PME 44 VIRTUAL

THE 44th CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

July 19-22, 2021

Hosted by Khon Kaen, Thailand

Virtually hosted by Technion, Israel Institute of Technology, Isreal

Proceedings

of the 44th Conference of the International Group for the Psychology of Mathematics Education

VOLUME 4

Research Reports (S-Z)

Editors:
Maitree Inprasitha, Narumon Changsri
and Nisakorn Boonsena

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PREFACE

We are pleased to welcome you to PME 44. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. The PME 44 Virtual Conference is hosted by Khon Kaen University and technically assisted by Technion Israel Institute of Technology. The COVID-19 pandemic made massive changes in countries' economic, political, transport, communication, and education environment including the 44th PME Conference which was postponed from 2020. The PME International Committee / Board of Trustees decided against an on-site conference in 2021, in accordance with the Thailand team of PME 44 will therefore go completely online, hosted by the Technion - Israel Institute of Technology, Israel, and takes place by July 19-22, 2021. A national presentation of PME-related activities in Thailand is part of the conference program.

This is the first time such a conference is being held in Thailand together with CLMV (Cambodia, Laos, Myanmar, Vietnam) countries, where mathematics education is underrepresented in the community. Hence, this conference will provide chances to facilitate the activities and network associated with mathematics education in the region. Besides, we all know this pandemic has made significant impacts on every aspect of life and provides challenges for society, but the research production should not be stopped, and these studies needed an avenue for public presentation. In this line of reasoning, we have hosted the IGPME annual meetings for the consecutive year, July 21 to 22, 2020, and 19 to 22 July 2021, respectively by halting "on-site" activities and shift to a new paradigm that is fully online. Therefore, we would like to thank you for your support and opportunity were given to us twice.

"Mathematics Education in the 4th Industrial Revolution: Thinking Skills for the Future" has been chosen as the theme of the conference, which is very timely for this era. The theme offers opportunities to reflect on the importance of thinking skills using AI and Big Data as promoted by APEC to accelerate our movement for regional reform in education under the 4th industrial revolution. Computational Thinking and Statistical Thinking skills are the two essential competencies for Digital Society. For example, Computational Thinking is related to using AI and coding while Statistical Thinking is related to using Big Data. Therefore, Computational Thinking is mostly associated with computer science, and Statistical Thinking is mostly associated with statistics and probability on academic subjects. However, the way of thinking is not limited to be used in specific academic subjects such as informatics at the senior secondary school level but used in daily life.

For the PME 44 Thailand 2021, we have 661 participants from 55 different countries. We are particularly proud of broadening the base of participation in mathematics education research across the globe. The papers in the four proceedings are organized according to the type of presentation. Volume 1 contains the presentation of our Plenary Lectures, Plenary Panel, Working Group, the Seminar, National Presentation, the Oral Communication presentations, the Poster Presentations, the Colloquium. Volume 2 contains the Research Reports (A-G). Volume 3 contains Research Reports (H-R), and Volume 4 contains Research Reports (S-Z).

The organization of PME 44 is a collaborative effort involving staff of Center for Research in Mathematics Education (CRME), Centre of Excellence in Mathematics (CEM), Thailand

Society of Mathematics Education (TSMEd), Institute for Research and Development in Teaching Profession (IRDTP) for ASEAN Khon Kaen University, The Educational Foundation for Development of Thinking Skills (EDTS) and The Institute for the Promotion of Teaching Science and Technology (IPST). Moreover, all the members of the Local Organizing Committee are also supported by the International Program Committee. I acknowledge the support of all involved in making the conference possible. I thank each and every one of them for their efforts. Finally, I thank PME 44 participants for their contributions to this conference.

Thank you

Best regards

M. Inprasitha

Associate Professor Dr. Maitree Inprasitha
PME 44 the Year 2021
Conference Chair

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RESEARCH REPORTS (S-Z)

USING SIGN LANGUAGE VIDEOS TO HELP DEAF STUDENTS UNDERSTAND AND SOLVE WORD PROBLEMS IN MATHEMATICS: RESULTS OF A SCHOOL INTERVENTION

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For students who are deaf or hard-of-hearing (DHH), word problems in mathematics are a particular challenge due to several reasons. One reason is that DHH students often have difficulties reading and understanding written information. In this paper, we present a school intervention with one class of DHH students (N=10) using sign language (SL). The aim of the intervention was to facilitate students' understanding of and work with word problems. We investigated if and how the use of SL videos can help for this purpose. In a mixed-methods design, analyzing data from the intervention qualitatively and from written solutions of the students in tests quantitatively, we found that SL videos in many instances appeared to help the DHH students in their understanding of the problem situation and accordingly to solve the word problems.

INTRODUCTION

Several studies indicate that students who are deaf or hard-of-hearing (DHH) tend to have severe mathematical difficulties (e.g., Blatto-Vallee et al., 2007; Marschark et al., 2013; Pagliaro, 2015; Qi & Mitchel, 2012; Traxler, 2000). Even before they start school, a majority of DHH students show difficulties in their mathematical development (Kritzer, 2009). DHH students often appear to have relative strengths, for example, in geometry. This may be related to their preference for visually presented information: "Geometry concepts and skills are developed sooner and/ or more quickly than those of other areas, perhaps influenced by their visual access to information" (Pagliaro, 2015, p. 183). Yet, in other mathematical areas (e.g., number sense) DHH students tend to have more difficulties (Spencer & Marschark, 2010). In particular, DHH students appear to struggle with word problems, which relates to the fact that they often lack conceptual understanding, for example, of operations (Zevenbergen et al., 2001), and that DHH students often have limited language and reading skills (Lederberg et al., 2012). As a consequence, word problems often appear to be particularly difficult for DHH students (Hyde et al., 2003): Extracting information from text and interpreting and applying their calculations linguistically constitutes a challenge (Swanwick et al., 2005). That is why DHH students find it rather difficult to understand and work with word problems (Spencer & Marschark, 2010).

In this paper, we present a school intervention with one class of DHH students, more precisely DHH students using sign language (SL), at grade 6/7. The aim of the intervention was to facilitate students' understanding of and work with word problems. To help students understand the word problems, we used sign language videos as a representation of the word problems: All students were able to watch these on individual

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tablet computers during their work on the word problems. We posed the research question, *Does the use of sign language videos facilitate students' work on word problems and how?*

STUDENTS WHO ARE DEAF OR HARD-OF-HEARING (DHH)

DHH students and sign language

People with a hearing loss can be referred to as deaf or hard-of-hearing (DHH), depending on their degree of hearing loss respectively hearing threshold. A distinction is made between mild, moderate, severe, or profound hearing loss (Davis & Hoffman, 2019). In many cases, both ears are affected. Hearing loss might lead to difficulties in hearing sounds and communicating with others. Moreover, congenital hearing loss has an impact on learning and social development (Spencer & Marschark, 2010).

SLs, such as German SL (DGS), differ from written and spoken language in their modality. While the modality of spoken language is auditory-vocal, SLs are based on a visual-gestural modality (Leonard et al., 2013). Further, there are many differences between spoken and SLs in linguistic aspects, such as lexicon, syntax, phonology, morphology, and pragmatics. For instance, spoken language signals follow each other linearly, whereas in SL different visual levels such as manual signs and visual expressions (e.g., for size or height) can be processed simultaneously (Wille, 2019). SL therefore is not a visualization of spoken language, but an independent language.

Difficulties of DHH students with word problems

The reasons for DHH students' difficulties in mathematics are complex. Nunes and Moreno (2002) argue that hearing loss is not a direct cause for problems in learning mathematics, but rather a risk. Despite delays in DHH students' language development, difficult access to spoken language or delayed access to SL have an influence on the development of informal mathematical knowledge and activities such as counting, naming shapes, or comparing quantities). Deaf children of deaf parents are at an advantage here (Freel et al., 2011; Svartholm, 2008), since their parents can provide SL early on. Interactions with parents play an important role not only in the development of language, but also in the acquisition of mathematical skills: Kritzer (2009) found for DHH children that "children who demonstrated higher levels of mathematics ability were found to spend a larger percentage of their day interacting with the adults around them and to experience more frequent and purposeful exposure to mathematically based concepts … at home" (p. 474). Besides, DHH students often show low reading abilities: On average, the reading skills of 18- and 19-year old DHH students are at about the same level as 8 to 9-year-old students without hearing loss (Kral & O'Donoghue, 2010). Reading comprehension is particularly important in word problems (Kail & Hall, 1999). Thus, adequate comprehension of word problems, understanding the problem, distinguishing relevant from irrelevant information as well as formulating answers might be particularly challenging (Swanwick et al., 2005).

THIS STUDY

The participating students

This exploratory study was conducted at a special school for children with sensory needs (hearing loss). It addressed ten DHH students (9 girls, 1 boy) of a class with a bimodal bilingual approach (SL and written language). Six out of the ten students additionally had general learning difficulties and had special educational needs in learning. The students were in mixed class of grade 6/7 and were 13 to 15 years old. The hearing loss was profound in seven cases, severe in two cases, and moderate in one case. Five students were bilaterally fitted with cochlear implants, although one student no longer wore them. Three were bilaterally fitted with hearing aids; here too, one student only wore them occasionally. Two students were bimodally fitted with a cochlear implant and a hearing aid. Seven students had a migration background. The languages spoken at home in these seven families were Kurdish, Turkish, Polish, and Arabic. German was not the first language in these seven cases, neither was SL. In a standardized reading comprehension test (ELFE1–6: Lenhard & Schneider, 2006), all students scored below average, which indicates difficulties in reading comprehension.

In their schooling, the students regularly communicated using SL: SL was the language of instruction and also the language the students used to communicate with each other. The students first came into contact with SL in pre-school or school, so they were not "native signers." This is in line with the situation overall, since about 90% of all DHH children have normal-hearing parents and, thus, native signers are a minority among DHH children (Mitchell & Karchmer, 2004).

The intervention

This intervention's aim was to support students in their understanding of and work with word problems. To do so, the students practiced extracting relevant information from the text in word problems, for example, by underlining and noting relevant information before starting to think about an operation. Additionally, during the intervention, the students were provided with SL videos of the word problems: Videos where word problems were translated to SL (see Fig. 1). During the intervention, the students always first were to read the word problems in text and to try to understand the situation. Their understanding of the word problem was also discussed in a plenary phase. Only after this, the SL video was shown to the students. During their individual work on the tasks, the students had the text as well as the video (on individual tablet computers) available.

The SL videos were created by two researchers (co-authors of this paper), under supervision of a SL lecturer, being a native signer himself. The researchers also conducted the intervention in the students' school. One example is illustrated in Figure 1. The according word problem was "Five friends are going on vacation together. Together, they are paying 212.50 € for the train. The hotel costs 535.00 € for all. How much does everyone need to pay individually?"

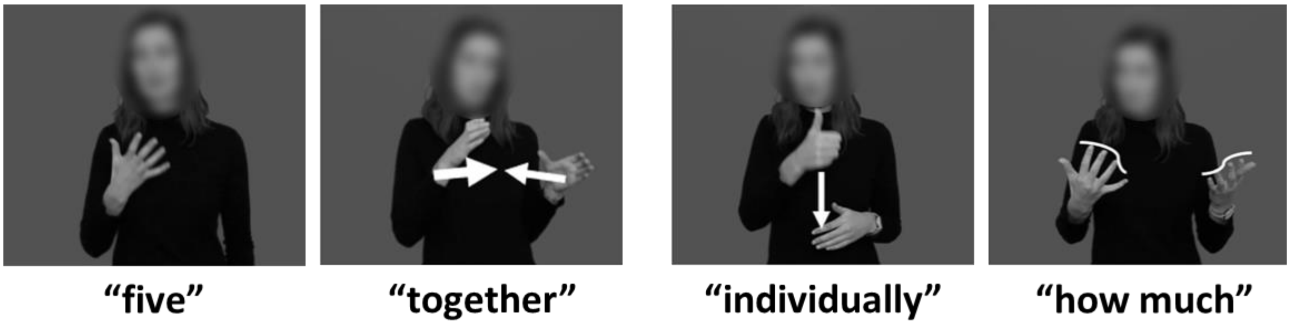

Figure 1: Screenshots from SL video (in order of appearance in the SL video)

The intervention had four sessions, each consisting of four school hours. The first session focused on filtering out relevant information from the word problems. In the second session, DHH students were to distinguish between relevant and irrelevant information in the word problems. The third session focused on formulating questions about the word problems and to understand the problem situation. Finally, during the fourth session, all content was repeated and consolidated.

Data and data analysis

For the evaluation of the school intervention, we used different data sources. For evaluating the process of the intervention, two researchers who were present during the intervention took field notes. Also, students' written work on word problems during the intervention was collected. These data were analyzed using qualitative content analysis (Mayring, 2014): In short, a summarizing procedure was used to find categories (themes during the intervention) with respect to the research question.

Additionally, a pre-test and post-test were conducted before and after the intervention, with four word problems each, addressing basic arithmetic operations. In the post-test, the students received the word problems in text without sign language first and were to solve them individually. In a second step, the post-test was repeated, now with the help of SL videos. No feedback on the correctness of the solutions was given in all tests. For analyzing students' work on the test, the solutions on paper were coded with respect to three aspects: (a) the identification of relevant information in the problem, (b) the modelling regarding the arithmetic operation applied, and (c) the interpretation of the results in the context. All three aspects were rated (three-step): For example, for (b) it was rated best if the correct operation was applied and all necessary steps were conducted (regardless of slips in calculation or similar), it was rated medium when a part was correct but relevant steps were missing, and it was rated lowest when the wrong operation was used or the relevant numbers were not used. These ratings (2, 1, 0) for the three aspects (a, b, c) were used to investigate differences between pre- and post-test (both text) and between the text- and SL-video-condition in the post-test, using the Wilcoxon-Test, a nonparametric statistical test suited for small sample sizes.

RESULTS

We posed the research question, *Does the use of sign language videos facilitate students' work on word problems and how?* In the following, we will first give insights into the results of the qualitative analysis of the intervention, then into the results of the pre- and post-test analyses and will finally enrich the findings through illustrations of the case of one girl, Mia (pseudonym). The qualitative analysis revealed the following results regarding the intervention:

General work on word problems. The students worked on the word problems intensely. They appeared to be motivated to identify relevant pieces of information and to find an answer that fits to the problem. They also discussed the word problems and the given situations vividly in collaborative work (with partners).

Students' work on the word problems (as text). When the word problems were given as text, the students often asked for the meaning of words (e.g., "What does cable mean?") and for what they were supposed to do. Two out of ten students appeared to understand the word problems also without SL videos: They used SL videos seldom and stated that they did not prefer SL videos over text (see below).

Students' work on the word problems (as SL videos). The students watched the SL videos regularly and often multiple times. They often asked for the video when it was not given directly. The students posed little questions regarding the meaning of words—they appeared to understand the words well in SL. The students tended to adopt the signs provided in the videos also for their own talk. After the video of the word problems was shown, they often signed "I see!", "It's easier now", or "Ah!", or they simply nodded. Generally, the students appeared to have less struggle to understand the problem situations in the SL videos than in text. When being asked about their perception of SL videos, eight students evaluated them positively. Two out of ten students said that they understood word problems equally well in text form or that they would prefer the task being read out loud.

When analyzing and comparing students' written work in the pre- and post-test in the text condition, we found that five out of eight students participating in both tests scored lower in the post-test than in the pre-test. Differences were not significant. This indicates that the intervention did not facilitate students' work with word problems given as text in general. When comparing students' work in the two post-tests—the text condition vs. SL video condition—we found significant differences (p=.011, r=.597, strong effect), with seven out of eight students scoring higher in the SL video condition than in the text condition. This indicates that the SL videos facilitated students' work with word problems. These results are in line with the qualitative results that the students appeared to benefit from the use of SL videos mainly.

We would like to use a case of one girl as an example to give further insights into the intervention and students' work with the SL videos. Mia was 14 years old. She had a profound hearing loss and was fitted bilaterally with hearing aids. In school, she communicated mainly in SL, at home in spoken language. She grew up trilingually (SL, German, Turkish). At the beginning of the intervention, Mia described that she was insecure with word problems, particularly about what needs to be calculated in word problems. She also often had difficulties understanding words such as "take/get" or "spend" in written text. Mia herself stated that she did not like either mathematics or reading. In the pre-test, when working on the "five-friends-problem" (see above), she multiplied (Fig. 2). She added (in text): "First I multiplied, then I calculated." and added as an answer "Every friend needs to pay 1,136,875,000 Euro alone."

pre-test (text) **post-test (text)** **post-test (SL video)**

Figure 2: Examples of written solutions by Mia for "five-friends-problem"

During the intervention sessions, Mia stated that she understood the word problems better through the SL videos than through text. We observed that when working with word problems presented in SL videos, she was more motivated in class and watched the SL videos several times in a row before deciding on how to calculate. According to her, without the SL video, some of the written words in the word problems were unknown to her. In the first post-test situation when the word problem was presented as text (not SL video), Mia subtracted to solve the "five-friends-problem" (Fig. 2) and wrote: "Five friend needs to pay 322.50€." (sic!) In the second part of the post-test with the SL video, although her calculation had a mistake, she used the correct arithmetic operation (Fig. 2) and answered: "Every friend needs to pay 149€ and 2.50€." Her case and in particular her work on this task illustrate that the SL in cases like hers helped the students to understand word problems and work with them accordingly.

DISCUSSION

The results indicate that the SL videos facilitated DHH students' understanding of and work with word problems in our intervention in many instances. The videos appeared to be an important support for those who had difficulties accessing written language. Furthermore, the students in this study described the work with the videos as motivating and appeared to be happy to understand the word problems through the SL videos ("Ah!!, "I see!").

The results of students' written work in the tests indicated that the students were more successful in working with word problems (identifying relevant information, applying an appropriate mathematical operation, and interpreting the results in the context) through the use of SL videos. However, the intervention with the videos did not generally help students to succeed with word problems: In the post-test, the students' results in the text condition (without SL videos) were not better than before the intervention. This hints at the need to support the students in their conceptual understanding of (in this case) arithmetic operations. Future interventions should include such support as well.

Our study is subject to several limitations. One obvious limitation is the small number of students, some of whom also had additional learning difficulties. The results of the statistical analysis can therefore serve as an indication of results of this particular intervention with these particular students only. The results of the study are not representative of the entire group of DHH students using SL.

For future practical work with DHH students, it is important to bear in mind that not all DHH students are fluent in SL, as some prefer oral communication. For DHH students with little SL skills, SL videos are unlikely to offer an advantage over other visual representations (e.g., a sketch) of the mathematical problem. Yet, the SL videos have the advantage that students can watch the same video several times and certain scenes in detail. This is comparable to a situation where NH students can read a written task multiple times (Wille, 2019). Here, SL videos have an advantage over or additional value to SL interpreters as persons.

In summary, SL videos in our intervention were found to support DHH students' understanding and work with word problems. We think that they offer a good opportunity to reduce barriers to understanding written instructions in word problems and hope that future interventions with DHH students can build on the results of our study.

References

Blatto-Vallee, G., Kelly, R. R., Gaustad, M. G., Porter, J., & Fonzi, J. (2007). Visual-spatial representation in mathematical problem solving by deaf and hearing students. *Journal of Deaf Studies and Deaf Education*, *12*(4), 432–448.

Davis, A. C., & Hoffmann, H. J. (2019). Hearing loss: rising prevalence and impact. *Bulletin of the World Health Organization*, 97, 646–646A.

Freel, B. L., Clark, M. D., Anderson, M. L., Gilbert, G. L., Musyoka, M. M., & Hauser, P. C. (2011). Deaf individuals' bilingual abilities: American sign language proficiency, reading skills, and family characteristics. *Psychology,* 2, 18–23.

Hyde, M., Zevenbergen, R., & Power, D. (2003). Deaf and hard of hearing students' performance on arithmetic word problems. *American Annals of the Deaf, 148*(1), 56–64.

Kail, R., & Hall, L. K. (1999). Sources of developmental change in children's word-problem performance. *Journal of Educational Psychology*, *91*(4), 660–668.

Kral, A., & O'Donoghue, M. (2010). Profound deafness in childhood. *The New England Journal of Medicine, 363*(15), 1438–1450.

Kritzer, K. L. (2009). Barely started and already left behind: A descriptive analysis of the mathematics ability demonstrated by young deaf children. *Journal of Deaf Studies and Deaf Education, 14*(4), 409–421.

Lederberg, A. R., Schick, B., & Spencer, P. E. (2012). Language and literacy development of deaf and hard-of-hearing children: Successes and challenges. *Developmental Psychology, 49*(1), 1–16.

Lenhard, W., & Schneider, W. (2006). *Ein Leseverständnistest für Erst- und Sechstklässler: ELFE 1-6.* Göttingen: Hogrefe.

Leonard, M. K., Ferjan Ramirez, N., Torres, C., Hatrak, M., Mayberry, R.I., & Halgren, E. (2013). Neural stages of spoken, written, and signed word processing in beginning second language learners. *Frontiers in Human Neuroscience*, 7, 332.

Marschark, M., Morrison, C., Lukomski, J., Borgna, G. & Convertino, C. (2013). Are deaf students visual learners? *Learning and Individual Differences, 1*(25), 156–162.

Mayring, P. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solution. Klagenfurt, Austria: Beltz.*

Mitchell, R. E., & Karchmer, M. A. (2004). Chasing the mythical ten percent: Parental hearing status of deaf and hard of hearing students in the United States. *Sign Language Studies, 4*(2), 138–163.

Nunes, T. & Moreno, C. (2002). An intervention program for promoting deaf pupils' achievement in mathematics. *The Journal of Deaf Studies and Deaf Education, 7*(2),120–133.

Pagliaro, C. M. (2015). Developing numeracy in individuals who are deaf and hard of hearing. In H. Knorrs, & M. Marschark (Eds.) *Educating deaf learners. Creating a global evidence perspective* (pp. 173–195). Oxford, UK: Oxford University Press.

Pagliaro, C. M., & Kritzer, K. L. (2013). The math gap: A description of the mathematics performance of preschool-aged deaf/ hard-of-hearing children. *Journal of Deaf Studies and Deaf Education*, *18*(2), 139–160.

Qi, S., & Mitchell, R. E. (2012). Large-scale academic achievement testing of deaf and hard-of-hearing students: past, present, and future. *Journal of Deaf Studies and Deaf Education*, *17*(1), 1–18.

Spencer, P. E. & Marschark, M. (2010). *Evidence-based practice in educating deaf and hard-of-hearing students*. Oxford, UK: Oxford University Press.

Svartholm, K. (2008). The written Swedish of deaf children: a foundation for EFL. In C. J. K. Bidoli & E. Ochse (Eds.) *English in International Deaf Communication. Linguistic insights* (pp. 211–250). Bern, Switzerland: Peter Lang.

Swanwick, R., Oddy, A., & Roper, T. (2005). Mathematics and deaf children: An exploration of barriers to success. *Deafness and Education International, 7*(1), 1–21.

Traxler, C. (2000). The Stanford Achievement Test, 9th edition: National norming and performance standards for deaf and hard-of-hearing students. *Journal of Deaf Studies and Deaf Education*, *5*(4), 337–348.

Wille, A. M. (2019). Gebärdensprachliche Videos für Textaufgaben im Mathematikunterricht – Barrieren abbauen und Stärken gehörloser Schülerinnen und Schüler nutzen. *Mathematik differenziert, 3*(19), 38–41.

Zevenbergen, R., Hyde, M., & Power, D. (2001). Language, arithmetic word problems, and deaf students: Linguistic strategies used to solve tasks. *Mathematics Education Research Journal*, *13*(3), 204–218.

IDENTIFYING STUDENT STRATEGIES THROUGH EYE TRACKING AND UNSUPERVISED LEARNING: THE CASE OF QUANTITY RECOGNITION

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Identifying student strategies is an important endeavor in mathematics education research. Eye tracking (ET) has proven to be valuable for this purpose: E.g., analysis of ET videos allows for identification of student strategies, particularly in quantity recognition activities. Yet, "manual", qualitative analysis of student strategies from ET videos is laborious—which calls for more efficient methods of analysis. Our methodological paper investigates opportunities and challenges of using unsupervised machine learning (USL) in combination with ET data: Based on empirical ET data of N = 164 students and heat maps of their gaze distributions on the task, we used a clustering algorithm to identify student strategies from ET data and investigate whether the clusters are consistent regarding student strategies.

INTRODUCTION

For researchers and practitioners (e.g., teachers) in mathematics education, it is important to not only evaluate student achievements, their results and products, but also to analyze students' thought processes and individual strategies leading to such products. In recent years, eye tracking (ET)—the recording of eye movements—has gained increasing importance in mathematics education research (Lilienthal & Schindler, 2019). Among others, it has proven to be valuable to analyze student strategies in different mathematical areas (e.g., Bruckmeier et al., 2019; Obersteiner & Tumpek, 2016), including quantity recognition in whole number representations (Lindmeier & Heinze, 2016; Schindler & Lilienthal, 2018). For example, Schindler et al. (2019a) analyzed student strategies in determining quantities in the 100-dot field and 100-abacus based on ET data: They used gaze-overlaid videos (videos of the scene with the eye gaze visualized as dot wandering around) to infer student strategies. However, such qualitative analysis of ET data is laborious: Analyzing ET data, which are rich by nature, is time-consuming and demanding (Klein & Ettinger, 2019). This calls for more efficient methods of analysis when bigger numbers of students are studied, and student strategies are to be inferred (Klein & Ettinger, 2019).

Our methodological paper explores the possibility to identify student strategies in whole number representations using ET data combined with unsupervised machine learning (USL). Based on data from $N = 164$ fifth grade students, we use a clustering algorithm (a specific instance of USL), to investigate the possibility to identify student quantity recognition strategies from so-called gaze heat maps (see Fig. 2). Broadly, we investigate what opportunities and challenges USL offers for identifying quantity

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recognition strategies. In particular, we ask the question: *Does the USL provide consistent clusters with respect to student strategies?*

Our paper illustrates with examples how a clustering algorithm, applied to heat maps, can be used to identify student strategies (“proof of concept”). We investigate the consistency of the clusters provided by the USL through qualitative interpretation using qualitative previous findings and elaborate on opportunities and challenges of USL.

EYE TRACKING IN MATHEMATICS EDUCATION RESEARCH

Eye tracking allows for a recording of spatio-temporal sequences of gaze points that indicate visual attention. The connection between gaze and visual attention exists due to an economic feature of the human eye, which concentrates a substantial fraction of the receptors on the retina in the small area of the *fovea*. Thus, in order to pay attention in detail, humans need to move their eyes constantly so that the area of interest is in line with the fovea, a process that can be tracked with ET devices unobtrusively by visually observing the pupils. ET is of interest for mathematics education research since the recorded sequences of gaze points do allow inferences about mental processes, though interpretation of gaze movements is not straightforward and bijective (Schindler & Lilienthal, 2019). ET is of growing interest since ET devices became increasingly affordable, advanced, and accurate (Lilienthal & Schindler, 2019); due to theoretical advances in interpretation (Schindler & Lilienthal, 2019); and since the required computational resources for partially automated analysis are available at low cost, which makes ET applications using (partially) automated analysis available for research and, in the future, also for mathematics education practitioners (e.g., teachers).

MACHINE LEARNING

The term *Machine Learning* (ML) refers to a set of methods for automated analysis of data, specifically “methods that can automatically detect patterns in data, and then use the uncovered patterns to predict future data, or to perform other kinds of decision making under uncertainty” (Murphy, 2012, p. 1). There are two major types of ML: *Supervised learning (SL)* algorithms learn a mapping between training samples and respective output. This means that each sample in the training set must be labelled. The learned mapping can then be used to make categorical or nominal predictions (Murphy, 2012). SL is thus also called *predictive learning*. SL is used, for example, in Schindler et al.’s (2019b) study, where the training samples are (as in this paper) ET sequences represented in the form of heat maps, with labels that specify each heat map to belong to a student with or without mathematical difficulties. After training, the SL algorithm can be used to classify previously unseen heat maps and predict whether the corresponding student has mathematical difficulties or not.

The second major type of ML is *unsupervised learning (USL)* where only training samples but no labels are given. The computer is then tasked to “find ‘interesting patterns’ in the data” (Murphy, 2012, p. 2). This is also called *knowledge discovery*. As Murphy (2012) notes, USL is a much less well-defined problem than SL. In this paper, we use *clustering*, a form of USL in which the set of samples (here: gaze heat maps) is divided ("clustered") into a number of groups. A clustering algorithm tries to find a meaningful division of the input data, but how a good division may look like and the "correct" number of clusters is not known *a priori*. To the best of our knowledge, USL has not been used on ET data in mathematics education research so far.

QUANTITY RECOGNITION IN WHOLE NUMBER REPRESENTATIONS

Whole number representations such as the 100 dot field or the 100 abacus (also called "100-frame"), which visualize substructures of 100 (50, 10s, 5s), are often used for students to learn the number range up to 100 (Gaidoschik, 2015). Previous research has shown that students, when perceiving quantities in such representations, make use of structures such as 10s (rows) and 5s (Obersteiner et al., 2014). While the analysis of student strategies in such representations is challenging (Obersteiner et al., 2014), recent studies have indicated that ET is a useful tool to identify strategies, e.g., from ET videos (Schindler & Lilienthal, 2018) or scan-paths, which indicate where the students looked at (Lindmeier & Heinze, 2016). Whereas such studies using ET to identify strategies are promising, the qualitative analysis of gaze patterns is demanding and time-consuming—especially for empirical studies with larger numbers of participants. Therefore, we investigate the opportunities that USL may offer to help identify student strategies based on their spatial gaze distributions on the task.

THIS STUDY

Participants. We use data from a study with 164 fifth-grade students (92 boys, 72 girls) in a German comprehensive school. The mean age was 10;9 (SD = 0;7). The study took place in the first weeks of fifth grade. Using a standardized arithmetic paper-pencil test, we identified 59 children as typically developing in mathematics, 69 children to encounter mathematical difficulties, and 36 to be "at risk" to have mathematical difficulties (see Schindler et al., 2019a;b for a detailed description of the test).

Tasks, procedure, and eye tracker. We used a digital version of the 100-dot field. We used the same numbers as in Schindler et al. (2019a), where student strategies were inferred from ET videos qualitatively (7, 15, 20, 31, 43, 54, 68, 76, 89, 92, and 100; in randomized order). The students were tested individually. We used Tobii x3-120, a remote eye tracker at a sampling rate of 120 Hz, which was mounted at the bottom edge of the 24'' full HD computer monitor. It was calibrated through a nine-point calibration. Before the students worked on the tasks, they saw a picture of the dot field and were to describe it. The students got two practice tasks (with numbers not used in further tasks). They were instructed to always name the number of dots as fast and correctly as possible. Before each task, the students were asked to fixate a star in the middle of the screen. The students did not receive a response on the correctness of their answers. We made audio recordings of verbal answers.

Heat maps. ET provides rich information and a large amount of data, reflecting that gaze patterns can differ in multiple ways. To find groups of strategies ("clusters"), we chose a representation of the recorded gazes to facilitate the subsequent analysis. This representation needed to reduce the amount of data the clustering algorithm has to handle while preserving the relevant features of the gaze patterns. Based on previous research that indicated a variety of student gaze distributions on the task sheets in quantity recognition tasks (Schindler et al., 2019a), we decided to use heat maps that show the spatial distribution of gazes over the presented digital task sheets integrated over the whole duration of a task. We used the Tobii Pro Lab Software to produce individual student heat maps. For clustering, we included only heat maps of correctly or inversely solved (common mistake in German, e.g., for 89: “ninety-eight”) tasks to assure that the students actually perceived the given information rather than guessed. In case of 89 on the dot field (focus of the *Results Section*), 90 heat maps were included.

Clustering. To automatically determine groups of similar heat maps, a *definition for the (dis-)similarity* between two heat maps is required. We use the Euclidean distance between the images: The sum of the squared pixel differences between two heat maps measures dissimilarity (Goshtasby, 2012). Calculating the Euclidean distance is a standard approach to determine similarity between images in digital image processing.

A second important choice concerns the *clustering algorithm* that assigns groups based on the similarity of heat maps. We use self-organizing maps (SOMs) (Kohonen, 2001), which are suited for explorative data analysis (Kaski, 1997). SOMs do not automatically determine the number of groups present in the data (which is a very hard problem) but require the number as input parameter. Since previous empirical work hinted at a set of five different kinds of strategies for quantity recognition in whole number representations (Schindler et al., 2019a), we use a structure with nine clusters, arranged in a 3x3 grid. Using nine clusters allows for the possibility that the algorithm would identify more strategies than previously found—or to differentiate them further. SOMs have the rather unusual feature that they assume an *a priori* topology over the relationship between the different groups. While this does not necessarily guarantee for optimal clustering results, the topology, usually a 2D grid (Fig. 1), provides an additional tool to interpret the clustering results: neighborhood indicates similarity. This study utilizes the SOM algorithm implemented in the Matlab Deep Learning Toolbox with a hexagonally connected 3x3 grid and default parameters. In the clustering process, each of the student heat maps is assigned to one of the nine clusters. These assignments are iteratively optimized until all similar heat maps are assigned to the same cluster, while highly dissimilar heat maps are assigned to different clusters on opposite ends of the 3x3 grid. As a result, each heat map is assigned to a group that contains its most similar peers. The implicit assumption here is that due to the similarity of the heatmaps in each group these groups represent particular quantity recognition strategies. For each cluster, we calculate a cluster prototype as the average of all heat maps assigned to that cluster (Fig. 1). These average heat maps help to draw conclusions about the quantity recognition strategy that every cluster may represent.

Analyzing the clusters. To answer the question if USL provides consistent clusters with respect to student strategies, for every task we regard each cluster of the SOM and qualitatively assign a tentative strategy based on the average heat map. We then analyze all single heat maps in each cluster: In particular, we qualitatively assign a strategy to each heat map, based on the set of strategies found through qualitative analyses by Schindler et al. (2019a): (1) *counting all*, where the students counted all dots shown, (2) *counting fives*, where the students counted groups of fives, (3) *counting rows*, where students counted all rows displayed, (4) *using 50 as unit*, e.g., when determining 76, they perceived 50 in one glance and counted only the further rows, and (5) *subtraction/last row*, where the students, e.g., in 89 looked at the missing 90st dot, or only on the last row of displayed dots. Note that in Schindler et al.'s (2019a) study, the design was alike to ours: This applies to the (identical) tasks, the procedure, ET, etc. The participants were at the same age and also at the beginning of fifth grade. The main difference is that Schindler et al. investigated only 20 students (whereof 10 were found to have MD). Because of the larger number of 164 students in our study, we assume that our data set may include all strategies found by Schindler et al. (2019a).

RESULTS

In the following, we will pursue the question: *Does the USL provide consistent clusters with respect to student strategies?* We do so by using one task as an example: 89 on the dot field. We use this particular task, since it affords a variety of strategies (Schindler et al., 2019a) and, thus, is an interesting case for the clustering.

For the task 89, the USL found four substantial clusters (Fig. 1), whereas five clusters remained effectively empty with only one member heat map that can be considered an outlier. Regarding the average heat maps of the clusters (Fig. 1, right), we tentatively assigned strategies to these four clusters: (7) *Counting Rows on the Right*, (9) *Counting Rows in the Middle*, (1) *Last Row/Subtraction*, and (3) *Counting Rows on the Left*.

