity assessment. However, implementing authentic assessment in classroom teaching or integrating it with regular evaluations is still a challenge for teachers. This holistic review was conducted using the EBSCO host Library. Authentic assessment is identified as a tool and as a task in mathematics education. The applied characteristics in PISA, STEM, and e-learning environment are also summarized to provide suggestions for mathematics educators in utilizing authentic assessment.

Introduction

Standardized test popular in the last century can provide evaluation information about students' learning performance in general. However, this form of assessment provides inadequate information on students' learning process, especially without giving diagnostic information to instructors (Allsopp et al., 2008). In contrast, some teachers found students responded naturally in authentic assessment because they did not realize that they were taking a "test" and felt authentic "tests" were easier than standardized tests (Gao & Grisham-Brown, 2011). According to Wiggins's study (1990), the assessment is authentic only when it directly examines students' performance on worthy intellectual tasks. Svinicki (2004) further explained that authentic assessment based on students' activities was close to real-life performance, instead of imitating by paper-and-pencil or even computer-drill-and-practice-type tests. In mathematics education, it is important for instruction to provide students more opportunities to transfer the knowledge or skills from previous learning experiences to new contexts in order to resolve meaningful tasks (Bottge, 2001). Authentic assessment could play such a crucial role. However, there still exists many obstacles for teacher to implement authentic assessment in teaching (Gao & Grisham-Brown, 2011; Hernández & Brendefur, 2003), such as teachers' understanding of authentic assessment, the times of planning and preparing authentic tasks, and the application of authentic assessments in current e-learning environment or science, technology, engineering and mathematics (STEM) education.

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Besides, authentic context is an imperative factor in design authentic activities or tasks. However, it can but not necessarily improve students' motivation since authenticity of a task sometimes is not apparent for learners, which means the task designer must foresee how students will recognize the authenticity of the context and the question, and whether they are capable of understanding in the context of the task (Vos, 2018). Thus, it is challenging for teachers to look for authentic and meaningful activities and apply authentic assessment to students (Cankoy, 2011). This study conducts a holistic review on authentic assessment-related literature and provides suitable suggestions for mathematics teachers or educators in the design and application of authentic assessment in practical instruction.

METHODOLOGY

The holistic review puts focus on authentic assessment in mathematics education. We search full-text articles though priors research terms: authentic assessment, authentic intellectual work, authentic instruction (Villarroel, Bloxham, Bruna, Bruna, & Herrera-Seda, 2018), and authentic task (Murphy, Fox, Freeman, & Hughes, 2017) along with "math*" using EBSCO host Library from 1988 to 2020. We have searched 113 articles in the initial review work. After removing the external words in the article title, such as "historical", "discourse intonation" and "geographical", there were 107 articles chosen in next phase. The researcher reviewed these articles to judge whether they suited for this review study. The topic including criteria is the research issues focus on mathematics education and authentic assessment. The former is easy to judge, but some articles study area is not clear, especially in some STEM studies. Articles which used "authentic" as a descriptor but did not actually focus on real-life contexts or assessment were excluded from the review. For example, Swaffield (2011) clarified the meaning of "authentic" as "genuine" but did not associate it with the term "authentic assessment". The frequency of the term "authentic" and its synonyms was generally low in these articles. Finally, there were around 70 articles retained after two rounds of screening.

THE NATURE OF AUTHENTIC ASSESSMENT

The term "authentic" was first used formally in the area of assessment by Archbald and Newmann (1988). In their studies, "worthwhile, significant, and meaningful tasks" are essential for ensuring the authentic assessment system's validity. In order to design authentic problems or activities, researchers proposed several standards. Wiggins (1998) provided six standards for authentic assessment problems. For example, authentic assessment problem shall be realistic; it requires judgement and innovation; it asks the student to "do" the subject; it replicates or simulates the contexts in which adults are "tested" in the workplace, in civic life, and in personal life; it assesses the student's ability to efficiently and effectively use a repertoire of knowledge and skill to negotiate a complex task; and it allows appropriate opportunities to rehearse, practice, consult resources, and get feedback on and refine performances and produces.

Based on previous literature, Herrington and Oliver (2000) identified nine elements for designing authentic learning. For instance, three authentic related principals of these elements are "*provide authentic contexts that reflect the way the knowledge will be used in real life*". "*provide authentic activities*", and "*provide for authentic assessment of learning within the tasks*" In summary, an authentic learning activity shall contain authentic contexts, authentic tasks (activities), and authentic assessment. Meaningful and realistic context (authenticity) is a prerequisite for authentic learning, and influences of authentic tasks (activities). For example, primary teacher can provide authentic context about money in the classroom, and the economic concept of interest can improve students' computation skills if the instructor design suitable authentic tasks and evaluation criteria (Althauser & Harter, 2016).

AUTHENTIC ASSESSMENT IN MATHEMATICS EDUCATION

Traditional mathematics education isolates content knowledge from real-life situations and mainly on tedious and repetitive mathematical problems solving (Kerekes, Diglio, & King, 2009). However, when learning in this way, children will always easily forget the mathematics knowledge and will not know how to apply mathematical knowledge to unfamiliar contexts (Freudenthal, 1973).

A possible way to link authentic assessment and mathematics teaching could be problem-solving approach, which has been advocated by National Council of Teachers of Mathematics (NCTM) in 1989. Instead of rote memorisation, "understanding mathematics" and "student-centred" learning is promoted with the reform in mathematics education since the NCTM documents of the 1980s (Suurtamm, 2004). To meet the demands in society for employable school leavers, more authenticity teaching has led to an increase in the demand for authentic assessment in mathematics (Lajoie, 1995). However, Hernández and Brendefur (2003) found that most teachers had a relatively shallow cognition of authenticity in mathematics education, tending to confine it to daily life and future occupation. The main obstacles include many aspects, such as time-consuming on planning and preparing tasks, insufficient participants, limited materials and funding, and outdated teacher professional development. which put more pressure to teachers, hence move them away from authentic instruction (Gao & Grisham-Brown, 2011). Only a few teachers could extend the authentic classroom beyond the connection with the real-life context and get the chance to problem-solving critically (Hernández & Brendefur, 2003). Below, we will interpret authentic assessment in mathematics education from three different perspectives.

Authentic assessment as a tool

As an assessment tool, authentic assessment can be carried out during the mathematics learning process. There are many different methods, such as cases, portfolios, exhibitions of performance, and problem-based inquiries are identified as tools to enhance students' learning and support teachers in the classroom (Darling-Hammond &Snyder, 2000). Reikerås, Løge and Knivsberg (2012) found that authentic assessment material may be a useful tool to cognize toddlers' mathematical competences for kindergarten teachers. Through structured observation performance of young children in their familiar environment, preschool teachers can distinguish stimuli of toddlers to acquire various abilities and provide professional help when needed. It also confirms the finding of Gao and Grisham-Brown (2011), teachers applied the authentic assessment to gather students' information and found that personalization did not reduce the accuracy of evolution results in authentic assessment. Children are relatively fragile in their relationship with adults. Hence, authentic assessment is selected for testing and evaluation to avoid mental and physical harm to young children, allowing them to fully express themselves in a comfortable environment (Reikerås, Moser, & Tønnessen, 2017).

Authentic assessment as a task

A situated context is an important element in designing authentic activities. If the context is related to daily life in problem-based learning environment, the instruction and assessment can improve the learners' mathematical thinking, understanding, and internalization (Kerekes et al., 2009). Authentic assessment results are consistent with students' learning processes from knowledge, competencies, and transfer. Thus, an authentic assessment could be regarded as a learning task to students. Through these tasks, the students' participation was enhanced in authentic assessment (Newmann, 1996). Some assessment tasks that allow students to construct knowledge related to real-world context are more meaningful than traditional assessment (Koh, 2011). In a cooperative learning environment, it is noted by Lowrie (2011), authentic assessment can be more complicated. Authenticity can be shown through solving problems combined with individual experience (Lowrie, 2011). Hernández and Brendefur (2003) found that when students have sufficient opportunities to explain generalizations, classifications, and relationships related to situations, problems, topics, or to defend their ideas, the course is considered more authentic. Consequently, emphasising the collaborative situation in authentic assessment is crucial.

Authentic assessment as a process in PISA

Although Organization for Economic Cooperation and Development (OCED) (2013) claims that the assessment materials of Programme for International Student Assessment (PISA) emphasize authenticity, and students will achieve authentic assessments. However, compared with authentic assessment standards (Wiggins, 1998), some open-end questions in PISA merely need students to answer few words or short sentences, which are less complicated than tasks in daily life (Koh & Chapman, 2018). In PISA 2012 (OECD, 2013), computer-based assessment of mathematics is provided optionally for participants because of the high usage of computer competence as a 21st-century skill of mathematics literacy, and improvement of interaction, authenticity and engagement.

AUTHENTIC ASSESSMENT IN CURRENT LEARNING ENVIRONMENT

With the rapid development of information technology (IT) and its close integration with education, the e-learning approach has been advocated at schools. Students are encouraged to integrate and apply knowledge and skills in practical situations to meet the changes in 21st century. In fact, since the mathematics curriculum reform in 2000, learning by using IT and across curriculum has been promoted. The foci of assessment forms have been changed from assessment of learning to assessment for learning, and assessment as learning. Authentic assessment is closely related to the real world and the learning process of students. Because of its characteristic, the authentic assessment is more valuable and meaningful under current learning environment.

Authentic assessment in e-learning environment

Shaffer and Resnick (1999) maintain that new media can create an authentic situation for learning in connectivity, authenticity, and epistemological pluralism. Through informative technologies, the communication avoids geographical limitations and changes the quality of learning experience; at the same time, these technologies provide an authentic environment for assessment (McLoughlin & Luca, 2001). However, authenticity in online learning makes teachers' instruction difficult because, in the digital world, regardless of academic status, teachers must continuously redefine themselves as lifelong learners and set an example for students (Barber, King, & Buchanan, 2015). Additionally, information and computer skills and task re-designing are also challenging for teachers (Barber et al., 2015). Instead of utilizing the software itself, Herrington and Oliver (2000) found it necessary to conduct an authentic assessment in multimedia as part of the learning environment.

Authentic assessment in STEM education

With the development of globalization, to rise the competitive competency, the study and careers of information technology and STEM get the world's attention (Koh & Chapman, 2018). STEM education is used to solve real-world situations through a design-based problem-solving process (Williams, 2011). Tan, Nicholas, Scribner and Francis (2019) noted the mathematics tasks would become more complicated after integrating into the STEM framework. Based on the student-centred instruction and meeting different students' needs, instructors who plan and construct authentic and interdisciplinary classroom activities should be encouraged, which is also adopted to foster students' 21st-century skills (Mahanin, Shahrill, Tan, & Mahadi, 2017).

CONCLUSION

This article summarizes and identifies the inherent reasons that hinder the application of authentic assessment in mathematics teaching. These reasons include student factors (e.g., students' characteristic), teacher factors (e.g., cognitive deviations about authentic assessment), and external factors (e.g., lack of time and funding, and insufficient related professional development training). Most of the relevant recommendations in the literature are for external factors. For example, Dennis and O'hair (2010) proposed that collaboration between teachers promotes teachers' professional competencies and saves the overall time for conducting an authentic assessment. Many research papers mention a combination of authentic assessment and on-going high-quality vocational training to assess instructor adapt to the classroom that facilitates assessment and have more chance bring benefits for students' learning (Guskey, 1994), and promote 21st mathematics classroom (Mahanin et al., 2017). This kind of effort will also alleviate teachers' perception of authentic assessment to a certain extent. Future research should focus on directly solving or reducing student factors that hinder the application of authentic assessment in mathematics education.

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INTRODUCING PROBLEM SOLVING TO A CULTURE OF LONGSTANDING HISTORY OF MEMORISATION

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In light of the ongoing radical national curricular reform agenda, this research aims at investigating Egyptian teacher readiness to embrace a shift away from traditional instruction and towards the integration of mathematical problem-solving in the context of their daily instruction. Grounded on the Task Analysis Guide, the research maps out results of four mathematics teacher focus groups all centered around mathematical problem-solving task classroom integration schemes. Results confirm the anticipated gap between the national vision of the reformed mathematics classroom and the likely implementation of the reformed curricular agenda as reported by the teachers. The research calls for a contextualized approach to the distribution of the reform agenda.

INTRODUCTION

Multiple scholars have contended a repeating historical pattern of educational reform initiatives proceeding from a time of political turbulence (Burde, Kapit, Wahl, Guven & Skarpeteig, 2017). Cohen and Ball (1999) argue that for such reform initiatives to be effective, it is important to fully capture the history and cultural identity of a given learning context (Cohen & Ball, 1999). This research particularly focuses on the case of the Egyptian system reform in mathematics education which has been recently introduced by the Ministry of Education (MOE). It explores how this reform is locally and contextually perceived.

In the recent years, ongoing national and international collaboration schemes are operating to serve the purpose of a complete system level reformation in the national schooling curriculum (MOE, 2019). Part of this reform targets to incorporate the element of problem-solving into the national mathematics curriculum (MOE, 2019). The reform is reported to present a radical shift away from the longstanding teaching culture of memorization (Megahed, 2017). This research seeks to investigate the readiness of ground level practitioners to embrace the reformation, which is mainly administered by the MOE. The study of ground level acceptance to hierarchically imposed reform in mathematics education is relevant because of its transferability to other high distance power relation learning contexts.

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LITERATURE REVIEW

The relationship between education and power has received substantial attention in scholarship. Multiple scholars (Apple, 1982; Arendt, 1993; Kupfer, 2015) have reflected on the Weberian view which envisions education as a tool of control and governance in the hands of policy makers. Literature on power and education in the Egyptian context has asserted the Weberian perspective (Hargreaves, 2006; Megahed, 2017). Naguib (2006) argues that educational schemes in Egypt serve the purpose of creating and re-creating a culture of despotism. In his works, he investigated the different hierarchical stages of the Egyptian educational system, arguing for the educational schemes to be created and communicated by decision makers at every level of the hierarchical ladder with the target of sustaining power distances and re-creating a culture of oppression (Naguib, 2006). According to Naguib (2006), school leaders lack the decision-making autonomy. This sense of oppression is cascaded all the way down across the schooling hierarchy.

In their reflections on mathematics curricular reform initiatives in the past three decades, Cohen and Ball (1999) acknowledge the incapacity of mere curricular reform and teacher training initiatives to capture the full contextual complexity needed in order to achieve enduring and sustainable improvement in students' mathematical classroom experiences. According to the Cohen and Ball (1999), schools are complex social institutions and a shift in the teaching mindset is only possible when buy in to the reform initiative is established at all levels of the schooling enterprise.

With these perspectives in mind, and in view of the national mathematics curricular reform launched by the MOE in Egypt, this research seeks to investigate the ground level contextual buy in of Egyptian teachers to a hierarchically channeled (Al-Ashkar, 2018) reform agenda. The research addresses the following research question:

- In light of the ongoing government-led mathematics education system reform initiative, which targets a shift towards mathematical problem-solving, how do Egyptian mathematics teachers relate to a problem-solving oriented lesson structure?

To study the integration of mathematical problem-solving into the classroom context, researcher found the work of Stein, Smith, Henningsen, and Silver (2000) to be of significant relevance. As part of a wider mathematics education system reform project, Stein et. al. (2000) observed mathematics classrooms, where problem-solving was incorporated into teachers' daily practices. Based on their observations, Stein et al. developed a schematic to evaluate this integration process. They differentiated between two stages of classroom integration; namely the setup stage and the implementation stage (Figure 1). The former refers to the timeframe prior to students' commencement of working on the task along with the approach the mathematics teacher adopts during this time. The latter refers to the timeframe where students are actively engaged with the task and the approach the teacher adopts during this time. Stein et al. (2000) argue that both stages of the mathematical problem-solving task integration are tightly related (Figure 1). In other words, it is unlikely for students to experience the mathematical task as a problem-solving task during the task implementation stage if already at the setup stage, the teacher has reduced the task complexity by, for instance, providing the necessary procedure for solving the task.

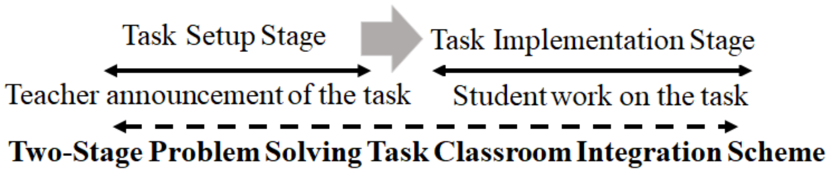

Figure 1. Two-stage problem-solving task classroom integration scheme

In their wider study, Stein et al. (2000) present the Mathematical Task Analysis Guide as a framework to evaluate mathematical tasks. They distinguish between four types of mathematical tasks, namely ‘memorization tasks’, ‘procedures without connection tasks’, ‘procedures with connection tasks’, and ‘doing mathematics tasks’ (Stein et al., 2000, p.20). The four task types vary in view of their cognitive level demand. At one end of the cognitive level demand spectrum, memorization tasks are mostly straightforward and simplistic. On the other end, the tackling of ‘doing mathematics tasks’ requires conceptual understanding, inquiry into the unknown, going through a process of constant re-examination of suggested solution approaches and pattern matching. Researcher chooses to refer to the ‘doing mathematics tasks’ as ‘problem-solving tasks’ for the purposes of this research, due to their high resemblance with described problem solving task features in literature (Callejo & Villa, 2009; Stylianides & Stylianides, 2014). In my investigation, researcher relies on Stylianides and Stylianides’s (2014) definition of problem-solving, which in turn relies of Callejo and Villa’s (2009) definition of a problem situation. In accordance with this view, the act of problem-solving is envisioned as finding a solution to a situation where there is little immediate access to a process that relates the data available to a required unknown.

In this research, researcher investigates how Egyptian mathematics teachers relate to the integration scheme of a given mathematical problem-solving task. Researcher bases the investigation on the depicted two-stage classroom integration scheme (Figure 1). Based on the analysis of the task setup and task implementation stages, researcher then utilize the Task Analysis Guide to map the likely mathematical task experience that would result from teachers’ reported classroom integration choices.

METHODOLOGY

This work adopts a multiple case study methodology, reporting results of a sample of four case studies: Each comprising a teacher focus group. Each focus group incorporates a group of teachers that teach mathematics at the same school. All schools that took part in this research are schools that adopt the national curriculum of mathematics and that are subjected to the suggested reform initiative reported earlier. All schools that took part in this research have been operating the highly centralized, highly traditional, memorization oriented national mathematics curriculum (Naguib,2006) at least for the past 30 years. The aim was to uncover how mathematics teachers, that are expected to radically shift from a longstanding history of traditional instruction, relate to mathematical problem solving when integrated in everyday practice. The wider study, from which this work is derived, also explores how group power dynamics – particularly in a collective Middle Eastern culture (Al-Omari, 2003)- influence the choice of buy in to problem-solving, being a foreign method of classroom mathematical task integration; hence the choice of focus groups as a data collection method.

The following three-stage data collection protocol was replicated across the four case studies. Firstly, teachers were provided with a mathematical problem-solving task and were given an open timeframe to grapple with the task. The task was then discussed in the group. Afterwards, researcher presented different approaches to tackle the task, each ascribing to a different level of mathematical cognitive demand as reported in the Task Analysis Guide (Stein et. al., 2000). Secondly, after being presented with the task and the various suggested approaches for addressing it, teachers were asked to each design their own lesson plan, outlining how they would integrate the presented task into the context of their daily instruction. Thirdly, the group was provided with three narrative scenarios, each presenting a different classroom integration approach of the same mathematical problem-solving task. The group discussion of the narrative scenarios was guided by a set of questions, which are theoretically grounded in the two-stage problem solving classroom integration framework (Stein et.al.,2000). The idea was to triangulate between an inductive and a deductive data collection activity (devising lesson plans and reflecting on narrative accounts of lesson plans) in order to ensure higher validity of results and to more holistically capture how teachers relate to a mathematical problem-solving task in view of their daily contexts of instruction.

The mathematical task, that the data collection activity has been centered around is reported by Stein et al. (2000) as being a highly demanding problem-solving task. Hence, in view of the problem-solving oriented national reform initiative, this task could be considered as a sample mathematical task which Egyptian teachers would be asked to integrate in their daily practices. The narrative scenarios present a culturally contextualized version of the narrative scenarios outlined by Stein et al. (2000). Setup and implementation of the same mathematical problem-solving task vary in each scenario, resulting in a different classroom experience of the task in every narrative.

DATA ANALYSIS AND RESULTS

Reported lesson plans as well as reported feedback to the three narratives were captured for each case study (schools S1-S4). Repeating trends were captured, coded, and mapped in view of the mathematical task features reported in Stein et. al.'s (2000) Task Analysis Guide (Table 1) which acts as an analytical framework of this research (Robson, 2000). Table 1 presents the resulting coding scheme. It adopts the frequency count analysis (Yin, 2009) to denote the most frequent patterns reported in the datasets of each case study with an X. The mapping against the Task Analysis Guide features targets to unravel the likely classroom task experience for every case study. To ensure validity of the results (Yin, 2009), multiple case study results (of schools with a similar history in a culture of memorization-based instruction) are cross compared. Table 1 also maps the results across the cases against the two-stage classroom integration scheme (task setup and task implementation) (Stein et. al., 2000).

	Mathematical Task Integration Stage	Findings from schools				Task Analysis Guide Mapping
	I. Setup Stage	S1	S2	S3	S4	
	I.1 What are students expected to do?					
1	Reproduce memorised knowledge					Memorisation
2	Adopt an already provided procedure	X		X	X	Procedure
3	Relate to presented conceptual knowledge		X			Connection
4	Develop a yet unknown approach					Problem Solving
	I.2 How are students expected to do it?					
1	Compete against each other to recall a memorised method	X			X	Memorisation
2	Follow the steps outlined by the teacher			X		Procedure
3	Make use of conceptual knowledge		X			Connection
4	Construct knowledge by connecting peers' input in order to collectively develop the approach to solve a mathematical problem					Problem Solving
	II. Implementation Stage	S1	S2	S3	S4	
	II.1 The target of working through the task					
1	The task is worked through with the target of recalling a memorised method	X				Memorisation
2	The task is worked through as a means to practise a studied procedure		X	X	X	Procedure
3	The task is worked through as an application of familiar concepts					Connection
4	The task is worked through as a challenge to discover new knowledge					Problem Solving

	II.2 The Meaning Making Process					
1	The teacher equips students with the capacity to use memorised procedures			X	X	Memorisation
2	The teacher equips students with as many mathematical procedures as possible	X				Procedure
3	The teacher draws on conceptual connections		X			Connection
4	The teacher facilitates a process of continuous inquiry to solve the task					Problem Solving
	II.3 The Task Ownership					
1	The teacher is the sole owner of the task	X	X	X	X	Memorisation
2	The teacher tightly controls the student exploration process					Procedure
3	The teacher guides students as they explore the task on their own					Connection
4	The teacher avoids offering input as it is viewed as disrupting students' independent exploration					Problem Solving
	II.4 The Time Management					
1	Any lesson time allocated for student exploration is considered wasted time	X			X	Memorisation
2	Minimal time is dedicated to independent student exploration			X		Procedure
3	The lesson time is split into teacher explanation followed by student application		X			Connection
4	Most lesson time is dedicated to independent student exploration					Problem Solving
	I. 3 What resources are the students provided with?					
1	The worked through sample task				X	Memorisation
2	The presentation of a mathematical procedure			X		Procedure
3	The presentation of a mathematical concept	X	X			Connection
4	The announcement of the problem without further supporting information					Problem Solving

Table 1: Mapping and across analysis of results

DISCUSSION

Table 1 shows how across the four schools, there were very little traces of the teachers setting up for- and implementing the task in such a way that its cognitive demand on problem solving (code 4) gets maintained and fully experienced in the classroom. Instead, teachers consistently, in almost all focus groups, seemed to prefer providing students with clear instructions on how to solve the task already at the setup stage (Table 1-I.1). Across all focus groups, teachers unanimously expressed that the implementation of the task was the role of the teacher and that students were viewed as passive recipients of the task implementation process (Table 1-II.3). Students were only given the chance to work on a mathematical task themselves once a similar task has already been presented by the teacher. In two of the focus groups (S1 and S2), there were traces of teachers emphasizing the importance of the establishing conceptual connections in relation to the mathematical task (Table 1-I.3). The wider focus group discussion revealed that in these cases, the task was used as a tool to explain a mathematical concept, hence the connection making element would be entirely performed by the teacher.

In line with Cohen and Ball (1999), the results in Table 1 confirm the incapacity of a reformed curriculum to alone penetrate the contextual barrier of school micro-cultures that have a longstanding history of traditional instruction. This research suggests that, beyond the efforts exerted in reforming the mathematics curriculum, the national roadmap needs to incorporate locally feasible implementation initiatives tailored to suit the unique and complex reality of every school microculture.

Beyond the case of Egypt, the research offers a framework for studying ground level buy in to mathematics reform initiatives in similar high-power contexts. It also sheds light on how problem-solving is perceived in the Middle Eastern culture. This understanding is particularly relevant, considering the growing global representation (Burde et al., 2007) of the Middle Eastern culture in classrooms around the world.

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THE SEMIOSPHERE LENS TO LOOK AT LESSON STUDY PRACTICES IN THEIR CULTURAL CONTEXT: A CASE STUDY

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This paper presents an experience of Lesson Study involving primary school teachers in a school in North Italy. Researcher will show how Lotman's Semiosphere construct can be used to analyze cultural and semiotic aspects of Lesson Study practices inserted in the Italian cultural context, as practices intended to enhance collaboration among teachers and their critical thinking on professional issues. Researcher will also show how this analysis may complement another analysis, performed in the perspective of the Chevallard's Anthropological Theory of didactics, and concerning the institutional aspects of the Lesson Study experience.

INTRODUCTION

To meet the new challenges of mathematical education related to changes in workplaces and more generally in society, the OECD "Teachers Matter" report defines "teacher quality" as the "most important school variable influencing student achievement" (OECD, 2005, p.2). In Italy, the National Plan for the Professional Development of Teachers, scheduled for 2016-2019 but still in force, considers the professional development of in-service (and pre-service) teachers "compulsory, permanent and structural" by law. In particular, the plan aims "to promote reflective thinking and collaboration" in all its forms.

The Lesson Study methodology (LS) can be considered one of the teachers' professional development methodologies suited to meet the Italian institutional requirements. Indeed, LS is "a teacher professional development approach, originating in Asia" (Huang, Takahashi, & da Ponte, 2019, p.3), that focuses on collaboration and co-responsibility. As Hummes, Font, and Breda (2018, p.69) exhaustively explain, we can consider LS as "a very broad and non-guided reflection phase of the professional development of mathematics teachers". A LS cycle is constituted of three consecutive moments: planning a lesson in a given class, teaching and mutual observation, and discussion. After this last moment, the LS working group can choose to start the cycle again for a new class. Within this cycle teachers are led, and in this free, to reflect for an improvement of the teaching and learning process of mathematics. We may observe that the need for a practice like LS is even stronger when teachers need to face the contradiction between current beliefs and new ideas. This contradiction must be resolved over time in a supportive community with mutual trust and respect. LS stimulates precisely an openly critical dialogue among educators about the teaching and learning processes collectively observed.

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Besides, international studies such as the OECD-PISA surveys encourage each of us to compare deal with the results of other countries, especially in the area of mathematics, and to study teaching and teacher professional development practices that are at the heart of the educational success of these countries, analyzing whether and how there can be a correlation between student learning and teacher professional development. However, studies like those by Kim, Ferrini-Mundy, and Sfard (2012) or Bartolini Bussi and Martignone (2013), suggest that teacher professional development is not the only element that affects the quality of student learning: We must take into consideration the cultural aspects that have an impact on teacher professional development, on teaching and and learning. In this perspective, my purpose is to study Lotman's Semiosphere construct (Lotman, 1990) as a theoretical lens – prospectively networking with others – to read the cultural aspects in teachers' practices. Hence, my research questions are as follows: How and which aspects of teachers' culture, relevant for their own professionality, are highlighted by the Semiosphere? Which elements of the Semiosphere are effective in analyzing teachers' practices?

The semiotic space is my main unit of analysis, specifically researcher will investigate how teachers' collective practice are observable in it within a LS experience: Since, as Geoffrey Saxe (2014) states, it is within collective practices where we can identify firmly the relationships between culture and cognition, and therefore between culture and reflection.

THEORETICAL FRAMEWORK

In the literature in mathematics education the systemic aspects (Font, 2002, pp.143-156) concerning the links between teaching and learning practices and organization and social constraints are the subject of important theoretical elaborations, in particular that of Yves Chevallard. His object of study is a ternary relation: the didactic system (students, teacher, mathematical knowledge), which cannot be understood except in relation to the (external) environment that surrounds it, the teaching system and society. The relationship between the system and its surroundings passes through the process of *didactic transposition* that converts scholarly knowledge, initially into knowledge to be taught and then into taught knowledge - and finally into learnt knowledge. The "intermediate area between the teaching system and society" is the space that Chevallard (1981, p.8) defined as "the noosphere: (the sphere where one thinks) about the teaching system". Bosch and Gascon (2006) warn us, however, that it can happen that the school may lose the logic of the knowledge to be taught, i.e. the questions that motivated the creation of this knowledge, stopping at the lowest *levels* of what Chevallard has defined as *didactic co-determination* (Figure 1).

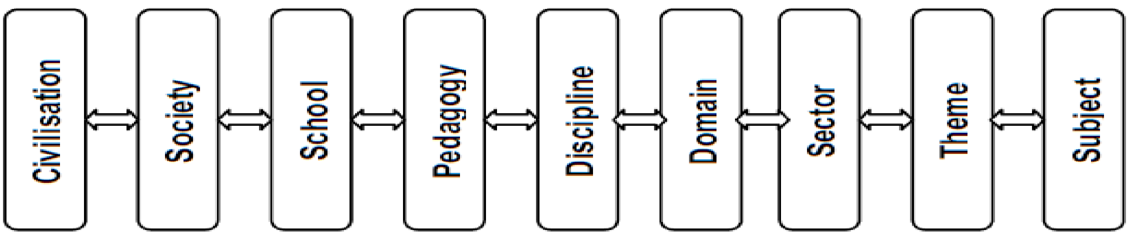

Figure 1: Didactic and mathematical co-determination levels (Bosch & Gascon, 2006, p.61).

In-service teachers' professional development, focused on critical reflection and on experiences of sharing, thinking, and collaborating, are aimed at awakening, or renewing the knowledge to be taught. Each choice, each element living within the didactic and teaching system, is dictated by one (or many) in-depth reflection on the way this content is structured and taking into account the conditions and constraints posed by the different levels of co-determination during the didactic transposition process. However, since I could not leave aside the cultural and semiotic aspects – since, furthermore, in mathematics the signs are themselves objects of mathematics –, researcher attempted to verify if in the perspective of Lotman's semiotics of culture there could exist theoretical tools suitable to account for these aspects.

According to Lotman (1990), semiotic knowledge is embedded in culture, that is a complex system of signs. Studying semiotic aspects, we study the correlation between the different sign systems that constitute culture. Moreover, the systems do not present elements in isolation, but are always immersed in a homogeneous semiotic continuum. In this way the idea of culture explains the necessary notion of dependence and reciprocity between systems in which the necessity of *the other* (another person, another culture) is fundamental. To express it, Lotman coins the term *Semiosphere*:

> *As an example […], imagine a museum hall where exhibits from different periods are on display, along with inscriptions in known and unknown languages, and instructions for decoding them; […] imagine all this as a single mechanism (which in a certain sense it is). This is an image of the Semiosphere. […] all elements of the Semiosphere are in dynamic, not static, correlations whose terms are constantly changing. (Lotman, 1990, pp.126-127)*

Lotman's semiotics differs from the others (Peirce, Eco, Greimas) because, instead of using unity (sign) as a primary element of study, he believes that only a global understanding of the culture system can lead to the recognition of the units that make it up. The smallest functioning mechanism of the process by which an expression takes on the value of a sign, the unity of semiosis, is not a separate element but the entire semiotic space of the culture in question. In particular a specific culture (e.g. the Italian culture of teaching-learning mathematics, in our case) is a semiosphere that lives immersed in the global "all cultures" semiosphere and it can exist as a system only in relation to the cultures with which it continuously exchanges cultural elements: In this sense "it seethes like the sun" (Lotman, 1990, p.150). The internal translation (in its semiotic meaning) currents express the *asymmetric character* of the Semiosphere. In fact, "besides the structurally organized language, [the semiosphere] is crowded with partial languages [semiotically asymmetrical – i.e. without mutual semiotic correspondences with the previous], [...] which can be bearers of semiosis if they are included in the semiotic context" (Lotman, 1990, p.127-128). Asymmetry between languages engenders the *dialogue*. In fact, the whole for Lotman consists of at least two texts, which dialogue with each other thanks to their constitutive asymmetry.

In potential continuity (to be further elaborated – see Discussion) with Chevallard's systemic-institutional approach, Lotman's Semiosphere might be considered as a dynamic (never motionless, always bubbling and exchanging) integration (considering also the semiotic and, more generally, cultural aspects) of noosphere.

METHODOLOGY AND DATA COLLECTION

A first experimental activity of LS cycles in a primary school near Turin was carried out starting from the LS experiences conducted in Reggio Emilia (Bartolini Bussi & Ramploud, 2018).

The working group is made up of six people: the researcher (Carola, a PhD student), a retired former teacher-researcher (Ezio) and four teachers who teach in different primary school classes of the same institute. Three are 1st-grade teachers: Michela is a support teacher for low achievers, Nicoletta teaches Italian in her class, Marcello teaches mathematics, science, history, geography, and English. Valentina, the fourth teacher, teaches mathematics and science in 3rd grade. The Italian school system is characterized by high flexibility in teaching in primary school. Teachers teach several subjects and even the support teacher, supporting the class in which there is the low achiever, can take charge of teaching subjects to the whole class, according to his skills, if the team deems it appropriate.

The first part of the experiment consists of three complete cycles in the three 1st-grade classes. The topic of the lesson is the introduction of the "plus" sign for the addition and its institutionalization. The specific goal for children is to understand the concept of addition as the sum of two quantities in its meaning of "putting together" and relate it to the signs of mathematical language. In the second part of the experience, consistently with the previous three cycle, a new lesson is carried out in the 3rd-grade. The designed activity is part of the educational path that includes the knowledge of weight measurements and the study of state transitions, via experiments. The aim is to accompany students in reinvesting their mathematical knowledge and argumentation skills with respect to the transversely of the disciplines. Each teacher implements the lesson in his or her class but in the total co-responsibility of the group, which is there in agreement with the school headmaster. During the lesson, the other participants play the role of active observers: in 1st-grade classes they interact with the students as "hand-lenders", i.e. they transcribe the thoughts of not yet writing-skilled children.

The experience, covering all four cycles, was carried out from November 2018 to April 2019. For a total duration of 24 hours of group work. All the design (4 hours of initial formation and 8 hours of design de facto, 2 per cycle) and discussion moments (8 hours, 2 per cycle), but also the classroom lessons (1 hour in each class – cycle –, for a total of 4 hours), were video-recorded. Some extracts from these recordings were then transcribed by the researcher. In addition, for each planned lesson, the group produced a Lesson Plan (Bartolini Bussi & Ramploud, 2018): a written document – a table – that collects the entire lesson planning, the objectives the group chose for the lesson, the positioning of the lesson within the long-term planning of the class, and the educational intentionality behind each choice of the group.

In the next section researcher will present a first analysis of a small transcription excerpt with the Semiosphere. Because of the nature of the Semiosphere, the excerpt is not self-sufficient: other extracts are required to grasp how the elements external to the teachers' and class' semiosphere are gradually translated and understood. Researcher uses the asymmetry within the semiosphere to grasp how teachers' collective practices evolve. Researcher looks at the teachers' discourses - i.e. words and vocabulary used, references to the institutional and cultural aspects of their professional background.

SOME EXCERPTS AND THEIR ANALYSIS

Here is a short extract from the exact beginning of the first meeting of the second LS cycle. Among all the data researcher has chosen to report just this because it represents a turning point for the teachers of the group: They have "now appropriated the methodology, understood its functioning and potential" (in Nicoletta's words, during the review of the first lesson held in her class), but they are still at the beginning of this professional development path. The lesson is still to be revisited and questioned in its details.

Nicoletta has already implemented and discussed the lesson. The LS group are now in the planning phase of the same lesson for Michela's class. Her children, also in Grade 1, have never worked in pairs. The lesson planned for LS includes an argumentation exercise in pairs on a double purchase: The children in a previous class bought a 12 cents card and 8 cents sample clips. Some of them paid with 20 cents. The key question is how and why they paid 20 if the prices tags were 12 and 8. The LS group is reflecting on what changes to make to the lesson for Michela's class. Here they are thinking about an introductory activity to the lesson, to experience the work in pairs for the first time.

Michela: Now I am talking nonsense. *I looked at the tests you did [referring to Ezio], the problem with the balloons*: For example, it could be... […] you give it to a couple. Because I wanted to rework that one anyway because… I saw it, it is really interesting... also the motivations the students gave. But it could be an idea!

Nicoletta: I believe we could also *do something about the comics* [introduced in the previous design]. […] The scheme is that one of this reasoning […] *that you should... that we want to* re-propose: take two reasonings, do what he [Marcello] said […] That is: what did the children who said "9 plus 6" think before? […] they are balloons, kids […] it's too similar with the LS lesson?

Marcello: [shoulders up] in the sense that it is!... in the sense that they *put together...*

In this brief dialogue we already note some essential aspects that can only be understood if we consider the Semiosphere in which the dialogue takes place:

- Tasks reported by a teacher [Ezio] recognized as an expert by the LS group are chosen instead of those reported in the textbook. The teachers had already declared from the beginning that they did not want to rely on the textbook.

To better understand the sphere in which this dialogue arises, researcher also report the following excerpt. Marcello describes his difficulty in relating to textbooks and institutional meta-didactic structures during the group's first planning meeting.

Marcello: [...] maybe you find a very fixed structure: Lessons, notebooks, but if you don't understand... [...] staring at the notebook and "making the notebook" is very far from me, even if it gives you a lot of [confidence]... I mean, I live a lot of anxiety, sometimes I get lost, because if you don't have a structure... but at the same time, I can't really get into it, because I am not interested in doing that. I think the best thing would be to meet [each other]. But of course, the times are what they are [...] I had to write all the subjects I do. Which is a lot. […] I want to talk to people, I want to see the practices, I want to confront myself directly [...] in my opinion the university is too tied to the book [...] seeing things together gives you a sense.

Using Chevallard we can say that the didactic transposition of some practices is not complete. At least in these teachers' beliefs, such practices have not passed through all the necessary levels of co-determination. There is a gap between academic and implemented knowledge. Lotman, analogously, could tell us that the Semiosphere of the group sees local institutional requests ("making the notebook") and national ones (mention of university practices) as external elements. They are currently "written" in a language that, Marcello and Michela declare, is not that of the group today. The identification of critical thinking as a practice of semiotic translation allows researchers, but also teachers themselves, to analyze these practices from a semiotic and not only institutional point of view: a semiospherical dialogue is created. The lens of the Semiosphere allows us to perceive the existing asymmetry between the current school reality (many subjects to teach, no time available) and the Italian university culture of prospective teachers, that is a training ground for the first personal beliefs.

- Graphic, material, and gestural visualizations are preferred to only written text: The idea is to propose to the children a drawing with comics and cartoon price tags. Then the group will choose a theatrical performance.

An excerpt from the implementation of the lesson underlines the embodied feature that the group sought to use.

Michela: So, now, kids, let us focus and work on what Valentina did in the sketch. […] She took a nice moment to think. She looked at the two prices and thought. Did you see Valentina thinking? […] She thought a little before giving me the coins. Okay? Good! So, I will put these two prices [on the board, so you can see them] ... Now, each of you will have a moment to think to what to say in the couple! Because I am asking you to say what Valentina is thinking right now. Nevi, you must do it in pairs! So, you, your thought will have to share it with Mattia, and Mattia will say his.

Teachers are aware of the Western culture in which they are immersed: Abstract thinking, for an Italian primary school child, must be approached gradually (Mellone, Ramploud, Di Paola, & Martignone, 2019, p.8). Grasping abstract thought requires time and continuity. A theatrical text, using bodily movements, returns the desired continuity: It realizes the act of thinking. Then, translating the action into the graphic text (the comics), the group keeps track of the signs that mark the passage from concrete to abstract.

- The shift of attention from the single teacher – who will enter alone in the classroom – to the co-responsibility of the group is the main objective, and for them the beauty, of this work with the LS methodology. A co-responsibility that is not in the usual Italian teachers' Semiosphere. Nicoletta says: [...] *that you should... that we want to re-propose the following time* [...]. Co-responsibility belongs to LS, but not to the Italian class culture. This asymmetry between the LS Semiosphere and that of the group allows a cultural transposition (Mellone et al., 2019) of the teachers' practices: During all the meetings the teachers bring themselves and their educational intentionality into play. They question their teaching practices with the group. Now educational intentionality and objectives are shared: They are meaningful for each member of the group.
- The exact words of the children are repeated. The expression "put together" was how the children of Nicoletta's class had referred to the idea of sum and therefore it becomes the pivot sign of the LS group for the institutionalization of the + sign.

To discuss this last point more thoroughly, researcher adds here a final excerpt from the implementation of the lesson in Michela's class. A student is responding to the problem posed.

Student:	Valentina thought a little bit about how she made the 20 cents.
Carola:	But she has not read 20 anywhere! [...] Was there a 20 written somewhere?
Student:	[…] because she puts them together [the two prices on the price tags].
Carola:	What is it that she put together?
Student:	20 uh... 12 cents and 8 cents. […] she counted in her mind... continually. And she realized that 20, uh... 12 cents and 8 cents make 20.

The task and its implementation guide the children within relational structures, going beyond simple calculation. They look at how the numbers relate to each other. The Semiosphere of the class sees the relational structure still external to itself, but through the pivot sign of "putting together", it translates its meaning.

DISCUSSION AND CONCLUSIONS

In the previous section researcher tried to observe what happened in the LS experience, through the Semiosphere lens. It is just one of the possible ways to look at a teachers' professional development practice.

Researcher can thus answer the research questions; in fact, it is now explicit that asymmetry is the effective element in analyzing collective teachers' practices. It allows us to read the changes in teachers' daily practices when introducing an element belonging to a different culture, such as LS. Semiosphere allows to keep together levels of signification culturally distant from each other. However, there is more, it allows to outline the internal structure of our practices: Our Semiosphere. Such a double look helps researcher and teachers to read the transpositions of the knowledge through the levels of co-determination. Here the critical dialogue and reflection of the teachers, if read through the Semiosphere, do not lose contact with the reality in which they are born. So, the problem of possible integration between Lotman and Chevallard lenses according to the Networking of Theories approach (Radford, 2008) arises spontaneously. The analysis of the institutional aspects and the levels of co-determination seems enriched by a dynamic interchange perspective, and vice versa this can be integrated with the institutional constraints typical of a school system governed by laws. Future studies could tell us about the connection of the two theories as lenses for professional development practices.

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THE EFFECTS OF A PROOF COMPREHENSION TEST ON COMPREHENDING PROOFS WITHOUT WORDS

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This study aims at exploring new ways to promote mathematical proof comprehension. It focuses on students' solutions to a proof activity based on a Proof Without Words (PWW). One hundred eighteen students worked in small groups to elaborate a proof, while having a PWW as an artifact at their disposal. Each student then individually wrote and submitted a proof-attempt and completed a proof comprehension test. We investigate the impact of the comprehension test on students' understanding of the proofs and their self-assessment. Findings show that students improved their proof-products in their answers to the test. They also reported higher proof-understanding after completing it and reduced their self-given-grades for their submitted proofs. We discuss the pedagogical roles the comprehension test served in the proving activity.

INTRODUCTION

Proofs Without Words (PWWs) or *Visual Proofs* are diagrammatical mathematical artifacts in which diagrams or graphs allude to the way of proving mathematical propositions or theorems (Nelsen, 1993). The diagram may contain some mathematical symbols, characters, and calculations to guide the observer, but the use of any written word is obstructed in this genre. PWWs are common in many mathematical domains; however, this paper focuses on a particular geometry PWW task shown in Figure 1 of the Pythagorean Theorem (Garfield's PWW - adapted from Nelsen, 1993, p. 7):

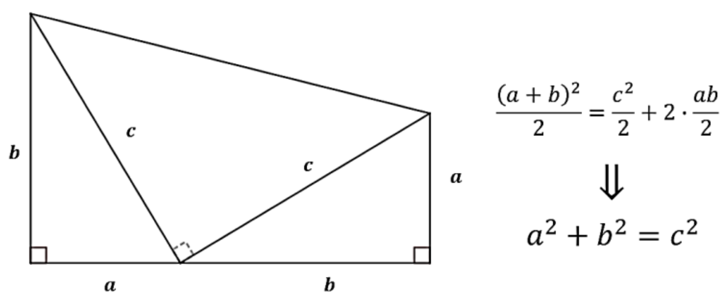

Figure 1: "discover and write down the proof implied by this diagram."

Since their appearance in American journals like the *Mathematics Magazine* and *The College Mathematics Journal* in the seventies, PWWs were designed primarily for mathematicians. When having a PWW at hand, experienced mathematicians can rapidly perform a set of epistemic mental actions to develop a valid proof: decode and transform diagrammatical information into verbalized expressions, construct a chain of warranted

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arguments, and fill in necessary gaps based on prior knowledge. Such mental actions are essential for elaborating an acceptable proof. As Nelsen, a mathematician and the author of three books on PWWs affirms: "Of course, 'proofs without words' are not really proofs" (Nelsen, 1993, p. vi). Nevertheless, mathematicians seem to enjoy and appreciate PWWs because of their elegance, mathematical beauty, and the insights they instigate (Arcavi, 2003). Nelsen goes further and suggests that math teachers share PWWs with their students (Nelsen, 1993, p. vii). But can high-school students use PWWs like experienced mathematicians to elaborate and comprehend proofs?

Research focusing on PWWs in mathematics education is scarce. Hanna and Sidoli (2007) referred to PWWs as a kind of mathematical visualization that represents a whole proof-process. Surveying attitudes towards the role of visualization in mathematics and mathematics education, they claimed that while the role of visualization as an essential aid to mathematical understanding is widely accepted, still, "there is room for more effort aimed at better ways to use visualization in this role" (p. 77). This study is one such effort, examining students' ability to (re-)construct a proof represented by a visualization of a PWW type. Two theoretical concepts are used to assess students' understanding: *proof-comprehension-test* (PCT) and the notion of *gap-filling*.

Proof comprehension and proof comprehension tests (PCTs)

According to Yang and Lin (2008), proof comprehension (PC) means "understanding proofs from the essential elements of knowing how a proof operates and why a proof is right" (p. 60). They developed a model for PC of geometry proof that includes six facets. Their work also emphasizes the immense challenges of observing and assessing PC. Based on Yang and Lin's model, Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff, (2012) suggested a general assessment model for constructing PCTs. They called researchers to *"*use our models to document specific types of comprehension benefits that their innovative proof presentations may have*"* (p. 17). We relied on Mejia-Ramos et al.'s model to generate a PWW Comprehension Test (PWW-CT), which is a specific PCT, to assess students' proof-comprehension of a PWW.

Since self-assessment methods help students better understand learning goals, be more accountable for their learning process, and improve their performance (e.g., Semana & Santos, 2010), we incorporated self-assessment items in the PPW-CT to stimulate students' reflective thinking and as another means to evaluate their understanding.

The notion of gaps and gap-filling

The idea of gap-filling was developed in literary theory, conceptualizing any text as a *system of gaps*, which the reader constantly needs to fill to construct meaning (Perry & Sternberg, 1986). Building on Perry and Sternberg's (1986) theory, Marco, Palatnik, and Schwarz (2021) suggested *gaps* and *gap-filling* as theoretical constructs in the domain of proofs in mathematics education. They define a *gap* as missing information in a proof-document, the filling of which is essential for comprehending the proof and making it acceptable in the eyes of a specific reader. Accordingly, *gap-filling* is achieved by performing any action that aims at identifying and closing a gap (Marco, Palatnik & Schwarz, 2021). In this study, the notion of gap-filling is used as an analytical tool to assess students' proof-products: the more gaps it rightfully fills, the higher the proof-product is evaluated.

Research question

In a first cycle of a design study, Marco et al. (2021) gave students several PWWs, including Garfield's PWW, and invited them to elaborate a proof and to inscribe it (Figure 1). They investigated students' proof-products and found that most of them managed to develop proof-attempts that included the key idea of area calculations. However, these proof attempts lacked some subtleties essential for being considered formal proofs. Their results suggest that if it is wished that high-school students construct valid proofs based on PWWs, then they need to be trained to do so. This paper describes one such pedagogical tool designed and included in a PWW-activity, A PWW-CT. The research question is: what are the impacts of a PWW-CT on students' written proof-attempts and their self-assessment.

METHODOLOGY

The data were collected in different cycles of a design-research program (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) on the pedagogical use of PWWs. Every cycle included three phases: designing, testing, and revising-and-redesigning. The analysis included questions such as which gaps in the PWW the students tended to identify and fill, and which task revisions could support more gap-filling and better proof-products.

The first design cycle was an exploratory case study (Marco & Schwarz, 2019). From the second cycle and on, the teaching experiment structure was as follows: (a) the students were given a PWW and collaborated in small teams to develop a proof based on it; (b) students individually wrote their proof attempt reflecting on their team discussions, and (c) completed an online PWW-CT developed to improve their proof-attempts. Students answered the PWW-CT via personal smartphones or laptops.

The PWW-CT was tailored for Garfield's PWW (Figure 1) but was based on the Mejia-Ramos et al.'s (2012) model for building PCT. It was used to inform future design cycles and as a didactical tool. It aimed at prompting students to reflect on their work, revise their written proofs, and attend gaps they might have overlooked. The PWW-CT included a mathematical and a reflective section. The mathematical section relied on the categories elaborated by Mejia-Ramos and colleagues in their model (Mejia-Ramos et al., 2012). The PWW-CT was revised several times after each testing phase, and in its final version, it consisted of six items of the type "meaning of terms and statements" (i.e., "what is the formula for trapezoid's area?"); two items of the type "justification" (i.e., "is the whole figure a trapezoid? How can you know it?"); one item of the type "summarizing via high-level ideas" (i.e.," 'what is the key idea of this proof in your opinion?"), and one item of the type "identifying modular structure" addressing the mechanism that endows the proof of its generality ("Do you think this proof shows that the Pythagorean theorem holds for all right triangles? Why?"). An open-ended item was added at the end of the mathematical section: "Which improvement would you incorporate into the proof you submitted?" After each testing cycle, the PWW-CT was revised to support more students' gap-filling.

The reflective section included two open-ended items and eight 10-point Likert scale items. Two of the 10-point Likert scale items are reported here: (1) "To what extent do you feel you understand the proof by now?" and (2) "What grade would you give yourself for the proof you submitted?" These two items appeared before *and* after the mathematical section, to check whether students' answers changed after completing the mathematical section.

Participants

A total of 118 tenth grade students from the second (N=37), third (N=10), and fourth (N=71) testing cycles answered the PWW-CT. Most of the students were high-achievers in mathematics, and only a small minority were average or low-achievers. Forty-two students refused to participate or did not submit the proof or the PWW-CT, and their data were excluded from the analysis.

Data gathering and analysis

Data from two sources were analyzed: the students' submitted proofs developed based on Garfield's PWW (Figure 1) and their answers to the PWW-CT. Two-tailed paired sample t-tests were undertaken to assess the mathematical section's impact on students' reported understanding of the proof and the self-given grades. The null hypothesis was that there would be no difference before and after completing the mathematical section of the PWW-CT. Effect sizes were calculated. A gap-filling rubric containing nine gap-filling actions was used to evaluate students' written proofs and the improvements found in the answers to the PWW-CT. Due to space limitations, this paper will focus only on the following three gaps:

- Gap-i - why is the middle isosceles triangle right-angled?
- Gap-ii - why is the whole figure a (right) trapezoid?
- Gap-iii - why is Garfield's PWW a general proof for the Pythagorean theorem?

Each gap was coded as fully filled, half-filled, or not filled. For each gap, the average rates of filling were calculated. Our method is exemplified in the next section.

An example of analysis

Dan and Shira (pseudonyms) are two students who participated in the fourth testing cycle. Figure 2 shows Dan's submitted proof developed based on Garfield's PWW (Figure 1), in which he did not fill gaps-i-ii-iii. Dan improved his proof-attempt in the answers to the PWW-CT, while Shira's responses did not exhibit much improvement.

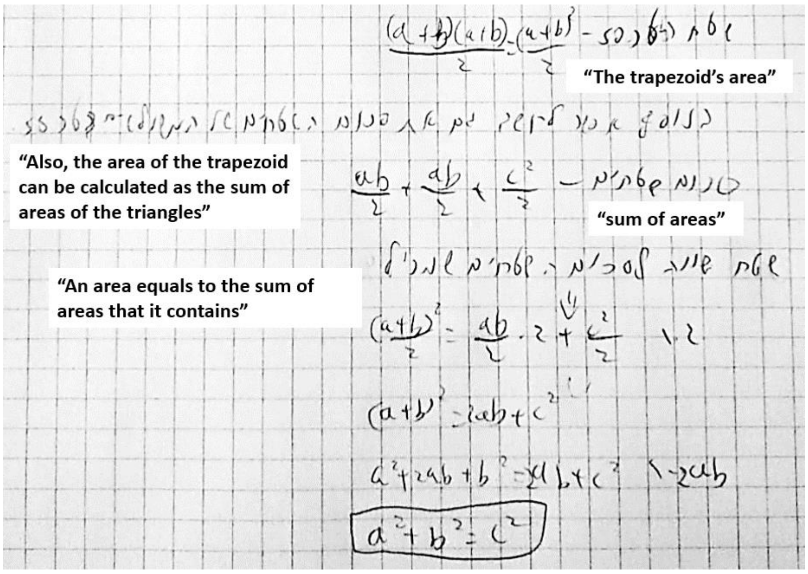

Figure 2: Dan's submitted proof (translation from Hebrew added by the author)

For the PWW-CT item "Can the middle triangle's bottom angle be known? If yes, how?" (In the version of Garfield's PWW that Dan and Shira had, this angle was not marked as right) Dan wrote:

Dan: Yes, it is 90° because the two triangles are congruent, and therefore their corresponding angles are congruent, and thus a straight angle minus two angles whose sum is 90° gives an angle equals to 90°.

We can see that through angle calculations, Dan fully filled gap-i and partially filled another gap - the congruence of two triangles, that did not appear in his written proof (Figure 2). For this same item, Shira wrote:

Shira: Yes, we can (calculate) because, in every right triangle, the angles are always $30°, 60°$, and 90°... So, we will assume that the small angle is 30° and the other one is 60°, so from both sides of this angle lie angles of $30°, 60°$. to complete it to 180° it must equal 90°.

Even though Shira mentioned the idea of subtracting two angles that complement to 90° from a straight angle, the gap was considered as not filled because her inference was based on a wrong assertion ("in every right triangle, the angles are always $30°, 60°$, and 90°"). For the item, "Is the whole figure a trapezoid? How can you know it?" Dan explained that the whole figure is a quadrangle with two parallel sides due to equal corresponding angles and received a top grade for filling gap-ii. Shira wrote: *"This is a* *trapezoid because it has two parallel sides and two sides that are not parallel."* Besides mentioning a correct definition of a trapezoid, Shira did not justify her assertion. Hence, we gave Shira half the points for filling gap-ii. To the item "Do you think this proof shows that the Pythagorean theorem holds for all right triangles? Why?" Dan received a maximum grade for responding*:* "Yes. Because we did not substitute any particular values for the sides nor the angles. We did it on a general case so that it will work for every right triangle". Shira wrote she thinks it is a general proof but did not reveal why, so she received no points for this answer.

General results

Dan's submitted proof (Figure 2) represents a general proof-product pattern in the data. In 44% of the submitted proofs, students expressed the key-idea of calculating the trapezoid area in two different ways. They simplified the equation to derive the theorem without filling any of the three gaps above. By completing the PWW-CT in the second (N=37), third (N=10), and fourth (N=71) testing, 62%, 70%, and 88% of the students gap-filled at least one of these gaps, respectively. Table 1 shows the average rates of filling the three gaps above in the written proof-attempts and in the PWW-CT's answers (if the gap was not filled in written proof). Revision of the PWW-CT before the 4^{th} testing cycle increased filling gap-i rates from 3% and 5% in the 2^{nd} and 3^{rd} testing cycles to 70% in the 4^{th}.

Testing cycle:	Gap-i - why the middle triangle is right? 2^{nd} 3^{rd} 4^{th}	Gap-ii - why the whole figure is a trapezoid? 2^{nd} 3^{rd} 4^{th}	Gap-iii - why is this PWW a general proof? 2^{nd} 3^{rd} 4^{th}
Rates of Gap-filling in submitted proofs	18% 10% 50%	24% 0% 45%	0% 0% 2%
Percentage of gap-filling in the PWW-CT*	3% 5% 70%	59% 55% 66%	31% 50% 54%

* From those who did not gap-fill this gap in their submitted proof.

Table 1: Rates of gap-filling actions in submitted proofs and answers to the CQ.

Table 2 shows the average self-reported proof understanding and self-given grade before and after completing the PWW-CT.

		Mean	SD	p-value	Cohen's d
"To what extent do you feel you understand the proof by now?"	Pre-CT	8.35	1.99		
	Post-CT	8.76	1.26	<.01	0.242
"What grade would you give yourself for the proof you submitted?"	Pre-CT	7.6	1.95		
	Post-CT	7.132	2.29	<.001	0.219

Table 2: Average self-reported proof understanding and self-given grades before and after completing the PWW-CT.

We see that the PWW-CT significantly increased the students' reported sense of understanding and significantly reduced their self-given grades. However, the effects of the PWW-CT on reported understanding and self-given grades were only minor.

DISCUSSION

This study contributes to the mathematics education literature on proof learning by pioneering empirical examination of students' activity around a PWW. It shows that students can develop proofs based on a PWW that includes its main idea but lacks essential details for being considered valid. A PWW-CT, accompanying the PWW activity, can improve students' proof-products, leading to better proof understanding, thus supporting proof learning.

The PWW-CT significantly increased students' sense of understanding and caused them to give themselves lower grades for their submitted proofs (both with small effect sizes). These results can be interpreted with the *Dunning-Kruger effect*; the more ignorant a person is in a subject, the more likely s/he is to overestimate her/his performance (Kruger & Dunning, 1999). Training in a task alleviates this effect and increases calibration – self-assessment accuracy. What substantiates this effect is that the knowledge necessary for self-assessment is the same knowledge that the person lacks. Reducing their grades after completing the PWW-CT reinforces the claim that they became more knowledgeable.

Pointing at the mechanisms that underlie the effectiveness of the PWW-CT requires further investigation. One possible explanation is that it functions as a didactical interview done by a teacher. It helps students doubt what seems apparent and recruit their mental resources to elicit uncertainties and fill gaps within their proof-product. The PWW-CT stir students' attention to identifying gaps they might have overlooked. Once students identify a gap, they are more likely to fill it. Repeatedly using PWW-CTs for different PWWs may develop students' ability to identify gaps by themselves.

The PWW-CT role could also be seen as setting proof-writing standards. In the collaborative learning part of the activity, some students may have filled some gaps that they did not write down in their submitted proofs. That might suggest they did not perceive these gap-fillings as essential details. If that is the case, the PWW-CT served as an indicator of the author's mathematical standards and expectations regarding the elements that students should elaborate on in written proof. To put it another way, it might be that the PWW-CT taught the students what should be written in a proof and did not necessarily help them generate new gap-filling actions. This way or the other, the data show that such PWW-CTs can facilitate high-school students' proving activity around PWWs and make PWWs more accessible and beneficial for them. The pedagogical approach of using such PCT can be well applied to proving activities based on other PWWs and other kinds of proof texts.

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UNDERSTANDING LENGTH, AREA, AND VOLUME MEASUREMENT: AN ASSESSMENT OF JAPANESE PRESCHOOLERS

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This study investigated how Japanese preschoolers (5–6-year-old children) understand the mathematical concepts of length, area, and volume, as mathematics is not part of the preschool curriculum. Researchers conducted structured clinical interviews with 32 children to investigate their mathematical content knowledge, skill, and problem solving ability. The interviews were statistically and qualitatively analyzed. The results indicated that children could perform direct comparisons in length, area and volume, but the challenge was found that they were not able to measure a few comparable objects with subtle differences correctly. Children's difficulty measuring correctly suggested that the future mathematical programs could enhance integrated learning, dealing with length, area, and volume simultaneously.

INTRODUCTION

Many studies have focused on elementary mathematics education for preschool and lower elementary school children (e.g., Brandt, 2013; Lin, 2013). However, measurement knowledge has not been sufficiently incorporated into these studies (e.g., Kotsopoulos, Makosz, Zambrzycka, & McCarthy, 2015; Smith, van den Heuvel-Panhuizen, & Teppo, 2011; Szilágyi, Clements, & Sarama, 2011; Tzekaki & Papadopoulou, 2017), even though it is an essential mathematical skill that is frequently used in everyday life (Clements & Sarama, 2009). Wilkening (as cited in Ebersbach, 2009) stated that preschoolers have been shown to have two-dimensional reasoning despite Piaget's earlier notion that young children could not reason beyond the number one; Ebersbach's study (2009) further supported this finding. Zöllner and Benz (2013) reported that four- to six-year-old children are competent in the normative aspects of indirect comparisons of mathematical concepts. This article, therefore, seeks to extend the research on multidimensional reasoning—especially when comparing length, area, and volume—among young children, particularly for those who do not study mathematics as a subject in school. The research question is: How do Japanese preschool children use multi-dimensional reasoning in a play-based setting?

Preschool Education in Japan

Teaching and learning in Japanese public preschools focuses on fostering three to six-year-olds' mental and physical development and is activity-based, not subject-based (Japanese Ministry of Education, Culture, Sports, Science and Technology [MEXT], 2018). Children are

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 238-244. Khon Kaen, Thailand: PME.

expected to develop through play. According to MEXT (2018), kindergarten teachers should focus on developing children's learning of numbers, quantities, and geometric figures, which comprise the foundation of elementary mathematics skills and should consider children's interests when teaching them.

The Project and its Theoretical Underpinnings

The article is part of our preschool mathematics education program in Japan. The aim of the project was for inexperienced preschool teachers to understand how to incorporate mathematical content into children's play and recognize activities that cause children to think and act mathematically. The theoretical and methodological underpinnings of the program were mathematical guided-play (Weisberg, Hirsh-Pasek, & Golinkoff, 2013) and the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982). In the project, fundamental mathematical programs were developed within a framework of learning activities, educational objectives, and the child's expected thinking and learning processes (Clements & Sarama 2009) to enable a smooth transition from preschool to elementary school (Matsuo, 2017).