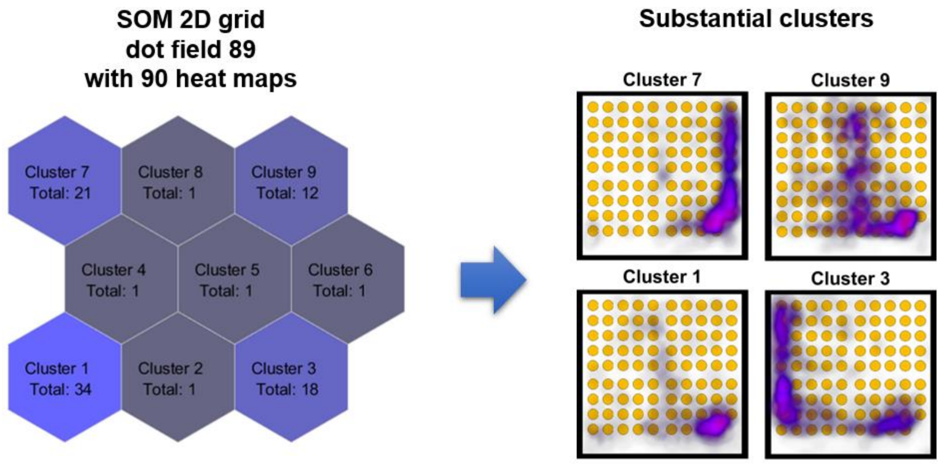

Figure 1: SOM for task 89 dot field (left) and all substantial clusters (with *n*>1) visualized through their average heat map prototype (right).

Cluster 7: "Counting Rows on the Right" (*n*=21). Of the 21 heat maps in this cluster, we identified 19 heat maps to indicate the strategy *counting rows*, which is consistent with the impression from the average heat map: The gazes are in every row, and the pattern indicates a counting process (Fig. 2). The heat maps indicate that these 19 students counted at the right edge of the rows. The remaining two heat maps in this cluster correspond to the strategy *using 50*: Here, there are no/few gazes on the upper half of the dot field, and the gaze patterns indicate that the students counted rows 6 to 9 at the right edge of the respective rows (Fig. 2). The similarity in appearance with a concentration at the right edge of the rows in the lower half of the dot field is likely the explanation why the USL put the two *using 50*-heat maps together with the 19 heat maps that indicate *counting (all) rows*. The clustering result is reasonable since in any instance there was presumably (at least some) counting of rows on the right side.

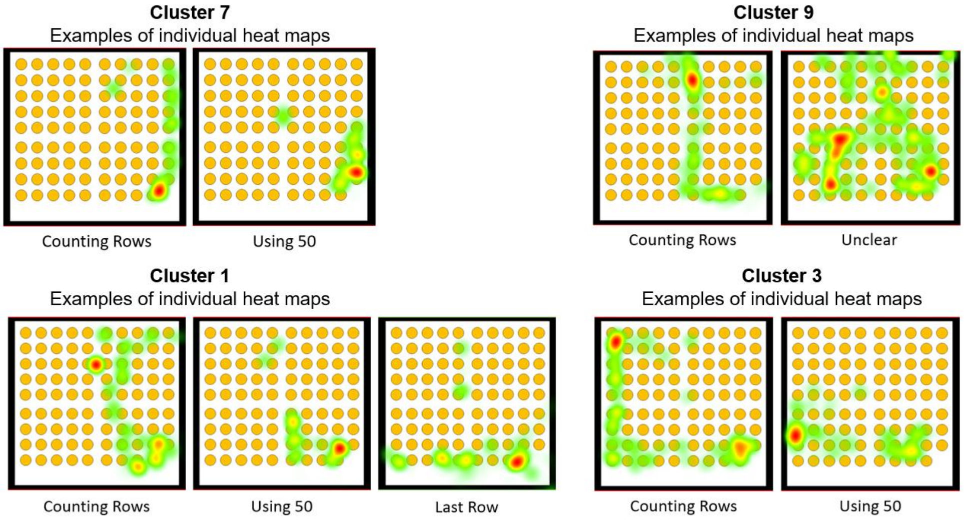

Figure 2: Examples of individual heat maps

Cluster 9: "Counting Rows in the Middle" (*n*=12). Of the 12 heat maps in this cluster, we found 7 heat maps to reflect the strategy *counting rows*, consistent with the assignment to the cluster prototype. The counting pattern is situated in the middle of the dot field, indicating that the students counted rows in the middle (Fig. 2). For the other 5 heat maps in this cluster, we are unable to identify a clear strategy. They were marked as *"unclear"* (Fig. 2): The gazes are spread over the task sheet, possibly reflecting a multitude of strategies. An indication that this cluster may contain a variety of different strategies is the rather noisy appearance of the cluster prototype.

Cluster 3: "Counting Rows on the Left" (*n*=13). 8 heat maps in this cluster indicate the strategy *counting rows*, with the gazes at the left edge of the rows (see Fig. 2). The other 5 heat maps indicate *use of 50*, since there are hardly any gazes on the upper 50 dots, but gazes that indicate that the students counted the rows from the 6th row onwards at the left edge. Similar to Cluster 7, this explains why these two kinds of heat maps were both included in the same cluster: The patterns were similar in a way that the gaze density at the left edge is high.

Cluster 1: "Last Row/Subtraction" (*n*=34). For this cluster, we found three different kinds of strategies: 7 of the heat maps indicated that the students *counted rows* (see Fig. 2). In 11 cases, we identified *using 50*: The students' gazes indicated that the students counted rows 6 to 9 (Fig. 2). Finally, 15 heat maps indicated that the students focused only on the *last row* displayed (Fig. 2) or that they focused only on the missing 90st point, indicating a *subtraction* strategy. We assume that this relates to the distance metric used, which regards the intensity of the gaze distribution: Since the areas of the dot field that are different between these strategies have a relatively low intensity (light green), but all heat maps in this cluster have a common feature, the "blob" in the right corner of the last row, which is intense (warm colors), this "blob" may be decisive here.

Answering the research question *if the USL provides consistent clusters with respect to student strategies*, we can say that the clusters found were—in the used example of 89 on the dot field—consistent in a certain way, but different from our previous qualitative analyses. For example, for the USL, heat maps reflecting *counting rows on the right* and *using 50* are similar and belong to one cluster if students when using 50 count the rows 6 to 9 on the right side. On the other hand, *counting rows on the right* and *counting rows on the left* belong to two different clusters. The clustering algorithm operates on visual similarity of heatmaps and inherently cannot cluster strategies together that manifest themselves very differently in the gaze distribution. A second important observation is that in cases where a student strategy involves different processes (e.g., grasping 50 in a glance and counting rows 6 to 9), clustering cannot evaluate what process is decisive for the strategy—as it was done in our previous study (Schindler et al., 2019a). Yet, given that the clusters found in our approach seldom involved more than two strategies, we find that they are—to a certain extent—consistent with respect to student strategies. So, if a student heat map belongs to one cluster, one can say that the student most likely had one or another strategy.

DISCUSSION

In this paper, we explore the possibility to identify student strategies in whole number representations using ET combined with USL. Based on ET data from N = 164 fifth grade students, we use the SOM algorithm for clustering and ask whether this automated analysis *provides consistent clusters with respect to student strategies*. Our question relates to a fundamental issue of USL: Compared to SL, where it is possible to quantify the performance of the trained algorithm for classification, there is no obvious error metric for USL (Murphy, 2012). As error metric from the application domain of mathematics education, we tested whether clustering identifies consistent groups regarding the strategies they represent. We found that this is true only to some extent. This is understandable: Our clustering of heat maps compares solely the visual appearance of the quantity recognition process as a whole and thus inherently cannot decompose strategies or give higher weight to certain features (e.g., the absence of gazes on the upper half). One would rather expect to find more clusters than possible strategies, since different combinations of strategies could result in additional, likely more consistent clusters. We did *not* observe such an “over-clustering” tendency and it will be subject of future work to evaluate whether other clustering algorithms and the use of other distance metrics result in a higher number and more consistent clusters.

We would like to stress that this paper gives an example of an empirical study in which Artificial Intelligence (AI) is used to support human researchers. Here, essentially, the AI component provides an independent view on a data set and makes suggestions about meaningful partitioning of the data. Human researchers interpret and verify these suggestions based on pre-studies with smaller numbers of participants and a principle understanding of the applied ML algorithms. Indeed, the clusters identified in this paper have predominantly a clear interpretation, which may be meaningful in some contexts and clearly provided an independent view from a different angle.

References

Bruckmeier, G., Binder, K., Krauss, S., & Kufner, H.-M. (2019). An eye-tracking study of statistical reasoning with tree diagrams and 2 x 2 tables. *Frontiers in Psychology, 10*, 632.

Gaidoschik, M. (2015). Einige Fragen zur Didaktik der Erarbeitung des „Hunderterraums“. *Journal für Mathematik-Didaktik, 36*(1), 163–190.

Goshtasby A.A. (2012). *Image registration.* London: Springer.

Kaski, S. (1997). *Data exploration using self-organizing maps*. Doctoral Thesis. Helsinki University of Technology: Helsinki.

Klein, C., & Ettinger, U. (2019). *Eye movement research: An introduction to its scientific foundations and applications.* Cham, Switzerland: Springer.

Kohonen, T. (2001). *Self-Organizing Maps* (3rd. ed.). Berlin: Springer.

Lindmeier, A., & Heinze, A. (2016). *Strategies for recognizing quantities in structured whole number representations – A comparative eye-tracking study.* Paper presented at 13th International Congress on Mathematical Education (ICME-13), 2016.

Lilienthal, A.J., & Schindler, M. (2019). Eye tracking research in mathematics education: A PME literature review. In *Proceedings of the 43rd Conference of the IGPME* (Vol. 4, p. 62). Pretoria, South Africa: PME.

Murphy, K.P. (2012). *Machine learning: A probabilistic perspective*. Cambridge, MA: MIT.

Obersteiner, A., Reiss, K., Ufer, S., Luwel, K., & Verschaffel, L. (2014). Do first graders make efficient use of external number representations? The case of the twenty-frame. *Cognition and Instruction, 32*(4), 353–373.

Obersteiner, A., & Tumpek, C. (2016). Measuring fraction comparison strategies with eye-tracking. *ZDM, 48*, 255–266.

Schindler, M., Bader, E., Lilienthal, A.J., Schindler, F., & Schabmann, A. (2019a). Quantity recognition in structured whole number representations of students with mathematical difficulties: An eye-tracking study. *Learning Disabilities: A Contemporary Journal, 17*, 5–28.

Schindler, M., & Lilienthal, A.J. (2019). Domain-specific interpretation of eye tracking data: Towards a refined use of the eye-mind hypothesis for the field of geometry. *Educational Studies in Mathematics, 101*, 123–139.

Schindler, M., & Lilienthal, A.J. (2018). Eye-tracking for studying mathematical difficulties
—also in inclusive settings. In *Proceedings of the 42nd Conference of the IGPME* (Vol. 4, pp. 115–122). Umeå, Sweden: PME.

Schindler, M., Schaffernicht, E., & Lilienthal, A. (2019b). Differences in quantity recognition of students with and without mathematical difficulties: Analysis through ET and AI. In *Proceedings of the 43rd Conference of the IGPME* (Vol. 3, pp. 281–288). Pretoria, South Africa: PME.

THROUGH THE EYES OF PROSPECTIVE TEACHERS: JUDGING TASK DIFFICULTIES IN THE DOMAIN OF FRACTIONS.

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Judging the difficulty of mathematical tasks is an everyday activity of teachers. However, empirical research about teachers' judgment process is still scarce. It is assumed that teachers use their knowledge in order to identify difficulty-generating task characteristics and to evaluate them in terms of their difficulty for students. This ongoing study uses eye-tracking technology and stimulated recall interviews to examine the assumed judgment processes with N = 25 pre-service teachers in the domain of fractions. In an experimental design, it is further investigated whether teachers' specific knowledge on difficulty-generating task characteristics influences identification an evaluation processes during the judgment.

INTRODUCTION AND THEORETICAL BACKGROUND

The difficulty of mathematics tasks is influenced not only by mathematical task characteristics (e.g., in the domain of fractions: like vs. unlike fractions; Padberg & Wartha, 2017), but also by instructional characteristics (according to cognitive load theory: e.g., split-attention vs. integrated task design; Sweller et al., 2011). When judging task difficulty, teachers should be able to identify difficulty-generating task characteristics and to evaluate them adequately in terms of their difficulty for students. Research results show, however, that teachers often make inadequate judgments about task difficulties (e.g., Anders et al., 2010; Karing & Artelt, 2013) and fail to adequately consider the task's instructional design (Hellmann & Nückles, 2013; Schreiter et al., 2021). It is assumed that in the genesis of diagnostic judgments, teachers use the information available in a diagnostic situation (e.g., difficulty-generating task characteristics) and process them on the basis of their knowledge to get to their result (cf. Loibl et al., 2020). Numerous studies show that teachers' knowledge of difficulty-generating task characteristics plays a significant role for the accuracy of teachers' diagnostic judgments regarding task difficulty (cf. Ostermann et al., 2017; McElvany et al., 2009). However, most of these studies focus on the result of the diagnostic judgment, and it remains unclear how teachers get to their result and what role knowledge plays in the judgment process (cf. Loibl et al., 2020). Diagnostic judgment processes, such as identifying and evaluating task characteristics, constitute internal cognitive processes that cannot directly be observed. An effective way to examine such cognitive processes is to collect eye-tracking data followed by eye-tracking stimulated recall interviews (ET SRI) (Schindler & Lilienthal, 2019). ET SRI is a research method to investigate cognitive processes by asking probands to retrospectively describe their own thoughts using a video sequence of their eye movements (Lyle, 2003). Numerous

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 17-24. Khon Kaen, Thailand: PME.

eye-tracking studies focusing on teachers' professional vision (for an overview, cf. Grub et al., 2020) revealed significant differences in gaze behavior between experts and novices: shorter fixation durations, for example, were found for experts in several studies and were interpreted as an indicator that experts are faster in encoding information. However, in most of these studies, experts were distinguished from novices only by the number of years of job experience, and other knowledge components of teachers were not considered (cf. Grub et al., 2020). It therefore remains unclear how and whether teachers' knowledge influences their gaze behavior during diagnostic activities.

THIS STUDY

This ongoing study aims to investigate prospective teachers' diagnostic judgment of task difficulties from a process-view: which task characteristics (mathematical vs. instructional) do pre-service teachers identify and evaluate when judging the difficulty of fraction tasks? In addition, there is a particular research interest in exploring the potential influence of specific knowledge about difficulty-generating task characteristics on identification and evaluation processes. Please note that this contribution constitutes a pre-report of an ongoing study and only covers a sample of the data and results of a larger data set.

Based on the above-mentioned findings of Hellmann and Nückles (2013) as well as Schreiter et al. (2021), we expect that a) mathematical task characteristics are identified and correctly evaluated more frequently compared to instructional characteristics (H1a) and b) instructional task characteristics are to a large extent not identified and correctly evaluated (H1b). Furthermore, building on the results of existing research on the influence of specific knowledge on judgment accuracy (e.g., Ostermann et al., 2017), we assume that specific knowledge enables pre-service teachers to identify and correctly evaluate more difficulty-generating task characteristics (mathematical and instructional) (H2). As eye-tracking studies focusing on teachers' professional vision revealed significant differences in gaze behavior between experts and novices (here: teachers with and without job experience; cf. Grub et al., 2020), we further aim to exploratively investigate whether specific knowledge influences the gaze behavior of pre-service teachers who have no job experience during their diagnostic judgment of task difficulties.

METHODS

Participants. The results reported here are based on data from N = 25 pre-service teachers of mathematics. Participants were assigned to two conditions: an experimental group (n = 11), that received a 90-minute intervention on specific difficulty-generating mathematical and instructional task characteristics and a control group (n = 14) that did not receive any treatment.

Material. Four fraction tasks were created. Between these tasks, difficulty-generating mathematical and instructional task characteristics were systematically varied. The mathematical difficulty of fraction tasks was varied by modifying the denominators (like vs. unlike), by mixing natural numbers and fractions, and by using mixed fractions (cf. Padberg & Wartha, 2017). The sample task (figure 1) requires students to add a natural number and fractions in mixed notations, which causes difficulties. The instructional difficulty was varied based on the *split-attention effect* and the *redundancy effect* (Sweller et al., 2011). Accordingly, the tasks' relevant information is presented either close or distant from each other. Furthermore, the tasks were created in such a way that a) one and the same information is presented by different information sources (redundancy 1) or b) additional information irrelevant for the solution is included (redundancy 2) or c) no redundant information is included. The instructional design of the sample task (figure 1) causes difficulties, as different information sources (the problem definition, the graphic, and the length information) are presented distant from each other and redundant information (the route from Philippshagen to Göhren) is included.

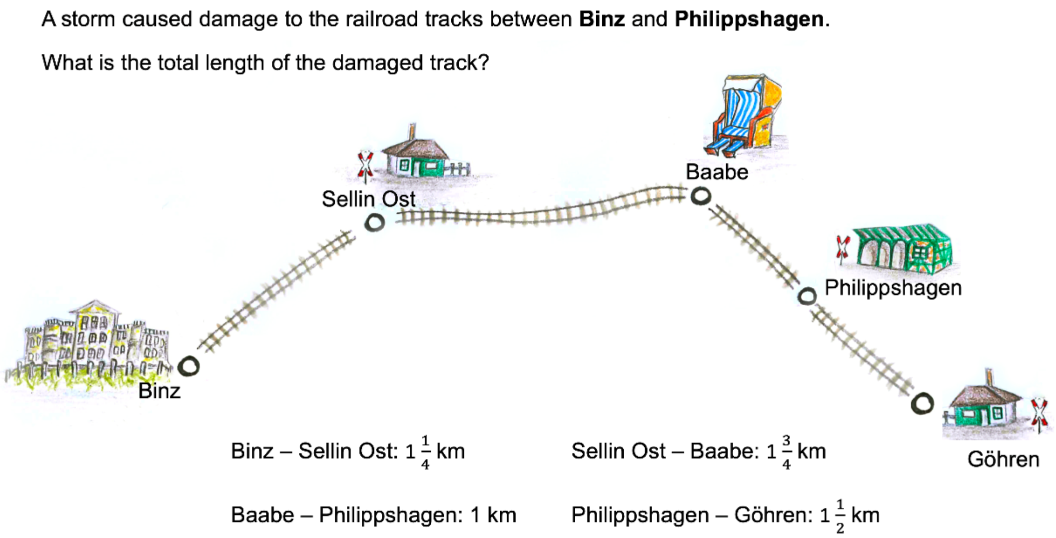

Figure 1: Sample for a fraction task with specific difficulty-generating task characteristics

Procedure. The diagnostic task consists of assessing four fraction tasks regarding the question "What makes the task easy/difficult for students?" The tasks are presented individually and in randomized order on a 24'' computer monitor. Eye-tracking data was collected using a monitor-based eye-tracker (Tobii Pro Fusion) that captures binocular eye movements at a sampling rate of 120 Hz. For adjusting the eye-tracker, a 9-point calibration was performed before each task. The time interval between the diagnostic task on the eye-tracker and the subsequent ET SRI was kept as short as possible to avoid loss of memory (approx. 1-3 min.). During the interview, subjects described what they did and thought during the diagnostic task, based on their shown eye movements. For the recording of the ET SRI, the software OBS was used, which records screen contents including sound, so that the videos of the eye movements with the corresponding comments of the subjects were available for the later analysis.

Data analysis. The Tobii Pro Lab software was used to analyze the eye-tracking data. In each task, specific Areas of Interest (AOIs) were defined around the varied mathematical and instructional task characteristics. The number of fixations and fixation duration were determined using the Tobii I-VT Fixation Filter. To determine the number of transitions between two AOIs, the videos of eye movements were visually inspected. A mixed-methods approach was used to analyze the ET SRI data: The ET SRI were first transcribed and coded deductively using qualitative content analysis according to Mayring (2015). The following category system was used and binary coded (in parentheses): difficulty-generating task characteristics can be *identified* (1), or *not identified* (0) when diagnosing a task. It turned out that some task characteristics are only identified when reflecting on one's own eye movements during the SRI. This resulted in another category *retrospectively identified*, which was evaluated as a subcategory of *not identified*. Identified task characteristics can be *correctly evaluated* in terms of difficulty for students (1) or *incorrectly / not further evaluated* (0). Task characteristics that are only evaluated during the SRI are assigned to the category *retrospectively evaluated* that counted as a subcategory of *not further evaluated*. Transcripts were coded by two raters with high interrater reliability (Cohen's Kappa = .88). The assigned codes were then integrated into a quantitative data set to examine differences across experimental conditions using variance analysis.

RESULTS

Identification and evaluation of difficulty-generating task characteristics

To test our hypothesis, an ANOVA was calculated with the within subject factor *task characteristics* (mathematical and instructional) and the between subject factor *condition* (experimental and control group). Figure 2 gives an overview of the average percentage of difficulty-generating task characteristics that were identified and correctly evaluated in terms of difficulty for students. The results show that there is a significant difference with high effect size between the identification and evaluation of mathematical vs. instructional task characteristics ($F(1,23) = 15.01$, $p < .001$, $\eta 2 = .40$). This effect is, however, dependent on the experimental condition. Bonferroni-corrected post-hoc tests show that differences between mathematical and instructional task characteristics can only be detected for participants of the control group (cf. figure 2). Here, significantly more mathematical task characteristics are identified and correctly evaluated compared to instructional characteristics (H1a). Instructional task characteristics are to a large extent not identified and adequately evaluated (H1b). Without specific knowledge, less than half of the instructional task characteristics are identified and correctly evaluated on average ($M = .42$, $SD = .12$).

A significant difference with high effect size could be determined between experimental conditions ($F(1,23) = 31.29$, $p < .001$, $\eta 2 = .58$). Bonferroni-adjusted post-hoc analysis reveal that participants of the experimental group identify and correctly evaluate a significantly higher number of both mathematical and instructional task characteristics (cf. figure 2). Specific knowledge about difficulty-generating task characteristics has thus enabled pre-service teachers to identify and adequately evaluate more difficulty-generating task characteristics when judging task difficulties for students (H2).

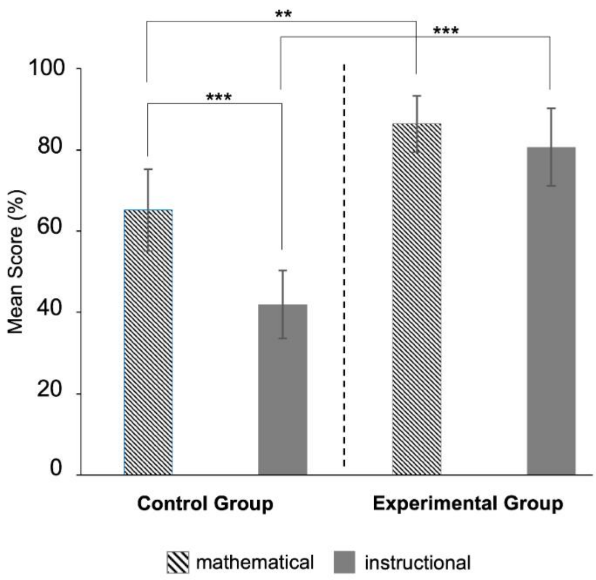

Figure 2: Means and Standard Error for identified and correctly evaluated difficulty-generating task characteristics (mathematical and instructional). $^{**}p \leq .01$, $^{***}p \leq .001$.

Analysis of gaze behavior

Table 1 provides an overview of the average number of fixations, transitions, and fixation durations (in s) to or between predefined mathematical and instructional AOIs. Furthermore, the table displays the total recording duration (in s), that is the time that was needed on average for the judgment per task.

Overall, the experimental group was found to have a lower number of fixations and lower fixation durations to predefined mathematical AOIs as well as shorter total recording durations compared to the control group. These group differences are statistically significant with high effect sizes (cf. table 1). Against the background that participants of the experimental group identified and correctly evaluated significantly more difficulty-generating task characteristics, these eye-tracking measures may indicate a more efficient approach to diagnosing with specific knowledge. For the instructional AOIs, however, no significant differences in gaze behavior between the control and experimental group could be determined.

AOI	ET measure	CG *M* (*SD*)	EG *M* (*SD*)	*t*	*p*	Cohens *d*
Mathem.	Fixation count	28.71 (12.71)	19.14 (7.84)	2.19	.039	0.91
	Fixation duration	8.31 (4.34)	4.89 (2.22)	2.55	.019	1.00
Instruct.	Fixation count	34.35 (6.95)	34.21 (11.03)	0.04	.972	0.02
	Fixation duration	6.92 (1.37)	7.68 (2.93)	0.79	.442	0.33
	Transition count	9.04 (3.90)	8.68 (4.24)	0.22	.830	0.09
Overall	Total record duration	118.62 (12.74)	102.07 (12.03)	3.30	.003	1.34

Table 1: Eye-Tracking measures for mathematical and instructional AOIs, each separately for the control group (CG) and experimental group (EG).

DISCUSSION

The main purpose of this ongoing study is to investigate which task characteristics (mathematical vs. instructional) pre-service teachers identify and evaluate when judging the difficulty of fraction tasks. Furthermore, a particular research interest is to explore the potential influence of specific knowledge about difficulty-generating task characteristics on identification and evaluation processes during the judgment.

In line with expectations, the results showed that mathematical task characteristics are more frequently identified and correctly evaluated compared to instructional task characteristics. Instructional task characteristics were to a large extent not identified and adequately evaluated in terms of their difficulty for students. These findings support existing research on diagnostic teacher judgments (e.g., Hellmann & Nückles, 2013; Schreiter et al., 2021), which showed that instructional task characteristics are insufficiently considered by teachers. However, if teachers do not consider the difficulty that is caused by the tasks’ instructional design, they risk creating cognitive overload in the students' learning process, which prevents successful learning (cf. Sweller et al., 2011). Therefore, the results reported here point to a need to promote the identification and evaluation of difficulty-generating instructional task characteristics in teacher education.

Furthermore, the results showed that specific knowledge enables pre-service teachers to identify and correctly evaluate more difficulty-generating task characteristics. This effect could be found for both mathematical as well as instructional task characteristics. These results are in line with existing research that highlights the importance of specific knowledge for the accuracy of teachers’ diagnostic judgments (e.g., Ostermann et al., 2017; Karing & Artelt, 2013). In our study, the collection of direct process indicators (eye movements, ET SRI) allowed to gain insights into the positive influence of specific knowledge on identification and evaluation processes that underlie teachers’ judgment. The analysis of eye movement measures further suggested that specific knowledge enables a more efficient approach to diagnosing: pre-service teachers with specific knowledge about difficulty-generating task characteristics look at mathematical task characteristics in particular and the entire task in general less frequently and for a shorter period of time. At the same time, they identify and correctly evaluate more difficulty-generating task characteristics. For instructional task characteristics, however, no significant group differences were found in terms of gaze behavior. It might be possible that there are no observable differences between persons who focus on task characteristics and process them fast and those who only pay little attention to the same characteristics. One possible explanation for our study finding could be that pre-service teachers without specific knowledge may have paid little attention to instructional task characteristics overall. These results complement existing research findings in the context of professional vision, which found indicators of faster information encoding processes among experts (here: teachers with job experience compared to teachers without job experience) (cf. Grub et. al 2020).

Regarding the results reported here, it should be noted that due to the small sample and due to the narrow focus on two areas of difficulty-generating task characteristics, solely in the domain of fractions, the results are only indicative. This report covers only a sample of the data and the results of an ongoing study with a larger data set. Additional data will be presented at the conference.

Overall, based on these findings, impulses for teacher training can be derived: An intervention on specific difficulty-generating task characteristics enables pre-service teachers to identify and adequately evaluate more difficulty-generating task characteristics and allows a more efficient approach to diagnosing. The question arises of what role other characteristics of the teacher, such as job experience, motivation, or other knowledge components, play in diagnosing task difficulties. These components are seen as further potentially relevant aspects for diagnostic judgments (cf. Loibl et al., 2020; Südkamp et al., 2012) and should also be investigated in future studies.

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References

Anders, Y., Kunter, M., Brunner, M., Krauss, S., & Baumert, J. (2010). Diagnostische Fähigkeiten von Mathematiklehrkräften und ihre Auswirkungen auf die Leistungen ihrer Schülerinnen und Schüler [Mathematics teachers'diagnostic skills and their impact on students' achievements]. *Psychologie in Erziehung und Unterricht*, 57, 175-193.

Hellmann, K., & Nückles, M. (2013). Expert Blind Spot in Pre-Service and In-Service Mathematics Teachers: Task Design moderates Overestimation of Novices' Performance. In M. Knauff, M. Pauen, N. Sebanz, & I. Wachsmuth (Eds.), *Cooperative minds: Social interaction and group dynamics; proceedings of the 35th Annual Meeting of the Cognitive Science Society.* Austin, Tex.: Cognitive Science Soc.

Grub, A.-S., Biermann, A. & Brünken, R. (2020) Process-based measurement of professional vision of (prospective) teachers in the field of classroom management: A systematic review. *Journal for Educational Research Online*. 12 (3), 75-102.

Karing, C., & Artelt, C. (2013). Genauigkeit von Lehrpersonenurteilen und Ansätze ihrer Förderung in der Aus- und Weiterbildung von Lehrkräften [Accuracy of teacher judgments and considerations on their improvement via teacher education]. *Beiträge zur Lehrerbildung*, *31*(2), 166–173.

Loibl, K., Leuders, T. & Dörfler, T. (2020). A Framework for Explaining Teachers' Diagnostic Judgements by Cognitive Modeling (DiaCoM). *Teaching and Teacher Education, 91*.

Lyle, J. (2003). Stimulated recall: a report on its use in naturalistic research. *British Educational Research Journal*, *29*(6), 861–878.

McElvany, N., Schroeder, S., Hachfeld, A., Baumert, J., Richter, T., Schnotz, W., Ullrich, M., et al. (2009). Diagnostische Fähigkeiten von Lehrkräften [Teachers' diagnostic skills]. *Zeitschrift für Pädagogische Psychologie*, 23(34), 223–235.

Ostermann, A., Leuders, T., & Nückles, M. (2017). Improving the judgment of task difficulties: prospective teachers' diagnostic competence in the area of functions and graphs. *Journal of Mathematics Teacher Education*, *21*(6), 579–605.

Padberg, F., & Wartha, S. (2017). *Didaktik der Bruchrechnung* (5. Auflage) [Didactics of fractions]. Mathematik Primarstufe und Sekundarstufe I + II. Berlin: Springer Spektrum.

Schindler, M., & Lilienthal, A. J. (2019). Domain-specific interpretation of eye tracking data: towards a refined use of the eye-mind hypothesis for the field of geometry. *Educational Studies in Mathematics*, *101*(1), 123–139.

Schreiter, S., Vogel, M., Rehm, M., & Dörfler T. (2021): Teachers' diagnostic judgment regarding the difficulty of fraction tasks: A reconstruction of perceived and processed task characteristics [Special Issue]. *RISTAL, 4*, 127-146.

Südkamp, A., Kaiser, J., & Möller, J. (2012). Accuracy of teachers' judgments of students' academic achievement: A meta-analysis. *Journal of Educational Psychology*, *104*(3), 743-762.

Sweller, J., Ayres, P., & Kalyuga, S. (Eds.). (2011). *Cognitive Load Theory*. New York, NY: Springer New York.

NOVICE FACILITATORS' PARTICIPATION PRACTICES IN DISCUSSIONS ABOUT ISSUES OF MATHEMATICS TEACHING

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Many professional development environments (e.g., courses, teacher communities) center on discussions about pedagogical and mathematical practices. These discussions usually go beyond "rights and wrongs" towards the formation of reflective arguments through interactions and dialogues. Thus, the productiveness of these discussions largely depends on facilitation skills, which only recently have become the object of thorough scrutiny. This study focuses on one aspect of facilitation of discussions: do the facilitators envision their role as a neutral "guide on the side" or rather as a participating colleague, or somewhere in between? The consideration of this question raises important theoretical and practical implications for the professional development of facilitators.

BACKGROUND, RATIONALE, AND RESEARCH QUESTIONS

Due to their crucial role in the dissemination of educational endeavors, there is a growing interest in facilitators of professional development (PD) courses for practicing mathematics teachers. Thus far, research has focused on facilitators' preparation (e.g., Maass & Doorman, 2013), the knowledge they require (e.g., Even, 2005;), and their practices (e.g., Coles, 2019; Karsenty et al., submitted; Prediger & Pöhler, 2019; van Es, Tunney, Goldsmith, & Seago, 2014). This study is intended to shed further light on a specific issue of facilitation, stemming from the common situation when PD facilitators are former or present proficient mathematics teachers. Though the knowledge and practices associated with teaching and facilitation may resemble and partially overlap, the knowledge required from facilitators "go beyond and look different than the knowledge that a typical mathematics classroom teacher holds" (Borko, Koellner & Jacobs, 2014, p. 165). For example, facilitators should skillfully lead in-depth discussions on issues of mathematics *teaching,* where actual problems of practice are surfaced and discussed, and create a safe place for teachers to reflect on practices, re-inspect them, and perhaps consider changing them (Horn, 2010). Leading such discussions is a challenge of facilitation in general, and even more so for novices who are not sure how to steer the conversations (Borko et al., 2014) and how directive they should be (Lewis, 2106). This challenge is also rooted in novice facilitators' multiple professional identities as facilitators, teachers, and colleagues (Knapp, 2017; Schwarts, 2020). The phenomenon of "expert become novice" (Murray & Male, 2005, p. 135), where experienced teachers are new to the norms and practices of facilitation, can lead them to feel "de-skilled" (ibid, p. 129). One of the issues that evolve in this

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situation, which has not yet been empirically explored, is how novice facilitators position themselves in PD discussions, when it comes to expressing or concealing their own opinions on issues of mathematics teaching. Most often, in the mathematics classroom, teachers are perceived as those who have the authority to determine which statements in the discussion are correct and which are not. This perception stems from both the inherent power relationship between teacher and students and the unique nature of the discipline of mathematics, in which every statement is perceived to be classifiable as right or wrong (Chazan, Callis & Lehman, 2009). In PD settings, however, it is less applicable to assign "correctness" to arguments in a discussion. Rather, PD discussions revolve around values, preferences, and orientations about teaching and learning mathematics, and different approaches do not necessarily converge into consensual stances. In other words, PD discussions are a means to deepen teachers' reflective views on their practice, and are not necessarily aimed at reaching closure, conclusions, or agreements. PD facilitators must therefore choose how to participate in such discussions and decide if and how their mandate as leaders allows them to expose their own opinions when working with PD participants. These decisions are at the core of establishing their new professional identity as beginning facilitators. Thus, the questions addressed in this report are the following:

How do novice facilitators, who are also mathematics teachers themselves, participate in PD discussions in terms of sharing their own opinions? How, if at all, do their participation practices change during the first year of facilitation?

CONTEXT

The above questions are explored in the context of a PD project called VIDEO-LM (Viewing, Investigating, and Discussing Environments of Learning Mathematics), developed at the Weizmann Institute of Science in Israel. The program aims to enhance secondary mathematics teachers' reflective skills, along with their mathematical knowledge for teaching (MKT; Ball, Thames & Phelps, 2008), via collective guided analysis and discussions of videotaped lessons of unfamiliar teachers. A six-lens framework (SLF) is used to focus participants' observations and analysis of the video (Karsenty & Arcavi, 2017). SLF includes *mathematical and meta-mathematical ideas* in the lesson; the filmed teacher's *goals*; the *tasks* used; the *interactions* in the lesson; the filmed teacher's *dilemmas and decision-making*; the filmed teacher's *beliefs*. An SLF-guided discussion shuns judgmental comments, and if they arise, for example when participants criticize a teacher's action, the facilitator redirects the conversation using questions such as "assuming the teacher acted in the best interests of her students, what could have been her goal for this action?". It follows that a prominent role of VIDEO-LM facilitators, besides choosing a video and appropriate lenses for each PD session, is to lead reflective discussions that shift evaluative comments into "issues to think about" (ibid, p. 438). In such discussions, teachers may reach insights about mathematics teaching that can catalyze reflective processes (Schwarts & Karsenty, 2018). During the project's upscaling, new facilitators were recruited, most of whom were experienced teachers who previously participated in a VIDEO-LM PD. The novice facilitators were prepared in a one-year course and were supported by a personal mentor and by facilitators’ group meetings during their first year of facilitation.

METHOD

This study is part of a broader research project, consisting of a multiple case-study investigating 7 novice facilitators during their first year of practice. Inspired by the notion of *problem-driven research* (Arcavi, 2000), the current report focuses on research questions that emerged during the data analysis.

Participants and settings.

We focus here on 5 novice facilitators (4 women and 1 man), all of whom led VIDEO-LM courses in the school where they teach. These yearly courses consisted of 21-30 hours, spread over 6-10 biweekly/monthly sessions.

Data collection.

To attain a broad picture of the facilitators’ practices, the following data were gathered: (1) pre- and post-facilitation questionnaires (beginning and end of year); (2) journals written by facilitators before and after each PD session; (3) videos of two PD sessions per facilitator, one early in the year and another towards the end of the PD; (4) videos of stimulated-recall interviews (SRIs) held with every facilitator a few days after each of the filmed PD sessions. In these SRIs the facilitator and the first author jointly watched the PD videos, and the facilitator was asked to stop the video whenever s/he noticed a decision or an instant to reflect on.

Framing the data analysis in the broad study: Searching for *issues of facilitation*.

To understand the nature of facilitation and the challenges it poses, we focus on macro-level descriptions of facilitation which we term *issues of facilitation,* in accordance with VIDEO-LM’s focus on *issues of mathematics teaching*. We define an issue of facilitation as a question, problem, phenomenon, or dilemma, which occurs during facilitation. Facilitators facing these issues must make decisions and take actions accordingly. Through an iterative interpretation of the entire corpus of data, 6 issues of facilitation were defined, refined, and validated in a 3-year process of analysis, observations, and discussions of our research group and a German group from Dortmund University who conducted parallel research using similar methods. This paper focuses on one of these issues – how facilitators participate in the discussions they lead, in terms of sharing their opinions.

Coding of the relevant data and identifying practices.

The basis for the following analysis is the data coded as “sharing opinion”. Employing thematic analysis (Brown & Clarke, 2006), five different practices were identified. The analysis includes two interconnected layers: the first is how facilitators act in sessions, and the second is what they say or write about their actions, i.e., how they interpret and reflect on their actions.

Searching for facilitators' changes in practices.

To understand changes in facilitation, facilitators' practices at the beginning and end of the year were compared. Although facilitators may perform different practices during one session, each of the two documented sessions of each facilitator was assigned to one of the five practices (see below) according to the most common practice identified in the session. Cases in which facilitators changed their behavior, or alternatively behaved in the same way but stated different reasons for this choice, were examined to pinpoint catalysts of change.

FINDINGS

We identified five practices associated with the issue of sharing an opinion: Neutrality, Emergence, Explicitness, Interweaving, Protectiveness. Below, we provide a short account of each practice and illustrate it with representative excerpts. Our basic premise is that every decision made by the facilitator carries a particular agenda, and indeed, opinions can be expressed indirectly, for example in choosing a question for discussion or a video to watch. However, we focus on the explicit expressions of the facilitators' opinions while managing discussions.

The practices. Figure 1 illustrates the identified practices (marked below as P1, P2, etc.). The letters A-E denote the five facilitators and the numbers "1" and "2" refer to the beginning and end of the first facilitation year, respectively. The position of circles under a certain practice indicates the prevailing practice in that session.

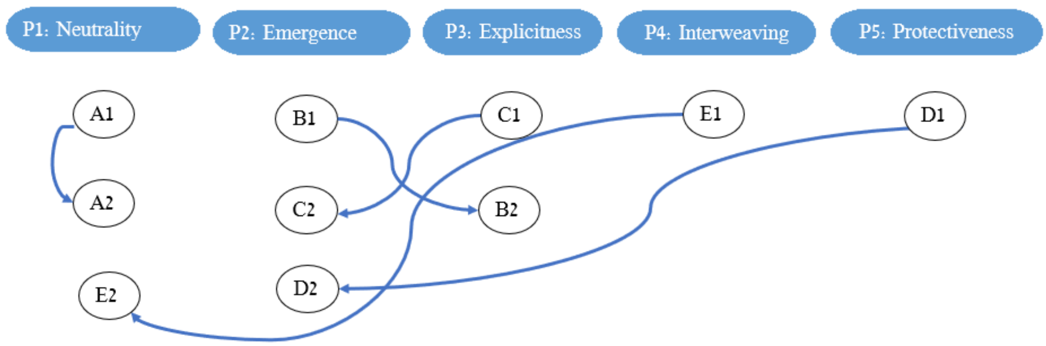

Figure 1. Practices of facilitators' participation in PD discussions

We now present the practices using examples from the data. Most of the excerpts are taken from interviews (labeled "SRI" with the transcript line), due to space limitations.

P1: Neutrality. This practice is characterized by the facilitator's complete avoidance of expressing personal opinions during discussions. Alternatively, the facilitation consists of asking questions that do not imply the facilitator's position, as Facilitator A described:

> My whole facilitation is made up of questions, I only ask questions, [...] even when they answer something, I reply with a question […] I'm not stating uh... my opinion or... What I think (SRI-A1-64, 68).

Facilitator A interprets his role of a VIDEO-LM facilitator as someone who should not take sides, someone whose job is to choose the PD activities and the questions to be asked. Nevertheless, he questions this perspective, framing it as a conflictual decision:

> I oppose to what Teacher H said, [but] I did not want a confrontation [...] If it was coming from the group, then fine, but if not, then no, I do not intend to express my personal opinion here [...] I am not sure what my role as a facilitator is, am I participating? The very fact that I come and bring up questions that interest me and raise issues that interest me. This is my conflict within the group, between a facilitator and a participant (SRI-A2-113).

The practice of staying neutral may have reasons other than avoiding confrontation or side-taking. For example, facilitators may use it as a strategy to elicit teachers' opinions freely, rather than in response to the facilitator's opinion. This was evident in the case of Facilitator E, who serves as head of the mathematics department in her school. Before she started the PD, she expressed her aim to encourage colleagues not to be intimidated by her position of authority:

> I want the teachers to participate, to speak, and that they will not be afraid to ask questions or reveal lack of knowledge (Pre-questionnaire).

P2: Emergence. Sometimes facilitators hold a position they want to promote without stating it explicitly, but rather by having it emerge collectively through the discussion. This practice is in line with the constructivist spirit that learners (in this case teachers) should be encouraged to articulate their own issues rather than receive them from an external authority. In the *emergence* stance (unlike in the *neutrality* discussed above), facilitators use subtle but directive moves to prompt an opinion or alternatively to counter another. The emergence practice was found, for example, in the case of Facilitator D toward the end of the year. It can be detected in her reflection below, on a facilitation move she enacted in response to a participant's criticism of the teacher in the observed video:

> I do not agree with what he said. And yet I asked, okay, so what's the gain? [in the filmed teacher's action]. And I'm glad I learned that subtle move. Like, okay, so he did that. So *why* did he do that? (SRI-D2-152).

This short excerpt reveals Facilitator D's opinion ("I do not agree with what he said") and how she addressed this disagreement. Rather than arguing or saying that she holds a different view, she used a learnt SLF discursive tool (“that subtle move”) to turn the disagreement into an issue for everybody to discuss (“So *why* did he do that?”).

P3: Explicitness. As the name of this category implies, this practice is characterized by explicit expressions of opinions by facilitators. For example, Facilitator C explains:

> When I ask questions, I first want to hear them. Then, if I have something that can add to the discussion, I think, like, if I was now sitting with the teachers around the table, I would also tell this story, I wouldn't have avoided it […] I feel that I shouldn’t refrain from expressing myself because I'm 'only a facilitator' and I'm 'only listening' (SRI-C1-33).

The multiple identities of facilitator C led her to participate in discussions both as a facilitator and as a colleague, thus she intentionally shared her views during the PD. In the case of Facilitator B, a similar decision was taken, but only towards the end of the year: as a teacher of advanced mathematics classes, at the beginning of the year she was reluctant to share her views about how to teach struggling students. Her decision-making changed owing to the encouragement of her mentor:

> I felt that with the struggling students, my teachers are the authority [...] they already have their knowledge and their expertise and I am not there, [so I] don't bring up my agenda. [And] Then, I had a conversation with Y. [the mentor], and she kind of "let" me, she thought I was "allowed" to say [what I think] and that I should bring that side into the school. Not to be only a moderator, but to share my input (SRI-B2-2, 4).

P4: Interweaving. This practice is manifested as the facilitator, in a casual yet deliberate way, interweaves normative statements while managing discussions. Rather than being explicit, the facilitator does not express a specific opinion directly; the opinion posed is blended into the discussion. One example can be seen in the following excerpt from a PD session led by Facilitator E at the beginning of the year, when the group watched a Calculus videotaped lesson:

> We all teach Calculus [...] and I think we all, I can say for myself for sure [...] we also got to teach [...] in a technical way [...] This lesson was clearly designed for an advanced class. Can we take something from it to other classes? (PD Session-E1-329).

The excerpt demonstrates how the facilitator interweaves her approach, namely, *teaching should not be done in a technical way*. Moreover, her question ("can we take something from it to other classes?") pre-assumes that teachers *need* to take something from the videotaped lesson. This practice may stem from Facilitator E's role as head of the mathematics department, and in general, from facilitators' desire to nurture clear values regarding acceptable ways of teaching. Alternatively, it may be rooted in a facilitators' belief that their own opinion is a convention shared by the rest of the group.

P5: Protectiveness. Observing videos of unfamiliar teachers is an activity that can elicit judgmental comments from viewers. To overcome this, the VIDEO-LM design of PD sessions utilized SLF in order to focus discussions on *issues*, as described earlier. However, participants still sometimes criticize the filmed teachers, which may lead facilitators to feel obliged to "protect" them. In such situations, the facilitator's opinion is communicated via this defensive reaction. Facilitator D reported:

> I had the need to protect the filmed teacher; that, too, is one of my problems. Instead of looking at it, not in this narrow way, [...] but more broadly [by asking] 'what do you think?', 'let's say the lesson is teacher-centered, what's the problem with a teacher-centered lesson?' 'did she achieve her goals, did she not achieve goals?' 'what goals?' 'how do you achieve your goals?' And develop [in the discussion] what is possible [to do]. But, I always feel the need to protect, to say – "you didn't watch the entire lesson". (SRI-D1-238)

The facilitator mentioned different questions she could have asked to redirect the criticism into a discussion of issues. Yet, she admits that her "need to protect" was stronger. For her, this practice changed by the end of the year (see P2 above, also the change D1⇨D2 in Fig. 1), following a facilitators' meeting where she received a direct feedback on this matter from her colleagues.

DISCUSSION

The analysis of novice facilitators' decisions regarding how they position themselves during PD discussions, reveals some of the challenges they confront as part of the epistemological shift embedded in their transition of roles – from teaching mathematics to facilitating discussions about mathematics teaching. Two findings of this study stand out: First, that facilitators' participation practices during discussions are notably diverse, and second, that the same practice can result from different and even conflicting reasons. The comparative analysis between beginning and end of year shows that most of the novice facilitators changed their participation practices in discussions (Fig. 1). These changes may result from different situational contexts, yet the findings indicate that at least some of them were intentional. Factors found as catalysts for change were associated with the support that facilitators received and the opportunities provided to reflect and refine their interpretation of the facilitator's role. We argue that a change in practices may be manifested in an informed and flexible decision-making process, taken after considering additional options for action, rather than in moving from one participating strategy to a "better" one. Further longitudinal investigations on facilitators can shed more light on evolutions in their practices and their underlying reasons.

Although the current study was based on a unique PD, it offers insights to other teacher education programs that focus on discussions among teachers. The contribution of this study is in unpacking another layer of the complexity that discussion management poses for new facilitators, in addition to those already recognized (Borko et al., 2014; Lewis, 2016), and in surfacing an issue that novice facilitators are concerned with, especially when establishing their new professional identity. Awareness to the space of possibilities spanning a facilitation issue may assist novice facilitators to choose a cognizant rather than spontaneous course of action. This awareness is crucial also for facilitators in training and for those who prepare facilitators and support them.

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References

Arcavi, A. (2000). Problem-driven research in mathematics education. *The Journal of Mathematical Behavior*, *19*(2), 141-173.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, *59*(5), 389–407.

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of professional development. *Journal of Mathematical Behavior, 33,* 149–167.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101.

Chazan, D., Callis, S., & Lehman, M. (2009). *Embracing reason: Egalitarian ideals and the teaching of high school mathematics*. Routledge.

Coles, A. (2019). Facilitating the use of video with teachers of mathematics: Learning from staying with the detail. *International Journal of STEM Education*, *6*(1), 1–12.

Even, R. (2005). Integrating knowledge and practice at MANOR in the development of providers of professional development for teachers. *Journal of Mathematics Teacher Education*, *8*(4), 343–357.

Horn, I. S. (2010). Teaching replays, teaching rehearsals, and re-visions of practice: Learning from colleagues in a mathematics teacher community. *Teachers College Record, 112,* 225–259.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education*, *20*(5), 433-455.

Karsenty, R., Pöhler, B., Schwarts, G., Prediger, S. & Arcavi, A. (Submitted). Processes of decision-making by mathematics PD facilitators: The role of resources, orientations, goals and identities.

Knapp, M. C. (2017). An autoethnography of a (reluctant) teacher leader. *The Journal of Mathematical Behavior, 46,* 251–266.

Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM – The International Journal on Mathematics Education, 48*(4), 527-540.

Maass, K., & Doorman, M. (2013). A model for a widespread implementation of inquiry-based learning. *ZDM – The International Journal of Mathematics Education, 45*(6), 887-899.

Murray, J., & Male, T. (2005). Becoming a teacher educator: Evidence from the field. *Teaching and Teacher Education*, *21*(2), 125-142.

Prediger, S., & Pöhler, B. (2019). Conducting PD discussions on language repertoires: a case study on facilitators' practices. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of 43rd Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 241–248). Pretoria, South-Africa: PME.

Schwarts, G. (2020). Facilitating a collaborative professional development for the first time. In H. Borko & D. Potari (Eds.), *Teachers of Mathematics Working and Learning in Collaborative Groups, Proceedings of the 25th ICMI Study Conference* (pp. 540-547). Lisbon, Portugal: ICMI.

Schwarts, G. & Karsenty, R. (2018). A teacher's reflective process in a video-based professional development program. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 123-130). Umeå, Sweden: PME.

van Es, E., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education*, *65*(4), 340–356.

PROSODIC REPERTOIRE OF MULTILINGUAL STUDENTS IN A NEW ZEALAND GEOMETRY CLASSROOM

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The paper explores 9 to 11-year-old multilingual students' discursive constructions of geometric shapes and their properties as they engage in classroom interactions in a New Zealand classroom. Undertaking the ethnomethodological approach, classroom interactions are construed as practical actions that facilitate the process of meaning-making of geometric shapes. Transcribed data of one excerpt from six audio-video recorded geometry lessons are presented in this paper. The analysis revealed that the intonational features of language use play a crucial role in making sense of geometric shapes as participants engage in classroom interactions.

INTRODUCTION

In present times, mathematics classes are becoming superdiverse in nature (Barwell et al., 2019). Research examining multilingualism and the role of language in mathematics education has often focussed on mathematical terminology (Adler, 2002; Moschkovich, 2007), as well as grammatical patterns of mathematical registers (Kotsopoulos, Cordy, & Langemeyer, 2015; Schleppegrell, 2007) for developing mathematical understanding. However, Ward (2019) argued that language is more than vocabulary and syntax. He explained that patterns of stress and intonation in language provide impact to the words, their meanings, and their social significance. The complexity of these prosodic patterns may increase with the presence of a variety of languages in a mathematics classroom. It is the ordered nature of indexicality of the language use that enables us to interpret what is said beyond the meanings of individual words (Barwell, 2003). Therefore, to acknowledge the existence of various languages in contemporary mathematics classes, this paper undertakes an ethnomethodological approach to explore how multilingual students discursively construct their understanding of shapes as they engage in classroom interactions.

THEORETICAL FRAMEWORK

As an ethnomethodological approach focuses on how reasoning and activities are organised as rational, identifiable events and occurrences within a culture (Heap, 1984). It aims to provide a detailed description of how members make sense of any event as it unfolds in its everyday manner. The circumstantial unfolding identifies the common norms underlying any practical action (such as request, command) that is "reflexively constitutive of the activities and unfolding circumstances to which they are applied" (Heritage, 1984, p. 109). The reflexive accountability of the action marks the practical reasoning that allows the construction of a world where the

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 33-40. Khon Kaen, Thailand: PME.

actors' actions are taken as accountable, intelligible, and sustainable in the course of development. This reflexive accountability of everyday actions also accounts for the deviations that occur from those normative actions.

In addition to accountability, Garfinkel and Sacks (1986) argue that reflexivity of participants' language use can be investigated using indexical properties of natural language. Indexical properties draw our attention to not only what is said but also to how it is being said. The characteristic of how something is said calls for an alternative understanding of language as practical action. Garfinkel (1996) argues that the understanding of an utterance is contingent upon the context in which it is being said. The context involves the background information about who said the utterance, where and when, in addition to what has been accomplished by making that utterance in light of other alternative utterances that could be made (Heritage, 1984). Moreover, a clear and detailed ethnomethodological account of activities provides the concrete evidence through which the cultural practices are displayed in the conversations as the participants engage in interactions. Therefore, the ethnomethodological approach is concerned with how people achieve interpersonal understanding through language. As a result, speaking/interactions in ethnomethodological studies are taken as a practical action capable of transformation. For this paper, the ethnomethodological approach provides moment-by-moment elicitation of classroom practices to explore the processes that contribute to meaning construction of shapes and their properties.

METHODOLOGY

This paper is part of a larger study that explores 9 to 11-year-olds multilingual students' discursive constructions of shapes and their properties in a New Zealand primary classroom. Six geometry lessons on shapes and their properties in one Year 5/6 classroom were observed, and field notes were taken. Informed and voluntary consent to participate in the study was sought from the participants following the ethical approval gained from the University of Waikato Division of Education Ethics committee. Participants included 15 students and their teacher. The school catered to the multilingual student population. Nine out of 15 students were multilingual. Data pertaining to students' languages were collected using a small questionnaire that was filled in by the parents.

The observed six lessons were also audio-and video recorded using two directional cameras, one eye gear, and five audio-recorders. One camera was kept in the front of the classroom, one at the back. The researcher also wore an eye-gear with an inbuilt camera to record a moment of interaction that caught the researcher's attention. The five audio-video recorders were kept on the tabletops to record students' interactions as they worked on group tasks. Each lesson lasted for 45 to 50 minutes each.

In addition to audio-visual data, three semi-structured teacher interviews were conducted. The interviews were audio-recorded. The purpose of the interviews was to seek clarifications about the lessons or activities that were used in the classroom. Each interview lasted 10-12 minutes. The interview was audio-recorded, transcribed and given back to the teacher for member checking. Focus group interviews with four groups of students (four students in each group) were also conducted after the unit on shapes had been taught. Each focus group interview lasted for 18 to 20 minutes. Students' work samples were also gathered.

This paper presents the interaction of one of the groups observed during the first lesson observed at a New Zealand school. The data presented in this paper is representative of the audio-visual data. During this group interaction, the group was engaged in identifying the shapes that they already knew in a task sheet with a picture from everyday life (see Figure 1). The task sheet was provided by the teacher. The group interaction was transcribed using a simplified version of Jefferson (2004) transcription conventions (see Appendix A for transcript key). The use of a few selected Conversation Analytic (CA) techniques enabled the researcher to analyse participants' use of prosody (intonation patterns including use of pitch, pauses, and volume), and its role in contributing to the process of meaning construction.

FINDINGS AND DISCUSSION

During this lesson, the teacher grouped students in groups of three to four students and provided each group with a task sheet to work collaboratively. The excerpt presented in this section displays a part of group discussion (due to limited space) of three 9 to 11-year-old multilingual students- Ozan, Tahi, and Garry. Data from questionnaire informed about the languages and ages of the student participants. Ozan is a male, 9-year-old, bilingual Somali student with beginner's proficiency in English. Tahi is a male, 11-year-old, Tongan student, with English as his second language; and Garry is a male, 11-year-old, Philippine student with advanced proficiency in English and Filipino/Tagalog as his second language. During this group interaction, Ozan identified a shape in picture A (see the circled shape in Figure 1), but could not remember the name of the shape.

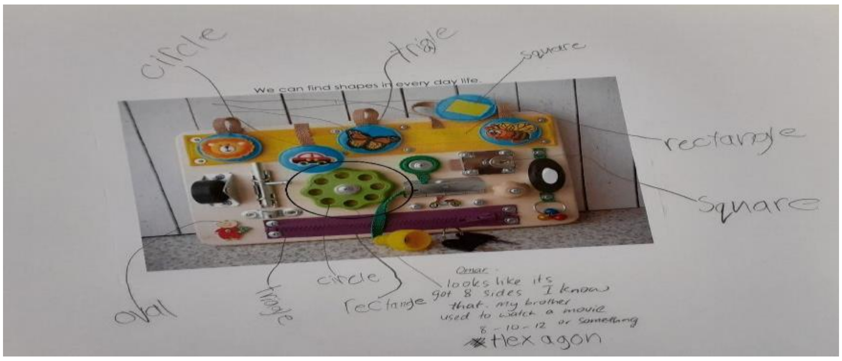

Figure 1. Work sample of a group with students Ozan, Tahi, and Matiu

The excerpt is selected as it displayed the use of prosody of multilingual speakers with different heritage languages.

205 Garry what sha:pes can you see <u>right</u> now

206 Tahi circ::les (1.5) squa:res

207 ((Garry takes the picture sheet and turn it over to put glue to paste it on large
208 white sheet as Tahi was still looking at it))
209 Ozan I see a lot of circles over there (3.0)
210 ((Ozan looks at the sheet while Garry and Tahi make faces towards the
211 camera))
212 Ozan okay(.) what is this shape called ((pointing to shape))
213 Garry ↑so ↑whats tha:t whats [that Tahi?
214 Tahi [squa::re
215 Garry thats a ↓rectangle
216 Tahi #square#
217 Garry then Ill say squa::re
218 Tahi ↑Squ°are::°(.8) °thats° a square
219 ((Garry writes square as Tahi speaks))
220 Ozan oh ↑I ↑SEE [One
221 ((Ozan looks at Garry who was given with the responsibility to write))
222 Tahi [he::re ((Tahi points to different shape and laughs))
223 Ozan THIS ONE ((points again to the shape))
224 Garry wha:ts that
225 Ozan I dont know what[it is called
226 Tahi [°circle thats a circle°
227 Garry cir(.)cle
228 Ozan not °this° (2.0) ((put his hand to his head to show that
229 it is not the shape that he was talking about))
230 I am talking about whole thing, like like (2.0)
231 ((drag his finger at the shape to show his imagination of sides))
232 (in jacks) (.5)what was it (2.0) [it=
233 Tahi =ohh (.) °I know there is this thingy like
234 this° ((points to another shape))
235 [theres like ((makes the shape with his finger on the sheet to
236 show the shape he implies))
237 Garry [there is: no thingy (you images)

Garry selected himself as the first speaker (line 205) and constructed his turn to put a question directed towards the other two group members. He could have identified a few shapes. However, he posed the question "what shapes can you see right now" (line 205). During the classroom observation (Fieldnotes, Lesson 1), it was noted that the teacher had explicitly stated that Garry would help Tahi and Ozan to write the names of the shape. During the first semi-structured interview conducted after the first lesson was observed, the teacher was asked why she asked Garry to help

Tahi and Ozan in writing the names of the shapes. The teacher informed that Garry was better English speaker than Ozan and Tahi as well as one of the high achievers in mathematics. Though Garry had not selected the next speaker, Tahi selected himself (line 206) and stated "circles" and "squares". He stretched the words circles and square along with a long pause of 1.5 seconds (line 206). It seems that Tahi used stretching as a way to hold the speaking floor, while Garry was writing these shape names on the task sheet. Tahi did not select the next speaker in his utterance; however, Ozan self-selected and stated that he saw circles in the picture (line 209). Through his utterance, Ozan stated that there are many circles in the task sheet. Ozan paused for three seconds after finishing his utterance. It is probable that he was waiting for Garry and Tahi to respond to his utterance. Schegloff (1982) has shown that speakers may fall silent in lack of supportive feedback in the form of the backchannel. Garry and Tahi did not respond to Ozan's last utterance; he again self-selected and pointed to the shape that he had identified (line 212). Ozan stressed 'what' and 'shape' as he used a slightly higher volume than the rest of the sentence to mark the focus of his utterance. In the Somali language, the focus of the utterance is often marked by the use of stress on the focus constituent in a particular utterance (Biber, 1984). It is possible that Ozan stressed the words to structure his utterance as a question without using rising intonation, which is often used to mark the statement as a question in the English language. Ignoring Ozan's utterance, Garry self-selected and again posed another question (line 213). This time, he selected Tahi as the next speaker. Tahi claimed that the shape that Garry referred to is square (line 214), a slight increase in volume at the syllable of "qua::" may be interpreted as a signal to the specificity of the shape, by stressing a particular syllable in a square. In Tongan intonation patterns, when a noun is made definite, the main stress falls on the second-last vowel (Anderson & Otsuka, 2006; Condax, 1989). That is, Tongan speakers directly feed the grammatical structures into the prosody they use instead of using definitive articles such as "the" as in the case of English (Ahn, 2011). Garry claimed that the shape was a rectangle with his low pitch (line 215), where he used the article "a" with a rectangle. Ward (2019) informs us that in English, speakers often use low pitch or creaky voice to show their authority over knowledge. Thus, it can be stated that Garry perhaps used his low pitch to show his dominance over knowledge. However, Garry's claim was rejected by Tahi in the next utterance (line 216). It is noteworthy that this time, Tahi used his creaky voice to claim his knowing.

Acknowledging Tahi's authority over his knowledge, Garry (line 217) accepted Tahi's claim and wrote the shape as a square. Tahi, in his following utterance, continued his claim (line 218). While the activity required students to discuss the shapes, it seems that Tahi and Garry were not considering Ozan's point of view to decide if the shape was a square or a rectangle. This act of neglecting Ozan's idea could be because Ozan is new to the class and has limited proficiency in English. Moreover, Ozan did not seem to bother about this; instead, he again attempted to draw his fellow students' attention to the shape (see the circled shape in Figure 1, green coloured object) that he identified (line 220). Tahi again self-selected and overlapped his talk with Ozan (line 222). Tahi pointed to a different shape that he had identified and noted on the sheet. Noticing that he was losing Garry's attention to his shape, Ozan used a loud voice and again pointed to his shape (line 223). Couper-Kuhlen (2004) argues that participants use loudness as a prosodic marker to mark the current turn as a new course of action. Thus, in this case, Ozan probably used a loud voice to begin a conversation about a new shape, instead of continuing the ongoing conversation about the different shape. However, when Garry (line 224) asked him about the name of the shape, Ozan claimed that he did not know the name of the shape. Hearing that Ozan did not know the name of the shape, Tahi again self-selected and claimed that the shape was a circle (line 226). Garry provides his agreement with the Tahi's statement that the shape is a circle (line 227). Ozan (line 230) again attempted to direct their attention to the shape as he dragged his finger on the shape to show them. Interestingly, in line 233, it seems that Tahi realised that there is such a shape. However, when Garry dismissed the possibility of such a shape (line 237), both Ozan and Tahi went along with Garry.

The analysis of this excerpt may suggest, one, that multilingual students may use a variety of prosodic features from their linguistic repertoire of different languages, and second, bilingual or multilingual students may perceive these prosodic cues differently in comparison to the English-speaking students. Therefore, the study suggests that the practice of language by multilingual speakers involves the use of prosodic features from their repertoire of multiple languages along with the words used; instead of just engaging in the practice of code-switching. The research focused on language as a resource perspective often ignores the role of the prosodic repertoire that contributes to the meanings conveyed through the utterances. Such a perspective may view English medium primary classroom such as one presented in this study as a monolingual classroom setting. Instead, the findings of this study support that the overt or covert presence of multiple languages in any classroom makes it a multilingual context (Barwell et al., 2016). The study adds to the knowledge base exploring multilingualism in mathematics education. It also suggests a need for further research exploring the use of prosodic features as part of language in classrooms where the utterances are apparently monolingual. Therefore, the present study calls for moving away from the monolingual bias evident in the teaching and learning of geometry concepts and mathematics in general (Charamba, 2020).

CONCLUSION

The paper aimed to explore 9 to 11-year-olds multilingual students' discursive constructions of geometric shapes in a primary school in New Zealand. An excerpt from audio-visual data from a larger study is analysed using selected conversation analytic techniques. The analysis drew attention to the role of subtle yet significant prosodic features that multilingual students may use to construct meaning during classroom interactions. The analysis revealed that multilingual speakers may use prosodic features from their repertoire of multiple languages. The study has the potential to contribute to the knowledge base in geometry education research. Moreover, the study suggests that the awareness of prosodic features may aid in developing a diverse knowledge base for teachers and researchers in developing effective teaching and learning practices.

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Appendix A: Transcript key

Convention	Use	Convention	Use
[text]	Overlapping speech	:	Stretch
(.)	Indicates the silence of one-tenth of a second	(text)	Unclear speech
(0.n)	n represents the n tenth of a second	> <	Faster speech
?	Rising pitch	=	Latching
↑	High Pitch	.	Falling tone
°	Whispering	((text))	context
Underline	Emphasis	#	Creaky voice

References

Adler, J. (2002). *Teaching mathematics in multilingual classrooms*. Dordrecht, Netherlands: Springer.

Ahn, B. (2011). Tongan relative clauses at the syntax-prosody interface. *Proceedings of the eighteenth meeting of the Austronesian Formal Linguistics Association (AFLA)*, 1-15. doi:https://ir.lib.uwo.ca/cgi/viewcontent.cgi?article=1033&context=afla

Anderson, V., & Otsuka, Y. (2006). The phonetics and phonology of" definitive accent" in Tongan. *Oceanic Linguistics, 45*(1), 21-42.

Barwell, R. (2003). A discursive psychology approach to the analysis of attention in mathematics classroom interaction. *Research in Mathematics Education, 5*(1), 3-15. doi:10.1080/14794800008520111

Barwell, R., Bose, A., Moschkovich, J., Phakeng, M., Planas, N., & Prediger, S. (2019). Researching mathematics education and language diversity: The theories we use and develop, and what they enable us to see, say or do. In M. Graven & H. Venkat (Eds.), *Proceedings of the 43rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 85-106). Pretoria, South Africa: PME.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., . . . Villavicencio Ubillús, M. (2016). Introduction: An ICMI study on language diversity in mathematics education. In R. Barwell, P. Clarkson, A. Halai, M. Kazima, J. Moschkovich, N. Planas, M. Setati-Phakeng, P. Valero, & M. Villavicencio Ubillús (Eds.), *Mathematics education and language diversity: The 21st ICMI study* (1st ed. 2016.. ed.). Cham, Switzerland: Springer International Publishing.

Biber, D. (1984). Pragmatic roles in central somali narrative discourse. *Studies in African Linguistics, 15*(1), 1-26.

Charamba, E. (2020). Pushing linguistic boundaries: translanguaging in a bilingual Science and Technology classroom. *Journal of Multilingual and Multicultural Development*, 1-15.

Condax, I. D. (1989). Tongan definitive accent. *The Journal of the Polynesian Society, 98*(4), 425-450.

Couper-Kuhlen, E. (2004). Prosody and sequence organization in English conversation. In *Sound patterns in interaction* (pp. 335-376).

Garfinkel, H. (1996). Ethnomethodology's program. *Social Psychology Quarterly, 59*(1), 5-21.

Garfinkel, H., & Sacks, H. (1986). On formal structures of practical actions. In *Ethnomethodological studies of work* (pp. 160-193): Routledge & Kegan Paul plc.

Heap, J. L. (1984). Ethnomethodology and education: Possibilities. *The Journal of Educational Thought, 18*(3), 168-171.

Heritage, J. (1984). *Garfinkel and ethnomethodology*. New York, N.Y.: Polity Press.

Jefferson, G. (2004). Glossary of transcript symbols with an introduction. *Conversation Analysis: Studies from the First Generation, 125*, 13.

Kotsopoulos, D., Cordy, M., & Langemeyer, M. (2015). Children's understanding of large-scale mapping tasks: an analysis of talk, drawings, and gesture. *ZDM, 47*(3), 451-463. doi:10.1007/s11858-014-0661-4

Moschkovich, J. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*(2), 121-144.

Schegloff, E. A. (1982). Discourse as an interactional achievement: Some uses of ‘uh huh’and other things that come between sentences. *Analyzing Discourse: Text and talk, 71*, 93.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly, 23*(2), 139-159.

Ward, N. G. (2019). *Prosodic patterns in English conversation*. Cambridge, MA: Cambridge University Press.

EXPLORING SECONDARY SCHOOL PRE-SERVICE TEACHERS' EXPERIENCES AND PERCEPTIONS OF TEACHING AND LEARNING PROBABILITY: AN INTERNATIONAL COLLABORATION

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There is a considerable and rich literature on students' intuitions and misconceptions in probability. However, less attention has been paid to the development of students' probabilistic thinking in the classroom. Based on literature, the first author developed a lesson sequence for teaching probability (see First Author, 2015). We decided to trial the lesson sequence with our junior secondary pre-service teachers. Our study was conducted in two phases. Phase one focused on exploring pre-service secondary teachers' experiences and perceptions on teaching and learning probability. Semi-structured interview data indicates that while some teacher experiences were similar, there were also some differences.

BACKGROUND

Probability is the measurement of uncertainty that is omnipresent in our everyday life situations. Moreover, probability straddles a number of disciplines (physics, economics and sciences) because of its wide range of applicability. In recognition of the importance of probability in both school and out of school settings, there has been a movement in many countries to include probability at every level in the mathematics curricula (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016). In western countries such as New Zealand (Ministry of Education, 2007) these developments are reflected in official documents and in materials produced for teachers. Probability is one of the three sub-strands in the curriculum document and seen as critical in the learning of mathematics. The use of meaningful contexts and drawing on students' experiences and understandings is recommended for enhancing the students' understanding of probability (Ministry of Education, 2007). Similarly, in the Pacific education context, some countries, such as Fiji, have included probability and statistics as an important aspect of mathematics curriculum from the early years of the school curriculum. (Fiji Ministry of Education, Heritage & Arts, 2017).

Given a strong presence of probability content in our school curriculum and importance of reflective practice (Batanero et al., 2016; Koparan, 2019), this study sought to explore pre-service secondary mathematics teachers experiences of learning probability and their views on how best to teach probability. The following research questions are answered in this paper:

What are pre-service mathematics teachers' experiences and perceptions of learning and teaching probability?

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 41-48. Khon Kaen, Thailand: PME.

Do pre-service mathematics teachers support the use of games in learning and teaching probability?

RELATED LITERATURE

There are different viewpoints on how best to teach probability so that students leaving school may be able to interpret probabilities in a wide range of situations (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016). If students are to develop meaningful understanding of probability, it is important to use effective ways to train teachers (Batanero, 2013; Koparan, 2019). Research and development in teacher education related to probability education is still scarce and needs to be fostered.