DESIGN AND METHODS

Research Participants

One *kodomoen*—a type of hybrid day care and preschool in Japan—took part in our pilot study. This *kodomoen* focuses on mathematical activities. A retired professional with experience in both kindergarten and elementary school settings supports the research and training of novice teachers in *kodomoen*, and she can help develop and disseminate early childhood informal mathematics educational activities. Children attending this school do not formally learn mathematics, although they engage in mathematical activities to a limited extent, but there are no measurement activities. In total, 32 of the children—18 boys, 14 girls, aged five and six years—were selected for participation from the graduating classes. Of them, 30 were Japanese and 2 were non-Japanese, and they were from middle- and upper-middle-class families.

Data Collection and Analysis

We evaluated the children's performance on tasks incorporating numbers, shapes, and measurement, focusing on how well they measured length, area, and volume, using both direct and indirect comparison. The researchers developed a structured clinical interview (Goldin, 1998) with 21 mathematical content items, 8 of which were on mathematical measurement quantified by 16 questions. The questions covered the following areas: Q1-7 on length, Q8-11 on area, and Q12-16 on volume. The questions were developed referring to the framework for early childhood mathematical curricula (Matsuo, 2017), with theoretical and methodological underpinnings from mathematical guided-play (Weisberg, Hirsh-Pasek, & Golinkoff, 2013) and the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982.) The interviews took place in January 2019 and each lasted about an hour. The interviewers were the first author and co-researcher of this paper. The children were randomly selected for interview order. Both interviewers sat in front of the child and took turns asking questions. The interviews were video recorded. The researchers analyzed the data statistically and qualitatively, scrutinizing each child's actions on the video as well as their transcript.

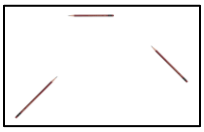
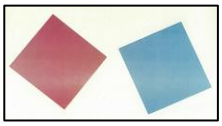
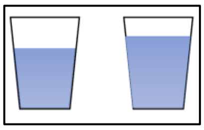
S/N	Questions	Mean score (standard deviation)
1	Which pencil is longer? (direct comparison of the lengths of two pencils)	96. 9 % (0.1740)
2	Why did you give that answer in Q1? (the reason why you selected a longer pencil in Q1)	75.0 % (0.4330)
3	Which pencil is the longest? Which pencil is longer? (indirect comparison of three pencils drawn on the paper) 	96.9 % (0.1740)
4	Why did you give that answer in Q3? (the reason you gave using mediation etc. in Q3)	31.3 % (0.4635)
5	Did you know how long 10 cm is? (knowledge of the word and meaning of 10cm)	65.6 % (0.4650)
6	Can you draw a straight line of 10 ㎝?	21.9 % (0.4134)
7	Can you measure an 8 cm line drawn with a ruler?	34.4 % (0.4750)
8	Which card is larger? (comparison between two cards (rectangles) drawn on paper) 	59.4 % (0.4911)
9	Can you compare two cards using the yellow card as an intermediary? (The area of yellow card is as same as the area of red card, but smaller than blue card.)	90.6 % (0.2915)
10	Can you put four cards of different sizes in order? (comparison among four cards (rectangles))	78. 1 % (0.4134)
11	Why did you give that answer in Q10? (the reason you gave is that the longer length means larger area while focusing on a particular part such as height in Q10)	84.4 % (0.3631)
12	Which cup of water has more water in it? (comparison between the volume of two cups of water drawn on paper) 	96.9 % (0.1740)
13	Why did you give that answer in Q12?	75.0 % (0.4330)
14	Did you compare the volume of the two cups of water in Q12 in terms of depth?	93.8 % (0.2421)
15	Do you know how much 1L of water is?	53.1 % (0.4990)
16	Can you select 1L of water from four different volumes of water? (The interviewer reminds children of a 1L carton of milk and asks them to choose from four options (500ml, 800ml, 1L, and 1.5L).)	28.1 % (0.4996)

Table 1: Question items and mean scores of correct answers

RESULTS

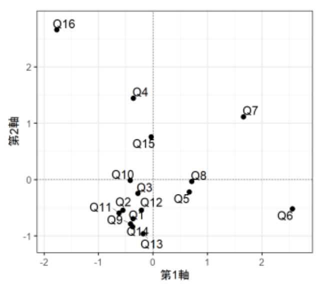
The interview questions were scored as 1 point for every correct response and 0 for incorrect response. The highest score possible was 16 points. The researchers calculated the participating children's total score and the average percentage for each question item, the mean score. The average total interview score was 10.81. Table 1 shows the question items with the mean score of the correct answers.

Correspondence and Cluster Analysis: The Relationship of Answering Patterns with Question Items

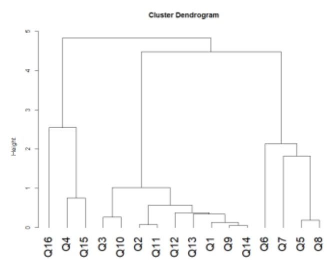
Semantic similarities among interview items were investigated using correspondence analysis and cluster analysis, this included all 16 question items with average percentages of correct answers. Figure 1 shows the results of the interview, the correspondence analysis is on the left and the cluster analysis on the right. The numbers correspond to the numbers of the interview items. The correspondence analysis shows the contribution ratio of the first and second axes are 28% and 23%. The cluster analysis shows three groupings, (G1) (Q4, and Q15-16); (G2) (Q3, Q10, Q2, Q11-13, Q1, Q9 and Q14); and (G3) (Q5-8). These groups show the differences in children's answer patterns. G1 and G3 show relatively lower achievement in their patterns.

In G1 and G3, the question items were related to the context of comparisons, moreover, G1 contains an item on explaining length and volume, and G3 includes length and area. Looking at G1 and G3 in detail, in Q15-16, twenty-three children did not identify the exact 1L amount by comparing it with the other quantities of water. Instead, on Q16, they selected the similar quantity, 1.5L, out of the four options (500 ml, 800 ml, 1L and 1.5L). This result indicates that children found it difficult to compare a particular volume with the other volumes of a given liquid. This could be resolved similarly to Q4. In Q4, children were expected to describe their reasoning when comparing the lengths of the three drawn pencils shown in Q3. It was not possible for young children to use a particular length, that is, a ruler or paper clip intermediary when they compared the lengths. Thus, making the comparison of similar quantities was challenging for children.

In G3, knowledge of length (Q5) and measurement of length (Q6 & 7) were related questions. Drawing a 10㎝ length and measuring the 8㎝ line were related to the skill of reading a measurement scale. Children also needed to capture the subtle differences between two areas of sheets of paper in Q8. They also had to make a comparison of the area of rectangles that were different in height (equal in width but slightly different in height). Thus, these three questions (Q6-8), which were categorized in the same group in the cluster analysis required children to show the correct skills for measuring things.



Correspondence analysis



Cluster analysis

Figure 1: Correspondence and cluster analysis

DISCUSSION

The statistical analysis did not show any differences in answering patterns between direct and indirect comparisons. Children's achievement in these two areas is crucial for their learning.

Direct Comparisons

There were 5 direct comparison questions (Q1, Q3, Q8, and Q12). Q1 required a direct comparison of length, Q3 required a comparison of length by judging appearance, Q8 required a comparison of area by judging appearance. Q12 also required a direct comparison of volume (a comparison of water in the same glass as drawn on paper; see Q12 in Table 1). Except for Q8, the number of correct answers for items related to direct comparisons was relatively high, showing that children performed direct comparison of length, area, and volume well (Zöllner & Benz, 2013).

Indirect Comparisons

The results varied when taking a closer look at existing knowledge applied across *concepts, and this was not an aspect discussed in the statistical analysis. First, the* research considered length measurement. Q4 required an indirect comparison of length. Children had to order the length of three pencils drawn on paper using another medium, such as a ruler, or a paper clip, and their performances were considerably poor. Moreover, in Q5-7, twenty-one children answered that they had heard of the standard unit of length, a centimeter, but they did not explain the concept well and could not measure or represent the length of an object in centimeters. Second, are measurement was considered. The result of Q8 showed that some children had a weak understanding of area measurement. However, when a medium for indirect comparison was provided (in the form of a yellow card the same size as the red square), ten more children succeeded in comparing area and answered correctly on Q9. This clearly indicated that simple interventions allowed children at these ages to be able to make indirect comparisons. Regarding Q10, twenty-five children answered correctly, but they may have judged the ordering of areas based on length without considering width. There was a possible misconception as it was difficult to arrange these cards in order of width: 12 cm × 12 cm, 11 cm × 11 cm, 9 cm × 15 cm, 9 cm × 9 cm; this result is consistent with that of a previous study (Skoumpourdi, 2015). Third, volume measurement was considered. As seen in the Q12 results, it seemed an easier task for children to compare the amount of water in a drawing on paper; they correctly identified the larger volume in the interview (Q14). Most children compared the volume of water by its depth (Q13); even when they looked at the same amount of water in different shaped glasses, they judged the difference in volume by comparing only the heights of the liquid in each glass. Similarly, in Q15-16, half of the children had heard of the standard unit, a liter, but could not express a concrete image of the volume.

Relating Mathematical Concepts

The analysis revealed that children were confused about the mathematical concepts of length, area, and volume (Skoumpourdi, 2015). Other previous finding (Nakawa, Watanabe & Matsuo, 2019) also found out the same results even with some interventions for two- and three-dimensional shapes. In particular, children were able to compare lengths well, but that knowledge was misleading when they applied it to area and volume measurement. When analyzing the performance of the 13 children who could not compare the area of two cards (Q8), only two of those children were able to explain, in Q4, how to compare length correctly using a medium ($p = .02246 < 0.5$). It can be inferred that children might not be able to compare the area correctly if they were not able to compare lengths. Furthermore, the number of correct answers was higher when comparing the volume of water in the same shaped glass than when comparing the area of cards. In the case of volume, children could compare the amount of water by its depth, but they could not in the case of area. Children were likely to compare area based on length, even if they noticed differences in the width of the cards. One reason could be that Japanese children often deal with height in day-to-day life. For example, teachers often measure the height of children or compare their height; they may teach children to observe plant growth and record height, and so on, but they get little experience with area or volume. Finally, even when children are able to act mathematically in specific activities (for example, Q12) in day-to-day life, they are not able to acquire the concepts for quantities such as 1L, 10 cm, and skills for describing the results of measurement. This should motivate better mathematics teaching.

CONCLUSION

The analysis showed that fifteen (47%) children in the interviews could perform direct comparisons of length, area, and volume. However, the analysis also showed that in cases where they did not do well on length comparison, it was also challenging for them to compare area correctly. In particular, it is difficult for children to make a comparison using a third vehicle or to make a detailed and accurate comparison. The comparison of length, size, and volume is closely related, and it was also revealed that children who could not compare lengths could not compare area. We need to develop children's understanding of measurement and consider how the individual concepts of length, area, and volume, which are currently learned separately, can be integrated with one another. Future research should analyze a larger number of children and develop an early mathematical educational program through which children can learn how to compare related mathematical concepts and gain better mathematical content knowledge and skills.

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SITUATIONAL AFFECT: THE ROLE OF EMOTIONAL OBJECT IN PREDICTING MOTIVATION FOR MATHEMATICS TASKS

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Situational affect (emotions and motivation) in first year mathematics courses in US high schools was studied using Experience Sampling Methods and Latent Path Analysis. Seven Hundred forty-six students from two US states were surveyed using Experience Sampling Methods. The results of a latent path analysis suggest that situational emotions can be distinguished by the objects to which they are directed, and that students interpret their engagement in mathematics tasks using both positive and negative emotions simultaneously depending on to whom or what they are directing their emotions. Finally, the objects to which emotions are directed are key determinants of the motivational outcomes they impact in mathematics classes.

INTRODUCTION

The role of emotion in students' motivation and success in mathematics is well established (Hannula, 2006; Hannula & Laakso, 2011). Emotions have been projected as the primary link between experiencing mathematics and one's motivation to engage (Hannula, 2006). The consensus in educational psychology and mathematics education, assumes that the role emotions play in self-regulated learning is complex. For example, in the psychological literature there is strong evidence that experiences promoting positive emotions improves student learning (Pekrun, Elliot, & Maier, 2009). Positive emotions (such as joy or hope) tend to be related to higher levels of self-efficacy and effort students are willing to expend on a task. But there is also evidence that positive emotions can have no effect or even negative effect on learning (Trevors, Muis, Pekrun, Sinatra, & Winne, 2016). Negative emotions, (such as anger or hopelessness) likewise, can have both positive and negative effects, depending on the constraints of the learning situation (Goldin, Epstein et al., 2011).

As a case in point, Muis, Pekrun et al., (2015) show that in a computer-mediated environment, students' emotions while engaged in the task impacted the self-regulated learning strategies they utilized in the task. Importantly, "positive" and "negative" emotions did not fall neatly into simple categories. Curiosity (a "positive" emotion) and anxiety (a "negative emotion") together, for example, contributed positively to more critical thinking in the assigned tasks. Likewise, confusion and curiosity together contributed positively to metacognitive self-regulation.

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One of the hypotheses that explains such conflicting evidence is that emotions, rather than being general affective responses to a task, instead are directed at particular objects in the environment. Middleton, Jansen, & Goldin (2017) propose that each emotion one experiences is directed at some *object* such as the teacher, oneself, one's peers, or the mathematics in which one is engaged. One can simultaneously be frustrated with the mathematics, but excited about the teacher or one's peers. There is some evidence that, given the multiple potential objects to which a student may direct their emotions, and given the multitude of emotions one may experience relative to those objects, there is no wonder emotional responses generate conflicting evidence related to learning outcomes. Mathematics and the teacher and peers in math class are prime objects. Recently, Vollstedt & Duchhardt (2019) show that students take both the academic and social features of their environment into account, and think about them separately when framing the meaning their mathematics engagement has for them (see also Botah & Hannula, 2019).

The pattern of emotions one displays in a learning environment and the subsequent personal meanings of the task one constructs through appraisal of the objects of one's emotions can be termed, *situational affect* in much the same way as the interest one experiences in a learning episode can be termed *situatonal interest*. In fact, situational interest can be thought of as one feature of situational affect. Other features such as mathematical self-efficacy, feelings of social inclusion, and personal relevance of the mathematics can also be seen as part of situation affect (Wiezel, Middleton, et al., 2019).

The distinctions between situational affect and longer-term, trait-like beliefs are reviewed thoroughly by Middleton, Jansen, & Goldin, (2017). Like situational interest, situational emotions can sustain persistence in the task, while longer-term moods and affective structures tend to direct whether or not one chooses to engage in mathematics-related activity in the future (e.g., Hannula, 2006). Together, aspects of situational affect, broadening the "catch and hold" theory of the transition of situational interest to personal interest (Mitchell, 1993), may contribute to long-term, trait-like approach orientations.

Recently, Wiezel, Middleton et al., (2019) found that positive and negative emotions were significantly associated with students' mathematical self-efficacy and interest, but not with other motivational factors. Moreover, both positive and negative emotions appear to exist simultaneously in students' experiences and contribute to their interpretation of their mathematical engagement. The present study builds on these findings. Whereas in that earlier study, we found that situational affect impacts situational motivation, the subtleties of emotional object, and measurement of situational affect were not measured at the task-level.

In the present study, our working hypothesis was that situational emotions, if distinguished by object (e.g., teacher, classmates, self, or mathematics) would manifest different effects across the situational motivational variables. Following Muis, Pekrun et al., (2015) we expected to see both positive and negative emotions contributing simultaneously to the motivational variables, with valences that would suggest that students may hold two or more emotions with different valences simultaneously. Evidence of these patterns would support the theory that: 1) Situational emotions are directed at objects; 2) Students may interpret a situation using both positive and negative emotions; and that 3) The objects to which emotions are directed will be a primary distinguishing feature of the motivational outcomes they impact.

METHOD

Participants

Participating students were recruited from 42 classes, taught by 17 teachers (8 in a Mid-Atlantic US State, and 9 in a Southwest US State). Forty-five percent of the students identified as Male, 52% identified as Female, and 4 % identified as "Other." Fifty-nine percent of our students identified as Hispanic/Latinx, 14% identified as /White, 14% identified as African-American, 2% American Indian/Alaska Native, and 2% identified as Asian, 2% identified as Other, and 8% identified with more than one category. All students were enrolled in a mathematics course designated as "first-year high school mathematics." The majority first year mathematics courses in the Mid-Atlantic sample were semester-long integrated mathematics focusing more heavily on Algebra, whereas in the Southwest, first-courses were year-long, traditional Algebra 1 content.

Measures

We assessed students' situational affect using Experience Sampling Methods (e.g., Shernoff, Kelly et al., 2016). A short survey (Wiezel, Middleton, Zhang, Tarr, Jansen, 2018) was designed to assess students' situational affect and motivation during a focal class activity. This instrument featured a set of situational emotion checklists including 16 emotions (*angry, anxious, ashamed, bored, confident, embarrassed, excited, frustrated, happy, hopeful, hopeless, interested, proud, relieved, satisfied,* and *worried*). Emotions were directed toward four possible objects in the students' mathematics class (*the math activity, themselves, their classmates,* and *their teacher*), yielding a matrix containing 16 emotions x 4 objects. Students were asked to check a box in the matrix corresponding each emotion they felt towards each object.

Factor analysis of these items revealed 4 separable factors: Negative emotions directed towards the teacher/class (4 items e.g., *Bored about/by Teacher*); Positive emotions directed towards the teacher/class (16 items e.g., *Excited about/by Teacher*); Negative emotions directed towards the mathematics (6 items e.g., *Anxious about/by Math*); Positive emotions directed towards the math (7 items e.g., *Excited about/by Math*). Chronbach's alpha for these 4 scales were 0.72 for Positive math emotions, 0.76 for Negative Math emotions, 0.37 for Negative class/teacher emotions, and 0.84 for Positive Math emotions.

The low reliability for Negative class/teacher emotions indicates that findings that show an impact of this factor on situational motivation should be taken with caution. There is current debate on what reliability means for experience sampling methods, wherein one "experience" may be quite different, qualitatively from another, even in the same mathematics class. We include this factor despite its low internal consistency, keeping this caution in mind.

Situational motivation variables were measured on the survey using a set of 5-point Likert items arranged in 5 factors (1 corresponding to a low rating and 5 being a high rating): Effort (4 items e.g., *How hard were you trying during the activity you were just working on?*), Self-Efficacy (2 items e.g., *I felt successful in the activity I was just working on*), Social Engagement (4 items e.g., *I felt like my contribution was respected during this activity I was just working on*), Perceived Instrumentality (4 items e.g., *How I performed on the activity I was just working on will affect my future success*); and Interest (2 items e.g., *I think the topic covered in the activity I was just working on is interesting*). Chronbach's alpha for these scales ranged from a low of 0.71 for Effort, and a high of 0.85 for Interest.

Procedure

Students were surveyed in the Fall of 2018 and Spring of 2019. Researchers asked teachers to identify an activity in their classroom that had the potential for engaging students in the mathematics. The entire class session was observed, and when the identified activity was completed, each student was asked to complete the ESM survey. Surveys were completed electronically, primarily, using the students' phones, or school laptops. Several classes did not have universal access to these devices so the surveys were administered using paper-and-pencil. Surveys took less than 5 minutes to finish. This resulted in 746 returned surveys, each assessing the student's emotions and motivations in the immediate situation they were studying.

RESULTS

Goodness of Fit Index	Estimate
χ^2	2013.86 (p<.0001)
df	1091
RMSEA	0.034 (0.031, 0.036)
CFI	0.85
TLI	0.83

Table 1. Goodness of fit indices for the Latent Path Model

We fit a hypothesized latent variable path model to the data using Mplus software (Muthén & Muthén, 2017). Each item on the survey was treated as either binary (for emotion items) or ordinal (for motivation items), and each scale was estimated as its own latent variable. To address the ordinal nature of these variables, polychoric correlations (which can handle both binary and ordinal data in the same analysis) and WLSMV estimations were used in our models.

Standardized Regression Coefficients	*+Class Emotions*	*+Math Emotions*	*-Math Emotions*	*-Class Emotions*
Interest	0.17***	0.54***	-0.25***	-0.28***
Self-Efficacy	-0.02	0.66***	-0.63***	0.17
Social Engagement	0.24***	0.34***	-0.25***	-.26***
Instrumentality	0.26***	0.25***	-0.06	-0.29***
Effort	0.08	0.07	0.28***	-0.50***

Table 2. Relationships Among Latent Variables

Overall, the model fit relatively well (see Table 1 for fit statistics). Although our chi square was significant [$\chi^2(1091) = 2013.86$, $p < 0.0001$], the rule of thumb that χ^2/df be less than 3 shows that our model has adequate fit. The RMSEA was 0.034 with a 90% confidence interval between 0.031and 0.036, which indicated very good model fit (Hu & Bentler, 1999). However, the CFI and TLI indices of 0.85 and 0.83 indicate moderately good fit.

In the hypothesized model, we found was a significant, positive standardized regression coefficient between emotion factors and all motivation outcomes. The pattern among the effects indicates that the object of the emotions determines, in part, the motivational variables impacted.

Positive Math Emotions, for example, is positively related to all motivational outcomes with the exception of effort. Effort expended in the task is predicted by experiencing *more* negative math-oriented emotions, but by having *fewer* negative teacher/class-oriented emotions. Positive emotions show non-significant relationships, regardless of object, towards effort students were willing to expend on the task.

The remaining significant effects in the model are as follows. Situational interest is positively associated with positive class- and mathematics-related emotions, and negatively associated with negative emotions. Task-based self-efficacy is positively associated with positive math emotions and negatively associated with negative math emotions. Emotions directed towards the class were not significantly associated with Self Efficacy. Social Engagement—the belongingness and support students felt in the task is positively associated with both positive class- and math-related emotions and negatively associated with both negative class- and math-related emotions. Perceived instrumentality appears to be associated with positive class and math emotions, and negatively related to negative class emotions. Perceived instrumentality was unrelated to negative math emotions. Figure 1 provides the full path model.

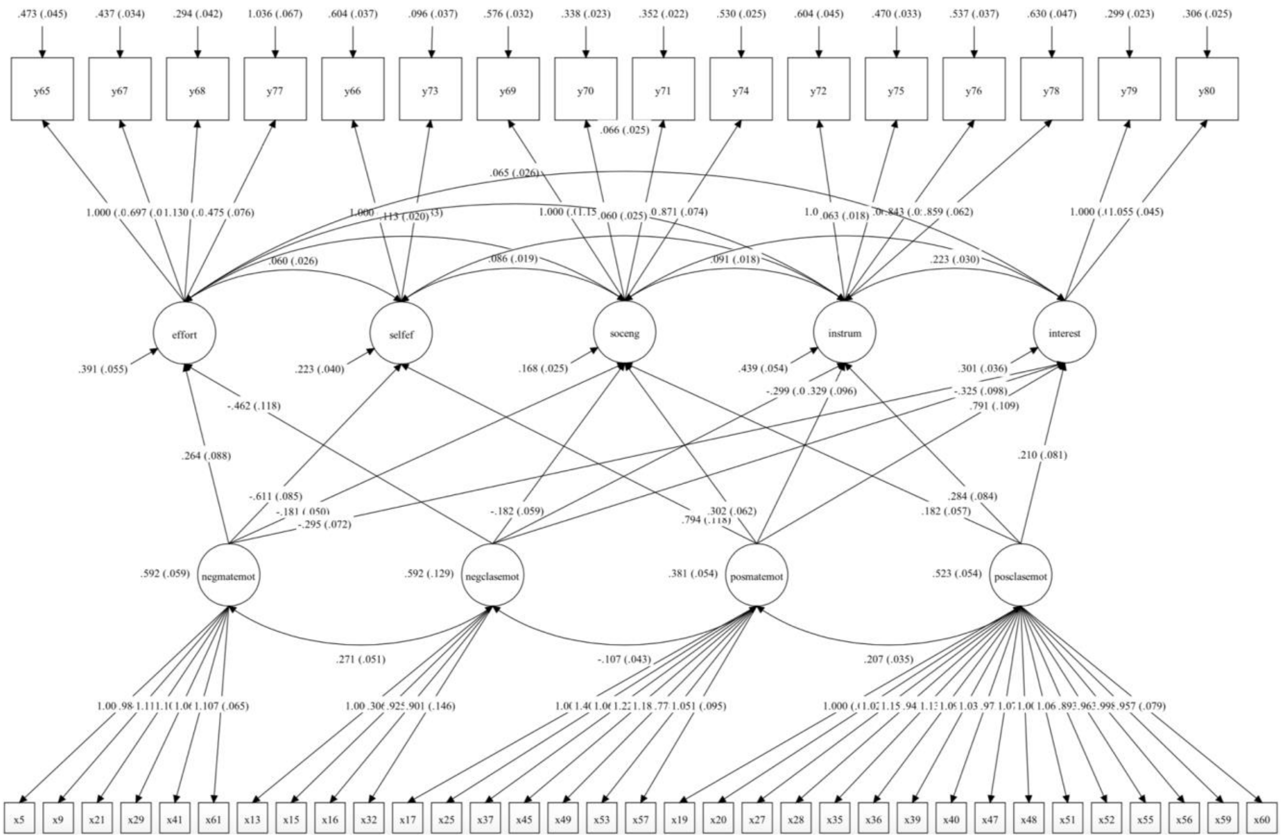

Figure 1: Latent Path Model of Situational Emotions by Object and their Hypothesized Relationship with Situational Motivation

DISCUSSION

Results confirm our hypothesis that situational emotions are directed at objects, and if these objects are taken into account when assessing situational affect, they explain some of the perplexing findings in the field related to seemingly conflicting emotions appearing simultaneously in the same experience. We found that, when we accounted for math-related and teacher/class-related emotions, the different emotional objects appear to impact aspects of situational motivation differently.

In particular, the role of "negative" emotions is fascinating, as negative math-related motivations appear to positively impact, or at least interact with, effort students are willing to expend in mathematics learning. The finding that positive emotions appeared to have little to no relationship with effort, but that negative emotions appear to *stimulate* effort is a fruitful area for followup research. Goldin, Epstein et al., (2011) in discussing affective structures make the point that frustration and other "negative" emotions may be interpreted by students as calls to expend more effort in order to learn or at least show some success in the task. How this plays out is uncertain, but our confirming evidence of this seems to indicate that challenging tasks, with support, are critical for students to feel that a task is worthy of effort.

Social-related emotions—those pertaining to the teacher/class as objects—are interesting for their part. Self-Efficacy appears to be fairly unrelated to social-oriented emotions compared to those focused on mathematics as the object. This makes sense, in that task-related efficacy beliefs are tied to feelings of success, and in the US, mathematics tasks are predominantly tackled individually, or even if done in groups, are assessed individually.

Some limitations to our work here must be noted, however: First, our negative class emotions scale had low reliability. This makes the findings related to that scale somewhat suspect. Further research and refinement of this scale is warranted. Second, our fit indices were inconsistent. Followup studies with larger samples should provide better estimates, particularly if the negative class emotions scale is improved.

In summary, this research contributes three important ideas to the literature worth further pursuit: 1) situational emotions are directed at objects; 2) students do tend to interpret learning situations in both positive and negative terms depending on to whom or what they are directing their emotions, and 3) the objects to which emotions are directed determine, in part, motivational outcomes of mathematics learning experiences.

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PROCESSES OF GEOMETRIC PREDICTION: PROPOSING A NEW DEFINITION AND A MODEL

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Building on Fischbein's notion of figural concept, this paper introduces a cognitive model of geometric prediction (GP) processes and proposes a revised definition of GP as the mental process of generation of a new geometrical object through the manipulation of figural elements that maintain invariant certain theoretical elements belonging to the solver's Theory of Euclidean Geometry. The model was developed during my doctoral study aimed at investigating cognitive aspects involved into processes of prediction in Geometry. I introduce the model and provide an example of the qualitative analyses that can be carried out through it. Specifically, I focus on the central role played by theoretical control and dynamism within the process.

INTRODUCTION

This paper reports on the main finding of my doctoral study, aimed at investigating cognitive aspects involved in a cognitive process carried out in the domain of Euclidean geometry: *geometric prediction* (*GP*). GP is a theoretical construct that was first presented at the 42nd PME conference (Miragliotta & Baccaglini-Frank, 2018). This was the very beginning of our studies on prediction processes. Although the theoretical construct of GP still needs further investigation, here I continue my discussion of GP within the PME community by reporting one of the main findings of the study: a revised definition and a model through which GP can be analyzed.

The study is situated within the wide domain of research on *visualization* in geometry, following Presmeg's definition (2006):

> [...] visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (p. 206).

Indeed, concerning the research on mental images involved in doing mathematics, processes of GP can be considered elaborations and processing of mental images. Moreover, when discussing the processes accomplished by a solver facing a geometrical task, Mariotti and Baccaglini-Frank (2018) highlight that it is rather frequent to use geometric prediction "to mentally manipulate a figure and imagine how it will change given certain constraints, that is, maintaining certain properties invariant" (p. 156). Since the authors highlight that this process is common among the solvers, this study is meant to contribute to this topic by providing a precise definition and a fine-grained description of GP.

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In order to gain a deeper insight into GP processes, building on the *Theory of Figural Concept* (Fischbein, 1993), we have constructed a model (Miragliotta, 2020) of the process, seeing it as a mental action of productively and dialectically combining figural and theoretical elements under the solver's theoretical control. The model that I will introduce was developed, refined, and tested, to ensure that it provides a suitable lens for observing GP processes and has the potential to explain a variety of students' difficulties. In this paper, I use the model to describe different kinds of GP processes, highlighting especially the role of theoretical control and manipulation.

THEORETICAL FRAMEWORK

Within the domain of geometrical reasoning, generating a prediction is a process that involves the solver's use of and interaction with geometrical objects. These are very particular mathematical entities. As highlighted by Hershkowitz et al. (1989), geometry as a mathematical theory is a cultural artifact, whose objects belongs to a logical system, and therefore are ideal; however, unlike other mathematical domains, geometry relies on a natural conceptualization of space, and its objects maintain a strong connection with material objects (solids or drawings). So, from an educational point of view, for observing and describing personal processes of prediction, we need a theory that describes this multifaceted nature of geometrical objects when they are used and conceived by a solver.

This is the key reason why we grounded our study on the *Theory of Figural Concepts* (Fischbein, 1993), which sees geometrical objects as composed by different – but strongly intertwined – aspects. Indeed, when a solver is approaching a geometrical task, s/he deals with *figural concepts.* A figural concept is neither a pure concept nor a pure image; it is the fusion between the conceptual and the figural components of a geometrical object as it has been thought by the solver. In Fischbein's words: *figural concepts* "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities – like ideality, abstractness, generality, perfection" (Fischbein, 1993, p. 143). So, from a psychological and educational point of view, when people use or refer to ideal *geometrical objects* they are thinking in terms of *figural concepts*.

As a consequence, the solver's figural concepts can be very close or quite far from the corresponding geometrical objects. Such a distance can be more or less wide depending on the solver's use of a system of control. Recently, building of Fischbein's work, Mariotti and Baccaglini-Frank (2018) have provided an explicit definition of *theoretical control* as the act of "mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (p. 156). Theoretical control has several functions (see, Mariotti, 1992, for further details). For example, when the solver needs to rearrange (through rotation, translation, reflection, …) figural components of a given drawing or imagined figure, s/he needs to theoretically control the figure in order to check the possibility of the transformation and its consistency within the mathematical reference theory (Mariotti, 1992); in this study the reference theory is the Theory of Euclidean Geometry (TEG). Generally speaking, the theoretical control supports the coordination between an image of a figural concept (the figural component) and its geometrical definition (the theoretical component).

The model of GP in a nutshell

In light of the theoretical framework and supported by data analyses, a new definition of GP was proposed in (Miragliotta, 2020) and here reported. *Geometric prediction* can be defined as: the mental process of generation of a new geometrical object through the manipulation of figural elements that maintain invariant certain theoretical elements belonging to the solver's TEG. Manipulation can be physically performed or only imagined. Depending on the solver's theoretical control, products of GP can be coherent or incoherent with respect to the given theoretical constraints within the TEG.

Data analyses revealed that GP processes are composed of general and stable components, following described. *Theoretical elements*, recalled by the solver through introduction of new elements or interpretation of the given ones; these are elements of the solver's productions that belong to the TEG; they include all the properties that solvers give to the figure (or part of it), theorems and mathematical results. *Figural elements*, on which the solver focused, and which can be manipulated; these are elements of the solver's productions that belong to the figural domain in a specific moment related to the image of the figure in front of the solver. Depending on how these components interact and come into play, they produce a process of GP with specific characteristics. Looking at the diagram (Fig. 1), the arrows make explicit the connections between the components and their possible features.

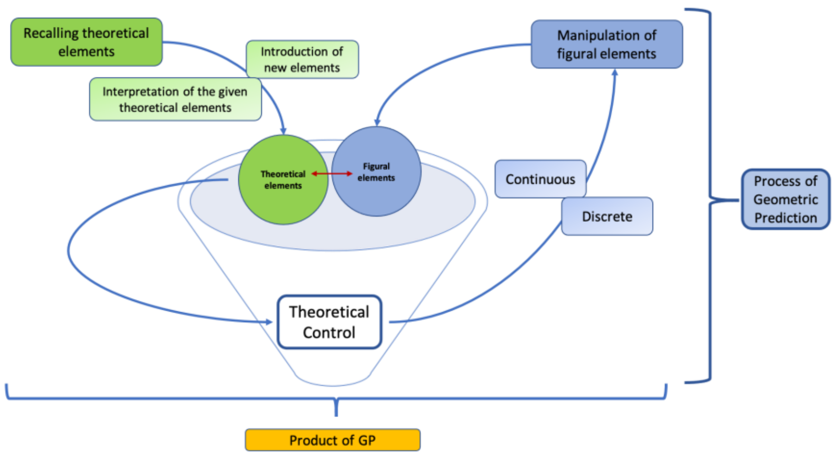

Figure 1: Visual diagram of the GP model

Starting from the upper left side of the diagram above, when a GP process starts, a geometrical figure is interpreted by the solver, who recalls or introduce theoretical elements that characterize certain figural elements. Figural and theoretical components are always intertwined (see the red arrow) and one can recall the other at any time. *Theoretical control* (Mariotti & Baccaglini-Frank, 2018) envelopes and harmonizes these two components (see the funnel in the centre). Moreover, using theoretical control the solver manipulates or decides to focus on particular figural elements (see the blue arrow on the right), allowing solvers to obtain new figural elements or additional theoretical properties of the figural elements. Manipulations can be accomplished in two ways: *continuously*, that is solvers can imagine, perform or mimic a continuous movement of one or more parts of the configuration (i.e., points, segments); *discretely*, that is solvers can locate these parts at a specific position on the plane and reconstruct the corresponding configuration. The cycle described can be repeated.

A GP process does not produce either a pure theoretical or a pure figural object: it produces an object that is a composition of the two. This object can be drawn out or conceived only mentally, according to the theoretical control that the solver exercises. The coherence of such a product within the TEG also depends on the theoretical control.

METHODOLOGY

The data collected for this qualitative study comes from 60-minute individual interviews in which solvers were assigned as many *prediction open problems* (i.e., open-ended task focused on prediction) as they could solve in the given time. Data consists of audio, video recordings, and drawings produced by each solver. The participants are from a convenience sample of 37 Italian solvers who volunteered to take part in the study. Among them there are 32 high school students (ages 14-18), 5 among undergraduate, graduate, and PhD students in Mathematics (ages 23-35). Tasks were proposed through *task-based interviews*. The first question was always the same; then there was a sequence of questions defined a priori and a set of stimuli to obtain solvers' comments or clarifications. In this paper I will refer to the following task:

"Imagine a triangle ABC. Consider the midpoint of the side AB and call it M. Imagine tracing the segment CM. Imagine that A and B are fixed. Make a prediction: is it possible that CM is congruent to CB?"

Solvers can reason in different ways. One of the possible resolution paths consists of recognizing CM as a median of the triangle ABC; focusing on the triangle CMB and interpreting the theoretical element "CM is congruent to CB" as the constraint "CMB is an isosceles triangle". Starting from a first prediction about a position for C so that CM is congruent to CB, the solver can imagine moving C to explore other positions of C that maintain the given constraints. This way, the solver can predict or recognize an entire locus for C. Instead, the solver may directly recall a figural concept, that is "the axis of a segment", and its definition as the locus of points equidistant from two given points (M and B).

When the model is used for describing GP processes, the researcher needs to make explicit all the theoretical and figural elements that are involved. The solver can explicitly mention some of these elements, but others need to be inferred; inferences can be drawn by looking diachronically and synchronically at all the solver's productions (i.e., utterances, gestures, drawings). Theoretical control can be inferred by looking at how theoretical and figural elements are blended within the process, more or less coherently with the mathematical reference theory.

CASE ANALYSIS

Among the data collected, I chose key excerpts from Sam's interview to be presented here, since they provide (in a quite short timespan) an example of different kinds of GP processes accomplished by the same solver. When the interview took place, Sam was a 10th grade student. In the following transcriptions, I make use of bold font and underlying to highlight **theoretical** and <u>figural</u> elements, respectively.

Right after the first question, Sam expresses his first product of GP, stating that CM does not be congruent to CB (GP_1) and explain why, as follows.

1 Sam: Because the **side** <u>CM</u> would represent, in any way, **the height of the triangle** that I imagined and therefore could not reach the length of the **side** <u>BC</u> anyway.

According to the model, initially, Sam introduces new theoretical elements: he sees CM as the height of the triangle ABC. His theoretical control is not strong enough to recognize the inconsistency (or at least the lack of generality) of this interpretation and to support manipulations of figural elements.

The interviewer asks Sam to make a drawing of what he has imagined (Fig. 2a), and then she repeats the question. Sam's answer is the following:

2 Sam: Because the **side** <u>CM</u>, that represents **the height**, is in any case... since it is **perpendicular to the sides** <u>AB</u> that are fixed points, it is **shorter** than the **side** <u>CB</u> since it is <u>slanted</u> instead with respect to the **side** <u>AB</u>.

The drawing supports Sam's first interpretation of the configuration and the perpendicular relationship between CM and AB is a new theoretical element, which is directly drawn from the theoretical element "height". This relationship is coherent within the solver's TEG and interpretation, but it does not coherently fit with the given constraints, since CM is given as a median of ABC. Until Sam conceptualizes CM as a height, he does not seem to be able to overcome this very fixed interpretation.

Now the interviewer asks Sam if he thinks he can move C so that CM is congruent to CB. Sam says that he does not think so, and he starts arguing why, but then he suddenly stops; after a long silence (8 sec) he states the following.

3 Sam: No! No, no, there would be one point. I should...I should move <u>C</u> **perpendicularly to the midpoint of segment** <u>MB</u> [*he points at a specific position, Fig. 2b*] and therefore these two, <u>these two</u> [*CM,CB*] **sides** would be identical.

He performs a drawing of the new configuration (Fig. 2c). Line 3 provides an example of a GP with different features. The interviewer's question seems to promote Sam's mental manipulation of figural elements (i.e., point C, segments CM and CB). He does not perform such a manipulation only figurally to reach a position for C so that CM "appears" equal to CB, but an important role is also played by theoretical elements. Indeed, Sam points at the position and, at the same time, provides theoretical elements (i.e., "midpoint", "perpendicularly") used to find this position. Looking at his utterance we can see that CM is no longer conceptualized as a height, but as a side (of a triangle). We can infer that Sam is activating theoretical control over the figure, which allows him to coherently manipulate figural elements, take into account all the theoretical constraints which characterize the figure, and drop the idea that CM must be the height. At the end of the process, Sam has reached a new product of GP: C on a perpendicular segment through the midpoint of MB (GP$_2$). The manipulation was discrete: Sam has pointed at a specific position within the plane and then mentally reconstructed the sides ("therefore would be equal"). Good theoretical control and discrete dynamic interaction with figural elements are the main features of this GP.

When the interviewer asks him for other ways for CM to be congruent to CB, he says:

4 Sam: Ehm... the other points would be moving point C **perpendicularly** to MB, I mean raising it [*he points at C and mimics a straight trajectory that is perpendicular to MB, see Fig 2d*] or lowering it [*he mimics the trajectory, see Fig 2e*] and, bringing to **the other side**, raising it or lowering it. Yes, moving point C only **perpendicularly** to MB.

Sam's utterances and gestures depict a new product of GP, by describing an entire locus for C. GP$_3$ (C on a line perpendicular to MB through the midpoint of MB) constitutes a refined version of GP$_2$. The dynamic dimension embedded into the process is evident: Sam continuously manipulates the figural element C for describing the locus that constitutes his prediction. This is a different way of accomplishing a GP.

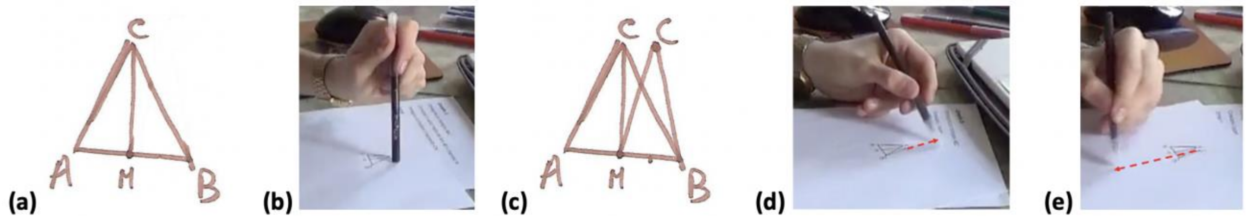

Figure 2: The drawings and gestures performed by Sam during the interview

Looking at the whole excerpt (lines 1 – line 4) we can notice how movement is more and more intergraded into the GP processes, realizing an evolution from a very fixed configuration to a continuous manipulation of figural elements, passing through a discrete approach. Supported by the interviewer's requests for alternative configurations that could realize the congruence between CM and CB, Sam undertakes three processes of prediction with different features. The first process is characterized by a misleading interpretation of the given constraints that reveals a solver's lack of theoretical control, leading to the fixedness of the whole figure. Sam's exploration is limited to the left part of the model; GP$_1$ is only the description of the very first triangle that Sam has imagined. The last two processes are both characterized by good theoretical control which supports the solver in completing the cycle of prediction. More precisely, the process that leads Sam to communicate GP$_2$ is characterized by the introduction of new figural concepts (the midpoint and the perpendicular segment). The theoretical elements that compose these figural concepts are verbally described; instead, the figural counterpart is performed gesturally (Fig. 2). The good theoretical control allows the solver to recall (eventually implicitly) the most suitable fragment of theory and ignore the first figural concept (the height of the triangle). A discrete manipulation of figural elements (i.e., C, CM, and CB) supports the prediction process. So, we can say that good theoretical control and (discrete) dynamism are the main features of this process. The last GP process manifests a prominent dynamic dimension, as well: seemingly, the locus is reached correctly because the solver is willing to manipulate figural elements (C, CM, and CB) continuously. While Sam is moving C, he constructs the locus; indeed, he starts moving C above the segment AB, and during this manipulation the idea of also moving C under the segment emerges. The last utterance suggests the intervention of the solver's theoretical control to maintain the coherence of the manipulation; indeed, a theoretical counterpart is added.

CONCLUDING REMARKS

Analyses reveal that there is not only one single process of prediction with certain fixed characteristics; more likely there are stable components (theoretical and figural elements, theoretical control) whose different interaction gives rise to a process of GP with specific features. As far as this report concerns, I focus particularly on the role played by theoretical control and dynamic dimension. The case study in this paper provides an example of three different processes of GP that were accomplished by the same solver as he solved a given task. The first process is characterized by a lack of both theoretical control and dynamism; theoretical control, coupled with dynamic interactions with figural elements (discrete and continuous, respectively), characterizes the last two.

Theoretical control is at the heart of the model; it has a crucial role in all the processes. For example, it is key for controlling the coherence of the manipulation of figural elements (line 4); for recalling suitable figural concepts (line 3); for taking into account all the theoretical constraints given by the problem or deduced from those (lines 3-4). Moreover, this report focuses especially on the function of a dynamic dimension as a catalyzer of GP processes: the possibility of considering several configurations that change dynamically (in a discrete or continuous way) has revealed to be central for reaching a coherent product of GP. This is consistent with Presmeg's findings on *dynamic imagery* (1997) which is revealed to be very effective and "especially powerful in facilitating the mathematical problem solving of the students who used them" (Presmeg, 1997, p. 305). Sam's excerpt also provides an example of a (quite) instantaneous breakthrough in the situation which allows the solver to undertake effective GP processes right because he is willing to explore the configuration dynamically. For other students who have interpreted CM as a height, the interviewer's request for other positions does not seem to trigger a more dynamic exploration, and they remain fixed in their first interpretation. Here is one of crucial roles played by the theoretical control: it makes the solver able to coherently reconceptualize the figure. In the excerpts, the interviewer's questions simply *activate* Sam's theoretical control.

The model provides insights into the GP process of a particular solver at a given moment while that solver is solving specific prediction open problems. Although findings from this study have no statistical ambitions because of the limited number of cases analyzed, the fine-grained qualitative analyses that were carried out have provided a richness in detail and depth which would not have otherwise been possible.

I see great educational value in tasks that explicitly ask students to make geometric predictions. Indeed, at the end of each GP process, solvers gain new insights into the initial geometric configuration; simultaneously such insight enriches the personal figural concepts with new figural and theoretical components that emerge during the process. Moreover, asking for a prediction sheds light onto the features of students' figural concepts, and consequently provides teachers insightful information for designing geometrical activities aimed at strengthening students' theoretical control. This is a new hypothesis that needs to be further investigated. However, in my view, considerations emerging from this paper can inspire educators to establish educational goals and "good practices" within the teaching and learning of Geometry.

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TEACHING–LEARNING STRUCTURES IN MATHEMATICS INSTRUCTION FROM A MULTIMODAL PERSPECTIVE

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This article focuses on structures of teaching and learning in mathematics classrooms, which are observed in everyday teaching. They are designed by the teacher. The goals and motivation for the design are to activate the learners' previous knowledge, to enable and to reflect mathematical explorations, as well as to structure and link new knowledge content. Different types of teaching–learning structures are analysed in more detail and examined for connections between their function in the teaching practice and the utterances of the teacher. With the help of the Content Structuring according to Mayring (2014), authentic teaching material is analysed. A characteristic use of modes is revealed for certain teaching–learning structures.

INTRODUCTION

In order to make mathematical learning as meaningful and supportive as possible, teachers vary teaching–learning structures and thus set different emphases for the design of the mathematical learning process (Steinbring, 2005; Barzel et al., 2013). At the center of these structural units is the object of learning, which is elaborated, illustrated, and made tangible, in the interaction between teacher and learners or in negotiation processes between learners, and can therefore lead to mathematical knowledge growth. Teachers design teaching–learning structures in different ways. The teacher's task is to prepare what is to be learned according to teaching–learning theories and to activate, enrich, and structure what has been learned in order to initiate mathematical learning processes. In this paper, the focus is on the teacher's design of mathematics lessons and its importance for the students' learning activities.

The teacher's instructions and the interaction situations are not only in speech, but are also accompanied by other modes, such as gestures and inscriptions. Modes are often not used separately or sequentially, but the subject matter is explained, structured, repeated, and applied in new contexts through multiple modes simultaneously (Alibali et al., 2014).

THEORETICAL FRAMEWORK

Multimodality

Interpersonal interaction in everyday situations, but also in the context of learning processes, is characterised by the use of varied signs. Kress (2010) focuses in his semiotic interpretation on the relation between the design of the sign, called form, and its meaning. The specific configuration of signs is produced in interaction and is thus part of the semiotic

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resources of a culture. This means, for example, for mathematical learning processes, that fractions are inscriptively represented in mathematics classes in two dimensions with numerator and denominator. A specialist mathematician, on the other hand, often chooses the one-dimensional pair representation. The respective cultural context, i.e., the mathematical subject culture or teaching culture, shapes the design of the sign.

> "Mode is a socially shaped and culturally given semiotic resource for making meaning. Image, writing, layout, music, gesture, speech, moving image, soundtrack and 3D objects are examples of modes used in representation and communication." (Kress, 2010, p. 79).

In the present research context of authentic mathematics teaching, the main observable modes are speech, gestures, writing, and typical iconic images used in mathematics and mathematics teaching (including layout).

In the context of mathematics education, speech is differentiated into technical language and academic language, which are used for interaction in mathematics lessons in addition to everyday language (Vogel & Huth, 2020). It can be stated that the academic language is shaped by the prevailing classroom culture, which in turn is strongly influenced by the teacher. The amount of technical language often varies greatly and depends on the teacher's professional self-image. Speech differs from writing primarily in its grammatical structure and medial implementation form (graphical and phonetical signs). In mathematics instruction, written language is often used in the form of a notebook entry or in the written formulation of mathematical rules (e.g., in a rule book).

Gestures as the result of hand movements in space (Goldin-Meadow, 2003) are described by Huth (2014, p. 151) as

> "stable components of the semiotic repertoire that it is probably almost impossible to suppress them for any length of time. But not only for the *producer* of gestures, but also for the *reader* or rather *interpreter* of them, they seem to be important in relation to the speech used, as indicated by Kendon (2004): 'The meanings expressed by these two components [gesture and speech] *interact* in the utterance and, through a reciprocal process, a more complex unit of meaning is the result.' (ibid. 108 f.) ".

In addition to written language, iconic signs (images) are significant in teaching contexts for the representation of mathematical relations or for the illustration of ideas that have become established in theories of mathematical learning. Such "written-graphical products" (Schreiber, 2013, p. 54), generated by learners and teachers alike, are called "insciptions" by Latour and Woolgar (1986).

> "Inscriptions are seen by Latour and Woolgar as a very ductile means of representation that is continuously changing and improving. " (Schreiber, 2013, p. 54).

This processual character of inscriptions can be used for mathematical learning processes in a special way. On the one hand, they are temporarily fixed and on the other hand they can illustrate transformation processes due to mobility and changeability.

Structures in mathematics instruction

The design of mathematics instruction is shaped by the teacher's intended learning goals, which are determined by the mathematical content of learning and the mathematical concepts relevant to the process of understanding. These mathematical foundations are framed by mathematics teaching principles of creating instructional situations and teaching–learning arrangements. Thus, Barzel et al. (2013) formulate "four main phases" (p. 287) to design teaching–learning arrangements that are oriented to the functions of learning and teaching and the learning activity initiated by it. For further analysis, the authors differentiate these into six teaching–learning structures.

By activating students' experiences from previous lessons and everyday experiences outside the classroom, existing knowledge is made accessible for the current learning content of mathematics lessons (*Activating*). Learners can refer to already existing knowledge and use it for the development of new learning content. While *Explorating*, learners discover and experience initial approaches and aspects of the new learning content (e.g., in authentic contexts). These findings can be shared informally within the learning group at first. By systematically observing the increase in knowledge, the learners become aware of the newly gained insights, so that an initial structuring is made possible (*Reflecting & initial Regularizing*). The pre-structured knowledge can be further abstracted, compared to previous mathematical knowledge, and linked to it (*elaborated Regularizing & Linking*). In this way, the experiences of the exploration are condensed to their mathematical essence. The knowledge condensate, in the form of regularized and linked knowledge, is documented in writing and thus fixed for later access (*Securing & Preserving*). The described structures of initial Regularization, Linking, and Securing are summarised by Barzel et al. (2013) as "organization of knowledge" (p. 285). The framework for practicing the newly acquired learning content and transferring it to new contexts is provided by the teaching–learning structure *Practicing*. The challenge for the teacher is to identify suitable tasks (Watson & Sulivan, 2008), to offer adequate support, and to ensure that differentiation potential is available for specific target groups.

These six teaching–learning structures can occur several times in the mathematics classroom or can be observed in different sequences. They are used variably by the teacher to enable an effective learning process for the students.

PURPOSES AND METHODOLOGICAL ASPECTS

The following research questions guide the analysis:

(1) Which teaching–learning structures can be identified in authentic mathematics instruction? (2) In which modes does the teacher utter in these teaching–learning structures? (3) Is there a connection between the function of the teaching–learning structures and the modes of the teacher's utterances?

The empirical findings presented refer to a lesson of 90 minutes. The lesson focuses on the introduction of addition and subtraction of like-named and unlike-named fractions in a 6^{th}

grade class of an integrated comprehensive school. The authentic lessons were videotaped and analysed using Content Structuring according to Mayring (2014, p. 104). For the analysis, two category systems were developed based on theory. One category system deals with teaching–learning structures and their importance for the instructional mathematical learning process. The following main categories are defined: ***Activating*** prior knowledge and experience (***a***), ***Exploring*** and elaborating knowledge elements (***e***), reflecting and initial ***Regularizing*** (***r***), elaborating Regularizing and ***Linking*** (***l***), ***Securing*** and preserving (***s***), ***Practicing*** in selected tasks (***p***), and other structures (***Other***). The main categories are differentiated into subcategories. The observable interaction structure (instructions by the teacher or interactions between the teacher and individual students or a group of students) is considered here. Except for the main category *Practicing*, pure instructional situations can be identified in which the teacher refers to previous learning content or explains new learning content. In addition, interactions with individual students, groups of students, or the entire class can be observed, each of which is coded in a subcategory. The category *Other* focuses on the teacher's organisational activities. A total of 502 time intervals of 10 seconds each are coded with this category system. The coding is done by two coders. An intercoder reliability of 0.89 (according to Holsti, 1969, p. 140) is achieved. The second category system deals with the semiotic resources (signs) used and their configuration (modes). For this purpose, the classical modes are used first: speech (sp), gesture (gs), and action (a). After coding, the authors decide not to consider the action mode.The actions observed by the teacher only have an organisational function and thus only have flanking significance for the learners' discussion of the content, e.g., opening the pen to write on the board or leafing through documents and the textbook in order to set further tasks. The inscriptional expression is split into two modes: written text (w) and signs, used in mathematics and mathematics teaching, as well as their arrangement, hereafter called image (i). For example, the two-dimensional representation of a fraction, an equals sign, or the pie model of fractions are likewise coded as image. In addition to the coding of single modes, combinations of modes are also included in the category system. Situations with signs that do not show any of the mode interpretations described above are coded as other (0). The intercoder reliability achieved here is 0.82 (according to Holsti, 1969, p. 140).

RESULTS

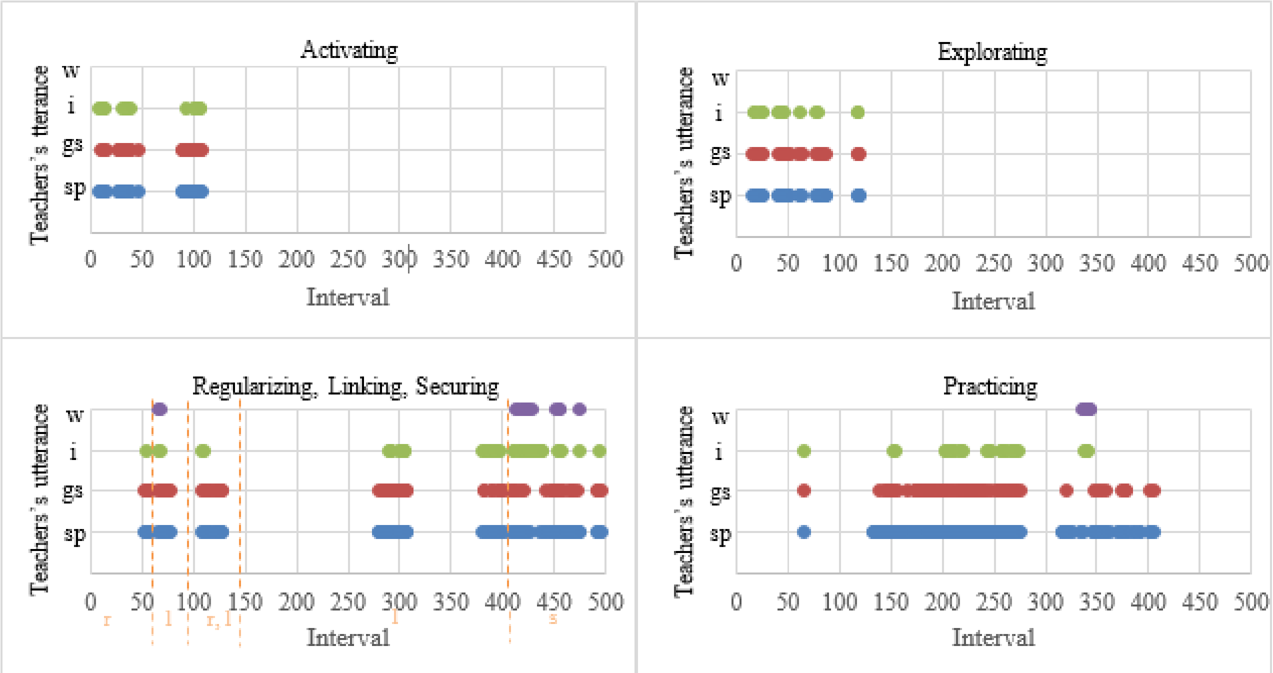

Figure 1: Modes of teacher’s utterance in progress of coding intervals, modes: speech (sp), gesture (gs), image (i), writing (w)

(1) Which teaching–learning structures can be identified in authentic mathematics instruction?

Each teaching–learning structure is condensing at certain times during the lesson. Activating previous knowledge can only be identified at the beginning of the lesson (see Figure 1, *Activating*). In the analysed introductory lesson, the teacher makes prior lesson content available by activating learners’ concepts of fractions (e.g., through iconic representations on the whiteboard). Early on, the teacher weaves explorations of the new learning content addition and subtraction of fractions with like and unlike denominators into the repetitive instructional segments of activation (see Figure 1, Exploring). This interlacing of previous knowledge (concept of fractions, comparison of fractions with the same name and fractions with different names, shortening and extending) and new knowledge elements is superseded by an interlacing of *Exploring* and *Regularizing* intervals (see Figure 1, *Regularizing*). In this process, new knowledge elements are provisionally reflected orally and increasingly systematised. The teacher therefore addresses the central mathematical aspects of the lesson at the beginning and systematises them successively in the teaching–learning structure, *Regularizing*.

Subsequently, the new knowledge content is further systematised by an alternating sequence of *Regularizing* and *Linking* intervals to support the embedding in the previous knowledge of the learners (see Figure 1, *Regularizing* & *Linking*). From interval 130 onwards, this combination of teaching–learning structures is replaced by an interplay of linking and Practicing intervals (see Figure 1, *Linking* & *Practicing*). While the students apply the new knowledge with textbook exercises, the teacher interrupts several times to discuss difficulties and errors observed by students' representation. Finally, the learning content is preserved by documenting it in the rule book (see Figure 1, Securing). This structure appears exclusively at the end of the lesson and picks up both the new learning content as well as connectable content from the previous lessons.

The subcategories of the category system of the teaching–learning structures show that the teaching practice is characterised by common dialogue between the learners and the teacher. This suggests that the teacher's spoken and gestural expressions concentrated on stimuli for the active participation of the learners in the mathematical lessons. At the same time, however, it is clear that the teacher very much moderates, complements, and drives the lesson and thus the teaching practice is very much directed by the teacher.

(2) In which modes does the teacher utter in these teaching–learning structures?

Across all coded intervals, speech emerges as an extraordinarily pronounced mode (approximately 91%), which substantially shapes the development of the mathematics lesson. Gestures are coded in more than half of the intervals, so that they are also attributed as a central part in the lesson's progress (see Table 1). Moreover, this introductory lesson reveals the close intertwining of speech and gesture, which is complemented by the inscriptional mode writing in the teaching–learning structures *Linking*, *Securing*, and *Practicing*.

(3) Is there a connection between the function of the teaching–learning structures and the modes of the teacher's utterances?

Similar modes can be identified in the teaching–learning structure *Activating* as well as in *Exploring* (see Table 1). In addition to speech and gestures, images such as mathematical signs, which are relevant for the mathematical learning process, as well as graphical highlighting (e.g., framing important examples) are used for visualisation and structuring (approximately 43%). In this way previous knowledge and new learning content are structured and networked for all learners. The teaching–learning structure *initial Regularizing* is intertwined with the teaching–learning structures *Activating* and *Explorating* and is characterised by its oral and gestural character (see Table. 1). In this first oral reflection and systematisation, the teacher refers gesturally to inscriptions on the whiteboard, created in *Activating* and *Explorating*. Based on the work with already existing inscriptions, the teacher is not dependent on the conception of new inscriptions. In the interplay of gesture and inscription, mathematical representations are compared with each other, and their respective characteristics are highlighted. Likewise, the teacher also uses fewer images (approximately 34%) in intervals of the teaching–learning structure *Linking*, with a comparable gesture percentage to the intervals of *Activating* and *Explorating* This is due to the gestural references of the teacher to already existing images on the board. Thus, the teacher works with the existing inscriptions, which are already familiar to the students from the other teaching–learning structures and uses them for the elaborated regularisation. During the *Securing* and *Practicing* of the learning content, the number of gesture codes decreases (approximately 40%), while the proportion of speech codes remains unexpectedly high. While the learners are working on tasks, the teacher comments on the further organisational workflow and gives content-related hints for the processing of the tasks. In doing so, the teacher mostly holds the solution book in his hand, which explains the relatively low number of gesture intervals (approximately 54%). As expected, the Securing, as a structure of written documentation, contains the highest proportion of written language.

	speech	gesture	image	writing	0	interval	
Activating	42	33	19	0	0	42	(8,37%)
Explorating	39	30	17	0	0	41	(8,17%)
Regularizing	18	16	4	2	0	18	(3,59%)
Linking	59	46	21	0	2	62	(12,35%)
Securing	53	26	24	13	6	72	(14,34%)
Practicing	194	97	30	7	12	215	(42,83%)
Other	52	21	3	4	0	52	(10,36%)
	458	269	118	26	20	502	(100,00%)

Table 1: Distribution of teaching–learning structures and modes among the intervals

FINAL REMARKS

Overall, the teacher's spoken utterances are omnipresent in the lesson under consideration. This is explained by the teacher's self-image as an expert, who models the process of exploration, reflection, and structuring in the sense of the "Cognitive Apprenticeship-Approach" by demonstrating how to work mathematically (Vogel, 2001). Intermodal and intramodal coherences of speech and gesture parts in the teaching–learning structures of Activating, Exploring, and Linking, as well as the high gesture part of the intervals of Regularizing, represent the initial point for further detailed analyses. Here, the focus will be on the respective concrete characteristics of the modes and their meaning for the mathematical content, and thus for the instruction and interaction between teacher and learners. The next step of the research will be to analyze and compare other teachers' introductory lessons in addition of fractions and written arithmetic operations in order to identify differences or rather similarities.

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MATHEMATICS ANXIETY AND SELF-EFFICACY OF MEXICAN ENGINEERING STUDENTS: IS THERE GENDER GAP?