Different authors (Batanero et al., 2016; Sharma, 2015) claim that many of the current programmes do not yet train teachers adequately for their task to teach statistics and probability. Even when many prospective secondary teachers have a major in mathematics, they usually study only theoretical (mathematical) statistics in their training. Few mathematicians receive specific training in applied statistics, designing sample collections or experiments, or analysing data from real applications. These teachers also need some training in the pedagogical knowledge related to statistics education, where general principles that are valid for other areas of mathematics cannot always be applied. Additionally textbooks and curriculum documents prepared for secondary teachers might not offer enough support. Sometimes they present too narrow a view of concepts (for example, only the classical approach to probability is shown) and definitions of concepts may be incorrect or incomplete (Batanero et al., 2016).

Lovett and Lee (2017) argue that teachers who have a weak content knowledge of statistical concepts may not feel confident to teach statistics as compared to other areas of mathematics and, therefore focus on procedural aspects of statistics. Furthermore, as their teachers missed to provide them with content and pedagogical content knowledge, they tend to teach their students in the same manner as they were instructed in their own school life.

It must be noted that non-cognitive aspects, such as beliefs and experiences towards statistics are essential to be considered in teacher education as they can affect the learning and teaching of statistics. Negative attitudes towards statistics may be related to low cognitive competence, and low motivation is caused by continuing instructional learning in statistics (Estrada, Batanero, & Lancaster, 2011). By contrast, when teachers have good learning experiences, they develop positive attitudes and appreciate the value of statistics. Therefore, beyond acquiring pedagogical content knowledge about probability and statistics, developing positive experiences towards statistics becomes a main goal in the education of future statistics teachers. Asking teachers to reflect on their learning can be a valuable strategy to educate teachers.

RESEARCH DESIGN

To conceptualise our study, we drew on design-based research theory (Cobb & McClain, 2004). Design research is a cyclic process with action and critical reflection taking place in turn. Our research utilised a comparative case study design to understand pre-service teachers' pedagogical perspectives and beliefs and experiences regarding the learning and teaching probability from two different educational contexts: New Zealand and Fiji. The larger study itself involved cycles of three phases: a preparation and design phase, a teaching experiment phase, and a retrospective analysis phase. Both mathematics educators were involved in the whole research process (posing questions, collecting data, drawing conclusions, writing reports and dissemination of findings). In this paper, semi-structured interview data from phase one are used to answer the two research questions.

RESEARCH CONTEXT AND PARTICIPANTS

The first phase of the research involved 24 pre-service teacher participants who were part of the Graduate Diploma in Teaching at the University of Waikato in New Zealand (UW) (n=10) and the Bachelor of Science and Graduate Certificate in Education (BSc GCEd) programme from University of the South Pacific (USP) (n=14) in Fiji. All the 24 pre-service teacher participants had mathematics as one of their teaching subject majors.

In terms of probability and statistics content knowledge, all the pre-service teachers from USP had studied high school probability and attained a pass in Year 13 examinations in their own country. A pass in year 13 mathematics is an official requirement to enrol in a mathematics major program. In addition, all the 14 participants had passed a 100-level undergraduate course in probability and statistics, ST130: Basic Statistics. The course outcomes emphasize collecting and analysing data, including introducing basic probability concepts. The course also covers design and analysis of experiments including elements of sampling. Similarly, seven of UW pre-service teacher participants had studied high school statistics and probability and five of these seven participants had taken undergraduate courses in probability and statistics.

Out of the 14 USP pre-service teacher participants, 10 were from Fiji and four from Kiribati. Of the UW pre-service teacher participants, six were New Zealanders and four international pre-service teacher participants.

After getting ethics approval from both the universities, all the pre-service teacher participants were notified about the research study through individual emails with a cover letter attached that clearly outlined the specifics of our study and the potential benefits that pre-service teachers could derive from this engagement. All the pre-service teacher participants had agreed to be part of this study and signed a written consent. Once their written consent was obtained, the participants took part in a one-to-one interview that lasted for approximately 20 to 30 minutes. The interviews were conducted by individual researchers at their respective universities. Prior to the interviews, a series of communication was held among both the researchers through the use of ICT such as Skype. These virtual meetings helped in revising the semi-structured interview schedule prior to the first phase of the study.

The purpose of the semi-structured interviews was to gather the participants' experiences and perceptions about probability teaching and learning as experienced during high school and university study. It was reasonable to assume that pre-service teachers would be able to recollect their most recent experiences about probability teaching and learning.

Data was analysed using a thematic approach (Braun, & Clarke, 2006). Four themes emerged from data analysis: experiences of high school probability; experiences of probability at university; teachers' perspective of most effective methods of teaching and learning probability; and pre-service teachers' views on the use of games in teaching probability. These themes are discussed in the next section. Pseudonyms are used to protect the participant identity. Participants are assigned letter names from A – X.

FINDINGS AND DISCUSSION

The section is presented according to key themes arising out of the analysis of teacher semi-structured interviews. The discussion is supported by the use of the participants' voice through direct quotes. Findings are linked to key literature.

Pre-service teachers' experiences of high school probability

Out of fourteen teachers interviewed, eight teachers reported that their high school teachers used traditional methods of teaching such as using text books and black boards to provide notes and examples. Students learnt by copying notes and examples and doing text book kind of activities as stated by Participant S "Like in high school we used the text books, notes and most probably our teacher used to discuss the examples from the notes".

Hence, for these pre-service teacher participants, high school learning probability and statistics was through the use of text books and blackboards only. A notable finding was that four interviewees stated that they learnt probability in high school using notes, books and blackboard. As said by one interviewee, "Teacher gave notes and we wrote in our books and then she gave activities" (Participant L).

Six pre-service teacher participants stated that their high school teachers used some form of games such as rolling a die, tossing a coin and picking a coloured candy. For example, one of the teachers said, "The way I leant probability in school was through the coins and the dice that our teacher used" (Participant B). Some teachers reported using other methods of games as stated by the interviewee, "Umm yes, he gave example like for dice and marbles or umm sometimes like small boxes like you colour them and then throw. You see how much red, yellow like that you get" (Participant F).

According to one teacher being, her high school probability and statistics teaching and learning was not good as the teachers did not had proper communication with the class. She said that "Our teacher was just a graduate from USP and the way he communicated and transferred the notes and ideas was really hard to understand" (Participant A). The participant claimed that the new graduate teachers enter the classroom with some experience from their practicum. However, the participant found that these new graduates find difficulty in lesson delivery and classroom management. In summary, there were mixed experiences reported, with relatively fewer pre-service teachers reporting high frequency use of games in probability teaching.

Of the 10 WU participants, two did not do any statistics at the high school level in their respective home countries as stated by Participant R: “In my country, they don't really introduce statistic a lot. That's like it's not even compulsory unit. So, I didn't learn much about statistic but after I came here, so you had to study a lot about it.”

Like USP cohort, four participants talked about learning from textbooks, definitions and doing calculations as stated by Participant P: “And the first thing I did learn was like the teacher telling us about events occurring and like probability trees”

Three of the participants remembered doing hands-on activities or using real data at school as reflected by Participant Q:

> My first experience is actually coming from a game that is very popular in Canada. So, it involves rolling dice. So, you had to roll the dice every time you want to go further. I always tried to find out how do I roll DICE to get larger number. But after I learned statistics, I noticed that it's random, so you can't really control that.

Overall, Fiji data suggests more of textbook and blackboard experiences, with a few mentioning some games. More of the NZ participants used hands-on activities and games to learn probability in high school.

Pre-service teachers experience of university probability

The most commonly reported change at university teaching was that the move away from text books and blackboard to PowerPoint presentations. A majority of the pre-service teachers said that their lecturer and tutors taught from the PowerPoint notes only, while at times using videos as well. From the total of fourteen interviewees, twelve teachers said that their lecturer used ICT tool to teach such as PowerPoint presentations and videos related to probability. Students learnt through PowerPoint notes presented in class and on Moodle. For example, one of the teachers said “All the lecturers in USP use PowerPoint notes and they just present. There was no learning that happened in an experimental way” (Participant H). There were no games introduced by lecturers while teaching however videos on probability was shown in the lecture. As said by one teacher “Umm yeah teaching in USP is based on PowerPoint notes only. No materials were used but videos were there. Videos like showing the dice and its outcomes and all” (Participant G). Lecturers in USP mostly rely on PowerPoint notes and no extra teaching aids such as marbles, coins, dice and cards are used. “They don’t use materials like in college. They rely on PowerPoint notes they have” (Participant D). Therefore, for most of the pre-service teacher participants, university probability learning was through the use of PowerPoint presentations only, with videos used sparingly.

From the total of fourteen interviewee, only two pre-service teachers said that their lecturer used ICT tools to teach such as PowerPoint presentations and materials such as coins and dice. For example, one teacher said “She would bring some objects. Some coins then tell how we can get the probability and then tosses the coin” (Participant M).

These two teachers think that USP is more interesting when it comes to learning and teaching. “I think the university has more good ways” (Participant F).

Unlike the Fiji cohort, only four NZ participants mentioned taking statistics at the university level. These participants stated covering topics in more depth as reflected by participant P “The university, you know, kind of same stuff but then more detail More conditional stuff and expected values of the dice.”

Pre-service teachers’ perspective of most effective methods of teaching and learning probability

All the fourteen USP teachers argued in favour of the use of demonstrations and materials such as coins and dice while teaching. They claimed that using objects makes learning more effective and interesting. It enables the students understand the concept in a better way. As said by one participant “I think it’s better to use objects or materials to show students” (Participant M). All pre-service teacher participants agreed that teachers should use materials such as coins and dice while teaching. None of the participants stated against the use of manipulatives such as coins, dice or any other teaching resource whilst teaching probability.

The WU participants suggested using hands-on activities, real life context, using cultural contexts and technology to engage and motivate students.

Do pre-service teachers support the use of games in teaching?

Similar to the findings reported above, thirteen out of the fourteen participants supported the idea of using games while teaching. They stated that using games is be really effective in learning and teaching. As said by one teacher “Yes, I think so because it will be really effective” (Participant L). Most of the interviewees who had undergone school-based practicum reported that they used games in their teaching. For example, one teacher said “Umm I just did my microteaching one on that topic. Ok when I was talking in terms of a die, I took a die and then I demonstrated it to the class and asked questions from that. I also took lollies. Three mentos and two red candy and I gave to the students if they tell the right answer” (Participant G). Teachers also mentioned some of the benefits of teaching through games. “Through games, students can learn a variety of important skills. There are countless skills that students can develop through game playing such as critical thinking skills, creativity and teamwork” (Participant J).

According to one teacher, showing students videos on probability is also effective. As said by a teacher “Umm yes. There are a lot of YouTube videos as well that tells how to make probability interesting and how students will like engage into it and be more interested into learning probability.” (Participant J). In summary, a majority of the pre-service teachers agreed with the idea of using games while teaching, and a few could even provide examples of how they have used games or videos in their practicum training.

On the contrary, from the fourteen teachers, one did not agree with the idea of using games in classrooms. According to the teacher, using games will shorten the lesson coverage time. As said by the teacher “No I don’t agree with it because if we use game than it will take time and we have to complete the coverage” (Participant A).

All WU participants supported the use of games for various reasons. For example, participant P stated: “Opportunity to use experiments instead of textbooks, by using real-life context they can also help students learn more actively.” I think it is engaging, as the game element brings in the chance to see probability in real life, as well as providing a change from bookwork

Like the USP interviewees, six of the WU teachers also reported that they had used games while on teaching practicum. They reported that they used games in their teaching. However, two participants added that using games in NZ classrooms could be challenging as stated by Participant V: “It may be hard to keep the students on task and not get distracted by the game.” Participant X stated “I suspect that many students may not establish a link between the game and the theoretical knowledge covered in the lesson because of a challenging and disruptive classroom environment and resulting lack of engagement.”

These participants realised that hands-on activities like games have the potential to engage students and contribute positively to their learning. However, if a classroom culture has not yet been effectively established, games can negatively impact on student engagement as suggested by participant X. This opinion is in line with the conclusions of Garfield and Ben-Zvi (2009) who called for the development of productive social norms for communicating mathematically.

In terms of probability teaching experience, Fiji cohort had little experience in the form of attending a 14-week school practicum, but majority support the use of games and other interesting ways of teaching probability and they agree to include games in their teaching post graduation. One way to explain this would be to say that from a socio-cultural perspective (Garfield, & Ben-Zvi, 2009), discussions among these group of young teachers may have helped them realise that traditional textbook approach to teaching is not a good way to learn probability. Their relatively negative experiences in learning probability in their high school and university teaching seems to suggest that these teachers are willing to teach differently (Koparan, 2019). NZ teachers seem to have experienced greater exposure to games and online resources such as kahoot. This is not a surprising finding given that NZ curriculum is designed to support such learning whereas Fiji one is exam-oriented. Overall, the findings resonate with the conclusions of Batanero and Díaz (2010) who stated that secondary school curricula in many countries is limited to few lessons and mainly taught in a procedural way. While several studies internationally (Batanero, 2013) have indicated that despite the emphasis in statistics and probability, teachers have limited awareness of promoting probabilistic reasoning, the pre-service teachers in the present study talked about using a range of specific resources consistent with research-based effective learning practice. Whether this was by virtue of prior learning in teacher education, or by experience in the collaborative setting in this study cannot be determined here, but this could be an area for future investigation.

References

Batanero, C. (2013). Teaching and learning probability. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 491–496). Heidelberg, Germany: Springer.

Batanero, C. Chernoff, E. Engel, J. Lee, H. & Sánchez, E. (2016). *Research on teaching and learning probability (ICME-13 Topical Surveys)*. Cham, Switzerland: Springer Open.

Batanero, C., & Díaz, C. (2010). Training teachers to teach statistics: What can we learn from research? *Statistique et Enseignement, 1*(1), 5–20.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101.

Cobb, P., & McClain, K. (2004). Principles of instructional design for supporting the development of students' statistical reasoning. In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 375–395). Dordrecht, The Netherlands: Kluwer.

Estrada, A., Batanero, C., & Lancaster, S. (2011). Teachers attitudes towards statistics. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Statistics in school mathematics. Challenges for teaching and teacher education* (pp. 163–174). Dordrecht, The Netherlands: Springer.

Garfield, J. B. & Ben-Zvi, D. (2009). Helping students develop statistical reasoning: Implementing a statistical reasoning learning environment. *Teaching Statistics*, 31(3), 72-77.

Koparan, T. (2019). Teaching game and simulation based probability. *International Journal of Assessment Tools in Education, 6*(2), 235-258.

Leavy, A. & Hourigan, M. (2014). Motivating Inquiry in Statistics and Probability in the Primary Classroom. *Teaching Statistics, 37*(2), 41-47.

Lovett, J. N., & Lee, H. S. (2017). New standards require teaching more statistics: Are pre- service secondary mathematics teachers ready? *Journal of Teacher Education, 68*(3), 299-311. doi: 10.1177/0022487117697918

Ministry of Education, Heritage and Arts (2017). *Maths guide for lower primary*. Suva, Fiji: Ministry of Education, National Heritage, Culture and Arts.

Ministry of Education. (2007). The New Zealand Curriculum. Wellington, New Zealand: Learning Media.

Sharma, S. (2015). Teaching probability: A socio-constructivist perspective. *Teaching Statistics, 37*(3) 78-84.

MELTING CULTURAL ARTIFACTS BACK TO PERSONAL ACTIONS: EMBODIED DESIGN FOR A SINE GRAPH

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The reification of the externally performed actions into internal schemes and representations is often considered as a key process of concepts mastering. In this paper, we present a radical embodied account of the process-object dialectics of mathematical concepts. Our empirical tryout of the elaborated embodied action-based design for trigonometry demonstrates how personal actions provide an opportunity to recognize new qualities of the cultural artifacts in the context of the enactment. From the radical embodied approach, the process of reification is related to reconsidering and creating cultural artifacts. The students do not interiorize external actions; instead, they learn to anticipate action possibilities as afforded by the artifacts.

INTRODUCTION

For several decades, mathematics education research on conceptual understanding has been stressing process-object dialectics—namely the idea that concepts emerge in an operational form and are later reified in a stable artifact—behind mathematical concepts (Sfard, 1991). The introduction of the object-process dialectics for mathematical concepts was initially meant to build on both Piaget's view on the development of cognitive structures, and on Vygotsky's perspective on the role of cultural signs (Sierpinska, 1994). Later, the incompatibility of these two approaches led to two theoretical tracks that appropriated the idea of object-process dialectics in different ways. The constructivist perspective would talk about automatization and condensation of processes and further encapsulation and reification of them in cognitive schemes of objects (Dubinsky, 2002; Sfard, 1991). The socio-cultural perspective, on the other hand, would stress a seemingly other process-object transition, namely crystallization (Radford, 2003) or reification (Wenger, 1998) of social practice in material products, such as symbols, definitions, and visual models in the development of mathematics as a cultural practice. In this paper, we propose that the radical embodied cognitive science allows reconsideration of cognitive processes in such a way that the two seemingly independent processes—the reification of actions in mental constructions and in material artifacts—become two sides of the same coin.

In this paper, we contribute to reconsidering the process-object transition in the conceptual development from the radical embodied perspective to cognition. In design research, we developed a sequence of tasks that lead to a conceptual understanding of sine function through the reification of embodied action in a material artifact of the sine graph. Our research questions concern (1) how we can understand the process-object dialectics from a radical embodied perspective and (2) how can embodied design help

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in the inclusion of mathematical artifacts into conceptual understanding of trigonometry.

THEORETICAL FRAMEWORK

Embodied approach to learning is a quickly developing approach in mathematics education and in cognitive science during the last two decades. A variety of researchers in both disciplines has been showing "how human thinking involves various parts of the body rather than just the Cartesian 'mind'" (de Freitas & Sinclair, 2013, p. 454). Here we propose an account of learning from radical embodied cognitive science perspective that joins enactive and ecological approaches. The detailed analysis of the advantages of this perspective in comparison with other embodied frameworks goes beyond the scope of this paper, as here we focus on what it can offer to theoretical reconsideration of process-object dialectics.

From a radical embodied perspective, cognition is not encapsulated in somebody's mind, instead it is deeply rooted in the material culture and therefore thinking is indispensable from enacting with things (Malafouris, 2018). Instead of operating with representations and internally reified objects, a person acts in the *rich landscape of affordances* that material culture provides, enabling a nested system of actions. The subject anticipates each action in the form of *action readiness* and persistently receives the *feedback* from the world (Kiverstein & Rietveld, 2018). A better grip—a better ability to act—is iteratively established in perception-action loops as a person comes to couple with the environment in fulfilling system of skilful and enactive intentionality. In actual doing these anticipations are involved in sustaining successful coordinated behavior in an ever-changing world. In the case of imagining (or thinking) the doing in the world is only pretended, while action readiness remains to coordinate multiple affordances without encountering feedback from external reality (Kiverstein & Rietveld, 2018).

Following this perspective, we propose that the process of conceptual learning lies in learning to recognize new affordances for the actions that resulting from the developing readiness to act with the material artifacts. This view on learning is very distinct from internalization of the external actions with artifacts. Instead, the artifacts themselves change how they look for the learner. While learning, reification appears to be a stabilization of actions by perceiving new affordances of the already provided cultural artifacts. While developing culture, reification appears to be a stabilization of actions by creation of new artifacts.

In the empirical part of the research we have used embodied action-based design genre as an analytical lens to elucidate theoretical proposal. At the same time, our theoretical ideas informed the design of the learning sequence.

EMPIRICAL RESEARCH

Design Research and Embodied Action-based Design

As our research aims to advance the theory of radical embodiment and create an effective learning environment following this theory, we conducted a design research that consisting of multiple cycles of theoretical considerations, design, and empirical tryouts of various sizes (Bakker, 2018). As a background design approach, we use embodied action-based design that was introduced for interactive teaching and learning of proportionality (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011). The main idea of this design genre lies in exposing mathematical concepts in the form of spatially articulated motor problems. The students receive continuous green-to-red feedback in response to their actions within the technological environment. As students aim for green feedback, they learn to perform the motor actions and further reflect on their performance in collaboration with an interviewer.

Our project aims to expand the embodied action-based design to the topic of trigonometry teaching and learning. In this paper, we provide a laboratory analysis of the next version within design research cycles of embodied design for trigonometry, which builds on the design reported at CERME conference (Alberto et al., in press). We conducted a micro-ethnographical analysis (Streeck & Mehus, 2005) on the videography and eye-tracking data from a clinical interview with a student working on the designed activities.

Embodied Design for Trigonometry

Unlike in the case of proportionality, where the embodied design was called to supplement the lack of visual imagination (Abrahamson et al., 2011), in teaching trigonometry, educators widely use a variety of mathematical artifacts such as a unit circle and graphs of trigonometric functions. In this paper, we focus on an embodied design for connecting the sine graph and the unit circle models of the sine function. The main mathematical artifact that students learn in this embodied design is a sine graph. However, the sine graph is not given to students in a ready-made form, but instead, it ought to emerge as the result of their actions. The design consists of three main phases each targeting a specific correspondence between the two models: (i) the students coordinate the *x*-value of the graph's points with the arc's length on the unit circle; (ii) the students coordinate the *y*-value of the graph's points with the sine value of the point on the unit circle; (iii) the students coordinate two previously established coordination as they disclose sine graph trajectory as joining *x*-value equal to arc's length and *y*-value equal to sine value. In each phase, immediate color feedback supports student's actions, thereby enabling new sensory-motor coordination. After reaching motor fluency, students are asked to describe to an interviewer their embodied experience. Here we focus on the selected episodes from phase 2 of the sequence. In phase 1, a student coordinated the *x*-value and the arc's length. In phase 2, this relation was now outsourced to the system that automatically generated the *x*-value of the point on the Cartesian plane in correspondence to the unit circle's point position. The students are tasked simultaneously to manipulate the point on the unit circle and to adjust the vertical position of the point on the Cartesian plane aiming to green feedback. Green feedback indicated that the *y*-value of the point on the Cartesian plane corresponded to the vertical positions of the point on the unit circle. The students, however, are required to discover the target relation effecting green feedback from their own activity.

Data Analysis and Discussion

We report here on the findings from one of the empirical tryouts of the multiple design cycles. A student (Dmitry, second year of multidisciplinary bachelor program of the Utrecht University Colleague) studied trigonometry at school approximately three years ago. This student's previous experience allows us to trace the advantages of embodied design for any student and reflect on the interlacing of embodied experiences with student's inevitably prior knowledge.

Episode 1. In trying to keep the feedback frame green, Dmitry is mostly looking at the frame with some brief glances to one or another hand. His hands are first moving up and then down in symmetry, which at first approximately matches the relation between unit circle and the sine graph, so the frame is green an essential amount of time. He continues to move his right hand further along the imagery sine graph as it corresponds to his prior knowledge. Already after 55 seconds from the start of the task Dmitry finds himself confused as he arrives to the minimum point of 3π/2:

S: I am just confused… because… it does not change color here at all [*indeed, the green feedback has some threshold of sensitivity to micro movements].* Because…because it should go further [down]. If I am moving this one [he iteratively moves the point on unit circle forward and backward Figure 1a]…

I: Should it go further?

S: There should be the minimum point, and it should be here [*he iteratively draws a pit down from the point, below the level of unit circle, Figure 1b, 1c]*

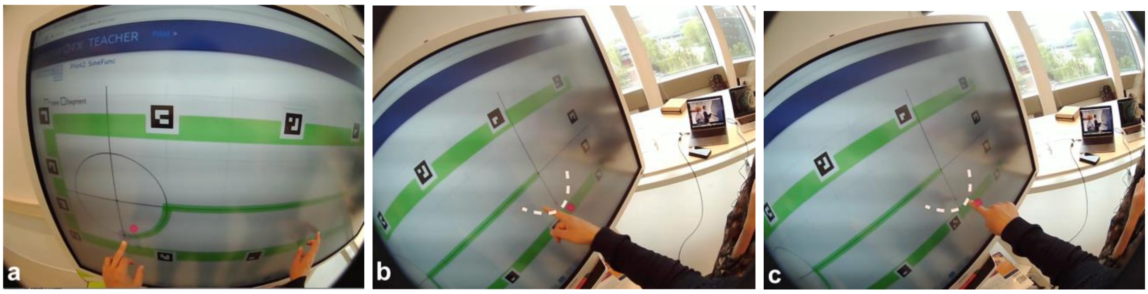

Figure 1. The student expects the sine graph to go below the magnitude –1. Here and further white line is overlaid to show the dynamics of the gesture

Dmitry might expect the existence of the minimum point on the graph, as it corresponds to both his previous experience with a reified sine graph and the movement of his left hand on the unit circle. His explanations confirm that his actions are determined by the anticipation of "wave, wavy function." However, an exact vertical position of the point on the enacted sine curve is not monitored or described. The sine graph does not seem to be connected to the unit circle as Dmitry expresses that he finds position of right hand "through guessing." He easily suggests the graph going below –1 magnitude of sine value, as the amplitude of the sine graph does not exist for him as an object or property for consideration.

Episode 2. As the student could not grasp the target relation between two points, a dashed segment connecting the two manipulated points was added to facilitate coordination and reflection. In about 10 seconds of enactment with the auxiliary segment, Dmitry for the first time makes two back-forward horizontal saccades between the point on the unit circle and the point on Cartesian plane. The next round of reflections brings forth the connection between the two sine function models.

I: So, you say this line…

S: It has to be straight. If it is straight it is green. So…If…

I: [in low voice] it has to be straight [revoicing the student].

S: [*moves the points to the initial position and finds green position near zero angle*] This has to be horizontal [*horizontal gesture*]. And if it is horizontal, it is green [*a gesture around the circle]*. That because…yea, because the point is aligned here, it has the same value. Yea…and this makes sense because… [*the student moves the point along the circle looking predicting the position of another point, and then he manipulates the point on the Cartesian plane, as if enacting entire performance once again]*.

S: If I would be about to draw the function, only using this…It will…, It will [*he enacts the motor action once again with two hands up and down]*. As, as, as this has, has to be, has to be horizontal. And…if you actually draw a line, you would see…how [*the student performs a sine-graph gesture with two hands, Figure 2]*…this nice movement [*the student again manipulates the point on the circle back and forward a few times]*.

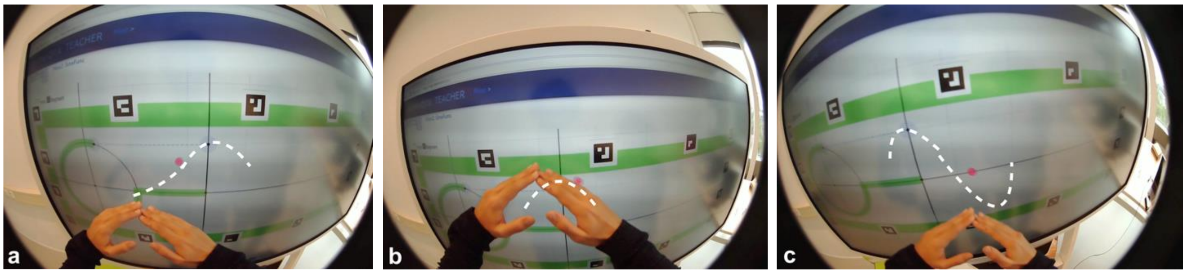

Figure 2. The student gestures the sine graph as it emerges from his enactment

The target position of the segment was expressed at first as “straight”, then as “horizontal” and then supported by a horizontal gesture. The motor action is now based not on the anticipated wavy function, but on maintaining the segment from the unit circle to the sine graph horizontal. The reified form of the sine graph as a ‘wavy function’ is now replaced by the description of a personal action “If I would be about to draw the function” and then by the description of this experience from second person perspective: “if you actually draw a line, you would see…” These references to personal experiences are supported by enactment and by the iconic gesture that tightly aligned with the possible enactment. The student now recognizes the sine graph as emerging from the movement on the unit circle. The discourse moves from first person perspective to the second person perspective, and the enactment is now changed to iconic gestures. These transformations are first steps in a new re-reification of the sine graph, which continues in Episode 3. The readiness for action, which at first maintained coordinated behavior with immediate color feedback, now serves as anticipation of the form of the sine graph.

Episode 3. After this reflection, the interviewer returns to Dmitry's idea that the sine graph needs to have maximum laying above the one-unit magnitude.

I: Could it go a bit higher? [*the interviewer tries to raise the point on the Cartesian plane above on the top position, just as the student gestured in Episode 1]*.

S: It cannot go higher than this point [*the student sets the point on the circle on the top position and gestures from it to the point on the plane, Figure 3a]*. It that's what I am…So…so this horizontal line [*the student two times gestures along this line with two index fingers together, Figure 3b]*, this is the maximum. So, it's all, it stays in the amplitude [*the student shows the amplitude as if gesturing a corridor from the maximum and minimum values on the unit circle, Figure 3c]*. That is given. With the circle [gesture along the y-axis of the circle].

Figure 3. The student explains how unit circle determine the amplitude of sine graph

The personal language ("It that's what I am") abruptly emerges as the student searches for an explanation of his immediate perception that the graph cannot go higher than the unit circle. Later, the student refers to the graph as separate from the enactment thing that "stays in the amplitude" and identifies the maximum point with the horizontal segment. The experience of maintaining the horizontal connection between the point on the unit circle and the point on sine graph opens a new affordance of the unit circle, namely determining of the sine graph's amplitude, maximum, and minimum points. The amplitude of the sine graph, which was missed at the beginning (Episode 1), is referred to as a separate reified object in the speech and tightly grounded in the visual material by corridor-like gestures (Figure 3c).

Episode 4. Further analysis shows that, contrary to our expectations, the student does not infer the correspondence between the sine values in two models from the horizontal connection of two points by an auxiliary segment:

I: Where is sine? It is a sine graph…Where is sine?

S: […] If it is a sine x function, then sine x…Well, sine x equals…So the thing that equals is to be found here [*the student repeatedly moves the point along the circle, but does not clarify how to find sine value]*. And the sine function should be here again [*he moves the point on the Cartesian plane to reach horizontal alignment]*.

The correspondence between the unit circle and the sine graph is now grounded in a horizontal alignment of two points: horizontality, which is visually easily assessed and maintained, helps to keep the green feedback. However, this special quality of the segment was never questioned further, hindering conceptualizations important for grasping the sine function. The vertical correspondence of sine values in two models was instead reified in the salient horizontal property of the auxiliary line, but never reflected by the student and thus alienated and forgotten (Wenger, 1998).

CONCLUSIONS

To conclude, we come back to our second research question and show how embodied design helps to include the artifacts in conceptual understanding of trigonometry, and also address the limitations that the current design version related to the ready-made auxiliary segment. Further, we return to the theoretical research question explain how empirical analysis corroborates our theoretical proposal of learning from the radical embodied perspective and how it influences the idea of process-object dialectics.

The example of the sine graph and unit circle, as the cultural artifacts, exemplify that embodied design provides an opportunity to re-enact these artifacts, which re-emerge for the student because of particular actions. This re-emerging sine graph now stands for the student as affording new action, namely the horizontal alignment with the unit circle. Simultaneously, the student comes to recognize a new affordance of the unit circle, as it is now capable of determining the sine graph's amplitude. Later, the affordances of the artifacts ground verbal description of their properties: the student describes the amplitude of the sine graph as given by the unit circle.

Adding a cultural artifact as ready-made (such as an auxiliary segment) substantially shifts students' sensory-motor coordination and conceptualization (as described by Abrahamson et al., 2011). However, there is always a risk of the artifact's alienation from its mathematical function: The horizontal segment connects two equal values; and, on the one hand, it pragmatically releases the performance, but on the other hand hides the core relations by outsourcing them. The embodied design allows us to "melt" the cultural artifacts and to provide students with an opportunity of reifying embodied actions in mathematical properties of re-emerging artifacts.

The empirical data corroborate our theoretical proposal that learning is progressing from direct enactment to the recognition of action affordances of the artifacts. As these affordances are expressed within multimodal discourse, they might be recognized as properties of the artifacts and as new objects. The amplitude of the sine graph is noticed because of moving the point along the unit circle, and a unit circle comes to be recognized as determining the sine graph's amplitude. The reified objects are not parts of the mental constructions, but systems of affordances, which are deeply and inherently grounded in the materiality of the artifacts. The process of reification, as a transition from enactment to stable objects, leads to a reconsideration of previous objects (such as a unit circle and a sine graph) and to the creation of new multimodal entities (such as the amplitude), which become new objects. The process-object dialectics from a radical embodied perspective is seen now as dialectics between direct enactment and further recognition of the artifact's affordance in the form of objective property.

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References

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge and Learning, 16 1*, 55–85.

Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (in press). Networking theories in design research: an embodied instrumentation case study in trigonometry In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th CERME*. Utrecht, the Netherlands.

Bakker, A. (2018). *Design Research in Education. A Practical Guide for Early Career Researchers*. London, United Kingdom: Routledge.

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics*, *83*(3), 453–470.

Dubinsky, E. (2002). Reflective Abstraction in Advanced Mathematical Thinking. In *Advanced Mathematical Thinking* (pp. 95–126). Amsterdam, Netherlands: Springer

Kiverstein, J. D., & Rietveld, E. (2018). Reconceiving representation-hungry cognition: An ecological-enactive proposal. *Adaptive Behavior*, *26*(4), 147–163.

Malafouris, L. (2018). Bringing things to mind: 4Es and material engagement. *The Oxford Handbook of 4E Cognition*, (October), 755–772.

Radford, L. (2003). On the epistemological limits of language: Mathematical knowledge and social practice during the renaissance. *Educ. Stud. in Mathematics, 52*(2), 123–150.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educ.Stud. in Mathematics*, *22*(1), 1–36.

Sierpinska, A. (1994). *Understanding in mathematics*. London, United Kingdom: Falmer Press.

Streeck, J., & Mehus, S. (2005). Micro ethnography: The study of practices. *Handbook of Language and Social Interaction*, (11237), 381–404.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York, United States: Cambridge University Press.

IDENTIFICATION OF GEOMETRIC SHAPES: AN EYE-TRACKING STUDY ON TRIANGLES

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Geometry is an important mathematical subdomain, in which the identification of geometric shapes, such as triangles, plays a central role in primary education. However, the identification of triangles is not trivial for children. Previous studies indicate that there are some triangles which cause grater challenges to children than others. These studies obtained their insights from interviews. Another promising tool for gaining insights into how learners perceive and process information, is eye tracking (ET). This paper presents a study that uses ET to investigate how N=174 fifth graders identify triangles. We investigated student strategies for the identification of triangles and found significant differences in strategy use between different types of triangles, indicating different levels of difficulty for the students.

INTRODUCTION

Geometry is a mathematical subdomain in which geometric shapes, such as triangles, play a central role, not only but also in primary education (e.g., MSW NRW, 2008). The identification of triangles is not trivial for children, as evidenced by the error rates for preschool children (36%; Clements et al., 1999). However even for older children, at grade 6, it is not trivial to identify the triangles (error rate of 19%; Clements & Battista, 1992a, cited in Clements & Sarama, 2000). Previous studies on the identification of triangles, using interviews, indicate that some triangles seem to cause greater challenges to children than others. For example, children identify prototypical triangles successfully (Aslan & Aktaş Arnas, 2007; Tsamir et al., 2008), but often mistakenly reject atypical triangles (Clements et al., 1999; Satlow & Newcombe, 1998; Tsamir et al., 2008). This indicates that non-critical attributes (such as the aspect ratio of a triangle) cause children to make mistakes in the identification of triangles. This study builds on these findings and investigates strategies for identifying triangles. For this purpose, ET is used whose potential to provide insights into learner's strategies was shown in several studies in mathematics education in different subdomains.

In this study, we investigate how students at grade 5 identify triangles. We analyze students' strategies from ET video data. We ask the following research question: *Are there differences in the students' use of strategies between different types of triangles?*

IDENTIFICATION OF GEOMETRIC SHAPES—TRIANGLES

Geometric shapes—triangles. One of the goals of teaching geometry at primary level is for students to be able to name geometric shapes and use technical terms such as

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"side" and "vertex" to describe them (MSW NRW, 2008). This is necessary for the understanding of mathematical concepts (for shapes). To highlight the role played by the individual's conceptual structure, the terms "concept image" and "concept definition" (Tall & Vinner, 1981) need to be distinguished. The formal "concept definition" is described as the words used to specify a mathematical term or concept that is accepted by the mathematical community. Beyond knowing a definition, the "concept image" plays an important role in understanding a concept: It is understood as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Over the years, the concept image develops and changes through experience. This process includes characteristics or properties of a concept—e.g., critical attributes of geometric shapes, by gaining an overview of the totality of all objects that are subsumed under a concept and by being able to point out relationships of the concept to other concepts (Weigand et al., 2018). For geometric shapes, there are critical attributes of being "closed" and having a "certain number" of "straight sides" (Satlow & Newcombe, 1998). In addition, there are non-critical attributes that can be changed without affecting the status as a valid representative. All representatives of the category "triangles" are mathematically equivalent, i.e., they fulfill the concept definition and contain all critical attributes (Hershkowitz, 1989). Differences between these representatives exist visually, in appearance. In this regard, some examples of geometric shapes are recognized more often than others, especially when they are typical and form the prototypical representatives of the category (for prototype theory see Rosch, 1973). The prototypical triangles are the equilateral and isosceles triangles (Tsamir et al., 2008). Prototypical triangles, in addition to the critical attributes that every triangle must satisfy, contain other specific, non-critical attributes that are dominant and attract the attention of the observer (Hershkowitz, 1989). In the case of equilateral and isosceles triangles, these are symmetry, aspect ratio, and orientation. All other triangles that deviate from these in the non-critical attributes—for example, through "skewness", i.e., the asymmetry or obliquity of a shape, the "aspect ratio" of the shape and the "orientation" of the shape—are understood as non-prototypical (Aslan & Aktaş Arnas, 2007; Clements et al., 1999).

Identification of triangles. Studies on the identification of triangles show that preschool children are predominantly visually oriented to the shape and quickly identify prototypical triangles as representatives (Aslan & Aktaş Arnas, 2007; Tsamir et al., 2008). They often mistakenly reject atypical triangles and mistakenly accept non-triangles as triangles, for example, if these look similar to a prototypical triangle but have rounded sides (Clements et al., 1999; Satlow & Newcombe, 1998; Tsamir et al., 2008). Rejecting atypical triangles as triangles indicates that mathematically irrelevant, non-critical attributes influence children's choices. In interview studies, it was for example found that children do not identify triangles correctly as triangles because they consider them to be too "long" (aspect ratio) or because they do not "point … at the top" (orientation) (Clements & Sarama, 2000,

p. 483). Other (prototypical) geometric shapes (e.g. squares) are confidently identified as non-triangles by children at preschool age (Aslan & Aktaş Arnas, 2007; Tsamir et al., 2008). As a trend, with children getting older the number of visual responses decreases and the number of attribute-based responses increases, and this is associated with more correct identifications of triangles (Aslan & Aktaş Arnas, 2007). Children at the end of the primary school years or at the beginning of secondary school reject non-triangles more often than atypical triangles, which indicates that they are increasingly oriented to critical attributes (Satlow & Newcombe, 1998). Error rates for the identification of triangles vary from 36% for preschoolers (Clements et al., 1999) to 19% for sixth graders (Clements & Battista, 1992a, cited in Clements & Sarama, 2000).

EYE TRACKING

ET is the technique to record a person's eye movements (Holmqvist et al., 2011). In the human eye, sharp vision is possible through the anatomy of the fovea. ET builds on this and captures foveal data, whereas extrafoveal vision—peripheral vision—is not captured by ET (Klein & Ettinger, 2019). ET has gained increasing significance in mathematics education research in recent years (Lilienthal & Schindler, 2019; Strohmaier et al., 2020). The potential of ET to offer insights into how students solve mathematical tasks was shown in several studies in mathematics education in different subdomains. Studies in the field of geometry (e.g. Schindler & Lilienthal, 2019) have also indicated that for this mathematical subdomain, the use of ET may provide insights into learners' strategies. Especially in mathematical tasks that require the perception and processing of information and through domain-specific interpretations, ET allows for inferences about mental processes (Schindler & Lilienthal, 2019).

One ET study on the identification of quadrilaterals of undergraduate and graduate students was conducted by Shvarts et al. (2019): These scholars used identification tasks in which four geometrical shapes were presented in each stimulus. The students were asked to search for a shape that corresponded to the concept that was named before each trial. Tasks using circle, square, triangle, and cross as stimuli showed extrafoveal solution of this task in most cases (77.5%). For tasks with four different rectangles, shapes in prototypical orientation "were mostly identified by extrafoveal vision, while in case of rotated exposure some foveal analysis was often required, especially in the cases of similarity between the target shape and distractors" (p. 128). Shvarts et al. conclude that the "prototypical phenomenon is not limited to distinguishing a weighted list of attributes, but can be seen even in the involvement of extrafoveal perception in different identification tasks" (p. 128).

THIS STUDY

Students. 174 students (87 girls, 87 boys) of a German inclusive comprehensive school participated in this study. The mean age was 10;8 (SD = 0;6) with ages ranging between 9;9 and 12;7. The study took place at the beginning of fifth grade, shortly after the students had just finished primary school.

Eye-tracker. Students' eye movements were recorded with the remote eye-tracker Tobii Pro X3-120 (120 Hz, average accuracy in our study: 0.8°, binocular, infrared). This is a very unobtrusive, stand-alone eye-tracker that can be attached to computer monitors. Stimuli were presented on a 24" Full HD computer screen. The distance of the students to the screen was about 50 cm.

Tasks and procedure. In individual sessions in a quiet room within their school, the students worked on 34 tasks: They were shown examples and non-examples of triangles (for a selection, see Fig. 1). They were asked if the shape presented was a triangle and were instructed verbally to answer "yes" or "no" as quickly and correctly as possible. In between the tasks, the students were instructed to fixate a star at the left side of the screen before the next task appeared. The students received no response whether their answers were correct. Verbal answers were recorded through an audio-recorder. The items were presented in a random order (the same for every student). The selection of these items was based on the primary school curriculum (MSW NRW, 2008). That is why at the beginning of grade 5, we could be sure that all students had dealt with triangles and their identification in prior schooling.

The development of the items was based on Satlow and Newcombe (1998), Clements et al. (1999), Aslan and Aktaş Arnas (2007), and Tsamir et al. (2008). The classification of the items was based on Tsamir et al. (2008). The prototypical triangles are the equilateral and isosceles triangles (orientation and aspect ratio). The non-prototypical triangles are atypical representatives that deviate from the prototypical triangles in non-critical attributes (orientation, skewness, & aspect ratio). The non-prototypical non-triangles have a triangle-like shape where critical attributes of a triangle are violated.

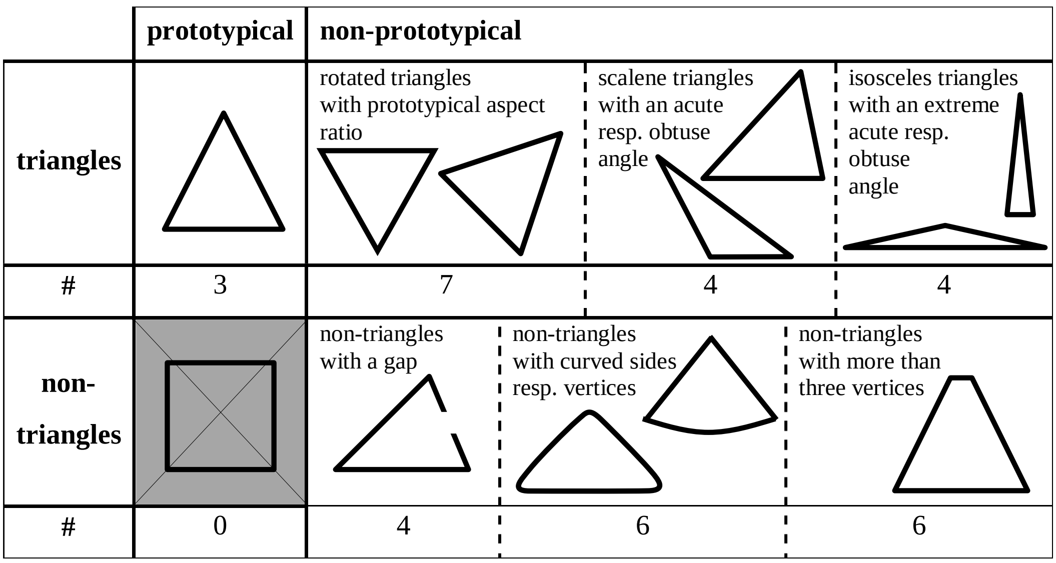

Figure 1: Selection of the items (# = number of items in the test)

The non-critical attribute "size" was not taken into account in the selection of the items, because the size of a shape has little influence on the identification of triangles even of preschool children (Aslan & Aktaş Arnas, 2007). Because preschool children already identify other prototypical geometric shapes (such as a square) reliably (Tsamir et al., 2008), these prototypical non-triangles were not included in this study either.

Our study covered 5'916 tasks (174 students × 34 tasks). Forty-one tasks (0.7%) were excluded due to student-related or technology-related data loss, where the eye-tracker did not validly record the children's eye movements, so that 5'875 tasks were analyzed.

Data analysis. For analyzing student strategies, we used gaze-overlaid videos provided by Tobii Pro Lab software: videos with eye gazes represented as a semi-transparent dot. The categorization of the gaze movements was carried out deductively, i.e. on the basis of the category system that had been developed inductively in a preliminary study by Schindler and colleagues. Categories are shown in Figure 2. For visualizing the strategies in this paper, we use gaze plots, although we used gaze-overlaid videos for analysis of strategies. For this study, 20% of the data (i.e., 1'175) were coded by one rater at two different points in time to investigate intra-rater reliability. We calculated the intra-rater reliability using Cohen's kappa. The agreement was 0.92, which can be considered almost perfect (Landis & Koch, 1977).

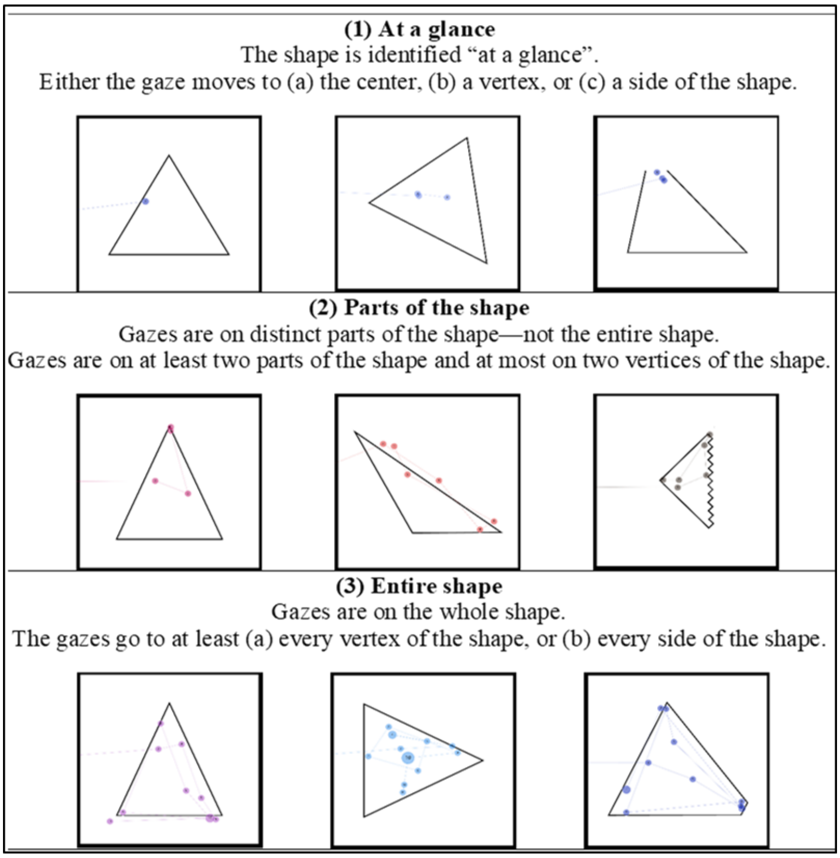

Figure 2: Categories of student strategies—examples of the eye movements

For the statistical analysis, the program SPSS 26 was used. To analyze differences in the students' use of strategies between the different types of triangles, we used a chi-square test. To compare strategy use between different types of triangles, we used the ratio of the number of observed strategies and the number of tasks belonging to the respective type.

RESULTS

In the following, we answer the research question: *Are there differences in the students' use of strategies between different types of triangles?*

A chi-square test with the accumulated strategies revealed significant differences in the distribution of strategies (see Fig. 3) between the three different types of triangles, χ^2 (4) = 52.23, p < .001.

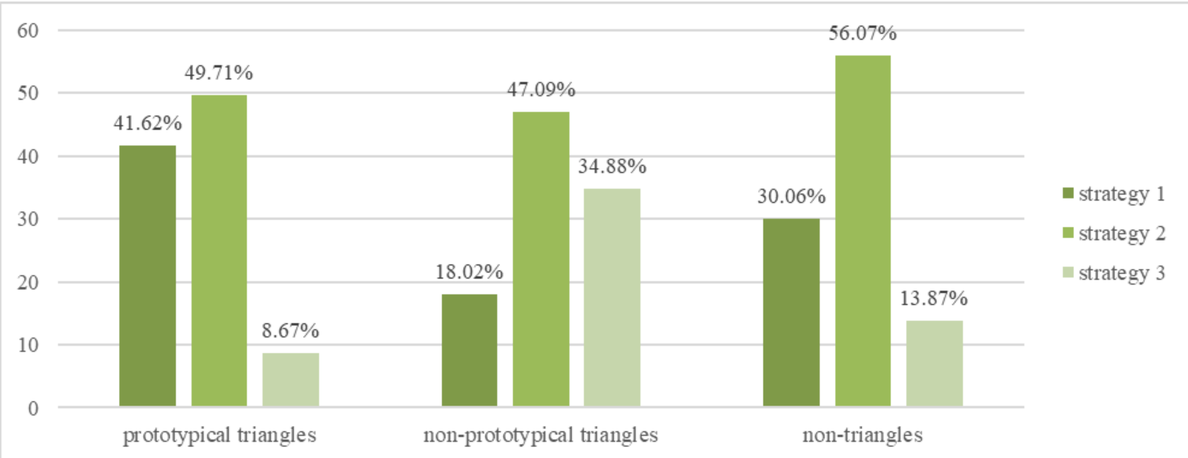

Figure 3: Distribution of strategies

Cell tests for the differences between the types of triangles revealed the following results: For *prototypical triangles*, the students used strategy 1 ("at a glance") significantly more often as compared to non-prototypical triangles (χ^2 (1) = 22.90, p < .001) and to non-triangles (χ^2 (1) = 5.51, p = .019). For *non-prototypical triangles*, the students used strategy 3 ("entire shape") significantly more often as compared to prototypical triangles (χ^2 (1) = 38.34, p < .001) and to non-triangles (χ^2 (1) = 24.63, p < .001). For *non-triangles*, the students used strategy 1 significantly more often as compared to non-prototypical triangles (χ^2 (1) = 5.96, p < .015).

Looking at the differences of specific types of non-prototypical triangles in more detail (and purely descriptively), the type of triangle where strategy 3 was used most often (52.87%) and strategy 1 at least (15.52%) is the *isosceles triangle with extreme acute resp. obtuse angles* (see Fig. 1). For the *rotated triangles with prototypical aspect ratio* the students also used strategy 1 very little (18.46%). The type of triangle where the students used strategy 3 second most often (33.53%) is *the scalene triangle with an acute resp. obtuse angle*.

DISCUSSION

Our analyses indicate that strategy use differed between different types of triangles: The prototypical triangles were identified *at a glance* more often than the other types of triangles, whereas when working on the non-prototypical triangles, the students looked at the *entire shape* more often as compared to the other types of triangles. This relates to the findings of Shvarts et al. (2019) on the identification of quadrilaterals: that shapes in prototypical orientation were mostly identified by extrafoveal vision, at the periphery, and that the rotation of shapes made identification tasks more difficult—with decreasing involvement of extrafoveal processes. Our study indicates that there are shapes, the prototypical triangles, which are easier to recognize at the periphery, while the non-prototypical triangles required stronger foveal processing.

Although identification of triangles is a mathematical topic of the primary level, the number of tasks that required extensive student gazes and the relatively high total number of errors (18.35%)—in line with Clements and Battista (1992a, cited in Clements & Sarama, 2000)—indicate that this is not necessarily an easy task for fifth graders. Our results indicate that non-critical attributes (skewness, aspect ratio, and size) have an influence on the students' use of strategies, which is in line with previous research findings on the reasoning of children for their decisions in identification tasks (e.g., Clements et al., 1999). This is interesting since the students in our study were already at the beginning of grade 5 and had dealt with triangles extensively following the German curriculum.

Our study also has implications for educational practice: Since our results suggest that students at the beginning of grade 5 still appear to be guided by non-critical attributes when considering whether a shape is a triangle or not, we think that teaching at grades 1–4 should focus even more on developing students' conceptions of triangles—and geometric shapes in general, on talking about attributes, and discussing them.

Future research should investigate, for example, whether our findings also apply to other geometric shapes. We hope that our study can be a springboard for further research in the area of identification of geometric shapes and can stimulate the discussion about this important topic in mathematics education.

References

Aslan, D., & Aktaş Arnas, Y. (2007). Three- to six-year-old children's recognition of geometric shapes. *International Journal of Early Years Education*, *15*(1), 83–104.

Clements, D.H., & Battista, M.T. (1992b). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: National Council of Teachers of Mathematics/Macmillan Publishing Co.

Clements, D.H., & Sarama, J. (2000). Young children's ideas about geometric shapes. *Teaching Children Mathematics*, *6*(8), 482–488.

Clements, D.H., Swaminathan, S., Zeitler Hannibal, M.A., & Sarama, J. (1999). Young children’s concepts of shape. *Journal for Research in Mathematics Education, 30,* 192–212.

Hershkowitz, R. (1989). Visualization in geometry — two sides of the coin. *Focus on Learning Problems in Mathematics*, *11*, 61–76.

Holmqvist, K., Nyström, M., Andersson, R., Dewhurst, R., Jarodzka, H., & Van de Weijer, J. (2011). *Eye tracking: A comprehensive guide to methods and measures*. Oxford: OUP.

Klein, C., & Ettinger, U. (2019). *Eye movement research: An introduction to its scientific foundations and applications.* Cham, Switzerland: Springer.

Landis, J.R., & Koch, G.G. (1977). “The measurement of observer agreement for categorical data”. *Biometrics, 33*, 159–174.

Lilienthal, A.J., & Schindler, M. (2019). Eye tracking research in mathematics education: A PME literature review. In M. Graven, H. Venkat, A.A. Essien, & P. Vale (Eds.), *Proceedings of 43rd Conference of the IGPME* (Vol. 4, p. 62). Pretoria, SA: PME.

Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen (MSW NRW) (2008). *Lehrplan Mathematik für die Grundschulen des Landes Nordrhein-Westfalen.* Frechen, Germany: Ritterbach.

Rosch, E. H. (1973). Natural Categories. *Cognitive Psychology*, *4*, 328–350.

Satlow, E., & Newcombe, N. (1998). When is a triangle not a triangle? Young children’s conceptions of geometric shapes. *Cognitive Development*, *13*, 547–559.

Schindler, M., & Lilienthal, A.J. (2019). Domain-specific interpretation of eye-tracking data: Towards a refined use of the eye-mind hypothesis for the field of geometry. *Educational Studies in Mathematics*, *101*, 123–139.

Shvarts, A., Chumachenko, D., Drenyova, A., & Krichevets, A. (2019). From prototypical phenomenon to dynamic functional system: Eye-tracking data on the identification of special quadrilaterals. In A. Shvarts (Ed.), *Proceedings of the PME and Yandex Russian conference: Technology and Psychology for Mathematics Education* (pp. 122–129). Moscow: HSE.

Strohmaier, A.R., MacKay, K.J., Obersteiner, A., & Reiss, K.M. (2020). Eye-tracking methodology in mathematics education research: A systematic literature review. *Educational Studies in Mathematics*, *104*, 147–200.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics, 69*(2), 81–95.

Weigand H.-G., Filler, A., Hölzl, R., Kuntze, S., Ludwig, M., Roth, J., Schmidt-Thieme, B. & Wittmann, G. (2018). *Didaktik der Geometrie für die Sekundarstufe I.* Berlin: Springer.

A COMPARATIVE ANALYSIS OF EYE TRACKING AND THINKING ALOUD IN NUMBER LINE ESTIMATION TASKS: A STUDY ON STUDENTS WITH AND WITHOUT MATHEMATICAL DIFFICULTIES

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For researchers and educators in mathematics education, it is of interest to gain insights into how students solve mathematical tasks. This holds also for students with mathematical difficulties (MD), whose strategies may be diverse. Eye tracking (ET) is a promising tool for analyzing student strategies, but its opportunities and limitations have only been explored for selected mathematical subdomains. This paper presents a comparative analysis of the opportunities that ET and thinking aloud (TA) may hold for analyzing student strategies in number line estimation tasks. The findings indicate that ET may offer more detailed insights than TA, especially for students with MD. However, a relatively high number of contradictions between the information obtained by ET and TA also indicates limitations of ET—and the need for further research.

INTRODUCTION

For mathematics education research, it is critical to not only look at student results, at product and outcomes, but also at the processes that lead there, and at their individual strategies when working on mathematical problems. Previous research has indicated that ET offers potential advantages for the analysis of strategies (e.g., quantity recognition, see Schindler & Lilienthal, 2018). The "inside view" provided by ET offers an alternative to the insights into thought processes gained by verbal survey methods—such as TA. Especially for children with MD, but also with difficulties in language acquisition, who may find it difficult to describe their strategies, ET seems to offer a greater informative content than TA (Schindler & Lilienthal, 2018). By informative content, we mean the strategy-related information provided through the analysis of the respective method. Yet, for other mathematical subdomains, such as number line tasks, it is not yet clear what potential ET may offer for gaining insights into student strategies. The number line is one of the essential representations in mathematics teaching and learning at primary level (e.g., Ernest, 1985) and students' number line estimation performance is of predictive nature for the general mathematical development (Booth & Siegler, 2008). Thus, it appears to be crucial to inquire into student strategies positioning numbers on the number line—and to evaluate what ET can offer to analyze the latter.

In this study, we compare the informative content of ET and TA for analyzing student strategies in number line estimation tasks and ask the research question (1) *To what extent does the informative content of ET and TA in the analysis of number line estimation strategies differ?* Due to the potential benefit of ET to gain insights into the

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 65-72. Khon Kaen, Thailand: PME.

strategy use especially of students with MD, we further ask (2) *Are there differences in this respect between children with and without MD?*

EYE TRACKING

ET is understood as technique to record a person's eye movements (see Holmqvist et al., 2011). Video-based ET has gained increasing significance in mathematics education research in recent years (Lilienthal & Schindler, 2019). Its potential was shown in several studies in mathematics education in different subdomains (e.g., geometry and arithmetic). Especially in controlled settings, such as visually presented cognitive tasks, and through domain-specific interpretations, powerful conclusions about mental processes are possible (Schindler & Lilienthal, 2019). ET research mostly draws on the so-called eye-mind hypothesis (EMH) (Just & Carpenter, 1976), which assumes that the eyes' fixations and the processing of information in the brain are closely related (Holmqvist et al., 2011). However, this hypothesis was developed in reading research and should not simply be transferred to the mathematical field without further reflection. Interpretation of eye movements is not trivial because ET data can be ambiguous (Schindler & Lilienthal, 2019) and the EMH does not always hold (Holmqvist et al., 2011). Another issue relates to peripheral vision: In the human eye, sharp vision is possible through the anatomy of the fovea (a small pit on the retina): The eyes need to move so that the objects of attention are perceived by foveal vision. ET makes use of this anatomical feature and measures these movements. However, also peripheral vision—which takes place in extrafoveal areas—can be involved in the processing of information, which is not captured by ET (Klein & Ettinger, 2019).

THINKING ALOUD

TA (see Ericsson & Simon, 1980) constitutes a well-established method in mathematics education research to explore individual thought processes from the students' perspective. According to Konrad (2010), the following forms of TA can be distinguished: (1) *introspection*, i.e. concurrent verbalization, (2) *immediate retrospection*, which follows directly after the work on the task, and (3) *delayed retrospection*, which takes place after all tasks have been solved or even days later. In contrast to introspective TA, where students verbalize their thoughts while performing a task and where the additional cognitive effort can be considerable (Ericsson & Simon, 1980), retrospective TA has the advantage of not interfering with the thought processes during the work on the task. However, difficulties can also occur with retrospective TA—for example in meta-cognition or in the verbalization of thought processes—and potentially limit the validity (Schindler & Lilienthal, 2018).

NUMBER LINE AND MATHEMATICAL DIFFICULTIES

The number line is one of the essential representations in mathematics teaching and learning at primary level (e.g., Ernest, 1985). On the number line, numbers are represented by their position in relation to other numbers. The ability to place numbers on the number line in accordance with their relative size is necessary to estimate the correct position of numbers (Sullivan et al., 2011). Research on number line estimation shows differences between children with and without MD. For example, the estimations of children with MD are often less accurate than those of children without MD (Landerl et al., 2017). By using the expression *students with MD*, we mean students who encounter difficulties both on a conceptual and procedural level: This includes, for example, basic arithmetic such as counting (e.g., counting by groups), (de-) grouping, the base-10 system, understanding place values, and basic arithmetic operations (see Moser Opitz et al., 2017).

When evaluating students' work on the number line, product-related results, such as the accuracy of the estimation, can be easily determined. However, so far there is only little known about the underlying processes and students' strategies in number line estimation tasks. First studies have used ET to investigate student strategies on the number line (e.g., Van Viersen et al., 2013; Van't Noordende et al., 2015) and hint at the potential that ET holds. Yet, there is no systematical analysis yet of the informative content that ET may provide as compared to TA. The above studies have illustrated that ET may be valuable to analyze number line estimation strategies especially of students with MD. However, previous research on number line estimation has not yet systematically compared the potential of ET and TA for students with MD.

METHOD

Students. 22 fifth-grade students (mean age: 11.5 years old; 11 girls) in a German comprehensive school participated in this explorative study. The study took place in the last weeks of fifth grade. Eleven children were found to have MD by the means of a standardized arithmetic paper-pencil speed test and qualitative diagnostics before the study took place. Five of the 11 students with MD also had special educational needs (in learning, social and emotional development, and physical development).

Eye tracker. The students' eye movements were recorded with the ET glasses Tobii Pro Glasses 2 (50 Hz). They have low weight (45 grams) and are relatively unobtrusive. Additionally, they recorded gestures, e.g., pointing, and verbal utterances, and synchronized these data with the eye movements. Stimuli were presented on a 24'' Full HD computer screen. The distance of the students from the screen was about 0.5 m. The students were tested in individual sessions in a quiet room within their school.

Tasks. The students worked on tasks, in which they were asked to estimate the position of a number on a number line with no markers except the endpoints zero and 100. This representation was chosen because on number lines with equidistant markers, students tend to use counting strategies, whereas on an empty number line their strategies are less pre-determined (Kaufmann & Wessolowski, 2006). Prior to the tasks, the number line was presented and introduced to the students. The students were asked to estimate the positions of the numbers on the number line in every task as fast and correctly as possible. The selected numbers were 40, 75, 90, 25, 10 and 50 (in that order). Stimuli in every task were presented as follows (Fig. 1): (1) The number appeared in the upper left corner. The students were asked to read the number aloud to ensure they had perceived it correctly. (2) The number line was presented, while the number remained visible. The students estimated the position of the number by pointing to it with the tip of a pencil. (3) A fixation star appeared, before (4) the students were asked to verbalize their thought processes retrospectively. This study covered 132 tasks (22 students x 6 tasks). Nine tasks were excluded due to data loss so that 123 tasks were analyzed.

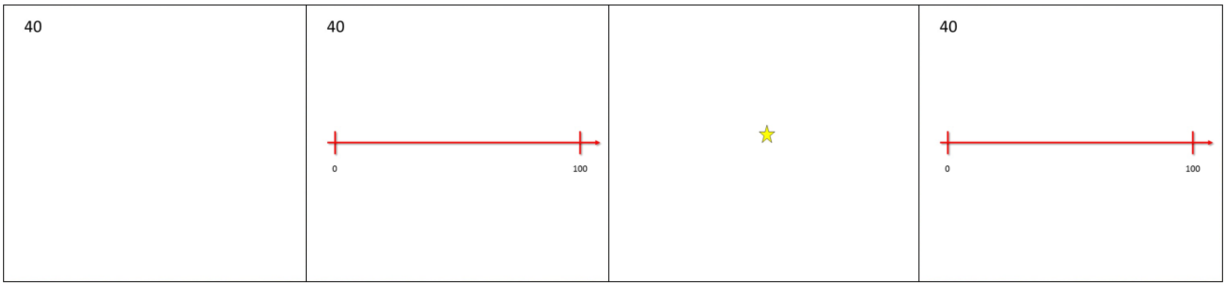


Figure 1: Series of stimuli during one task (40)

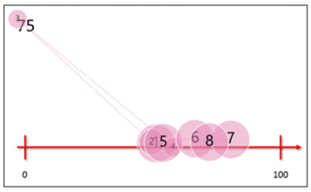
Data analysis. We analyzed gaze-overlaid videos (videos of the ET glasses, where the student gazes were augmented through a dot wandering around) through inductive category development based on Mayring's (2014) qualitative content analysis: The eye movements were first *described*. Afterwards they were *interpreted/paraphrased*. In a subsequent *category development* process, strategies with according descriptions were assigned. The categorization of all data was followed by a *category revision* step, which involved a partial recategorization. For analyzing TA, we transcribed the students' utterances and gestures (e.g., pointing). Afterwards, the same steps as in the ET data analysis were followed with the transcripts: interpretation/paraphrasing and assignment of categories. Category assignment was conducted by two raters independently and resulted in an interrater reliability of 0.94, which can be considered very high. Finally, the informative content was compared as follows: For every task for every student, we regarded if the strategies assigned through ET and TA were the same. In case they were, we noted the "match". We then analyzed the paraphrases (both in the ET and in the TA analysis) for the detailed information they provided on the student strategy—and then decided whether ET or TA provided more detailed information on the student strategy, or if their informative content on the strategy was equal. On the other hand, if the categories assigned based on ET and TA did *not* match, we marked the respective task as contradictory case (see *Results Section* for examples).

RESULTS

In the following, we answer the research questions (1) *To what extent does the informative content of ET and TA in the analysis of number line estimation strategies differ?* and (2) *Are there differences in this respect between children with and without MD?* together (see Fig. 6 for an overview). For visualizing gaze patterns in this paper we use gaze plots, although we used gaze-overlaid videos for analysis of gaze patterns.

In our study, ET and TA were *equally informative* in 51.22% of the cases (MD: 36.67%; TD: 65.08%): The students' verbal descriptions corresponded with their eye movements. Figure 2 shows an example where the student's eye movements and utterance are similar regarding the information contained. The gazes indicate that the student looked into the middle of the number line, looked again at the number 75, looked again at the middle, then looked two "steps" to the right, and then a small step back to 75. The utterance also reflects the two steps. (Note: All verbal data presented in this paper (in the Figures) is translated from German.)

Eye movements during number line estimation



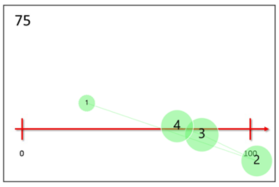
Utterances in immediate retrospective TA

„Here is the 50 certainly *(pointing to the middle of the number line)*. Then one can go, like, two forward: here approximately *(indicating two steps with the finger)*. Around here, it'd be then *(pointing to the place of 75 on the number line)*."

Figure 2: Example of ET and TA being equally informative

In 19.51% of the cases (MD: 30.00%; TD: 9.52%) ET *was more informative* than TA: The information provided by the two methods was consistent with each other and the ET contained more detailed or more extensive information. Figure 3 shows an example. The student's utterance provides two different strategies. Because of her difficulties in describing her strategy, it could be assumed that she used the first mentioned reference point (100) to estimate the number. However, this cannot be determined clearly. Her eye movements showed an orientation of 100 backwards. Thus, ET offers a closer insight into the strategy that the student had initially.

Eye movements during number line estimation



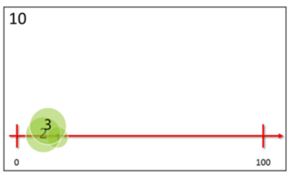
Utterances in immediate retrospective TA

"Uh, cause it is very close to 100. Well, not super close, but … I cannot explain it better… Uhm, cause 50 is just in the middle. And then just 20 further there."

Figure 3: Example of ET being more informative than TA

In 7.32% of the cases (MD: 10.00%; TD: 4.76%), TA *was more informative*: The information provided by the two methods was consistent and TA contained more detailed or more extensive information. An example can be seen in Figure 4. The student's gazes show movements around 10, which cannot be assigned a clear strategy, whereas the student's utterance suggests an orientation from the beginning of the number line step-by-step forward. Thus, TA offers a closer insight into the strategy.

Eye movements during number line estimation



Utterances in immediate retrospective TA

„Cause here is 5 *(pointing to 5)* and next is 10 *(pointing to 10)*."

Figure 4: Example of TA being more informative than ET

However, the information obtained from TA and ET was contradictory in 21.95% of the cases (MD: 23.33%; TD: 20.64%). Figure 5 shows an example where the student's eye movements indicate a forward counting from the beginning of the number line to

50, whereas TA suggests that the student immediately estimated the position of the number on the number line.

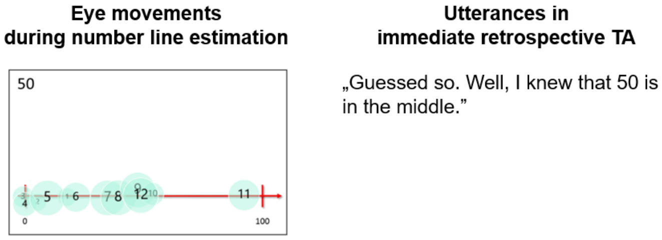


Figure 5: Example of ET and TA being contradictory

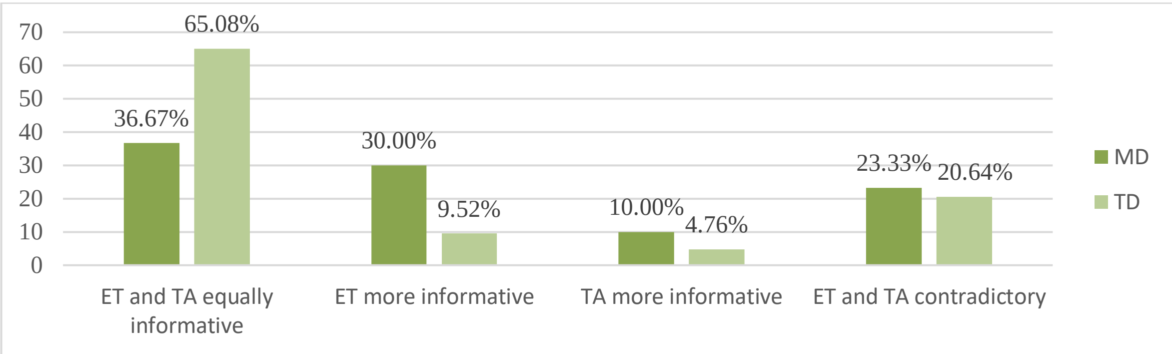


Figure 6: Comparison of the informative content of ET vs. TA
for students with MD and TD students; relative frequencies

For the statistical analysis, the program SPSS 26 was used. To investigate the differences between the informative content of the two methods with regard to the two groups (research question 2), the Freeman-Halton test was used. This is the extension of Fisher's exact test, which is "a way of computing the exact probability of the chi-square statistics" (Field, 2013, p. 724) and suitable for small sample sizes. As extension of Fisher's exact test, it involves contingency tables larger than 2x2—e.g., for two samples regarding a k-stepped (here: triple-stepped) characteristic. Since the informative content of each method was under investigation (ET vs. TA), the contradictory cases, where ET and TA did not match and no overall strategy could be determined, were not included. The null hypothesis for this test was that the informative content provided by the respective method is the same for the students with MD and the TD. The analysis revealed that the null hypothesis was rejected ($p < .01$). Accordingly, the informative content of the methods was different for the two groups.