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This research explores engineering students' math anxiety and math self-efficacy levels aiming to determine if there is a gender gap for this specific population. Data were collected from 498 students using adapted items from existing validated surveys. These items were translated to Spanish and validity tests were used to establish content validity and reliability. Student t-test and analysis of covariance (ANCOVA) were used to determine possible differences between male and female math anxiety and math self-efficacy levels. Male engineering students reported higher self-efficacy and lower math anxiety levels. However, the ANCOVA results showed that only the math test anxiety construct was significantly different. These results could help engineering educators to design strategies aiming to ameliorate female students' math anxiety feelings.

INTRODUCTION

It is clear that enough engineers must be educated to be able to address society's most relevant problems through developing new technologies. Many countries are employing strategies to attract and retain engineering students, and there is a special interest in attracting more females to engineering majors due to their under-representation in this field (Chubin, May, & Babco, 2005).

This research focuses on two factors that have been shown to be relevant in students' decisions to pursue and successfully complete an engineering major: mathematics self-efficacy (Hackett, 1985; Lent, Lopez, & Bieschke, 1991) and mathematics anxiety (Maloney & Beilock, 2012; Suinn & Winston, 2003). Self-efficacy refers to "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (Bandura, 1986, p. 391). Richardson & Suinn (1972) defined math anxiety as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic settings" (p. 551). Analyzing students' reactions and behaviors in math-related activities through the lens of their math self-efficacy and anxiety levels could help engineering educators to develop strategies to motivate and guide students in completing mathematics courses in engineering curricula. This is important because the lack of adequate mathematical preparation of engineering students as they complete their first semesters of engineering education is a challenge

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for educators; struggling to understand math topics and failing math courses is one of the main reasons for students leaving engineering majors (Kokkelenberg & Sinha, 2010). Strategies designed with students' math self-efficacy and math anxiety in mind could have additional positive effects if they specifically address the needs of students from under-represented populations in engineering such as women (Marra, Rodgers, Shen, & Bogue, 2009).

The literature shows a gender gap in students' math self-efficacy and math anxiety, however most studies addressing this issue usually report analysing precollege populations (Chen & Zimmerman, 2007; Hill et al., 2016). The purpose of this research is to document the math self-efficacy and math anxiety levels of students after their high school experiences, focusing on engineering students and possible differences based on gender. This research aims to expand current literature about engineering students' math self-efficacy and math anxiety due to the lack of focus in the gender gap of this specific population. This study was led by the following research question: Is there a gap between math self-efficacy and math anxiety levels of female and male engineering students in Mexico?

MATHEMATICS SELF-EFFICACY

Literature suggests that high math self-efficacy is related to increased interest in pursuing a math-related major (Hackett, 1985). One study suggests that students with low math self-efficacy are more likely to avoid math-related activities, making it more difficult to overcome struggles they may experience in their math courses (Cooper & Robinson, 1991). Conversely, students with high math self-efficacy are more likely to perform well in their math courses and persist in math-related tasks even if they experience struggles learning complex math topics (Williams & Williams, 2010). The relevance of math self-efficacy beliefs and their influence on students' performance in math courses is well-documented, and it is consistent for different contexts, cultures, and populations (Marra et al., 2009). But there is very little research focusing on how math self-efficacy can affect engineering students' performance in math courses and their motivation to successfully complete their major. Math self-efficacy is also relevant to students' decisions to leave STEM majors, with current literature suggesting that students' math self-efficacy was lower for students leaving STEM majors; this factor was more significant for students leaving college during their first semesters (Eris et al., 2010). Retaining students that are enrolled in engineering degree programs is important for industry to meet its global technological work force needs (Chen & Soldner, 2013). Math self-efficacy beliefs could play a key role in engineering students' motivation for successfully completing their major, influencing students' performance in math-related courses that have shown to be significant predictors of student retention in engineering (Middleton et al., 2015).

Previous studies have found that math self-efficacy beliefs of males are significantly stronger than those of females, and these findings seems to be similar in different contexts and populations (Gwilliam & Betz, 2001). Female students usually report lower math self-efficacy beliefs than male students even after performing better in their math courses, getting similar or even better grades than their male peers (Reis & Park, 2001). This math self-efficacy gender difference favoring male students could lead female students with similar math abilities as their male peers to perform at different levels, making them more likely to face difficulties performing math and jeopardizing their possibilities to complete the math courses required for an engineering major.

MATHEMATICS ANXIETY

Math anxiety is linked to students' perceptions of low math ability, prior unsuccessful experiences, and lack of studying or test preparation skills (Hoffman, 2010). Math anxiety comprises a set of feelings that affect students' performance in math that may lead to avoidance of math courses and math-related activities (Pajares, 1996). High math anxiety has been identified as a significant predictor of poor math performance, and is also negatively correlated with the decision to pursue a math-related major (Maloney & Beilock, 2012). Although moderate anxiety levels may actually facilitate performance and motivate students academically, high anxiety may hinder students' performance and interest in certain academic activities (Skemp, 1986). Students' math anxiety levels could be influenced by the importance they attribute to performing well in math, especially if their academic success involves math calculations (Wigfield & Meece, 1988).

Although male and female students seem to place equal importance on math-related activities and courses in their academic preparation, female students have shown to be more likely to report feelings of stress and anxiety when they perform math (Karimi & Venkatesan, 2009). This math anxiety gender gap is more evident with college students, with female college students reporting higher math anxiety levels than males. In contrast, research with younger populations such as primary school students rarely report gender differences in math anxiety (Harari, Vukovic, & Bailey, 2013). These findings suggest that the math anxiety gender gap emerges in the secondary level, showing that female students experience more anxiety when performing math-related activities than males when they are facing higher educational demands (Hill et al., 2016). Higher math anxiety could negatively influence females in their decisions to take math courses or get involved in math-related activities, making them less likely to pursue a math-related major like engineering.

METHODS

Participants for this study were selected from a Mexican university with only engineering majors. All participants were first year students taking the first math course required in their engineering curriculum during the Fall 2018 semester (n=498), with 203 female (41%) and 295 male students (59%).

Participants answered a paper-based survey during their math class time. The survey was an adapted version of the Mathematics Self-Efficacy Survey (MSES) developed by Betz and Hackett (1983) and the 30-item Mathematics Anxiety Rating Scale (MARS 30-item) developed by Suinn and Winston (2003). The MSES assesses math self-efficacy with 52 items within three constructs, asking participants to rate their level of confidence performing math-related activities on a scale from 1 ("no confidence at all") to 10 ("complete confidence"). The MARS 30-item assesses math anxiety levels with Likert-type items considering two constructs with scores from 1 ("not anxious at all") to 5 ("very anxious"). Items from these surveys were translated to Spanish and adapted to the Mexican engineering students' context to represent relevant daily activities and problems related to the math topics in their first semester math course. The adapted items were presented to professors and students at the research site for face and content validity. Minor adjustments were made based on to their feedback. Maximum likelihood exploratory factor analyses were conducted on each survey to demonstrate validity, with three constructs for the MSES items: 1) math problem-solving, 2) everyday math activities, and 3) math courses; and with two constructs for the MARS 30-items: 1) math test anxiety and 2) math activities anxiety. An oblique or promax rotation was used due to correlations found among factors within constructs. Items showing a correlation factor below 0.40 for each construct were deleted from the final version of the survey. The final version of the MSES retained nine items for math problem-solving, nine items for everyday math activities, and six items for math course self-efficacy; for MARS, seven items for both math test anxiety and math activity anxiety constructs were retained. Additionally, Cronbach's alpha values were used to evaluate the internal consistency reliability for all constructs for this specific population; all showed a Cronbach's alpha value above 0.8.

Averages of all the items within the three math self-efficacy constructs (on a scale of 1-10) and two math anxiety constructs (on a scale of 1-5) were calculated, and total math self-efficacy and math anxiety averages were determined for all participants. Seven Student t-tests were conducted to identify possible differences between the math self-efficacy and anxiety levels of male and female engineering students: three for the math self-efficacy constructs, two for the math anxiety constructs, and two for the total averages. Additionally, an analysis of covariance (ANCOVA) was conducted to determine if there was a difference between the constructs of math self-efficacy and math anxiety levels (7 independent variables) of male and female engineering students. The ANCOVA model was used for testing significant differences among all the constructs while controlling for confounding variables, and compare these results with the t-test analysis. All statistical tests were run using the statistical software R.

RESULTS

The math self-efficacy levels of male engineering students were higher than those for females for all constructs (see Table 1), and t-tests showed significantly higher average math self-efficacy scores for male versus female students. No significant differences were found between males and females for math problem-solving and math courses constructs; everyday math activities scores were significantly higher for male students.

Math self-efficacy	Female (*n*=203)	Male (*n*=295)	*St. Dev.*	*p*-value
Math problem-solving	6.09	6.33	1.69	Not Sig.
Everyday math activities	8.05	8.37	1.43	0.015
Math courses	6.88	7.17	1.95	Not Sig.
Math self-efficacy (total)	7.01	7.29	1.37	0.022

Table 1: Scores (mean and standard deviation) for math self-efficacy constructs for male and female engineering students (*n*=498), and t-tests results (*p*-values)

Math anxiety levels of female engineering students were higher than those of males for both constructs (see Table 2), and the t-tests results showed that the difference between the average math anxiety was significantly higher for female students. While the math activities math anxiety construct was not significantly different between female and male students, the math test anxiety construct showed a very significant difference with higher levels for female students (see Table 2).

Math anxiety	Female (*n*= 203)	Male (*n*= 295)	*St. Dev.*	*p*-value
Math test	3.87	3.43	0.90	< 0.001
Math activities	2.03	1.98	0.74	Not Sig.
Math anxiety (total)	2.95	2.70	0.66	< 0.001

Table 2: Scores (mean and standard deviation) for math anxiety constructs for male and female engineering students (*n*=498) and t-test results (*p*-values)

The ANCOVA analyzed the math self-efficacy and math anxiety constructs together, and showed that higher math self-efficacy average for male engineering students was not statistically significant, while the math test anxiety difference was significantly higher for female students (see Table 3). No significant differences were found between male and female students for the other math self-efficacy and anxiety constructs, indicating that math anxiety is a covariate with math self-efficacy.

Construct	F-ratio	*p*-value
Self-efficacy - Math problem-solving	2.61	Not Significant
Self-efficacy - Everyday math activities	3.39	Not Significant
Self-efficacy - Math courses	0.87	Not Significant
Math self-efficacy (average)	3.71	Not Significant
Anxiety - Math test anxiety	24.12	< 0.001

Anxiety - Math activities	1.32	Not Significant
Math anxiety (average)	0.78	Not Significant

Table 3: Results of ANCOVA identifying significant differences between male and female student math self-efficacy and math anxiety levels

DISCUSSION AND CONCLUSIONS

There are significant differences between male and female engineering students' scores on two of the math self-efficacy constructs based on t-test results, with males showing higher math self-efficacy for everyday math activities and total the total average. However, these difference drop out from the results of the ANCOVA, indicating that t-test results may have been subject to a Type 1 error due to multiple t-tests on the same data set. The average math self-efficacy for both male and female engineering students was above 7 on a scale of 1 - 10, suggesting that engineering students are confident they can perform math-related activities well and successfully complete math courses. Although math self-efficacy scores were high for both male and female students, male students tend to have higher self-efficacy for performing everyday math activities such as calculating the mileage that a car can travel with 15 gallons of gasoline. Math problem-solving self-efficacy was the lowest math self-efficacy construct for both male and female engineering students.

The math anxiety levels of male and female engineering students were relatively low, with an average math anxiety below 3 on a scale of 1 – 5. Although math anxiety average was significantly higher for female students based on t-test results, like math self-efficacy, this difference dropped out based on ANCOVA results. The math test anxiety construct was higher for both male and female students, and both statistical tests showed a strong significant difference between male and female engineering students, with female students reporting higher math test anxiety. A significantly higher math test anxiety could be a reason why female engineering students experience more stress and anxiety performing math in general. Experiencing higher math test anxiety than their male peers could be negatively affecting female students' math performance, and this could be seen at high school level where a gender performance gap starts to emerge (Ganley & Vasilyeva, 2014), with male students traditionally outperforming females in achievement and selection tests.

When math self-efficacy and math anxiety were tested in the same statistical model (ANCOVA), average differences were no longer significant. This indicates that these two factors could be covariates, or interacting with each other. This agrees with prior literature; according to Bandura (1986), math anxiety has an inverse relation to math self-efficacy levels. Any effort aiming to increase math self-efficacy may help students to decrease feelings of anxiety when they perform math-related activities. Math and engineering educators should consider developing environments that help students to learn math topics without generating feelings of stress and anxiety. Developing strategies to motivate female engineering students to feel more confident about their math abilities could help them deal with their high math anxiety and avoid negative feelings towards math courses. Decreasing math anxiety feelings could have a positive impact in female students' math performance, making them more likely to get involved in math-related activities and increase their interest and persistence in math-related majors such as engineering, which ultimately could help close the gender gap in these fields.

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THE DEVELOPMENT OF IDEAS IN GEOMETRY OPTIMIZATION PROBLEM AS REVEALED WITH EYE TRACKING

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We use eye tracking to investigate the development of ideas in mathematical problem solving as three students equipped with eye trackers attempt to solve a geometrical problem. The problem has a rich solution space, and the eye trackers allow us to follow the students' thinking as they generate plausible solutions to the problem and as they look at the different pictures representing the solutions they have already generated. We syntehsized the student eye movements into scanning signatures. *These signatures revealed two of the students shifting their attention from more optimal to a less optimal, but more complex solution that served as a stepping stone on the way to the optimal solution.*

INTRODUCTION

There is a rich body of literature on eye tracking, cognition, and expertise. Gaze is an early indication of the ultimate solution as that attentional gaze is more likely to be directed towards preferred (i.e., ultimately chosen) than unpreferred (i.e., rejected) options (Tsai, Hou, Lai, Liu & Yang, 2012). Experts more expediently locate relevant visual features than novices when performing a task (Gegenfurtner, Lehtinen, & Säljö, 2011). Novices display significantly more attentional transitions than more advanced problem-solvers (i.e., experts) while less advanced participants also use longer gaze sequences (called scanpaths) for each problem-solving task, compared with more expert counterparts (Kim, Aleven & Dey, 2014). Knoblich, Ohlsson, and Raney (2001) point out that even novices succeed more often if they attend to relevant features. Specifically in the context of geometry problems, Kim et al. (2014) highlight the value of attentional transitions.

The question arises whether eye-tracking data can provide information about cognitive processing during mathematical problem solving. In some cases, mathematical tasks have multiple possible solutions. Leikin and Lev (2007) identify alternative solution spaces for such problems. *Expert solution spaces* are the collection of solutions that expert mathematicians can suggest to the problem. *Personal (available) solution spaces* include those solutions that one is able to present independently. *Potential solution spaces* include also such solutions that once can produce with help. *Collective solution spaces* include solutions that a group can generate. In our study, we are interested in

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personal, potential, and collective solution spaces and especially how the potential solutions enter the realm of individual and personal solution spaces. In other words, how people create new solutions. We also acknowledge that the new solution can be either incrementally improved from an earlier solution (refining an idea) or a qualitatively novel solutions (creating an idea). The distinction between these two processes would be similar to, just less radical, than the difference between gradual development and conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008).

In this study, we used a geometrical optimization problem that has a large potential solution space, even though there is a unique optimal solution. Moreover, we consider some ideas qualitatively different from some other solutions. In this article, we study the development of ideas during a collaborative mathematical problem-solving task using eye tracking. We analyze the evolving visual attention to different solution attempts using the "scanning signatures" (Garcia Moreno-Esteva, Kervinen, Hannula & Uitto, 2020) of each problem solver before a significant novel idea is accepted.

METHODS

Participants

Three mathematics teacher students Meri, Tiina, and Toni, volunteered for wearing eye-tracking glasses in their mathematics education class in a Finnish university. They worked together as a group with a fourth student who did not wear eye-tracking glasses.

Apparatus

We recorded the group's work using audio recording and two stationary video cameras. The eye-tracking device had two eye cameras, a scene camera, and simple electronics attached to 3D-printed frames (see, Toivanen, Lukander, & Puolamäki, 2017). Toivanen built the glasses for our project. The data was recorded on laptop computers carried in backpacks allowing subjects to move freely.

A priori task analysis

In this study, the students were solving a four-point Euclidian Steiner tree problem. The teacher asked the students to find the shortest way to connect the four cities located on the corners of a square with fiber-optic cable. Here, we present the solutions collective solution space (Leikin & Lev, 2007), with the highest relevance to the group's problem solving process. There were also other solutions, but they received less attention from the students. The Figure 1 shows the most relevant solutions that students generated during the process. These solutions have varying degrees of optimality and complexity.

Figure 1. The student solutions that are relevant in this analysis are, from left to right, C, Z, T, H, X, odd Z, and double Y.

With optimality, we refer simply to the total length of the path. The shorter the connecting path, the more optimal the solution is. We are only interested in the order of solutions by their optimality. Hence, we use the path length as an ordering criterion, not as a continuous measure. We classified the solutions in five categories of optimality. Two solutions (Z and T) belong to the least optimal category, C and H belong to the second category, then follow odd Z, X and double Y, each in their own category.

As the measure of complexity, we use the number of nodes in the graph. In the task, there are four points to connect, so the lowest complexity solutions have four nodes. The optimal solution (double Y) and some suboptimal solutions (H and odd Z) have six nodes. There are several five-node solutions but we discuss only the optimal of these that connects the corners along the diagonals (X). While adding complexity makes it possible to find a shorter path, increasing complexity does not automatically make the solution more optimal. For example, the five-node solution X is shorter than any of the four node solutions, but the six-node solution H is as long as the four node solution C.

It would be expected that the solutions C, Z, T, and X are the easiest to find, as they simply connect the given points with direct lines in different ways. Of these, C and X are obviously shorter than Z or T, and the comparison between C and X lengths might be possible purely visually or, at least, through measurement or calculation.

H is relatively similar to C, with only one line positioned differently, so this might logically follow as the next solution idea. The optimal solution could be seen either as a modification of the solution X or of the solution H. As the solution H has already six nodes, this might better serve as a springboard towards the optimal solution. However, if the students already have concluded that X is shorter than C or H, their path could rather follow from there. The solution idea odd Z could be seen as a combination of the ideas X, Z, and H.

Procedure for data collection and analysis

The course teacher agreed to engage his class with a problem presented above. Students first worked on the problem individually, then with a partner, and then in groups of four. The fourth student in the group did not wear an eye tracker.

As students tried to solve the problem, they invented several possible solutions, concluding that X was the best solution. However, they continued to work together, comparing each others' solutions and generating alternative solutions for approximately ten minutes as their teacher challenged them frequently to search for a better solution. Both the odd Z and double Y solution were first produced by Tiina. In our analysis, we examined how the student solutions developed in complexity and optimality through an analysis on the students' visual behavior during this time.

The software used the information from the camera's focused on the eyes to produce a marker of the gaze location on each frame of gaze video. The researchers and a research assistant then annotated the targets of each student's every gaze manually.

From the temporal dwell sequences, we computed scanning signatures (Figure 3) (Garcia Moreno-Esteva, Kervinen, Hannula & Uitto, 2020). In a scanning signature, nodes in a network represent the areas of interest (AOI), i.e. the relevant features of the scene. The relative size of the node tells the frequency with which a student was attending to that feature during the process. If student gaze moved directly from one feature to another feature, it is represented as an arc connecting the respective nodes. The thickness of the arc reflects the frequency of a such transitions during the process. Finally, we use colors to represent the temporal order in which features are observed and in which feature transitions (which indicate visual comparisons) take place on average.

The scanning signature thus tells how much attention, relatively speaking, the student devotes to each solution (size of the node), how common are transitions (which indicate comparison) between different solution attempts (width of the arrow), and in which temporal order these nodes and transitions occur (colors from red to blue). Moreover, we computed the average order in which the students attended to different gaze targets (Figure 2).

Results

In Figure 2, we present how the focus of attention for all three students developed during the problem solving session, using the "average order" measure of scanning signature. One thing to notice is how two students moved from a shorter solution to a more complex solution that is longer. This suggests that it is important to introduce complexity even if it leads to solutions that might not be near optimal in order to make the cognitive leap to discover solutions of higher complexity, potentially leading to the optimal solution. We see that the students have individual patterns before focusing on the odd Z solution. After attending to odd Z, all students progress to focus on the optimal solution. This reflects the observable progress in the group where they figured out the optimal solution after discussing the odd Z solution together.

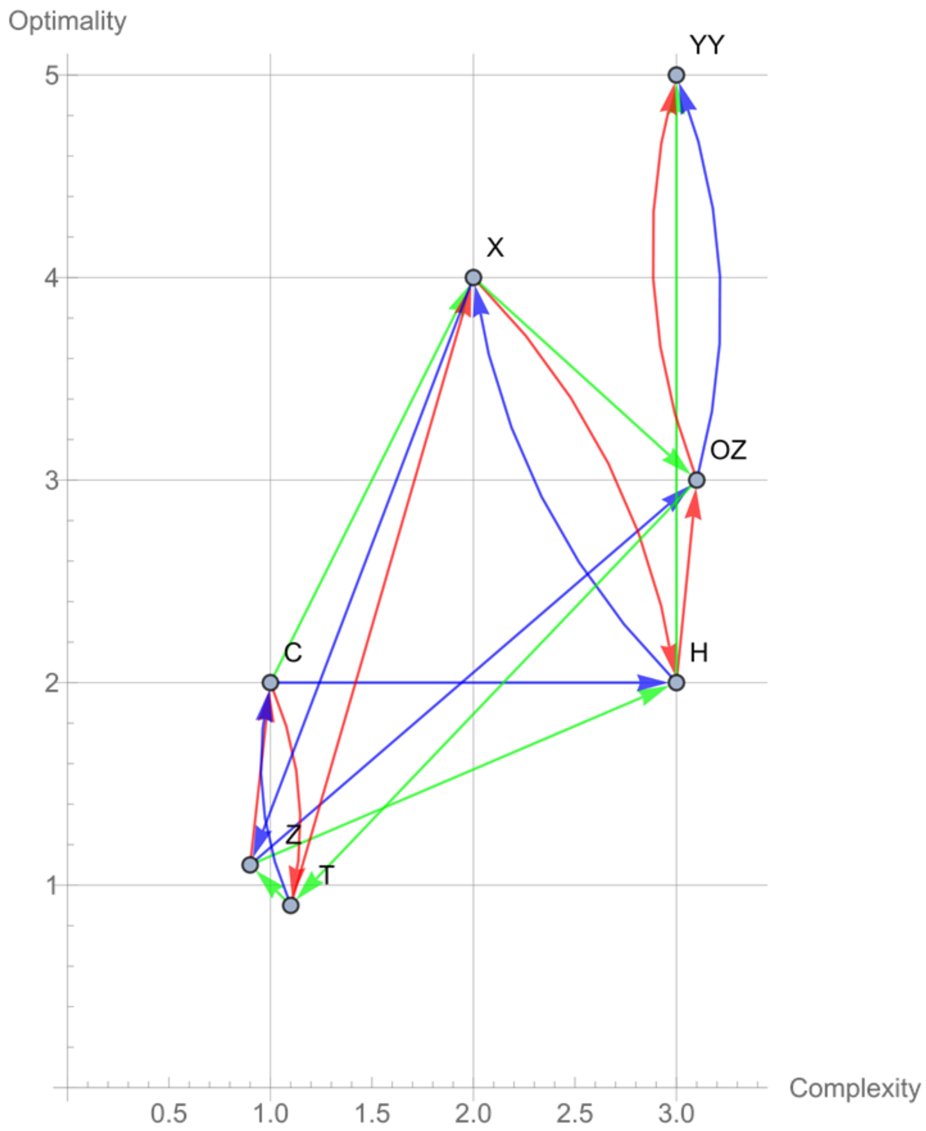

Figure 2. The gaze trajectory of each student: Toni/Red {Z,C,T,X,H,OZ,YY}; Tiina/Green {C,X,OZ,T,Z,H,YY}; Meri/Blue {T,C,H,X,Z,OZ,YY}.

The scanning signatures of the three students (Figure 3) before discussing odd Z indicate differences in how they analyzed the solution space in their search for a novel idea. All three have spent most of their attention on X, which they considered as their best solution. Toni had compared it with Z and to some extent also with H. Meri had compared X with C and H. Tiina, on the other hand, had examined rather evenly all other solutions, including a half-completed double Y, which she had drawn herself. Unlike Toni and Meri, her visual process reveals her not only comparing other solutions with X, but also comparing these other solutions to each other.

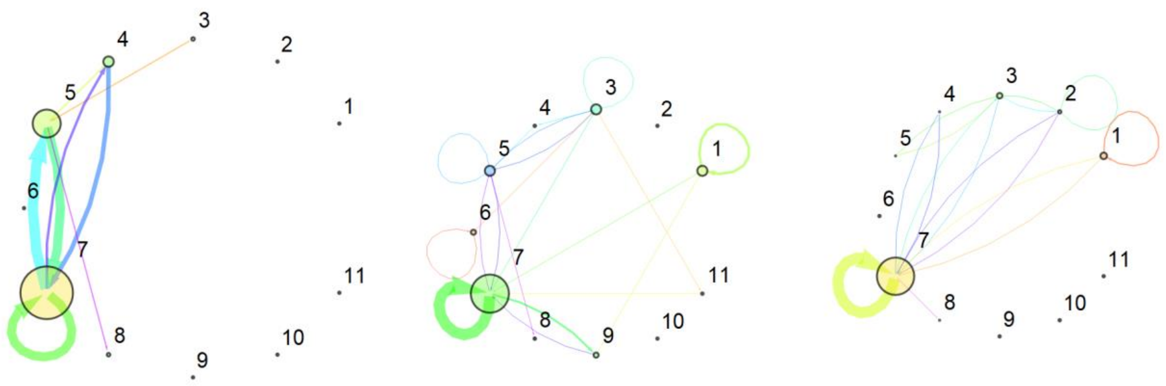

Figure 3. Scanning signatures for Toni, Tiina, and Meri before they discuss the solution odd Z. In this figure, 1 is the problem statement, 2 are some presolutions (e.g two of the four dots connected), 3 is C, 4 is H, 5 is Z, 6 is T, 7 is x, 8 is the odd Z, 9 is the double Y, 10 are variants of the double Y, and 11 are areas with computations.

DISCUSSION

Our study sheds light on mathematical problem solving as a process. There are two different observations that we make from this case study that highlight the importance of opening up solution space when searching for novel solutions. First, the pathway for all students in the group from a suboptimal solution (X) to the optimal solution (double Y) went through an intermediate solution (odd Z) that was less optimal than X, but had the same, higher level of complexity, as the optimal solution. Second, the student who was successful in generating novel solutions used a different visual strategy for working with existing ideas than the two students who were not successful. While the unsuccessful students focused on the so far best solution (x) and mostly compared other solutions to it, the successful student was comparing solutions in a more uniformly distributed way.

While our study is a small-scale case study, it presents how the scanning signatures can synthesize the complexity of eye-movement behaviour in a meaningful way. The time series representing a visualization process is a complex data set with thousands of data points for each student. Hence, it is challenging to synthesize the eye-movement process in a phenomenologically meaningful way, which is both, intuitive, as well as quantitative. Unless the data can be quantified, it cannot be analyzed further with mathematical or statistical tools.

When analyzing eye-tracking data, it is natural to ask the following questions:

a) What are the salient features in the scene that the subjects observe?

b) Are there salient feature comparisons?

c) Is there an overarching temporal order in which the different features of the scene are observed?

Scanning signatures are designed to synthesize the eye-movement information both, in an intuitive and visual way, and in a quantitative way allowing further mathematical and statistical analysis. The information in a scanning signature helps us answer questions such as

a) What did the observer look at first? What did she look at towards the end? And during the rest of the observation process?

b) On what features of the scene did the observer focus her attention?

c) Did the observer compare certain features? At what stage of the observation process did this happen mostly?

In our study, the subjects were mostly looking at diagrams or drawing plausible solutions to the problem that was given for them to solve. They scanned different solutions, and while they discussed some of their ideas, the verbal information covers only a fraction of what we see in the eye-tracking data. Scanning signatures help us to understand how the participants navigated the solution space of already produced solutions as they attempted to find an even better solution that the teacher hinted there to be. Each of the representations draws upon a particular idea for its production. It is in this sense that navigating the visual space provided by the drawings reflects a trajectory in a cognitive space of ideas required to produce the representations as plausible solutions to the problem. And thus, we are able to analyze meaningfully, the flow of ideas that leads to the discovery of the optimal solution of the problem.

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MATHS RECOVERY CAN IMPROVE EARLY NUMBER LEARNING IN THE SOUTH AFRICAN CONTEXT

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This paper reports on findings from a study focused on developing early number skills in a context of widespread low learner attainment in number. Positive learning gains achieved through Wright et al.'s (2006) Maths Recovery (MR) programme provided the rationale for a grouped intervention trial with comparison individual intervention cases incorporated. Learner pre- and post-test data around the relatively short intervention indicated comparable learning gains made by a sample of middle- attaining Grade 2 learners across the grouped and individual intervention formats. Gains made by the intervention group also compared favourably to those made by a matched control group. This is an important finding in any context of highly inequitable learning outcomes coupled with 'lags' in number learning on the ground.

INTRODUCTION

The number skills children learn during their first years of schooling are considered by many researchers as the key foundation upon which all understandings of mathematics are built (Reys et al., 1999). Most of the mathematical concepts children learn during primary school are based on a sound understanding of number (Perry & Dockett, 2008). Poor number skills in young children, which are evident as early as pre-school, have far-reaching consequences as these learning 'deficits' tend to snowball rather than decrease over time (Spaull, 2013), thus the need for early intervention. Researchers agree that early intervention for children who fall behind the mainstream in hierarchical subjects like mathematics yield better learning gains over time than intense input during later years of schooling (NCTM, 2000). The importance of a solid foundation in early number skills, coupled with evidence of substantial differences in these skills among children in the early grades, prompted the development of Mathematics Recovery (MR).

THE MATHEMATICS RECOVERY (MR) PROGRAMME

The MR programme has been used successfully in many developed countries to improve low attainers' early number learning (Wright et al., 2006). Wright and colleagues make a strong case for individualised intervention that grows out of initial, comprehensive assessment of the child's current number knowledge which then forms the basis for instruction that is aimed 'just beyond the cutting-edge of the child's current knowledge'. The widespread success of this programme has caused it to evolve over time to the place where it is now also used as a daily classroom numeracy programme, addressing the needs of all ages and levels of ability.

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The Learning Framework in Number (LFIN) is a key part of the MR programme which guides teaching and assessment and provides a trajectory charting children's learning. The LFIN enables summary profiling of children's number knowledge which is an important first step for intervention and forms the basis for instruction. Learner profiles are created using several interrelated aspects of early number delineated in the framework: Stages of Early Arithmetical Learning (SEAL); Base-Ten Arithmetical Sequences; Forward Number Word Sequences and Number Word After; Backward Number Word Sequences and Number Word Before and Numeral Identification (Wright et al., 2006). An explanation of the SEAL and Base Ten aspects follow as the rest of the LFIN aspects are fairly self-explanatory.

Strategies for Early Arithmetical Learning (SEAL) focuses on children's most sophisticated strategies for solving additive tasks. The SEAL profile is the most important part of the LFIN and is the only aspect where progression is described in terms of stages. In every other aspect of the LFIN, progression is described in terms of levels. In MR, every new *stage* is likened to a "plateau" and is characterised by "a qualitative advancement in knowledge" while a *level* is regarded as "a point on a continuum" (Wright et al., 2006, p.190). In this interpretation, progressing from one SEAL stage to another involves a bigger shift in conceptual understanding than moving across levels in other LFIN aspects. The SEAL stages are:

SEAL	Description of learners' additive strategies
Stage 1	can only count perceivable items
Stage 2	solves additive tasks using the 'count all' strategy
Stage 3	uses curtailed counting strats like 'count on' and 'count-down-from'
Stage 4	uses 'count-down-from' and 'count-down-to' efficiently
Stage 5	solves 2-digit tasks using 'non-count-by-one' strategies like derived facts, N-10 and compensation without the support of materials

Table 1: Description of SEAL stages

Using base-ten involves the organisation of numbers and calculations into 1s, 10s, 100s and 1000s and is premised on children seeing/using ten as a unit. 'Base-ten thinking' draws on children's ability to reason in tens and ones which is linked to informal place value knowledge. Children's ability to skilfully structure numbers and calculations using 'base-ten thinking' helps them to get a handle on working with numbers larger than twenty (Wright et al., 2012). A focus on developing 'base-ten thinking' emerged during intervention when learners struggled to increase and decrease a quantity (shown with bundling sticks) by ten without counting-in-ones. Placing learners at particular Base-Ten levels requires determining their facility with using ten as a unit to solve additive tasks (Wright et al., 2006). The Base-Ten levels are:

Base Ten	Description
Level 1	solves additive tasks involving tens by counting-in-ones
Level 2	uses ten as a unit to solve additive tasks involving 10s and 1s but needs a tens-based setting (e.g. rek-en-rek) to do so
Level 3	solves additive multidigit tasks using ten as a unit, without materials

Table 2: Description of Base Ten levels

Children's understanding of ten as a collection of ten ones and as a single unit of ten is crucial for solving 2-digit by 2-digit number problems (Steffe et al., 1988). So, to achieve the highest SEAL stage – which involves solving 2-digit additive tasks mentally without a concrete tens-based setting – learners need an understanding of base-ten. This means that learner's facility with using 'base-ten thinking', as opposed to counting-in-ones, is fundamental to their progress along the LFIN SEAL trajectory.

EARLY NUMBER LEARNING IN THE SOUTH AFRICAN CONTEXT

Researchers rightly describe mathematics learning in South Africa as being in a state of 'crisis' (Fleisch, 2008) when seen in light of national, regional and international comparative studies. An over-emphasis on counting-in-ones, the use of concrete materials and the lack of progression from counting to calculating strategies underlie the poor early number development seen on the ground (Hoadley, 2012). Challenges that impede children's early number learning in this context are found to apply more to poorer or working-class learners who attend no-fee public schools – who make up roughly the bottom 60% of children attending public schools (Ramadiro & Porteus, 2017). Individualised intervention in a context where the majority of learners in public schools are falling behind curriculum attainment targets (Ramadiro & Porteus, 2017) is impractical. Highly structured remedial interventions are considered key in addressing wide-spread low number attainment (in the midst of limited pedagogical content knowledge) in South Africa (Spaull, 2013). In light of the importance of developing children's early number skills and the scale of the problem in South Africa, I considered investigating whether short, structured intervention using Mathematics Recovery for grouped and individual intervention could be effective in developing the early number learning of a sample of Grade 2 learners in the South African context.

METHODOLOGY

Twenty Grade 2 learners from a public primary school were purposively selected to make up the control and intervention groups (10 learners each). Results from Annual National Assessments and an adapted Leverhulme test (Brown et al., 2008) conducted with Grade 1s at the end of 2013 were used to select 35 middle-attaining Grade 2 learners at the start of 2014. Middle-attainers were selected because they were more likely in the country's skewed international performance to gain from a short-term structured intervention than low-attainers. These 35 learners were then tested using a shortened version of MR Assessment 1.1 to select 20 matched learners for the intervention and control groups. Intervention learners were equitably split into two quartets (4 per group) and two 'singletons' based on test results and pragmatic reasons. Each group and each singleton were withdrawn from class twice a week for 18 intervention sessions of 40 min, which totalled 12 hours. A 'teaching experiment' was the approach used and the 'test-teach-test' model was followed: individual pre-test interviews using MR assessments followed by 18 grouped/individual intervention sessions based on Maths Recovery and then the same individual post-test interviews.

All intervention sessions and interviews were video-recorded (and later transcribed) so that focus could be placed on learners' verbal responses and gestures. Thus, two sets of transcriptions were generated. Based on Cobb & Yackel's (1996) Emergent Theory, which is framed by a social constructivist paradigm, the original study coordinated an individual constructivist perspective with a social interactionist perspective to analyse the different data sets. For this paper I only focus on participants' pre- and post-test interviews which were coded using the LFIN descriptors outlined earlier. Through this individual constructivist analysis a pre- and post-test summary profile was created for each participant. While Wright and colleagues do not ascribe numerical stage/level descriptors in the LFIN, for the purpose of exploratory comparative analysis I followed Venkat & Weitz (2013) in interpreting levels attained on the LFIN as scores. The SEAL stage achieved was doubled for use as a score (hence the superscript x2) to reflect the qualitative difference between stages and levels in LFIN noted earlier. My dual roles as intervention teacher and researcher created an expected tension that was managed through 'reflective practice' (Schwab, 1959).

RESULTS

The results presented here only focus on the SEAL and Base-Ten aspects of learners' (named using pseudonyms) pre- and post-test LFIN profiles. This is because of the importance of SEAL stages in the LFIN and the emergent focus on 'base-ten thinking' during intervention. Learners' scores, shown according to the mode of intervention, are shown side-by-side in Tables 3, 4 and 5 (post-test scores in bold). The highest possible score for each LFIN aspect in learner profiles are: SEAL – 5^{x2}, Base-Ten – 3, and a total score of 13.

	Khanye		Josh		Total Gains
	Pre	**Post**	Pre	**Post**	
SEAL	3^{x2}	$\mathbf{5^{x2}}$	2^{x2}	$\mathbf{3^{x2}}$	6
Base-Ten	2	**3**		**1**	2
Totals	8	**13**	4	**7**	
Gains	+5		+3		+8

Table 3: Pre- and Post-test SEAL and Base-Ten scores for Singletons

The 'singletons' scores in Table 3 show that Khanye, who started from a stronger base, made a higher gain in the SEAL aspect of the LFIN – an improvement of 2 SEAL stages – while both learners improved by 1 level on Base-Ten. Khanye made the highest overall gains for these aspects (+5) and the mean gain for singletons was +4.

	Bongi		Mali		Jenna		Sibu		Gains
	Pre	**Post**	Pre	**Post**	Pre	**Post**	Pre	**Post**	
SEAL	3^{x2}	$\mathbf{5^{x2}}$	3^{x2}	$\mathbf{4^{x2}}$	3^{x2}	$\mathbf{4^{x2}}$	2^{x2}	$\mathbf{4^{x2}}$	12
Base Ten	1	**3**	1	**2**	1	**2**		**2**	6
Totals	7	**13**	7	**10**	7	**10**	4	**10**	
Gains	+6		+3		+3		+6		+18

Table 4: Pre- and Post-test SEAL and Base-Ten scores for Group 1

In Group 1 the highest SEAL and Base Ten gains were made by the strongest and weakest members of the group, namely, Bongi and Sibu – each improved by 2 SEAL stages and 2 Base-Ten levels. By way of example, in his pre-test Sibu found solutions to tasks like 5+4 (both addends screened) by counting from ‘one’ to ‘five’ on one hand, then counting from ‘one’ to ‘four’ on the other and finally counting all his raised fingers from ‘one’ to ‘nine’. This ‘count all’ strategy placed Sibu at SEAL Stage 2. During his post-test Sibu used ‘count-down-from’ to solve removed items tasks (9-4 and 15-3) and ‘count-down-to’ for missing subtrahend tasks (12-☐=9 and 15-☐=11). His judicious use of these advanced strategies put Sibu at SEAL Stage 4 for the post-test. The average overall gain for Group 1 was +4,5 which was slightly better than the singletons’ mean gain.

	Julie		Khosi		Kyle		Kgomo		Gains
	Pre	**Post**	Pre	**Post**	Pre	**Post**	Pre	**Post**	
SEAL	4^{x2}	$\mathbf{5^{x2}}$	3^{x2}	$\mathbf{5^{x2}}$	2^{x2}	$\mathbf{4^{x2}}$	3^{x2}	$\mathbf{3^{x2}}$	10
Base Ten	1	**3**	1	**3**		**1**	1	**2**	6
Totals	9	**13**	7	**13**	4	**9**	7	**8**	
Gains	+4		+6		+5		+1		+16

Table 3: Pre- and Post-test SEAL and Base-Ten scores for Group 2

The highest SEAL gain in Group 2 was achieved by Kyle and Khosi. For Base-Ten the two stronger attainers at the time of the post-test (Julie and Khosi) gained 2 levels each. For example, in her pre-test Khosi counted-on in ones (using fingers) to solve 13+10 and 40+10 posed with dot strips. She was thus placed at Base Ten level 1. But, solving additive problems in her post-test using base-ten strategies without the support of a tens-based setting qualified Khosi to be at Base Ten Level 3. For example, Khosi correctly solved all eight additive and subtractive 2-digit horizontal number sentences (like 38+24 and 43-15) in her post-test using mental jump (N10) and split (1010) strategies. The mean gain for Group 2 was the same as that made by singletons: +4.

Next I compare SEAL and Base Ten results between *intervention* and *control* groups. For a nuanced comparison of the gains made in these aspects I will look at what was invariant between groups at the pre-test and what changed at the time of the post-test.

Intervention: Top 4 learners	Pre-Test	**Post-Test**	Control: Top 4 learners	Pre-Test	**Post-Test**
Julie	1	**3**	Yazu	1	**1**
Kamo	2	**3**	Rila	2	**2**
Bongi	1	**3**	Nozi	1	**1**
Khosi	1	**3**	Olsen	1	**1**

Table 4: Intervention and Control Top 4 learners' Base-Ten scores

Table 4 shows that in the pre-test 1 of the top 4 learners in each group could use ten as a unit with the support of a tens-based setting (level 2) while 3 of the 4 could not use ten as a unit (level 1). Post-test results show that while all 4 intervention learners could now use ten as a unit without the support of a tens-based setting (thus reaching Base Ten level 3), the control groups' Base Ten results remained unchanged.

Intervention learners at SEAL 3 for Pre-test	Post-Test SEAL Stages			Control learners at SEAL 3 for Pre-test	Post-Test SEAL Stages		
	3	4	5		3	4	5
Khanye			Khanye	Nozi		Nozi	
Bongi			Bongi	Olsen	Olsen		
Khosi			Khosi	Ntlako		Ntlako	
Mali		Mali		Sipo		Sipo	
Kgomo	Kgomo			Bonang		Bonang	
Jenna		Jenna		Rhaka		Rhaka	
6	1	2	3	6	1	5	0

Table 5: Intervention and Control Group SEAL Comparison

Both groups had the same number of learners who achieved SEAL Stage 3 (the 'counting-on' stage) at the time of the pre-test, i.e. 6 learners each. At the time of the post-test, 1 of these learners from each group remained at SEAL Stage 3, i.e. Kgomo and Olsen. The biggest difference in groups' post-test results was that half of the 6 intervention learners at SEAL Stage 3 at the time of the pre-test had achieved SEAL Stage 5 in the post-test (i.e. they could use various non-count-by-one strategies to solve additive tasks). But, none of the control learners who were at the 'counting-on' stage at the pre-test reached SEAL Stage 5 at the time of the post-test.

Discussion and Implications

The biggest gain for intervention learners was made in the SEAL aspect of the LFIN. This result may seem unsurprising as intervention centred on improving learners' strategies for solving additive tasks. Yet, the SEAL gains certainly surprised me as I did not expect such a relatively short intervention to produce the cognitive reorganisation required for learners to make big shifts. But, in the absence of a delayed post-test one cannot be certain to what extent these gains were sustained over time.

The biggest individual gain across these two LFIN aspects was made by Bongi, Sibu and Khosi – who all received grouped intervention. These learners had very different pre-test profiles. Both Bongi and Khosi were at the 'counting-on' SEAL stage but could not use ten as a unit when solving multi-digit tasks. Sibu, not placed on the Base-Ten model, solved additive tasks using 'count-all'. The significance of this result is that gains were possible to achieve across the attainment range in the sample.

Khosi had the highest overall gains for SEAL and Base Ten in Group 2, despite having begun intervention ostensibly from the same place on the LFIN framework as Kgomo who made the least gains in Group 2. This result shows that children's acquisition of early number is an individual process even in the midst of the social construction of knowledge taking place within the microculture of a group.

Both intervention and control groups had similar pre-test Base Ten results for their top four learners but the post-test results differed considerably – with higher attainment for intervention learners. This result speaks to the value of 'base-ten thinking' as a key contributor to the use of more sophisticated mental strategies for solving additive tasks.

CONCLUSION

The comparable mean learning gains made by learners who received one-on-one intervention and those who received grouped intervention is an important result as this outcome points to the efficacy of both intervention formats. This is an encouraging outcome in our current context of highly inequitable learning outcomes and the scale of the maths crisis on the ground (Schollar, 2008) where it is reported that 75-80% of children in South African public schools are two to four grade-levels below the grade-appropriate levels outlined in the curriculum (Spaull, 2013). Further, the higher gains made by intervention learners in the SEAL and Base Ten aspects compared to matched control learners is a promising outcome from short-term intervention using Maths Recovery which suggests that a broader trial into the use of MR for intervention in similar contexts is warranted.

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I DON'T UNDERSTAND THESE EXERCISES:

STUDENTS INTERPRETING STORIES OF FAILURE

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In this paper we focus on students' view of difficulty in solving mathematical exercises in order to study how the "learning to learn" competence may occur in the specific case of learning mathematics. Preliminary results give insights into "learning to learn" competence in mathematics and suggest directions for intervention in case of difficulty. Moreover, results show the importance of co-responsibility between teacher and student in learning and the crucial role of the teacher in helping students to overcome learning difficulties.

INTRODUCTION AND BACKGROUND

The European Union recommendations (Education Council, 2006) identify "learning to learn" competence and mathematical competence as two of the eight key competencies for lifelong learning.

"Learning to learn" competence is defined as "*the ability to pursue and persist in learning, to organise one's own learning, including through effective management of time and information, both individually and in groups*" (European Council, 2006 annex, paragraph 5). Such a competence consists in being aware of one's learning process and needs and being able to overcome obstacles, also through external guidance, with the final aim of learning. In such a definition, motivation to learn (especially a specific discipline) and self-confidence are crucial.

Mathematical competence is defined as "*the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations, with the emphasis being placed on process, activity and knowledge*" (European Council, 2006). The notion of mathematical competence is crucial yet problematic, as outlined by Maracci & Martignone (2016) who point out that mathematical competence can be considered a "boundary object" across the institutional context, pedagogy and mathematics education. In the field of mathematics education, Niss (2003) refers to mathematical competence as "the ability to understand, judge, do and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (p.7) and takes into account a set of components: "*thinking mathematically, posing and solving mathematical problems, modelling mathematically, reasoning mathematically, representing mathematical entities, handling mathematical symbols and formalisms,*

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communicating in, with, and about mathematics, and making use of aids and tools" (Maracci & Martignone, 2016, p. 263). This definition applies to specific mathematical activities and refers to all mathematical fields and all school levels. However, as Maracci and Martignone point out, it doesn't take into account metacognitive abilities and affective and volitional resources. On the contrary, we take as a starting point the relevance of affective factors in mathematical activity (McLeod, 1992). As Goldin (2002) put it: "*When individuals are doing mathematics, the affective system is not merely auxiliary to cognition, it is central*" (p. 60). Hence, we propose to consider attitude towards mathematics as a key component of mathematical competence. Di Martino and Zan (2011) propose a multidimensional model for attitude towards mathematics characterized by three interconnected dimensions: emotional disposition towards mathematics, vision of mathematics, perceived competence in mathematics.

In this paper we are interested in the way "learning to learn" competence may be described in the special case of mathematics and its possible links with attitude towards mathematics and connected affective constructs. More specifically, we investigate the way secondary school students interpret a situation of difficulty in mathematics, and how they are (or are not) resilient to difficulty and failure. The study may provide new insights to help teachers in helping students in difficulty, from the perspective of instructional design and formative assessment of learning.

As pointed out by Lutovac (2019), research rarely addressed the issue of failure from a qualitative and subjective perspective. Lutovac analyzes pre-service teachers' narratives of failure, showing that the concept of failure is highly subjective and is shaped by students' goals and expectations. Lutovac claims that failure resilience should be addressed as "*an adaptive process, including the variety of strategies that allow students to not only cope with failure, but also adapt and respond to it in ways that allow them to maintain their identities and continue learning, developing and changing*" (p. 243).

In order to study students' interpretation and reaction to a situation of difficulty, we also refer to two constructs. The first one is causal attribution, that is what subjects perceive as cause of the successes and failures (Weiner, 1980; Clarkson & Leder, 1984). Weiner (1980) categorizes attributions and perceptions of success and failure along three main dimensions: locus (internal vs. external), stability (stable vs. unstable), controllability (controllable vs. uncontrollable) of the causal agent. The second one success theories, that is what students believe to promote success and understanding in mathematics. For instance, students (especially those that are task-oriented) think that effort and cooperation with peers may promote success (Nicholls, Cobb, Wood, Yackel, Patashnick, 1990).

METHOD

As noted by Di Martino and Zan (2015), interpretive methods are more and more used in mathematics education research, thus shifting the attention from "measuring" or explaining in terms of causal relationships to understanding complex phenomena. In this paper we adopt an interpretive approach. An element of originality is the starting point, which is "fictional". Students were proposed a comic strip that tells a story of difficulty in mathematics (Figure 1). The last drawing is empty and the students were asked to fill it, according to their personal interpretation of the story. Data consist in written texts for the last strip (in the following, we'll refer to them as "answers"). This means that, rather than asking directly students' views about difficulty and failure, students were involved in a fictional narrative situation of difficulty and failure and led to give it a sense.

Figure 1 – Translation of the strip: *1. Ms Swanson, I don't understand the fourth exercise. 2. Obviously, I don't understand very much also the other three exercises. 3. To tell the truth, I don't understand mathematics at all. 4 Empty balloon to be filled*

The written answers were collected during a university orientation workshop (in collaboration with ALISEO - the regional Student Orientation Agency) for secondary school students, aimed at presenting them different options for their university career. More specifically, the data collection took place in front of the information desk for the science of education department. Totally, 355 secondary school students provided their answer. They came from secondary schools with different orientation. They could answer in an anonymous way and chose voluntarily to take part into the study.

Data were initially coded adopting two perspectives: 1. "learning to learn" competence (code assigned by author 3, researcher in education) 2. attitude towards mathematics and connected affective constructs (code assigned by authors 1 and 2, researchers in math education). For each answer, a minimum of 1 code and a maximum of 5 codes per perspective were attributed. For instance, answer 110 ("*I don't feel like doing the exercises, I give up, I don't like*") was coded as "drop out" and "no motivation" from the "leaning to learn" perspective, and as "negative emotional disposition" for the perspective of attitude towards mathematics and connected affective constructs. Once coded all the answers, we looked for co-occurrences between couples of codes or code groups (one referring to the "learning to learn competence", one referring to affective constructs) that seemed to be interesting in order to study how the "learning to learn" competence may be related to mathematics learning and to difficulty and failure in mathematics. Results in terms of occurrences of codes and co-occurrences of codes are discussed in the subsequent paragraph.

RESULTS

As already mentioned, we collected 355 answers. Only 8 of them were out of topic and were not taken into account for our study. This is a good result, since almost all students accepted to "put themselves" into the situation of the comic strip. The answers were analyzed using the textual analysis software Atlas.ti (v. 8), which generated the relevant Code-List, Code-Books and Code-Manager files. Totally, we used 143 codes (98 referring to "learning to learn", 45 referring to attitude towards mathematics and connected affective constructs). Since for each answer it was possible to use up to 10 codes, we found 1499 occurrences of codes (790 referring to "learning to learn", 709 referring to attitude towards mathematics and connected affective constructs). To organize results, we decided to group codes into 17 code groups. For instance, the group-code "internal strengths", referring to those resources that can promote "learning to learn" competence, comprehends the following codes: "engagement", "awareness", "motivation", "improvement", "collaboration", "communication", "complicity", "resolution", "self-confidence".

First of all, we report on those code groups that were most represented. For the "learning to learn" competence, the relevant code groups are "**help request**" (71 occurrences) and "**internal strengths**" (114 occurrences). Being aware of one's needs and searching for external help are important components of the "learning to learn" competence. Concerning "help request", we note that only 2 answers refer to a peer/classmate.

Concerning **attitude** towards mathematics and other related affective constructs, we mention the group-code "**low perceived competence in mathematics**" (172 occurrences). We may note that, within that group-code, the most represented code is understanding (36 occurrences). The group-code "**view of mathematics**" occurs 24 times, while "**negative emotional disposition**" (the third component of the tripartite model of attitude, according to Di Martino & Zan, 2011) occurs only 9 times. The situation of failure seems to call into action the perception of one' competence, rather than the other two components of attitude.

Two connected constructs that emerge are "causal attribution" and "success theories". "**Causal attribution**", that is the process of looking for causes for failure, appears 35 times. Interestingly, only 8 occurrences represent external causal attribution (for instance, in answer 279 the student Patty says to Mrs. Swanson "*Maybe you should learn how to explain better*"), while in all the other occurrences the students identify an internal cause for failure (for instance, in answer 27: "*I should listen to the lesson more carefully*"). Being aware of a possible cause of failure is a good starting point for overcoming the difficulty, if the cause is perceived as controllable (e.g., attention during the lesson).

220 students try to identify some strategies to overcome difficulty. We coded such answers as "**success theories**", since they give us some insight into what is perceived by students as way to promote learning and reach success. The group-code "success theories" contains for instance the following codes: "looking at future", "engagement", "explain again", "remedial course", "sense of the task", "wake up". The most represented codes within the group-code "success theory", apart from "looking at future" (73), that is a general disposition to overcome difficulty by changing some habit, are "**explain again**" (59) and "**engagement**" (50). Indeed, those are not necessarily very efficient strategies. "Explain again" refers to a mere repetition of the lesson. Engagement in terms of willingness to listen again the same explanation is not necessarily efficient.

In front of difficulty and failure, also "**math avoidance**" is widespread (72 occurrences). Math avoidance (Di Martino & Morselli, 2006), as the decision to choose the kind of school/university that provides the least amount of hours of mathematics in order to avoid difficulty (e.g. answer 3: "*Then, scientific-oriented secondary school is not for me*"), can be seen as the extreme way of giving up and not trying to overcome difficulty.

Overall, the analysis of occurrences gives us a first glance on "learning to learn" competence and affective constructs, in front of failure. In order to gain more insight on what students need in order to overcome difficulty, we also study co-occurrences. For our aim, we confine our analysis to specific couples of code groups and codes.

The first couple of code groups we take into account is "success theories" and "learning". The most relevant co-occurrence of "success theories" is with the specific codes "**after school tutoring**" (61 occurrences) and "**explain again**" (26 occurrences). Hence, we may say that students recognize to after school tutoring a key role in overcoming difficulties. Students seem to perceive the class as the place of "explaining": if the action of explaining does not lead to understanding, a new explanation is asked - not different or better, but simply a "replay". Understanding often is reached at home with the help of another adult ("after school tutoring"). Knowledge construction and recovery after failure don't take place, as one might think, at school (also understood as an interpersonal relationship with teacher and classmates) but out of school, with a tutor, who totally assumes the responsibility of the student's learning.

If we look at the couple "success theories" and "learning strategies", we find that the specific code "looking at future" (to face failure situation) has interesting co-occurrences with "**understanding**" (23), "**reflection**" (14), "**method of study**" (12) and "**classroom concentration**" (11). More in details, two specific codes belonging to "success theory", that is to say "**engagement**" and "**explain again**", have good co-occurrence with the code "understanding". Once again, "engagement" and "explain again" seem to be fundamental, also with respect to learning strategies. In reference to the method of study, we wonder which kind of resources students identify to overcome difficulty. If we look at resources for learning, we find that "success theory" has a good co-occurrence with "engagement" (69), "motivation" (39), "**awareness**" (33) and "attention" (28). This shows that students are aware of the importance of internal resources. At the same time, they do not seem to have at disposal useful strategies to activate both engagement and motivation. In other words, students are willing to engage, but do not know how to "direct" such engagement. Indeed, "engagement" is associated with "attention in the classroom" (10), "collaboration" (1), "concentration" (3), "engaging in class" (1), "listening" (1), "practicing" (4), "doing exercises" (7), that are not necessarily efficient in terms of learning. Even "after school tutoring" is strongly associated with "engagement" (28 occurrences), but it is not well explicated what engagement means. Only 5 students interpret the engagement in terms of "**study**" (e.g. answer 58: "*You can see that I did not study, next time I'll have to study more*"). Here we see the crucial role a teacher should play in promoting engagement and providing strategies to overcome difficulty.

If we focus on the **method of study**, which is a relevant part of the "learning to learn" competence, only 9 students mention it. Moreover, if we look in details, we find that only 1 of these really refers to a method of study in terms of metacognitive awareness (answer 331: "*Couldn't you help me or advise me what to do?*"). Only in this case, we note a co-construction of learning, that is a relationship between teacher and student in order to construct learning.

PRELIMINARY CONCLUSIONS

In the present study we investigated students' views of difficulty and failure in mathematics, as a preliminary step to identify their needs and, consequently, approaches to help them in class. To this aim, we studied students' ways of addressing a fictional story of difficulty and failure in mathematics. The analysis of their answers provided information on students' attitude towards mathematics, as well as their learning to learn competence in the special case of mathematics.

We found that students link failure to a low perceived competence, rather than to the other two components of attitude towards mathematics (view of mathematics and emotional disposition). In connection, we also evidenced a widespread phenomenon of mathematics avoidance. Those cases represent a first type of reaction to failure, that is to say just "assuming" difficulty, without any attempt to overcome it. Another reaction to failure consists in searching for causes, that is the process of causal attribution. Interestingly, we found that most causal attribution is internal. For all the aforementioned reactions to failure, we find a confirmation of the fact that an intervention at affective level could be efficient, as already discussed in the literature (Di Martino & Zan, 2011).

Most students complete the comic strip with a final sentence that refers to something that can be done to overcome the situation of difficulty and failure. Most students look for an external help, which is in principle a good approach. However, some elements are striking. First of all, the kind of help that is required is a mere "explanation". There is no reference to alternative ways to reach understanding, nor to class activities different from a frontal lesson. It is significant that only two students ask the help from a classmate. Moreover, many students refer to after school tutoring as the way of overcoming difficulty. This sheds light on their view of the mathematics lesson at school, that is the moment when explanation is given, versus home tutoring, the moment when difficulty is treated. Hence, the teacher is seen as the mere provider of explanation, rather than a mediator in the process of learning. Students don't require the co-responsibility for their learning with the teacher and such a responsibility is given to another adult, the tutor, outside the school. Thus, the students seem to reject the implicit pact of co-responsibility in learning that should be present in school.

Finally, many students are aware of the importance of relevant resources, such as engagement and motivation, but do not seem to have at disposal useful strategies or methods of study. Engagement, without having at disposal alternative learning strategies, risks to be non-relevant. Hence, internal resources are not used to support some learning strategies that are deemed to be effective for overcoming failure in mathematics.

As a preliminary conclusion, our study highlights the crucial role of the teacher in creating a learning environment where students learn together, with the mediation of the teacher, and are actively involved in doing mathematics, not just listening to explanations (Goos, 2004). In such an environment, alternative ways of learning are promoted and each student is at the same time responsible of his learning and resource for the classmates, as advocated in the formative assessment approach (Black & Wiliam, 2009). We mention here the project by Lee & Johnston Wilder (2013), where students are involved as co-researchers and led to experience alternative ways of learning and models of resilience. In this way, students are given the opportunity to express their voice and become "active, resilient participants in the learning process" (p. 175).

In this way, the teacher could also address students' perceived competence in mathematics. We mention here the study by Heyd-Metzuyanim (2013) who shows the crucial role of the teacher in the construction of an identity of math failure. Hence, we argue that the teacher can also have a role in turning difficulty into a success story.

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CONTRIBUTION OF ACADEMIC MATHEMATICS STUDIES TO TEACHING TOPICS FROM THE HIGH-SCHOOL CURRICULUM

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This study investigates the contribution of academic mathematics studies to classroom teaching of topics from the high-school mathematics. Data sources included interviews with five teachers who taught high-school mathematics before, during, and after their academic mathematics studies. All teachers exemplified changes in classroom teaching of various topics, which they linked to knowledge they acquired in the academia. The domain of analysis and the topic of integrals were central in the reports. Use of the Essence-Doing-Worth framework revealed that the contribution of academic mathematics studies to classroom teaching was mostly related to: essence of the topic and topic-related calculations; the worth of the topic was seldom mentioned.

INTRODUCTION

In many countries, the education of high-school mathematics teachers traditionally includes a strong emphasis on academic mathematics courses. The assumption underlying this tradition is that academic studies of mathematics are relevant to teaching high-school school mathematics. Yet, the scholarly literature on the contribution of knowledge of academic mathematics to school teaching is rather limited. It mainly comprises theoretical contemplations of potential contribution, and offers only limited empirical research-based information (Even, 2020). Review of the literature suggests that teachers often do not regard their studies of academic mathematics as having any contribution to their teaching. When they do, they tend to refer to contribution to their knowledge about the nature of the discipline of mathematics (e.g., Hoffmann & Even, 2019; Baldinger, 2018), and sometimes mention responding to students' questions or enriching topics taught (e.g., Even, 2011; Shamash et al., 2018). Reports on contribution of knowledge of academic mathematics to teaching specific high-school topics are rather rare. This is the focus of our study.

The literature that centers on the contribution of academic mathematics studies to teaching specific mathematical contents mainly focuses on identifying and making explicit connections between specific contents in academic mathematics and contents in school mathematics (e.g., Wasserman, 2018). Most of this literature centers on abstract algebra and to some extent analysis. For example, connections between group axioms studied in abstract algebra and the conventional procedure for solving equations in secondary school (Suominen, 2018), or between group axioms and the concept of inverse function (Zazkis & Kontorovitch, 2016).

Whereas the study of connections between specific contents in academic mathematics and in school mathematics is growing in recent years, it is not clear yet, how knowledge of such connections contributes to the actual teaching of mathematics in secondary school. The scarcity

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of teachers' reports on contribution of their knowledge of academic mathematics to their teaching of specific secondary school topics may reflect the marginal role that such knowledge plays in teaching such topics. Yet, results of recent intervention studies that aimed at supporting teachers in connecting university to secondary mathematics (e.g., Wasserman et al., 2019) suggest that knowledge of connections between specific contents in academic mathematics and in school mathematics has the potential to contribute to the teaching of mathematics in secondary school, when academic mathematics courses purposely focus on helping teachers notice these connections.

Thus, an alternative explanation of the scarcity of teachers' reports on contribution of academic mathematics studies to the teaching of specific secondary school topics could be that such contribution may exist even when academic mathematics courses do not purposely focus on helping teachers notice these connections. It could be that teachers are simply unaware of this contribution. The limited teaching experience of prospective teachers may hinder their capability to recognize such contribution. And practicing teachers, who commonly start teaching after finishing their academic studies, may take their knowledge for granted, making it difficult for them to notice the contribution of their knowledge of academic mathematics to their classroom teaching of specific topics in school mathematics.