To investigate group differences for every condition (ET and TA equally informative, ET more informative, TA more informative) in detail, we calculated effect sizes using *Cramérs V* (Tab. 1). Differences with *Cramérs V* > .30 (medium effect) revealed that ET and TA tended to be equally informative more often for TD students (than for MD students) (.36) and that ET tended to be more informative more often for students with MD (than for TD students) (.31).

	ET and TA equally informative	ET more informative	TA more informative
MD	47.83	39.13	13.04
TD	82.00	12.00	6.00
Cramérs V	.36	.31	.12

Table 1: Percentage of cases of informative contents of ET and TA and *Cramérs V* for the comparison of MD and TD

Summarizing, regarding the first research question, it can be said that the informative content was equal for ET and TA in about half of the cases. In about a fifth of the cases, ET was more informative. TA was in very few cases more informative. Regarding the second research question, ET appeared to be more informative for students with MD. ET and TA tend to be equally informative more often for TD students.

DISCUSSION

Our findings indicate that ET may offer more detailed insights than TA, especially for students with MD. This result points to a potential added value of ET as research method. However, the rather high number of contradictions between the information we gathered through using ET und TA raises questions. On the ET side, peripheral vision probably plays a role. Since ET captures foveal data, perception by means of peripheral vision (e.g., orientation at known beginning- and endpoints) may not be represented in the ET (Klein & Ettinger, 2019). With the number line used in this study (no changes during task processing), the students were able to use peripheral vision for orientation on the number line—so, not all mental processes were reflected in the ET, which only captures foveal vision. Furthermore, the question arises as of when and how the EMH applies in number line estimation activities. Studies that have shown that the analysis of eye movements provides reliable information about student strategies predominantly use mathematical tasks, which require to a high degree a perception of information displayed in the task (e.g., numbers of dots that need to be perceived). Yet, number line estimation tasks also require the retrieval of mental representations of numbers and numerical relationships for the students to find the right position of numbers on the number line (Sullivan et al., 2011). Further research should explore the contradictions between ET and TA in more detail and pursue the question how eye movements and mental processes are linked in number line estimation tasks. Using stimulated recall interviews may shed light on how the displayed stimuli and the recalled information play together (see Schindler & Lilienthal, 2019). Prospectively, it would be valuable to investigate how ET data can be interpreted in this kind of tasks, even without TA.

References

Booth, J. L. & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development*, *79*, 1016–1031.

Ericsson, A. & Simon, H. A. (1980). Protocol analysis: Verbal aspects as data. *Psychological Review, 87*, 215–251.

Ernest, P. (1985). The number line as teaching aid. *Educational Studies in Mathematics, 16*, 411–424.

Field, A. (2013). *Discovering Statistics Using IBM SPSS Statistics*. London: Sage.

Holmqvist, K., Nyström, M., Andersson, R., Dewhurst, R., Jarodzka, H., & Van de Weijer, J. (2011). *Eye tracking: A comprehensive guide to methods and measures*. Oxford: OUP.

Just, M.A. & Carpenter, P.A. (1976). Eye fixations and cognitive processes. *Cognitive Psychology, 8*, 441–480.

Kaufmann, S. & Wessolowski, S. (2006). *Rechenstörungen.* Seelze, Germany: Kallmeyer.

Klein, C., & Ettinger, U. (2019). *Eye movement research: An introduction to its scientific foundations and applications.* Cham, Switzerland: Springer.

Konrad, K. (2010). Lautes Denken. In M. Günter & K. Mruck (Eds.), *Handbuch Qualitative Forschung in der Psychologie* (pp. 476–490). Wiesbaden, Germany: VS.

Landerl, K., Vogel, S. & Kaufmann, L. (2017*). Dyskalkulie: Modelle, Diagnostik, Intervention.* Munich, Germany: Ernst Reinhardt.

Lilienthal, A.J., & Schindler, M. (2019). Eye tracking research in mathematics education: A PME literature review. In M. Graven, H. Venkat, A.A. Essien, & P. Vale (Eds.), *Proceedings of 43rd Conference of the IGPME* (Vol. 4, p. 62). Pretoria, SA: PME.

Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solution.* Klagenfurt, Austria: Beltz.

Moser Opitz, E., Freesemann, O., Prediger, S., Grob, U., Matull, I., & Hußmann, S. (2017). Remediation for students with mathematics difficulties: An intervention study in middle schools. *Journal of Learning Disabilities*, *50*, 724–736.

Schindler, M. & Lilienthal, A.J. (2019). Domain-specific interpretation of eye-tracking data: Towards a refined use of the eye-mind hypothesis for the field of geometry. *Educational Studies in Mathematics*, *101*, 123–139.

Schindler, M. & Lilienthal, A.J. (2018). Eye-tracking for studying mathematical difficulties – also in inclusive settings. In *Proceedings of the 42nd Conference of the IGPME* (Vol. 4, pp. 115–122). Umeå, Sweden: PME.

Sullivan, J.L., Juhasz, B. J., Slattery, T.J. & Barth, H.C. (2011). Adults' number-line estimation strategies: Evidence from eye movements. *Psych. Bull. & Rev.*, *18*, 557–563.

Van Viersen, S., Slot, E. M., Kroesbergen, E. H., Van't Noordende, J. E. & Leseman, P. P. M. (2013). The added value of eye-tracking in diagnosing dyscalculia: A case study. *Frontiers in Psychology, 4*, 679.

Van't Noordende, J. E., Van Hoogmoed, A. H., Schot, W. D. & Kroesbergen, E. H. (2015). Number line estimation strategies in children with mathematical learning difficulties measured by eye tracking. *Psychological Research, 80*, 368–378.

HOW BELIEFS SHAPE THE SELECTION OF PROOFS FOR CLASSROOM INSTRUCTION

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When implementing argumentation and proof in classrooms, selecting specific proofs for a claim is an important teacher activity. Although mathematics-related beliefs are discussed as shaping the selection process, there is so far only limited quantitative data for these claims. The present research thus examined how teacher students' endorsement of six mathematics-related beliefs influenced their selection of experimental, operative, and formal-deductive proofs. Data from N = 183 participants suggest that their endorsement of mathematics-related beliefs only partially impacts the selection of different types of proof. Moreover, after controlling for proof construction skills, only effects related to mathematics as a process of inquiry were significant. Effects of beliefs thus appear to be less profound then indicated by prior qualitative studies.

INTRODUCTION

Argumentation and proof are fundamental for mathematics as a science and thus part of secondary school classrooms worldwide (e.g., KMK, 2003; NCTM, 2000). However, curricula only set the framing conditions for argumentation and proof in classrooms. The actual implementation of proving in class is influenced by various aspects, relating to i) characteristics of the *task and proof*, ii) characteristics of the *class*, iii) characteristics of the *teaching and learning situation*, and, of course, to iv) characteristics of the *teacher*. Still, there is currently little knowledge about factors determining teachers' selection of proofs for class. Firstly, evidence is mostly from qualitative or descriptive studies (e.g., Brunner & Reusser, 2019; Furinghetti & Morselli, 2011) and results are often hard to interpret and compare as contexts and classrooms within these studies vary naturally. To address this research gap and better understand factors that play a role in the selection of proofs, the BABS I project used an experimental design based on questionnaires, which systematically controlled and varied these factors, and examined teacher students' selection of proofs for class.

One of the teacher characteristics discussed as important for lesson planning and teaching, both generally and in the context of proof, are *mathematics-related beliefs* (Furinghetti & Morselli, 2011; Philipp, 2007). This paper thus focuses on prospective teachers' mathematics-related beliefs and evaluates their impact on the selection of different types of proof for classroom instruction, also behind the background of other, possibly confounding variables such as their mathematical proof construction skills.

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 73-80. Khon Kaen, Thailand: PME.

THEORETICAL BACKGROUND

Mathematical Proof in Secondary Education

Although there are no generally accepted definitions for *argumentation* and *proof* (Reid & Knipping, 2010), mathematical proofs are commonly interpreted as mathematical argumentations that additionally satisfy certain socio-mathematical norms (Yackel & Cobb, 1996). These may, for example, relate to the permitted types of inferences within proofs, the completeness of the argumentative chain within a proof, or other aspects of proofs (see further Sommerhoff & Ufer, 2019). However, there is no generally accepted list of criteria for the acceptability of proofs in mathematical practice and those for formal mathematical proofs or derivations are rarely satisfied in everyday mathematics and in school mathematics. To address this issue, alternative definitions for proof in school contexts have been formulated (e.g., Stylianides, 2007) and multiple 'didactical types of proof' have been introduced. These types of proof (see e.g., Healy & Hoyles, 2000; Wittmann & Müller, 1988) often distinguish *experimental proofs* consisting of multiple examples, *operative proofs* that use a concrete operation, manipulation, or visual representation to show the validity of a claim, and *formal-deductive proofs*, which are embedded in a strictly deductive theory to allow more general conclusions. Although especially experimental and operative proofs may not be able to guarantee absolute validity that a certain claim holds, they are still seen as useful in mathematics classrooms (e.g., Hanna & Jahnke, 1996), in particular as i) easier to understand precursors of formal-deductive proofs and as ii) ways to focus on other functions of proof than verification (de Villiers, 1990). A common distinction, which is made in this context, are *proofs that prove* and *proofs that explain*. When preparing their mathematics classes, teachers should thus ideally consider these different types of proof, weigh up their advantages and disadvantages, and select one or multiple proofs for class. This process can be shaped by multiple factors, for example by mathematics-related beliefs, appears plausible.

Selection of Proofs for Classroom Instruction

Currently, it is mostly unclear, what teachers' selection of proofs for class is based upon. As teachers' selection of proofs be *decision-making* processes, related frameworks (e.g., Blömeke, Gustafsson, & Shavelson, 2015; Schoenfeld, 2010) have been used to outline and structure different factors that may influence teachers' selection of proofs. Prior research has related the selection of proofs to teachers' prior content and pedagogical content knowledge, to their proof skills, and to their (leading) beliefs (e.g., Brunner & Reusser, 2019; Furinghetti & Morselli, 2011).

Beliefs

Goldin (2002, p.59) regards beliefs as "multiply-encoded, cognitive/affective configurations, to which the holder attributes some kind of truth value". They are neither purely cognitive nor completely affective and can be compared to lenses that shape how we see the world (Philipp, 2007). Beliefs, belief systems as organized clusters of beliefs, as well as so-called leading beliefs are assumed to guide an individual's actions also teachers' actions in classroom. They are thus considered in many current theoretical conceptions, such as the framework by Blömeke et al. (2015) or the concept of competence (Weinert, 2001). In the context of proof, Furinghetti and Morselli (2011) investigated how teachers treat proof in their classrooms and how this is shaped by their beliefs. Based on interviews with 10 highly experienced teachers, they identified "main poles around which the instructional practice of proof revolves" (p. 597) and identified teachers' leading beliefs as "definitely" shaping teaching practices related to proof.

Based on theoretical conceptions by Ernest (1989) and by Grigutsch, Raatz, and Toerner (1998), beliefs can be distinguished into several aspects, relating for example to conceptions of the nature of mathematics or the process of learning mathematics. A large-scale study, which picked up these different facets of mathematics-related beliefs was the Teacher Education and Development Study in Mathematics (TEDS-M) (see further Tatto et al., 2012). Based on the theoretical conceptions by Ernest (1989) and Grigutsch et al. (1998) as well as existing instruments to measure beliefs, TEDS-M included scales on beliefs about *the nature of mathematics*, about *learning of mathematics*, as well as about *mathematics achievement* (Tatto, Rodríguez, Reckase, Rowley, & Lu, 2013). Beliefs about the nature of mathematics included scales on *mathematics as rules and procedures* (i.e., mathematics is a set of procedures and strict rules, which have to be learned and applied) and mathematics as a *process of inquiry* (i.e., mathematics is a tool for inquiry and discovery). Beliefs about the learning of mathematics included subscales on *teacher direction* (i.e., learning of mathematics should be teacher centered and students follow instructions) and *active learning* (i.e., students must do mathematics on their own to learn effectively). Finally, beliefs about mathematics achievement included a scale on *fixed ability* (i.e., school mathematics is accessible only to those students with according ability and mostly inaccessible to others). Results of TEDS-M showed that future teachers in their last year of training from most countries (in particular from Germany) mostly endorsed mathematics as a process of inquiry and learning of mathematics through active learning, did less agree with mathematics as a set of rules and procedures, and did not endorse learning of mathematics by following teacher direction and mathematics as a fixed ability. Moreover, Tatto et al. (2012) report a general tendency for positive correlations between mathematics as a process of inquiry and learning of mathematics as active learning with future teachers' content and pedagogical content knowledge as well as negative correlations between mathematics as a set of rules and procedures, learning as teacher direction, and mathematics as a fixed ability and future teachers' content and pedagogical content knowledge.

Research Questions

Prior research ascribes an important role to teachers' beliefs when planning their classes. Firstly, qualitative studies also report that they shape teaching practices related to proof, thus they can be expected to shape the selection of different types of proof for classroom instruction. Teachers endorsing mathematics as a process of inquiry and learning through active learning might choose experimental or operative proofs more frequently. However, prior studies did not systematically control participants' further characteristics, for example their proof construction skills, which might be confounding variables as data from TEDS-M suggests.

The present research thus addresses the following research questions: (RQ1) How do teacher students' beliefs about *the nature of mathematics*, about *learning of mathematics*, and about *mathematics achievement* relate to their proof construction skills and their higher education entry qualification? (RQ2) How are teacher students' beliefs related to their selection of different types of proof for classroom instruction, especially when controlling for their mathematical proof skills?

METHOD

Data to answer the research questions was gathered at a large German university. A total of 183 students (78 m, 104 f, 1 NA) from a teacher training program for secondary school participated in the research. Each participant had attended lectures in mathematics, mathematics education, education, and psychology. Participants received a questionnaire with six sections. The first section contained two claims from elementary number theory suitable for 8th grade teaching. Participants were asked to prove both claims. Each of their proofs was scored on a scale from 0 to 4. Results were aggregated to a proof construction score (scale 0-1). In Section 2 and Section 3, students received multiple didactical proofs for these claims based on different types of proof (Healy & Hoyles, 2000; Wittmann & Müller, 1988), systematically varying factor *i) proof characteristics* that may influence the selection probability of a proof for class. Students were then asked to select one or multiple proofs for their teaching in an 8th grade classroom. For this, participants were given a detailed description of the class and classroom setting, thus setting, and controlling factors *ii) characteristics of class* and *iv) characteristics of teaching and learning situation*. Section 4 then assessed students' appraisals for the presented proofs (Sommerhoff, Brunner, & Ufer, 2019), followed by Section 5, which assessed students' mathematics-related beliefs (Likert-type items; scale 1-6) about the nature of mathematics (12 items; subscales: rules and procedures, process of inquiry), about learning of mathematics (14 items; subscales: teacher direction, active learning) as well as about mathematics achievement as a fixed ability in general (8 items) and in the context in proof (2 items). All belief scales were taken from TEDS-M (Tatto et al., 2013), only the scale for beliefs about mathematics achievement in the context of mathematical proof was self-constructed ("Students are not able to construct proofs themselves.", "Students are not able to understand and validate proofs themselves"). Reliabilities were acceptable ($\alpha_{Mean} = .63$). Finally, Section 6 gathered demographic data, including students' higher education entry qualification.

Before addressing the research questions, descriptive statistics for students' beliefs and their proof construction skills and higher education entry qualification were calculated. To answer RQ1, correlations between participants' endorsement of the six mathematics-related beliefs, their proof construction skills, and their higher education entry qualification were calculated to estimate their relationship. To address RQ2, generalized linear mixed models with a logit link function were calculated using the selection of a proof as dichotomous dependent variable and participants' endorsement of the belief scales and the type of proof as independent variables (direct and interaction effects). Moreover, participants' proof construction skills were introduced as an additional independent variable. Finally, dependencies between answers of the same person were considered by including a random intercept. To analyze interaction effects, that is to evaluate i) if students' selection of a specific type of proof was influenced by their endorsement of the beliefs and ii) if the magnitude of this influence significantly varied between the types of proof, we calculated planned contrasts for each belief.

RESULTS

University Students' Beliefs

Descriptive results for students' beliefs underline that participants endorsed the nature of mathematics as a process of inquiry ($M_{NoM\_pi} = 4.63$, $SD_{NoM\_pi} = 0.62$), whereas they agreed less to mathematics as rules and procedures ($M_{NoM\_rp} = 3.76$, $SD_{NoM\_rp} = 0.59$). Still, they appeared to rather accept both aspects of the nature of mathematics. Further, participants strongly endorsed learning of mathematics as active learning ($M_{LoM\_al} = 5.10$, $SD_{LoM\_al} = 0.46$), whereas they clearly did not see learning of mathematics as teacher directed ($M_{LoM\_td} = 2.48$, $SD_{LoM\_td} = 0.48$). Moreover, participants did neither endorse mathematics achievement as fixed ability in general ($M_{MA\_fa} = 2.63$, $SD_{MA\_fa} = 0.67$) nor in the context of proof ($M_{MA\_fap} = 2.61$, $SD_{MA\_fap} = 0.94$). Finally, differences between both scales for mathematics achievement as a fixed ability were non-significant ($t(164) = 0.08$, $p = 0.935$), suggesting that beliefs about achievement in the context of proof do not differ from beliefs about achievement in mathematics in general. Overall, descriptive results are in line with those from TEDS-M.

Finally, descriptive analyses of students' proof construction revealed average values $M = 0.53$ ($SD = 0.37$; scale 0-1) and their average higher education entry qualification was $M = 2.08$ ($SD = 0.54$; scale 1-6, 1 = best).

Relation to Proof Construction and Higher Education Entry Qualification

Correlational analysis (Table 1) showed weak but significant *negative* correlations between students' proof construction skills and their endorsement of mathematics as rules and procedures, learning of mathematics as teacher direction, and mathematics achievement as a fixed ability in the context of proof.

		Nature of Mathematics		Learning of Mathematics		Mathematics Achievement	
		NoM_rp	NoM_pi	LoM_td	LoM_al	MA_fa	MA_fap
Proof construction	r	-0.16*	0.08	-0.19*	0.16*	-0.02	-0.20*
	p	.039	.289	.014	.035	.842	.015
HE entry qualification	r	0.04	0.04	0.03	-0.14	0.01	0.05
	p	.628	.622	.682	.062	.930	.534

Table 1: Correlations between participants' beliefs and their proof construction skills and higher education entry qualification (significant results highlighted)

In contrast, beliefs about learning of mathematics as active learning correlated weak and significantly, but *positively* with students' proof construction skills, whereas the correlation with mathematics as a process of inquiry and mathematics as a fixed ability in general did not reach significance. None of the belief scales correlated significantly with students' higher education entry qualification.

Influence of Beliefs on the Selection of Different Types of Proof

The generalized linear mixed models, calculated to evaluate if students' endorsement of the beliefs influenced their selection of proofs, showed a significant direct effect only for learning of mathematics as active learning (B = 0.38, p = .033). That is, a higher endorsement of this belief led to the selection of more proofs for class. Including interaction effects for different types of proof and evaluating them using planned contrasts revealed that this direct effect was mainly based on a significant impact of students' endorsement of learning of mathematics as active learning on the selection of experimental proofs (B_{exp} = 0.38, p = .033), whereas the effects on both other types of proof were positive, but insignificant (B_{fd} = 0.29, p = .099; B_{op} = 0.01, p = .970).

Further, detailed analysis of the interactions between type of proof and beliefs revealed a significant impact of the endorsement of the nature of mathematics as a process of inquiry on the selection of formal-deductive proofs (B_{fd} = 0.28, p = .033), whereas the effects on both other types of proof were insignificant (B_{exp} = 0.11, p = .388; B_{op} = -0.16, p = .220). Furthermore, comparing the influences of the beliefs for each type of proof showed that selecting a formal-deductive proof ($\Delta B_{\text{fd\_op}}$ = .44, p = .042) was related to the nature of mathematics as a process of inquiry in a significantly stronger way than for operative proofs.

Finally, including proof construction skills in the models weakened the effects observed for learning of mathematics as active learning (B = 0.33, p = .069; B_{exp} = 0.33, p = .069), however the observed indirect effects of the nature of mathematics as a process of inquiry remained significant (B_{fd} = 0.29, p = .034; $\Delta B_{\text{fd\_op}}$ = .44, p = .045). Across the various models, participants' proof construction skills generally showed higher effects on the selection of different types of proof than the various beliefs.

DISCUSSION

Parallel to prior qualitative research (e.g., Furinghetti & Morselli, 2011) our data mirror the importance of mathematical beliefs, as the selection of different types of proof appears to depend on the degree of students' endorsement of the beliefs analyzed in this research. In particular, data supports a significant influence of mathematics as active learning (direct & interaction effects, before controlling for proof construction skills) and for the nature of mathematics as a process of inquiry (interaction effects, even when controlling for proof construction skills) on teacher students' selection of proofs for classroom instruction. However, the according effects do not appear as pronounced as expected by qualitative research and only isolated effects reach significance.

As these results appear to question prior research regarding the magnitude of the impact of beliefs, their interpretation must be done carefully. A key difference between prior research and this research is the sample. Whereas prior research on the impact of beliefs on teaching proof in classroom has mostly focused on in-service teachers with multiple years of experience, this research focused on mathematics teacher students. Given, that the beliefs of pre-service teachers and teacher educators in TEDS-M did not differ greatly (Tatto et al., 2012, p.160) and our descriptive results are generally in line with those of TEDS-M, it appears at least surprising that the impact of beliefs should change so profoundly between university and school. Moreover, a systematic review of empirical research on teachers' decision making by Stahnke, Schueler, and Roesken-Winter (2016) mirrors our data, reporting weak correlations between beliefs and decision making as well as regression coefficients for beliefs of r = .3. Moreover, the minor influence of beliefs in our data and in the review by Stahnke et al. (2016) is also consistent with weak to insignificant correlations between beliefs and participants' content and pedagogical content knowledge in TEDS-M (Tatto et al., 2012, chapter 5) as well as to their proof construction skills in our research. It thus appears feasible that there is an influence of beliefs on the selection of proofs, but that the influence is rather weak and likely falls behind the influence of other, cognitive variables. This also fits to conceptions by Blömeke et al. (2015) or Weinert (2001), which see affective-motivational aspects and beliefs as one variable among many others that drive (teachers') behavior.

Overall, the BABS I project aimed at examining the impact of multiple factors on the selection of proofs for secondary school classrooms. The analysis above suggest that beliefs do play a role in this process, but that the impact of the considered beliefs is rather small. Thus, to better understand the selection of proofs for class, other aspects such as cognitive characteristics of the teacher or characteristics of the class should be examined more closely. Moreover, to get even more conclusive data, future studies should evaluate the impact of beliefs on teachers' behavior while systematically controlling for as many variables suggested by Blömeke et al. (2015) or Weinert (2001) as possible, in order to determine the impact of beliefs even more accurately.

References

Blömeke, S., Gustafsson, J.-E., & Shavelson, R. J. (2015). Beyond dichotomies. *Zeitschrift für Psychologie, 223*(1), 3-13. doi:10.1027/2151-2604/a000194

Brunner, E., & Reusser, K. (2019). Type of mathematical proof: Personal preference or adaptive teaching behavior? *ZDM, 51*(5), 747-758. doi:10.1007/s11858-019-01026-y

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-254). London, United Kingdom: Falmer Press.

Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: Hows and whys in the teaching of proof. *ZDM, 43*(4), 587-599. doi:10.1007/s11858-011-0316-7

Goldin, G. A. (2002). Affect, meta-affect, and mathematical belief structures. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59-72). Dordrecht, Netherlands: Kluwer Academic.

Grigutsch, S., Raatz, U., & Toerner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. [Mathematics teachers' epistemological beliefs about the nature of mathematics]. *Journal für Mathematik-Didaktik, 19*(1), 3-45.

Hanna, G., & Jahnke, H. N. (1996). Proof and proving. In A. Bishop, M. A. K. Clements, C. Keitel-Kreidt, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 877-908). Dordrecht, Netherlands: Springer

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 21*(4), 396-428.

Kultusministerkonferenz. (2003). *Bildungsstandards im Fach Mathematik für den mittleren Schulabschluss*. [Educational Standards in Mathematics for Middle Schools] Bonn, Germany: (Beschlüsse der Kultusministerkonferenz).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school Mathematics*. Virginia, United States: National Council of Teachers of Mathematics.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. *Second Handbook of Research on Mathematics Teaching and Learning, 1*, 257-315.

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education. Research, learning and teaching*. Rotterdam, Netherlands: Sense.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision-making and its educational applications*. New York, United States: Routledge.

Sommerhoff, D., Brunner, E., & Ufer, S. (2019). Appraisals for different types of proof. In M. Graven, H. Venkat, A. A. Essien, & P. Vale (Eds.), *43rd Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 101). Pretoria, South Africa: PME.

Sommerhoff, D., & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by pupils, university students, and mathematicians in the context of teaching. *ZDM, 51*(5), 717-730.

Stahnke, R., Schueler, S., & Roesken-Winter, B. (2016). Teachers' perception, interpretation, and decision-making: A systematic review of empirical mathematics education research. *ZDM, 48*(1-2), 1-27. doi:10.1007/s11858-016-0775-y

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M., . . . Rowley, G. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM)*. Amsterdam, Netherlands: IEA.

Tatto, M. T., Rodríguez, M., Reckase, M., Rowley, G., & Lu, Y. (2013). Scale development and reporting: Opportunities to learn, beliefs, and mathematics knowledge for teaching. In M. T. Tatto (Ed.), *The Teacher Education and Development Study in Mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics in 17 Countries. Technical Report* (pp. 161-174). Amsterdam, Netherlands: IEA.

Weinert, F. E. (2001). Concept of competence: A conceptual clarification. In D. S. Rychen & L. H. Salganik (Eds.), *Defining and selecting key competencies* (pp. 45-65). Washington, United States: Hogrefe & Huber

Wittmann, E. C., & Müller, G. (1988). Wann ist ein Beweis ein Beweis.[When is a proof a proof?] In P. Bender (Ed.), *Mathematikdidaktik: Theorie und Praxis: Festschrift für Heinrich Winter* (pp. 237-258). Berlin, Germany: Cornelsen.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

PRE-SERVICE TEACHERS' ENTHUSIASM FOR SCHOOL AND UNIVERSITY MATHEMATICS

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Teachers' enthusiasm for teaching mathematics has shown positive effects on affective student characteristics and achievement, yet effects of enthusiasm for mathematics were at most mixed. However, research has so far focused mostly on enthusiasm for mathematics in general, likely obscuring effects of more nuanced facets of mathematics-related enthusiasm. Based on person-object theories, teachers' enthusiasm can be expected to vary between mathematical contexts (school vs. university), subjects (e.g., calculus, algebra, probability, geometry) and may change over time. To generate evidence for these differences and confirm the person-object theory for enthusiasm, N = 232 pre-service teachers were surveyed. Results revealed significant differences between contexts and subjects as well as significant changes during university studies.

INTRODUCTION

Within the last decades, various teacher characteristics have been identified as predictive for effective classrooms and student learning. This is mirrored in a multitude of research frameworks and results, both from domain-general research (e.g., from educational psychology; Hattie, 2008) and from mathematics education research (e.g., Baumert & Kunter, 2013). Besides the long-lasting focus on cognitive teacher characteristics (e.g., teachers' pedagogical-content knowledge), research is today increasingly considering affective-motivational characteristics such as teachers' motivation, emotions, or enthusiasm (e.g., Frenzel, 2014; Hannula et al., 2016; Kunter et al., 2008, 2013; OECD, 2020). Although teacher enthusiasm has been repeatedly discussed to have several aspects such as emotional expressivity during teaching, teacher enthusiasm is often interpreted as a personal characteristic that can be seen as a "trait like, habitual, reoccurring emotion" (Kunter et al., 2008, p. 470) that relates to a certain object, for example to the domain of mathematics. Research has repeatedly shown that enthusiasm is an important element of effective, high quality teaching. In particular, there is evidence for positive direct effects of enthusiasm on affective-motivational student characteristics (e.g., Frenzel et al., 2009; Frenzel et al., 2019; OECD, 2020; Patrick et al., 2000) as well as partially mediated, positive effects on student achievement (e.g., Baier et al., 2019; Kunter et al., 2013; OECD, 2020). However, these effects mostly refer to teachers' enthusiasm for teaching mathematics, while their enthusiasm for the subject mathematics has shown at most mixed effects.

In prior research, teacher enthusiasm has so far mostly been conceptualized as a construct relating either to teaching mathematics or mathematics in general (e.g., Keller et al., 2014), without considering a more nuanced structure of mathematics. However, based on person-object theories, which are widely presumed for many affective-motivational variables such as interest (Krapp, 2002), it appears reasonable that examining only these two facets (teaching mathematics vs. mathematics) can be an

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oversimplification. A more nuanced analysis of enthusiasm in future research thus appears necessary to validly assess the effects of teachers' enthusiasm and not bias effects based on too general notions of enthusiasm. In this context, it is of particular interest, if teachers' enthusiasm differs between mathematical subjects (e.g., calculus, algebra, probability, geometry) as relatively distinct sub-areas within mathematics and between contexts (school *vs.* university), possibly mirroring the often-reported differences between school and university mathematics (e.g., Clark & Lovric, 2009; Tall, 2008). Finally, from a teacher education perspective, it would be interesting to investigate the stability and development of teachers' enthusiasm, for example during pre-service teachers' university studies.

To address these questions, pre-service mathematics teachers in their 1st to 9th semester of a university teacher education program were surveyed regarding their enthusiasm for four mathematical subjects in the contexts of secondary school and university.

THEORETICAL BACKGROUND

Enthusiasm for mathematics

In the context of teaching, enthusiasm is often connected to a certain, positively valued style of teaching, which may display a high passion or interest for a subject, be expressed via certain verbal or nonverbal behaviors, and is generally seen as a desirable characteristic of teachers (Keller et al., 2014; Kunter et al., 2011). The conceptualization as a 'characteristic of teachers', however, stresses the fact that enthusiasm goes beyond a specific instructional behavior or expressivity, but should be considered as a latent personal trait, much like trait emotions or interest. In their work, Keller et al. (2014) combine these – so far mostly disconnected – perspectives on teacher enthusiasm and provide evidence for the conceptualization of enthusiasms as a latent trait underlying teachers' positive affect and emotional expressivity.

Dispositional teacher enthusiasm, in the sense of Keller et al. (2014), has been focused by several researchers and research projects in the context of mathematics education. In this regard, Kunter et al. (2008, 2011) conceptually and empirically distinguish two facets, *enthusiasm for the subject mathematics* and the *enthusiasm for teaching mathematics* (the latter which in subsequent research is either interpreted as 'enthusiasm for teaching in general' or as 'teaching a specific subject'). Although teachers' enthusiasm for teaching mathematics proved to be among the best predictors for student achievement and mathematics enjoyment (Kunter et al., 2013), teachers' enthusiasm for the subject mathematics was found "independent of characteristics of the classes taught" (Kunter et al., 2011), that is showing low predictivity.

However, it remains unclear if the conceptual differentiation between enthusiasm for i) mathematics and ii) teaching mathematics, which is somewhat equivalent to the differentiation of i) content knowledge (CK) and ii) pedagogical content knowledge (PCK), is sufficient to capture the effects of teachers' enthusiasm in learning contexts. Like the measurement of CK and PCK, which usually focusses on CK and PCK *about a specific topic* and not regarding mathematics in general, it may be purposeful to distinguish different facets of enthusiasm for mathematics, for example based on different contexts like school and university mathematics (see also; Ufer et al., 2017) or based on mathematical subjects like calculus or geometry. This more specific focus may help to accurately analyze the impact of teacher enthusiasm as these do not obscure effects by overly broad constructs like “enthusiasm for mathematics”.

Person-object Theories

The relevance of assessing teacher enthusiasm and its effects on teaching and learning with a more specific focus, for example on different contexts or mathematical subjects, can be based on the person-object theory of interest (POI) and similar conceptions (see Krapp, 2002). These suggest that (dispositional) teacher enthusiasm should be conceptualized as a relation between the teacher and the object the enthusiasm refers to. First, this leads to the conclusion that enthusiasm is intrinsically content-specific and that research on enthusiasm thus has to consider an adequate focus and specificity of enthusiasm. Second, the conception allows for a (positive or negative) development of enthusiasm (for a specific object) over time, based on the changing relationship. For example, the introduction of formal mathematics, rigor, and proof (e.g., Tall, 2008) and the lectures and seminars on various mathematical subjects during pre-service teacher training may be drivers of such a development. They might lead to changes in pre-service teachers’ enthusiasm and, in particular, also to an increasing separation between enthusiasm for school mathematics and university mathematics.

RESEARCH QUESTIONS

Prior research has highlighted positive effects of teachers’ *enthusiasm for teaching mathematics* for student learning, while results for *enthusiasm for mathematics* have been mixed. However, it is unclear, if a more specific conceptualization of *enthusiasm for mathematics* would lead to more conclusive results and if teachers’ enthusiasm actually differs between mathematical subjects and, for example, between school and university mathematics, as implied by person-object theories. Moreover, it is unclear, how stable more nuanced facets of enthusiasm are and how they develop over time.

The present study sheds first light on these topics in the context of pre-service mathematics education by answering the following research questions:

RQ1 How does pre-service teachers’ enthusiasm differ between mathematical subjects, contexts, and during the course of university studies?

RQ2 What is the relation between pre-service teachers’ enthusiasm for the mathematical subjects calculus, algebra, probability, and geometry in the contexts school and university?

Based on the prior findings and person-object theories, we assumed that participants’ enthusiasm would differ between the examined mathematical subjects and both contexts. In particular, we assumed that participants as future secondary school teachers would be quite enthusiastic about the subjects in the context of school, however less enthusiastic about the subjects in the context of university. Moreover, we assumed that their enthusiasm would at least partially differ between the examined semesters and may show a positive or negative development, without a more specific hypothesis regarding the direction or magnitude of these developments.

METHOD

Sample

To address these research questions, 232 pre-service mathematics teacher students for upper secondary level schools from a German University were surveyed using a short online questionnaire in winter 2020. Students were recruited during regular lectures and mostly answered the questionnaire during the lecture; however, the provided link could also be used to participate after the lecture. Participation was voluntary. Information about students' distribution to different semesters in their degree program is presented in Table 1, demographic data regarding age or sex were not gathered based on data protection regulations and the absence of related research questions.