Our study addresses this issue by focusing on a special group of teachers, who are likely to have greater capability to notice contribution of their knowledge of academic mathematics to their teaching of specific secondary school contents. These are teachers who began their studies of academic mathematics after several years of teaching high-school mathematics, and continued to teach in parallel with, and after their studies, at the same grade levels they taught prior to their academic studies. The research question is: *How do teachers, who taught high-school mathematics before, during, and after their academic mathematics studies, regard the contribution of academic mathematics studies to teaching topics from the high-school mathematics curriculum?* To address this question, we adapt the EDW (Essence-Doing-Worth) conceptual framework developed by Hoffmann and Even (2018, 2019), and examine the contribution of academic mathematics to teaching specific mathematics topics, using three aspects: (1) essence of the topic, (2) topic-related calculations, and (3) worth of the topic.

METHODS

Five high-school teachers participated in the study. All of them started teaching high-school mathematics, with no academic education in mathematics. After several years of teaching, these teachers enrolled in a B.Ed. program in mathematics education, that included academic mathematics courses whose scope and level were parallel to B.A or B.Sc. university mathematics programs. The teachers taught in Girls' high-schools (single-gender education) the same grade levels before, during, and after their academic mathematics studies.

Data sources included individual semi-structured in-depth interviews with the teachers. The aim was to learn how the academic mathematics courses contributed to their teaching. The interview consisted of two main open-ended questions: (1) Has there been any change in the teacher you were before the program and the teacher you are today? and (2) Has there been any contribution of the mathematics courses you studied to your work as a teacher? Following each question, the interviewees were asked to explain their responses and to give examples from their experiences in the program and their teaching. In addition, the teachers were shown artifacts that we had expected would help to refresh their memory, such as, the academic courses syllabi and list of topics in the national mathematics high-school curriculum.

The interviews were transcribed and then analyzed, employing the method of directed content analysis (Hsieh and Shannon 2005). First, we marked all interview excerpts in the transcripts that included reports on changes in classroom teaching of mathematics topics that were explicitly linked to new knowledge acquired in the academic mathematics courses. Second, we marked the mathematics domain and the specific topic of each reported change, and using the EDW framework (Hoffmann & Even, 2019) we coded each change as contribution to teaching pertaining to: (1) essence of the topic, (2) topic-related calculations, and (3) worth of the topic. The analysis was iterative and comparative, and included peer validation. Finally, we focused on the topic mentioned the most in the interviews – the integral – and conducted a similar in-depth analysis of the changes reported regarding its teaching.

FINDINGS

Below, using the three EDW aspects, we briefly describe the contribution of academic mathematics to teaching different mathematics domains and topics. Then we elaborate on the topic of integrals that was rather central in the teachers' reports.

Contribution to teaching different mathematics domains and topics

Analysis of the interviews revealed that all five teachers reported on changes in their classroom teaching of selected mathematics topics, which they explicitly linked to new knowledge they acquired in the academic mathematics courses. These changes in classroom teaching were associated with a variety of topics in different mathematics domains. All five teachers reported on changes in their teaching of selected topics in analysis; four teachers in selected topics in each of the domains of algebra and probability; and three teachers in selected topics in geometry. The topic of integrals was mentioned by four teachers; all other topics were mentioned by 1-3 teachers.

The teachers reported mainly on contribution related to two aspects: *essence of the topic,* and *topic-related calculations.* The aspect, *worth of the topic* was hardly mentioned in the interviews.

Regarding the aspect *essence of the topic*, the teachers reported changes associated with frequently used mathematical concepts and topics, such as, extremum point, derivative and integral in analysis, conditional probability and complementary events in probability, solution of equations in algebra, and Euclidean geometry in geometry. The teachers explained that following their academic studies, they modified the way they have presented these concepts in class. Such changes, they said, helped them to present concepts as they are in mathematics, and to overcome difficulties they had previously encountered in their teaching when using inaccurate definitions. In the following we exemplify changes Teachers A and D made in the definitions of extremum point of a function that they had previously used in their teaching (which did not cover all possible cases of extremum points) after learning at the academia a definition that was based on the notion of neighborhood of a point.

Teacher A's original definition was based on finding points of "transition from increasing to decreasing" and vice versa. In the interview excerpt below, she explains the reason for the change she made:

Teacher A: Let's say extremum points. Then, in the past, in high school, I could have said, point of change. Today I will never say that... that an extremum point is a transition from increasing to decreasing. [Today] I will say, this is a point that is the highest or the lowest in its neighborhood. The definitions would be more accurate today.

Interviewer: Why?

Teacher A: Because there are endpoints that are extremum points. So, when we get to the root function, I will not tell them that it is at the end. Then, I will define in advance that in my neighborhood, the point that is the highest in its neighborhood.

Teacher D's original definition was based on finding points for which the derivative was zero. In the interview excerpt below, she explains why she changed it:

> Can I say that extremum is everywhere the slope is zero? No, this is not the definition of extremum. It is a point that is the highest or the lowest in its neighborhood. Then all questions regarding functions with absolute value automatically disappear: Is this peak… is it an extremum point? It is not differentiable, then, is it not an extremum point? [Or] is it an extremum point? If the definition is precise and correct, then there won't be questions regarding these places, which actually it is unclear what happen with them, the points which are not differentiable. If it were before I learned all that I learned [in college], then for me, [to find] an extremum point, okay, then I do derivative equals zero, [and this way I] find the extremum points… No, there could be extremum points which I didn't find: extremum points which are endpoints… [and] as we say, in absolute value functions.

Regarding the aspect *topic-related calculations*, the teachers mainly reported changes associated with justifications of various methods of calculation, such as, differentiation and integration rules in analysis, and computing binomial distributions in probability. There were also a few reports related to learning new methods of calculation in the academic courses, which the teachers then incorporated into their classroom teaching. The teachers explained that their academic studies enable them to justify in class methods of calculation, which previously they presented to students as rules to follow only. The following excerpt from the interview with Teacher E demonstrates her changed practice that includes now justification of differentiation rules:

Interviewer: Has there been any change in the teacher you were before the program and the teacher you are today?

Teacher E: I really give them proofs from calculus… Derivative, proof, these are actually things that I got from the college… I was not used to get to this… proving derivatives… You start with $y=x^2$, and demonstrate the whole process on the board, with the limits, that is, with the approaching to zero… of the limit of the slopes of the secant lines… With the more complicated derivatives I also demonstrate it to them...

Contribution to teaching the topic of integrals

Four teachers reported on changes in their classroom teaching of the topic of integrals, and explicitly linked these changes to new knowledge they acquired in the academic mathematics courses. All four teachers mentioned contribution of their academic studies to their teaching of the topic of integrals in relation to two aspects: *essence of the topic* and *topic-related calculations*. None referred to the aspect *worth of the topic*.

The main contribution reported by the teachers regarding the topic of integrals was related to the connection between two forms of the integral: the indefinite and the definite integral. The teachers explained that prior to their academic studies they did not really know what an integral is. They described their teaching of this topic as focused on how to calculate areas using integral as a formula. Following their academic studies, which included formal definitions of central concepts and proofs of key theorems, three teachers reported that they made substantial changes in their teaching of the topic of integrals (the fourth teacher made only a minor change). They incorporated into their classroom teaching the use of limit of sums of rectangles in order to justify the calculation of an area bounded by the graph of a function defined on an interval and the x-axis (definite integral), and then connected it to the antiderivative function (indefinite integral) and the Newton-Leibniz formula ($\int_a^b f(x)dx = F(a) - F(b)$).

In the following interview excerpts, Teacher D demonstrated this change in her classroom teaching, which she connected to the knowledge she acquired in her academic studies. The first excerpt describes her teaching of the topic of integrals before her academic studies:

> And before I studied calculus at the college, I also didn't know about the deep connection between area and integral. I didn't really understand. I thought, okay, to find area, this is what I am supposed to do [$\int_a^b f(x)dx = F(a) - F(b)$], and this is what the girls [students] need to do. So, I automatically taught technically what needs to be done. Not where it came from, and what it is, and so on. And that it doesn't have to be area, the integral, like it can be many other things. And this is why an integral can be negative, the definite integral, it can be negative and not only positive...

She then moved to describe the change she made in her teaching practice following her academic studies, and explained how she works now with her students:

> The goal was, okay, how do I find area without knowing the concept of the integral. How do I find the area between the function and the x-axis?… And then we really started to break it [the area] down into rectangles, together with the girls [students], and take the maximum of the areas of the rectangles, and take the minimum... So, the area should be in the middle. And then of course I introduced them to the theory of limits … And we did algebraic operations, and suddenly they see that the derivative function was created… And then they say: Ah, okay, then the area should be in the range between this and this… we [then] reached the inverse operation of derivative. And that was it, this is where it came from. And then we also proved the fundamental theorem of calculus, and so on.

As demonstrated above, this change in classroom teaching was related to both aspects: *essence of integral* and *integral-related calculations*. The teachers used the knowledge they acquired regarding the connection between the indefinite and the definite integral to explain in class what an integral is; an explanation which served also to justify the way the definite integral is calculated. In addition to this significant change in classroom teaching, two teachers reported on three less substantial changes related to the indefinite integral, which were associated with the aspect *integral-related calculations*. Each of these changes was mentioned by one teacher.

Two of these changes were mentioned by Teacher C. She described how before her academic studies, she used to teach her students a large number of formulas for finding the antiderivative function. During her academic studies she realized that quite a few of these formulas are actually specific cases of general formulas. This insight led her to change the way she presents and uses in class formulas for finding antiderivative functions. Instead of providing her students with a long list of formulas, she switched to presenting only general cases and to guide her students in using these general formulas in specific cases. For instance, $\int f(x)^n dx = \frac{f(x)^{n+1}}{(n+1)\cdot f\prime(x)}$ when the function $f(x)$ is linear, is used in her class as a general case for finding the antiderivative of the function $g(x) = \frac{1}{(3x+5)^2}$.

This teacher mentioned also that learning about the notion of integrability in her academic studies enable her to respond to students' common claim that it is possible to find the antiderivative for all functions:

> Because often girls tell me: 'It is possible to find the antiderivative of every function.' Then I show them that there are many more functions that don't have antiderivatives. And many more functions for which is it very complicated to find their antiderivative, or that it is known that they have antiderivatives, but not how to find it.

Another specific change was mentioned by Teacher D. She was introduced in her academic studies to the method of integration by substitution for calculating integrals. Even though this method is not part of the high-school curriculum, she decided to teach it to her students because it is useful: “They use it a lot, and they also understand where it came from… why I can switch *dx* with *du*.”

DISCUSSION AND CONCLUSION

The starting point for our study was the intriguing scarcity of secondary school teachers’ reports on contribution of their academic mathematics studies to their teaching of secondary school mathematics topics. We posited that a reasonable explanation for this situation could be not the absence of such contribution, but instead, teachers’ restricted capabilities to recognize such contribution. Consequently, we focused in our study on a unique group of teachers who we expected to have greater capability to notice contribution of their academic mathematics studies to their teaching of secondary school topics. These were teachers who began their studies of academic mathematics after several years of teaching high-school mathematics with background in high-school mathematics only, and then continued to teach in parallel with, and after their academic mathematics studies, at the same grade levels they taught prior to their academic studies.

And indeed, all five teachers who participated in our study provided tangible examples of contribution of their academic mathematics studies to their teaching of various topics from the high-school mathematics curriculum. The domain of analysis in general, and the topic of integrals in particular, were central in the teachers’ illustrations of changes they made in their teaching. Thus, the results of our study add important insights to current literature, which mostly deals with the domain of algebra (e.g., Wasserman, 2018).

The use of the EDW framework (Hoffmann & Even, 2018, 2019) revealed that the contribution to teaching mentioned in the teachers’ reports was mostly related to two aspects: *essence of the topic* and *topic-related calculations*. The changes in classroom teaching reported by the teachers were mainly related to presenting mathematics concepts more accurately (e.g., extremum point, integral), and to justifying procedures used in class (e.g., rules for finding derivatives and definite integrals). Examination of these changes in light of the national high-school curriculum (Ministry of Education, 2019) suggests that these changes reflect a broader interpretation of the high-school curriculum, which previously was interpreted at face value.

In contrast with the findings of another study (Hoffmann & Even, 2019) which used the EDW framework to study the contribution to secondary school teaching of academic mathematics studies in a different program, the aspect *worth of the topic* was seldom mentioned by the teachers in our study. This discrepancy between the results of our study and that of Hoffmann and Even might be related to the nature of the mathematics courses included in each of the different programs. More research is needed to better understand this discrepancy.

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MATHEMATICS GAME-PLAYING IN SECONDARY SCHOOLS: AN OPPORTUNITY TO LEARN

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This research focuses on 9th grade students' learning processes while playing mathematical games. We adopt the commognitive lens and ask: how do the discursive processes around a board-game about quadrilaterals afford explorative participation? We found four main routines followed by the participants: two relating to the mathematical objects and two relating to the rules of the game. These routines were organized in patterned ways where the mathematical routine usually preceded the game-playing routine. In addition, subjectifying discourse was relatively neutral in the mathematical discourse while competitive and identifying in the game-discourse. In this way, the mathematical routines served as a gateway and a resource for the main goal of winning the game.

INTRODUCTION

Several studies have shown that mathematics game-playing can promote motivation and engagement (Orim & Ekwueme, 2011; Parsons, 2008). These studies join the contemporary enthusiasm about gamification and game-based learning in education (Gee, 2008; Hays, 2005). However, not much is known about the processes of learning happening through these games. Even less is known about how students learn mathematics through game-playing. Having a long experience with teaching mathematics by game-playing in middle-school mathematics classrooms (2nd author), we designed a study that aims at identifying the learning opportunities that students are provided with during mathematics games. In the current paper we report on students' discourse around a board-game about quadrilaterals, and how the organization of this discourse affords explorative participation.

GAMES IN CLASSROOMS AND LEARNING THROUGH GAMES

Teaching through games is well supported by the constructivist theory since games are natural activities for children (Vankúš, 2005) and provide them with opportunities to explore their ideas and create strategies through trial and error without being concerned with outcomes in real life. That is, mistakes which are done during games will not influence the players' real lives when compared with mistakes done in a test (Hays, 2005). With no outcomes in reality, students are more motivated to explore the subjects through the game. Their social skills and emotional regulation can be improved as well as their ability to better understand and remember what they learn (Gee, 2008; Hamari, Koivisto & Sarsa, 2014;

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 309-316. Khon Kaen, Thailand: PME.

Stott & Neustaedter, 2013). Various studies detail possible contributions of game-playing to mathematics learning. Those include: improving mental calculations, providing an environment in which students generate their own mathematics questions and problems along the game (Parsons, 2008), helping to conceptualize mathematics problems, improving students' attitude towards mathematics, motivating students to practice in an enjoyable way, learning mathematical vocabulary and ideas, improving math reading (Henry, 1973) and improving logical thinking (Orim, Ekonesi & Ekwueme, 2011). It could also enhance social and communicational skills and competencies (Gee, 2013; Higgins & Chaires, 1980). Literature relates to different types of games. In this study we focus on students playing one game which complies with the following definition: "A competitive interaction based on mathematical principles. This interaction must include at least two players that are committed to the same set of rules and are free to choose their own strategy and moves based on mathematics and luck". This definition is based upon various other definitions for games (e.g. Gough, 1999; Orim & Ekwueme, 2011).

While studies about game-playing in educational settings have been growing in number and scope, studies about the learning processes involved in learning with mathematical games is still lacking. For studying such processes, a discursive lens, which focuses on students' communication while playing, can be beneficial. Adequate methodological tools would have to be able to analyze mathematical discourse among students and the way their identities and emotions are expressed during the games. Commognition theory (Sfard, 2008) has proved useful in the past for examining processes of learning, and in particular, for examining affective aspects of communication as they interact with mathematical activity (Heyd-Metzuyanim & Sfard 2012). Next we detail about the commognitive framework.

LEARNING AS BECOMING AN EXPLORATIVE PARTICIPANT

According to the commognitive framework, learning mathematics is a change in one's mathematical discourse, towards becoming an explorative participant in the discourse (Lavie, Steiner and Sfard, 2019). Such participation is characterized by the students' authoring of narratives about mathematical objects. The learner chooses between alternatives according to the task at hand and considers alternative processes. She is expected to make independent decisions on the way. This, in contrast to ritual participation which is characterized by rigidly following rules according to others' authority and focusing on the doing and not on the goal.

To learn about the type of student participation while playing in the mathematics classroom, whether more ritual or more explorative, we look at the routines that they perform. Routines are patterns that are repeated in similar situations. We identify routines as pairs of tasks and procedures: *the task one sees herself performing together with the procedure she executed to perform the task* (Lavie, Steiner and Sfard, 2019). That is, when a person faces a new task, she is able to act thanks to her previous experience, past situations that she interprets as sufficiently similar to the present one to justify repeating what was done then, whether it was done by herself or by someone else.

As mentioned above, game-playing is often characterized as promoting motivation and emotional engagement (Hamari, Koivisto & Sarsa, 2014; Parsons, 2008). Taking a discursive lens we use the terms of subjectifying and mathematizing to analyze these affective aspects in students' communication. By subjectifying we mean communication about the participants of the discourse, including statements about one's actions, thoughts, feelings and general attributes. These communicative actions are often accompanied by a distinct *emotional hue* – that aspect of the utterance that is interpreted by the audience as indicating how the communicator is feeling. By mathematizing we mean communication about mathematical objects. Previous studies (e,g, Heyd-Metzuyanim & Sfard, 2012) have found that subjectifying can interfere with mathematizing. For example, when students are focused on subjectifying negatively about each other (e.g. in relation to who is smarter, who is better understood, etc.), the effectiveness of the mathematical communication may suffer. These studies, however, did not relate to playful or game-playing situations. There, subjectifying may be intense due to the competitiveness of the game (the individual's goal to win), yet it is unexplored whether such subjectifying is beneficial or detrimental for learning.

The goal of the present study is to examine students' communication around game-playing to better understand the affordances of this teaching practice. Therefore we ask: *How do the discursive processes around a board-game about quadrilaterals afford explorative participation?*

METHOD

In this paper we closely look into students' playing of one geometric game, the "quadrilaterals Totem game", inspired by a game named Totem. It is a board game with 10x10 ellipses (Figure 1). On 96 ellipses, a name of a quadrilateral is written. The other 4 ellipses are empty. The game includes a stack of property cards. On each property card, a geometrical property is written, for example "two adjacent angles sum to 180". The players' goal is to move their 4 plastic chips from one side of the board to the other according to the property card. In each turn it is possible to move one chip in one direction through as many ellipses that comply with the written property. For instance, if the property card says, "all sides are equal", a player can move in one turn one plastic chip in one direction but it can pass through all sequential ellipses that comply with the property. Once a student reaches an empty ellipse, players draw a new card.

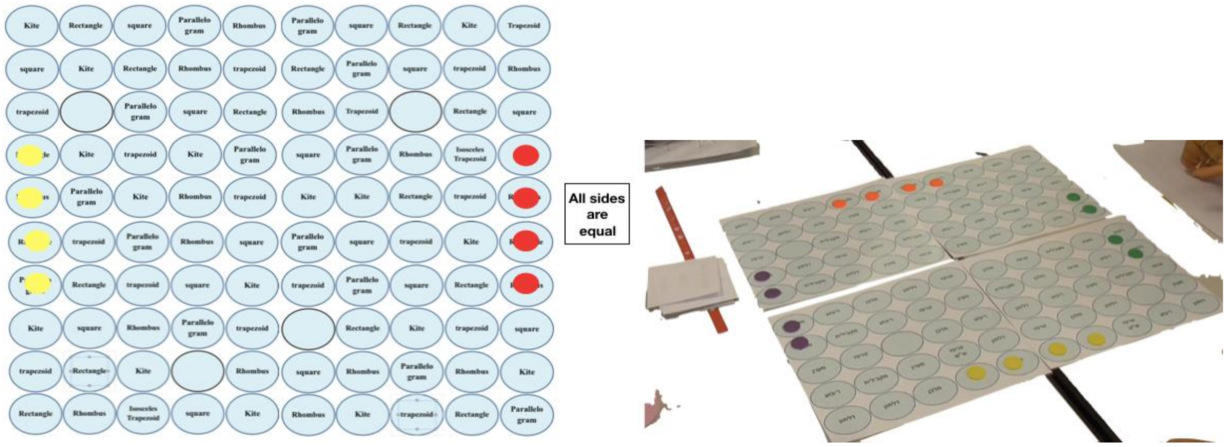

Figure 1: The Totem game board – a sketch and in reality

The data for this paper includes a videotaped 9th grade mathematics lesson in which students were playing the quadrilaterals Totem game. We focus on two groups of four students each. Both groups played for the whole lesson (about 30 minutes). One group played eight rounds and the other – five. The videotapes were transcribed and analyzed in three steps. First, utterances were identified as either relating to the mathematical discourse (talk about mathematical objects) or to game-discourse (talk about the game). Second, we identified routines that the participants performed in each of the discourses, by identifying the tasks that they set before them. To identify the task, we look at what the participants do and consider the possible task that she tries to perform. For example, if a player says: "it's a rhombus because it has four equal sides", we could interpret her task to be "justify your identification of shape". Third, we identified all subjectifying talk and identified its emotional hue.

FINDINGS

Finding 1. We found that while playing, students talked either about the game or about the mathematics. That is, they participated in two discourses – game and mathematics discourses. Only 1% of students' talk was about other issues. This implies that students were engaged with the game and did not talk about other topics.

Finding 2. When looking at students' organization of the discourse in all rounds, we found that after picking a card and reading the property, students first jointly discussed which of the quadrilaterals comply with the property and which do not. They also clarified words that were written on the card. Later, students negotiated about specific mathematical narratives produced – they challenged narratives formerly endorsed by the group and justified narratives about the shapes. In those parts, students raised doubts, tried to convince others and addressed their peers' questions. After mapping the shapes as complying / not complying with the given property, players shifted to talk about the game. Here, students talked about game strategies and negotiated options and choices of moves that they could or should take. We conclude that during the game students' discourse included the following four routines of participation: (for each routine we specify the task that defines the routine). Math_a: *Laying the mathematical ground*. This routine includes the task: to produce mathematical narratives that are required to play and win the game. This routine was initiated in every round right after reading the property card. Math_b: *Negotiating the mathematical narratives*. Task: to challenge and reason about formerly produced mathematical narratives. This was initiated when a certain shape was not checked earlier (after starting to play) or when an agreement about a shape was questioned. Game_a: *Game strategy*. Task: to identify the best strategy/moves to win. This was initiated after mapping which shapes comply (or not) with the property. Game_b: *Negotiating game-moves*. Task: to determine which moves are allowed according to the game rules.

Episode 1 illustrates the Math_a routine.

Episode 1: Laying the Mathematical Ground

Turn	name	what was said (what was done)
196	H	(Picks up a card and reads) Every pair of adjacent angles sums 180
197	N	Every pair of adjacent angles
…		
201	S	Ahh in a rectangle? What is adjacent?
204	N	Adjacent is, one (angle) next to the other (using hand gestures). Those are 180
205	S	So also in a square, in a rectangle, in a parallelogram
206	H	Almost every shape
207	S	In parallelogram?
208	H	Almost every shape except
209	S	Trapezoid
…		
214	H	Everything that includes, let's say, kite and rhombus
215	S	Great, my turn

The routine is initiated by picking up a card and reading the property (196). The task that students were performing during this episode was to produce mathematical narratives that are relevant to moves required in the game. The sub-tasks that students performed towards this goal were: (a) clarify mathematics words that are unclear (201, 204), and (b) determine the shapes that comply (or not) with the given property (201, 205-209, 214).

Episode 2: Negotiating the Mathematical Narratives

The next episode illustrates routine Math_b. The routine was initiated when a player challenged narratives that were formerly accepted by the group. The students' mathematical discourse during this routine usually included mathematical explanations and justifications. This episode refers to the property card “All opposite sides are equal” and was initiated when student M challenged a narrative formerly accepted by the group, that "in a rhombus all opposite sides are equal".

Turn	Name	what was said (what was done)
196	M	In a rhombus opposite sides are eq(ual) not equal
197	T	In rhombus all sides are equal.
198	M	But it doesn't mean that they, like the opposite have to be equal, (but just) the adjacent are equal.
199	A	But all sides are equal
200	T	But all sides are equal in a rhombus, it's like a square
201	R	Listen if it (the rhombus sides) were four four four four (each side equals 4 length units), so four equals to four (pointing at opposite sides), four equals four
202	M	Well. So, OK

Here, students T, A and R justify the formerly accepted narrative by using three justifications: (a) if all sides are equal then a sub-set of them would necessarily be equal (197, 199, 200); (b) making an analogy to a different shape (a square, (200)) that was formerly accepted as being complying with the property due to a similar reason (all sides are equal); and (c) by giving an example by attributing specific length (four) to the sides (201).

The sub-tasks that students perform here are "challenge a formerly accepted narrative", "justify a mathematical narrative" and "accept a suggested mathematical narrative".

Finding 3. Performing routines Math_a and Math_b resulted in students' production of multiple narratives. For example, during the performance of routine Math_a for the property-card: Every pair of adjacent angles sums 180 (episode 1), students authored the following narratives: (1) adjacent angles are next to each other (204); (2) Each pair of rectangle's adjacent angles sums 180 (201, 205); (3) Each pair of a parallelogram's adjacent angles sums 180 (205); (4) Each pair of a square's adjacent angles sums 180 (205); and (5) In a trapezoid, not every pair of adjacent angles sums 180 (208-209).

Finding 4. In all 13 rounds that were analyzed, the activity structure that we identified was Math_a - Game_a - Math_b OR Math_a - Math_b - Game_a. The Game routines never started the round, and Game_b did not always occur. In cases in which it did, it followed Math_b or Game_a.

Finding 5. In both discourses - the mathematics and game, subjectifying utterances were found. However, most subjectifying took place the game discourse, that is, while performing routines Game_a and Game_b. We found that 26% (76/287) of the utterances in routine Math_a included subjectifying, and 22% (68/305) in routine Math_b. In contrast, 78% (208/267) of the utterances in Game_a and 64% (190/299) of utterances in Game_b included subjectifying.

Finding 6. Subjectifying differed not only in quantity but also in quality: whereas routines Math_a and Math_b included neutral or positive subjectifying and call for collaboration, routines Game_a and Game_b included negative (mostly humorous) subjectifying narratives.

Episode 3: Negotiating Game-moves

Turn	name	what was said (what was done) [emotional hue]
321	M	You can't come back to where I am [Worried about T's move]
322	T	I can do whatever I want to and you won't tell me what to do [annoyed]
323	A	So eat him (plastic chip), eat him [excited]
324	R	T is getting into a game mode (laughing) [cynical]
…		
328	M	No, where are you going? [surprised]
329	T	BOOM! No (lands on M's chip and takes it backwards). [Pride and slight aggression]
330	A	Your turn (turns to R)
331	T	You don't mess with T! [Arrogance and pride]

In episode 3 we can see that the subjectifying during game discourse is mainly about winning, competing and teasing each other. This can be seen through the aggressive and exited emotional hue within the game-discourse, that was totally absent during the math-discourse.

DISCUSSION

Our question in this research was how do discursive processes around the game of "quadrilaterals Totem" afford explorative participation in mathematical learning. We found that students' routines of participation were organized in patterned ways, where the mathematical routines usually preceded the game-playing routines. This pattern promoted explorative participation since every time students wished to reach the actual game-playing (routines Game_a and Game_b), they needed to pass through the mathematics (routine Math_a). Also, while participating in the game, students authored multiple mathematical narratives: they determined whether to produce a narrative, which narratives to produce, whether to question a narrative stated by others or when and how to justify narratives. Also, they often did not have a ready-made procedure to follow while producing or justifying narratives. In addition, the mathematics discourse hardly included subjectifying narratives, unlike the game-discourse. This allowed relatively free engagement with authoring mathematical narratives devoid of the potential negative consequences of authoring wrong mathematical narratives. At the same time, the motive to win, and the emotional enjoyment related to that was channeled to the game-playing discourse.

The use of mathematical games in class may divert traditional narratives such as "being good/bad in mathematics" to two discourses with new narratives: the game discourse which is mainly about the narrative of "I want to win", and the mathematical discourse which is about collaborating to produce mathematical narratives that will serve the main narrative of the game. That is, the mathematical routines served as a gateway and a resource for the main goal of winning the game.

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YOUNG JAPANESE CHILDREN'S SUBJECTIFICATION AND OBJECTIFICATION THROUGH THE LENSE OF JOINT LABOR

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The article reveals how the processes of subjectification and objectification proceed in a mathematical activity at a Japanese preschool and how the roles of finger gestures change for children during these processes. The theoretical construct used was joint labor as proposed by Luis Radford throughout his work. We relied in part on his methodology and in part on a microgenetic approach to analyze a scene of addition involving children and a teacher in a Japanese preschool. The analysis captured children obtaining help from the teacher to reconstruct the meaning of fingers as tools for solving quizzes rather than preserving the meaning through unprompted practice. The analysis also showed that the role of finger gestures was reconstructed in class to solve a conflict between children's differing solutions to an addition problem.

INTRODUCTION

In recent decades, studies examining young children's mathematical abilities and skills in various areas from a constructivist point of view have accumulated (e.g., Duncan et al., 2007; Lin, Tsamir, Tirosh, & Revenson, 2013). Some have investigated children's gestures during geometrical problem-solving (Elia & Evangelou, 2014; Elia, Hadjittoouli, & van den Heuvel-Panhuizen, 2014). Moreover, socio-cultural issues, which are not intensively discussed in constructivist research on young children, have been examined in the context of research on preschool children in recent studies (Dijk, Oers, & Terwel, 2004; Radford, in press), drawing attention to the socio-cultural nature of the early development of mathematical abilities. Gestures, as well as other embodied actions with verbal languages, were regarded by Radford (2012), as the integral part of children's cognitive functioning. From this point of view, the purpose of this paper is to reveal finger gestures' mathematical roles, especially in the context of socio-cultural settings. To explore such roles, we refer to Radford's theoretical construct of joint labor (2016a, 2016b) and analyze Japanese preschool children's mathematical behaviors from a socio-cultural perspective. The Japanese Ministry of Education, Culture, Sports, Science and Technology's Course of Study for kindergartens (2017) does not explicitly define school subjects, including mathematics, and expects groups of same-aged children to acquire mathematical concepts

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and skills through integrated play. Individual preschools are responsible for designing mathematical (and other) activities. The authors believe that observing a mathematical group activity in a Japanese preschool provides an opportunity to analyze the mathematical roles of finger gestures.

THEORETICAL FRAMEWORKS AND RESEARCH QUESTIONS

In the process of illustrating his theory, Radford (2016b) proposed the idea of joint labor—where students and teachers work together to create common work—as a key theoretical construct. The theory is built on a Vygotskian view of activities, the aim of which are

> the dialectic creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical practices and ponder and deliberate on new possibilities of action and thinking. (Radford, 2016a, p. 4)

Radford (2016a, 2016b) calls such specific activities joint labor, arguing that subjectification and objectification are two sides of the same coin. These processes occur simultaneously during an activity:

> Learning can be theorized as those processes through which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and action. Those processes are termed processes of objectification (Radford, 2015, p. 551, italics in the original)

> [O]bjectification is more than the connection of the two classical epistemological poles, subject and object: it is in fact a dialectical process—that is, a transformative and creative process between these two poles that mutually affect each other […] Subjectification is the making of the subject, the creation of a particular (and unique) subjectivity that is made possible by the activity in which objectification takes place. […] [L]earning is both a process of knowing and a process of becoming (Radford, 2015, p. 553)

The concept of joint labor, thus, reconceptualizes teaching. A mathematics teacher both objectifies a new aspect of the mathematical concept to be taught and subjectifies herself as two sides of the product of a collaborative activity with her students. Radford (2016a) views:

> [T]eaching and learning not as two separate activities but as a single and same activity: one where teachers and the students, although without doing the same things, engage together, intellectually and emotionally, toward the production of a common work. Common work is the sensuous appearance of knowledge (e.g., the sensuous appearance of a covariational algebraic or statistical way of thinking through collective problem posing and solving and discussion and debate in the classroom). […] The joint labor-bounded encounters with historically constituted mathematical knowledge materialized in the classroom common work are termed processes of objectification. (p. 5, italics in the original)

Based on the abovementioned theoretical frameworks, our research questions are as follows: 1) How does the process of subjectification and objectification proceed in a mathematical activity at a Japanese preschool? and 2) How do mathematical roles of finger gestures for preschool children change in the process?

METHOD

Research Design

As mentioned, Japanese preschools design and implement the annual plan for activities on an individual school basis. The present authors have an interest in the development of children's mathematical abilities during activities implemented as a part of the curriculum. The first author of the current paper has collaborated with a Japanese private preschool on the development of a mathematics curriculum. That school participated in the study.

From the perspective of joint labor, we focus on an activity featuring mathematical quizzes where students and teachers quiz each other regarding the number of bananas belonging to a monkey. The episode of activity presented in the paper is short-term; longitudinal research on joint labor is recommended for documenting processes of objectification and subjectification (cf. Radford, 2015, 2011). Instead of the longitudinal track of studying children's mathematical development, we adopted a microgenetic approach to capture the processes of developmental changes themselves in short-term episodes rather than only milestones, or snapshots of development (Lavelli, Pantoja, Hsu, Messinger, & Fogel, 2008). Indeed, even Radford reported a set of short-term episodes as parts of the historical processes of objectification and subjectification (e.g., Radford, in press, 2016b, 2011). Accordingly, we recorded the entire session and then selected a salient segment to analyze in terms of joint labor. We did not follow the remaining procedures proposed by Radford (2015) because our focus on joint labor is a relatively new application of his theory, and its means of identification have not yet explicitly emerged in his methodological formulations (2015). Instead, our analysis was inspired by the steps used in the microgenetic approach (Lavelli et al., 2008): (1) roughly identify stable and changing components of the relationship between children and teacher through repeatedly watching a clip; (2) transcribe the clip chronologically; (3) divide the transcription into several frames; and (4) create a storyline synthesizing the frames to explain the stable and changing components.

Procedure (4) for creating the storyline is further divided into four steps: (4.1) identify joint labor based on stable and changing components of the relationship between children and teacher; (4.2) interpret the algebraic knowledge which emerges through the joint labor; (4.3) interpret how it emerges, i.e., how it is objectified; and (4.4) interpret how the children subjectify themselves.

RESULT

Children's Activity

In the implemented activity, after the children watched a video clip on wild animals, the teacher introduced an activity about monkeys and their lives. When the children and teacher sang a song about monkeys, she introduced questions about 1-digit addition and subtraction. Children were encouraged to use their fingers, with each finger representing one of the monkey's bananas. As she quizzed the children regarding the number of bananas, they calculated their answers by watching and counting the teacher's presenting fingers. After that, the children indicated that they wished to share their own quizzes by raising their hands. The teacher selected students one by one, and each in turn came to the front of the class and took on the teacher's role. Their questions followed the sentence format for verbal expressions provided by the teacher, a common pedagogy at the school. Children provided simple addition and subtraction problems, including 10 - 5, 4 - 1, 10 - 8, 10 - 5, 10 + 2, and 11-3, to their seated peers, who listened and answered the quizzes together by counting the presenting student's fingers. Student presentations were followed by additional presentations from the teacher, followed by presentations from the remaining students who quizzed their peers on 10 - 9, 10 - 7, 5 + 4, and 9 + 1.

The focal scene comprised the penultimate question asked by a student. Following the third procedure for microanalysis, the transcription was divided into six frames. All the conversations were in Japanese and were translated into English by the authors. All children's names are pseudonyms.

Frame 1 features a male child's question.

182	Yu:	(Singing a song.) Quiz, quiz
183	SS&T:	What is the quiz?
184	Yu:	(Singing.) Quiz of answering the number of bananas.
185	SS&T	What is the question?
186	Yu:	Here are 12 bananas. A monkey put 5 more bananas. How many bananas are there altogether?

After Yu's question, the teacher wryly smiled, likely due to the large size of the number, considering the ages of the children, but decided to continue the game.

187	T:	It seems more difficult than before. Okay, okay. Let us try.
188	SS:	(Raising their hands.) Yes, me.

Frame 2 shows the teacher's confirmation of the content of the question.

189	T:	How many bananas were there in the beginning?
190	Saki:	12.
191	T:	12. And how many bananas were added?

Frame 3 shows a conflict between children.

192	Yu:	(Pointing to a child, saying her name.) Hiroko.
193	Hiroko:	17.
194	Yu:	Wrong.

Though Hiroko correctly and quickly answered, Yu did not seem to correctly understand what she said. After denying her answer immediately once he put a troubled look on his face.

Frame 4 shows the conversations between Yu and teacher to reconfirm the question.

195	T:	Let us think together.

196 Yu: Wrong.
197 T: Wasn't it right? What was the answer?
198 Yu: 16.
199 T: 16. Well, let's think together. It became more challenging than before.
200 Yu: There were 12 bananas and the monkey added 5 more bananas, and then…

Frame 5 shows the start of a collaborative discussion using finger gestures.

201 T: Well, in the beginning, the monkey had 10 and 2 bananas. (Showing her 10 fingers and using 2 of Yu's right fingers.)
202 T: So, there are 12 and it added 5 more… (Showing 10 with her hands and letting Yu show 7 more with his hands.)
203 Konoha: There are 16 bananas.
204 T: 16 bananas?
205 Yu: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17.
206 T: Thus? Right! Wonderful! (Clapping) It became difficult gradually.

The teacher demonstrated the use of finger gestures, and the children followed her lead, actively counting using finger gestures.

Frame 6 shows Hiroko's reasoning about the solution to 12 + 5. Immediately after recognizing that her friends obtained the same answer she had, 17, by using finger gestures, she saw the connection between 12+5 and 2 + 5 and argued the point.

207 Hiroko: Well, 2 + 5 = 7. (Showing her 2 and 5 fingers together)
208 T: Wow, well, you counted! I see, so let us do the final one? Please.

Created Storyline

The activity presented constitutes joint labor between children and teacher. We regarded it as joint because its production was considered historically contingent. The quiz was given by Yu, a child, not by the teacher (Frame 1). Following her song format he invented a quiz by selecting the numbers 12 and 5 by himself. Although his quiz was relatively difficult for most children besides Hiroko, they finally obtained the answer '17' with the assistance of the teacher (Frame 5). The role of the teacher was to provide neither the quiz nor the answer. She only provided the format of the activity and demonstrated the use of finger gestures. That is, her role of supporting the children did not change when they presented quizzes. Instead, the children were free to decide (Frame 5) whether to count on their fingers as the teacher had modeled. The children's focuses actively changed following their interactions with the teacher. Their joint labor was visible in that particular scene.

During the joint labor, three kinds of algebraic thinking emerged as common work. First, Yu proposed new numbers: 12 and 5. The teacher's smile indicated that she did not expect her children to use a number as large as 12. However, contrary to her expectations, Yu realized that he could use larger numbers for quizzes. In the beginning, the teacher controlled the rules, but ultimately such role was delegated to the children, which might have prompted Yu to further develop the scope of the questions. In this process, the children objectified the new numbers as part of the quiz and gradually subjectified themselves as new quiz producers.

Second, the children used finger gestures to determine the number of bananas. They did so under the teacher's facilitation. For example, since the number 17 was too large for the children to quickly count, Konoha counted her fingers in error (Frame 5). The finger expressions of the number, however, spatially maintained the initial assumption of the quiz that there were twelve and five bananas and mediated the children's repeated and careful counting. Therefore, adding and counting, kinds of algebraic thinking, are re-embodied and re-mediated by the artifactual use of the fingers by the teacher. The children re-objectified finger gestures as tools for solving the conflict over the solution and re-subjectified themselves as the finger gesture users in solving the quizzes.

Third, Hiroko realized that 12 + 5 was separable into 10 and 2 + 5. Since she immediately answered the quiz, she might have already known how to calculate this way before the joint activity. Only the teacher recognized what Hiroko asserted; the other children, including Yu, did not respond to her. Her separating strategy was difficult for the others, who depended on the finger gestures. She objectified the separating strategy as a tool for solving the quizzes, but her subjectification could not be determined from this observation. When she uses the strategy again in the future, her subjectification might be gradually determined, depending on the responses from members of her community.

DISCUSSION AND CONCLUSION

Our observation of the children's reuse of finger gestures shows that spatial and numerical structures are linked in accord with Radford's (2011) claim that algebraic thinking is by nature embodied and mediated by artifacts. On the other hand, the children needed the teacher's suggestion to finger gestures. Although they repeatedly used finger gestures before the focal scene, they did not themselves propose to use them to resolve Hiroko's and Yu's conflicting solutions. This fact does not completely fit into Radford's (2008) theoretical assumption that humans on their own preserve artifacts' meanings. The children appeared to obtain help from the teacher to reconstruct the meaning of fingers as a tool for solving quizzes rather than demonstrating the preserved meaning in practice. Although the ability to preserve the meaning of artifacts might be built into human beings by nature, we may need to be taught to demonstrate such ability.

However, that the children did not use finger gestures should not be construed negatively. Instead of focusing on the intermediate process of the finger gestures, they seemed to focus on input and output. This could be an origin of flexible thinking, also called proceptual thinking (Gray & Tall, 1994), which is based on a focus on the relationship between input and output. It is natural and mathematically appropriate for finger gestures to lose their artifactual meaning for children as they master adding two numbers mentally.

We agree with Radford's (in press) argument that social rules and mathematical content in classrooms are part of the fabric of children's subjectivities. Our interpretation and analysis corroborate this. We draw one possibly important implication: the role of the knowledgeable other, in this case the teacher, in solving conflicts between learners' idiosyncratic rationalities. The observed children showed their own valuable abilities: Yu's ability to generate a new quiz, Hiroko's strategy for addition without counting, and other children's focus on the input-output relationship. However, these are still potential abilities and are not always performed in appropriate situations. Teacher intervention may potentially show them when to perform their abilities. We argue, therefore, that the traditional constructivist focus on learners' own idiosyncratic rationality (Confrey, 1991) can be more widely investigated from Radford's theoretical perspective.

Let us finally answer our two questions. First, in accordance with Radford's theory, the subjectification and objectification proceeded in the scene of preschoolers' and teacher's conversations regarding addition; joint labor in a classroom activity offers a valuable opportunity to investigate these processes. Second, through teacher mediation, the role of finger gestures was reconstructed to solve a conflict over a mathematical problem. In addition to Radford's assumptions about the ability of human beings to preserve the meanings of artifacts, we suggest that for young children, learning may be part of that process. As our methodology is, at this stage, suggested, our interpretations will be refined each time we obtain new empirical data.

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CROSSING THE BOUNDARIES OF MATHEMATICS ASSESSMENT THROUGH SUMMATIVE SELF-ASSESSMENT

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The idea of 'Assessment for Learning' is widely encouraged in education, but mathematics assessment lags behind. In Finland, mathematics is mainly assessed through exams. Implementing alternative assessment practices might cause resistance, from both teachers and students. The present study, conducted in the context of undergraduate mathematics, introduces summative self-assessment that includes the element of self-grading as an assessment model that violates the norms of mathematics assessment. Utilising the discursive framework of boundaries, it was observed whether students were able to cross the boundaries of mathematics assessment.

INTRODUCTION

"Cultures of assessment" are often advocated as beneficial for students, but they might actually hinder learning. This might be true in university mathematics, where the assessment culture is heavily based on examinations and students even prefer these traditional ways of assessment (see, e.g., Iannone & Simpson, 2015; Nieminen, 2020). The situation does not seem to differ too much at lower levels of education, either. A recent Finnish national report (Atjonen et al., 2019) revealed that mathematics teachers mainly assessed learning through traditional methods such as exams, although this is clearly against the ethos of the National Curriculum that calls for "Assessment *for* Learning". For example, the STEM subjects, with mathematics included, scored the lowest of all of the school subjects in the use of peer-, self-, and group-assessment. It seems that if assessment is to truly "reflect the mathematics that is important to learn and the mathematics that is valued" (Suurtamm, 2016, p. 5), there would need to be a substantial rethink of the culture of mathematics assessment.

The present study introduces an attempt to challenge the culture of Finnish university mathematics assessment that as such reflects the current international culture of undergraduate mathematics education (Iannone & Simpson, 2015; Nieminen, 2020) and the culture of lower levels of Finnish mathematics education (Atjonen et al., 2019). The aim of this study is two-fold: By reporting on an empirical study of an innovative assessment model, the study seeks both to highlight the outlines of the usual norms of mathematical assessment culture and to reimagine them. Following Ben-Yehuda and colleagues I argue that "a norm becomes explicit and most visible when violated" (2015, p. 183). Here, student perceptions of an implementation of a summative self-assessment

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model are reported (see Nieminen, Asikainen, & Rämö, 2019). In a linear algebra course we asked students to self-grade, therefore challenging the usual norms of summative assessment in mathematics. I call this a "shaking up method", since rather than simply interviewing students about the norms of mathematics assessment these norms were shaken up by self-grading to make their outlines more visible.

In the field of higher education, it has been suggested that mathematicians are to be held responsible for their resistance to developing new assessment practices (Burton & Haines, 1997). However, as they have been shown to favour traditional assessment methods in university mathematics (Iannone & Simpson, 2015), the students themselves might also show resistance towards non-traditional assessment practices. Furthermore, implementing alternative assessment methods in mathematics would require students to adapt to these practices by changing their concept of learning (see Martínez-Sierra et al., 2016). However, exactly how this is achieved is rarely covered in the literature. The present study approaches the concept of mathematical assessment culture by utilising the theoretical framework of boundaries, aiming to understand how students both challenge and co-create the cultural norms of assessment.

THEORETICAL FRAMEWORK: BOUNDARY CROSSING

The present study conceptualises the assessment culture of mathematics through the framework of boundaries. The approach draws on discourse analysis by defining cultures of assessment as the outputs of discursive practices. This theoretical lense allowsme to capture the two-fold nature of assessment cultures, both as the overarching artefact of the way assessment is done and simultaneously as a factor influencing the assessment practices and students' perceptions of those (Fuller & Lane, 2017). Connecting the framework of boundaries with that of discursive practices shifts the focus from "what are the boundaries of mathematical assessment culture?" to "how are the boundaries of mathematical assessment culture constructed?"

Boundaries have been defined as 'socio-cultural differences leading to discontinuity in action or interaction' (Akkerman & Bakker, 2011, p. 132). Therefore, boundaries are constructed to maintain sameness and continuity through categorisation. These categorical boundaries organise social life and maintain social order (Lamont & Molnár, 2002). A person's transition between different fields has been characterised as *boundary crossing* (Engeström, Engeström, & Kärkkäinen, 1995). Boundary crossing involves entering a new territory through negotiation of the boundaries themselves – for example, crossing the boundaries of mathematics assessment through self-assessment might leave the students feeling unqualified to assess their own learning if the transition is not truly made. *Boundary objects* (Star, 1989, cited in Akkerman & Bakker, 2011, p. 133) are used to bridge various fields in the process of boundary crossing. Here, these objects refer to concrete actions and artifacts that are conducted to help students cross the boundaries of mathematics assessment. In the present study, personal boundary crossing is defined to have happened when the new norms of mathematical self-assessment are internalised in the discourse of a student.

In the present study, the boundaries of different assessment cultures are not taken as given; rather, they are actively constructed by various actors in the field through *boundary-work* (Lamont & Molnár, 2002). It is notable that boundary-work is not always purposeful. As Tan (2012) has argued, both teachers and students bring their previously learned assumptions and roles into the assessment process. Since both may have been conditioned to these roles in assessment, boundary-work done by students occurs only within the restricting effects of the assessment culture itself. However, the present study aims to see students not as passive recipients of the assessment culture but as active agents co-constructing the boundaries of assessment cultures through their own boundary-work (see Raaper, 2019). Finally, boundary-work does not always lead to boundary crossing (Akkerman & Bakker, 2011). The present study utilises a ‘micro perspective’ as suggested by Akkerman and Bakker to investigate whether boundaries are crossed in students’ discourses, and how this is conducted.

THE OBJECTIVE OF THE STUDY

The present study uses summative self-assessment as a concrete example of an assessment model that demands that students cross the usual norms and boundaries of mathematics assessment. By observing university students’ discourses, the study aims to understand students as active co-constructors of the assessment culture of mathematics. The research questions were formulated as follows: What kind of boundary-work did the students take part in when they negotiated the boundaries of mathematics assessment after taking part in summative self-assessment? What kinds of boundary objects did they use to cross these boundaries?

METHODOLOGY

The course design: Summative self-assessment in action

This study utilises the concept of summative self-assessment (Nieminen, 2020; Nieminen et al., 2019) to refer to a self-assessment model that builds on formative self-assessment but also includes the element of self-grading. Most often higher educational studies recommend using self-assessment as a formative tool for learning that would help students to monitor their own learning (see Brown & Harris, 2013; Panadero et al., 2016). This means that during the learning process, the teacher would provide some kind of self-assessment tasks that would prompt self-reflection on one’s actions, therefore leading to a better quality of learning (Brown & Harris, 2013). However, it has been argued that effective self-assessment models would not just allow students to compare their skills and knowledge with teacher-generated criteria, but would give them power over their own grade. Hence, in the summative model the students can decide their own grade, but only after a longer process of engaging in practicing self-assessment.

The study took place at a large undergraduate mathematics course (313 participants) in a research-oriented university in Finland. The proof-based course addresses linear algebra and matrix computations and is usually one of the first courses students take in their mathematics studies Assessment in the mathematics department of this university is heavily based on individual exams. On this course, the traditional course exam was replaced with summative self-assessment; the students graded themselves on a scale of 0 ('fail') to 5 ('excellent'). During the course, self-assessment was practised through formative self-assessment. Digital feedback on students' self-assessments was offered, and students could reflect in writing how that feedback represented their skills and knowledge. The feedback processes were designed as dialogic and sustainable, and students were prompted to act on the feedback they received from their peers, tutors and themselves (cf. Carless et al., 2011). Self-assessment was based on a learning objective matrix (rubric), making the learning objectives transparent. For further details about the course arrangements, see Nieminen et al. (2019; also Nieminen, 2020; Nieminen & Tuohilampi, 2020).

Data collection and analysis

In total, 26 students were interviewed after the course about their experiences of summative self-assessment. Eight of the participants were majoring in mathematics and the rest studied, for example, computer science and chemistry. None of the participants reported to have any previous experience of self-assessment practices in mathematics.

The analysis draws on discourse analysis. First, the interview data was reduced through thematic analysis, using in vivo coding to capture the words and meanings by the students themselves. After the data had been structured into meta-themes, further discourse analysis followed.

FINDINGS

Crossing the boundaries: "Finally studying for myself"

Many students described how summative self-assessment enabled them to study in a way that was not aimed at succeeding in an exam but rather at gaining personal mathematical knowledge. These accounts were coded to reflect boundary crossing, since the students needed to internalise the new cultural norms of summative self-assessment.

> Student: It supported independent studying [studying through summative self-assessment]. Acknowledging that encouraged me to think that there's some sense in assessing yourself, and that was inspiring.
>
> ***
>
> Student: I took more responsibility of my learning. Now I didn't need to stress about any exams, but I could challenge myself with a good feeling.

Quite soon in the analysis it became evident that boundary objects were needed to support students in their boundary-work. As one student put it:

Interviewer: Will you continue assessing your own mathematical skills after this course as well?

Student: I guess I should (laughs). It would be bad to just leave it here. But it's another thing if you are offered tools for that.

The most frequently described boundary object was the detailed rubric with exemplars. A frequent theme in the data was that students felt that self-assessment was strange and complicated at first, but could be conducted after becoming familiar with the transparent learning objectives of the course. Overall, the formative self-assessment tasks were described as important boundary objects that taught goal-setting and self-reflection skills. In particular, the feedback offered from the digital self-assessment tasks was described as a crucial boundary object.

Student: In the beginning of the course it [self-assessment] felt more like, well let's just write something here according to my feelings. The learning objective matrix included many confusing concepts that you couldn't even define at the time. But during the course you digged deeper to it, you could learn those skills.

Reflection (Akkerman & Bakker, 2011) was the discursive practice that was connected with boundary crossing while the students negotiated the boundaries of mathematics assessment. Summative self-assessment forced the students to reflect on why it challenged the usual norms of mathematics assessment. Furthermore, this process enabled them to critically reflect on those usual norms. Thus, summative self-assessment did not just generate simple boundary crossing, but rather encouraged students towards critical boundary-work.

Student: If I'm studying for an exam, I often feel like now I'm studying for that exam. And for the fact that I would get a good grade. Now I felt more like I would have been learning to be able to use these skills in the future.

Student: I feel it was so useful and should be used in studying everywhere. Once in a while you stop and think about what you really know and what you don't know.

Strengthening the boundaries: "Self-assessment belongs to humanistic disciplines"

Across the whole dataset, summative self-assessment was largely described as a new, strange, and even radical kind of assessment method. Not all of the students' discourses reflect boundary crossing - many of them were not willing or able to cross the boundaries as the course teacher wanted them to. In these cases, the boundaries of mathematics assessment with their usual norms were strengthened further. Two discursive practices were identified when the students resisted internalising the norms of summative self-assessment: Naturalisation and illegimatisation.

Naturalisation was identified when the students leaned on simplifications of the assessment culture of mathematics; when socially constructed discourses and practices were taken as natural and even connected to the nature of mathematics itself. For example, the students often naturalised the traditional practices, framing the use of exams as a given. These accounts underlined that mathematics must be assessed with exams. This was seen as the nature of mathematics assessment: Because summative self-assessment does not belong to this nature, it should not be used.

> I think self-assessment belongs to humanistic disciplines. Somehow in mathematics I'm used to the fact that knowledge has to be assessed brutally.

> The teaching culture of mathematics has never before guided us towards self-assessment, so I can't say it would have felt natural.

Illegitimating the process of summative self-assessment was also a common discourse aiming to frame it as an inadequate assessment model. Many students thought that self-grading is not an adequate way of determining one's grade since it might not reflect one's real skills and knowledge. Often, the legitimacy of summative self-assessment was only doubted in certain contexts or with certain student groups. For example, it was pondered whether this assessment model was suitable for 'lazy', 'young' and 'mathematically weak' students. Self-assessment was constantly compared to exams, and many students thought that the validity of summative self-assessment could be improved with an external exam.

> Maybe this kind of self-assessment would be suited better for advanced mathematics courses later on in the studies. So that in the beginning of your studies you would get a certainty of the level of your knowledge by doing an exam.

> I would combine the methods of self-assessment and exams. Just because an exam would really show whether you have really learnt or not.

DISCUSSION

The present study investigated university students' discursive boundary-work after taking part in a mathematics course drawing on summative self-assessment. Seeing students as active negotiators of the assessment culture (Nieminen, 2020; Raaper, 2019), this study sought to understand how students either crossed the boundaries of mathematics assessment and adjusted to summative self-assessment or resisted this by further boundary-work that re-established the frontiers already existing.

The results underline the importance of offering adequate boundary objects (e.g. rubrics) to students when asking them to negotiate the boundaries of the culture of mathematics assessment. None of the students reported having any earlier experience of self-assessment in mathematics, which calls for a careful scaffolding of self-reflection. These findings reflect earlier research on self-assessment in higher education; for example, studies highlighting the importance of practicing self-reflection skills based on transparent learning objectives (Brown & Harris, 2013; Panadero et al., 2016). I argue that these kinds of concrete support systems are especially important in the exam-driven culture of mathematics. Here, the students were not only required to learn new mathematical content but to adjust to new cultural norms as well, each of which is quite demanding in themselves.

Through carefully designed boundary objects some students were able to critically *reflect* (Akkerman & Bakker, 2011) on the boundaries of mathematics assessment. As noted above, a certain structure was needed to support students in this process. However, the results of the present study emphasise the power of offering students alternative experiences of mathematics assessment. I argue that this method of ‘shaking up’ the boundaries of assessment offers a powerful tool not only for practice but for future research as well. The present study showed that summative self-assessment was able to generate reflection; would formative self-assessment, added on top of external summative testing, truly challenge the boundaries the assessment culture of mathematics in the same way?

Not all of the students were able - or willing - to cross the boundaries of mathematics assessment. Two discourses were identified as boundary work: naturalisation of the usual norms of mathematics assessment and illegimitation of summative self-assessment. These findings remind us that students are active co-constructors of assessment cultures. Their perspective must be considered while re-imagining mathematics assessment, and especially while evaluating whether boundary crossing has actually occurred. It is notable that naturalisation and illegimitation can be made visible to the students themselves through reflection (Akkerman & Bakker, 2011). However, the two-fold nature of assessment cultures (Fuller & Lane, 2017) creates a challenge, as students both co-construct the cultural norms and are restricted by them. We call for future research to tackle this methodological issue by further understanding assessment practices as *discursive* practices; as shown here, the framework of boundary crossing can offer an adequate tool for this.

CONCLUSIONS

Finally, I argue that mathematics educators have an ethical responsibility to actively try to reconstitute the exam-driven assessment culture of mathematics. Even though the present study examined the viewpoint of the students, the teachers - and researchers! - need to do boundary-work as well. It is argued that the least mathematics educators could do is to avoid strengthening the boundaries of mathematics assessment through offering alternative discourses such as those identified in this study. If new frontiers are not reached in the field of summative mathematics assessment, it might be that external testing and validation will, by default, keep dominating what is seen as ‘valued mathematics’ (Suurtamm et al., 2016). Finally, there is a need for future research that would boldly initiate innovations in mathematics assessment. A vast amount of literature on sustainable assessment practices already exists - it is time to take that knowledge into mathematical classrooms.

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THE USE OF QUANTITIES IN LESSONS ON DECIMAL FRACTIONS

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As numbers and quantities are closely related, it is sometimes recommended that situations including quantities are used so that students can understand numbers and their operations well. The purpose of this paper is to analyze the lesson on decimal fractions in order to get insights into such a use of quantities in learning numbers. The analysis using the framework, which distinguishes two different relationships between numbers and quantities, showed that while the use of quantities helped the students develop their own explanations, it enabled them to bypass the reasoning process critical for achieving the goal of the lesson. This result suggested that the use of quantities should reflect the structures of numbers necessary for learning of the lesson.

INTRODUCTION: NUMBERS AND QUANTITIES

As younger students' understanding of numbers is closely related to their experiences of quantities (Irwin, 2001; Krajewski & Schneider, 2009), it is natural that some researchers recommended using situations with quantities and measurement in learning numbers (Astuti, 2014; Rahayu, 2018).

Nunokawa (2001) extended the scheme of mathematical modeling and introduced the "Manipulations in Real World" component (Figure 1a). When applying this extended scheme to situations including quantities, we can get the scheme in Figure 1b.

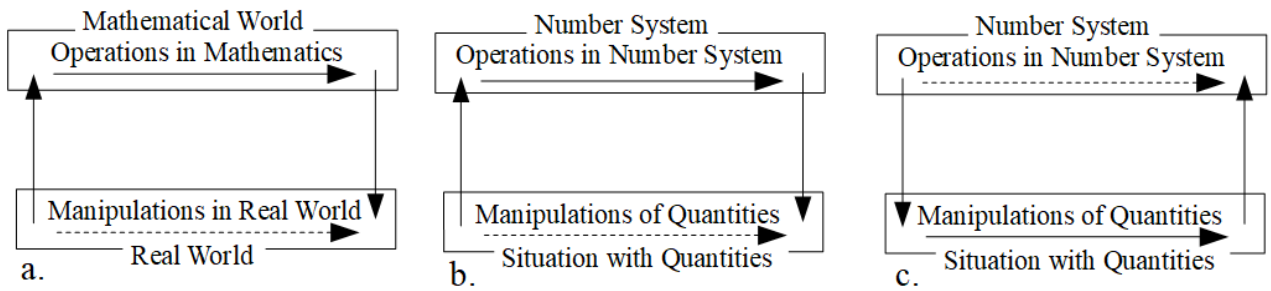

Figure 1: Relations between numbers and quantities

It shows the following process: (a) some relations among quantities in a situation are expressed by arithmetic expressions; (b) calculations in the expressions are performed and their answers are found out; (c) new information about the situations is obtained based on those answers. The dotted line indicates manipulations of quantities. The results of these manipulations can be inferred, however, by the operations in the number system even when relevant quantities are not actually manipulated.

But, when students begin to learn decimal fractions, they do not yet know how to

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perform operations on decimal fractions. That is, the arrow in the Number System box is not yet established and will be established through lessons on decimal fractions. In the lessons where teachers utilize realistic situations with quantities, it may be expected that students construct operations on decimal fractions (e.g. 0.4+0.5) by referring to manipulations of quantities and their results (combining 0.4 L- and 0.5 L-milk). The scheme in Figure 1c shows this process: (a) an operation on numbers, represented by the dotted line, is temporally considered in a realistic situation; (b) corresponding quantities are manipulated; (c) the steps and the result of the operation of numbers are established by referring to the steps and the result of the manipulation of quantities.

Nunokawa (2019) used these schemes to analyze textbooks and found that there are some pitfalls from the perspective of the relationship between numbers and quantities in learning fractions, which might provoke the difficulties students often face. It may be also possible to find such pitfalls in mathematics lessons on fractions if we analyze lessons using the above-mentioned relationship between numbers and quantities.

The purpose of this paper is an attempt to analyze a mathematics lesson from the perspective of the relationship between numbers and quantities in order to get insights into the use of quantities in learning numbers.

THE LESSON ANALYZED

The lesson analyzed here was recorded in a third-grade classroom (8 or 9 year-old), whose goal was to understand how to calculate additions of decimal fractions. For this goal, the teacher posed the following problem to the class: "The teacher's family drank 0.4 L milk at breakfast and 0.5 L milk at lunch. How many liters milk did the family drink in all?" According to her lesson plan, the teacher planned to ask her students to explain why the answer of 0.4+0.5 is 0.9 so that they became able to understand how to add two decimal fractions deeply. She expected that through devising their explanations, the students would pay attention to 0.1 as a unit of decimal fractions and understand that they can reduce the addition of decimal fractions (0.4+0.5) to an addition of natural numbers (4+5) by considering the numbers of this unit 0.1. In other words, it might be expected that her students would not understand the procedure of additions instrumentally: Removing 0 and decimal points, adding resulting integers, and bring back the removed 0 and decimal points.

The outline of the lesson

At the beginning of the lesson, the teacher reviewed the knowledge of decimal fractions, which the students had learned before and they could use to make their explanations in this lesson. The teacher asked how many liters the drawings in Figure 2a show, how many deciliters 0.1 L and 0.2 L are equal to (Figure 2b), and how many 0.1s 0.7 and 0.3 are equal to (Figure 2c). The students could respond to these questions immediately.

After the review, the teacher posed the above problem. She put two one-liter measures on a table (see Figure 3a) and poured 0.4 L- and 0.5 L-milk in each one-liter measure. Showing these one-liter measures, she wrote the problem on the blackboard.

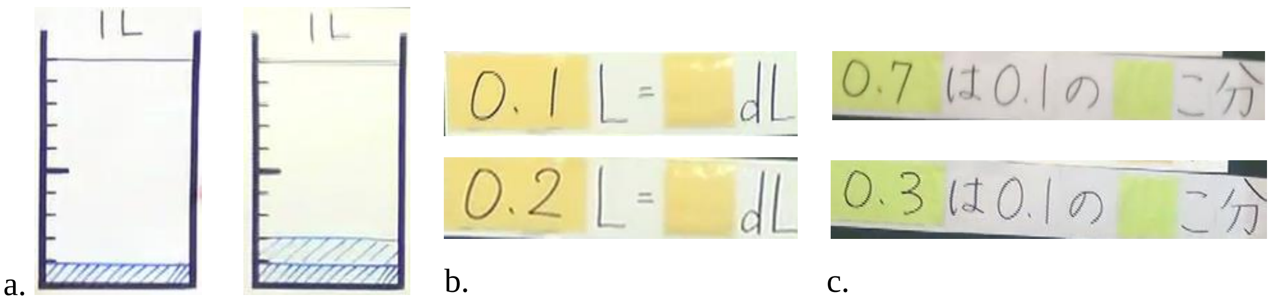


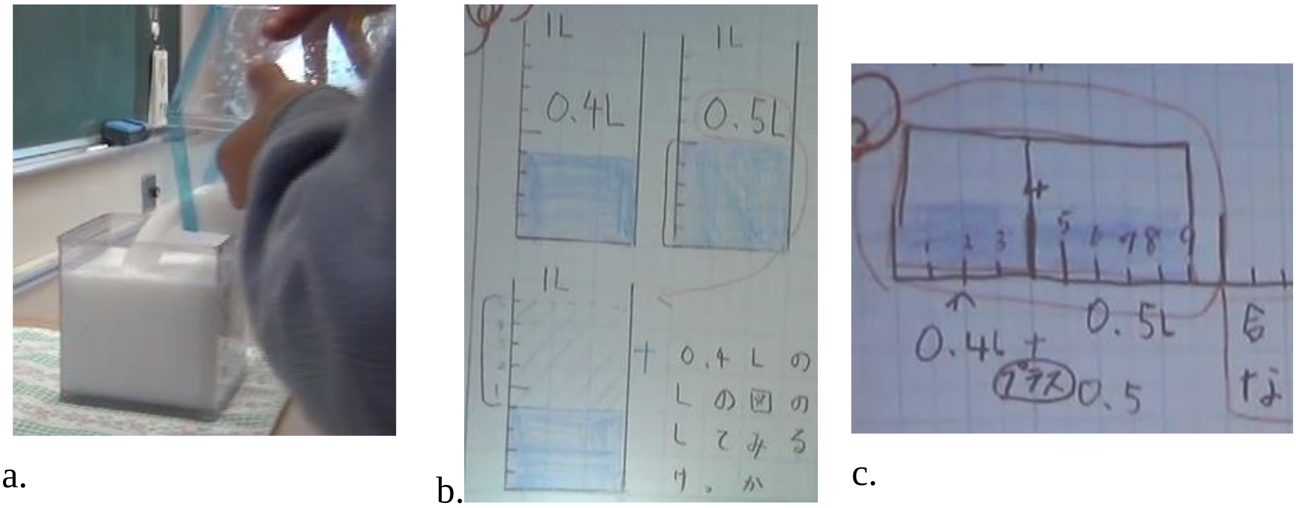
Figure 2: The review at the beginning of the lesson

The students noticed soon that they could find out the answer by addition 0.4+0.5. Many students also told that the answer probably became 0.9 L. The teacher invited her students to explain this probable result and required them to explain why combining 0.4 L and 0.5 L makes 0.9 L. The teacher wrote on the blackboard the numerical expression 0.4+0.5=0.9 and the goal of the activity as follows: “Let us explain the reason why 0.4+0.5 becomes 0.9.”

When the teacher asked them about ideas usable for explanations, the students proposed the following ideas: (a) Converting L to dL; (b) Actually combining 0.4 L- and 0.5 L-milk; (c) Using drawings; (d) Using a number line. The teacher pointed to Figure 2c and asked whether they could use this idea. Then, one student mentioned using the number of pieces after dividing it into 10 equal pieces (the idea (e)).

All the above five ideas were observed when the students attempted to build their explanations for the reason why combining 0.4 L and 0.5 L makes 0.9 L. There were students who first went to the teacher’s desk and poured the milk into another one-liter measure (Figure 3a). Some of them tried to represent what they had observed using drawings or number lines. Many other students also used drawings or number lines. The students who used a drawing represented 0.4 L-milk in a one-liter measure and then found the drawing of the combined milk by adding 0.5 L-milk to it on the drawings (Figure 3b). The students who used number lines marked the fourth tick mark on the number line and found the tick mark corresponding to the answer by taking five steps forward (Figure 3c). Some students used the idea (a). They converted 0.4 L and 0.5 L into 4 dL and 5 dL respectively and then converted 9 dL into 0.9 L to answer the “How many liters” question (Figure 3d). Most of those students also drew drawings of one-liter measures. Only a few students used the idea (e) (Figure 3e).

After the students worked individually for about 10 minutes, the teacher asked them to present their explanations to their neighbors. Finally, the teacher chose four students and asked them to present their explanations to the class. The explanations of those students were the ones based on the idea (c), (e), (a), and (d) respectively. Figure 3b, 3e, 3d, and 3c are the worksheets which these students presented to the class. The teacher also reviewed these four explanations after the four students presented their explanations (Figure 4). As most of the students came to know the explanations new to them, the teacher encouraged them to try the new types of explanation by themselves.



a. b. c.

Figure 3: The strategies the students adopted

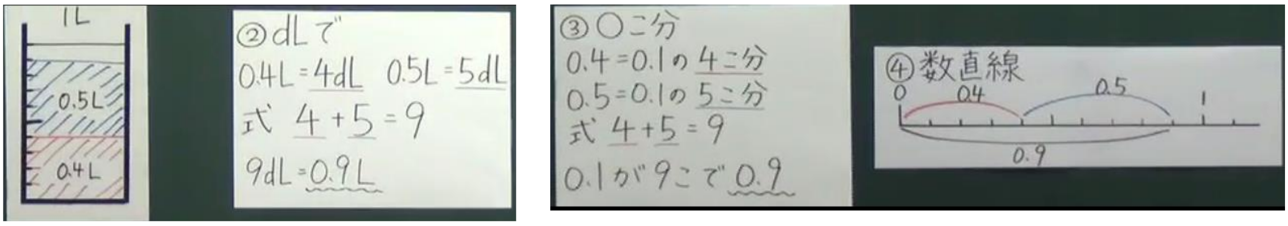


Figure 4: The teacher's review of the four explanations

At the end part of the lesson, the teacher attempted to summarize the students'explanations first. She asked the students what a kind of computation they performed when finding the answer of 0.4+0.5. The students immediately said, "An addition." When the teacher questioned them what plus what, the students told 4 plus 5. The teacher checked that all the four explanations included the addition 4+5 as a part of them. In order to clarify the main point of this lesson, the teacher asked the students what to think when we add decimal fractions. One student answered spontaneously that they should change an addition of decimal fractions into an addition of integers. Following this student's answer, the teacher began to write the summary of this lesson on the blackboard as follows, by interacting with the students: "An addition of decimal fraction gets easier by thinking of it using integers." Here the phrase of "gets easier" was proposed by some students. The students first proposed "by changing it into integers," instead of "by thinking of it using integers." As the teacher seemed unsatisfied with that proposal, the students suggested other expressions and the teacher adopted the phrase "by thinking of it using integers." When she finished writing the above summary, however, the teacher tilted her head and still looked unsatisfied.

Then the teacher asked the students to calculate 0.2+0.7 and 0.5+0.1 as a practice. The students seemed able to calculate them easily. The lesson ended with individual reflections of the lesson.

QUANTITIES IN THE LESSON

On the one hand, in this lesson, the students could translate the situation with quantities (i.e. drinking milk) into the arithmetic expression 0.4+0.5. Moreover, they could find out the answer 0.9 L and develop their own explanations of the reason why the answer became 0.9 L. On the other hand, although the teacher expected that the students would pay attention to 0.1 as a unit of decimal fractions and understand how to calculate additions of decimal fractions focusing on the numbers of the unit 0.1s, the idea of the numbers of the unit 0.1s was not mentioned by the students when the class made up the summary of this lesson and the teacher looked unsatisfied with this summary. That is, this lesson seemed successful on the one hand but also seemed unsuccessful on the other hand. In the rest of this paper, this consequence of the lesson will be analyzed from the perspective of the relationship between numbers and quantities.

Quantities' support for students' thinking

The teacher's use of the situation with quantities in this lesson enabled her students to adopt at least three strategies for thinking about the addition of decimal fractions: (a) Converting L to dL; (b) Actually combining 0.4 L- and 0.5 L-milk; (c) Using drawings representing milk in one-liter measures. Because they understood well that 0.1 L is 1 dL and 0.2 L is 2 dL and so on (Figure 2b), the students could translate 0.4 L and 0.5 L into 4 dL and 5 dL respectively and find the resulting quantity by calculating 4+5. Here, the students transformed the original situation a little by converting the units so that they could apply to this situation the mathematical knowledge they had already learnt (Figure 5a). Two one-liter measures and a pack of white-water representing milk provoked the idea of pouring one measure of milk into another to combine them, which is a manipulation of quantities. The students who used this idea could confirm the resulting quantity by the direct manipulation of quantities (Figure 5b).

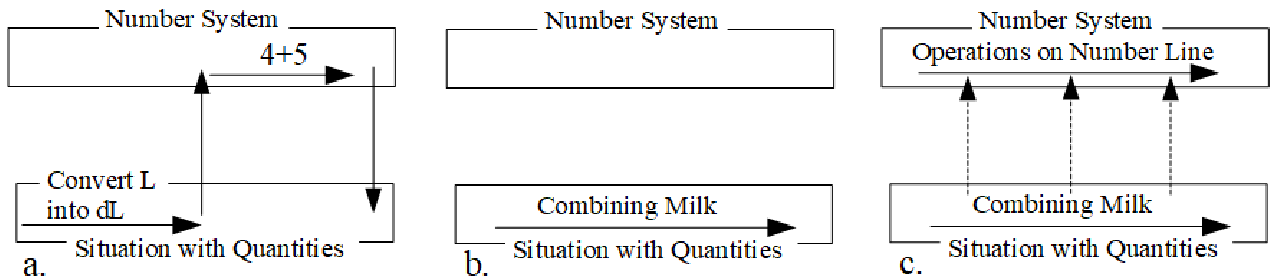


Figure 5: Strategies supported by the use of quantities

When they used the drawings, the students drew the pictures of milk in one-liter measures, the quantities in the situation. The student who presented her explanation based on the drawings (Figure 3b) to the class explained that she "added the tick marks of the 0.5 L-milk measure (i.e. 5 ticks) to the drawing of the 0.4 L-milk measure." Her explanation implied that she simulated the quantitative manipulation of pouring milk on her drawings. That is, her strategy was based on the manipulation of quantities (Figure 5b).

The operations on number lines can be considered operations in a number system. Seeing the number line in Figure 3c, which was drawn by the student who presented her explanation based on the idea (d) to the class, however, even number lines might be influenced by the use of quantities in this lesson. This number line has the following elements: (1) the picture of one-liter measures (boxes) and liquid (pained parts in the boxes); (2) natural numbers 1, 2, …, 9 written at the tick marks instead of 0.1, 0.2, …, 0.9; (3) the quantities 0.4 L and 0.5 L written at each interval on the number line instead of decimal numbers 0.4 and 0.5. In other words, this number line has the feature similar to the drawings discussed above. This suggests that the reasoning similar to the drawing strategy might be used even when the students used number lines, and their use of number lines was supported by the use of quantities in this lesson (Figure 5c).

The situation with quantities might remind the students of their experiences of manipulating quantities, and such experiences could help the students develop their own explanations of the reason why the answer became 0.9 L.

Quantities' interruption in students' thinking

The student who presented her explanation based on the idea (e) to the class wrote the following explanation without using drawings and number lines: "0.4 is four 0.1s, 0.5 is five 0.1s, when combining four 0.1s and five 0.1s, because 4+5=9, then the answer is 0.9L." This explanation is based only on the structures of decimal numbers 0.4 and 0.5, which consists of four and five 0.1s respectively, and on the operation on those structures: $4\times0.1+5\times0.1=(4+5)\times0.1$. In this sense, her explanation can be schematized as Figure 6. And this is the explanation the teacher seemed to expect her students to attend to at the end of this lesson, as her lesson plan shows. In fact, the teacher pointed to Figure 2c and tried to remind her students of the structures of decimal fractions. And it might be the reason why the teacher asked how many 0.1s 0.7 and 0.3 are equal to at the beginning of this lesson.

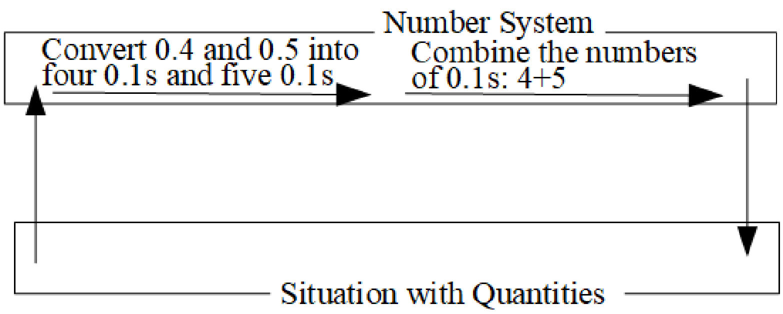


Figure 6: Explanation in number system

Contrary to the teacher's expectation, the students did not mention the numbers of 0.1s in making the summary of the lesson. The students paid their attention to the additionof natural numbers 4+5, but did not focus on 0.1 as a unit of decimal fractions.

Seeing the students' explanations shown in Figure 3, all of them except Figure 3e did not include 0.1s as a component of the explanations. That is, most of the students did not use the idea of 0.1 as a unit of decimal fractions in constructing their explanations. The students who combined milk or used drawings or number lines attempted to explain the resulting quantity on the basis of the number of the tick marks. The students who converted L into dL attempted to explain it referring to the numbers of 1 dL, instead of the numbers of 0.1 L. Even when they calculated 4+5=9, these students used this calculation to find the total number of ticks or 1 dL. The students could pay attention to the natural numbers and their addition, which represented the numbers of ticks or dL. That is the reason why the students focused on the idea of "thinking of [an addition of decimal fraction] using integers" in summarizing the lesson, but could not pay attention to 0.1 as a unit of decimal fractions.

Furthermore, the students could develop such strategies because they could resort to reasoning about quantities. Adopting manipulations of quantities, the students could bypass the conversion of 0.4 and 0.5 to four 0.1s and five 0.1s (Figure 7a). The situation with quantities allowed the students to use their knowledge and experiences about quantities to make their explanations, and those knowledge and experiences in turn made it possible for the students to convert L into dL or translate the quantities of milk into the numbers of tick marks. In other words, the use of quantities in the lesson might facilitate bypassing the conversion of decimal fractions to multiples of 0.1s.

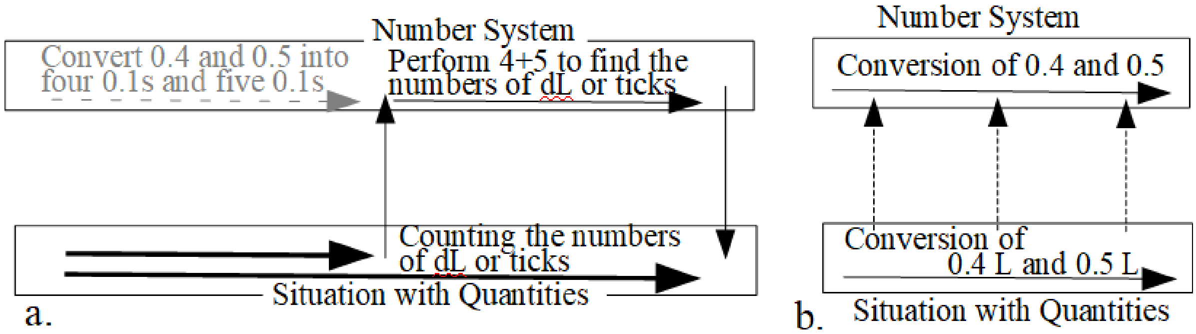


Figure 7: Bypass of conversion of 0.4 and 0.5

In fact, the use of quantities can facilitate converting decimal fractions into multiples of 0.1s. In order to make the use of quantities facilitate this conversion, the way in which quantities are used should reflect this conversion of decimal fractions. That is, 0.4 L and 0.5 L should be converted into four 0.1 L and five 0.1 L, instead of 4 dL and 5 dL. Even when 0.4 L and 0.5 L are represented by drawings or number lines, it should be highlighted that a tick mark represents 0.1 L, and the numbers of 0.1 L, rather than the numbers of tick marks, should be focused on. This conversion of 0.4 L and 0.5 L might be able to facilitate the conversion of 0.4 and 0.5 into four 0.1s and five 0.1s by the translation of the structures of quantities into the structures of numbers (Figure 7b). The fact that this conception of 0.4 L and 0.5 L, i.e. four 0.1 L and five 0.1 L, was not highlighted during the lessons can be considered the reason why the students did not mention the numbers of 0.1s in making the summary of the lesson.

Furthermore, seeing the activities at the beginning of this lesson, the students could convert 0.7 and 0.3 into seven 0.1s and three 0.1s without much trouble. If the students could carry out this conversion without referring to quantities, it should be examined whether it was really necessary to adopt the learning process shown in Figure 1c. The learning process shown in Figure 1b could have been also adopted in this lesson. That is, a situation with quantities is used only to introduce additions of decimal fractions and students attempt to develop their explanations why 0.4+0.5=0.9, without referring to 0.4 L+0.5 L=0.9 L, based on the structures of 0.4 and 0.5, i.e. consisting of four and five 0.1s respectively.

CONCLUDING REMARKS

The above finding has some implications for teaching numbers using quantities. First, while the use of quantities can be helpful for learning numbers, the ways of using them must correspond to the ways of seeing numbers which are necessary for achieving the goals of lessons. If students are expected to use a certain structure of numbers in a lesson, a teacher should introduce the situation which enables students to attend to the aspects of quantities corresponding to that structure.

Second, the use of quantities does not necessarily lead to learning of numbers which teachers expect to occur in students. Unless using quantities makes students deal with necessary structures of numbers more easily than directly dealing with numbers and manipulations of quantities can substantially support students' operations on numbers (Figure 1c), it may be unnecessary to use quantities in learning numbers and operations on numbers. The relationship between quantities and numbers which can be helpful for the learning expected in a lesson should be examined in advance and highlighted during the lesson.

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FACTORS AFFECTING MIDDLE SCHOOL SIXTH GRADE STUDENTS' PROBLEM-POSING PROCESSES

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This study aimed to examine middle school sixth grade students' thinking processes during posing problems and to discover the determinants of their problem-posing processes. Eight students were asked to write three problems for each of the two semi-structured problem-posing tasks, and then interviews were conducted with them. The data were analyzed by means of the thematic analysis method. The findings of this study indicated that students were affected by the mathematical structure of the activity and their own situations when they were posing problems. In addition, eight codes were determined under these themes, and it was observed that students tended to use multiple codes together for different problems.

INTRODUCTION

Problem posing (PP) has interested mathematics education researchers for more than 30 years. Silver (1994) defines PP as producing new problems or reformulating an existing problem. Kilpatrick (1987) argued that "the experience of discovering and creating one's own mathematics problems ought to be part of every student's education" (p. 123). Although students are interested in engaging in PP activities, our knowledge of the processes involved when generating problems is still limited (Cai & Leikin, 2020). In support of these results, Lee (2020), who reviewed the articles published in 13 mathematics education journals, found that only a small portion were related to PP (62/17456 about 0.4%), and only four examined the students' thinking processes. This study is stimulated by these limitations and aimed to examine the middle school sixth grade students' PP processes and thus to discover the determinants of their PP processes.

THEORETICAL BACKGROUND

Based on previous studies, Kontorovich et al. (2012) presented a theoretical framework regarding the complexity of students' PP in small groups. According to this framework, the PP process is influenced by task type and organization, mathematical knowledge, and individual considerations of aptness, as well as PP heuristics and schemes, and group dynamics and interactions. Stoyanova and Ellerton (1996) classified PP activities as free,

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 341-348. Khon Kaen, Thailand: PME.

semi-structured, and structured, and studies have indicated that PP performance differentiates in line with these activity types (e.g., Silber & Cai, 2017). PP also requires strong mathematical content knowledge (Harpen & Presmeg, 2013). This is because the data in the activity should be analyzed in depth and the problem should be associated with the added mathematical concepts to write mathematically valid and complex problems (Leung & Silver, 1997). Kontrovich et al. (2012) also stated that the process of posing a problem within the scope of individual considerations of aptness was influenced by the intrinsic satisfaction of the problem poser about the nature of the problem, how other individuals would evaluate this problem, or whether it was suitable for potential solvers. For example, Lowrie and Whitland (2000) indicated that third grade students took into consideration the interests of the problem solvers in addition to the mathematical structure of the problem (e.g., number magnitude and operation complexity) when they were posing problems for second and fourth grade students.

Christou et al. (2005) pointed out two important cognitive processes that characterize successful PP performance: editing (generating meaningful problems on the tasks by organizing their data) and selecting (selecting the data for a specific answer). The researchers indicated that these two processes were used more dominantly in PP tasks for open-ended stories or pictures. Silver and Cai (1996) asked elementary school students to pose three problems for an open-ended story (see Figure 1, task 2). The results of the study showed that students posed their second and third problems based on the first problem in a chained manner, and the complexity of problems tended to increase.

As a result, the presented literature indicates that there are various factors affecting PP performance. However, the current literature does not provide any conclusive data in detail about the factors students effectively use (particularly for editing cognitive process) for open-ended PP tasks and how they use them. This study aimed to clarify this situation by adopting qualitative approaches.

METHODOLOGY

Participants

This study was carried out with the students of a teacher experienced with PP, who had taken a master's course and was also conducting a thesis on PP. The school was located in the rural areas of the city center and reflected the middle and low socioeconomic groups. There was only one classroom for sixth grade students, and this was composed of nine students. Although the researchers planned to conduct the interviews with all students, since one student did not want to attend the study, it was completed with eight. The mathematics teacher rated the math achievement of four students as high and the others as moderate or low. She stated that high-achieving students were able to establish strong connections between newly learned and previous content, show high performance in solving problems, provide alternative solution strategies to the problems, and have low error rates in arithmetic operations compared to others. She also indicated that tasks such as PP for open-ended stories were used in her lessons, but PP tasks for symbolic operations were used more frequently. Each student was assigned a pseudonym.

Data collection and analysis

In this study, semi-structured PP situations (Stoyanova & Ellerton, 1996), which gave opportunities for students to pose various problems based on their knowledge and experiences (see Figure 1), were used. Both tasks were related to daily life. The tasks differed from each other in terms of their mathematical structure (quantitative data and the relationship between these data). In the first task, the variables (prices for pizza, cherry juice, and balloons) were not expressed in terms of others. When constituting the second task, on the other hand, the relational data, in which some quantitative data in the problem were described over others, were presented. In this context, a widely used PP task in different studies was applied to the students in this study.

An incomplete problem sentence is given below. Imagine (consider) you wrote down the problem and brought it up to this part. You can complete the rest of the story as you wish. Write three problems for each task by using your mathematical knowledge and experience.

Task 1: Pizza, cherry juice, and balloons are ordered for the birthday party. The prices of small, medium, and large pizzas are 18 Turkish Liras (TL), 24 TL, and 32 TL, respectively. The price of one liter of cherry juice is 3 TL, while a two-liter cherry juice is 5 TL. The price of each balloon is 5 TL.

Task 2: Mehmet, Ali, and Hasan went on vacation in a car together. During the trip, Hasan drove 80 km more than Ali. Ali drove twice as many kilometers as Mehmet. Mehmet drove the car for 50 kilometers.

Figure 1: PP tasks

The tasks with each of the eight students were held on different days. First, the students were given 30 minutes for the PP test and asked to write three problems for each task. Then, a break was given. The teacher examined the problems posed by the students. After the breaks, interviews were conducted ranging from 11 minutes to 22 minutes with an average of 17 minutes. The focus of the interviews was on how students thought and considered different problems. In this context, interviews were conducted around two questions, and these questions were directed to the students for all the posed problems: i) Can you explain what you thought when posing the problem? and ii) How did you differ this problem from your previous problem(s)? (What do you think about the difference between this problem and your previous problem(s)?)

The data were analyzed according to Braun and Clarke's (2006) thematic analysis process, which "minimally organizes and describes your data set in (rich) detail" (p. 79). First, the interviews were transcribed. Then, the two researchers, who were the authors of this study, independent of each other, read the transcripts many times to become familiar with the data and create the initial codes. Each of these two researchers created the initial codes by considering the factors in the related literature affecting PP performance. Then, they came together and compared the coding. They composed the codes and themes, which reflect the entire data set. A total of eight codes were determined, and they were presented under two main themes (see results section).

RESULTS

The codes regarding the factors that students took into consideration when posing problems were categorized under the following themes: *considering the task structure* (CTS) and *problem poser's own situation* (PPOS). There were three codes under the CTS theme: i) focusing on the operations evoked by the data, ii) separating the data set into independent parts, and iii) increasing the amount of data. Five codes were observed under the PPOS theme: i) experience, ii) interest area, iii) mathematical knowledge limitations, iv) testing the mathematical knowledge, and v) understanding the task (some codes are explained in this paper, but others will be presented at the conference due to page limitation).

Students focused on mathematical data in the task and the nature of the connections between these data in the CTS theme. The code *focusing on the operations evoked by the data* was frequently considered by the students, and it was emphasized that the presented data encouraged students to write some problem types. For example, providing the prices of the objects in the birthday task encouraged them to ask the amount of money to be paid if a certain number of them were purchased. For example, the problems posed by Kerem were as follow:

Problem 1: According to this, how much money will he pay if he purchases 2 large size pizzas, 3 cherry juice and 1 balloon?

Problem 2: According to this, how much money will he [the buyer] pay if he purchases 4 medium size pizzas, 6 cherry juice and 2 balloons?

Problem 3: According to this, how much money will he pay if he purchases 3 small size pizzas, 2 cherry juice and 3 balloons?

A script from the interview with Kerem was as follows:

Researcher:	Can you explain how you posed your first problem?
Kerem:	As seen, a text is provided. The prices of the pizzas, cherry juice and balloons were given. The pizzas are given in small, medium and large sizes. We are asked to write three problems. In the first problem, I wrote how much money will be [the buyer] paid if 2 large pizzas, 3 cherry juice and 1 balloon are purchased.
Researcher:	Ok. After reading the task, what attracted your attention when writing your problem?
Kerem:	There are pizzas, cherry juice and balloons. The prices were given there [data on the tasks is meant]. I thought it would be a good result by summing them up.

Since the students were asked to write three problems, in *the separating the data set into independent parts* code, they separated the data in the task into three independent parts and used each part in a different problem. This approach was observed in Ayşegül's paper. Three problems were posed by Ayşegül in the birthday task, and a script from the interview with her was as follows:

Problem 1: Ahmet brought a small, a medium, and a large pizza to Ayşe's party. The small pizza costs 18 TL, while the medium pizza costs 24 TL and the large pizza costs 32 TL. How much money in total does she pay?

Problem 2: Sultan will buy cherry juice for Ayşe's party. If a one-liter container of cherry juice costs 3 TL and a two-liter cherry juice costs 5 TL, what is the total amount of money to be paid for 4 one-liter and 4 two-liter cherry juices?

Problem 3: If Erol buys 100 balloons, how much will he pay?

Researcher:	Can you explain how you posed your third problem?
Ayşegül:	In fact, I wanted to write it first. But since I went in order, I wrote this at the end. It was an easy problem as I solved it in my mind.
Researcher:	Why did you use the story by separating it into parts?
Ayşegül:	Because, how can I say, it says three problems here. The first problem could be pizza. Or it could be cherry juice or a balloon. That's why I wrote pizza in the first problem. I wrote cherry juice on the second problem and balloons on the third problem. I went in order.
Researcher:	You could write a problem by using them all. In this way, was there a reason to use it in order?
Ayşegül:	First, I wrote pizza since it was in the first place. I went like that. I used the order. I also like to follow the order.

In *increasing the amount of data* code, students tried to make the problem difficult by focusing on increasing the volume of the data. By adopting this code frequently, students tended to pose their problems from simple to complex. In this context, they either increased the number of the operations in the posed problems, changed the magnitude of the numbers in the task, or tried to add different mathematical concepts, such as fractions. For example, Erol explained the rationale for writing difficult problems as follows: *Difficult problems become more different and beautiful. The simple problems are generally the things we have learned. They do not look very beautiful. It is better to struggle and find something difficult.* Within the scope of this code, students' second problems commonly contained more data than their first problems. However, they experienced difficulty in the third problem by adding more data than their second problem. For this reason, students either left the third problem empty or opted to write an easier problem than their second problem. Beril's opinion reflecting this situation was as follows: *This* [her second problem] *was the hardest one for me. I wrote the second problem with difficulty. I thought what I could write differently. I did not think of anything more to write. It was simpler than this* [her third problem].

Within the scope of PPOS, some students also considered their *mathematical knowledge limitations* when posing their problems. For example, Yağmur considered her limitations in the mathematical knowledge in the vehicle task. Yağmur indicated in her explanations that the unit of kilometer was very large, and she had not been able to understand this concept since primary school. She also stated that she had difficulty in understanding this concept, and therefore, she focused on posing short problems for this task. A part of her explanations was as follows: *I have been confusing kilometer problems since primary school years. That's why I always wanted to write short... When I think of kilometers, I think of huge lengths. I'm afraid of doing operations with them.* She asked the difference between the lengths of the trip that Mehmet and Ali took in the first problem, while in the second problem, she asked three times about the total of the trip Ali and Mehmet traveled. Parallel to her explanations, it was understood that the problems she posed were in a form that did not require analyzing the data set.

Some students, on the other hand, posed problems that tested the limits of their mathematical knowledge or improved their understanding of the connections among the data set in the task. For example, Engin posed two problems for birthday tasks and indicated that the second problem was more difficult. Engin stated that the purpose of posing a difficult problem was to see how hard the problem he could pose. Therefore, he indicated that he did not pose a third problem because he could not think of a more difficult problem. A part of the interview conducted with Engin was as follows:

Researcher:	You stated that you thought of posing a longer problem in the second problem. Why did you think like that?
Engin:	I wanted to test myself, whether I could solve my own problem or not.
Researcher:	Why did you not write your third problem?
Engin:	Third problem. Since I got confused in the second problem, the third one did not come to my mind.

Understanding the task code was observed only in Engin's explanations about why he posed his first problems. In his problem, Engin asked the total amount of money to be paid if two large-sized pizzas, two two-liter cherry juices, and 20 balloons were purchased. Engin emphasized that this problem was a simple one. He indicated that this kind of simple problem helps to create different problems. A part of Engin's explanations was as follows: *First, I decided to write simple.... I said two large pizzas because they had many friends. Again, two of two liters of cherry juice. And balloons, too. I said there should be 20 balloons. I did that way.... A simple problem. Because sometimes when I pose a simple problem, I think of other problems.*

DISCUSSION

The results of this study support the fact that PP is a multifaceted and complicated task (e.g., Leavy & Hourigan, 2019). This study reveals important results about students' PP processes. First, it was determined that the students looked at the data in the task from the perspective of their own mathematical understanding, interests, and experiences. In particular, it was pointed out that how students perceive their mathematical understanding was an important factor affecting PP products. These results underline the effect of metacognitive factors on PP processes. Second, previous studies indicate that participants consider the problem solver's or problem-solving group's interest areas when posing problems (Lowrie & Whitland, 2000; Kontorovich et al., 2012). This situation was not observed in this study. This might be due to the differences between the directives of the PP tasks used in the present study and in other related studies (e.g., pose problems for your friends). Third, students also frequently tended to make the problem more difficult in this study. They tried to add more operations to their second problems compared to their first. When some students thought that they could not write more complex problems than the ones they posed in their second problem, they either did not write the third problem or posed a simpler one. Therefore, the tendency to increase the difficulty of the problems could not be maintained systematically. These results differ from the studies identified in the literature (e.g., Silver & Cai, 1996) that students' second and third problems tended to be more complex than their first. Fourth, the tendency to increase the difficulty of the problems was also a factor in the number of posed problems. The quantity indicating the number of problems posed for the tasks is an important component in PP analyses (Silver & Cai, 2005). The results of this study revealed that having a small number of problems stemmed from students' inability to produce complex problems. From this point of view, this study indicated that some students gave more importance to posing complex problems than the number of problems they posed.

"Although interest in integrating mathematical problem posing into classroom practice is growing, implementing this integration remains a challenge" (Cai et al. 2020, p.2). We believe that the results of this study, which aims to expand our knowledge about what students do and how they think in PP tasks, will contribute to the efforts to make PP an important component of mathematics classrooms.

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ENHANCING THE LEVEL OF GEOMETRIC THINKING THROUGH LEARNING THE INSCRIBED ANGLE THEOREM

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This case study investigates how ninth-grade students enhanced their geometric thinking from van Hiele's third level toward the fourth level through lessons on the inscribed angle theorem. We found that students' reasoning went beyond the third level when they discussed how to construct case proofs, compared them with each other, and integrated them into one overall proof as follows: 1) developing the proof using semiotically guided analogical reasoning, 2) standardizing symbol use, 3) identifying the case proofs as being the same based on their identical linguistic descriptions, 4) conceptualizing the proof as a description package when comparing the case proofs with each other, and 5) interpreting all of the case proofs in terms of one description (by introducing zero degree and negative degree angles).

INTRODUCTION

The basic understanding of a geometric proof corresponds to van Hiele's third level of thinking (in his revised model incorporating levels 1 to 5). However, how do students who have achieved this level progress toward the fourth level, and what type of learning environment do they require? Van Hiele (1985, p. 48) describes these levels as follows: "Properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students" (third level), and "Thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions" (fourth level). Numerous studies on proofs and proving have focused on how students can achieve the third level of thinking, but only a few studies have been conducted on the learning necessary to enhance students' geometric thinking beyond the third level (e.g. van Hiele, 1986; Battista, 2007), although there have been related studies on levels of proof writing (Yang and Lin, 2008; Zazkis & Zazkis, 2016).

What type of learning environment helps students to progress beyond the third level of thinking? In this study, we analyzed ninth-grade students' learning of the inscribed angle theorem (IAT) and its proof in the classroom setting. The proof of the IAT involves combining the proofs of three separate cases based on the relationship between the inscribed angle and the middle of a circle. In exploring the IAT, it is expected that students will compare the proofs of the three separate cases and investigate them from a higher point of view. If we categorize proofs by cases when solving the problem, this enables us to identify all appropriate situations necessary to solve the problem (Boero, 2011). Thus, a dilemma arises: the cases are determined only after completing the proof, although the proving begins only after deciding on the cases.

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We believe that the point at which a person understands the structure of proofs by cases is achieved through a mutual process of deciding on the cases and constructing the proof of each case.

The main purposes of this study are to clarify how ninth-grade students display their geometric thinking beyond van Hiele’s third level through learning the proof of the IAT and to theoretically explain their thinking.

THEORETICAL BACKGROUND

Understanding the proof of the inscribed angle theorem

The IAT is proved using the following three cases (see Figure 1), which differ in terms of the relationship between the inscribed angle and the middle of a circle.

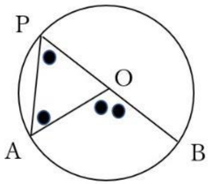
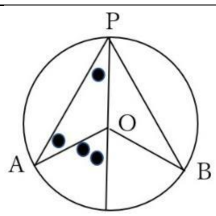
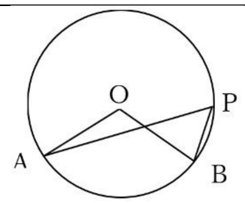
Case 1	Case 2	Case 3
		

Figure 1: The three cases used to develop the proof of the inscribed angle theorem.

To understand the proof of the IAT, students need to overcome two key difficulties.

First, the proof of case 3 is more difficult to construct than those of case 1 and case 2. Case 1, the fundamental case, can be proved by using the relationship between the isosceles triangle composed of two radii and the exterior angle. Case 2 can be proved by first drawing the auxiliary line and then applying the proof of case 1 twice. The proof of case 3 is basically equivalent to that of case 2. However, the figure configuration of case 3 may be harder to see in terms of where the auxiliary line is drawn, where two isosceles triangles exist, and how the angles can be dealt with. In particular, it is necessary to obtain the proof of case 3 using subtraction of the angles, while that of case 2 is obtained by addition of the angles.

Second, it may be difficult to understand why investigating just three cases satisfies all of the phenomena related to the IAT. For example, students may ask whether it remains true if the central angle changes, or whether it is true if the point on the circumference falls within the arc of the central angle. Furthermore, they may wonder why it is necessary to compare the three cases.

We assume that the students’ new level of thinking can emerge when they discuss and overcome these difficulties. In the following section, we use two theories to capture their learning process: relational analogical reasoning and semiotic change.

Relational analogical reasoning and the integration of symbolic expressions

Analogical reasoning refers to the transfer of structural information from one system (the base) to another system (the target). English (1997, p. 198) stated that “this transfer of knowledge is achieved through matching or mapping processes, which entail finding the relational correspondences between the two systems. It is this emphasis on corresponding relational structures that has significant implications for mathematics learning." We examine how the relational structure of the proof is transferred from one deductive system (proof as a base) to another system (proof as a target).

In this respect, we refer to the process of semiotic change in terms of the diverse representations and the mutual changes among them that are involved in mathematics learning. Duval (2008) calls the semiotic change within the same kind of representation "treatment," and that between the different kinds of representation "conversion." Treatment is the transformation guided by the potentials and rules that are intrinsic to the representation, based on which some explanation or proof is achieved. This transformation includes not just the algorithmic change of symbols, but also the change between drawings or the change from one figure configuration to another. Conversion provides a means of truly distinguishing the difference in meaning between two statements that look alike and determining which statements' meanings are mathematically relevant (Duval, 2008). Duval notes that thinking in mathematics involves synergy between at least two mobilized registers, even when a person works explicitly in only one register.

We can observe how a proof is semiotically constituted by relational analogical reasoning. First, the proof of case 1 may be used for the construction of the proof of case 2. Then, a figurative or graphical argument is useful for supporting the construction of proofs. Mason and Pimm (1984) presented a generic example of a mathematical representation that presents the context and powerfully influences problem solving. From this viewpoint, the figurative pattern of case 1 may work as a generic example for the construction of the other proofs. However, the transition from the figurative pattern of case 2 to that of case 3 may be difficult for students. One option may be to construct the proof of case 3 after students first consider the analogical reasoning in relation to the differences in symbolic-linguistic representations between cases 2 and 3, followed by the conversion of case 3 from a symbolic-linguistic representation to a figurative pattern. Regardless, we need to examine how students' semiotic changes are implemented within/between representations (symbolic-linguistic and figurative) in the analogical relationships between the case proofs.

Furthermore, it is necessary to synthesize or integrate the proofs constructed for each case into a single overall proof. Students may then focus on the analogical and semiotic correspondences among the proofs of the cases and reflectively abstract the essential components necessary for proof construction. We consider this process as the beginning of the formal deduction necessary for understanding proofs and proving.

METHOD

In this study, we used data obtained from two ninth-grade classrooms, I and II, in a junior high school attached to a national university. There were three one-hour lessons in each classroom in which the IAT and its proof were dealt with. The teacher had 19 years of teaching experience, and the students had a higher level of scholastic ability than public school students because they had already mastered the skills involved in proof construction using basic knowledge such as the properties of a triangle and the conditions for congruence/similarity among triangles before learning the IAT. However, the students had not experienced the process of developing a proof after dividing the conditions of the theorem into various cases.

We asked the teacher to first prepare a lesson on the proof of the IAT that enabled the students to freely divide the phenomena in the theorem into several cases that they felt were necessary to prove the IAT and to construct the proof for each case, and then to discuss whether all of the cases they had created were necessary, and if so, why.

We videorecorded the lessons and created transcripts of the discussions. Then, we undertook qualitative analysis using these transcripts, our research notes, and the students' worksheets. Our analysis focused on classroom episodes in which it seemed that the students had progressed beyond van Hiele's third level of thinking during discussions on generating and evaluating their proof for the IAT, and analyzed whether and how their thinking could be perceived through the theoretical lenses of semiotic change and relational analogical reasoning.

RESULTS

We identified 26 episodes in classroom I and 27 episodes in classroom II, in which we found four different types of episodes related to the students' thinking that indicated progression beyond van Hiele's third level. Below, we discuss these episodes, labeled episodes 2 to 5, in addition to episode 1, which involved the students' initial selection of cases. We used data from classroom II for episodes 1 to 3 and data from classroom I for episodes 4 and 5, as our aim is not to identify the process of understanding, but rather to identify the aspects of the students' thinking regarding the proof.

Episode 1: The students' initial selection of cases for proving the IAT

In the first lesson, the students' initial selection of cases necessary for the proof of the IAT resulted in five different cases (see Figure 2) after the teacher asked them to draw the figures necessary to prove the IAT.

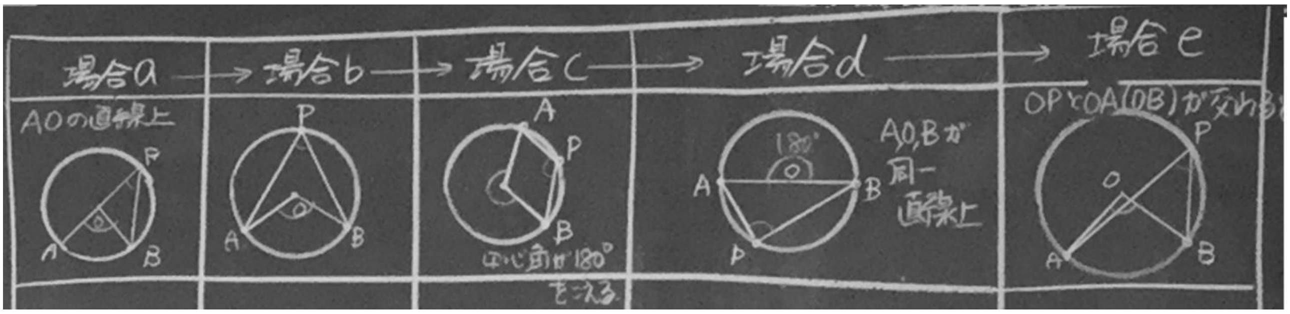


Figure 2: The five cases initially selected for proving the IAT.

The cases were characterized in terms of the position of points and straight lines as follows. Case a: points A, O, and P are on a straight line; case d: points A, O, and B are on a straight line; and case e: line AP intersects line OB. Furthermore, cases b, c, and d differed in terms of the size of the central angle. In case b, the central angle was less than 180 degrees, in case c, the central angle was greater than 180 degrees, and in case e, the central angle was exactly 180 degrees. The students were given two tasks: to construct the proof for each case, and to determine whether the proofs of all of the cases were necessary to prove the IAT.

Episode 2: Developing the proof for the case with a complex composite figure

In the second lesson, the teacher first requested the students to develop proofs of all of the cases in alphabetical order. They obtained the proof of case a by using the isosceles triangle *POB* and the relationship between the two base angles and the exterior angle. In this proof, the students perceived the configuration as representing a person's slipper, which provided the figurative generic example used to generate the proof (see Figure 3). They were also able to use this idea to obtain the proofs of cases b, c, and d.

Figure 3: the figurative generic example perceived as a person's slipper.

However, approximately 50% of the students could not construct a proof for case e. For example, although Nana could find the isosceles triangle *AOB* and mark the base angles, she had difficulty finding the other isosceles triangle, either *OAP* or *OBP*. We believe that this was primarily caused by the students' difficulty in extracting the appropriate figures from the complex composite figures. In addition, because two slippers overlapped, it may have been harder to find them, and thus the figurative generic example was not helpful in this case.

Conversely, we found that the students who could find the two isosceles triangles, *OAP* and *OBP*, for case e referred to the proof descriptions of the other cases. In particular, they either viewed case e in terms of an analogy with cases b, c, and d, or related all cases with each other. Thus, we believe that it may be difficult to successfully focus on the slippers as the figurative example to generate the proof for case e if the students have not paid attention to the symbolic-linguistic descriptions of the other cases.

Episode 3: Standardizing symbol use across all cases and identifying some cases as the same in terms of their symbolic-linguistic description

In the third lesson, when the teacher asked the students to discuss the proofs of the five cases, they tried to conceptualize the proofs in a consistent manner. Most of the students who completed the proof for case e adopted the same use of symbols, labeling the angle *OPA* as a and the angle *OPB* as b. Then, because the angle *APB* was represented by $-a + b$, the angle *AOB* was represented by $-2a + 2b$, and thus case e was proved.

Figure 4: A different figure configuration.

One student, Kura, proposed a proof of case e by labeling the angle *OPA* as a and the angle *APB* as b (see Figure 4). While several students accepted his proof after examining it, they suggested that he should standardize his use of symbols, stating that "the way you have used the label b differs from the proofs for the other cases."

Then, the teacher again asked if the five cases were necessary for the entire proof. Several students responded that just three cases, a, b, and e, were sufficient for the entire proof because cases b, c, and d had the same description.

Episode 4: Conceptualizing the proof as a package of symbolic-linguistic descriptions

In the third lesson in classroom I, when the teacher asked the students if all five cases were necessary for proving the IAT, several students responded that all of the proofs had almost the same description. Then, one student, Yuwa, stated that all of the cases could be understood using just one proof, that of case b. Then, the teacher confirmed that the descriptions in relation to case a had been used twice for case b, and that the proof of case a could not be used as a substitute for the proof of case b (see Figure 5). Yuwa responded by stating that "The opposite is possible." A comparison of the proofs showed that cases b, c, and d were identical in terms of their descriptions. Here, it was clear that the proof of case a was conceptualized as one package of description, and that the necessary and sufficient conditions were contained in the other proof descriptions, as Yuwa had suggested.

Figure 5: the proof as a package of symbolic-linguistic descriptions.

Episode 5: Interpreting the proofs for all of the cases in terms of one description

The teacher and the students continued to discuss Yuwa's idea using the figures for case a and case b. Then, Yuwa again proposed the idea of integrating the phenomena of case a into case b by considering the angle *PBO* as an angle of 0 degrees.

1 Yuwa: If we just write the proof of case b, we have the proof for all of the cases.

2 Teacher: All of the cases? But in the figure for case a, the angle *AOC* does not exist.

3 Yuwa: Uh…if we consider the angle PBO in case b as being 0 degrees, … If point C in the figure for case b arrives at point B, we can regard the situation as the same as in case a. (Note: point C is the intersection of line PO and the circumference.)

In addition, the idea of integrating case e into case b was discussed. Yuwa tried to integrate them by considering the angle *PBO* as a variable, saying that "If we consider the angle *PBO* as a negative angle for case e, the proof of case e can be regarded as being the same as that of case b." Moreover, Kana explained that "While the expression is $a + b$ for cases b, c, and d, in case e it becomes $b - a$, and thus the sign changes." Here, we note that her idea of integrating the proofs by changing from a plus sign in cases b to a minus sign in case e was based on the symbolic-linguistic descriptions.

At the end of the third lesson, the teacher summarized the activity by stating that the proof of the IAT consisted of the proofs of three cases, a, b, and e, based on the students' considerations. Even Yuwa finally realized that there were limitations to his idea of integrating all of the cases using just one proof because of the complexity involved in Kana's proposal to incorporate negative angles.

DISCUSSION

Five aspects of proof understanding were identified in the process of learning the proof for the IAT: 1) developing the proof using semiotically guided analogical reasoning, 2) standardizing symbol use throughout all case proofs, 3) identifying those case proofs with identical linguistic descriptions, 4) conceptualizing a proof as a package of symbolic-linguistic descriptions, and 5) interpreting all of the case proofs in terms of one description.

The first aspect refers to connecting the case proofs with each other in terms of the relational correspondences between the symbolic-linguistic descriptions and between those and the figure configurations. Approximately 50% of the students could not construct a proof for case e, while they could indicate van Hiele's third level of thinking for cases a and b, as shown in episode 2. The reason seemed to be the lack of relational analogical reasoning in moving from case b to case e. We believe that students who were able to derive the proof for case e had achieved a level of proof understanding that enabled them to use the links among the proof descriptions.

We identified two of the abovementioned aspects in episode 3. The first (aspect 2) involves being able to identify the unification of symbol use throughout all of the case proofs, with the students able to recognize another proof involving different use of the symbols. We believe that the ability to identify consistencies among various cases goes beyond merely being able to construct the proof for a case. The other aspect (3) involves being aware of the proof descriptions and being able to integrate cases that were initially seen as different because of differences among the figure configurations. Because the figure configurations for cases b, c, and d had very different appearances as a result of changing the central angle, the students initially regarded these cases as requiring different proofs.

The fourth aspect involves conceptualizing a proof as a package of symbolic-linguistic descriptions when comparing proofs. This was seen in episode 4, in which the description of case a was twice used for the proof of case b, while acknowledging that the proof of case a could not be used as a substitute for the proof of case b, even though the reverse was possible in terms of the necessary and sufficient conditions contained in the proof descriptions. Regarding the fifth aspect, one student attempted to unify the interpretation of all of the case proofs using only case b, as seen in episode 5. Then, he introduced the concept of a zero degree angle for case a and negative degree angles for case e, while another student symbolized these angles as $b - a$.

In summary, we consider that the students' reasoning identified in this case study is based on two kinds of correspondences. One is between the symbolic-linguistic description and the figure configuration within each case proof, and the other is between either the figure configurations or the symbolic-linguistic descriptions of the various case proofs. Specifically, we believe that the students conceptualized the proof as an object for their thinking, focused on the proof structure, compared the proofs semiotically, and then tried to integrate them into one proof. The idea of connecting the relationships within and between the case proofs appears to be a new level of proof understanding, one that seems to move beyond van Hiele's third level. In particular, we believe that understanding the proof structure is an important first step in progressing toward van Hiele's fourth level. We recognize that these five aspects of thinking are not sufficient for students to attain van Hiele's fourth level of thinking. Nevertheless, we believe that they signify progress by the students beyond merely constructing the proof. We reiterate that these aspects of thinking occurred when the students discussed how the cases were determined after developing the proofs for the various cases. In other words, these would not have occurred if the teacher had treated the cases as pre-determined before constructing the proof.

In future research, we aim to investigate new qualities in relation to proof understanding regarding other content and to further clarify the learning process necessary to attain Hiele's fourth level of thinking.

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THE INTERPLAY BETWEEN DIGITAL AUTOMATIC-ASSESSMENT AND SELF-ASSESSMENT

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It is well established that digital formative assessment can support student learning, for example by means of digital automatic-assessment of students' work in rich digital environments. However, at the same time self-assessment is regarded as important in order to support students' meta-cognitive skills and to put learners in a key position where they develop responsibility and ownership of their learning. Yet, little is known about combining automatic- and self-assessment. In this pioneering research study, we investigate the interplay between automatic-assessment and self-assessment in the context of Example-Eliciting-Tasks. Based on quantitative and qualitative data we demonstrate the potentials of combining self- and automatic-assessment and outline obstacles that can inform design principles for combining both forms of assessment.

INTRODUCTION & THEORETICAL BACKGROUND

The Interplay between Self-assessment and Automatic Digital Assessment project (ISAA) is a design-based research project that aims at scrutinizing how students' self-assessment and digital automatic-assessment can be combined in order to support student learning in the context of formative assessment. The research reported in this paper reports on results from the first design-cycle of the project.

Formative assessment

Formative assessment can be conceptualized as "all those activities undertaken by teachers, and or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (Black & William, 1998, pp. 7–8). With respect to formative assessment, it is well established that feedback is essential, and that the effectiveness of formative assessment will therefore depend to a great extent on the nature of the feedback (Hattie & Timperley, 2007). Feedback can range from simple verification feedback, which merely provides information about whether or not the student's answer is correct to more elaborated forms of feedback. For example, "attribute isolation feedback" presents information regarding central mathematical attributes of the student solution. The meta-analysis by Van der Kleij et al. (2015) shows that elaborated forms of feedback are more effective for higher-order learning outcomes in mathematics. We view feedback not only as a singular event but rather as a more holistic, bi-directional ongoing process of interaction between the student and various parts of the activity in which

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 357-364. Khon Kaen, Thailand: PME.

both sides of the interaction (e.g., student, technology) are affected and modified by actions of the other side.

Digital formative assessment with example-eliciting-tasks

Technological tools can support formative assessment by providing immediate and automatic feedback to students. For example, Harel et al. (2019) use "attribute isolation elaborated feedback" (AIEF) to support students learning when working with "Example-eliciting-Tasks" (EET). EETs are a special kind of formative assessment tasks that are built on the notion that the examples that students generate are indicative of the students' mathematical reasoning (Zaslavsky & Zodik, 2014). For example, the EET depicted in figure 1 was designed by Harel et al. (2019) to support the students in raising conjectures, as part of a guided inquiry activity. In this EET students investigate the relations between two non-constant linear functions and their product function in a multiple linked representation (MLR) interactive diagram. For this, students can dynamically drag points on the linear functions to create multiple examples and can display the product function by pressing a button. Then, they are asked to formulate a conjecture about which types of product functions can be obtained and to construct three examples that support their conjectures. Subsequently students receive AIEF that provides feedback on whether certain predefined characteristics are fulfilled in their examples (see figure 1, characteristics that are fulfilled are highlighted in yellow). Harel et al. (2019) show, that iteratively working with the EET task and the AIEF can support students in improving their conjecture.

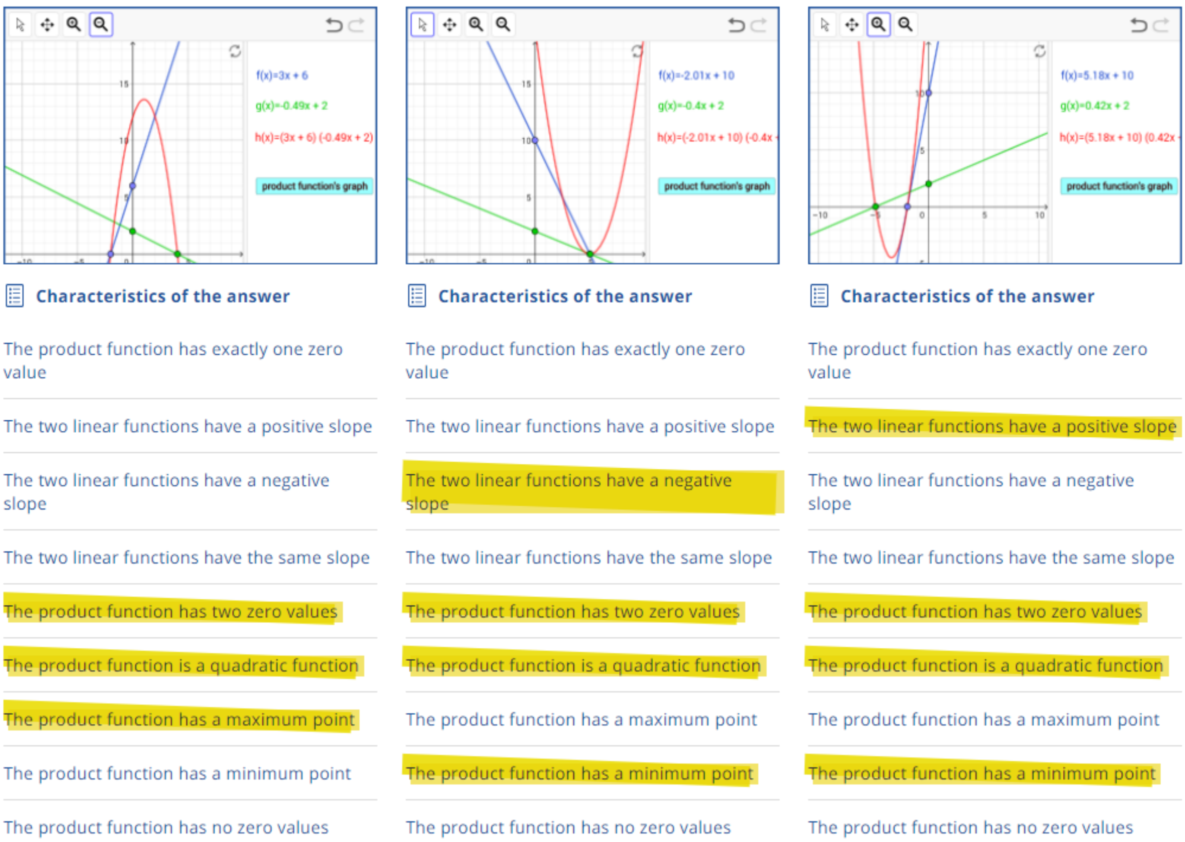

Figure 1: Example of an EET with attribute isolation elaborated feedback (characteristics that are fulfilled by the examples are highlighted in yellow)

Self-assessment

Besides automatic-assessment, student's self-assessment has intensively been researched as an essential element of formative assessment (Black & William 1998; Cizek, 2010). For example, Cizek (2010) highlights that current usage of the term formative assessment "equally, if not to a greater extent, highlights the notions of student engagement and responsibility for learning, student self-assessment, and self-direction" (Cizek, 2010, p. 7). This reflects more holistic approaches to formative assessment where assessment is "not just 'done' to students, but rather something in which they participate and have some element of ownership" (Bull & Mc Kenna, 2004, p. 13). This could be achieved with the self-assessment being part of an ongoing feedback process. In particular, self-assessment is regarded as important as it can support students to develop meta-cognitive skills which are urgently needed if students should become self-directed learners in a fast-changing and complex world. Despite this, in many cases self-assessment is rarely implemented in classrooms (e.g., Kippers et al. 2018).

RESEARCH QUESTIONS & METHODOLOGY

Research questions

While self- and automatic-assessment both carry much potential to enable rich formative assessment the lack of integrated research that investigates the *combination* of both forms of assessment in order to support student learning is stunning. Little is known about how to best integrate both forms of assessment in a learning environment and which opportunities and obstacles arise from an amalgam of both forms. This study addresses this research gap by exploring the interplay between self- and automatic-assessment in the context of an EET learning environment. In particular the study addresses the following research questions (RQ):

RQ1: To what extent are students' able to self-assess the EET characteristics?

RQ2: Does the self-assessment of the students improve when combining self- and automatic-assessment with EETs?

RQ3: What potentials and obstacles can be identified with respect to the interplay of self- and automatic-assessment of EET characteristics?

Methodology

To answer these questions, we used the Seeing the Entire Picture (STEP) platform (Olsher, Yerushalmy & Chazan, 2016) which is a digital environment that supports example-eliciting tasks. We combined self- and automatic-assessment using the EET described before (see figure 1) in the following way:

<u>A) Creating examples and formulating conjectures:</u> Students first created three examples and elaborated about the types and characteristics of quadratic functions that can be obtained from multiplying two non-constant linear functions.

B) Self-assessment: Instead of receiving the automatic-assessment right away, students were now asked to self-assess whether the characteristics depicted in figure 1 were fulfilled or not. Besides the categories “fulfilled” and “not fulfilled”, we included the categories “unsure” and “I don’t understand the characteristic”. The students were provided with a pre-structured paper sheet where they could mark which characteristics were fulfilled for each of their three examples.

C) Comparing self-assessment and automatic-assessment: After finishing the self-assessment students received the automatic-assessment report. The report indicated which characteristics were fulfilled in the examples that were submitted by the students (part A), by highlighting them in yellow (see figure 1). Students were now asked to compare their self-assessment (part B) with the automatic-assessment.

In order to gain more insight into how students experienced parts A-C, students subsequently answered various multiple-choice questions about their experience. For example, one question asked students whether the comparison between self- and automatic-assessment brought new insights or was surprising. Another question captured whether students preferred the self-assessment, the automatic-assessment or a mixture of both. After finishing these questions students entered a second cycle working through the previously described parts once more.

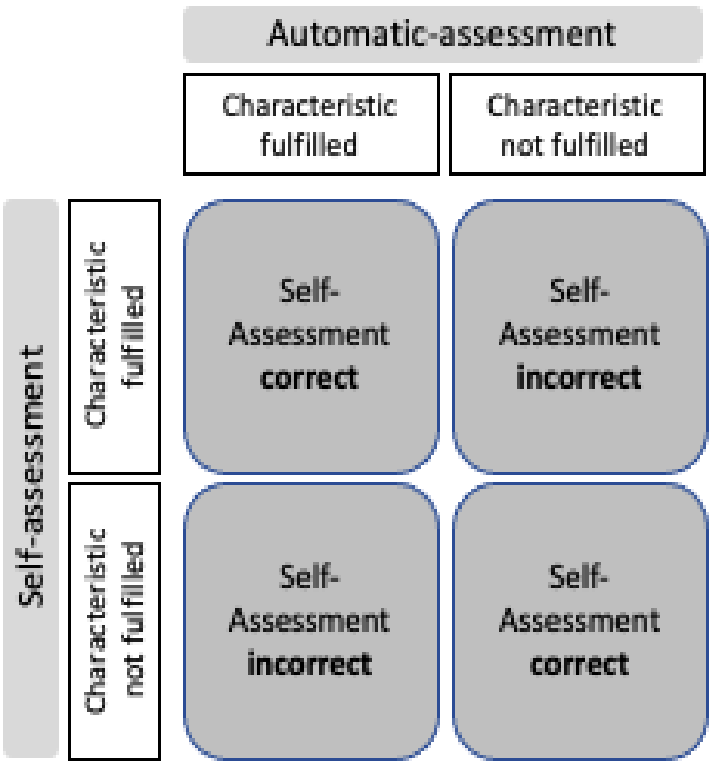

Figure 2: Self- and automatic-assessment when working with EET

Nine pairs of students from grade 9 of a German upper secondary school were video-recorded. Data was analysed in two ways. To answer research questions 1 and 2 we captured how often students' self-assessment was correct, and how often students choose the options “unsure” or “I don’t understand the characteristic”. Students' self-assessment was rated correct if a characteristic was fulfilled in the submitted example and the student realized this in their self-assessment or if a characteristic was not fulfilled in the submitted example and the student realized this in their self-assessment (see figure 2). To identify potentials and obstacles with respect to the interplay between self- and automatic-assessment (research question 3) we analysed the parts of the video-recordings where students self-assessed themselves and where they compared their self-assessment with the automatic-assessment. Analysis was done in an exploratory manner with a focus on how students engaged in self-assessment and comparing both types of assessment.

RESULTS

Quantitative results

Table 1 shows the frequencies on how often students’ self-assessment (SA) was correct, incorrect and how often students chose the option “unsure” or “I don’t understand the characteristic”. As we had nine pairs of students, nine characteristics per example and three examples submitted per cycle, there were in total 486 (=9·9·3·2) characteristics that the students had to self-assess.