Instruments

To measure students' enthusiasm for different mathematical subjects in the context of secondary school and university, parallel sets of items for the school and university context were developed. Based on national as well as (inter)national standards for school and university mathematics education, the subjects calculus, algebra, probability, and geometry were selected to survey participants' enthusiasm. Each subject was covered extensively in school and corresponds to typical university lectures. However, participants in the first semester had only participated in university lectures on calculus and (linear) algebra (see also discussion).

For each subject, students were asked to rate their enthusiasm on a 11-point likert scale item, ranging from 0 ("absolutely not enthusiastic") to 10 ("absolutely enthusiastic"). Items for each context were clearly introduced as focusing on a secondary school or university context and each item's formulation additionally included the context (e.g., "Please rate your enthusiasm for the following subjects within school mathematics.").

Statistical Analyses

First, descriptive data was calculated for participants' enthusiasm distinguishing between contexts, mathematical subjects, and semester. Additionally, repeated-measures *t*-tests were calculated to illustrate that participants' enthusiasm differed significantly between different combinations of contexts, mathematical subjects, and semesters. Following up on these differences, a repeated-measures ANOVA was calculated using mathematical subject and context as within person factors and semester as a between person factor to determine the significance and effect of each factor as well of their interactions (RQ1). To determine the relationship between students' enthusiasm for a subject in the secondary school context and the university context (RQ2), pairwise correlations were calculated for each subject.

RESULTS

Descriptive overview of Pre-Service Teachers' Enthusiasm

The gathered data show that participants' enthusiasm for the subjects calculus, algebra, probability, and geometry is generally quite high (overall mean $M = 7.00$, $SD = 1.51$). However, descriptive data (see Table 1) show that pre-service teachers' enthusiasm differs heavily between the contexts school and university, different mathematical subjects, and between different semesters.

Semester	N	Context	Calculus M (SD)	Algebra M (SD)	Probability M (SD)	Geometry M (SD)
1	56	School	8.79 (1.71)	8.05 (2.10)	7.00 (3.08)	9.50 (1.69)
		University	3.45 (3.48)	6.59 (2.88)	2.93 (3.56)	3.23 (3.87)
3	82	School	8.77 (1.98)	7.74 (2.20)	6.60 (2.95)	9.34 (2.07)
		University	6.44 (2.73)	6.88 (2.41)	4.13 (3.30)	6.22 (3.30)
5	17	School	8.94 (2.36)	7.53 (2.65)	6.29 (2.59)	9.59 (1.54)
		University	7.29 (2.20)	6.47 (2.90)	4.41 (3.30)	7.59 (2.35)
7	44	School	9.32 (1.57)	7.98 (1.80)	6.45 (2.67)	9.55 (1.37)
		University	7.50 (2.14)	6.32 (2.42)	5.50 (2.65)	7.80 (2.17)
9	33	School	9.12 (2.01)	9.15 (1.39)	5.85 (3.33)	9.15 (2.17)
		University	7.39 (2.67)	6.18 (3.03)	4.88 (2.68)	7.70 (2.60)

Table 1: Overview about pre-service teachers' enthusiasms.

For example, first semesters' enthusiasm for calculus in school contexts was rather high ($M = 8.79$), whereas their enthusiasm for calculus at university was rather low ($M = 3.45$), corresponding to a significant difference ($t(55) = 10.8$, $p < .001$, $d_{\text{Cohen}} = 1.45$). Data also suggest that first semesters differentiate between subjects, as, for example, participants' enthusiasm for algebra ($M = 8.05$) was significantly ($t(55) = -4.49$, $p < .001$, $d_{\text{Cohen}} = -0.6$) lower than for geometry ($M = 9.50$). Finally, although the study's design was only quasi-longitudinal, data suggests a development in pre-service teachers' enthusiasm, as, for example, their enthusiasm for probability in the school context decreased monotonously ($M_{\text{1. semester}} = 7.00$, $M_{\text{9. semester}} = 5.85$). Results for other semesters and mathematical subjects are similar.

The impact of context, subject, and semester on pre-service teachers' enthusiasm

To more specifically analyze the impact of the factors context, mathematical subject, and semester, a repeated-measures ANOVA was calculated, using context and mathematical subject as within person factors and semester as between person factor. Results (see Table 2) reveal highly significant effects of all within and between factors and even highly significant two- and three-factor interactions. Focusing on the effect sizes, the factor context shows the largest impact on participants' enthusiasm ($\eta_p^2 = 0.474$), followed by mathematical subject ($\eta_p^2 = 0.233$), and semester ($\eta_p^2 = 0.109$).

Within Person Factors	*df*	*Mean Square*	*F*	*p*	η_p^2
Context	1	1848.92	204.6	<.001	0.474
Context * Semester	4	127.81	14.14	<.001	0.200
Subject	3	472.14	68.79	<.001	0.233
Subject * Semester	12	24.16	3.52	<.001	0.058
Context * Subject	3	29.23	10.25	<.001	0.043
Context * Subject * Semester	12	26.24	9.20	<.001	0.140
Between Person Factors					
Semester	4	114.4	6.96	<.001	0.109

Table 2: Repeated-Measures ANOVA with main and interaction effects.

Post-Hoc analyses regarding context, mathematical subject, and their interaction showed a multitude of significant differences (even when applying a Tukey-correction). Although semester showed a smaller effect than the factors context and mathematical subject in the repeated-measures ANOVA, its significant effect suggests a development of enthusiasm in the course of university studies. More specific analyses of participants' enthusiasms for the different subjects in the contexts school and university resulted in significantly different developments (see Figure 1).

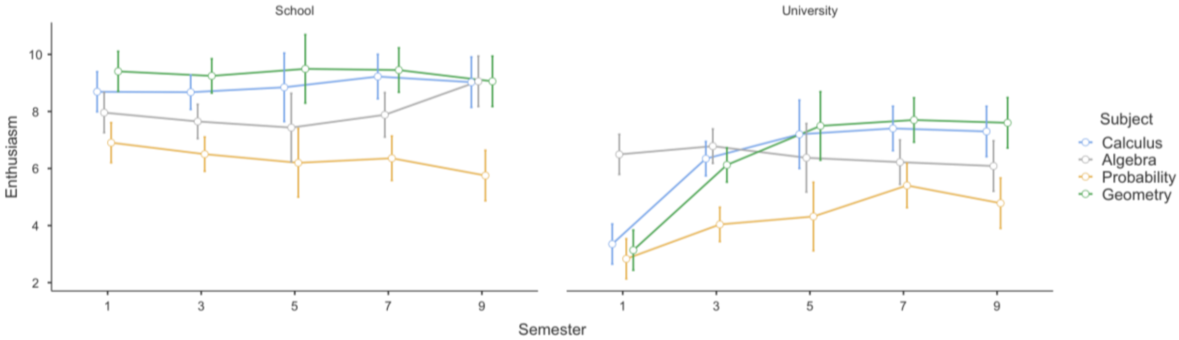

Figure 1: Quasi-Longitudinal development of participants enthusiasm for each of the four subjects in the contexts school and university from semester 1 to 9.

The relation of teachers' enthusiasm for mathematical subjects between contexts

Participants' enthusiasm in the school context (M = 8.23, SD = 1.24) was significantly (t(231) = 16.0, p < .001) higher than in the university context (M = 5.76, SD = 2.40), leading to an effect of $d_{\text{Cohen}} = -1.05$. That participants distinguished between both contexts is also mirrored by correlations between their enthusiasm for the subjects in the school and university context, each which was highly significant but at most moderate ($r_{\text{Calculus}} = .407$, $r_{\text{Algebra}} = .277$, $r_{\text{Probability}} = .413$, $r_{\text{Geometry}} = .217$).

DISCUSSION AND OUTLOOK

Based on so far mixed results on the effects of teachers' enthusiasm for mathematics (Kunter et al., 2011) and person-object theories of interest (Krapp, 2002), the present study focused on differences in pre-service teachers' enthusiasm based on i) a school or university context (Tall, 2008; Ufer et al., 2017), ii) different mathematical subjects (calculus, algebra, probability, geometry), and iii) their semester in their university teacher education program. As expected, results confirm highly significant differences between the contexts, the subjects, and even their various interactions. This can be seen as evidence that enthusiasm confirms to a person-object theory. This is also supported by the relatively low pairwise correlations between pre-service teachers' enthusiasm in school and university context as well as by the significant differences that could be observed for pre-service teachers' enthusiasm for the examined subjects from 1st to 9th semester. The latter differences are less pronounced for subjects in the school context than in the university context, which appears reasonable based on person-object theories. The pre-service teachers surveyed in the lower semesters had i) much more experience with school mathematics than with university mathematics, leading to an already consolidated relation, and ii) were predominantly exposed to university mathematics during their university studies, thus allowing their enthusiasm in the university context more potential to change and develop.

Beyond confirming the person-object theory, descriptive data also gives a first impression of the extent of pre-service teachers' enthusiasm towards different mathematical subjects in school and university. Although their enthusiasm towards the subjects in the context of school is generally quite high, it appears interesting that there is only little development in this regard. Apparently, the contents of their university studies do not relate sufficiently to these subjects in the context of school to increase pre-service teachers' enthusiasm for these. Thus, their later enthusiasm for these subjects as novice teachers is mainly based on their school experiences. In contrast, pre-service teachers' enthusiasm for the subjects in the university context changes significantly over time, likely mirroring i) that students in their first semester only had a faint person-object relation to these subjects in the university context (due to the lack of experience) and ii) the exposure to these subjects during the course of their studies. However, results may mirror the typically high drop-out for mathematics programs in the first year and thus be an artefact of a selection process with those students with less enthusiasm dropping out. Although, the quasi-longitudinal nature of this study does not allow for a conclusion on this matter, the finding may be interesting in the context of predicting drop-out and supporting students with a high drop-out probability.

Concluding, results clearly show that 'enthusiasm for mathematics' may be a too broad construct that has likely obscured effects of mathematics-related enthusiasm facets on classrooms and students. However, results of this study do not imply the existence of any effects, these will have to be confirmed by future research. In particular, it should not be expected that mathematics-related enthusiasm facets show the same indirect effects via learning support and classroom management (Kunter et al., 2013). They may, however, impact students' emotions and enthusiasm and thus be 'contagious'.

References

Baier, F., Decker, A. T., Voss, T., Kleickmann, T., Klusmann, U., & Kunter, M. (2019). What makes a good teacher? The relative importance of mathematics teachers' cognitive ability, personality, knowledge, beliefs, and motivation for instructional quality. *Br J Educ Psychol, 89*(4), 767-786. https://doi.org/10.1111/bjep.12256

Baumert, J., & Kunter, M. (2013). The COACTIV model of teachers' professional competence. In *Cognitive activation in the mathematics classroom and professional competence of teachers* (pp. 25-48). Springer.

Clark, M., & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Int J Math Educ Sci Technol, 40*(6), 755-776.

Frenzel, A. C. (2014). Teacher Emotions. In L. Linnenbrink-Garcia & R. Pekrun (Eds.), *International Handbook of Emotions in Education* (pp. 473-493). Routledge.

Frenzel, A. C., Goetz, T., Lüdtke, O., Pekrun, R., & Sutton, R. E. (2009). Emotional transmission in the classroom: exploring the relationship between teacher and student enjoyment. *Journal of Educational Psychology, 101*(3), 705.

Frenzel, A. C., Taxer, J. L., Schwab, C., & Kuhbandner, C. (2019). Independent and joint effects of teacher enthusiasm and motivation on student motivation and experiences: A field experiment. *Motivation and Emotion, 43*(2), 255-265.

Hannula, M. S., Di Martino, P., Pantziara, M., Zhang, Q., Morselli, F., Heyd-Metzuyanim, E., Lutovac, S., Kaasila, R., Middleton, J. A., & Jansen, A. (2016). *Attitudes, beliefs, motivation and identity in mathematics education*. Springer Nature.

Hattie, J. (2008). *Visible learning: A synthesis of over 800 meta-analyses relating to* *achievement*. Routledge.

Keller, M. M., Goetz, T., Becker, E. S., Morger, V., & Hensley, L. (2014). Feeling and showing: A new conceptualization of dispositional teacher enthusiasm and its relation to students' interest. *Learning and Instruction, 33*, 29-38.

Krapp, A. (2002). Structural and dynamic aspects of interest development: Theoretical considerations from an ontogenetic perspective. *Learning and Instruction, 12*(4), 383-409.

Kunter, M., Frenzel, A., Nagy, G., Baumert, J., & Pekrun, R. (2011). Teacher enthusiasm: Dimensionality and context specificity. *Contemporary Educational Psychology, 36*(4), 289-301. https://doi.org/10.1016/j.cedpsych.2011.07.001

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teachers: Effects on instructional quality and student development. *Journal of Educational Psychology, 105*(3), 805—820.

Kunter, M., Tsai, Y.-M., Klusmann, U., Brunner, M., Krauss, S., & Baumert, J. (2008). Students' and mathematics teachers' perceptions of teacher enthusiasm and instruction. *Learning and Instruction, 18*(5), 468-482.

OECD. (2020). Teacher Enthusiasm. In *PISA 2018 Results (Volume III): What School Life Means for Students' Lives*. OECD Publishing. https://doi.org/10.1787/b9ed2d5d-en

Patrick, B. C., Hisley, J., & Kempler, T. (2000). "What's Everybody So Excited About?": The Effects of Teacher Enthusiasm on Student Intrinsic Motivation and Vitality. *The Journal of Experimental Education, 68*(3), 217-236. https://doi.org/10.1080/00220970009600093

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5-24.

Ufer, S., Rach, S., & Kosiol, T. (2017). Interest in mathematics = interest in mathematics? What general measures of interest reflect when the object of interest changes. *ZDM, 49*(3), 397-409. https://doi.org/10.1007/s11858-016-0828-2

WHERE DO STUDENTS FOCUS THEIR ATTENTION ON SOLVING MATHEMATICAL TASKS? AN EYE TRACKER EXPLORATIVE STUDY

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Several international studies recognize the central role of the understanding in problem solving in the mathematics teaching-learning process. Interdisciplinary studies have shown how the type of text affects student's reading and, consequently, its performance. It emerged that "selective- reading", through which specific attention is paid to certain textual elements, often involves a lack of understanding of the problematic situation. The aim of our research is to understand how some structural and textual aspects influence the understanding of a mathematical text. This research, conducted using the eye-tracker tool, shows the results of the first phase of a larger study.

RATIONALE

Researches on mathematics education highlight the central role of the mathematics texts' understanding in the undergraduate students' learning of mathematics (Barton et al., 2004). A discussion of the different approaches to the study of the impact of the formulation of a task on the performance of the students can be found in Bolondi, Branchetti & Giberti (2018). It is recognized how attitudes related to the didactic contract in the sense of Brousseau (1988), such as the "selective readings" (Zan, 2012), in which the student focuses attention only on certain textual elements, often lead to a lack of understanding of the problematic situation. The aim of our investigation is to understand how some textual aspects and the graphic and textual arrangement influence the understanding of a mathematical text and, therefore, the students' performance. Attitudes such as the identification of isolated sentences or key words highlight widespread inability to use skills acquired in transversal areas; language training and text interpretation, rather than tools that help in the representation or communication of information, are transformed into indications of procedures to be performed (Ferrari, 2001; Radford, 2000). Some of the problem-solving processes activated in problem solving, especially in reference to the comprehension of the mathematical text and the identification of the resolution strategy, highlight these behaviors and therefore require constant attention and monitoring. And it is precisely in this direction that our research is moving, in which the processes of understanding mathematical texts are analyzed with the support of the eye-tracker tool.

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In this paper we show the results of the first phase of a larger experimental study. This phase involving 8 university students from the Faculty of Education of the University of Bozen-Bolzano. In recent years, eye-tracker technologies have become an increasingly effective tool for analysing students' learning process. The results obtained provide information on eye movement and, therefore, on the choice and catalyzation of the attention of students of different school levels during the mathematical activities and are therefore significant from an interpretative point of view of the activated resolution processes.

THEORETICAL PERSPECTIVE

The eye-tracker in the panorama of Mathematics Education

By its nature, the use of technology in the learning and teaching processes of mathematics requires an interdisciplinary approach. In Mathematics education, several studies have been conducted with the eye-tracker, studies that have also involved knowledge from other fields of study, such as computer science, neurology, biology, sociology and cognitive psychology. Cognition is closely related to body actions and the position of the body in space and time (Lakoff & Núñez, 2000). Eye movements are part of sensory experience and, following the Radford approach (2010), Their relationship with mathematical representation can shed light on how humans access mathematical knowledge. Several studies in the field of eye tracking have shown that there is a correlation between what one "looks" at and what one "thinks" (Rayner, 1998; Yarbus, 1967). These results also agree with other research which support the existence of a correlation between ocular fixations and cognitive information processing (Latour, 1962). Consequently, there has been a growing interest in eye tracking in educational research (Scheiter & van Gog, 2009). As far as the path of mathematical learning is concerned, these eye-tracking experiences seem to be in line with Duval's idea, which, starting from the famous statement "there is no noesis without semiosis" (Duval, 2006), highlights how the understanding of a concept is born from the relationship between the signifier represented by a sign, a representation and the meaning or the mathematical object. Ferrara and Nemirovsky (2005) argue that all perceptual-motor activities, related to changes in attention, consciousness and emotional states, contribute to the understanding of a mathematical concept. In the research in mathematics education, many studies (e. g. Ferrara & Nemirovsky, 2005, Andrà et al. 2009, 2015; Holmqvist et al., 2011) highlight interesting data on students' approach to the reading of mathematical text, on the transformations between different representations (formulas, graphs, words) to understand the meaning of a text. It was also pointed out that there are quantitative and qualitative differences between beginners and experts in the approach to reading of a mathematical text and precisely for these reasons in our experimentation were involved participants of different school grades with different mathematical skills.

Duval frame

The contribution of the Duval ideas to the experimentation conducted is fundamental, in particular for the fundamental role attributed to representation and visualization in understanding mathematics.

Representation refers to a large range of meaning activities: steady and holistic beliefs about something, various ways to evoke and to denote objects, how information is coded. In contrast, visualization seems to emphasize images and empirical intuition of physical objects and actions. (p.3, Duval, 1999). These two processes play a fundamental role in the process of learning mathematics and, even more, as regards the cognitive architecture concerning the apprehension of geometric concepts. Thus, in geometry it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expression of magnitude and the other for visualization. What is called a "geometrical figure" always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required. Then, students are expected to go to and from between the kind of representation that is explicitly put forward and the other that is left in the background of this discursive/visual association that forms any geometrical figure." (p. 108, Duval, 2006).

The discursive/visual association is complex by a cognitively point of view: oftentimes a contrast between this association and the common association between words and shapes and because its use goes against the perceptual obviousness (Duval, 1998). Each activity in geometry involves the use of at least three registers: the natural language register, the symbolic language register and the figurative register. Figures therefore play an important role, because they represent a concept or situation extensively. They are often much more rapidly receivable than verbal representations and various "gestaltic" mechanisms cause figure to convey information both analytically and synthetically. But not every design can function effectively as a "geometric figure". There are usually four levels of understanding of a geometric figure: perceptual, sequential, discursive and operational (Duval, 1995; 1999). A drawing acts as a geometric figure when it activates the level of perceptual understanding and at least one of the others. The perceptual level involves the ability to recognize figures (for example, distinguish shapes) and to identify components in a figure (recognize sides or other elements). The epistemological function of the perceptual level is identification. Therefore, as regards the understanding of the geometric figures, the activation of the perceptual level is fundamental. As we will see in the analysis of the experiment carried out, the observation of eye movements with eye tracking provides useful information to investigate the activation, or not, of the perceptual level of the geometric figures within the resolution of geometric problems.

THE STUDY

The results of international research have shown that the number of fixations is a reliable and sensitive measure that can provide valuable information on the attention flow of participants during mathematical activities (e.g., during the resolution of equations, Susac et al., 2014). In particular, the data from Susac et al. (2014) show positive correlations between the number and "positions" of the student of the "fixations" and the efficiency of the participants in finding the solution of the mathematical activities, suggesting that the participants who behaved well adopted winning strategies in terms of "knowing where to look for information useful for the resolution".

It is from these and other evidence from national and international literature that we have designed our study. Experimentation was conducted with the eye-tracker tool on geometric task that involved 8 university students at the Free University of Bozen-Bolzano.

The aim of the research is to understand if and how much students are aware of the strategies they put in place when faced with a mathematical question in the geometric field with an image present in the text. The eye-tracker tool has allowed us to investigate what are the students' eye movements in the resolution of mathematical tasks and to study the link between them and the students' performance. To do this we analyzed both the movements of the eyes during the resolution of the mathematical tasks (to understand the order of the fixations we used gaze plot videos), both if and to what extent the structure of the tasks affects the place where the students focus their attention (to understand if some elements of the text captured the attention more than others we dwelt on the fixations).

The questions all have the same basic characteristics: they are geometric, they have an image in the text and they have been taken from international standardized survey OECD-PISA.

The chosen tasks have been built on the quantitative results of this survey and are focused not only on the investigation of how much and how the structure of the task affects the resolution procedures, but also on how much it actually affects the students' performance.

The design and implementation of this experimentation are the result of reflections and evidence highlighted by a first pilot study conducted in collaboration with the University of Bozen-Bolzano and presented at the 14th International Conference on Technology in Mathematics Teaching (Bolondi & Spagnolo, 2019).

The items administered in this phase were constructed from the quantitative results that emerged from the international standardized administration of mathematics of the OECD PISA 2015 (OECD-PISA, 2016).

Let us observe Figure 1:

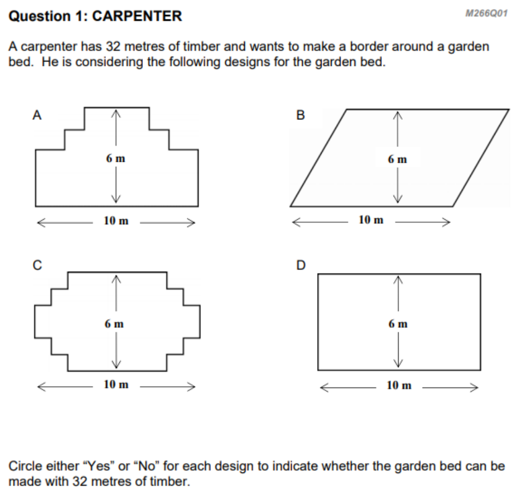

Figure 1. Question 1, OECD-PISA 2015.

Below we highlight the characteristics of the question that allowed us to implement Phase 1 of our study conducted with the eye-tracker tool. The difficulty level of the item is 6 on the overall literacy scale in mathematics. The student, in fact, is required to model a complex situation, developing a strategy in an unfamiliar context. He must show a good mastery of geometry and apply it in a real context. The concepts and mathematical knowledge fundamental to the resolution of the question are flat figures and their properties, while the fundamental mathematical skill is the following: to answer correctly it is necessary to be able to deduce, from the data provided, the lengths of the unknown segments. The question is of the type "complex multiple choice", because for each question you must select an answer between two possible (yes/no). The main difficulties are encountered in determining the overall length of the "vertical sides" of the individual figures. It is particularly difficult to determine the perimeter of Figure B, because the information on the lengths of the sides is not directly inferred from the stimulus. It is therefore necessary a good reasoning ability and a good mastery of basic Euclidean geometry to understand that the oblique sides of the parallelogram of Figure B are longer than the vertical components of the sides of the other figures (whose perimeter is exactly 32 m). Only 12.3% of Italian students were able to provide four correct answers, while 30% were able to identify three. Among the questions of geometric scope with an image in the text, this question was chosen because of the difficulties highlighted internationally. Afterwards the question was prepared for administration with the eye-tracker. In order to obtain information about the resolution processes implemented by the students, also in relation to the structure and textual characteristics of the task, the question in Figure 1 was presented in four different stimuli. This also increased the readability of the task and the calibration of the eye-tracker. The application is unchanged, but the drawings of the projects were shown to the students individually and no longer all together.

In addition, the student was asked to briefly motivate his answer. This variation was made in order to be able to observe the students' eye movements while they explain their solving strategy and retrace the process of solving the problem. It was specified to the students that they could look at the situation for as long as necessary before responding and that, in responding to a situation, they could also refer to reflections inherent to the previous situations displayed. In this way, it was possible to detect, through the recording of eye movements, the focus and permanence on certain structural and textual elements and therefore, to investigate the resolution strategies.

Eight students from the 2nd year of Primary Education at the University of Bozen-Bolzano were involved. In particular, the choice fell on students of educational science as they are not only students but also teachers in training and for this reason, they also pay attention to any educational consequences. These students were identified on a voluntary basis and each of them was involved in the task for two hours. In addition to the resolution of the question, the task included an unstructured and in-depth interview during which the student was confronted with the results of the eye-tracker tool. In this way it was possible to compare what the student thought he had looked at and what he had actually looked at.

THE RESULTS

The data was collected by the Tobi pro-lab software. Before proceeding with the analysis, we specify that to allow an analysis with the eye-tracker, the task must refer to only one stimulus at a time and be visible without scrolling the page.

The following figure (Figure 3) shows the first results of the analysis referring to stimulus 1.

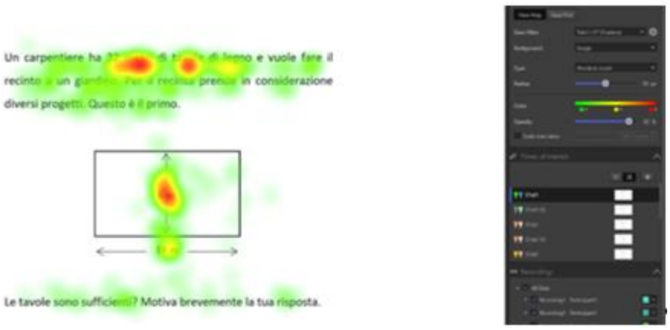

Figure 3: First results of the analysis related to stimulus 1.

In the figure we can see the part of the text where the student lingered for less time coloured in green, the part where the student lingered for a little longer coloured in yellow, and the part where the student lingered for a little longer coloured in red. The student, both in the text and in the image, focuses the attention on the numbers (red part) and the little attention given to the text can be an indication of "selective reading" (Zan, 2012). We can interpret this evidence with some of the categories of the didactic contract in the sense of Brousseau (1988).

We find a similar behaviour also in other situations. The focus on the movements of the eye when a student is reading a mathematical text provided us with insights on the solution processes and provide us outcomes framed within the structural and functional approach to semiotic (Duval, 2006). These results highlight a meaningless: students confuse the graphic elements and number representations (signifiers) with the mathematical object (signified) and they are not able to establish the correct semiotic reference to the mathematical object. The incorrect relation between graphs, numbers and sense confirms the Duval's (1995) cognitive paradox that impels students to identify semiotic representations with the mathematical object.

REMARKS.

The results of this first experimental phase provide rich insights into the structural and textual elements of mathematical tasks that capture students' attention and how much these choices affect their mathematical performance. We investigated which textual and structural characteristics influence -at least in part- the solving procedures and therefore, the students' performance. The eye-tracker tool was used to confirm the students' eye movements while solving mathematical tasks.

The students themselves recognised how their attention is often catalysed by elements that are useless for the solving procedure; the data collected, and the analyses carried out in these first phases made it possible to outline the design and implementation of a new step of the broader research in which this experimental study is inserted.

References

Andrà, C., Arzarello, F., Ferrara, F., Holmqvist, K., Lindström, P., Robutti, O., & Sabena, C. (2009). How students read mathematical representations: An eye tracking study. In *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49-56). PME Thessaloniki, Greece.

Barton, B., Chan, R., King, C., & Neville-Barton, P. (2004). The mathematical discourse of advanced undergraduate mathematics. In *Proceedings of 27th Mathematics Education Research Group Conference* (pp. 79-86).

Bolondi, G., Branchetti, L., Giberti, C. (2018) A quantitative methodology for analyzing the impact of the formulation of a mathematical item on students learning assessment. *Studies in Educational Evaluation*, vol.58, 2018, 37-50.

Bolondi, G., Spagnolo, C. (2019), "An eye-tracking language-mathematics comparative study on argumentation". *Proceedings of 14th International Conference on Technology in Mathematics Teaching (ICTMT14),* Essen, Germany.

Brousseau, G. (1988). Le contrat didactique: le milieu [The didactic contract: the milieu]. *Recherches en Didactique des Mathématiques*, 9 (3), 309-336.

Duval, R. (1995). Geometrical Pictures: Kinds of representation and specific processing. In R. Suttherland and J. Mason (eds.), *Exploiting Mental Imagery with Computers in Mathematics Education*. Springer, Berlin, pp. 142–157

Duval, R. (1998). Geometry from a cognitive point a view. In *C. Mammana and V. Villani* (eds.), Perspectives on the Teaching of Geometry for the 21st Century, Kluwer Academic Publishers, Dordrecht, pp. 37–52.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–26). Cuernavaca, Morelos, Mexico.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational studies in mathematics*, 61(1-2), 103-131.

Ferrara F., & Nemirovsky, R. (2005). Connecting talk, gesture, and eye motion for the microanalysis of mathematics learning. In: H.L. Chick & J.L. Vincent (Eds.), *Proceedings of PME 29* (Research Forum) 1, 137-142. University of Melbourne, PME: Melbourne, Australia.

Ferrari, P. L. (2001) Understanding Elementary Number Theory at the Undergraduate Level: A Semiotic Approach, in Campbell, S.R.&R.Zazkis (Eds.) *Learning and Teaching Number Theory: Research in Cognition and Instruction*, Westport (CT, USA): Ablex Publishing, pp.97-115

Holmqvist, K., Andrà, C., Lindström, P., Arzarello, F., Ferrara, F., Robutti, O., & Sabena, C. (2011). A method for quantifying focused versus overview behavior in AOI sequences. *Behavior research methods*, 43(4), 987-998.

Lakoff, G., & Núñez, R. E. (2000). Where mathematics comes from: How the embodied mind brings mathematics into being. *AMC, 10*(12), 720-733.

Latour, P. L. (1962). Visual threshold during eye movements. *Vision Research*, 2, 261–262.

OECD. (2016). PISA 2015 Technical report. Paris, FR: OECD Publishing.

Radford, L. (2000) Signs and meanings in students emergent algebraic thinking: a semiotic analysis, *Educational Studies in Mathematics*, 42: 237-268.

Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the learning of mathematics*, 30(2), 2-7.

Rayner, K. (1998). Eye movements in reading and information processing: 20 years of research. *Psychological Bulletin*, 124(3), 372–422.

Scheiter, K., & Van Gog, T. (2009). Introduction: Using eye tracking in applied research to study and stimulate the processing of information from multi-representational sources. *Applied Cognitive Psychology*.

Susac, A. N., Bubic, A., Kaponja, J., Planinic, M., & Palmovic, M. (2014). EYE MOVEMENTS REVEAL STUDENTS’STRATEGIES IN SIMPLE EQUATION SOLVING. *International Journal of Science and Mathematics Education*, 12(3), 555-577.

Yarbus, A. L. 1967. *Eye movements and vision (B. Haigh, Trans.)*, New York: Plenum Press.

Zan, R. (2012). La dimensione narrativa di un problema: il modello C&D per l'analisi e la (ri) formulazione del testo. *L’insegnamento della matematica e delle scienze integrate*, 35(2), 107-126.

AESTHETICS IN EARLY GEOMETRY

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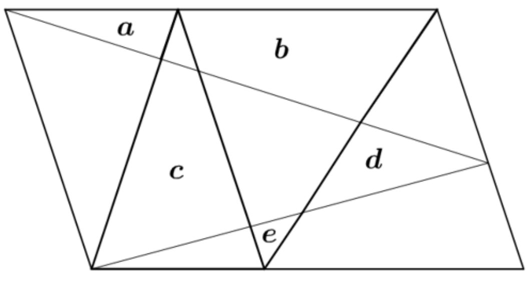
Aesthetic mathematical solutions involve clarity, simplicity, conciseness, an element of surprise, and more. Although the elaboration of aesthetics may develop mathematical thinking, the amount of studies of aesthetics in ordinary teaching is limited. Perhaps this is due to a conception that aesthetics is only for experts and the gifted. This paper presents a different view. The paper displays 7th graders' solutions of geometrical area calculations and students' comments on their solutions. The solutions involve auxiliary constructions and structure decompositions. Both cumbersome and aesthetic solutions are displayed. Some of the cumbersome solutions were erroneous. Aesthetic solutions were simple, very concise, and somewhat surprising. Students who offered different solutions expressed appreciation of the appealing characteristics of aesthetics.

INTRODUCTION

Mathematics is a science of patterns (Schoenfeld, 1992). The discovery of illuminating patterns yields an emotional excitement of the beauty of mathematics. Poincaré said: "The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful" (in Huntley, 1970, p. I). Beauty is correlated with aesthetics, which according to Webster (1973) is "a branch of philosophy dealing with the nature of the beautiful". What is mathematical aesthetics? Can non-mathematicians be delighted as well? Can it be advocated? Before addressing these questions, we begin with a short illustration.

Given the following parallelogram and the values of the areas *a, b, c,* and *d*, can you tell the area of *e*? (There are no special assumptions on angles or lines.)

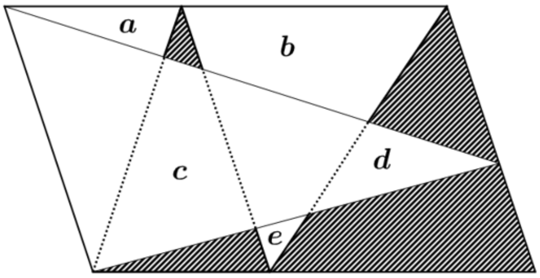


The problem involves basic geometry. We posed it to mathematics teachers, who found it challenging and engaging. (The reader is welcome to try it before turning the page.) Some teachers noticed that the area of half of the parallelogram may be specified in two 'vertical' ways (each with two triangles) and in two 'horizontal' ways. They named the unnamed areas and developed two equations – one with

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 97-104. Khon Kaen, Thailand: PME.

equality between the two 'vertical halves' and one with equality between the two 'horizontal halves'. At this point, some felt at loss, but a few reached an illuminating pattern.

If you equate the two 'vertical' triangles that include the areas c and d and the two 'horizontal' triangles that include a, b, and e, the two sides of the equation 'intersect' at the grey areas below, which may therefore be discarded. You get: $c+d = a+b+e$.



The key is the recognition of the illuminating intersection. Some may argue that this is straightforward. In our experience, this recognition is not immediate, and sometimes not reached. Many of the teachers expressed surprise from the illuminating observation. Its clarity and brevity were appealing.

Clarity, brevity and surprise are regarded as (some) aesthetic values. Poincaré (1956) indicated that a sudden illumination may surface during problem solving, from the unconscious to the conscious, and underlie the aesthetics of a solution process. Such a phenomenon may profoundly affect one's emotional sensibility.

Hofstadter (1979) believed that as engaging as it is, aesthetics of a mathematical argument cannot be defined in an inclusive or exclusive way. Papert (1980) argues that aesthetics plays the most central role in mathematical thinking; yet he did not offer a clear definition of aesthetics and did not draw a clear line between aesthetic and logical. Birkhoff (1956), on the other hand, offered to measure aesthetics with a formula, which relates order and complexity (in opposite ratios).

Halmos (1981) recognized clarity and structural brevity as primary values of the elegance of thought. Dienes (1964) underlined the power of a single, illuminating argument, or step; and Hardy (1940, in Johnson & Steinberger, 2019) offered the aesthetic values of: seriousness, generality, depth, unexpectedness, inevitability, and economy (related to brevity).

Dreyfus and Eisenberg (1986) related to these values, and added a few more. They suggested the chain of: clarity → simplicity → brevity → conciseness → structure → power → cleverness → surprise. They underlined the relevance of as little prerequisite knowledge as possible, a notion that is tied to clarity and simplicity.