	1st Cycle n=243	2nd Cycle n=243	Difference %
SA correct	150	194	29.33
SA incorrect	43	31	-27.91
Unsure	20	6	-70
Don't understand	24	12	-50
Missing	6	0	-100

Table 1: Students self-assessment (SA) across the two cycles

With respect to research question 1, it can be seen that in the first cycle roughly only 61% (150/243) of the self-assessments were correct, indicating that self-assessment was not an easy endeavor for students. However, the self-assessment improved remarkably (research question 2) as the number of correct self-assessments increased substantially in the second cycle to 194 which corresponds to an increase of roughly 30 percent. These trends will also be statistically analyzed for significant differences.

Cycle \ Pair	**S1**	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**	**S9**
C1	20	20	14	22	17	20	6	6	25
C2	27	21	20	21	24	26	14	16	25

Table 2: Number of correct self-assessments in cycle 1 (C1) and cycle 2 (C2) for each pair of students (S1-S9)

Table 2 shows that almost all pairs of students improved with respect to the number of correct self-assessments. The increase in correct self-assessments was not only due to a reduction of incorrect self-assessment: In addition, students reported in fewer cases that they were unsure about whether a characteristic was fulfilled or not, or that they do not understand a characteristic.

In line with these results, the answers of the students to the questions that were asked after the completion of each cycle, indicate the great appreciation of the combination of self- and automatic-assessment by the students. Eight out of nine pairs stated that comparing the self- and automatic-assessment brought new insights or encouraged them to think differently and that they prefer a mix between self- and automatic-assessment. Interestingly five groups stated that they were surprised that there were any differences between their self-assessment and the automatic-assessment.-These findings can be further elaborated using the qualitative results presented in the following section.

Qualitative results

The analysis of the segments where students self-assessed their work (part B) and where they compared their self-assessment with the automatic-assessment (part C) brought to the forefront the following four central aspects related to the potentials and opportunities of combining self- and automatic-assessment (research question 3).

High cognitive activation during self-assessment

During the self-assessment (part B) students often discussed whether a certain characteristic was fulfilled or not. As mentioned before, many times students did not self-assess the characteristics correctly (see table 1) which was often due to students limited concept images. For example, one pair of students held the concept image that a quadratic function is always opened upwards and therefore the parabola in their example which was opened downwards could not be a quadratic function. Another pair of students noted that the two linear functions that they had produced have slightly different slopes, however they argued that the slopes are sufficiently similar to say that they are actually the same (characteristic 4).

Comparing self- and automatic-assessment can lead to new insights

The combination of self- and automatic-assessment (part C) has the potential to lead to new insights as illustrated by the following example from the first cycle. Two students had generated the linear functions $f(x)=-2.01x+10$ and $g(x)=-0.4x+2$. In the self-assessment the students had marked the characteristic "The product function has exactly one zero point" as fulfilled. When they compare their self-assessment with the automatic-assessment they realize that the automatic-assessment has marked this characteristic as *not* fulfilled. They look at the graph and are surprised because the product function appears to have only one zero point (see figure 1, second example). Then one student started to investigate the graph, zooms into the zero point and concludes that they probably have not looked close enough.

Hence students encountered a cognitive conflict, and resolved it by reanalyzing their example.

High cognitive load when comparing self- and automatic-assessment

Even though some cases appeared where students investigated conflicts between self- and automatic-assessment (part C)) these investigations were quite rare. A possible reason for this was that before investigating possible differences between self- and automatic-assessment students had to identify whether their self-assessment was aligned with the automatic-assessment or not. Figure 2 highlights the complexity of this evaluation process as students have to distinguish between four cases. While most students managed to identify their self-assessment as aligned if a characteristic was fulfilled in self- and automatic-assessment (upper left square in figure 2), the other cases were considerably more difficult for students to evaluate and led to high cognitive load just to manage the evaluation of the self-assessment. This cognitive demand was additionally increased by the fact the automatic-assessment was displayed on the screen while the students' self-assessment was done on paper. Students had to constantly move back and forth between screen and paper which made the comparison between self- and automatic-assessment quite tedious.

Making graph and algebraic expressions easily accessible

Another reason that impaired student's investigation of the differences between self- and automatic-assessment was that the tablet which students used could not display the automatic-assessment and the submitted examples of the students (e.g., the graphs) at the same time. Most students scrolled down to easily oversee the yellow highlighted characteristics but did not bother to scroll back and forth between seeing the characteristics and the graphs of their examples.

SUMMARY AND DISCUSSION

The Interplay between Self-assessment and Automatic Digital Assessment project (ISAA) aims at scrutinizing how students' self-assessment and automatics assessment can be combined in order to support student learning. The results of the first design-cycle show that students' self-assessment did improve remarkably throughout the two cycles (table 1 and 2). This is particularly striking since this improvement was not moderated or scaffolded by any teacher intervention. Rather, one of the likely reasons for the improvements was the fact that students worked in pairs which allowed interactions and discussions between the group members. The qualitative analysis revealed how the comparison of differences between self- and automatic-assessment can create cognitive conflict that can lead to new insights. However, we also identified several challenges that can inform the design of learning environments that combine self- and digital automatic-assessment. First, evaluating the self-assessment with respect to the automatic- assessment was not easy for students and created high cognitive load which impeded a deeper engagement with the differences between self-and automatic-assessment. A possible way to reduce cognitive load would be to embed the self-assessment into the digital environment, and automatically highlight differences between self-assessment and automatic-assessment within the digital environment. This would allow students to immediately investigate the difference between the two forms of assessment. Another aspect would be to increase the simultaneous accessibility to all relevant information for example by presenting the interactive diagram on the same screen as the report without having to scroll between them.

Self- and automatic-assessment carry both tremendous potentials to support formative assessment. We have shown that *combining* self-assessment and automatic-assessment has the potential to enhance students learning and outlined important design considerations. However, while we gained many important insights, the results of this study are somewhat limited by the small number of students that were investigated. The next design cycle of the ISAA project will comprise a larger group of students, a further development of the technological environment with the goal of supporting an easier and deeper engagement with the differences between self- and automatic-assessment. Furthermore, we will investigate whether the combination of self- and automatic-assessment increases the quality of students conjectures in the EET.

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PERSPECTIVES OF HIGH SCHOOL STUDENTS FROM THE PHILIPPINES AND TAIWAN ON FACTORS CONTRIBUTING TO EFFECTIVE MATHEMATICS TEACHING

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This study used exploratory factor analysis to explore and compare Filipino and Taiwanese students' perception of effective mathematics teaching behaviors, which help students understand, focusing on the dimensions of representation, teaching methods, and problem-solving strategies employed by the teacher. Every dimension yielded two factors. On representation: Filipino students endorsed "formal and symbolic" and "concrete and real-life" similarly while Taiwanese students endorsed the second more. On teaching methods: "teacher-supported student active learning activities" and "teacher-led heuristic inquiry opportunities," and on problem-solving: "lecture centered approach" and "teacher-facilitated multiple self-discovery opportunities" where both countries endorsed the second factor higher than the first.

INTRODUCTION

The past years have seen the Philippines participate in various international assessments for mathematics given to students. Unfortunately, results from both assessments, PISA (2018) and TIMSS (2019), show the Philippines at the bottom of mathematics competency standings. In contrast, Taiwan and its other East Asian neighbor countries have always been consistently placed on top of these assessments. This prompted cross-country-comparative research on students' various preferences of effective mathematics teaching behaviors that may give hints as to why they learn well or not. For example, Hsieh et al. (2020) found that students in high achieving countries, China and Taiwan, endorsed formal and symbolic representation to a higher degree than their teachers and concluded that this might relate to students' pursuit of academic achievement. It is thus interesting and helpful to see how Filipino students fare compared to their neighboring country and see commonalities or differences that might explain the gap between achievement in international mathematics assessments. This might influence educators' view as to what teaching behaviors to tweak to ensure effective mathematics teaching, as reflected in students' mathematics understanding.

Effective mathematics teaching

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Many studies about what can be considered as effective mathematics teaching between countries have been done in the past decade. This range from cultural contexts (Cai & Wang, 2010) to belief systems (Purnomo, 2017; Yu, 2009) to values (Seah & Peng, 2012) and to many other topics in between. Many pertinent studies consider effective teaching as whether the instruction done by the teacher cultivates student understanding of mathematics. In a study done by Correa et al., (2008) it was found out that US teachers tend to favor hands-on activities, whereas Chinese teachers tend to favor linking mathematics to real-life situations in their instruction. Others asserted that it might have something to do with one's race, like the premise of the study done by Jerrim (2015). Other studies focused on teaching behaviors and the role of teachers in mathematics instruction. In their paper, Ismail et al. (2015) stressed the vital role of teachers in ensuring classroom instruction effectiveness. They further mentioned that for mathematics teaching to be effective, a teacher must be both an explainer and an inquirer, highlighting the interplay of teacher-led and student-centered instruction. Thus, it shows that the teacher's teaching behaviors are a great contributor to developing and nurturing a students' mathematics understanding. A consensus exists in the literature that effective mathematics teaching measures may consist of a. ways that teachers present mathematical concepts and ideas through representation (Capraro et al., 2010) and teaching methods (Cai & Wang, 2010), and b. ways to help students understand problem-solving solutions (Ambrus & Barczi-Veres, 2015). Thus, in this study, the conceptual framework focused on the teachers' use of representation and teaching methods to help students understand mathematics content and strategies and methods in teaching problem solving to help students understand solutions (see Figure 1).

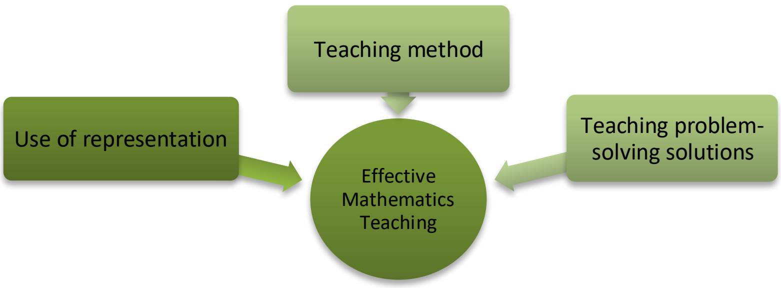

Figure 1: The framework of this study

Research questions

This research aims to answer the following questions:

1. What factors of teaching behaviors are preferred by students from Taiwan and the Philippines to better understand mathematics?
2. Are there commonalities and significant differences in the preferred factors that help mathematical understanding between students from Taiwan and the Philippines?

METHODOLOGY

The survey instrument

The research proponents adopted an instrument used by Wang & Hsieh (2017) for gauging the perspectives on effective mathematics teaching behaviors of Taiwanese high school students. The dimensions tested for this study were on representation, teaching methods, and how teachers teach problem-solving. A common prompt of what they think helps them understand mathematics better is used throughout the study. Use of representation was composed of 11 items, teaching methods 17 items, and problem-solving 15 items. A sample of survey questions is found below in Figure 2.

(read the statements carefully and check the box before the statement if you think that it is applicable)

When **representing mathematical concepts and ideas**, I think I would understand better if my math teacher would …	When **instructing (teaching) students' mathematics**, I think I would understand better if my math teacher would …
☐ 1. Use graphs	☐ 1. Present incorrect ideas or solutions for us to figure out
☐ 2. Use stories	☐ 2. Provide hands-on activities for us to understand mathematics in class
☐ 3. Use a formal approach (for example, definitions)	☐ 3. Ask us to solve problems on the board to enable us to learn how others solve the same problems

Figure 2: Sample questions used in the instrument

Participants and data collection

An initial pilot study consisting of 100 high school students and 56 mathematics teachers was conducted to determine the instrument's validity and reliability in the study as it applies to the Philippine setting. Reliability was computed using the KR20 formula for dichotomous variables in SPSS and yielded a result of 0.929, indicating that it is suitable for use in the Philippine setting.

A formal survey was conducted with students from four different high schools in three cities in Metro Manila. The total number of participants for the two countries combined was 1819, Philippines: 600, Taiwan 1219. Kaiser-Meyer-Olkin Measure of Sampling Adequacy result was 0.895, indicating that the data is suitable for factor analysis to be conducted. Exploratory factor analysis was done using MPlus software on each dimension to identify the factors contributing to students' perception of effective mathematics teaching. Fit statistics, including CFI, TLI, RMSEA, and SRMR, were used.

RESULTS AND DISCUSSION

On the use of representation

Teachers' use of representation in presenting and explaining mathematical concepts and ideas showed two factors extracted after running the EFA. Fit statistics for the data are as follows: CFI = 0.969, TLI = 0.950, RMSEA = 0.061, and SRMR = 0.059. All the factor loadings reached an adequate level of greater than 0.3.

Figure 3 shows the percentages of picking (POP) of the items of the two factors extracted, F1: *Formal and Symbolic representation* and F2: *Concrete and Real-Life Representation. Formal and symbolic representation* involves using formulas, symbols, proofs, and other abstract means to present new concepts or ideas. *Concrete and real-life representation* consists of manipulatives, graphs, or stories, tangible, concrete, and related to real-life situations. The POPs of F1 and F2 are 29% and 28% and 27% and 50% for Philippines and Taiwan. From the figure, it can be seen that Taiwanese students overwhelmingly preferred the second factor with a 50% percent of picking and with five (*R8*: *use of graphs*, *R11: use of examples, R6: use of appropriate metaphors, R9:use of things in real-life, and R10: use of demonstrations*) out of the six descriptors for the factor clustered in the top five slots as opposed to their Filipino counterparts where the descriptors were scattered, and where the first factor was endorsed more than the second (29% vs. 27%). The *R11: use of examples* was both highly favored by Filipino and Taiwanese students, where it ranked first and second place, respectively, at 50% and 60% POP. On the other end, the *R1: use of abstract symbols* (POP: 12%, 9%) and *R7*: *use of stories* (POP: 25%, 8%) were both not highly favored. Differences can be seen in the preference for the *R6: use of metaphors*, where Taiwanese students endorsed it at 55%, whereas it is one of the least preferred by their Filipino counterparts at only 9%. Filipino students supported the *R5: use of formulas* highly at 48% (ranked 2nd), whereas their Taiwanese counterparts endorsed it at only 22% (ranked 10th). The *R4: use of the formal approach* is also supported more by Filipino students at 39% (ranked 3rd) as opposed to the 34% (ranked 8th) given by their Taiwanese counterparts. Results indicate that Filipino teachers prefer to use a more straight-forward approach to teaching instead of using other forms like stories or graphs. In contrast, Taiwanese students prefer alternative representations that are not purely numerical and symbolic. The reason may relate to the different difficulty levels of curriculum or examinations.

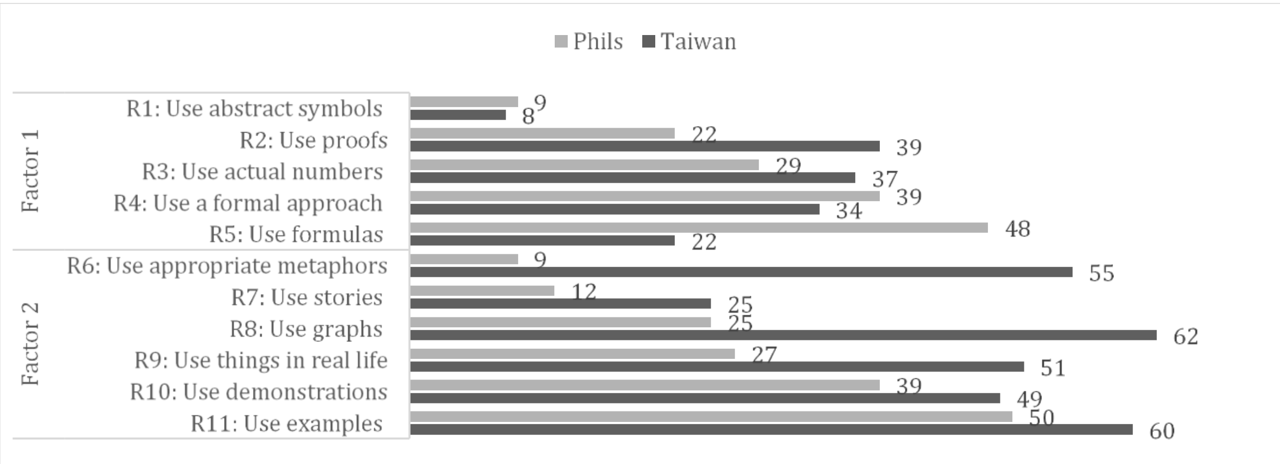

Figure 3: Percentages of pickings for the use of representation

On teaching methods

The EFA had to be done twice because, in the first run, one of the indicators (*Adopt lecturing as the main teaching form to avoid wasting unnecessary time*) did not load to any of the two factors extracted. After deleting this item, the second EFA still had the same number of extracted factors and had desirable model fit statistics as follows: CFI = 0.984, TLI = 0.979, RMSEA = 0.041, and SRMR = 0.041. All the factor loadings reached an adequate level of greater than 0.3.

Figure 4 shows the POP of the items of the two factors extracted: Factor 1: *Teacher supported student active learning activities,* and Factor 2: *Teacher-led heuristic inquiry opportunities.* Factor 1 is more student-centered; the teacher plays more of a supportive role in providing students with worksheets, learn through games, work in groups, and employ various teaching methods in a class. Factor 2, on the other hand, is centered more on the teacher as he/she leads the students to self-discovery opportunities by facilitating inquiry from students, clear explanations in class, and emphasizing critical ideas in class.

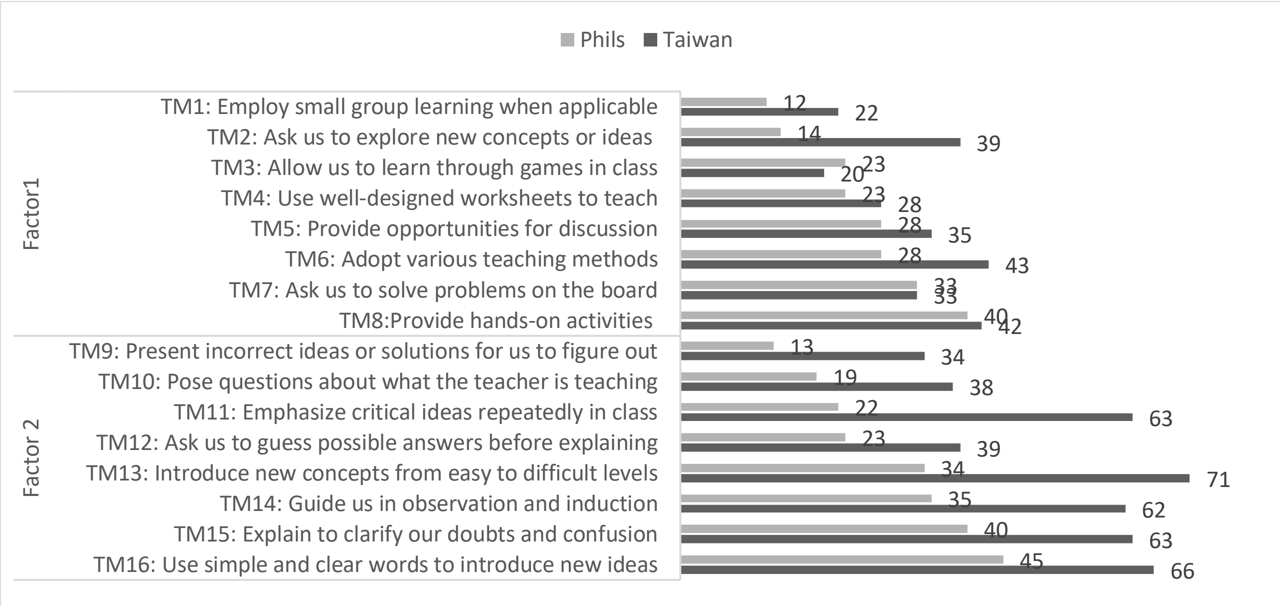

Figure 4: Percentages of pickings for teaching method

Both countries endorsed the second factor (Average POP: Taiwan - 55%, Philippines – 29%) higher than the first factor (Average POP: Taiwan - 33%, Philippines – 25%). Both countries students also shared four items that are on the top five of their preferences (*TM13*:*introduce new concepts from easy to difficult levels, TM16: use simple and clear words to introduce new ideas*, *TM15:explain to clarify our doubts and confusions*, *and TM14: guide us in observation and induction to develop our concepts*). Students do not prefer teachers *TM1: employing small group* learning in both countries, ranking it 15th and 16th for Taiwan and the Philippines, respectively. Completing their top 5 preferences, Filipino students included *TM8: provide hands-on activities for us to understand mathematics in class* (40%) ranked 2nd while Taiwanese students have *TM11: emphasized critical ideas repeatedly in class* (63%) ranked 3rd. The indicators *TM4: use well-designed worksheets to teach*, and *TM3: allow us to learn through games in class*, are part of the bottom four of preferred teaching methods by Taiwanese students (rank 14th and 16th). In contrast, the Filipino students ranked it relatively higher (rank 10th and 11th) based on their POP values. The most significant difference in preference is on *TM11: emphasizing critical ideas repeatedly in class*, with Taiwanese students ranking it 3rd at 63%, whereas their Filipino counterparts ranked it at 12th at 22%. On the other hand, Filipino students favored if they were *TM7: asked to solve problems on the board to enable them to learn* *how others solve the same problems,* which they ranked 6th at 33% compared to rank 14th (also at 33%) by their Taiwan counterparts. Results would show that indeed students from both countries favor teacher-led heuristic inquiry opportunities over that of teacher-supported active learning activities. Results may indicate that many students still rely heavily on or feel confident in teacher guidance and instruction in both countries. Students would rather want to be taught that way than making the discoveries for themselves via prepared worksheets or small group activities to play a more significant part in their learning. It is also evident that there is a rather large gap with Taiwanese students' preference between the factors. In contrast, it is not as heavily pronounced in their Filipino counterparts.

On teaching students problem-solving

Teachers' ways of teaching students how to solve mathematical problems in class, as shown in the EFA results in Figure 4, showed two factors extracted. Fit statistics for the data are as follows: CFI = 0.993, TLI = 0.991, RMSEA = 0.028, and SRMR = 0.034. All the factor loadings reached an adequate level of greater than 0.3.

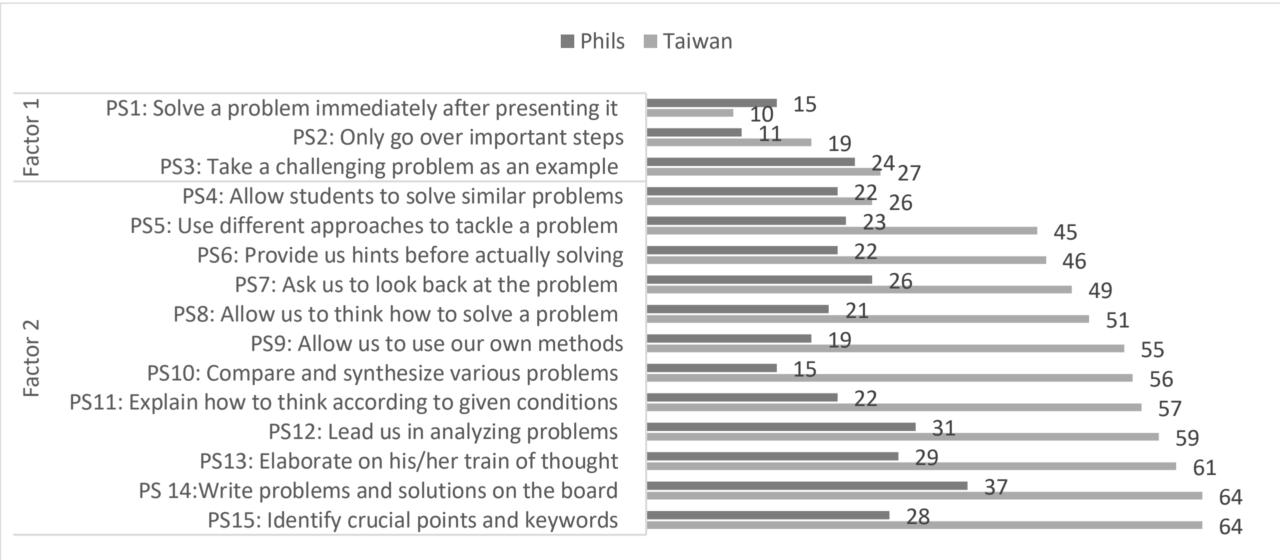

Figure 5: Percentages of pickings for teaching problem solving

Figure 5 shows the POP of items of the two factors extracted: Factor 1: *Lecture centered approach*, and Factor 2: *Teacher facilitated multiple self-discovery opportunities*. The first factor deals with teachers immediately solving problems on the board once presented, going over only on the important steps rather than every single one when solving a problem, and using challenging problems as examples and using it for examples, as can be seen heavily teacher-centric. Factor 2, on the other hand, involves the students more as they go through the problem-solving process; they are allowed to work on the problems first, elaborate their ideas, work on similar problems, look back at the way the teacher solved the problem to see how it was done, this is more student-centered, and needs more attention to detail on the part of the teacher. In this dimension, the differences between the students' preferences from the two countries are more evident. Factor 1 had averages of 17% and 19%, whereas factor 2 had averages of 23% and 49% endorsement from the Philippines and Taiwan. Among the top choices, both countries endorsed the indicators *PS15: identify crucial points and keywords of a problem when solving it* and *PS7:ask us to look back at the problem after he/she solves it to see how it is done* with the Filipino students ranking it at 4th and 5th, while Taiwan students rated it 1st and 2nd based on POP. However, consideration must be given to this dimension as the gap between the POP values is great, with the highest of 64% and 64% for the two items from Taiwanese students as opposed to 28% and 26% from the Filipino students. Filipino and Taiwanese students differ significantly on teachers *PS14: writing problems and solutions on the board to let the students know what he/she is talking about* and on teachers *PS13: elaborating on his/her train of thought in detail*, where the former ranked it at 1st and 3rd. In contrast, the latter ranked it 14th and 15th. On the flip side, Taiwan students endorse it when teachers *PS9: allow them to use their own methods to solve problems* at 56% ranked 6th, compared to their Filipino counterparts with 19% ranked 12th. The findings indicate that both countries' students place a premium on seeing the detail of the technique rather than the teacher just going over the essential steps in a problem. Results also show that Taiwanese students are more accustomed to using their methods and solutions to solve problems. In contrast, Filipino students tend to follow how the teachers solve the problems and mimic them. Taiwanese students are also generally given a chance to think about how a problem could be solved before the teacher explains how to solve it, contrary to what Filipino students experience.

CONCLUSION

The study results show that Taiwanese and Filipino students share some common teaching methods that they prefer to learn mathematics. Some indicators are regarded highly by Taiwanese students but generally not endorsed by their Filipino counterparts. Results could show that there are still underlying differences in how mathematics is taught in schools. One sensible explanation of the Filipino students' endorsement of *teacher-led heuristic inquiry opportunities* to a higher degree than *teacher supported student active learning activities* as well as their endorsement of *formal and symbolic representation* and *concrete and real-life representation* to a similar degree is that students prefer those teaching methods they are accustomed to in class. This may result from their teachers' not exhibiting those teaching behaviors or not doing it well. This may mean that teachers from the Philippines might not be using varied teaching mathematics approaches in the classroom. This could be caused by limited resources, lack of training, too much content to cover, and many other factors that could be studied further based on the results found in this research. Results have also shown that Taiwanese students are generally accorded more freedom in their ways of tackling problems. However, they generally still want more of the teacher, leading them towards self-discovery in their learning. Filipino students are more exposed to the teacher-centered approach, whereas Taiwanese students show a semblance of a student-centered approach to mathematics learning.

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STUDENTS' ACADEMIC PERFORMANCE IN MATHEMATICS AND PROBLEM-POSING: AN EXPLORATORY STUDY

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This contribution focuses on start exploring possible relations between students' academic performance in mathematics and their creativity in a semi-structured problem-posing task, and between students' academic performance and the quality of their posed problems in terms of solvability and syntactic complexity. Findings suggest that students with a good academic performance seem to generate more creative mathematical problems, while in terms of solvability and syntactic complexity no difference is evident between students with a higher or lower academic performance. The data analysis scheme developed should represent a prototype for analysing various aspects of problem-posing, such as creativity and syntactic complexity, in different mathematical domains.

INTRODUCTION

Problem-posing is a process through which the importance of creativity and critical thinking are emphasized (NCTM, 2000), representing an essential skill for the future. Indeed, in this perspective students can actively construct meaning in both the natural and simulated worlds in classrooms, making connections between mathematics and their real lives (Kopparla et al., 2019), supporting in this way the development of more democratic and critically thinking members of society (Singer, Ellerton & Cai, 2015).

Although mathematical problem-posing has great importance in mathematics education practice and research, it has received little attention by students, teachers and researchers (Silver, 1994, 2013; Van Harpen & Sriraman, 2013). Lee (2020) remarked how in literature still persists little research on problem-posing, finding that in the last thirty years only the 0.4% of papers in the most important journals in Mathematics Education focused on mathematical problem-posing. Further research is needed for the future (Singer, Ellerton & Cai, 2015) particularly on: (i) developing problem-posing skills for pre-service and in-service teachers' education; (ii) analysing possible connections between problem-posing and creativity; (iii) supporting students' learning trough problem-posing.

The aim of this contribution consists in exploring possible relations between students' academic performance in mathematics and their problem-posing performance. Specifically, the results of an exploratory study conducted in a 12th-grade class (age 17) are presented, together with an analytic scheme used to analyse students' creativity in problem-posing and the quality of the posed problems in terms of solvability and syntactic complexity.

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THEORETICAL FRAMEWORK

In this contribution problem-posing is considered as the process by which students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems (Stoyanova & Ellerton, 1996). In the specific, the focus will be on a semi-structured situation, where students are provided with an open situation and are invited to explore its structure and to complete it using their personal previous mathematical experience.

The theoretical arguments supporting the importance of problem-posing in school mathematics are supported by a growing body of empirical research. Several studies focused on the relations between problem-posing and problem-solving (Silver, 1994; Van Harpen & Presmeg, 2013) and/or between problem-posing and creativity (Leung, 1997; Bonotto, 2013; Xie & Masingila, 2017), developing several analytic schemes to assess students' or teachers' problem-posing performances.

A first analytic scheme to examine problem-posing of middle school students was developed by Silver and Cai (1996). Students' problem-posing responses were firstly categorized as *mathematical questions*, *non-mathematical questions* or *statements*. Then, mathematical questions were divided in *solvable* and *non-solvable*. Specifically, problems were considered to be *non-solvable* if they lacked sufficient information or if they posed a goal that was incompatible with the given information. The last step involved examining the complexity of the posed problems. Complexity was considered under two perspectives: *syntactic complexity* and *semantic complexity*. Syntactic complexity consisted in the presence of *assignment*, *relational* or *conditional* propositions. Semantic complexity involved the number of the semantic relations used, taken from the five categories: *change*, *group*, *compare*, *restate*, *vary*.

Creativity started receiving attention in the fifties, when Guilford (1959) proposed three categories characterizing creativity in his model for divergent thinking: *fluency, flexibility* and *originality.* Fluency in thinking referred to the quantity of output; flexibility in thinking referred to a change of some kind (meaning, interpretation, use of something, strategy); originality in thinking meant the production of unusual, remote or clever responses. Researchers created and applied different schemes to evaluate students' creativity in problem-posing. For example, Bonotto (2013) used Guilford's categories to examine the relationships between problem-posing and creativity with students engaged in problem-posing activities implemented involving the use of real-life artefacts. More recently, Xie and Masingila (2017) proposed a scoring rubric to assess prospective teachers' problem-posing performances. In particular, in relation to a given problem, they analysed teachers' posed problems in terms of creativity assigning 3 points if the posed problem was completely different respect to the given one, 2 points if it was somewhat different, 1 point if it was comparable.

RESEARCH QUESTIONS

Students who are good at solving routine mathematical problems or taking routine mathematical tests might not be good at posing good quality mathematical problems, as indicated in Van Harpen and Sriraman (2013). Moreover, Bonotto (2013) suggested the necessity of studies with the aim of investigating a possible relation between students' academic performance in mathematics and their performance in creativity. The aim of this study is to start exploring possible relations between students' academic performance in mathematics and their creativity in semi-structured problem-posing situations, and between students' academic performance and the quality of their posed problems in terms of solvability and syntactic complexity. As a consequence, the research questions of this study are the following:

1. Is there a relationship between students' academic performance in mathematics and students' creativity in problem-posing?
2. Is there a relationship between students' academic performance in mathematics and the quality of their posed problems in terms of solvability and syntactic complexity?

Another goal of this study was to develop and use an analytic scheme to examine students' creativity in problem-posing. In order to find first answers to the research questions, the results of an exploratory study are reported.

METHOD

Participants and procedure

The exploratory study was conducted in a twelfth-grade class (age 17) composed by twenty-one students. The classroom had never participated to a problem-posing activity before the study. At the moment of the intervention, the official mathematics teacher, used to teach in a traditional way, was working on probability. With students' academic performance in mathematics, in this study the choice was to consider the mathematics level of each student in result of a mathematics test performed before the problem-posing activity. The test consisted of three problems involving various mathematical topics already covered in previous mathematics lessons by the teacher. For each problem the maximum score was 2, so the total maximum score was 6. In order to examine the relationship between students' performance in the test and their problem-posing performance, two extreme groups were formed on the basis of the results from the test, in analogy with Silver and Cai (1996): the high-test group, composed by students whose score was from 4 to 6 (48%, 10 students over 21); the low-test group, composed by students whose score was from 0 to 3 (52%, 11 students over 21). The two groups had substantially different levels of success in the test, indeed the high-test group had a significantly higher mean test score than the low-test group (MeanHigh=5.0, MeanLow=2.5, $z= -3.84$, $p<.001$).

After the initial test, each student completed a semi-structured problem-posing task in a single class period of approximately 40 minutes. The task was administered by the regular mathematics teacher during a mathematics class. The context chosen for the activity was the Italian game of *tombola*, that students were used to play with their teacher every year before Christmas holydays. As a consequence, the context was meaningful for students, in addition of being rich in mathematical stimulus to generate probabilistic problems. To complete the task, students were provided with a sheet of paper with the rules of tombola, and the request to pose at least three problems starting from that game dealing with probability.

Data coding

A summary of the data coding developed in this study is provided in Figure 1. The scheme is an adaptation of Silver and Cai (1996) scheme. The novel contribution concerns the characterization of creativity, that is explained in this section.

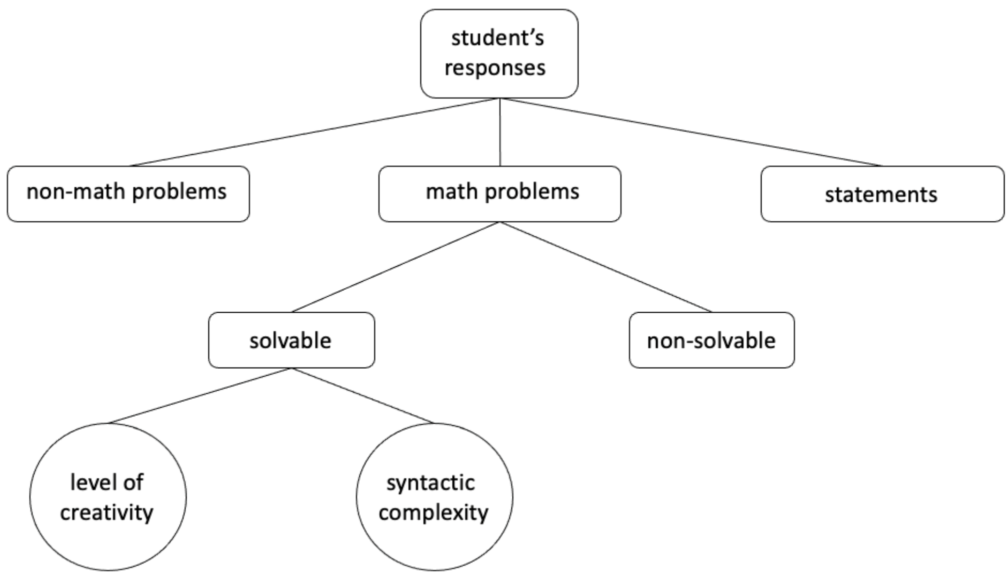

Figure 1: Data coding scheme

The first step of the coding consisted in classifying each student's responses as *mathematical problems*, *non-mathematical problems* or *statements*. The next step involved categorizing the mathematical problems as *solvable* or *non-solvable*. The following step consisted in associating to each student a level of creativity. To do this, an analytic scheme was developed starting from the one proposed by Xie and Masingila (2017). Let P_i^n be the i-th solvable problem posed by the n-th student, so for three problems $i = 1, 2, 3;\ n > 0$. For every n, consider the first posed problem P_1^n and compare it with the second posed problem P_2^n: if P_2^n and P_1^n are comparable, then $(P_2^n, P_1^n) = 0$; if P_2^n and P_1^n are somewhat different, then $(P_2^n, P_1^n) = +1$; if P_2^n and P_1^n are completely different, then $(P_2^n, P_1^n) = +2$. Then consider the third posed problem and compare it with both the second and the first one: if P_3^n and P_2^n are comparable, then $(P_3^n, P_2^n) = 0$; if P_3^n and P_2^n are somewhat different, then $(P_3^n, P_2^n) = +1$; if P_3^n and P_2^n are completely different, then $(P_3^n, P_2^n) = +2$; if P_3^n and P_1^n are comparable, then $(P_3^n, P_1^n) = -2$; if P_3^n and P_1^n are somewhat different, then $(P_3^n, P_1^n) = -1$; if P_3^n and P_1^n are completely different, then $(P_3^n, P_1^n) = 0$. In the end, calculate the sum $c \coloneqq (P_2^n, P_1^n) + (P_3^n, P_2^n) + (P_3^n, P_1^n) \in \{-2, -1, 0, 1, 2, 3, 4\}$. If $c \leq 0$, the student has a *low level* of creativity, if $c \in \{1,2\}$, the student has a *medium level* of creativity, if $c > 2$, the student has a *high level* of creativity. The scheme can easily be extended to activities in which students pose a number of problems *d* greater than three, with $c := \sum_{j=2}^{d} \left(\sum_{k=1}^{j-1} \left(P_j^n, P_k^n\right) \right)$, where $\left(P_j^n, P_k^n\right) \in \{0, +1, +2\}\; if\; j - k = 1$, $\left(P_j^n, P_k^n\right) \in \{0, +1, +2\}\; if\; j - k > 1$.

The final step of the data coding involved examining the syntactic complexity of the posed problems. In the specific, every problem of each students was classified as an *assignment problem*, a *relational problem* or a *conditional problem*. In agreement with Mayer (1992), the presence of relational or conditional problems was considered as an indication of complexity. For this reason, to each student was associated a level of complexity as follows: *low level* if s/he posed only assignment problems; *medium level* if s/he posed an assignment plus a relational problem, or an assignment plus a conditional problem, or only a relational problem, or only a conditional problem; *high level* if s/he posed an assignment plus relational plus conditional problems, or a relational plus a conditional problem.

Concerning inter-rater reliability of the scoring, the coding was performed separately by two different researchers. Rates of agreements on the classifications of levels of creativity and complexity were highly acceptable: concerning creativity, agreement of 85% with Cohen's k of 0.83 (almost perfect agreement); concerning complexity, agreement of 85% with Cohen's k of 0.74 (substantial agreement).

MAIN RESULTS

Students provided a total of 65 responses. About the 95% of the responses were classified as mathematical problems, about 3% were statements and 2% were non-mathematical problems. More than 87% of the mathematical problems posed by students were judged to be mathematically solvable.

Concerning creativity, a level of creativity was calculated for every student, applying the developed coding scheme to students' solvable problems (Figure 2). Then, distributions of students respect their level of creativity were calculated. The 42% of students had a high level, the 29% a medium level and the 29% a low level of creativity.

The syntactic complexity of the posed problems was determined by examining the presence of assignment, relational or conditional problems among all the solvable problems posed by students. The 50% of the solvable problems were assignments problems, the 13% relational problems and the 37% conditional problems. Some examples are shown in Table 1. Concerning levels of complexity, the 10% of students had a high level, the 67% a medium level and the 23% a low level, showing that the majority of students (74%) reached a medium-high quality level in terms of complexity of the posed problems.

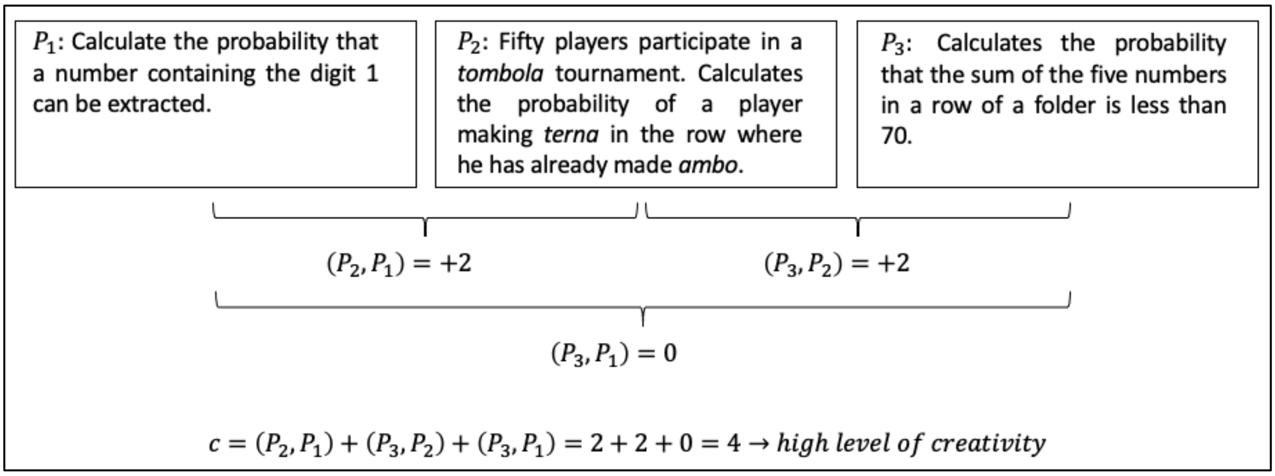

Figure 2: Example of calculation of a student's level of creativity

Assignment problem	In the game of bingo, what is the probability that a number is drawn which is power of 2 and multiple of 3?
Relational problem	Luca and Marco are playing tombola. Marco bought 2 folders while Luca 4. Each folder had 15 different numbers that did not appear in the other folders. What is the probability that Marco makes *tombola* before Luca?
Conditional problem	Suppose the croupier tricks the game before its start, by removing all the two-digit numbers starting with the digit 4. Which is now the probability of doing *ambo* in a given folder where one of the removed numbers was present?

Table 1: Examples of assignment, relational and conditional problems

In order to answer to the research questions, similarities and differences among the high-test group and the low-test group of students were examined, in terms of solvability, levels of creativity and complexity. Results are reported in Table 2.

Number of math pr.			Solvability			Levels of creativity			Levels of syntactic complexity		
			solv	non-solv		low	med	high	low	med	high
high-test group	27	count % within math pr.	25 93	2 7	students' number row percentage	1 10	2 20	7 70	2 22	6 67	1 11
low-test group	35	count % within math pr.	29 83	6 17	students' number row percentage	5 45	4 36	2 18	3 25	8 66	1 9

Table 2: Results from the data analysis

DISCUSSION AND CONCLUSIONS

In the present study students were asked to pose at least three problems dealing with probability from a given context. The students involved in the study, despite the novelty of the task, were able to generate a large number of mathematical problems, precisely the 95% of their responses.

The findings of this exploratory study provide first insights into some aspects of the relationships between students' academic performance and creativity in problem-posing, and between students' academic performance and the quality of their posed problems, in terms of solvability and syntactic complexity. To pursue this goal, students were divided in two groups respect to the results of an initial test: high-test group and low-test group. The majority of mathematical problems posed by the two groups was represented by solvable problems (93% and 83% respectively, table 2). As a consequence, a good mathematical level in terms of test performance should have no direct consequences in terms of solvability of the posed problems. However, some differences can be observed concerning creativity. The 70% of students in the high-test group showed a high level of creativity, and only the 10% a low level of creativity. In the low-test group, instead, almost a half of students (45%) had a low level of creativity, and only the 18% reached a high level of creativity. These findings suggest that students' who are good in solving mathematical tests reaching a high academic performance, seem to have a more creative disposition during problem-posing activities. Besides creativity, students' academic performance was compared also with the complexity of the posed problems. In this case, differently from the analysis of creativity, students' in the high-test and low-test group showed a comparable behaviour. Indeed, students' distributions respect to their level of complexity in the two groups was almost equal: for high-test group the 22% of students had a low level, the 67% a medium level and the 11% a high level of complexity; for the low-test group the 25% of students had a low level, the 66% a medium level and the 9% a high level of complexity. Therefore, answering to the research questions, results from this exploratory study suggest that students with a good academic performance seem to generate more creative mathematical problems, while in terms of the quality (solvability and syntactic complexity) of the posed problems no difference is evident between students with a higher or lower academic performance. However, due to the small-scale sample, at this stage results cannot be generalized.

Besides the previous results, the study also offers an approach to problem-posing through an analytic scheme that may be used by practitioners or researchers. Teachers should use the scheme to evaluate the effectiveness of their problem-posing-oriented instruction, or to measure students' progress in the problem-posing process. Researchers, instead, might see at this scheme as a prototype for analysing various aspects of problem-posing, such as creativity and syntactic complexity, in different mathematical domains. However, there are limitations to the scheme used, and improvements are necessary. Indeed, other measures should be assessed, concerning for example the originality (Guilford, 1959) and semantic complexity (Silver & Cai, 1996) of the posed problems. Another limit, already remarked, consisted in the small sample size. For the future other problem-posing activities are requested, in order to support (or not) the findings of this exploratory study and increase the power of statistical analysis, combining to the descriptive also an inferential analysis.

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THE IMPORTANCE OF UNDERSTANDING EQUIVALENCE FOR DEVELOPING ALGEBRAIC REASONING

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While many researchers have highlighted the importance of algebraic reasoning for middle-years students some have suggested that students should develop computational procedures using the algebraic idea of equivalence to integrate their learning of whole numbers and fractions. This paper focuses on four tasks from a paper and pencil assessment instrument used in a larger study which investigated the links between fractional competence and algebraic reasoning. The tasks were developed to encourage students to move beyond using the equals sign as meaning 'give an answer' to a relational understanding of the equals sign which focused on the equivalence of the expressions on both sides of the equals sign.

INTRODUCTION

Many researchers have highlighted the importance of algebraic thinking for middle-years students. Kieran (2004) described algebraic thinking as: analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting. Empson, Levi and Carpenter (2010) suggest that students should develop and use computational procedures using relational thinking to integrate their learning of whole numbers and fractions. Other researchers have noted that three distinct aspects of algebraic thinking include students' understanding of equivalence, transformation using equivalence, and the use of generalisable methods (Jacobs, Franke, Carpenter, Levi, and Battey, 2007; Stephens and Ribeiro, 2012). Knuth et al. (2008) suggest that: "helping students acquire a view of the equal sign as a symbol that represents an equivalence relation between two quantities many, in turn, help prepare them for success in algebra (and beyond)" (p.518).

Jones, Inglis, Gilmore and Evans (2013) highlighted three different conceptions of the equals sign: operational, sameness-relational, and substitutive-relational. The operational conception of the equals sign is described as the expectation that the equals sign indicates that the student needs to 'give an answer' (Jones et al., 2013, p. 36). The sameness-relational conception of the equals sign involves seeing the equals sign as meaning 'is the same as' (Jones et al., 2013, p. 34). This encourages students to see the sameness of the expressions on both sides of the equals sign thus seeing the equivalence when comparing each expression. The substitutive-relational conception involves students thinking that the equals sign also means 'can be substituted for' (Jones et al., 2013, p. 35) and enables students to use arithmetic rules, such as commutativity, to change the arithmetic expressions on either side of the equals sign but retain the equality. Jones et al (2013) suggested that students' understanding of both the sameness-relational and substitutive-relational conceptions of the equals sign are important for algebraic thinking.

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 381-388. Khon Kaen, Thailand: PME.

Researchers such as Knuth et al. (2008) believe that students' dependence on the operational conception of the equals sign hinders both arithmetic and algebraic calculations.

THE CURRENT STUDY

The key question of our current research is to investigate the links between fractional competence and algebraic thinking as middle-years students solved mathematical tasks using whole numbers, fractions, decimals and pronumerals. More than 600 Australian students from Years 5 – 9 (10 – 16 years) completed two paper and pencil assessments: The Fraction Screening Test (Pearn, Pierce, & Stephens, 2017; Pearn & Stephens, 2018) and the Algebraic Thinking Questionnaire (Pearn & Stephens, 2016) and 45 students were interviewed (Pearn, Stephens, Zhang & Pierce 2019) using a semi-structured interview. This paper focuses on four tasks from The Algebraic Thinking Questionnaire (ATQ).

DEVELOPING THE ALGEBRAIC THINKING QUESTIONNAIRE

The ATQ (Pearn & Stephens, 2016) built on research that identified specific features of the transition from arithmetic or calculation-based thinking to thinking about number sentences as mathematical expressions with algebraic features that include:

- Keeping a number sentence in its uncalculated form and viewing the number sentence as a group of numbers in relation to each other according to the operations involved (Britt & Irwin, 2011; Jacobs, Franke, Carpenter, Levi, & Battey, 2007).
- Utilising the idea of equivalence to solve missing number sentences (Kaput, Carraher, & Blanton, 2008).
- Exploring variation, compensation and equivalence, and identifying numbers that stay the same and numbers that vary in equivalent expressions (Britt & Irwin, 2011).
- Identifying rules that underlie relationships in equivalent expressions and expressing these relationships in the form of a generalisation (Stephens & Ribeiro, 2012).

The purpose of the ATQ tasks was to ensure that students needed to move beyond an *operational* conception of the equals sign in order to successfully solve tasks. It was hoped that the tasks would encourage them to use either of the two relational conceptions of the equals sign: *sameness-relational* or the *substitutive-relational* which Jones and colleagues (2013) indicated were important for algebraic reasoning.

The ATQ has two distinct parts that include whole numbers, fractions, decimals and pronumerals. Part M focuses on multiplication and Part D focuses on division. Each part is divided into two sections: Question 1 and Question 2. Question 1 has one box for an unknown response while Question 2 includes two boxes for two unknown responses.

Figure 1 shows the first of four tasks of Question 1 for multiplication (Task M1a) and division (Task D1a). Later Question 1 tasks include decimals and fractions.

For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$36 \times 25 = 9 \times \square$ $\qquad\qquad$ $3 \div 4 = 15 \div \square$

Figure 1: Examples of whole number tasks from Question 1, Parts M and D

Figure 2 shows the first of five tasks for Question 2 for multiplication (Task M2a) and division (Task D2a). Question 2 tasks were designed to focus students' attention on the relational features of equivalence while a follow up question requires them to describe that relationship: "When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?" Later tasks in Question 2 expected students to apply the same thinking to similar questions for pronumerals and fractions.

In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$5 \times \underbrace{\square}_{\text{Box A}} = 10 \times \underbrace{\square}_{\text{Box B}}$ $\qquad$ $3 \div \underbrace{\square}_{\text{Box A}} = 15 \div \underbrace{\square}_{\text{Box B}}$

Figure 2: Examples of initial tasks from Question 2, Parts M and D (ATQ)

To be successful with Question 1 and Question 2 tasks students would need to use the *sameness-relational* or *substitutive-relational* understanding of the equals sign.

RESULTS AND DISCUSSION

Question 1 (ATQ): Equations with One Unknown

Multiplication Task (M1a): Students used a variety of strategies to successfully complete Task M1a (left-hand side of Figure 1). Student A (left-hand column of Figure 3) used arithmetical calculations that demonstrate the *sameness-relational* understanding of the equals sign and ensured that the expressions on both sides of the equal sign were equivalent. This student calculated the left-hand side of the expression as 900 and then determined that the missing number in the expression on the right-hand side was 100. Student B (right-hand column of Figure 3) used relational thinking that demonstrate the *substitutive-relational* understanding of the equals sign. This student recognised that 36 divided by four is nine and that, in order to maintain the equality of the two expressions, multiplied 25 by four to get 100.

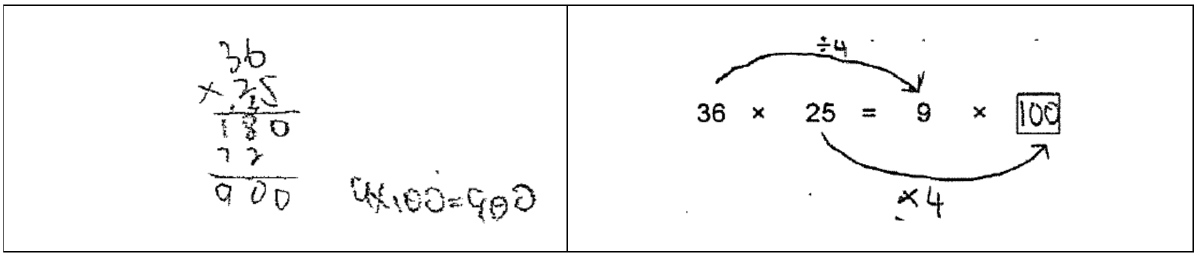

Figure 3: Students' responses to Task M1a

Students who successfully solved Task M1a (left-hand side of Figure 1) used either a *sameness-* or *substitutive-relational* approach. Column T (Table 1) shows the results for Task M1a while Column *E* shows the results for the explanation for Task M1a. Fifty-six percent of all students gave a correct response for Task M1a. There was an increase in the successful responses from 45% at Year 5 to 63% at Year 6. Fewer Year 8 students were successful than those at Year 6, with 55% at Year 8, while 87% Year 9 students correctly answered Task M1a (Column T).

While 56% of all students gave a correct response for Task M1a only 45% explained their solution (Column *E*). Forty-one percent of all students used arithmetical calculations while 4% used relational thinking. Of the Year 5 students who gave a correct response 94% used arithmetical calculations and 6% used relational thinking similar to that shown in Figure 3. Ninety-six percent of successful Year 6 students used arithmetical calculations and 4% used relational thinking while 82% of Year 8 students used arithmetical calculations and 18% used relational thinking to successfully respond to Task M1a. Six of the 13 successful Year 9 students used arithmetical calculations and seven (54%) used relational thinking.

Responses	Year 5 (n = 195)		Year 6 (n = 175)		Year 8 (n = 122)		Year 9* (n = 15)	
	T	*E*	T	*E*	T	*E*	T	*E*
Not attempted	18	*41*	11	*33*	7	*23*	0	*0*
Incorrect	37	*25*	26	*17*	38	*32*	13	*13*
Correct	45	*35*	63	*50*	55	*45*	87	*87*

Table 1: Percentage of responses and explanations by year level for Task M1a (ATQ)

Many students gave a correct response to Task M1a but did not show their solution method, so it was not possible to determine whether they used arithmetical calculation or relational thinking.

Division Task D1a: Some students used their knowledge of equivalent fractions to respond to Task D1a (right-hand side of Figure 1). Student C (left-hand column of Figure 4) used equivalent fractions and then simplified the answer to check his response. Student D used relational thinking and indicates the relationship between the numbers on either side of the equals sign using arrows.

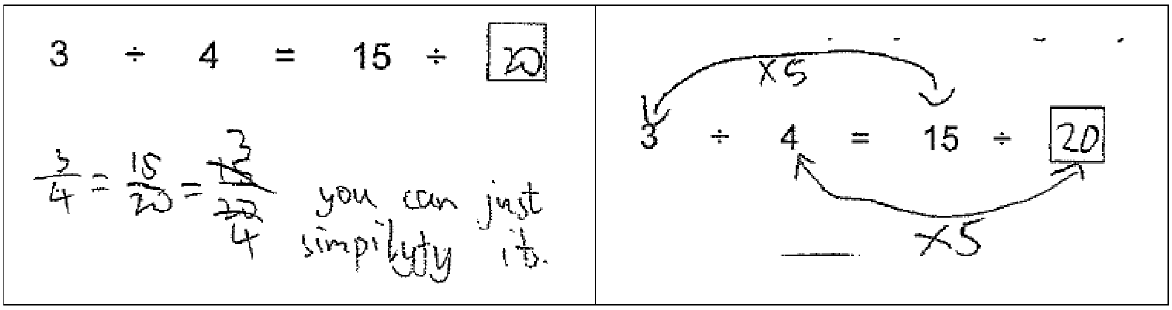

Figure 4: Two students' correct responses for Task D1a

As shown in Row 3 (Table 2) there was an increase in the percentage of successful responses from 20% at Year 5, 35% at Year 6, 69% at Year 8, while 100% Year 9 students correctly answered Task D1a (Column T).

Responses	Year 5 (n = 195)		Year 6 (n = 175)		Year 8 (n = 122)		Year 9* (n = 15)	
	T	*E*	T	*E*	T	*E*	T	*E*
Not attempted	41	*84*	29	*73*	16	*48*	0	*20*
Incorrect	39	*7*	36	*7*	16	*7*	0	*0*
Correct	20	*9*	35	*20*	69	*45*	100	*80*

Table 2: Percentage of responses and explanations by year level for Task D1a (ATQ)

Table 2 shows the percentage of students who explained or presented the method they used to solve Task D1a (Column *E*). Many students did not show their solution method, so it was not possible to determine whether they used an arithmetical calculation or relational thinking. Nearly 40% of all students successfully explained their responses for Task D1a. Three percent of all students successfully used an arithmetical calculation including 4% of primary students (Years 5 and 6) and 2% of Year 8 students. Five percent of successful Year 5 students, 16% Year 6, 43% Year 8 and 80% of Year 9 students used relational thinking to explain their solution for Task D1a.

While the primary students were less successful solving the division task (Task D1a) than the multiplication task (Task M1a) the secondary students were more successful with the division task (Task D1a) than the multiplication task (M1a).

Question 2 (ATQ): Equations with Two Unknowns

Some students started by substituting numbers in the two empty boxes and checked if both sides were equal as for the *sameness-relational* conception of the equals sign. However, in the following question students needed to reason that, whatever numbers were substituted for Box A and Box B, the relationship between the numbers was the same. While this may not be quite the same as *substitutive-conception* of the equals sign, used by Jones et al. (2013), there is a clear sense that students need to realise that whatever numbers are used the relationship between the numbers in Box A and Box B must remain the same.

Multiplication Task M2a: Figure 5 shows two students' correct responses for Task M2a (left-hand side of Figure 2). It is difficult to determine how Student F solved Task M2a (left-hand side of Figure 5) as there was no working shown, but Student G (right-hand side of Figure 5) has used arrows to demonstrate the relational thinking.

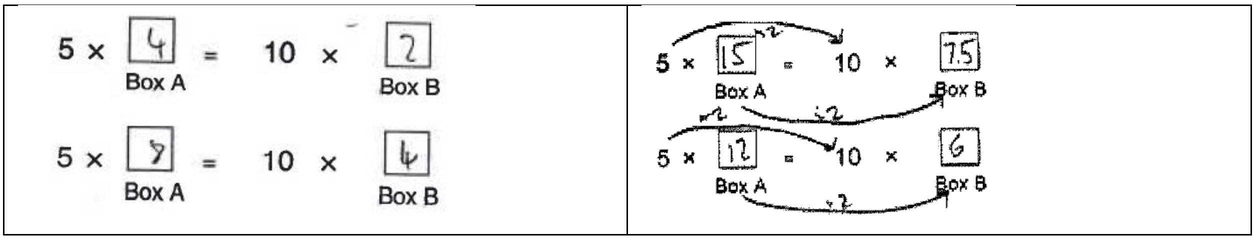

Figure 5: Two correct responses for Task M2a (ATQ)

Table 3 shows the results in percentages for the students by year level. Column M shows the results for Task M2a while Column *D* shows the results for Task D2a. Overall, 64% of students gave two correct pairs and another 16% gave one correct pair for Task M2a. As shown in Row 4 (Table 3) 54% Year 5, 67% Year 6, 73% Year 8 and 93% Year 9 students gave two correct pairs for Task M2a. Another 19% Year 5, 10% Year 6, 13% Year 8 and 7% Year 9 students correctly gave one pair of numbers.

Altogether 39% of all students explained the relationship between the two numbers in Task M2a. Thirty-two percent of Year 5, 36% Year 6, 55% Year 8 and 93% Year 9 students were able to explain the relationship between the pairs of numbers using a similar response to that of Student G: "Box A is two times as much as Box B".

Responses	Year 5 (n = 195)		Year 6 (n = 175)		Year 8 (n = 122)		Year 9* (n = 15)	
	M	*D*	M	*D*	M	*D*	M	*D*
Not attempted	19	*46*	9	*33*	6	*43*	0	*0*
Incorrect	8	*31*	9	*31*	8	*19*	0	*0*
One correct response	19	*11*	16	*11*	13	*7*	7	*0*
Two correct responses	54	*13*	67	*25*	73	*32*	93	*100*

Table 3: Percentage of responses by year level for Task M2a and Task D2a (ATQ)

Division Task D2a: Overall 26% of students gave two correct responses for Task D2a similar to Student H's response (left-hand side of Figure 6), an additional 10% gave one correct response, while 27% gave no written response. However, some students used an incorrect inverse relationship as shown in the right-hand column of Figure 6.

As shown in Row 4 (Table 3) 13% Year 5, 25% Year 6, 32% Year 8 and all Year 9 students correctly wrote two appropriate pairs of numbers for Task D2a. An additional 11% of Year 5 and Year 6, and 7% of Year 8 students correctly gave one pair of numbers. However only 9% Year 5, 19% Year 6, 25% Year 8 and 93% Year 9 students stated the correct relationship symbolically or verbally using multiplication e.g. Box B is five times Box A or division e.g. Box A is one-fifth of Box B or Box A equals Box B divided by five.

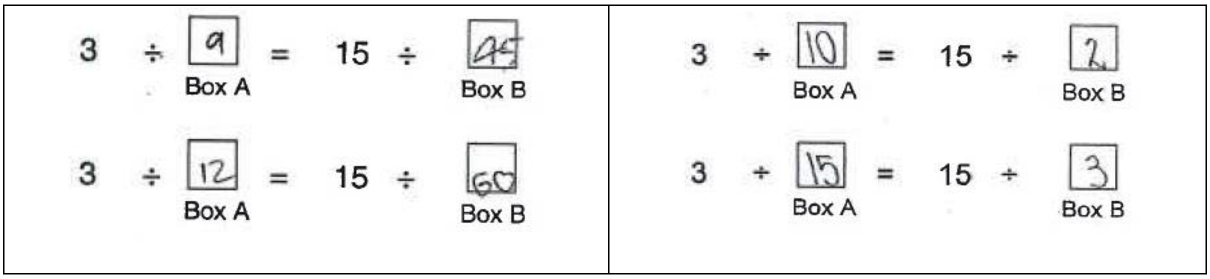

Figure 6: Two responses for Task D2a (ATQ)

The primary and Year 8 students were more successful with the multiplication task with two unknowns (Task M2a) than the division task (Task D2a). All Year 9 students gave a correct response for the division task (Task D2a) while one student gave an incorrect response for the multiplication task (Task M2a).

CONCLUSION

Many students found the ATQ tasks difficult with a large number of no attempts. While some students relied heavily on arithmetical or computational methods some began to use equivalence-based relational thinking as evidenced by the use of arrows in their responses. Question 1 and Question 2 tasks using division appeared to be considerably more difficult for primary students as the percentage of "no attempts" and incorrect responses increased. The percentage of correct responses for Task D2a are much lower for Year 5, Year 6 and Year 8 than the results for Task M2a. Year 8 and 9 students correctly solved more Question 1 division tasks than multiplication tasks.

The four whole number tasks described in this paper permitted students to use a *sameness-relational* or *substitutive-relational* understanding of the equals sign which focused on the equivalence of the expressions on both sides of the equals sign. The *operational* understanding of the equals sign was not appropriate for these tasks. While only four tasks were described in this paper the results are similar for subsequent tasks involving fractions and pronumerals (Pearn & Stephens, 2016).

In answering our key research question, many students in the middle years are not confident using equivalence based algebraic thinking for either multiplication or division tasks. In order to develop algebraic reasoning, teachers will need to determine the types of strategies their students are using to solve mathematical tasks, and then develop tasks that will encourage their students to demonstrate their understanding of the equals sign and equivalence which will enable them to use generalisable methods.

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PROFESSIONALIZATION OF FACILITATORS IN MATHEMATICS EDUCATION: A COMPETENCY FRAMEWORK

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Facilitators play a crucial role when scaling up continuous professional development (CPD). They have to design and conduct programs to initiate the process of teachers` professionalization. This requires competencies about adult learning and the specific knowledge and needs of mathematics' teachers, which are much broader than teachers' competencies. The aim of the study was to specify facilitators' competencies in a framework mainly for the design and use in the context of qualification. Following a Delphi method in the context of the German Center for Mathematics Teacher Education (DZLM) network, experts (researchers, facilitators, stakeholders) were interviewed in several development cycles. This resulted in a competency framework for facilitators who train mathematics teachers in teaching and learning.

INTRODUCTION

There is a great diversity of terms used in literature to describe the profession of the group of people who are initiating and leading processes to professionalize teachers, e.g. facilitators, teacher trainers, multipliers, coaches or teacher educators. We prefer the term facilitator to emphasize the cooperation aspect of learning, it being rather a give-and-take than a one-sided teacher-pupil relationship. When it comes to the breadth of terminology or the lack of clarity in the qualification or task description of the facilitators (Thomas, 2004), it is clear that their competencies are also not directly identified.

Kunina-Habenicht and colleagues (2012) highlighted the necessity of a competency framework instead of just formulating a curriculum on the level of teachers' professionalization. They regard the concretization of competencies as an important step to develop adequate quality standards in the field. This is relevant on the facilitators' level, as well, especially when regarding that many facilitators were trained primarily as teachers, and often act "self-made" in their new profession as facilitators (Zaslavsky, 2008, p. 93). Usually, there are no uniform career paths or qualification programs for these persons. Coming up with a clear competency framework emphasizes the importance of a specific qualification for facilitators.

There is a research gap about what facilitators need to know and how to act to be able to adequately support teachers (Sztajn, 2011). It is, therefore, the goal of this study to develop a competency framework for facilitators in mathematics education and thus to use it in the

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context of qualification within the DZLM. In order to follow the aim of reaching to competencies of facilitators in mathematics education, the perspective of general adult education and of mathematics-focused facilitations were taken into account.

THEORETICAL BACKGROUND

Teacher competencies in Mathematics Education

At the level of teachers, research on professionalisation has already resulted in differentiated frameworks, which in particular also include mathematic-specific aspects (Ball et al., 2008; Baumert & Kunter, 2013). Baumert and Kunter (2013) are often cited in this context and provide a summary of the existing literature, which relates to teacher competencies in mathematics education. The aim of this overview was to organize the different approaches to the CPD of teachers, which brings together, integrates and empirically tests knowledge from different research areas. It is relevant to include the facilitators' perception on teachers' competencies, their beliefs on mathematics, teaching of mathematics (Grigutsch et al., 1998) and their self-awareness (Hattie, 2009; Thurm, 2019). For professional knowledge, Shulman's (1987) proposal has largely prevailed – content knowledge (CK), pedagogical content knowledge (PCK) and pedagogical knowledge (PK). At the level of teacher competencies, the professional knowledge was supplemented by organizational knowledge (e.g. school organization, school quality, according to Fried, 2002) and the advisory knowledge on which professionals in communication with laypersons are dependent (Bromme & Rambow, 2001). These two knowledge areas are rather largely content-independent.

Facilitators in General Adult Education

One can also find numerous competency frameworks in general adult education, which usually refer to company structures and therefore define "competencies" as authority, rights and duties, also as the ability to cope with complex requirements in certain situations (see Weinert, 2001). An overarching framework covering the competencies of teachers and facilitators in general adult education has been developed within the framework "Basics for the Development of a Cross-Provider Recognition Procedure for the Competences of Teachers in Adult and Continuing Education" (germ. abbrev. GRETA; Lencer & Strauch, 2016). The GRETA framework has been developed on the basis of a literature review and by conducting a Delphi method (ibid.). The Delphi method is a multi-level qualitative survey procedure aiming to combine the knowledge of several experts in order to arrive at a forecast for the future. It takes the competency framework of Baumert and Kunter (2013) into account and includes all areas of adult education. In addition to the close cooperation between science and practice, this framework offers orientation for the first time on competency requirements for adult and continuing education. Next to professional values and beliefs and professional self-monitoring, professional knowledge is specifically divided into didactic and methodological areas that take adult learning into account.

Facilitators in Mathematics Education

Usually, the typical requirements that facilitators have to accomplish, are described in a non-specialized way. In the literature, there are already a number of approaches to systematize the competencies for facilitators. Smith (2005, p. 182-183) mentions a number of relevant aspects for qualified facilitators in any subject: They should be self-confident, reflective on their actions and they should have comprehensive, in-depth knowledge based on theory and practice. Facilitators could benefit from being involved in curriculum development and in research, and at least they should be good teachers with experiences teaching different age groups. In addition to a comprehensive understanding of the educational system, a high degree of professional maturity would be useful. Zaslavsky (2008, p. 95), who focussed on facilitators in mathematics, adds further aspects such as adaptivity and conscious selection of methods and media.

For the CK-elements that facilitators should have at their disposal, different emphasis is placed in various studies, even though they all accentuate that the knowledge required from facilitators must go beyond the knowledge of teachers, since they must impart new knowledge in a similar way as teachers do to their students (Borko et al. 2014, p. 165). This becomes vivid by looking at the "three-tetrahedron model (3TM) of professionalization research" for the content-related PD research, where the individual levels are described and related to one another (see in more detail Prediger, Leuders & Roesken-Winter, 2019). A first bundling of the mathematical specific competencies for facilitators is developed by Borko and colleagues (2014). With their competency framework, they refer to the work of Ball and colleagues (2008) but they make explicit that there is a specific math related knowledge needed to cover PCK with teachers as learners and they call it "Mathematical Knowledge for Professional Development (MKPD)".

Going beyond the classroom-level, expanded knowledge refers not only to the new knowledge of mathematical content and the relevant didactic aspects aimed at continuing education, but also to the didactic knowledge of adult education. This includes, for example, knowledge about mentoring, about existing teacher practices (Even, 2008) or current views on teacher education (Borko et al., 2014). Cochran-Smith and Lytle (1999) also stress the knowing about conceptions of teacher learning as a relevant aspect for facilitators. The following practices must be taken into consideration when designing a PD module: practical knowledge generated in the teacher's professional action ("knowledge-in-practice"), theoretical knowledge delivered to the teacher from outside ("knowledge-for-practice") and theoretical knowledge generated by the teacher's own reflection on professional actions ("knowledge-of-practice").

Even though facilitators should have a more extensive knowledge than teachers, it should be emphasized that there are also knowledge elements that are relevant for teachers, but not for facilitators (Beswick & Chapman, 2015). These include, for example, detailed background knowledge about individual students. For facilitators, only general knowledge of educational standards and curricula is important, as well as relevant empirical findings (ibid.).

RESEARCH QUESTIONS

Taking into account the research gap concerning a comprehensive competency framework for facilitators in mathematics education, the following research questions arise:

(1) Which competencies are relevant for facilitators with a focus on mathematics teacher professionalization?

(2) How can these competencies be structured and categorized to give a good orientation for the design and research of qualification programs?

METHODOLOGY

Based on the findings of the literature we conducted a research process following the Delphi method (Linestone & Turoff, 2011). We involved 33 researchers, 28 key stakeholders and teachers with experience in CPD and realized three cycles of further development. All researchers involved are experts in the field of CPD in mathematics education for primary and secondary level and were asked to use this expertise to point out key competencies of facilitators. The teachers' and the stakeholders' perception was important to gasp the systemic processes and necessities by reflecting their practical experiences in the field.

Different instruments were used to collect the ideas and experiences – a questionnaire and individual written statements and discussions in small groups (recorded). Due to the fact that the discussions were not anonymous, but were still supported by anonymous surveys, one could rather speak of a quasi-Delphi method. Nevertheless, several cycles have been carried out in the frame of networking conferences, which were documented and evaluated. The aim of a Delphi method is not necessarily to reach consensus, but rather, in a first step in particular, to identify the breadth of the field so that all relevant aspects can be covered.

The three examination surveys were carried out as follows:

Survey 1:

The first development step was carried out by a team of DZLM researchers, which consisted of 4 persons - the authors of this paper. The literature was reviewed, and discussions were held within this team with the aim to come up with a first framework combining the essential competencies from adult education and from the perspective of mathematics education. The GRETA competency framework (Lencer & Strauch, 2016) was considered as a good structure to categorize the broad range of diverse competencies of facilitators: Professional Values and Beliefs, Professional Self-Monitoring, Competency on the Professional Development Level, Competency on the Classroom Level. In addition, the substructuring in aspects and facets was also picked up from the GRETA framework and adapted to mathematics education. This first framework was presented to 20 DZLM researchers and reflections in group work were initiated. This phase was characterized by the discussion of the topic and the central questions "What is missing? What can be omitted?".

Survey 2:

The results from the first survey of the Delphi method and the further adaptation were again presented to the DZLM network with 26 experts. This discussion was followed by an online questionnaire, which was made accessible to 33 DZLM experts. With a response rate of 34%, this can be assumed to be very satisfactory.

After the results had been incorporated, it was established that there were no more significant differences of opinions, so that a new round in the DZLM network was no longer necessary.

Survey 3:

The results of survey 2 were presented to 11 further researchers and 28 stakeholders. The stakeholders come from 12 different federal states and are in their region responsible for the qualification of facilitators in mathematics education. This whole sample included representatives from various universities, (pedagogical) state institutes, ministries of education, schools and special authorities for teacher training.

This third survey was conducted in two steps. In a first round, everyone wrote down a personal statement to the framework in general. These statements were discussed in small groups resulting in written group statement. All these written statements were analysed and included in the further development of the framework.

In a second round we focussed on the usability of the framework. For this the stakeholders reflected the framework by applying it to a specific PD-topic from their experiences. They were asked to think of whether the single competencies have been addressed and how and if they could structure all the competencies according to their importance.

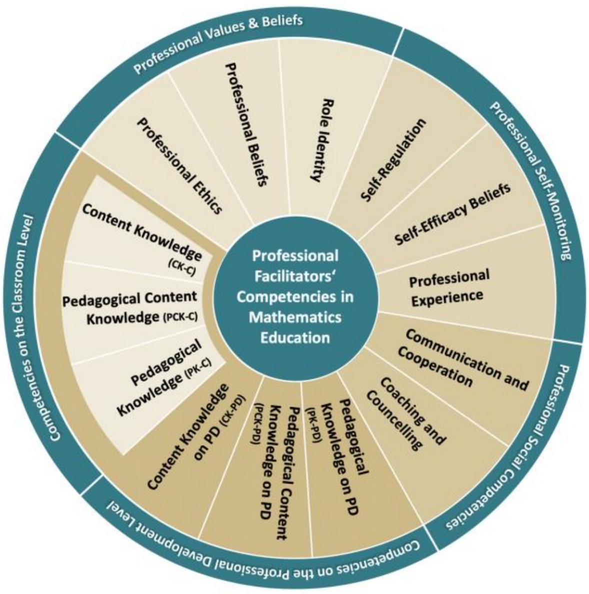

Figure 1: Competency framework for facilitators in mathematics education

Based on these findings and exchange processes in the DZLM, a final competency framework for facilitators in mathematics education has now emerged (Figure 1).

RESULTS

All the steps of the Delphi method were characterized by strong discussions resulting in important points for the further development of the framework. To give an exemplarily insight into the whole process and the emerged framework we highlight the most important aspects, which easily converged or led to strong debates.

The key issue through all surveys was the question to figure out differences and similarities between the teachers' competencies on classroom level and the facilitators' competencies on PD-level. What can be lifted from classroom- to PD-level and what is new on PD-level? These were the questions, which emerged at different steps in the process and led to the specification in the two areas about competencies on the classroom and on the PD-level. It was an intensive debate, whether these two areas are separated as equal areas or if knowledge on classroom level is an integral part of knowledge on PD-level. The final agreement was that content knowledge on PD-level (CK-PD) covers all aspects of teachers' knowledge. Lifting the well-established specification CK, PK and PCK from classroom- to PD-level, you must specify PK-PD and PCK-PD for facilitators (Wilhelm et al. 2019). Both take into account the specific orientation on teachers as learners, either from a general view on adult education (PK-PD) or as PCK-PD in a subject-related way (Prediger, 2019). PCK-PD covers all "the abilities to engage teachers in purposeful activities and conversations about those mathematical concepts, relationships and to help teachers gain a better understanding of how students are likely to approach related tasks" (Jacobs et al., 2017, p. 3). It also includes learning hurdles when teaching mathematics (Rösken-Winter et al., 2015).

The clear structure of the framework in key competency areas was appreciated from the beginning. The four areas (Competencies on PD Level, Competencies on Classroom Level, Professional Values and Beliefs, Professional Self-Monitoring) were changed to five by supplementing "Professional Social Competencies". The reason was to point out that "Communication and Cooperation" has to be considered on all levels and is relevant between all players (teachers, facilitators, stakeholders).

The segments "Professional Values and Beliefs" and "Professional Self-Monitoring" have been restructured. Above all, the dual role and one's own understanding of their role as a facilitator also had an increase influence. Therefore, the facet "Role Identity" was included alongside professional beliefs and ethics. The concept of motivational orientation has been replaced by self-efficacy beliefs, as this is the more relevant aspect in the field of PD (Bandura, 1999; Thurm & Barzel, 2020). For stakeholders it was important to include "Professional Experiences" explicitly to foster appreciation for teaching practice.

What became apparent, however, was that the developed framework did not directly cover the systemic dimension such as school development processes, which are rather in the background, behind the aspects. Nevertheless, on the basis of the evaluation of the documents, the participants found the idea, that embedding the systemic dimension has to be always taken into consideration, sufficient.

CONCLUSIONS

The development process of this competency framework showed the importance of involving the different players who are responsible for scaling up CPD-process in mathematics education. The Delphi method was highly relevant to reach a consensus in structure, the detailed competencies and in terminologies. This offers a sound basis when designing cooperatively processes in scaling up in CPD.

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ENHANCING THE DEVELOPMENT OF MULTIPLICATIVE REASONING IN EARLY CHILDHOOD EDUCATION: A CASE STUDY

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We report a case study intervention pilot-testing a program of activities aiming at enhancing pre-primary children's multiplicative reasoning competences. The program treated discrete and continuous quantities in a unified manner; provided learning experiences pertaining to three fundamental multiplicative operations, namely iteration of a quantity, equi-partitioning, and counting with composite units / measuring with fractional units; and introduced terms for multiples and submultiples. Four pre-primary children participated in an intense 4-day intervention. The program of activities was well within their range of abilities and enhanced their competences in terms of their ability to discern and express verbally multiplicative relations; and to tackle multiplicative situations and explain their strategies.

THEORETICAL FRAMEWORK

Research-based evidence indicates that young children perceive, at a rudimentary level, multiplicative/proportional relations (Mix, Huttenlocher, & Levine, 2002), and are able to tackle simple multiplicative situations between discrete as well as continuous quantities (e.g., Hunting & Davis, 1991; Kornillaki & Nunes, 2005). For example, 4-5 year-olds identify pictures of imaginary creatures that are magnified proportionally among others that are not (Sophian, 2000). Provided that a sufficient number of area models of a part-whole relation are presented, 6-year-olds can select a model that represents the same relation albeit via different shapes with respect to kind or size (Goswami, 1989). Children who have not been taught multiplication or division (6-7 years of age) can recognize simple multiplicative transformations of discrete and continuous quantities and predict the effect of the transformation on a different quantity (McCrink & Spelke, 2016). Five to seven-year-olds deduce the principle "more recipients, smaller share" that underlies fair-sharing situations (Kornillaki & Nunes, 2005), for discrete as well as continuous quantities. As could be expected, these early competences manifest themselves in a limited range of contexts and conditions. In addition, there are inter-individual differences with respect to these competences. It is nevertheless important to note that early multiplicative reasoning is enhanced when children are exposed to relevant informal or formal learning experiences (Hunting & Davis, 1991; Van den Heuvel-Panhuizen, & Elia, 2020).

Early education has not capitalized yet on such evidence. For example, an analysis of the latest Greek early mathematics curriculum (K-2) showed that learning objectives pertaining to additive reasoning precede and are far more than the ones for multiplicative reasoning,

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for discrete as well as continuous quantities (Vamvakoussi & Kaldrimidou, 2018). In the Kindergarten curriculum, in particular, all learning objectives for multiplicative reasoning are limited to discrete quantities; no linguistic or other tools for expressing multiplicative relations are mentioned, not even the word "half". Nevertheless, children are intended to familiarize themselves with multiplicative situations pertaining to multiplication, partition, and quotition that call for three fundamental multiplicative operations, namely iteration of a quantity, equi-partitioning, and counting with composite units. These operations are applicable also for continuous quantities, if "counting with composite units" is replaced by "measuring with fractional units". This fact is not exploited in instruction.

These limitations in the Greek kindergarten early math curriculum indicate that early multiplicative reasoning competences are not adequately supported in early instruction, especially in the context of continuous quantity. The lack of terms for multiplicative relations is also important, given that linguistic tools are indispensable for prompting children to attend to the relations embedded in multiplicative situations and recognize the same relation in different contexts (Hunting & Davis, 1991). Indeed, vocabulary pertaining to multiplicative relations in the first grade has been found to uniquely predict proportional reasoning abilities in the second grade (Vanluydt, Supply, Verschaffel, & Van Dooren, 2021).

It could be argued, however, that introducing terms for multiples and submultiples in the first years of instruction as well as extending the multiplicative situations that children are intended to explore to continuous quantities as well, might be beyond the range of abilities of young children.

We designed an program of activities addressing discrete and continuous quantities in a unified manner (see Steffe, 1991, for a relevant recommendation); providing learning experiences pertaining to all three aforementioned operations; and introducing terms for multiples and submultiples. To the best of our knowledge, there is no similar intervention reported in the literature targeting young children. We report results from a case study intervention investigating whether this program a) was within the range of abilities of pre-primary children, b) would enhance children's multiplicative reasoning competences, in terms of their ability to discern and express verbally multiplicative relations; and to tackle multiplicative situations and explain their strategies.

METHOD

The present study is a quasi-experimental case study. (Pre- test/ Intervention/ Post-test without a control group).

Participants

The participants were 4 children (mean age 5 years 7 months, one girl) who had just graduated from kindergarten and were familiar with fair-sharing situations involving discrete quantities (two recipients, no remainder).

Experimental tasks

Pre- and Post-test were conducted via individual interviews. Three tasks were used (A, B, C, 4 trials each) targeting the relations 1:2 and 2:1 for discrete and continuous quantities (represented by concrete materials). Task A was an analogical task where the intended relation (X/Y) was exemplified, with the information that "X matches Y". The children were asked to find the quantity Z matching a new quantity W. Task B was framed as a fair-sharing problem. The children were given the initial quantity and asked to find the share (1:2); and vice versa (2:1). In task C children were explicitly asked to find "half" and "double" of given quantities, and to explain what the terms "half" and "double" mean. Overall, there were 3 trials for 1:2 for discrete quantities, 3 trials for 1:2 for continuous quantities; and similarly for 2:1.

An additional task (D), similar to C, albeit for 1:3/ 3:1 was added in the post-test. The children were given 5 alternatives for each trial and were asked to explain their answers.

Procedure

The children participated in the intervention as a group, during four consequent days (one session per day, about 45΄ each). The pre- and post-test took place one day before and two days after the intervention, respectively. Children and their parents consented to participate in the study. The intervention was carried out by the first author of this paper, a qualified kindergarten teacher.

The intervention

We designed two types of activities. Both types addressed discrete and continuous quantities represented by concrete materials; were embedded in story-based scenarios; and required iteration of a quantity, equi-partitioning, and measuring with different units (composite units for discrete quantity, fractional units for continuous quantity). The first type of activity was based on simple and proportional sharing. The children worked with 24 such multiplicative situations during the first two days. According to the scenario, the children were asked to help imaginary creatures (represented by rectangular bars of cardboard with equal width, but different length) to share candies (discrete) or chocolate bars (continuous) proportionally to their length. The relations between the lengths were 1:1 (fair-sharing), 1:2, and 1:3 (proportional sharing). Depending on what was asked (number of recipients, quota, or the shared quantity), different operations were required.

The second type of activities addressed multiplicative change situations. We employed "fractions machines" (Hunting & Davis, 1991), producing multiples of given quantities from one side (2, 3, 4), and the corresponding submultiples (1/2, 1/3, 1/4) from the other. The children worked with 23 such problems during the last two days.

During the intervention, the researcher modeled the operations and introduced the new terms. The terms for multiples (double, triple, quadruple) were introduced in the context of iteration of a quantity. The children were familiar with the term "half" in fair-sharing situations (equi-partitioning), so the same context was used for the introduction of other terms for submultiples (one third, one fourth). Because equi-partitioning continuous quantities, in particular in three parts, was challenging for the children, fractional pieces of the quantities were available for them to choose from and examine how many times they fit in the given quantities. Thus, the children had to estimate the magnitude of the part first, and then to verify their estimate by measuring the quantity with its part.

RESULTS

Children's response to the intervention

The intervention tasks were challenging for the children. Indeed, most of the tasks were unfamiliar to them, in particular the tasks pertaining to multiplicative change, which was a novel multiplicative situation for them. The children collectively came up with effective strategies for some of the unfamiliar situations (e.g., dealing for fair-sharing to more than two recipients, folding for equi-partitioning a continuous quantity in two parts). More importantly, they appeared capable of adopting and using the intended strategies and vocabulary introduced by the researcher; and to transfer them to the novel situation of multiplicative change. To illustrate this point, we present two episodes that occurred on the fourth day of the intervention. In the first episode, the researcher presents for the first time the children with the 1:3/3:1 "fraction machine" for discrete quantities. The capital letters in the brackets refer to Figure 1, illustrating the use of materials by the researcher and the children.

Researcher: This machine works with candies. If I put this candy in here, it will produce three candies out of its big side [*illustrates with the materials, A]*

Child 1: Triple. And if you put two candies in, it will make them six [*mentally*].

Researcher: How do you know this?

Child 1: Because it will repeat three two times [*illustrates with the materials, B*]

Child 4: No, it will repeat two three times. Because there are two candies [*illustrates with the materials, C*]

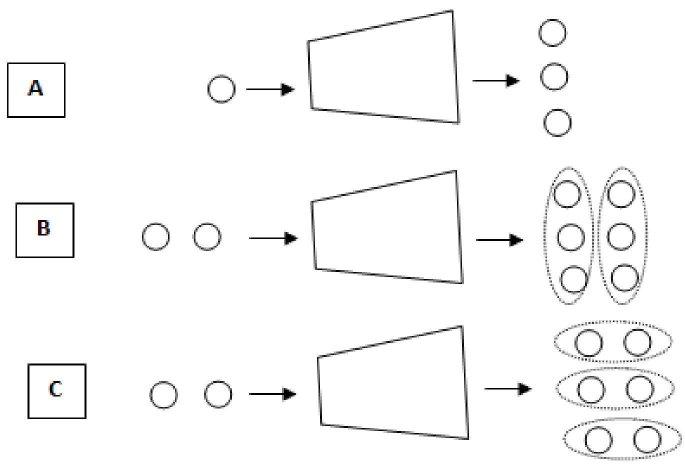

Figure 1: Introducing the 1:3/3:1 "fraction machine" (discrete quantity, multiple)

Child 1 discerned and verbalized the intended relation already with the first example (Figure 1, B), and offered an additional example. His answer, although numerically correct, did not model accurately the given situation. Child 1 presumably relied on one-to-many correspondence (one candy -> three candies, one + one candies -> three + three candies), a strategy that was not presented in the intervention. Child 4 recognized and corrected the "misstep" using the intended strategy (Figure 1, C).

In the second episode, the researcher had already introduced the 1:3/3:1 "fraction machine" for continuous quantities and the children had worked with tasks regarding the increase of the length of stick candies by a factor of 3. The researcher then asked about the inverse process:

Researcher: Now let's see what happens if I put this stick candy into the big side of the machine. What do you think will come out from the small side?

Child 3: It will share it [*sic*].

Child 1: Yes, three times [*sic*]

Child 2: Where are the little pieces? [*tries with two smaller stick candies (1/2, 1/3) checking whether they fit three times into the given one*].

As this excerpt indicates, the children were able to anticipate what the machine would do, and also to use the intended strategy in order to find the outcome. However, none of them used the term "one third". More generally, regarding submultiples, the children used spontaneously only the word "half" during the intervention. On the contrary, they adopted and used the terms for multiples (e.g., Child 1 in the first episode), and also attempted to generalize them. For example, Child 3 invented a word similar to "sixtuple" to refer to "the one that makes everything six times bigger".

Quantity,	Pre-test					Post-test				
Relation	Ch1	Ch2	Ch3	Ch4	Total	Ch1	Ch2	Ch3	Ch4	Total
D, 1:2 (n=3)	1	3	1	1	6	2	2	2	3	9
D, 2:1 (n=3)	0	0	1	0	1	2	2	2	3	9
C, 1:2 (n=3)	0	1	2	1	4	1	2	1	3	7
C, 2:1 (n=3)	2	0	1	1	4	3	3	3	3	12
Total	3	4	5	3	15	8	9	8	12	37
D, 1:3 (n=2)	-	-	-	-	-	1	0	1	0	2
C, 3:1 (n=2)	-	-	-	-	-	0	1	0	1	2

Table 1: Total of correct responses per type of quantity, per relation, and per child in the pre- and post-test.

Performance

Children's correct and incorrect responses in the pre- and post-test were scored by 1 and 0, respectively. Table 1 presents the frequencies of correct responses in the trials corresponding to each relation (1:2, 2:1) across the tasks A, B, C, for discrete (D) and continuous (C) quantities. For the common part of the pre- and the post-test, this results in three responses per relation and per type of quantity; and similarly to two responses in the additional task (Dask D) of the post-test.

Table 1 shows that there was a considerable increase in correct responses after the intervention, at group as well as at individual level. For the common part of the pre- and post-test, the percentage of correct answers in the total of trials involving discrete quantities has increased from 29,2% to 75%; and the percentage of correct responses in the trials involving continuous quantities has increased from 33,3% to 79,2%. In Task D, each child responded correctly to one out of four trials.

Explanations

Children explanations during pre and post- test can be roughly categorized in two different types. The first type (Non-valid explanations) includes null explanations (e.g., "I don't know" or "I saw it"); pseudo-explanations that were relevant to the general context but not to its quantitative aspects (e.g., "Because he wants to eat chocolate after dinner"); and inadequate quantitative explanations, typically based on absolute quantity, rather than on quantitative relations (e.g., "because it's small", "because there are three").

The second type of explanations (Valid explanations) includes the cases where children expressed verbally and/or non-verbally (e.g., with gestures) a valid strategy that they used to make or verify their choice. We also included in this category explanations indicating that the children employed the principles underlying the situation at hand. In the following excerpts we present three examples of valid explanations, two in the context of continuous quantity, and one in the context of discrete quantity.

In the first example, Child 3 explained how he found half of a "chocolate bar":

Child 3: We cut in the middle and we got one half [*passes his hand over the "chocolate bar*]. And the other piece that remains is also half. It is the same as this one [*points to the correct alternative*]. Here, look! [*picks up the correct part and shows it fits two times into the "chocolate bar*].

In the second example, Child 2 was presented with a "chocolate bar" and was asked to find the one that was double (in length) than the given.

Child 2: Which chocolate bar should I choose for this small one? I know! That one! [*points to the correct alternative*]

Researcher: Why do you think it's this one?

Child 2: Because if you try to fill the big chocolate bar, you have to have two like the small one [*mentally*].

In the third example, Child 4 was told that the researcher gave some candies to "Helen and her little sister" and that the two girls shared the candies fairly. Then he was presented with the two candies that Helen took.

Researcher: Look, Helen took two.

Child 4: And her little sister another two [*mentally*].

Researcher: And how many candies did I have in the beginning?

Child 4: Four. And then they became two for each girl.

In this excerpt, Child 4 showed a quite principled understanding of the situation at hand: First, he appeared to employ the principle that in fair-sharing situations, the shares must be equal. He then used the two equal shares to compose the initial quantity, while also referring to the inverse process.

Table 2 presents the frequency of valid and non-valid explanations in the total of the trials in the common part of the pre- and the post-test (12 trials per test) and in the additional task of the post-test (4 trials), per child. There was a considerable increase in the number of valid explanations after the intervention, at group as well as at individual level. Overall, in the common part of the pre- and the post-test the percentage of valid explanations increased from 6,3% to 62,5%. In the additional task (task D) of the post-test, two of the children gave no valid explanations.

Tasks	Explanation type	Pre-test					Post-test				
		Ch1	Ch2	Ch3	Ch4	Total	Ch1	Ch2	Ch3	Ch4	Total
A, B, & C	Non-valid	12	11	12	10	45	4	5	7	2	18
	Valid	0	1	0	2	3	8	7	5	10	30
	Total	12	12	12	12	48	12	12	12	12	48
D	Non valid	-	-	-	-	-	3	4	2	4	13
	Valid	-	-	-	-	-	1	0	2	0	3
	Total						4	4	4	4	16

Table 2: Total numbers of valid and non-valid explanations in the pre- and post-test, per child.

CONCLUSION –DISCUSSION

We designed a program of activities introducing three fundamental multiplicative operations in a variety of situations, and terms for expressing multiplicative relations, across discrete and continuous quantity. We pilot-tested this program with a case study intervention, with four pre-primary children. The intervention was short and, arguably, very intense in terms of the amount of work required from the children.

Nevertheless, the results were promising. The program of activities was well within the children's range of abilities. By the 4th day, the children implemented the intended strategies; invented their own strategies; discerned and verbalized multiplicative relations and anticipated the outcome of multiplicative transformations. These competences were fairly stabilized for 1:2/2:1, as indicated by children's performance in the post-test in terms of correct answers and valid explanations. Children's ability to tackle relations beyond 1:2/ 2:1 was evident during the intervention, but did not reflect in their performance in the post-test. This is not an unexpected result, since 1:2/ 2:1 are more accessible to young children (Hunting & Davis, 1991) and the participants already had some relevant experience. The short duration of the intervention should also be taken into consideration.

A long-term, systematic intervention, with a larger and more diverse sample, is required to investigate whether a program of activities with the specific features can substantially enhance young children' multiplicative reasoning. In particular, it is worth investigating whether children who have acquired vocabulary relevant to multiplicative relations in early instruction (possibly, at kindergarten) and can use it in a variety of multiplicative situations are more competent in multiplicative reasoning in the long run (Vanluydt et al., 2021).

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RELATIONS BETWEEN INDIVIDUAL INTEREST, EXPERIENCES IN LEARNING SITUATIONS AND SITUATIONAL INTEREST

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Students' individual interest in mathematics decreases in the first semester at university which leads to demotivation and even dropout. One way to prevent this could be to foster students' situational interest. Thus, the present study aimed to take a closer look to individual and situational factors that influence the emerge of situational interest in university mathematics courses. 150 first-semester students filled in a questionnaire concerning individual traits and stated three times in a lecture their situational interest as well as their experience of competence and autonomy. Linear mixed models indicate that individual interest in university mathematics as well as experience of autonomy and competence strongly relate to situational interest. These findings could contribute to a better support of students' interest development.

INTRODUCTION

Transition to Advanced Mathematics Courses

A high dropout rate of study programs in mathematics indicates serious students' problems, especially in the first year of study (di Martino & Gregorio, 2018; OECD, 2010). Researchers postulate two changes of the learning environment at the transition from school to university: a shift in the character of the learning domain, mathematics, and a change of the learning opportunities from guided to self-regulated learning (cf. Rach and Heinze, 2017).

Advanced mathematics includes mathematics as an academic discipline based on concept definitions and deductive proofs (Gueudet, 2008). This character of mathematics strongly shapes teaching at university, e. g., by a strong focus on the Definition–Theorem–Proof structure (Engelbrecht, 2010). In contrast to that, teaching mathematics in school primarily focused on the goal of general education: Mathematical concepts and procedures are useful tools for describing the world and solving real world problems (e. g., Gueudet, 2008). Specifically, in Germany, the transition from school to university therefore coincides with a shift of the learning domain from an applied-oriented form of mathematics to advanced mathematics.

In addition, the formal organization of learning opportunities and the individual learning strategies necessary for an effective use of the learning opportunities differ between school and university. In Germany, the Linear Algebra course for students in a mathematics or a

2021. In Inprasitha, M, Changsri, N., Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 405-412. Khon Kaen, Thailand: PME.

teacher education program consists of three different learning opportunities a week: two 90-minutes lectures, a set of approx. four challenging exercises as obligatory homework (self-study phase alone or in small study groups) and a 90-minutes tutorial per week (see Rach & Heinze, 2017). In lectures, a lecturer presents fundamental definitions, theorems, and proofs which students apply when solving the exercises. In tutorials, students discuss the exercise solutions with a tutor.

Role of Situational Interest in Learning Processes

Students' problems in the first year of study are mainly based on insufficient performance and the lack of motivation. One important motivational variable is interest, often conceptualized as a person-situation-relationship (Hidi & Renninger, 2006; Krapp, 2002). It is assumed that high interest leads to a frequent use of deep learning strategies (Willems, 2011) and to more engagement in learning situations (Dietrich, Viljaranta, Moeller, & Kracke, 2017) because an interested learner seeks to find out more about the learning content. However, there is only little empirical evidence for this chain of effects. One reason for this difference between theoretical argumentation and empirical results is that many studies analyse the relation between individual interest and the use of learning strategies and not between interest in a specific learning situation and the use of the learning strategies in this situation. In contrast to individual interest as a trait, situational interest is defined as "a temporary state aroused by specific features of a situation, task, or object (e. g., vividness of a text passage)" (Schiefele, 2009, pp. 197-198) and may influence learners' behaviour in concrete learning situations. Situational interest consists of two components, a feeling- and a value-related one: "The feeling-related valences refer to positive experiential states while being engaged in an interest-based activity" (Krapp, 2002, pp. 389) and "The value-related valences refer to the assumption that an interest has the quality of personal significance" (Krapp, 2002, pp. 388). In line with these works, I define situational interest as follows: *Situational interest is a motivational state which is characterized by a feeling- and a value-related component. Situational interest results of an interaction of learners' and situational features.*

Other psychological models that describe feeling- and value-related valences are expectancy-value-models (Eccles & Wigfield, 2002). Especially, value concerning an object is closely connected to situational interest because e. g. value is decomposed in intrinsic value (similar to the feeling-related component of interest) and utility value (similar to the value-related component of interest). Gaspard and colleagues (2015) divide the utility component further on in utility for career, school, daily life etc. Since these two concepts are very similar to each other, results from both research directions will be considered. In the following, I will summarize in which way situational interest resp. value supports learning processes which individual and situational characteristics may influence the emerge of situational interest resp. value.

Previous studies show that situational interest resp. value is related to engagement in learning situations (e. g. Dietrich et al., 2017; Linnenbrink-Garcia et al., 2013) and a long-maintained situational interest can grow into individual interest (Hidi & Renninger, 2006). Individual interest is an important trait that has an impact on the choice of study program or career (Ufer, Rach, & Kosiol, 2017).

Several studies indicate that individual interest influences situational interest (Ferdinand, 2014, 10th grade, social sciences; Linnenbrink-Garcia, Patall, & Messersmith, 2013, science summer program for adolescents; Willems, 2011, 8th grade, mathematics). Until now, there are divergent result concerning the role of prior knowledge on situational interest: Whereas Schukajlow and Rakoczy (2016) didn't identify any relation in a study with 9th graders in mathematics, Rotgans and Schmidt (2011) reported a small correlation between prior knowledge and situational interest in a university economic lecture.

Divergent results of these studies can be explained because the reported studies vary in the learning context, e. g. in the learning content and the learners' age. Thus, characteristics of the learning context should be taken into account. As summarized above, learning situations in the first year of university are characterized by specific features, especially by distinguishing academic mathematics from school mathematics. However, in many questionnaires items like "I enjoy doing mathematics" are applied when measuring individual interest in mathematics. For students, it is not clear which character of mathematics they should refer to when estimating such items – mathematics at school or advanced mathematic at university. So, I recommend for studies in tertiary mathematics programs to use instruments that differentiate between mathematics at school and at university (Ufer et al., 2017).

Not only individual characteristics but also person-situation-interactions can influence the emerge of situational interest. A prominent approach to describe learners' experiences of a learning situation is the self-determination theory, especially the concept of basic needs (Deci & Ryan, 2002). The mode of action may be that only if the basis needs are fulfilled, then learners perceive situational characteristics of the learning context that foster learners' interest. Empirical studies partly support this assumption: Situational interest is partly connected to the experience of competence (Ferdinand, 2014; Willems, 2011; see also Linnenbrink-Garcia et al., 2013) and to the experience of autonomy, especially to the fit of personal wishes (Ferdinand, 2014; Willems, 2011). Based on these studies, the experience of relatedness seems to be less important for the occurrence of situational interest.

To increase the experience of autonomy, competence and thus situational interest, situational characteristics of the context could be helpful. Hulleman and Harackiewicz (2009) found that students in a science class with low expectations profit from activities which encourage them to connect course materials to their lives. These learners with low expectations reported a higher interest in science than students who didn't think about the relevance of the course content. Gaspard and colleagues (2015) use a similar approach for a relevance intervention in mathematics classrooms.

In sum, it is essential to foster situational interest and situational interest is probably closely connected to individual interest. However, at the beginning of tertiary courses in mathematics, the character of mathematics change and it is an open question which motivational traits concerning which character of mathematics influence situational interest in this situation. I expect that a more differentiated measure of individual, motivational variables could explain the emerge of situational interest better. In addition, the role of prior achievement and experiences of competence for the emerge of situational interest is not fully clarified.

RESEARCH QUESTIONS

To gain a closer insight into students' situational interest, I focus on the following questions:

- How does students' situational interest relate to their individual charac-teristics, especially their study program, prior achievement and facets of individual interest?
- How does students' situational interest relate to their experience of autonomy and competence?

METHODS

Participants were 150 students (94 male, 56 female) in a first-semester-course. The majority of these students (72%) were enrolled in a teacher education program for secondary schools; the others in a mathematics (13%) or in a computer science or physics program (15%). Almost all students of the lecture participated in this study.

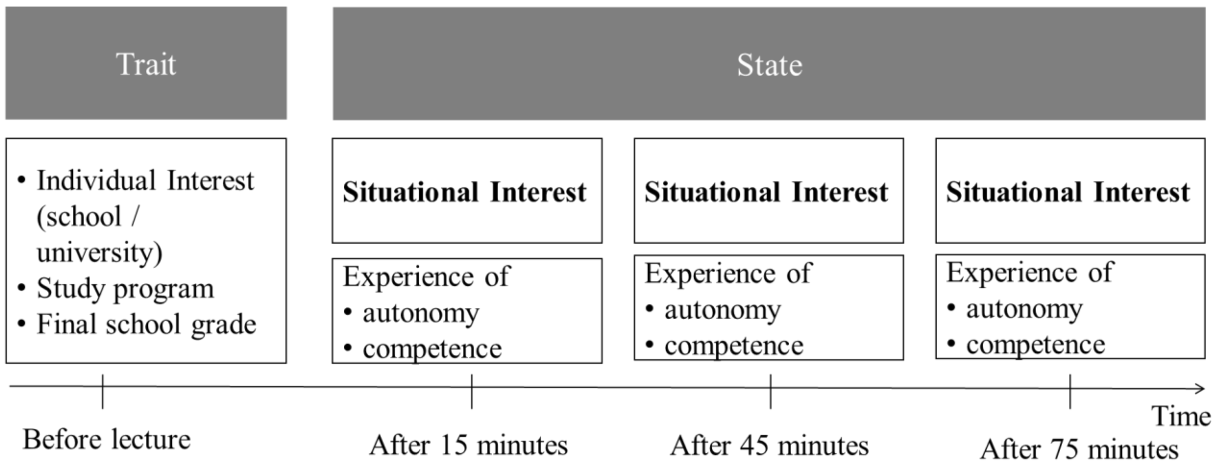

Figure 1: Design of the study

The study took place in a first-semester lecture "Linear algebra" in the second week of semester. This course was an advanced mathematics course and typical topics like vector spaces, bases, linear equation systems etc. were included. Students voluntarily reported at the beginning (after 15 minutes of lecture), in the middle (after 45 minutes of lecture) and at the end (75 minutes of lecture) their situational interest (Figure 1). The applied questionnaire is based on the instruments of Gaspard et al. (2015) and Dietrich et al. (2017) and contains the feeling- as well as the value-related component of situational interest (see Table 1). When stating their situational interest, students also rated how autonomous and self-competent they felt in the last 15 minutes of the lecture (adapted from Willems, 2011, see Table 1). As I only wanted to interrupt the lecture smoothly, I used single-items to measure the experience of competence and autonomy. Students rated all items on a four-point-likert scale from agree (4) to disagree (1). The descriptive analyses don't give hints for floor or ceiling effects, the reliability for the scale "situational interest" is still acceptable (Table 1).

Scale	Sample Item	*M* / *SD*	α
Situational interest, feeling- and value-related component (4 items)	I like these contents. The content is important for my study.	2.89 / 0.57 2.82 / 0.61 2.89 / 0.64	.64 .66 .69
Experience of competence (1 item)	I have the feeling that I can understand difficult content.	2.65 / 0.82 2.40 / 0.86 2.45 / 0.88	-
Experience of autonomy (1 item)	I have the feeling that the lecture is as I wish.	2.59 / 0.81 2.52 / 0.89 2.50 / 0.91	-

Table 1: Scales with sample items, means (*M*), standard deviations (*SD*), and cronbachs' α for all three times.

Before the lecture started, students filled out a questionnaire concerning their individual interest in school and their interest in university mathematics (each scale à five items, $\alpha = .74$ and $\alpha = .81$, item example for interest in school mathematics: "I am interested in the kind of mathematics that I learned at school", see Ufer et al., 2017). The correlations between these two interest facets are small to middle: $r = 18$, $p < .05$. To use it as an indicator for prior achievement, the students reported their overall final school grade (reversed scale, from 1.0 sufficient to 4.0 very good).

RESULTS

This design is a multi-level design because situational interest is measured three times in a lecture. So, to investigate which factors situational interest predict and to control the standard errors, I use a linear mixed model (see Bates, Mächler, Bolker, & Walker, 2015). By this model, one can explore the relation between fixed factors like study program or facets of individual interest as well as random factors, such as times.

In the first step, I analyse if the person or the time is more relevant to explain differences in situational interest. It turns out that situational interest differs more between persons than between time. In the second step, I examined which individual and situational factors predict the emerge of situational interest. Whereas the final school grade doesn't explain any variance in situational interest, the study program does: Students enrolled in a teacher education program reported less situational interest than students from the mathematics study program (Table 2, Model 1). By integrating facets of individual interest into the analysis, the prediction power of the study program decreases because individual interest in university mathematics relates strongly to situational interest, in contrast to interest in school mathematics (Table 2, Model 2). In the last model, interactions of students and the learning context are focused: Experiences of competence as well as of autonomy relate to situational interest. As individual interest and experiences are measured on the same likert-scale, one can compare the strength of the relations: Individual interest and experience of autonomy are similar connected to situational interest, while experience of competence shows a smaller relation to situational interest than the other two factors (Table 2, Model 3). With the last model, 41% of the variance of situational interest can be explained.

Variable	Model 1	Model 2	Model 3
Study Program	-.61***	-.37***	-.30**
Final school grade	.03	.02	.00
Individual Interest School		-.09	-.05
Individual Interest University		.41***	.21**
Experience of Competence			.12***
Experience of Autonomy			.19***
R^2	.18	.31	.41

Table 2: Results (unstandardized regression coefficients) of a mixed-linear regression model to predict situational interest; *** $p < .001$, ** $p < .01$.

DISCUSSION

The starting point of this contribution are the enormous dropout rates in mathematics study programs which are partly based on students' demotivation of studying (academic) mathematics. The present study aims to take a closer look on factors which influence students' situational interest in university learning situations. In line with previous research (Linnenbrink-Garcia et al., 2013), individual characteristics, especially individual interest in university mathematics, strongly influence situational interest. The results also provide additional support for the relation between experience of autonomy resp. of competence and situational interest (see Ferdinand, 2014; Willems, 2011). In addition, the results of this study show that situational interest differs between learners of different study programs. Students in a teacher education program report less situational interest comparing to students of the mathematics program in this study. Following the results of Dietrich et al. (2017), these students will be less engaged in learning academic mathematics. Ufer and colleagues (2017) support this observation when they reported that students in a teacher education program are much more interested in school mathematics than in academic, university mathematics. For these students who want to become teachers in the future, it is probably important to understand the relevance of university mathematics for their future career as a school teacher to improve their learning engagement.

The interpretation of this study is limited because the reliability of the situational interest scale is only acceptable. This is probably due to the broadness of the construct. In addition, I investigated situational interest at three times in only one lecture. Future research could replicate these results in other study programs and deeply examine the influence of situational interest on the engagement and success in university learning situations by a longitudinal study.

The results of this study still can contribute to our understanding of the emerge of situational interest and can help to construct activities that foster the development of situational interest. According to the study of Hulleman and Harackiewicz (2009), university courses could offer activities which encourage students to make connections between the learning content and their future career. One possible activity builds on the so called “Schnittstellenaufgaben” which link school and advanced university mathematics (Bauer & Kuennen, 2017) and could be helpful to motivate students in teacher education programs. Besides supporting students in their learning process, these intervention studies could deeply explain the relation between situational interest and the experience of autonomy and competence in concrete learning situations.

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TEACHER PREPARATION FOR PRIMARY MATHEMATICS IN A BILINGUAL, RURAL CONTEXT IN SOUTH AFRICA

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In South Africa, although school language policy expects that the majority of early grade teachers should teach mathematics in an African language, initial teacher education programmes are offered in English. To investigate bilingual sense making of mathematics, students in their first year of a teacher training programme wrote the same mathematics test in two languages: English and isiXhosa, as part of the national assessment. Data from 88 student teachers revealed that their performance was better in the English version of the test. The majority of the students (69%) preferred the English version of the test. The study recommends that universities need to pay far greater attention to the teaching of mathematics in isiXhosa for the early grades.

INTRODUCTION

Globally mathematics classrooms are becoming increasingly diverse in terms of their linguistic resources (Barwell, Wessel & Parra, 2019). There are a diminishing number of classrooms where all the children in the class speak the world language which is used as the medium of instruction for mathematics. It can no longer be guarenteed that the mathematics teacher shares a home language with all the members of her class. This is a result of globalisation and increased migration and movement of displaced people; and the growth of small groups advocating for the rights of indigineous people and those in the linguistic minority in a country to access learning of mathematics in their mother tongue (Barwell, Barton, & Setati, 2007). According to Moschkovich (2002: 189)

> if mathematics reforms are to include language-minority students, research needs to address the relation between language and mathematics learning from a perspective that combines current perspectives of mathematics learning with current perspectives of language, bilingualism, and classroom discourse.

A systematic review of research in multilingual classrooms in South Africa (Setati, Chitera, & Essien, 2009) found an insignificant number of publications focusing on multilingual classrooms; and described the research on the role of language in early grade mathematics (Grades R–4) as insignificant (compared to research on language in mathematics at higher levels). This gap has been partly addressed by a more recent review which focuses explicitly on the early grades and has South Africa, Kenya and

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Malawi with their multilingual mathematics classrooms (Essien, 2018). In this review Essien identifies three themes pertaining to this corpus of research: (1) research on the impact of language policy framework on curricula in relation to teaching and learning of mathematics; (2) research in teacher education and professional development; and (3) research on pedagogic/language practices. Essien (2018) notes that, in relation to theme 2, there is a paucity of research in general, and clear gaps in terms of how teachers are trained to teach early grade mathematics in multilingual contexts; and that there are rarely quantitative or mixed methods studies.

This paper seeks to contribute to filling this gap by offering a mixed methods research design focusing on language and mathematics competencies of student teachers in an initial teaching programme. As home language speakers of a African language it is assumed that teachers will be able to translate their understanding of mathematics, taught in English at university level, into their African language when teaching in an early grade classroom. We investigated the assumption be comparing student teachers' attainment in mathematics tests (written in English and isiXhosa); as well as their perceptions of this experience of the tests and needs in relation to teaching mathematics in an African language.

THEORETICAL FRAMING

South Africa with 11 official languages, all being used to teach mathematics in the early grades, provides a rich lab for reflections on multilingualism and mathematics. Its language context is complex, and reflects both its colonial and Apartheid past. There are two languages of the colonisers (English and Afrikaans) and those may be considered world languages. The other nine languages comprise of sign language, and eight indigenous African languages (some of which stem from the same linguistic branches).

The use of African languages for the teaching of mathematics has been an area of debate for a long time in South Africa (see Setati & Adler, 2000). While mother tongue instruction in mathematics is considered to be the most valuable for learning; such learning is allusive when the language in question is not yet fully developed as an academic language; and when this language competes with the hegemony of a world language. Essien (2018: 52) describes the language in education policy in South Africa as being: "Based on non-discriminatory language use, ...which calls for the promotion of multilingualism in the education sector through the use of languages". The policy therefore advocates that learners choose their ‡Language of Learning and Teaching (LOLT) and that school governing bodies need to stipulate how the school will promote multilingualism in the school. Most (public) schools use the mother tongue as the LOLT in the foundation phase (Grade R–3) (Essien 2018: 52).

Setati (2008) describes the ‘freedom to choose’ the language of learning of teaching in schools as advocated in policy, as a chimera, given the economic, social, political and ideological factors underpinning the world language (English). So while the majority of children in the early grades learn mathematics in their home language, by Grade 4 they are expected to learn mathematics in English. They continue to learn an African language (at home language level) and English (as a first additional language).

In order to find solutions to the mathematics under-achievement in all levels of schooling, particular importance was given to primary school mathematics teaching and learning and the national Primary Teacher Education (PrimTEd) was established. This had a dual focus: on mathematics and on language (where the primary concern was African languages).

METHODOLOGY

The setting for this study was the rural province of Eastern Cape, were the dominant African language is isiXhosa. The study was conducted in a rural university where the majority of student teachers were isiXhosa home language speakers, and had passed their National Senior Certificate with isiXhosa as a subject at home language level. They are also relatively fluent in English (having been taught in English from Grade 4 onwards and passed their National Senior Certificate being examined in English).

The research question addressed in the paper is: How did student teachers perform in, and respond to, the same mathematics test administered first in isiXhosa and then in English (or vice versa). Quantitative data was gathered from the performance trend by the primary teacher education student teachers in PrimTEd mathematics test (Alex, Roberts, Hlungulu 2020).

The PrimTEd mathematics test items were translated from English into isiXhosa by an accredited isiXhosa language translator from the rural university. The translation was reviewed by the isiXhosa speaking primary mathematics lecturer. The test comprised of 50 questions worth a total of 50 marks. Each item was allocated 1 mark and there was no partial marking as a result of the single answer and multiple choice format of items. The test was written using pen and paper, in one of the lecturer session of the university, under normal test conditions. The scripts were marked by the second author, using a memorandum.

The first year students (n = 96) voluntarily took part in the study (after giving informed consent). They were randomly assigned into two equal groups. Half of the class wrote the English version and half of the class wrote the isiXhosa version. After a week, the test was administered the other way around. After checking the number of student teachers who wrote both tests, the final data set comprised of 88 students in two groups: ‘isiXhosa first’ group (n = 44) who wrote Test 1 in mother tongue and Test 2 in English; and the ‘English first’ group (n = 44) who wrote Test 1 in English and Test 2 in mother tongue). It was assumed that student teachers would apply roughly the same effort to both tests (see Alex, Roberts, Hlungulu 2020). The results were considered in relation to the four categories:

		Group	
		'English first' group	'isiXhosa first' group
Language of the test	**English**	Test 1	Test 2
	isiXhosa	Test 2	Test 1

Table1: Four categories of groups of participants

Firstly, the test was analysed using descriptive statistics for overall attainment across all the items. Each student had two data points: Test 1 and Test 2. The mean result and standard deviation were calculated for each category. Paired t-tests were conducted to establish whether the observed differences in the means from Test 1 to Test 2 were significant or not. Secondly, for each student their change in result (delta) from Test 1 to Test 2 was calculated. They were each coded as having 'improved', 'stayed the same' or 'declined' from Test 1 to Test 2.

A short structured questionnaire was designed. This was administered after the second test, in English (which is language of instruction of the university). The following questions were posed: (1) What do you feel about the tests? (2) Which version of the test you felt easier to answer? English/ IsiXhosa? Why do you say so? (3) Based on your performance in the test, what would you like us to do for you? (4) Any other comments you want to share with us? The responses from the students, in their groups, were categorized into themes.

FINDINGS

In the first attempt (Test 1), the isiXhosa version of the test ($\underline{x}$ = 29%, SD = 7%) was found to be more difficult than the English version of the test ($\underline{x}$ = 34%, SD = 12%). A paired t-test assuming equal variance found this difference to be significant, $t\ (43) = 1.99$, $p < 0.05$. Calculating Cohen's D gave an effect size of 0.47 (medium). At the second attempt (Test 2) the isiXhosa version of the test ($\underline{x}$ = 34%, SD = 9%) was found to be more difficult than the English version of the test ($\underline{x}$ = 35%, SD = 13%). A paired t-test assuming equal variance found this difference was not significant. So while in the first attempt the English results were better than the isiXhosa results, (by a significant difference, of medium effect size), this difference was no longer evident at the second attempt.

From T1 to T2	**isiXhosa first**	**(%)**	**English first**	**(%)**
Improved	32	73%	27	61%

Stayed the same	3	7%	0	0%
Declined	9	20%	17	39%
Total	**44**	**100%**	**44**	**100%**

Table 2: Percentage of student teachers where results improved, stayed the same or declined from Test 1 and Test 2 for each group

A greater proportion of 'isiXhosa first' students improved (73%), than the proportion of 'English first' students who improved (61%). We see a complementary trend for the proportions of student teachers whose results declined from Test 1 to Test 2: There was a lower proportion (20%) of 'isiXhosa first' students, than the proportion (39%) of 'English first' students whose results declined. There was lower attainment on the mother tongue version of the test than the English version of the test.

Drawing on the qualitative data, we found that 61 (69%) of the student teachers said that the English version of the test was easier to answer; 23 (26%) of them found the isiXhosa version easier, and 4 (5%) student teachers responded that both versions were easy. Most of the open ended responses on why found the English version easier to answer were "I understand it in English better", "isiXhosa words are difficult", "English explains better than isiXhosa", and "I was taught in English".

In relation to the question: Based on your performance in the test, what would you like us to do for you? The main theme that emerged out of the student teachers' responses from both groups was that there are content issues and language issues need to be addressed for them to do better in the mathematics. Some of the content issue were general in nature: "Help me learn more Maths"; "We need to do revision" and "I need extra lessons", while others were more specific such as the mention of 'fractions' as an area of difficulty. It was positive to see that the student teachers realised that they need help with the topics and directly request help with isiXhosa vocabulary and discourse. The typical responses on language issues were "I need help with isiXhosa terms" and "Teach in English not in isiXhosa". There was a strong plea from the students to change the language of future mathematics tests to English. On a more general note, the students requested for the use of calculators in doing the mathematics tests in future.

The open ended question on 'any other comments', resulted in similar responses in terms of language and content. The language issues were stronger this time as student teachers in both groups recommended that "Maths should be taught in English" and a small minority group of students suggested that "Maths should be taught in home language and English". "Make sure learners understand the language" and "Teaching Maths in isiXhosa won't work" also echoed in the responses. The major concern from the student teachers was that Mathematics is more difficult when it is in isiXhosa as they were not exposed to it in their own schooling years and in their teacher training. The majority of the student teachers suggested that to support their teaching of Mathematics in isiXhosa required the provision of relevant resources.

DISCUSSION

The assumption that as home language speakers of an African language, teachers will be able to translate their understanding of mathematics, taught in English at university level, into their African language was found to be false. Despite the student teachers being home language speakers of isiXhosa, and expected to teach mathematics in isiXhosa, their attainment in the isiXhosa version of the PrimTEd mathematics test was significantly worse than their attainment in the English version of the same test. In addition, the students expressed a preference for the English test. They were less able to make sense of the mathematics, when it was expressed in isiXhosa, and felt that they lacked the isiXhosa vocabulary and discourse to communicate mathematics in isiXhosa.

The findings affirm the conclusion by Ramollo, (2014) who identified from two Initial Teacher Education programs that both lecturers and student teachers were concerned about the lack of vocabulary and syntax to communicate some mathematical concepts in African languages and the poor preparation which student teachers receive in the teaching of mathematics in an African language.

CONCLUSION

This paper sheds light on the complexity of the South African language and mathematics education situation. In a rural province, teachers are expected to teach early grade mathematics in isiXhosa and bring isiXhosa as their home language resource. Yet they perform better in the mathematics tests administered in English than in isiXhosa. In addition, when reflecting on the experience, the majority of the student teachers indicated their preference for the English version. This is a serious problem considering the policy expectation that they should teach mathematics in isiXhosa to early grade isiXhosa speaking children in their communities. The impact of the language policy – on their own mathematics development – where they last communicated in formal written mathematics using isiXhosa when they were ten years old; has not adequately prepared them for their role as a mathematics teacher in a multilingual classroom. As a result, it is clear that universities need to pay far greater attention to the teaching of mathematics in isiXhosa when preparing teachers for the early grades.

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WHAT MIGHT BE CRITICAL IN A CRITICAL EVENT?

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The goal of this study is to examine how different definitions for a critical event may reflect what is found to be critical by pre-service mathematics teachers. We describe and analyze two critical events that were identified by two pre-service mathematics teachers from two different classroom observations. In each event, the pre-service teacher identified an instance where the teacher could build his or her instruction based on the student's mathematical understanding. Data analysis is used to identify the similarities and differences between the two events. Implications are discussed.

RATIONALE

What is "critical" in a critical event that occurs during a mathematical lesson? It seems that the answer is in the eye of the beholder. Different researchers define, characterize and use critical events differently (e.g., Goodell, 2006; Rowland, Thwaites, & Jared, 2015; Stockero, Leatham, Ochieng, Van Zoest, & Peterson 2019). Here we refer to critical events as moments that hold the opportunity for the teacher to build on the student's thinking, to make connections within mathematics and extend students' mathematical horizons beyond the immediate task (Stockero et al., 2019).

The common thread between the different definitions and characterizations is that they all use critical events as a tool to foster teachers' learning and professional development in teachers' education programs (e.g., Jacobs, Lamb & Philipp, 2010; Karsenty, Arcavi & Nurick, 2015; Stockero et al., 2019; Van Es, Cashen, Barnhart & Auger, 2017). Here, we suggest using critical events that were chosen by pre-service teachers (hereinafter PTs) from their lesson observations as a "research tool that provides a 'window' into a learner's mind" (Zazkis & Leikin, 2007; p.15). In contrast to the shared use of critical events in teachers' education, where the critical events are brought to the PTs for interpretive discussions by the researchers (e.g., Karsenty et al., 2015; Stockero et al., 2019; Van Es et al., 2017), here the PTs choose critical events for interpretation from their lesson observations.

Similar to Zazkis and Leikin's (2007) use of examples that were generated by teachers, we believe that the critical events PTs choose "mirror their conceptions of […] their pedagogical repertoire, their difficulties and possible inadequacies in their perceptions" (p.15). Therefore, characterizing PTs' critical events will enable researchers' learning. In this study, we will examine how the different existing definitions reflect what is found to be critical by PTs in order to suggest an initial characterization.

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LITERATURE REVIEW

The research of critical events is usually done within the context of research on mathematics teachers' professional development programs, and mathematics PTs' preparation programs, in which teachers' educators bring critical events for interpretive discussion to enhance teachers' learning. For example, Karsenty et al. (2015) examined the way interpretive discussions regarding observed videotaped events enhanced the development of mathematical knowledge for teaching within the context of a teachers' professional development program. Van Es et al. (2017) examined the development of PTs' noticing of ambitious mathematics pedagogy while using videotaped critical events as the focus of the university course.

However, critical events are addressed differently by different researchers. The different names: *MOST* (Sotckero et al., 2019), *contingency moments* (Rowland et al., 2015), and *critical incidents* (Goodell, 2006) somewhat reflect the different characteristics of the different definitions. Each definition brings to the forefront some different features of critical events. For example, Stockero et al. (2019), who framed critical events as MOST, characterized them at the intersection of *student mathematical thinking*, *significant mathematics*, and *pedagogical opportunity*. S*tudent mathematical thinking* means that the student's statement concerns mathematics. *Significant mathematics* is defined as mathematics that is appropriate for the students' mathematical development level and central for the students' learning goals. *Pedagogical opportunity* is framed by the opening to build on that student's thinking and the timing for taking advantage of that opening. In this definition, Stockero et al. (2019) emphasized the mathematics of the event and the opportunity to deepen the student's mathematical understanding. Rowland et al. (2015) characterized critical events according to the triggers that led the teacher to deviate from the lesson plan: (1) Responding to student's idea – whether a response to a teacher's question or a spontaneous statement; (2) A teacher's insight that provoked her/him to modify their instruction 'in the moment'; (3) The teacher's response to the availability of tools and resources (for example, a digital resource that was central to the lesson plan is unexpectedly unavailable). Rowland et al. (2015) characterization emphasizes the gap between the planning of the teacher and the actual occurrences of the lesson. Goodell's (2006) use of critical events highlights the learning potential for PTs who use mathematical and non-mathematical critical events while learning to teach.

With the overarching goal of suggesting critical events that were chosen by PTs as a tool for researchers' learning, the goal of this study is to examine how different existing definitions for critical events may be reflected in what is found to be critical by PTs. As an initial step, in this study, we will compare and contrast two critical events that were chosen by PTs from lesson observation in order to identify what might be characterized as critical in the critical events. The specific research question is: What are the similarities and differences between the two critical events?

THE STUDY

Data Collection

The data source for the research project was 48 critical event reports submitted by 13 participants in the context of a pre-service high-track secondary school mathematics teachers' university course, ACLIM-5 (a Hebrew acronym for "clinical training for unique 5-unit (high track) mathematics teaching"). ACLIM-5 is a part of the PTs' field-based preparation in which critical events identified by the PTs during classroom observations served as a focus for interpretation.

In the study, PTs were directed to choose critical events and to submit written reports in which they described and interpreted events according to a structured framework based on Jacobs et al., (2010). The reports included prompts for describing the critical event, the mathematical context, what the student said and/or did, and what the teacher said and/or did and their thoughts about the students' and teacher's actions. PTs were instructed that a critical event is an instance where a student says something that involves mathematics and which holds the potential for further mathematical learning.

In this study, we chose to analyze two critical events, Felice's and Nora's. Felice and Nora had their field-based preparation in different schools. Choosing two critical events that were identified in different classrooms and were taught by different teachers allows us to compare and contrast the events to examine how the different existing definitions are reflected in the critical events. Additionally, both Felice's and Nora's critical events were exemplary in the sense that their events represented an instance where a student's mathematics holds the potential for further mathematical learning.

Data Analysis

To examine what might be characterized as critical in the critical events, we used the following characteristics: The mathematics of the event, student's learning opportunity (both taken from Stockero et al., 2019), the teacher's response to student's idea that could lead the teacher to deviate from the lesson plan (taken from Rowland et al., 2015) and, the learning potential critical events hold for the PTs (taken from Goodell, 2006).

FINDINGS

Felice's Story of a Critical Event

The place: 10^{th}-grade high track gifted mathematics classroom.

Lesson topic: Complex numbers. Up to this class, the students learned to add/subtract/multiply complex numbers in the Cartesian form. The lesson goal, as I [Felice] see it, was to expose students to the concept of the absolute value of Complex numbers as preparation for transitioning from the Cartesian form to the Polar form. The situation lasted no more than 5 minutes.

[…] One situation caught my attention, […] when she [the student] did not understand how $(6)^2 = 36 = (-6)^2$ but $\sqrt{36}$ maybe only 6. In response to her question, the teacher said: "Look at the equation $x^2 = 36$ and the function $y = \sqrt{x}$ for $x = 36$. The equation has two solutions ± 6, and this can be checked by substitution. While the function $y = \sqrt{x}$ corresponds to $x = 36$ a single value (y value), since this is the definition of a function, each x corresponds to a single value of y. Also, the 'numbers' world' (domain) of the function is only the non-negative number set; that is why there is a difference; an equation does not act as a function."

This event occurred in the 10th-grade high track mathematics classroom while learning the absolute value of Complex numbers. A student's question regarding $(6)^2 = 36 = (-6)^2$ vs. $\sqrt{36} = 6$ seems unconnected to the lesson's topic, complex numbers. Felice tried to settle this gap between the mathematical content of the lesson and the mathematical content of the student's question in several ways. First, she addressed this gap in her interpretation of the event: "Throughout mathematics, students meet quite a few times with square roots, powers and absolute values (for example, the distance between a line and a point). In my experience, I see quite a few students experiencing this situation at the very end of their track in the 12th-grade." In this excerpt, besides acknowledging this gap, Felice also legitimized it. According to the Israeli context, the question of $(6)^2 = 36 = (-6)^2$ vs. $\sqrt{36} = 6$ is usually dealt with in the previous year while learning about the square root function. However, Felice relied on her experience and did not consider this a unique phenomenon. Second, she framed the context and gave an interpretation for the goal of the lesson as a way to bridge the gap between the mathematics of the lesson and the mathematics that the student's question entailed. She wrote: "The lesson goal *(from my perspective)* was to expose students to the concept of the absolute value of complex numbers as preparation for transitioning from the Cartesian form to the Polar form." In the Israeli context, the absolute value of the complex number is a usual intermediate phase in the transition between the Cartesian form and the Polar form. A typical approach will be to start with $|x + yi| = \sqrt{x^2 + y^2}$ and then to give some numeric examples such as $|-1 + 6i| = \sqrt{(-1)^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37}$.

In terms of the opportunity to deepen the student mathematical understanding, Felice wrote: "I think it was a great question that would benefit the whole class for several reasons: It first sharpened the understanding of what equation is and what function is and what it means to solve an equation and what it is a value of a function, it further sharpened the meaning of the equal sign." From her perspective, the question of $(6)^2 = 36 = (-6)^2$ vs. $\sqrt{36} = 6$ held the opportunity to differentiate between the multiple mathematical concepts, and she saw this as an opportunity to benefit the whole class mathematical understanding.

When addressing the teacher's response to the student's idea Felice wrote: "I think the teacher was not surprised by the question, in particular, this question was not a surprise in this specific class (Because they seem to understand things so deeply, how can it be that one student got confused by such a matter)." Felice's explanation for the reasons why the teacher was not surprised by the student question is characterized by the class traits, as the students of this class are gifted, they have internal motivation for a profound understanding of the mathematics. She wrote: "[…] Students in this class (gifted) do not take things for granted, it is very difficult to 'sell' them things as they are, they strive for deep understanding." Consequently, from Felice's point of view, the teacher did not deviate from his plan, and there was no gap between the planning of the teacher and the actual occurrences of the lesson.

The learning potential that this event held for Felice was articulated by her, as the following interesting question: "I am debating whether the teacher should introduce students to this question and its solution in advance […] or give them the possibility that they will come to a 'something here does not work out' moment and then provide an answer?" Felice's question can be framed in the broader context of the teacher's role. Is the teacher's role to attempt to address questions and mathematical contradictions in advance, or is his role to support students as they struggle to understand mathematics?

Nora's Story of a Critical Event

> The lesson was about trigonometric functions analysis. […] The teacher asked to analyze the function $y = sin\,(2x - \pi\,/\,3)$. The class found extremum points, calculated the axis intersection points, and drew the graph. After they were finished, the teacher asked them if they could guess the graph without doing the analysis. One of the students said that we got this function after contracting $sin(x)$ by two and then moving it to the right by $\pi\,/\,3$. Later the same student asked what would the graph of $sin\,(x)\; + \;cos\,(x)$ look like, will it be one graph at the end or two graphs? The teacher explained to her that, in the end, we would get one graph and suggested using the function analysis.

This event occurred in a trigonometric function analysis lesson at the beginning of the lesson as the teacher was checking homework. The class was requested to analyze the function $y = sin\,(2x - \pi\,/\,3)$. After the class used the procedure of function analysis, the teacher asked the follow-up open question: "if [they] can guess the graph without doing the analysis." One student used function transformation as a way to draw the function without analyzing it. The student's response followed by her question: "how would the graph of $sin\,(x)\; + \;cos\,(x)$ look like, will it be one graph at the end or two graphs?" This question can be seen as an elaboration on function transformation. As Nora wrote: "The student developed a method to understand the behavior of a function without doing function analysis, so in my opinion, she divided the functions into groups, and each group has a parent function. And this led her to ask about the function I mentioned. How, by the same logic, can you guess the graph or understand behavior without analyzing it?" Nora saw the student's question as an attempt to generalize a mathematical idea of graphing without analyzing. If multiplying is expansion and contraction and adding and subtracting from the function argument leads to shifting the function, then what happened in terms of function transformation when adding functions? Nora saw the students' question as an outcome of the teacher's question: "I really admire his method, which does not settle for one solution and always wants to hear more ideas which *lead* students to think further." It seems that from Nora's perspective, the teacher's question was the opening for the student's question.

In that sense, she reconciled for herself the gap between the mathematics of the lesson and the mathematics of the student. It was the teacher's open question which led the student's "what if?" follow up question.

In terms of learning opportunities for the students, Nora addressed the learning that this event held for the entire class and not just the student who asked the question. She wrote: "I would […] use desmos, for example (i.e., dynamic software) and show them how the two graphs look and try to conclude from the graphs and their features.". Nora suggest for the whole class to explore possible connections, difference and similarities, between $sin(x)$ and $cos(x)$ and $sin(x) + cos(x)$. This mathematical exploration is beyond the scope of the curriculum, and it can broaden the students' horizons.

Nora did not elaborate on the teacher's response to the student's idea. However, from the teacher's answer to the student's question, we can assume that he did not deviate from his plan as he did not elaborate on this instance. Furthermore, even the teacher's open question, which, in a sense, triggered the student's question, aligns with the context of the Israeli curriculum. This type of question, relying on function transformation after conducting function analysis, is apparent in all recent matriculation exams. Therefore, the teacher's open 'follow up' question and the student response can be viewed as a usual instance in function analysis lesson. It seems that there was no gap between the teacher's plan and the actual occurrences of the lesson.

The learning potential that this event held for Nora was articulated by her: "During high school, I would solve all the function analysis questions using the standard procedure as we all did, but this event and the teaching courses I take have given me an option to look at things from a different perspective." Here it seems that she referred to the opportunity to graph a function based on function transformation. However, she did not articulate if it was an opportunity for her to learn about the connection between function transformation and graphing a trigonometric function, or an opportunity for her to learn that this is a legitimate way to learn and to teach students to graph a function. From her suggestion to explore $sin(x)$ and $cos(x)$ and $sin(x) + cos(x)$, it seems that Nora learned more about teaching mathematics than the mathematical content itself.

Compering Felice's and Nora's Events

The analysis indicated two similarities between the events: (1) the gap between the planning of the teacher and the actual occurrences of the lesson and, (2) the learning opportunities the critical events held for the PTs. In both events, it seems that the teacher stuck to his plan. Even if the teacher did not plan to discuss the question of $(6)^2 = 36 = (-6)^{,2}$ vs. $\sqrt{36} = 6$, he addressed it with a simple explanation, and as Felice wrote, the teacher was not surprised. In that sense, it seems that there was no gap between the teacher's plan and the actual occurrences of the lesson. In terms of the learning opportunities the events held for Felice and Nora, both articulated insights about teaching and learning mathematics. Felice has discussed the teacher's role and Nora the possibility to use function transformation while teaching function analysis.

The analysis indicated two differences between the events: (1) the gap between the mathematics of the lesson and the mathematics of the student and (2) learning opportunities for the students. First, in terms of the gap between the mathematics of the lesson and the mathematics of the student, in Felice's critical event, it seems that the student's question was not connected to the lesson's topic. In fact, the student's mathematics is connected to a previous topic (square root function). In Nora's critical event, the student's question can be seen as an attempt to generalize a mathematical idea of using function transformation to graph a function without analyzing it. This attempt can be seen as a step forward from the lesson's topic. Thus, both events had a gap between the mathematics of the lesson and the mathematics of the student but not the same gap. Second, in terms of learning opportunities for the students, Felice's event held the opportunity to underscore the distinction between two key mathematical concepts studied formerly. Nora's event held the opportunity to elaborate the mathematical meaning of the concept by going into new mathematical territory (Lampert, 2001). In that sense, Felice's event had the opportunity to review old mathematics and Nora's event had the opportunity to broaden the students' horizons by learning new mathematics.

DISCUSSION AND POSSIBLE IMPLICATIONS

The differences between the two events can be viewed in terms of the gap between the mathematics of the student and the mathematics of the lesson and the learning opportunities for the students. These parameters in which the two events differ can be seen as connected. The magnitude of the gap between the mathematics of the student and the mathematics of the lesson influences the learning opportunities for the students. When there is a low magnitude between the mathematics of the student and the mathematics of the lesson - for example, when a student tries to make sense of a mathematical concept that was used in the lesson, as in Felice's critical event - the teacher aligns the unclear concept with mathematical concepts the student is already familiar with (e.g., Stockero & Van Zoest, 2013). When there is a high magnitude between the mathematics of the student and the mathematics of the lesson - for example, when a student tries to generalize the mathematical concept, as in Nora's critical event - the learning opportunity for the students can be an exploration of that generalization by going into new mathematical territory (Lampert, 2001), which enhances the mathematical knowledge of the student.

Furthermore, the parameters in which the two events are similar - the gap between the planning of the teacher and the actual occurrences and the learning potential critical events held for them - are also connected. Although Felice and Nora did not perceive a gap between the planning of the teacher and the actual occurrences, it seems that they both learned from the teacher teaching while continuing with his plan. Felice raised the question whether the teacher should tackle in advance issues that the students might perceive as contradictory, or wait until the contradiction arises from the student, as happened in her critical event. Nora embraced the idea of function transformation as a legitimate teacher's question that prompts student thinking.

Despite the limitation of generalizing from these particular examples, we propose this initial characterization as a step toward the emergence of a model for critical events that will not only allow us to analyze critical events but will also allow pinpointing what is critical in the critical event.

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ARE YOU JOKING? USING REFLECTION ON HUMOR TO TRIGGER DELIBERATIVE MINDSETS

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In this study with 401 pre-service teachers, inspired by a study by Van Dooren et al., we show that presenting a humoristic situation significantly increases critical thinking as compared to a context in which routine word problems are given. We interpret our findings in the context of the mindset theory by Gollwitzer: reflecting on incongruences in jokes triggering a deliberative mindset that enhances reflective thinking.

INTRODUCTION

At PME 41, Van Dooren et al. (2017, 2019) presented a study, in which 6th graders had to solve four so-called problematic word problems (P-items). Such word problems are tasks in which realistic considerations must be taken into account to solve them reasonably (cf. Greer, 1993), for example

> When Calvin goes to school, Hobbes sometimes takes a swim. His best time to swim 25 m is 20 seconds. How long does it take Hobbes to swim 500 m?

A non-realistic answer to this P-item would be "20 times 20 seconds = 400 seconds," whereas a realistic answer would be "probably more than 400 seconds, because he can't keep up the pace" (cf. Van Dooren et al., 2019).

The intriguing result of this study is that students who were presented with jokes (and had to comment on them) in between tasks did significantly better in giving realistic answers than students who had to solve routine tasks (standard or S-items) instead. The jokes on the one hand and the routine tasks on the other hand were framed similarly:

Humor Condition: Calvin says: 'If I have one melon in one hand, and two in the other, what do I have, Hobbes?' Hobbes answers: 'Very large hands, and strong muscles!	**Word Problem Condition:** Calvin and Inge are doing an excursion with the class. The bus drives on the highway at 80 km/h. After how much time will the bus have travelled 120 km?

Van Dooren et al. (2019), drawing on Attardo (1997), explain this result with the fact that "[h]umor may also stimulate children to see the problem situation from a different perspective," focussing on the incongruity theory of humor with their interpretation:

> The incongruity relief mechanism is specifically relevant for our study as it focuses on the perception of the same situation from two different, seemingly incongruent,

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> perspectives. Only one of these two interpretations of the situation is considered to be the more plausible one, and this one will occur in the listener/reader when the situation is presented. Humor originates when it becomes clear that the alternative, less plausible interpretation – the one the listener/reader did not think of initially – ultimately turns out to be true. The reaction to this unexpected experience of incongruity is one of laughter. (ibid., p. 99)

In their article, Van Dooren et al. (2019) conclude that “the addition of humor could be tested on a variety of items, taking into account different item characteristics” (p. 103). In the study at hand, we will follow two goals: Firstly we test the “humor effect” in a different context: we use critical thinking items instead of P-items and ask pre-service teachers instead of 6th graders, thus testing for the replicability of the effect. Secondly, we introduce a theoretical explanation for the effect of humor, which draws on more general considerations on human reasoning. In our interpretation of the study by Van Dooren et al., the experience of incongruence produced by reading the jokes triggered a specific “deliberative” mindset in the students, making them aware of realistic considerations, whereas solving routine tasks triggered another “implemental” mindset, making them ignore such considerations. Thus, drawing on the mindset theory by Gollwitzer (2012), the *rationale* of our study is to further explore the humor condition as an implicit trigger for a deliberative mindset, compared to an implemental mindset triggered by the routine tasks (S-items).

In the following, we elaborate on theoretical foundation of our approach and then present an empirical study to support our interpretation.

THEORETICAL BACKGROUND

The ability to solve problems is certainly influenced by someone’s intelligence as well as other characteristics (Guilford, 1967). However, psychologists like Kahneman (2011) have demonstrated convincingly, that humans do not always make rational choices. Decision processes are influenced by heuristics and biases (ibid.). Such factors – favorable as well as unfavorable – are usually seen as *traits* that characterize a person, which can be measured with special tests like IQ tests or test for critical thinking. In addition to such traits, decisions and the ability to solve tasks can also be influenced by *states*, which should be visible in randomized studies with specific manipulation conditions. (The differentiation between states and traits was introduced by Cattell and Scheier, 1961.) A theory which allows for interpreting the solution behavior described above from the perspective of states is the aforementioned theory of mindsets:

Mindsets

Gollwitzer defined *mindsets* as mental procedures and cognitive orientations that influence the ways in which people act intellectually (see Gollwitzer, 2012, for a summary). He distinguishes between a *deliberative* and an *implemental* mindset; the former facilitating a heightened receptiveness to all kinds of information (open-mindedness), and the latter facilitating a focus on processing information and realizing previously set goals (closed-mindedness). (Please note: Do not confuse Gollwitzer's theory with Dweck's (2006) theory of fixed and growth mindsets. The latter is regarding a trait rather than a state.)

In several studies (see Gollwitzer, 2012, or Weinhuber et al., 2019), it could be shown that mindsets tend to remain active beyond the triggering situation (cf. ibid.). For example, Gollwitzer had one group of test persons deliberate on unresolved personal problems and another group plan chosen goals. This way, the first group was placed into a deliberative and the second in an implemental mindset, respectively. Afterwards, these groups acted differently in a second, seemingly unrelated task. Persons who were placed in a deliberative mindset showed more planning and reflecting on reasons that persons who were placed in an implemental mindset.

In two studies with 62 pre- and 54 in-service teachers, Weinhuber et al. used comics to induce mindsets: a comic showing a "math-club" in which secondary students are shown to debate alternative approaches to differential calculus problems vs. a comic showing a "math-test-prep" course in which the teacher shows consecutive steps of solving a complex differential calculus task. In both studies, the participants were asked to subsequently draft explanations about an extremum problem. On the one hand, participants primed with a math-club comic (i.e. deliberative mindset) generated more principle-oriented and less procedure-oriented explanations. On the other hand, participants primed with a math-test-prep comic (i.e. implemental mindset) generated more procedure-oriented and less principle-oriented explanations.

Critical thinking

To investigate the effect of humour on task solutions within mindset theory, we chose to use problems with specific characteristics: (i) they should reflect mathematics-specific solution processes but should *not* require higher level mathematics, and (ii) require a reflective component of reasoning and judgment when solving a task or evaluating the solution (cf. Rott et al., 2015). Our choice fell on problems from the heuristics and biases literature, and especially from the Cognitive Reflection Test (CRT) by Frederick (2005) as those are predictive of decision making and cognitive ability (ibid., p. 26) and have mathematical content (ibid., p. 37). Typical participants – pre-service teachers, in our case – should be able to solve such problems correctly as they have all knowledge and skills necessary. However, in multitude of studies, such problems have proven to have solution rates that are far from perfect (e.g., Frederick, 2005; Rott & Leuders, 2016).

For the analyses of such problems and the construction of the CRT, researchers differentiate cognitive processes into two types or systems: subconscious or "Type 1" processes are characterized as fast, automatic, and emotional, whereas conscious or "Type 2" processes are described as slow, effortful, logical, and calculating (cf. Kahneman, 2011; Stanovich & Stanovich, 2010).

A problem that requires type 2 thinking would be 123 × 456 as no solution comes to mind spontaneously. Without a calculator, an algorithm is needed to arrive at the solution 56.088. By contrast, the famous bat-and-ball problem (that is part of the CRT; Frederick, 2005) does produce a spontaneous, automatically generated answer: "A bat and a ball cost $ 1.10 in total. The bat costs $ 1 more than the ball. How much does the ball cost?" Almost everyone thinks of "10 cents" as a first response. Finding the correct answer – "5 cents" – requires a critical reflection of the "intuitive" answer. In this sense, *critical thinking* encompasses the conscious checking and regulation of intuitive answers (cf. Frederick, 2005; Stanovich & Stanovich, 2010).

Research questions

Against this background, we assume that the solution rates of critical-thinking tasks can be increased by influencing the state of the participants:

- Do explicit awareness prompts increase critical thinking?
- Does triggering a deliberative mindset by humor situations (compared to an implemental mindset in routine-problem situations) increase critical thinking?

METHODOLOGY

Design and material

Drawing on the model of thinking by Stanovich and Stanovich (2010), and using the CRT by Frederick (2005) as inspiration, Rott and Leuders (2016) have developed a pool of items for measuring mathematical critical thinking (CT). For the study at hand, ten CT items have been selected that have proven their worth in qualitative and quantitative studies; i.e. they measure critical thinking and not arithmetic skills. Sample items (without the bat-and-ball problem, see above, that is also part of the test) are presented in Table 1. The items are rated dichotomously with 1 point for a correct solution and 0 points for a wrong solution.

Item	Examples of Responses
Water lilies grow in a lake and their surface area doubles every week. At the beginning of the growth phase only ¼ m² of the lake surface is covered, but already after 12 weeks, the whole lake is overgrown. After how many weeks was half of the lake covered?	CR: 11 weeks MR: 6 weeks (linearity assumed)
In a gamble, a regular six-sided die with four green faces and two red faces is rolled 20 times. You win € 25 if a certain sequence of results is shown. Which sequence would you bet on? • RGRRR • GRGRRR • GRRRRR	CR: RGRRR MR: GRGRRR (i.e. most occurrences of "G")
A frog falls into a 20 m deep well. During the day, it climbs up 4 m, at night it slides down 2 m again. After how many days does the frog climb out of the well?	CR: 9 days MR: 10 days (= 20:2)

For a 56 cm high mural, you need 6 ml of paint. How many ml of paint is needed for a 168 cm high mural?		CR: 54 ml MR: 18 ml (= 6 ml × 3)

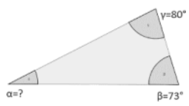
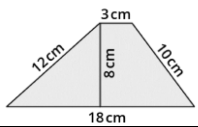
Table 1: CT items with correct responses (CR) and typical misresponses (MR)

To investigate the possible influence of mindsets and their triggers, we developed the three-group design that is presented in Table 2. (Please note that we do not think that it is possible to differentiate between an explicit and an implicit trigger for an implemental mindset; therefore, it is not a four-group design.)

Trigger / Mindset	**Implemental**	**Deliberative**
Explicit	(1) Mathematical routine procedures (comparable to S-Items by Van Dooren et al.)	(2) Awareness prompts
Implicit		(3) Humor (cf. Van Dooren et al.)

Table 2: Three-group design of the study

For each of the three groups, a different test booklet has been created: The ten CT items were spread over three pages in the test booklet. On top of each of these three pages, an environment trigger (1), (2), or (3) was presented to the participants (see Table 3).

Env.	**Trigger**
(1)	(i) What is the size of angle α?  (ii) Solve the following equation: $x^2 - 4 = 0$ (iii) What is the area of the shown trapezium? 
(2)	(i) ATTENTION, check your results! (ii) THINK CRITICALLY! (iii) REMINDER: THINK TWICE!
(3)	(i) Teacher: “75% of all students in this class have no idea of percentages.” Students: “Teacher, we’re not that many!” (ii) The teacher gives a simple task: “If five birds sit on the roof and you shoot one, how many are left?” Student: “None, teacher, the others have flown away!” (iii) Why does a mathematician who is afraid of terrorist attacks smuggle a bomb on board an airplane?

	Because the probability of two bombs being smuggled into an aircraft independently of each other is almost zero.

Table 3: Environment (env.) triggers that differentiate the three test booklets

Participants

In May 2019, the test was administered in three different lectures at the University of Cologne, addressing mathematics pre-service teachers for primary and special education schools (cf. Noeding, 2019). Those lectures were “B1: Fundamentals of Mathematics” for the first Bachelor semester, “B2: Fundamentals of Mathematics Education” for the second Bachelor semester, and “B5: Didactics of Arithmetic” for the fifth Bachelor semester. Participating in this study was voluntary and not participating did not have any consequences for the students.

In total, 401 university students (338 or 84.3 % female, 61 or 15.2 % male, and 2 or 0.5 % not specified; B1: 198 or 49.4 %, B2: 179 or 44.6 %, and B5: 24 or 6.0 %) with a mean age of 21.8 years (standard deviation 4.0 years) submitted a completed booklet (filled in on paper), which took them less than 20 minutes. By distributing the booklets in no specific order, the students were randomly assigned to the three environments.

Additionally, all participants reported their Abitur (university entrance degree) grade (median 2.0 on a scale from 1, the best, to 6, the worst) as well as their final mathematics degree from school (median 2.0 on the same scale).

RESULTS

On average, the students solved 3 problems (median = 3; mean = 3.20) with a minimum of 0 and a maximum of 8 points. The sizes, mean values and standard deviations sorted by the lecture in which data was gathered are given in Table 4. Compared to our previous experiences with CT tests (e.g., Rott et al., 2015; Rott & Leuders, 2016), these results are on the lower end of the spectrum, but not untypical, especially since almost 95 % of the participants are in their first or second semester and students with a low number of semesters normally do worse than students with a high number of semesters.

B1	**B2**	**B5**	**total**
N = 198	N = 179	N = 24	N = 401
M = 2.94	M = 3.44	M = 3.63	M = 3.20
SD = 1.84	SD = 1.87	SD = 1.58	SD = 1.85

Table 4: Mean values (M), standard deviations (SD) sorted by semester

The data regarding the mindset environments (Table 5) show that group (2) – explicit, deliberative mindset, awareness – performed better than group (3) – implicit, deliberative mindset, humor – which in turn performed better than group (1) – implemental mindset, S-items. A one-way ANOVA confirms significant differences between those groups with post-hoc tests revealing significant differences between groups (1) – (2) and (1) – (3), but not (2) – (3).

As expected, the groups in which a deliberative mindset was triggered, outperformed the group in which an implemental mindset was triggered. A state highlighting open-mindedness (i.e. the deliberative mindset) is favourable for working on CT problems compared to a state inducing closed-mindedness (i.e. the implemental mindset). Interestingly, both the explicit (awareness) as well as the implicit (humor) environments seem to trigger the deliberative mindset, which strengthens our initial interpretation of Van Dooren's humor environment in the light of the mindset theory.

(1) routine	**(2) awareness**	**(3) humor**		(1)	(2)	(3)
N = 148	N = 135	N = 118	(1)		p = 0.002	p = 0.024
M = 2.86	M = 3.51	M = 3.29	(2)			p = 0.172
SD = 1.79	SD = 1.98	SD = 1.72	(3)			
One-way ANOVA: F = 4.64, p = 0.0102				Post-hoc t-tests (one-sided)		

Table 5: Mean values (M), standard deviations (SD) per environment

DISCUSSION

The data confirm the result by Van Dooren et al. that a humoristic environment is favorable for solving problems demanding reflection as compared to a word problem environment. Presenting jokes in the test booklet has an effect in the same direction as direct awareness prompts. Using the notion of mindsets by Gollwitzer (2012), reading the jokes might have placed the students in this specific environment in a deliberative mindset, whereas students in the environment with routine tasks were placed in an implemental mindset.

However, there are *limitations* to this study: Even though the differences between the three groups are significant, the effect is rather small – especially, compared to the study by Van Dooren et al. (2017), in which pupils in the humor condition "gave almost twice as many realistic reactions compared to the word problem condition" (p. 4-303). This might be due to the different problems (CT items instead of P-items), the different participants (university students instead of pupils), or differences in the conditions: Unlike Van Dooren et al., where pupils were asked to compare the jokes, our tests persons were only presented with the jokes and with a lower number of jokes. This way, the humor environment provided only a very "soft trigger" for a deliberative mindset. In other studies using mindsets (e.g., Gollwitzer, 2012; Weinhuber et al., 2019), the triggers are more intense. Therefore, in future studies, we plan to make our participants reflect upon jokes for a presumably even stronger effect.

Implications of this study are introducing the theory of mindsets to the PME community and as a consequence the awareness for considering states in performance testing compared to only considering traits (like knowledge and beliefs).

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“I DON’T WANT TO BE *THAT* TEACHER”: ANTI-GOALS IN TEACHER CHANGE

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This paper uses the theory of goal-directed learning to examine anti-goals that arise as teachers implement change in their mathematics practice. Findings suggest that anti-goals develop as teachers begin to recognize who they **do not** *want to be as a mathematics teacher. Accompanying anti-goals are emotions that can be useful in measuring progress towards anti-goals (fear and anxiety), and away from anti-goals (relief and security). Furthermore, acknowledging anti-goals allows mathematics teachers to focus on the cognitive source of their difficulties rather than be overwhelmed by the emotional symptoms.*

INTRODUCTION

In *Intelligence, Learning, and Action*, Skemp (1979) describes a thought experiment in which we are to imagine two events. Firstly, we strike a billiard ball causing it to move across the table into a pocket. Secondly, we are in a room with a child who, upon our command, moves across the room to sit in a chair. Superficially, these are two similar events: we have ‘caused’ the child to cross the room and we have ‘caused’ the billiard ball to roll into the pocket. This is basic stimulus and response in which an object remains in a state of rest, or uniform motion in a straight line, unless acted upon by an external force. Now imagine inserting an obstacle into the pathways of both the ball and the child, what a strange billiard ball it would be if it could detour around the obstacle and continue its path. However, a child will do this with no change in stimulus — perhaps by going around the obstacle, hopping over it, or even moving it. Skemp suggests that, unlike those of a billiard ball, the child’s actions are goal-directed, and necessary to reach her goal state (sit in the chair).

Recognizing that many human activities are goal-directed is essential if we want to understand their actions. In other words, we need to attend to their goals with the same importance as we do their outwardly observable actions. Skemp offers the example of someone crawling around on their hands and knees on the office floor. There is no point in asking them what they are doing — we can see that for ourselves. A better question might be “Why are you doing that?”, which might elicit a reasonable answer such as “I’m looking for the cap of my pen”.

Let us try another thought experiment. Imagine observing a secondary mathematics teacher in her classroom. The teacher is standing by an overhead projector demonstrating how to solve a problem while the students sit quietly at their desks and take notes. Another adult

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sits with a notepad at the back of the room. Students who talk are met with a polite reminder to raise their hand if they wish to speak. What might the actions of the teacher suggest? An immediate response might be the teacher is teaching, albeit in a somewhat traditional manner. Now imagine talking with the teacher after the lesson as she describes her frustration with having to conform with school norms regarding effective teaching of mathematics to successfully pass a probationary evaluation. To observe that the teacher was teaching traditionally is accurate, but incomplete. To make sense of her actions also requires consideration of her goal — she was teaching traditionally to achieve her goal of maintaining employment.

The aim of this study is to make sense of teachers' actions through consideration of their goals. As the pursuit of a goal is an emotive experience (Skemp, 1979), I begin the next section by describing some of the literature regarding emotions in teaching. To connect emotions with goals, I then outline Skemp's theory of goal-directed learning.

EMOTIONS, ACTIONS, AND GOALS

Liljedahl (2015) describes emotion as the "unstable cousin" of beliefs yet there is much to be learned from their study; it is the erratic cousin at the family dinner who is most likely to blurt out uncomfortable truths. Fortunately, while pursuit of these uncomfortable truths was once the least researched aspect of teaching practice over the past two decades. There has been an increased focus on what emotion reveals to us about teaching and its implications for teacher change (Zembylas, 2005). As Kletcherman, Ballet, and Piot (2009) suggest, "A careful analysis of emotions constitutes a powerful vehicle to understand teachers' experience of changes in their work lives" (p. 216). The unstable cousin is being heard.

Prior to this, teacher cognition had been the primary focus of research on teachers (e.g., Richardson, 1996) and its underlying assumption was that teacher actions and behavior were strongly influenced by cognition. More recently has come the recognition that emotion and cognition are inseparable and that emotions may provide insight into the relationship between a teacher and the socio-cultural forces that surround her (Van Veen & Sleegers, 2009). According to Hannula (2006), emotions (along with attitudes and values) encode important information about needs and may even be considered representations of them. While cognition is related to information, Hannula understands emotions as affecting motivation and therefore as directing behavior by affecting both a person's goals and choices. Emotions constitute a feedback system for goal-directed behavior, and thus shape a person's choices. Hence, emotions, cognitions, and actions turn out to be strongly intertwined and inseparable; it is necessary to consider these factors and their relationship to understand teachers' actions.

Skemp's (1979) theory of goal-directed learning considers these connections between emotions and actions. His framework was built on fundamental ideas in psychology and links emotions to goals which a learner may wish to achieve, and to anti-goals which a learner wishes to avoid. Goals can be short-term, such as the desire to learn a procedure for solving a routine problem or long-term, as in the desire to be successful in mathematics. A short-term anti-goal may be to avoid failing a test, while a long-term anti-goal may be to avoid future mathematics studies altogether. Skemp emphasizes that goals and anti-goals are not simply opposite states, rather, a goal is something that increases the likelihood of success, while an anti-goal is something to be avoided along the way.

For Skemp, emotions come into play as they provide information about progress towards either goal state in two distinct ways (see Figure 1). First are the emotions experienced as one moves towards, or away from, a goal or anti-goal (pleasure, unpleasure; fear, relief). For example, moving towards a goal brings pleasure, while moving towards an anti-goal result in unpleasure. Although the four emotions bear similarities, there are subtle differences. Consider relief and pleasure; the relief one feels upon not failing a test is a different feeling from the pleasure one experiences upon learning the correct procedure for a problem. The second aspect of emotions concerns one's sense of being able to achieve a goal or, conversely, avoid an anti-goal (confidence, frustration; security, anxiety). For example, believing one can achieve a goal is accompanied by confidence while believing that one is unable to move away from an anti-goal state induces anxiety.

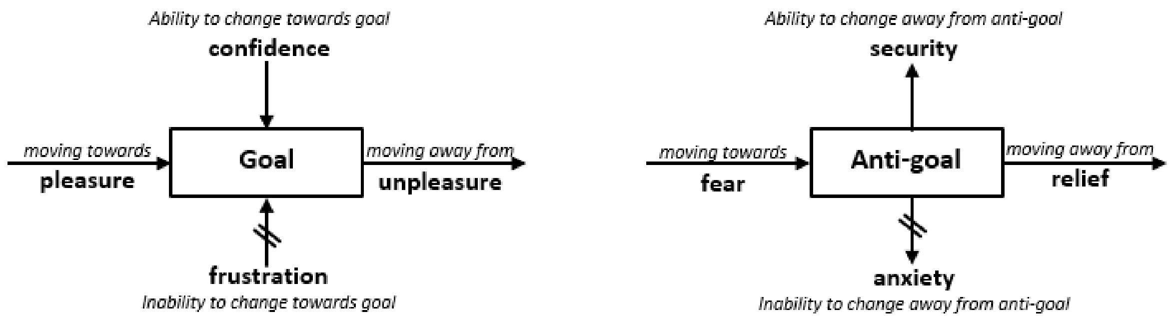

Figure 1. Emotions associated with goal states (adapted from Skemp, 1979)

Although initially designed to examine goals related to the learning of mathematics, in this study, researcher uses the Skemp's theory to examine goals related to the teaching of mathematics, particularly those of teachers who are trying to change elements of their practice. In doing so, researcher follows Jenkins (2003) who suggests that change is not just to make different, but, like learning, it is also to continually improve in skill or knowledge. Specifically, researcher uses Skemp's notion of goals and anti-goals to better understand the actions of teachers involved in changing their mathematics practice.

METHODOLOGY

McLeod (1992) suggests that detailed, qualitative studies of a small number of subjects allow for an awareness of the relationship between emotions, cognitions, and actions that large-scale studies of affective factors overlook. Accordingly, in this study, researcher adopt an exploratory and qualitative approach that focuses on documenting that relationship. Data for analysis was taken from a larger study involving 15 teachers whose teaching experience ranged from 1 to 16 years. The data was created during semi-structured interviews that ranged from 40 to 60 minutes. The interviews were audio-recorded and then fully transcribed. The structure of the interview aimed at letting emotions emerge organically through a narrative rather than by direct questioning. For example, the teachers were asked to describe the changes they had implemented in their mathematics practice without explicitly asking them to describe the emotions they felt. This allows for richer descriptive data of personal experiences that leading questions may inhibit. With DeBellis and Goldin (2006), researcher aware that emotional meanings are often unconscious and difficult to verbalize but, with Evans, Morgan, and Tsatsaroni (2006), researcher believes that textual analysis of teachers' narratives allows the identification of emotional expressions that function in teachers' positioning. As such, the transcripts were scrutinized for utterances with emotional components such as "I was worried…" and then re-examined for their potential connections to goals. Due to space limitations, researcher reported only on those findings related to anti-goals.

THE DEVELOPMENT OF AN ANTI-GOAL

For the teachers in this study, the decision to implement change in their mathematics classrooms stemmed from dissatisfaction with their current practice. Most had learned mathematics as learners in traditional mathematics classrooms and had simply gone on to replicate that for their own students. As Kelly recalled, *"There was nothing during my journey to becoming a mathematics teacher that made me think of another way to teach math."* Their collective desire to move away from the notion of teaching as telling and learning as listening (and remembering) so permeated their interviews that I originally coded these excerpts as 'That Teacher'. However, it was this tension between who they were and who they wanted to be that led to change in their mathematics practice, as who they wanted to be as a teacher became their goal, while who they had been, or wanted to avoid becoming, became an anti-goal.

Many of the teachers described similar situations where tension with their teaching practice drove them to seek out professional development. For example, the development of Amy's anti-goal began with the feeling that *"I was boring, like they just weren't getting from me what they needed."* It coalesced into an anti-goal as she realized her practice was harming, rather than helping, her students:

> "*My practices resulted in increased anxiety and frustration amongst my students; damaged their mathematical confidence; removed their desire to think deeper and search for understanding; as well as robbed my students of conceptual experiences. Valuing speed and accuracy comes at a great cost for students and gives them little mathematical benefit.*"

To alleviate the anxiety that caused her, Amy sought out professional development *"for some new ideas".* Instead she experienced a student-centred teaching style that *"completely transformed my pedagogy."* No longer content with her product-oriented mathematics classroom where students worked individually to develop fluency with procedural skills, she turned to a process-oriented model that valued conceptual understanding and collaboration. In searching for relief from her anti-goal, Amy found security in the new practices she implemented.

For other teachers, it was attending professional development that caused the development of an anti-goal. Kelly described the same sort of experiential learning from professional development as Amy but added, *"I never questioned it* [her practice] *until my eyes were opened — when I saw another way. Since then, I have felt my teaching pedagogy do a complete 180° shift."* Although she had willingly attended the professional development session, it was not due to tension with her own practice; it was more a matter of convenience and opportunity: *"It was our district Professional Development and it was a mathematics topic. I was there because I was a mathematics teacher."* Describing herself as a typical, traditional mathematics teacher, the experience provoked a desire to implement changes in her teaching as she noted:

> "*It was confounding to learn that something I was doing in my class was taking away from students' learning. It really makes you think about and reflect on what you are doing as a teacher.*"

Like Amy, the traditional teacher she once was became her anti-goal as she emphasized, *"I knew I never wanted to be that teacher."*

For both Amy and Kelly, their use of figurative language like *"transformed"* and *"eyes opened"* suggests the core of who they were as a teacher had been unexpectedly altered and the result was the development of an anti-goal. They may have set out to change some things about their practice but ended up changing themselves. For other teachers, this alteration appeared to be a more purposeful decision. Sam spoke of being at a *"crossroads"* where anxiety with his teaching style caused him to ponder two choices: seek out professional development or quit teaching. In the end he chose the former as he explained, *"I'm going to try out for one more year and I'm going to become better."* No mention of transformational experiences, this was a deliberate response to relieve the pressure of an anti-goal: he was not happy with who he was as a teacher and he set out to change that. This sense of deliberation appears again in David, a new teacher assigned to teach mathematics. Having never planned to be a mathematics teacher, he first turned to colleagues for advice on what to do:

> "*I asked them, how do you teach mathematics? How can I make this fun? And they are...like, I hate to say it, but they are older teachers, and they have very traditional views on mathematics, and the kind of do it like how I was taught in mathematics. They just work on the problem on the board, show them how it is done, and get them to practice, practice, practice until they get it. And I knew that is not how I want to do it. That is not who I want to be.*"

Although David had not yet developed a mathematics pedagogy, he knew who he did not want to be as a teacher. This anxiety led him to sign up for a series of professional development sessions that focused on progressive teaching practices in mathematics. Over time he implemented the strategies he learned in his classroom. Again, there is less a sense of an unexpected transformation and more of a determined decision to avoid an anti-goal.

Like the others, Corey had implemented new practices in her classroom that required changes not only in the physical movements of her students but for herself as well. She mentioned, *"Physically the vertical learning can be challenging for me. I struggle to stand for the whole day, so I have to make sure I am doing a mix of things throughout the day."* During the interview, she let this thought be and then came back to it unexpectedly about 10 minutes later as she further explained, *"I just don't want to be that teacher."* When asked to clarify, she added:

> "*Because I struggle to stand. I do not ever want to be that teacher that sits at the desk all day, because that's not effective at all. I think if it is this bad, I am 43, what am I going to do five years from now? Six years from now? How is it going to look? That is something that keeps me up at night. How am I going to best serve these kids when I cannot move around the room? So, yeah, it is a concern. That is one of the reasons I might not always be a classroom teacher; it might not be an option for me physically, to do a really good job of it."*

There are two things to note here. Like Kelly and David, Corey's use of the adjective/noun combination '*that* teacher' suggests she has developed a schema of what a teacher is and is not. This sets up an anti-goal as she knows what kind of teacher she does not want to be, and despite the tension that results from worries over her physical limitations, she does not veer from that. Second, it is interesting that while Corey does later mention solutions such as a *"motorized scooter"* or *"mixing things up",* moving away from the new practices that are taxing her physically is not mentioned. Like Sam, it seems she would rather leave the profession than move towards her anti-goal state.

RECONNECTING WITH AN ANTI-GOAL

Traditional mathematics practices comprise universally accepted norms such as teacher-led examples, individual seat work, and silent practice that are especially difficult to displace. Such a strictly controlled environment offers the illusory appeal that serious learning is taking place. This notion is embedded in the mathematical backgrounds of the teachers in my study for whom the pull of traditional practices lingered. This created anxiety and fear for those attempting to suppress these desires and for those who succumbed. Lily recalled that in her early teaching career she believed that *"The quieter the class the more I thought learning was happening."* She had come to recognize that this is not the case, yet acknowledged:

> "*I do on occasion go back to this method because of a bad day or I am not prepared. When I do go back to this traditional method, I am aware that it was not a good teaching day for me or the students.*"

This created anxiety as she realized that her decision, while satisfying her immediate needs, had unintended consequences for both her and her students. Interestingly, this notion of being unprepared appeared to be the impetus for several others who also return to traditional practices to satisfy their own needs. As Kelly recounted:

> "*So today I sort of reverted. I have not been feeling great and I needed something quick and easy to put together for a lesson. I started the class with a review/notes of all the topics we have been doing. We did some examples together on the board then I gave them a worksheet. This class has rarely come into the room to see desks and chairs set out that are available to sit in. But today I caved. I was hoping for some quiet time while they worked.*"

This backfired for Kelly as she later admitted, *"For the most part I spent the rest of class going from one student to another with hands up helping them with problems."* Like Lily, her anxiety lay in knowing that her decision to 'revert' had had unintended consequences for both herself and her students. It appears that the challenge of implementing change can occasionally nudge teachers towards which was once familiar and therefore seen as easier. Hoping for a respite, they instead experience the emotion that accompanies a move towards an anti-goal.

This return to the familiar also occurred for several teachers not because it was easier but rather, they missed the reassurance of traditional teaching. This created anxiety for them as they struggled to suppress this need. Linda mentioned wanting to be sure she was covering the content since she implemented the changes in her classroom:

> "*I still occasionally like to start by demonstrating something new and then having students do similar problems or problems connected to what was demonstrated. This comforts the 'conventional' teacher in me, but I do feel like it is cheating or missing the point.*"

This need for reassurance is also apparent in Diane who mentioned occasionally returning to her previous teaching practices:

> "*I really want to make sure that everybody's learning. When they are quiet, and they are all looking at me I know I have their attention. I am not sure if everybody is paying 100% attention when they are working in the problem-solving groups.*"

When speaking later of year-end assessments she added, *"I know I do not need to do it* [teach traditionally]. *I know I should not. They all did so well that it solidified for me that the way I was doing it was already working."* This suggests that anti-goals serve another purpose. Teachers, like Diane, might purposefully move towards an anti-goal to experience the relief it brings when they move away. They are reconnecting with their anti-goal in order to affirm the changes they are making in their mathematics practice.

CONCLUSION

Anti-goals develop during teacher change as teachers come to recognize and articulate who, and how, they *do not* want to be in the mathematics classroom. For some, this process occurs during change, for others this recognition propels them to seek out ways to change. In either instance, researcher suggests anti-goals are useful in three ways. Firstly, having teachers reflect on the emotions they feel may be useful in reasoning why they felt this way and how they might use this knowledge to their advantage. Doing so allows teachers to focus on the cognitive source of their difficulties rather than being overwhelmed by the emotional symptoms. For example, a teacher who can connect the anxiety she experiences to the action she is undertaking, can take steps to alter the action. Secondly, I suggest that recognizing what one *does not want* to be brings into sharp relief what one *does want*. Having that clarity might enable teachers to seek out the actions and changes that will help them reach that goal. For example, a teacher who realizes she does not want to be *that* teacher who only uses unit tests for assessment may look for learning opportunities that broaden her assessment practice. Finally, anti-goals also prove useful in keeping change alive. Teachers who find themselves pulling back from the changes they have implemented, find in the emotional reconnection with their anti-goal encouragement or reinforcement needed to continue with the change. As Zembylas (2005) suggests, "Teaching practice is necessarily affective and involves an incredible amount of emotional labor" (p. 14). Harnessing that emotion during teacher change may prove valuable for teacher educators.

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SELECTING AND SEQUENCING STUDENTS' IDEAS: TEACHERS' SOCIAL CONSIDERATIONS

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Teacher participants were asked to choose three from a set of eight solutions to a task to feature in a whole-class discussion. Participants were asked to indicate whom they would invite to present each solution, from among higher-achieving girls and boys and lower-achieving girls and boys. Participants often indicated that they would first have students present the direct model or an error, that they would invite a lower-achieving girl to present a direct model, and that they would invite a higher-achieving student to present an error. Most indicated that they would invite a higher-achieving boy to present the solution with an unexpected geometric representation, usually at the end of the sequence. Participants' explanations reveal how gendered beliefs about mathematics shape this aspect of classroom instruction.

INTRODUCTION

A lesson format featuring whole-class discussions begins with the posing of an open-ended problem, allotment of time for problem-solving, and culminating in a teacher-facilitated discussion during which specific students present their solutions. Stein et al.'s (2008) *5 Practices* (anticipating, monitoring, selecting, sequencing and connecting) are seen to support the quality and content of the whole-class discussion in this lesson model. Part of that model, the selecting and sequencing of solutions, indicates the inviting of students to present mathematical ideas. In doing so, teachers publicly position students as *authors* of noteworthy ideas. Here, we focus on teacher decision-making regarding whom to select to present a mathematical idea.

In particular, we focus on social considerations of gender and mathematics achievement. We selected gender as a social construct because it carries meaning in mathematics learning contexts (Walkerdine, 1998). Solely focusing on gender, however, would essentialize and over-extend the meaning of this single dimension of identity. Mathematics as a school domain tends to be highly stratified by perceptions of ability level, thus we explore teachers' decision-making with respect to the intersection of gender with mathematics achievement. That is, in the context of multiple solutions to a mathematical task, if a given solution is available to be presented by higher-achieving and lower-achieving girls and boys, whom do teachers invite to present which solution and why?

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CONCEPTUAL FRAMEWORK

Our understanding of gender and mathematics ability are as socially constructed, rather than fixed characteristics. We acknowledge a problematic reproduction here of outdated binary notions of gender but opt for a political analysis in a context in which the predominant social understanding of gender in schools remains as binary and fixed. Perceptions of students' mathematics abilities are socially constructed as well and represent a significant dimension of a student's status in and outside of the classroom. Teachers' perceptions of students' mathematical abilities interact with conceptions of ability as innate and as context independent (Snell & Lefstein, 2018).

Decisions about whom to invite to the classroom's public floor are not only about resolving a problem's solution. Encouraging students to share ideas, as well as prompting the class to evaluate the mathematical ideas of others, are considered to be pedagogical moves that distribute mathematical agency and mathematical authority (Gresalfi & Cobb, 2006). Inviting a student to present their ideas could signal the teacher's assignment of competence, in a form of public validation (Cohen & Lotan, 1997). Affordances, however, of inviting a student to present their ideas to the whole class might depend on the classroom's norms (Cobb et al., 2008). In some classrooms, for example, being invited but among the first to present could signify the teacher's positioning of an idea as the *least* sophisticated. In this paper, we explore the question: whom do teachers select to present which mathematical solutions and why?

RELATED RESEARCH

Gender-based differences in mathematics achievement have narrowed over time, unlike girls' and women's lower self-concept in mathematics, which persists and around the world (Sax et al., 2015). This is an outcome of social systems that position mathematics as masculine and reinforce boys as better suited for it, rather than evidence of deficiencies innate among girls (Walkerdine, 1998). How students see themselves relative to mathematics is shaped, in part, by their teachers' gender-related beliefs and their impact on classroom interactions (Gunderson et al., 2012). Even when teachers perceive girls as equally capable in mathematics, they might attribute that success in different, gendered ways. That is, teachers might attribute the success of high-achieving girls to their eagerness to learn and compliant classroom behavior, but attribute the success of boys to their spontaneity and independent thinking (Robinson-Cimpian et al., 2014). This example is part of a broader, extensive set of interrelated and unequal binary oppositions -- like objective/subjective, analytic/emotional, certain/fleeting, innately able/ hard-working, -- wherein one in every pair is more valued *and* associated with masculinity. Gender, though, does not carry meaning or operate in an isolated way, and our focus is on its intersection with mathematics achievement. Prior studies show that achievement, not gender, determines classroom participation (Myhill, 2002). By accounting for achievement with gender, Myhill revealed how, as students increase in age, high-achieving boys become less involved in classroom participation, whereas high-achieving girls remain compliant and willing to participate. This pattern is interpreted by Myhill as not serving girls well, in how boys' success then becomes positioned as a product of their intellect, in contrast with girls' success as produced by their attentiveness and effort.

Sequencing & selecting solutions

Prior research presents a range of pedagogical principles that might guide teachers' selecting and sequencing of solutions (Ayalon & Rubel, under review). This research indicates that teachers tend to value accessibility over mathematical storyline considerations. Livy et al. (2017) found that most explained that they would sequence the solutions in a progressive order, beginning with what they considered the weakest solution and then progress to what they viewed as the strongest. Meikle (2014)'s results similarly show that participants tend to sequence solutions either by starting with an erroneous solution and building to a correct solution; starting with an incomplete solution and building to a complete one; or starting with a direct-model strategy and building to an abstract one. Participants in both studies justified their choice of these progressions in terms of an assumption that such progressions best support students that might be struggling. These studies focus on which mathematical ideas teachers value for inclusion in a whole-class discussion but do not extend to teachers' considerations about which students are invited to present what kinds of ideas on the public floor. Our research question is: *With respect to lower and higher-achieving girls and boys, who do teachers select to present which kind of solution and why?*

METHODS

Forty-two Israeli teachers participated, in the context of a university-based course. All but four participants identify as women. We posed this version of the Handshake Problem (in Hebrew):

> There were 9 players at the first basketball practice of the season. The coach told the players to introduce themselves by shaking hands with one another. Assuming that every player shook hands exactly one time with every other player, how many handshakes occurred?

In written form, we provided participants with the problem and eight solutions (Figure 1) and prompted them to interpret the mathematical thinking underlying each. We draw attention here (because of their dominance in our results) to how solution H is a direct-model; solution C uses a geometric representation; and solutions D and G are errors. We asked participants to consider a hypothetical top-track 7th grade class of 32 students, and to consider that each solution was produced by four students: a higher-achieving girl or boy and a lower-achieving girl or boy. We asked participants to indicate the three solutions to be presented to the class, in what sequence, which student they would choose to present from among the four categories, and to explain their choices.

Analysis

We tabulated the selections, according to position (Table 1) and the assignment of each solution by gender and mathematics achievement (Table 2). We compiled the written justifications for each solution, separated them according to the gender-achievement attributes and then analyzed those justifications using qualitative thematic analysis (Braun

& Clarke 2006). This process yielded three main categories of justifications, which we describe in the results section.

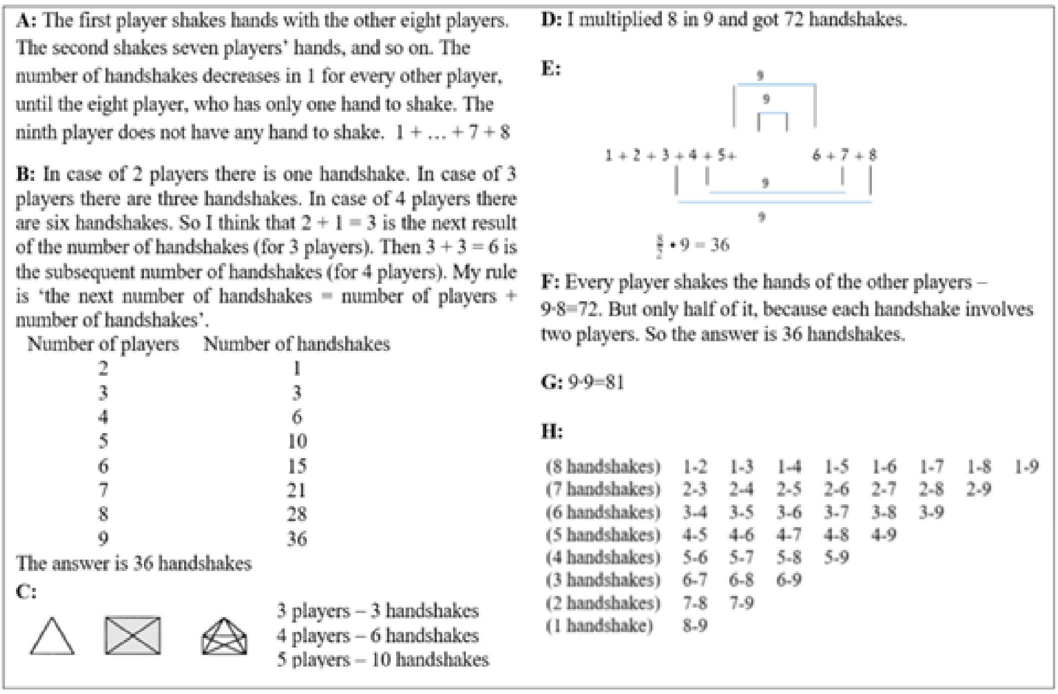

Figure 1. Provided Solutions

RESULTS

As shown in Table 1, participants frequently chose one or more from among three solutions: C, the geometric representation; H, the direct model, or D/G, errors. Those who selected H, assigned it to the first or second position, explaining that H is the most accessible, comprehensive, and confirms the correct answer. Those who selected an error (D or G), also usually in one of the first two positions, explained this as a way to fix the error or to normalize error-making. Those who chose C assigned it nearly always as the last in the sequence, explaining that it is "non-routine" and "creative."

Participants commonly cited accessibility considerations in justifying the sequencing of an error or the direct model, citing needs of "struggling" or "weaker" students. Fewer continued to attend to accessibility with regards to the third position, even though they often culminated their sequence with what they perceived to be more complex. Instead, in justifying their third choice, most often the geometric representation, participants commonly cited an inclination toward a variety of representations and approaches. Our analysis of their justifications about which solutions and in what sequence is presented elsewhere (Ayalon & Rubel, under review).

Although participants selected boys or girls roughly evenly, there are patterns in terms of how they assigned each solution, as shown in Table 2, according to three main trends: 1) Solution C to a higher-achieving boy; 2) Solution H to a lower-achieving girl; 3) an error to higher-achieving students. We note that Solution A was assigned only to lower-achieving students but was seldom chosen.

Solution	1st choice	2nd choice	3rd choice	Total
A	2	9	0	11(26%)
B	0	2	4	6(14%)
C	0	4	30	34(81%)
D or G	21	8	0	29(69%)
E	1	3	1	5(12%)
F	2	6	1	9(21%)
H	16	10	6	32(76%)

Table 1: Selection of Solutions in Sequence

	B-Low	G-Low	B-High	G-High	Total n(%)
Solution A	4.5*	6.5	0	0	11(26%)
Solution B	1	0	2	3	6(14%)
Solution C	5	2	18.5	8.5	34(81%)
Solution D or G	6.5	3.5	10.5	8.5	29(69%)
Solution E	3	1	3	1	5(12%)
Solution F	2.5	0.5	0	3	9(21%)
Solution H	5.25	20.25	2.25	4.25	32(76%)
	27.75	33.75	36.25	28.25	

*Responses such as lower-achieving boy or girl were halved in this table.

Table 2: Assignment of selected solutions by gender and math status

Participants explained their assignment of a solution to a student because of (1) characteristics of individuals in that group, pertaining to perceived mathematical reasoning or communication skills; (2) to encourage participation, of the designated student or others in the class; or (3) to attend to issues of self-confidence, status, or other existing gender-achievement stereotypes. Here we present our analysis of justifications of Solutions C, H, and D/G, because these were the most commonly selected and because their assignment was not distributed equally across the indicated social categories.

Participants assigned errors more often to higher-achieving students, for three reasons connected to the above themes: (1) Because of an expectation that higher-achieving students will be able to identify and explain the errors; (2) To normalize error-making in mathematics, primarily to encourage participation. For example, P24 explained, “I want everyone to learn that everyone makes mistakes. This will support and motivate all of the students”; and (3) because participants perceive these students, particularly boys, as invulnerable as a result of self-confidence perceived to be high enough to easily withstand criticism. Several participants, in contrast, noted that girls might be vulnerable in this situation. For example, P37 explained “It would embarrass a girl to show this or to start a discussion.”

Participants most often assigned Solution H to a lower-achieving girl. In many cases, participants explained that they felt the direct model was more likely to have originated with a girl, according to gendered beliefs about mathematical reasoning. For example, P27 explained, “Girls usually choose the long and safe way, different from boys who look for short-cuts and clever ways." Many indicated a trust in a girl’s ability to present H in a clear and organized way for the class. For example, P29 explained “Girls are more organized than boys and can present better than boys.” Many others explained that inviting a lower-achieving girl would represent an opportunity to broaden participation among others in the class and strengthen her self-confidence. Assignment of Solution H to lower-achieving boys was far less common, and in those instances, justified by how solution H is clear and not hard to explain.

Participants most often assigned Solution C to the category of higher-achieving boy. Most commonly, they justified this around expected characteristics of individuals in that group, with respect to mathematical reasoning or communication skills. Participants attributed characteristics to these boys such as above-average, creative, and special. For example, P26 explained that “on the basis of experience, boys like busy-work less than girls, and mostly try to find other ways to solve a problem. In addition, boys have better spatial abilities than girls, so this solution is better suited to a boy than a girl, in my opinion.” In contrast, when participants assigned solution C to a higher-achieving girl (this occurred much less often), they tended to explain this as an opportunity to remedy those girls’ lack of self-confidence. For example, P8 explained that a high-achieving girl would “explain well. Girls don't have self-confidence, and are shy, so I chose a girl because boys have self-confidence, to go to the board and to talk in front of the class.”

DISCUSSION

An important limitation of our study is that it is oriented around a single mathematics task. In addition, we have flattened broad social diversity into only two social constructs (gender and achievement) and separated those continua into artificial, binary distinctions. Neither of these variables actually operate in binary terms and clearly intersect with other variables, such as race, language status, physical appearance, social popularity, and more, depending on the context. Despite these limitations, our results contribute to a growing knowledge-base about classroom interactions with respect to an essential component of structured problem-solving lessons, the selecting and sequencing students’ ideas for the subsequent whole-class discussion (Livy et al., 2017; Meikle, 2014; Stein et al., 2008). Our results suggest that social considerations play a significant role in teachers’ decision-making about whom to invite to present which kind of solution.

Participants often selected lower-achieving girls to present the direct-model solution (H), and explained this in several ways. In many cases, they invited a lower-achieving girl because of their perceived lesser mathematical ability, even though this was stated to be a top-track class. Inviting them to present mathematical work represents, according to many participants' justifications, an opportunity to boost what they perceive to be their lower self-confidence. It is important to remember that in most cases, they designated solution H as the first or second to be presented, and only in rare instances as the third. If there is a classroom norm around supporting accessibility by always beginning with the easiest example or least sophisticated solution, as prior studies indicate, then inviting the lower-achieving girl to present first or early on presents a signal to the class (and to the girl herself) about the low value of her solution. Thus participants' indications about inviting a lower-achieving girl to present her mathematical ideas to increase what they perceive to be girls' low self-confidence might, instead, undermine that intention and reinforce a position of low self-confidence.

With respect to solution C, on the other hand, participants tended to invite higher-achieving boys to present the more unusual solution, with a geometric representation (C). They explained this primarily in terms of these students' mathematical and communication skills. Here, too, the sequencing is significant, as in most cases, participants designated Solution C as the last to be presented. Again, if there is a classroom norm around the sequencing of examples or ideas, as the prior research indicates, then who is invited to present later in the sequence could signal to the class the value of this solution, as most sophisticated.

The fact that there were no gender differences around the assignment of errors is telling. That participants more often assigned errors to higher-achieving students to present is evidence of their recognition that this could be risky for lower-achieving students in terms of that student's self-confidence or willingness to further participate. This indicates that participants were paying attention to the potential impact on a student of being invited to present a particular kind of solution. It is striking, therefore, that our results did not include more instances of their seizing opportunities to assign competence, either to lower-achieving students or girls, by having them present more sophisticated solutions.

Prior studies outline a range of pedagogical principles that guide teachers in selecting and sequencing of solutions for a whole-class discussion. These studies focus on the mathematical and pedagogical considerations, without addressing how teachers consider which students to invite to present which kinds of solutions on the classroom's public floor. Our findings suggest that gender and achievement play a significant role. Further research could explore this question relative to other mathematical tasks. Additional research could study how such practices around sequencing of solutions or examples are enacted to determine how students interpret these norms.

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