Poincaré and Hardy discussed aesthetics for mathematicians, who are acquainted with mathematical beauty and its elicited emotional sensibilities. Polya, Gardner and others exemplified to laymen mathematical beauty and its appealing solutions.

Should aesthetics be shown and advocated to mathematics novices? Papert argued four decades ago that this should indeed be the case. He expressed his frustration from its absence. Dreyfus and Eisenberg strongly argued for its relevance, and indicated that its appreciation is "… yet to be achieved in the mathematics classroom". They offered several suggestions for its embedment, including comparisons between cumbersome and elegant solutions, which they demonstrated in their work.

Bishop (1991) claimed that mathematics education should include the teaching of values of the discipline, including aesthetics. Sinclair (2009) underlined mathematical enculturation, and argued that "… aesthetic values should be explained and shared in the classroom level". She added that although aesthetics is most commonly viewed while applied to finished products, it can also arise during exploration and inquiry. This approach was also advocated by Dreyfus and Eisenberg.

Sinclair (2001) conducted some initial work in this direction, with middle school students, who used a colour calculator in exploring patterns of fractions and decimals. She observed that students turned to aesthetic values in activities of choosing problems, generating conjectures, and evaluating their solutions. The students related to values such as fruitfulness, visual appeal, and surprise (Sinclair, 2009). De Freitas and Sinclair (2014) indicated in a later work that aesthetics is often associated with mathematicians and gifted students, despite efforts to 'democratize' its access and experience.

Johnson and Steinberger (2019) examined laypeople's ratings of landscape paintings, music pieces, and mathematical arguments. They observed that "even laypeople share an intuitive sense of mathematical aesthetics" and that "this sense sharpens with mathematics training". The rating tasks they posed involved impressions and judgements, but not problem solving.

In this paper we examine students' aesthetics illuminations in common middle school problem solving. We relate to accepted aesthetic values offered by earlier studies. In the next two sections we display our study's methodology and findings. We then discuss the findings and advocate the relevant role of aesthetics teaching in an ordinary mathematics class.

Our objective is two-fold: 1. to show that students may reach aesthetic solutions even at the very basic level of geometry; and 2. to show that students may appreciate aesthetic solutions and become aware of their appealing characteristics. Such appreciation may encourage teachers to underline aesthetic values in their teaching.

METHODOLOGY

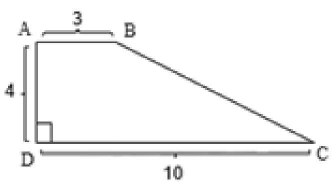
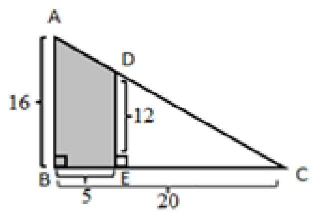
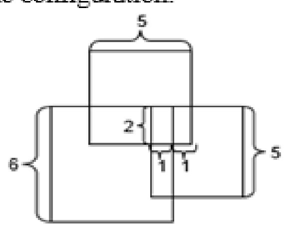
The study presented here involves an examination of students' area calculations of elementary geometrical structures. Our primary intention was to examine facets of utilization of the heuristic of decomposition. The problems posed to the students were very basic, and we did not originally expect unique, stimulating solutions. However, in an earlier pilot study we were surprised to see some stimulating, unanticipated solutions. Following this phenomenon, we decided to widen the perspective of our examination of decompositions solutions, and add a component that focuses on aesthetics. The results presented below reveal findings of this component.

Sample

The study's sample included 68 7th grade students, from four junior-high schools. The 7th graders were all acquainted with the fundamental geometrical structures of triangle, rectangle and square, and the terms circumference, area, diagonal, and angle. During their studies they have seen diverse structure decompositions and auxiliary constructions.

Tools

The study's questionnaire included 11 problems of geometrical area calculation. The problems involved geometrical structures constructed from compositions of the generic structures of triangle, rectangle and square. The table below displays three of the 11 questionnaire tasks, for which we display aesthetic solutions.

Calculate the following area.	Given the two right triangles ABC and DEC, calculate the area of ABED.	Three squares are placed one on top of the other. Calculate the total area of the configuration.
 Fig 1.	 Fig 2.	 Fig 3.

The compositions in the questionnaire problems involved: *concatenation* of generic structures (i.e., 'glued' structures), *inclusion* (such as the trapezoid inside a triangle in Figure 2), and *interleaving* (such as the intersected squares in Figure 3).

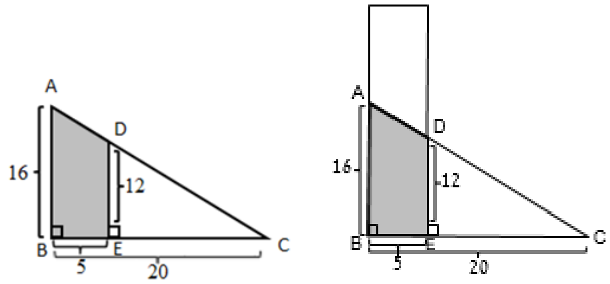
Process

The students were given 90 minutes to solve the whole questionnaire. Following their written answers, about 20% of the students were interviewed about their solutions. We focus here on solutions to the three problems above, for which we received aesthetic solutions. We examined solutions using aesthetic values offered in previous studies, particularly those indicated by Dreyfus and Eisenberg (1986). This includes: clarity, simplicity, conciseness, brevity, and surprise. In addition, Dreyfus and Eisenberg indicated the relevance of observing differences between less (or non-) aesthetic solutions and aesthetic ones. We relate to this as well.

FINDINGS

The different solutions to the three problems are displayed with short statistics and student sayings. We begin with the problem in Figure 2 above.

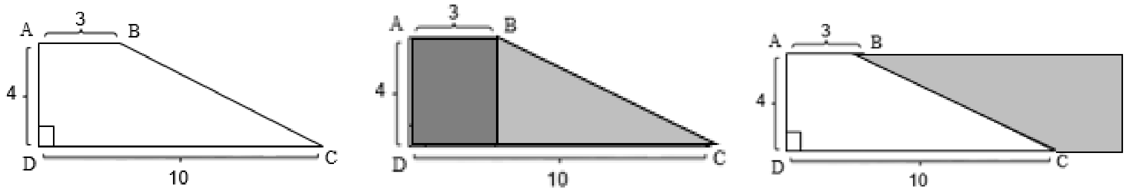
The students were not acquainted with the trapezoid area formula. About half of them (52%) calculated the difference between the two triangles in the given figure displayed on the left below. A fifth of them erred in their calculations, mostly with subtraction operations. Interviewed students indicated that this solution was immediate for them. About a quarter (26%) of the students decomposed the trapezoid into a rectangle and a triangle. Other students offered direct trapezoid area calculations. More than half of them were erroneous. One student was very creative and offered the solution on the right below. Notice that this very simple and elegant solution does not use all the given data. It involves no triangle calculations and no subtractions; only a single rectangle calculation.



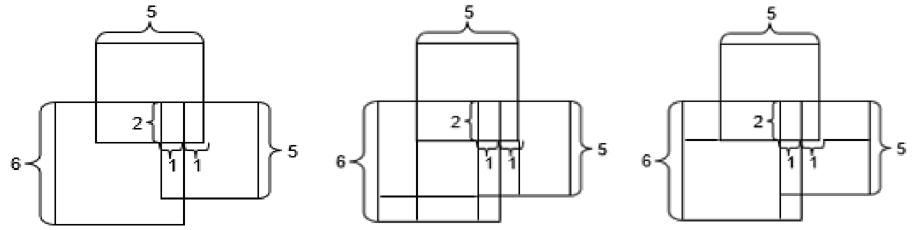
We regard this solution as aesthetic. It adheres to aesthetic values described earlier. It is clear, simple, concise, and somewhat surprising. Its auxiliary ‘doubling’ construction is powerful, and encapsulates a glimpse of brevity. It involves symmetry, a notion tied to aesthetics (Sinclair, 2004). And it is less error-prone.

This solution was shown to several students who offered the common solution. They found it appealing. One said: “This is beautiful … nicer than mine … it involves a kind of complete structure”. Another indicated: “This is surprising. I have not seen such a solution before … we did not learn in this way; we learned to decompose into parts”. A third noted: “This is interesting. I thought there is only one solution. I see that there are other ways … next time I will also seek ways to which I am not used …”.

The solutions of the problem in Figure 1 were similar in nature to those of Figure 2. This time the common solution involved an implicit inner construction and decomposition into a rectangle and a triangle. The given structure is on the left below, the common solution is in the middle, and the aesthetic solution is on the right.

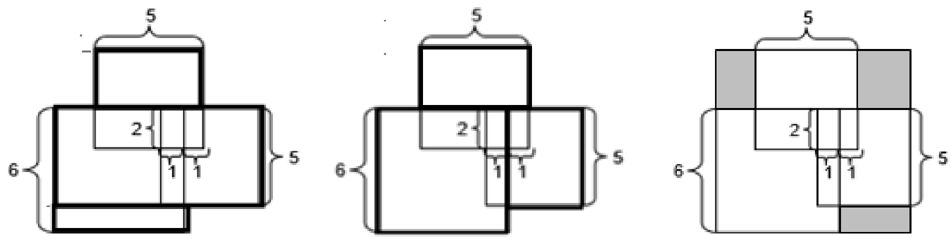


The problem in Figure 3 involved several aesthetic solutions. It was challenging to the students. About one third (30%) of the students did not solve the problem. About a sixth (15%) attempted inclusion and exclusion calculations. Only two of them provided the correct answer. One third (34%) of the students turned to cumbersome decompositions into many sub-structures. Below is the original structure of the posed problem (on the left), and two (out of several) decompositions into many fragments.



Only one quarter of the students who offered these decompositions provided correct calculations, as careful subtractions/additions are needed. Students said that the problem was difficult. In follow-up interviews they said: "... it seemed hard by looking at the figure ..."; "... the fact that the total area is divided into many parts is confusing ... "; "... too many parts and calculations; I calculated twice, to be sure ...". It seemed that a primary reason for these feeling was the difficulty to identify suitable ‘non-atomic’ sub-structures, which include inner lines that should be ignored.

About one fifth (18%) of the students capitalized on ‘non-atomic’ sub-structures and offered the two solutions on the left below, which decompose the original structure into only 3 concatenated parts. Their calculations were correct.



These solutions express clarity, simplicity, conciseness and brevity. They involve an element of abstraction, as ‘noisy’, unnecessary lines are ‘masked’. We regard these solutions aesthetic. The middle one involves only lines of the original structure, and requires only two subtractions. The left one involves an additional inner construction.

Interviewed students, who offered other solutions and saw these, expressed surprise. One said: “Wow, I do not believe that I made it so complicated … I did not see that”. Another indicated: “I realize that I could decompose the area into much fewer parts”.

A student who offered one of these aesthetic solutions mentioned a sudden change of point of view (as subconsciously occurs with aesthetics (Poincaré (1956)): "... at first I was focusing on subtracting intersected areas … suddenly I saw this solution …”. Another followed a different course: “I tried to separate parts with my eyes … and gradually got there”. It seems that he was aware of his problem solving process, and sought a simple and elegant solution.

Three students (5%) provided the creative, outer auxiliary construction on the right above. This solution embeds a clever, elegant utilization of the notion of complement. Complement encapsulates an element of surprise. One student who offered it indicated that she was seeking a simple, illuminating solution. Her awareness of simplicity and illumination express a tendency to aesthetics.

DISCUSSION

We believe that mathematical aesthetics should be a relevant notion for students. Aesthetics does not require extra knowledge and resources. Its beauty may stem from its simplicity. It may be embedded in school learning materials; and when it comes to geometry, it is relevant with calculations (in addition to proofs).

Our findings reveal three observations: 1. aesthetics is relevant in problem solving of common middle school problems; 2. young students are capable of reaching aesthetic solutions; and 3. both students who reach, and students who do not reach elegant solutions appreciate aesthetic values and can become aware of them.

One theme of aesthetic solutions indicated by Dreyfus and Eisenberg (1986) is that of little (or no) pre-requisite knowledge. This was apparent in our initial illustration, as well as in the aesthetic solutions in the finding. Such a theme facilitates the exposure of aesthetics to everyone. Clarity, simplicity, conciseness and brevity are compelling when comparing aesthetic solutions to cumbersome ones, as was apparent here.

Many of the cumbersome solutions evolved from the first ideas that came to mind. Those who went for the first idea indicated in interviews that they could have done better had they been more aware, explored alternatives, and progressed more carefully.

The students who offered aesthetic solutions demonstrated three essential geometry problem solving elements – competence in turning to concise and relevant 'non-atomic' sub-structures; outer auxiliary constructions; and capitalization on the notions of complement and symmetry.

Students who offered cumbersome solutions often overlooked relevant 'non-atomic' sub-structures, and mentioned "too many lines", that led to a "blurred" picture. Many turned to inner constructions, and did not attempt outer ones. Yet upon seeing the aesthetic solutions, they appreciated their appealing characteristics. They indicated that in the future they will seek an additional, illuminating perspective. Awareness of alternative solutions is advocated in problem solving, and elaborates creativity (e.g., Levav-Waynberg & Leikin, 2012). The findings here show that awareness and competence may not be expected only from gifted students, but also from students of ordinary classes.

The findings correlate with finding of Johnson and Steinberger (2019) of recognition of aesthetic values by non-experts. Johnson and Steinberger suggested to elaborate aesthetic awareness with geometry proofs. We displayed it here with geometry calculations. Teachers should be encouraged to underline aesthetic values through appealing, and possibly surprising solutions such as those presented here.

References

Birkhoff, G.D. (1956). Mathematics of Aesthetics. In J.R. Newman (Ed.), *The World of Mathematics*, 4, 7th edition. New York, NY: Simon & Schuster, 2185-2197.

Bishop, A. (1991). *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

de Freitas, E. & Sinclair, N. (2014). The politics of the mathematics aesthetic: curricular con(sens)us and acts of dissensus. *Philosophy of Mathematics Education Journal*, *28* [Available in http://people.exeter.ac.uk/PErnest/pome28/].

Dienes, Z. (1964). *Building Up Mathematics* (2nd edition). London: Hutchinson Educational Ltd.

Dreyfus, T. & Eisenberg, T. (1986). On the aesthetics of mathematical thought, *For the Learning of Mathematics*, *6*(1), 2-10.

Halmos, P.R. (1981). The heart of mathematics, *The American Mathematical Monthly*, *87*(7), 519-524.

Hofstadter, D.R. (1979). *Goedel, Escher, Bach: An Eternal Golden Braid.* New York, NY: Basic Books

Huntley, H.E. (1970). *The Divine Proportion: A Study in Mathematical Beauty*. New York, NY: Dover.

Johnson, S.G.B. & Steinberger, S. (2019). Intuitions about mathematical beauty: A case study in the aesthetic experience of ideas. *Cognition*, *189*, 242-259. https://doi.org/10.1016/j.cognition.2019.04.008.

Levav-Waynberg, A. & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry, *Journal of Mathematical Behaviour*, *31*(1), 73-90.

Papert, S. (1980). *Mindstorms: Children, Computers and Powerful Ideas*. New York, NY: Basic Books.

Poincaré, H. (1956). Mathematician's Creation. In J.R. Newman (Ed.), *The World of Mathematics*, 4, 7th edition. New York, NY: Simon & Schuster, 2041-2052.

Schoenfeld, A. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In Grouws, D. A. (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. New-York, NY: Macmillan, 334-370.

Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics, 21*(1), 25-33.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, 6(3), 261–284. https://doi.org/1207/s15327833mtl0603_1.

Sinclair, N. (2009). The aesthetic as a liberating force in mathematics education. *ZDM*, *41*(1), 45-60.

Webster's New Collegiate Dictionary (1973). Springfield, Massachusetts, USA.

BEGINNING UNIVERSITY MATHEMATICS STUDENTS' PROOF UNDERSTANDING

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Research has highlighted that students of all age have difficulties with mathematical proof, many of which can be traced back to a limited understanding of proof. Despite various research focusing on aspects of proof understanding, a generally accepted framework for proof understanding, systematizing its various aspects, is missing so far. We thus outline a framework for a persons' proof understanding along several important perspectives, foci, and aspects, for example distinguishing concept-oriented and action-oriented foci. To substantiate the latter distinction and the value of the framework, a first explorative empirical study was conducted with N = 72 beginning mathematics university students', indicating that concept-oriented and action-oriented methodological knowledge can be distinguished.

INTRODUCTION

The concept of proof and handling proofs adequately (e.g., constructing, comprehending, validating) play a central role in mathematics as a scientific discipline and in mathematics education in school and university (Hanna & Jahnke, 1996). However, as schools – for good reason – do not focus on mathematics as a deductive, axiomatic system, it is challenging to allow learners to get an authentic image of scientific mathematics and to get an adequate understanding of the concept and use of mathematical proof. Adding to this is the fact that the concept of proof is not consistently defined in mathematical practice and partially differs between mathematical communities. This is, for example, mirrored in different acceptance criteria for validating mathematical proofs (Sommerhoff & Ufer, 2019). Overall, empirical research focusing on school and the transition to university has confirmed that students have problems with mathematical proofs (Healy & Hoyles, 2000; Kempen & Biehler, 2019; Sommerhoff & Ufer, 2019). Many of the reported results suggest, that students' do not possess an adequate understanding of proof.

Research from mathematics education and philosophy of mathematical practice (see further Hamami & Morris, 2020) has focused on proofs and handling proofs from a variety of perspectives. These include research on the level of mathematics as a discipline, for example regarding norms and values for proof (Dawkins & Weber, 2017) or functions of proof (De Villiers, 1990). In contrast there is also research on the level of individual persons and their individual understanding of proof. Here, one approach is to examine how persons handle (e.g., construct, validate) exemplary mathematical proofs and to infer, for example, students' proof schemes (Harel & Sowder, 1998) or their methodological knowledge (Heinze & Reiss, 2003) from

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these actions. In contrast, other approaches focus on proofs on a conceptual level – without using specific examples – for example to examine the acceptance of proofs in mathematical journals (Andersen, 2018).

However, the concept of a persons' understanding of proof has so far not been extensively addressed by research and only selective evidence on some aspects of proof understanding exist. In particular, there is currently no generally accepted framework of proof understanding outlining the various aspects of proof understanding, which would, for example, allow to systematize future research.

The present paper thus outlines a framework for systematizing different aspects related to proof understanding. For this Research Report, it subsequently empirically focuses on one aspect of the framework, namely methodological knowledge. In this regard, methodological knowledge focusing on proofs on a conceptual level as well as methodological knowledge focusing on actions with exemplary proofs, are empirically compared in the context of beginning mathematics university students.

THEORETICAL BACKGROUND

A framework for a persons' proof understanding

Undoubtedly, the concept of mathematical proof and its understanding are essential for mathematics and for mathematics education (Hamami & Morris, 2020; Reid & Knipping, 2010). Based on prior research, a **disciplinary perspective** on proofs and handling proofs can be taken (e.g., Dawkins & Weber, 2017; in the context of school e.g., Stylianides, 2007). This perspective includes **philoscientific aspects** that relate to an ideal view of mathematics and proof, for example outlining which methods are allowed or which rules and principles must be fulfilled for an ideal proof. At the same time, it includes **socioscientific aspects** that relate to mathematical practices within (various) mathematical communities, for example outlining the factors for the acceptance of proofs in practice or the use and functions of different types of proof.

This disciplinary perspective on proofs and handling proofs is essential as basis and orientation for an **individual-psychological perspective** on proofs and handling proofs that addresses individual persons' proof understanding. This perspective includes i) **concept-oriented aspects**, referring to the individual understanding of mathematical proof outside of mathematical action contexts, which can for example be examined by using questionnaires or evaluating statements regarding the concept of proof (e.g., Andersen, 2018), and ii) **action-oriented aspects**, referring to an individuals' proof understanding while handling concrete proofs, which can be examined by focusing on activities like proof validation or construction (e.g., Healy & Hoyles, 2000; Heinze & Reiss, 2003). However, in prior research both aspects of the individual-psychological perspective on understanding of proof are not clearly distinguished and mostly used equivalently.

Obviously, the individual-psychological perspective is related to the disciplinary perspective, as the disciplinary perspective should (at least to a large degree) correspond to what is taught about mathematics and mathematical proof in school and university and thus be the basis for what a person develops as an individual-psychological understanding of proof.

Methodological knowledge as a facet of a persons' proof understanding

The term methodological knowledge was first introduced by Heinze & Reiss (2003) and refers to "understanding and knowledge of correct mathematical proof procedures" (Heinze & Reiss, 2003, p. 2). Based on their theoretical and empirical analysis, methodological knowledge comprises three central sub-aspects *proof scheme*, *proof structure*, and *chain of conclusions*. Proof scheme refers to knowledge about the nature arguments used in a mathematical proof, for example, that only deductive arguments are valid. Proof structure refers to the overall arrangement of arguments within a proof and its in-principle suitability to prove the given claim. For example, a circular arrangement of arguments would not be valid. Finally, chain of conclusions neither focuses on individual arguments nor on the proof as a whole, but on the sequencing of individual arguments within a proof, in particular regarding possible gaps, unwarranted conclusions, or erroneous warrants.

Methodological knowledge clearly relates to an individual-psychological perspective on proof understanding, as it relates to a person's knowledge. Here, it could relate i) to a concept-oriented focus, that is on the mere availability of methodological knowledge (e.g., measured by the item "Please evaluate the following statement: *Mathematical proofs that use the statement to be proved as a premise are particularly elegant.*"), or ii) to an action-oriented focus, that is on the availability and use of methodological knowledge for specific tasks (see Figure 1).

Ben has to prove the following proposition:
The sum of three consecutive natural numbers is divisible by 3.
Ben's purported proof:
I know this from school. Our textbook contained a proof that this is valid for every natural number. There, it was shown that: $3+4+5=3+3+1+3+2=3\cdot 3+3$
This proves the proposition.

Please rate the following statements:

a) The steps in the above argumentation build logically on each other.
b) The argumentation only uses valid arguments.
c) The argumentation presupposes what has to be shown.

Figure 1: Task for action-oriented methodological knowledge focusing on the sub-aspect proof scheme (translated).

But although methodological knowledge can refer to (and be measured) both regarding a concept-oriented and action-oriented focus, prior research has mostly examined action-oriented methodological knowledge (Healy & Hoyles, 2000; Heinze & Reiss, 2003; Sommerhoff & Ufer, 2019). In the presentation of the findings, the emphasis was often put on the concept-oriented aspect, but the measurement itself corresponded to the action-oriented aspect. Thus, either the measurement of concept-oriented methodological knowledge should be considered as indirect (via action-oriented methodological knowledge), or concept-oriented and action-oriented methodological knowledge were implicitly considered as closely related. Still, this relation is questionable, as, for example, it appears reasonable that the sub-aspects of action-oriented methodological knowledge are linked but in-principle independent of each other (Heinze & Reiss, 2003), as persons' may be good at identifying circular reasoning but not in finding gaps in proofs. However, for concept-oriented methodological knowledge, this independence may be less evident.

Overall, evidence for a close relation of concept-oriented and action-oriented methodological knowledge as well as on possible differences regarding the relation of the three sub-aspects within either concept-oriented or action-oriented methodological knowledge is still missing.

RESEARCH QUESTIONS

Based on the outlined framework for a persons' proof understanding, the present paper investigates beginning university students' proof understanding from an individual-psychological perspective. For this, it analyzes similarities and differences of concept-oriented and action-oriented methodological knowledge, thus creating evidence for the relevance of distinguishing both aspects in the framework for proof understanding. For this, the following research questions were focused:

> (RQ1) How do the sub-aspects proof scheme, proof structure, and chain of conclusions relate to each other either within concept-oriented or within action-oriented methodological knowledge?
>
> (RQ2) How do concept-oriented methodological knowledge and action-oriented methodological knowledge relate to each other with regard to the sub-aspects proof scheme, proof structure, and chain of conclusions?

METHOD

Sample

To answer these questions, $N = 72$ (46 m, 26 f) future students from a German university, enrolled in a degree program with a focus on mathematics (i.e., mathematics, computer science) were surveyed in an online study. The sample consisted of beginning students only who had not heard any content of their studies at the time of the survey.

Instruments

The questionnaire used in the study consisted of multiple sections focusing on understanding of proof as well as on more general information about the participants, for example demographic data. For the above research questions, only three sections are of relevance: demographic data, concept-oriented methodological knowledge, and action-oriented methodological knowledge.

Concept-oriented methodological knowledge was assessed using 16 statements focusing the three sub-aspects proof scheme (10 items), proof structure (2 items), and chain of conclusions (4 items). The students were asked to evaluate the statements on a 6-point Likert scale with answers ranging from “Not true at all” to “Totally true”. One item used to measure the sub-aspect chain of conclusions of concept-oriented methodological knowledge was “In a mathematical proof, each step can be concluded from the previous”. The concept-oriented items were analyzed based on an ideal concept of mathematics/proof.

To assess action-oriented methodological knowledge, a task format based on the tasks used by Healy & Hoyles (2000) was used, in particular focusing on the activity of proof validation. The students were presented six purported proofs, all of which contained errors, and were then asked to judge each purported proof regarding each of the three sub-aspects of methodological knowledge (see Figure 1). For each sub-aspect of methodological knowledge, a 6-point Likert item was used ranging from “strongly disagree” (1) to “strongly agree” (6).

The purported proofs focused on secondary school mathematics and both proofs and contained errors are representative for proofs and errors the participants should have experienced at school. According to norms in school, the contained errors should clearly be identifiable as errors in proof. Thus, item difficulty was assumed to be reasonable. Of the six presented purported proofs, four contained an error regarding the proof scheme. One proof uses a circular argument and thus contained an error regarding the proof structure. In the last proof, one step cannot be concluded from the previous steps, thus leading to an error regarding the chain of conclusions.

Statistical analyses

For concept-oriented methodological knowledge, statements corresponding to each of the three sub-aspects of methodological knowledge were combined using mean scores. For action-oriented methodological knowledge, a score for each sub-aspect of methodological knowledge was created by combining the corresponding statements of the six purported proofs to mean scores. The answers were (re-)coded in such a way that high values correspond to a correct validation of the purported proofs with regard to the sub-aspect (possible values for each statement ranged from 1 to 6).

Subsequently, Pearson correlations were calculated between the calculated mean scores in order to draw conclusions about the relations between sub-aspects of methodological knowledge both within (RQ1) and between (RQ2) concept-oriented and action-oriented methodological knowledge.

RESULTS

Descriptive Statistics

Table 1 shows the descriptive statistics for the three sub-aspects of concept-oriented and action-oriented methodological knowledge. Regarding the sub-aspects of concept-oriented methodological knowledge, participants scored highest for chain of conclusions, followed by proof scheme, and proof structure. Still, the standard deviation for proof scheme was far higher than for the other sub-aspects. Regarding action-oriented knowledge, participants scored highest for proof scheme, followed by proof structure and chain of conclusions.

	Concept-oriented		*Action-oriented*	
	M	*SD*	*M*	*SD*
Proof scheme	4.37	0.72	4.04	0.70
Proof structure	4.23	0.13	3.68	0.80
Chain of conclusions	5.01	0.09	3.30	0.76

Note: Possible values ranged from 1 to 6.

Table 1: Descriptive statistics for the three sub-aspects of concept-oriented and action-oriented methodological knowledge.

Relation of the three sub-aspects within Either Concept-oriented or Action-oriented Methodological Knowledge (RQ1)

The correlations for each of the three sub-aspects of either concept-oriented or action-oriented methodological knowledge are displayed in Table 2. Results highlight that correlations were significant for concept-oriented and action-oriented methodological knowledge. For concept-oriented methodological knowledge, all correlations were positive and moderate. Results for action-oriented methodological knowledge were less uniform, as the correlations between the two sub-aspects chain of conclusions and proof structure as well as between the two sub-aspects chain of conclusions and proof scheme were negative and moderate. In contrast, proof structure and proof scheme showed a weak positive correlation.

	Concept-oriented		*Action-oriented*	
	Chain of conclusions	**Proof structure**	**Chain of conclusions**	**Proof structure**
	r_{Pearson} (p)	r_{Pearson} (p)	r_{Pearson} (p)	r_{Pearson} (p)
Proof structure	.47 (< .001)		-.47 (< .001)	
Proof scheme	.57 (< .001)	.55 (< .001)	-.61 (< .001)	.25 (< .031)

Table 2: Correlations between the three sub-aspects of concept-oriented and action-oriented methodological knowledge.

Relation between Concept-oriented and Action-oriented Methodological Knowledge (RQ2)

Focusing on the relation between action-oriented and concept-oriented methodological knowledge, pairwise correlations for each sub-aspect revealed heterogeneous results. The sub-aspect chain of conclusions showed a significant negative, weak correlation (r = -.24, p = .047), while the sub-aspect proof scheme showed a significant positive, barely moderate correlation (r = .31, p = .010). Finally, the correlation between concept-oriented and action-oriented proof structure was not significant (r = .15, p = .212).

DISCUSSION AND OUTLOOK

In this report, we have outlined a framework that allows to systematize aspects of proof understanding that correspond either to an individual-psychological or the more general disciplinary perspective on proofs and handling proofs. Focusing on the first, we also introduced the distinction between action-oriented and concept-oriented aspects of proof understanding, which have not been systematically addressed and distinguished in research so far (Healy & Hoyles, 2000; Heinze & Reiss, 2003). The latter distinction was then examined more closely for of methodological knowledge of beginning university students, in particular regarding the three sub-aspects proof scheme, proof structure, and chain of conclusions. For this, correlations of the three sub-aspects within concept-oriented and within action-oriented methodological knowledge, as well as pairwise correlations of these sub-aspects between concept-oriented and action-oriented methodological knowledge were calculated.

Focusing first on concept-oriented methodological knowledge, the observed correlations between the three sub-aspects were positive and moderate. Although there is little prior evidence on the relation of these three sub-aspects under a concept-oriented focus, the results appear plausible. They give a first indication that concept-oriented methodological knowledge is rather homogeneous regarding the three sub-aspects. In contrast, the correlations between the three sub-aspects of action-oriented methodological knowledge are intriguingly heterogeneous. One may have expected a positive relation between the sub-aspects similar to concept-oriented methodological knowledge or perhaps also no significant correlation, highlighting that the sub-aspects are not closely related (Heinze & Reiss, 2003). However, correlations with chain of conclusions are negative, moderate and highly significant. Although the presented study does not provide sufficient data to thoroughly explain these negative correlations, one explanation could be based on participants split attention to three different criteria while validating proofs. It may thus be, that these participants – as novices – were only able to focus either on the chain of conclusions or another criteria, but not together at the same time. In this regard, it would be highly interesting to see, how the relation between the three sub-aspects of action-oriented methodological knowledge develops over time and increasing expertise, possibly allowing the parallel validation of proofs regarding all three sub-aspects. Overall, data suggests different relations between the three sub-aspects of concept-oriented and action-oriented knowledge, adding evidence to the distinction introduced in the framework for proof understanding. This is further substantiated by the small, rather inconsistent, and partially insignificant correlations between the sub-aspects of concept-oriented and action-oriented methodological knowledge.

As with every research, the presented study has some limitations, for example regarding the number or participants, the sole examination of beginning university students, the focus on proof validation for examining action-oriented methodological knowledge, or the low reliability of the 2-item scale for concept-oriented proof structure, which will have to be addressed in future research. Still, data give a first indication that the outlined framework for a persons' proof understanding is beneficial and that the distinction in concept-oriented and action-oriented foci of understanding of proof is valuable. Although it is conceivable that students may have concept-oriented methodological knowledge while not having action-oriented methodological knowledge, prior research did not systematically distinguish both aspects, probably leading to partially conflicting results. However, also in this regard it would be interesting to see if the relation between concept-oriented and action-oriented aspects of proof understanding improves over time, mirroring the encapsulation of both aspects with increasing expertise.

Although promising, the presented research can only be regarded as a first step in the substantiation of the outlined framework for proof understanding and a systematic research agenda regarding the latter. For this, it will be important to include other aspects than methodological knowledge, for example about different types of proof, different activities with regard to the action-oriented focus and examine different populations with different expertise. Finally, questions about the benefits of i) the framework beyond its use as an analytical tool for research and ii) the broad conception of understanding of proof it conveys, will have to be answered.

References

Andersen, L. E. (2018). Acceptable gaps in mathematical proofs. *Synthese*, *197*(1), 233–247. https://doi.org/10.1007/s11229-018-1778-8

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics, 95*(2), 123–142.

De Villiers, M. (1990). The Role and Function of Proof in Mathematics. *Pythagoras*, *24*, 17–24.

Hamami, Y., & Morris, R. L. (2020). Philosophy of mathematical practice: A primer for mathematics educators. *ZDM*, *52*(6), 1113–1126.

Hanna, G., & Jahnke, H. N. (1996). Proof and proving. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde, *International Handbook of Mathematics Education* (pp. 877–908). Springer.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, E. Kaput, E. Dubinsky, & T. Dick (Eds.), *Research in collegiate mathematics education* (pp. 234–283). AMS.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *21*(4), 396. https://doi.org/10.2307/749651

Heinze, A., & Reiss, K. (2003). Reasoning and proof: Methodological knowledge as a component of proof competence. *Proceedings of the Third Conference of the European Society for Research in Mathematics Education*, *4*, 1–10.

Kempen, L., & Biehler, R. (2019). Fostering first-year pre-service teachers' proof competencies. *ZDM, 51*(5), 731–746. https://doi.org/10.1007/s11858-019-01035-x

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education. Research, learning and teaching*. Sense Publishers.

Sommerhoff, D., & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by school students, university students, and mathematicians in the context of teaching. *ZDM, 51*(5), 717–730. https://doi.org/10.1007/s11858-019-01039-7

Stylianides, A. J. (2007). Proof and Proving in School Mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

WORKING ON GRAPHS IN ELEMENTARY SCHOOL – A PATHWAY TO THE GENERALIZATION OF PATTERNS

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Pattern generalization is a key element of early algebra. However, it is also an area that causes significant problems for students as well as teachers, as it has proved challenging for elementary school students to understand the meaning of generalization. To address these problems, an intervention was done to introduce the graph and functions in relation to pattern generalizations in Grades 1 and 6. Working on graphs was new for these teachers because, in Sweden, graphs are normally not introduced in school until Grade 7. The results show that the introduction of graphs became a tool to understand and talking about a pattern generalization. As a result, their teaching on linear functions and patterns changed, and the implications of the results on mathematics education in elementary school are discussed in this paper.

INTRODUCTION

Algebra learning includes the ability to express and generalize relationships among quantitates. One way of introducing young students to generalizing and functional relationships is through pattern generalizations (e.g., Blanton et al., 2019; Radford, 2010; Wilkie, 2019). The main problem with generalizing in early grades is that the meaning and activity of generalizing in mathematics has proved challenging to understand (Stylianides & Silver 2009). Therefore, generalizing in early grades warrants further investigation. This paper presents results from a Swedish educational design research study on linear functions and the generalization of patterns in elementary school. The intervention was conducted in close collaboration between three teachers and one researcher (the author). This paper focuses on the three teachers' learning during the nine months of the intervention. More explicitly, the focus is on how these teachers come to implement graphs in their teaching of generalization of patterns and how they reflected upon their changed teaching when using graphs. In the intervention, the teachers initially expressed that they lacked the words to describe and teach the generalization of patterns, and they found it hard to discern when the generalization had been realized (Sterner 2019). This is in line with Blanton's et al. (2019) and Wilkie's (2019) statement that teachers lack awareness about functional thinking, and the authors point to the graph's contributing to functional thinking and pattern generalization.

In this paper, I will present how the introduction of graphs made the teachers aware of the generalization of patterns, and how, as a result of this awareness, their teaching

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of functions and patterns changed. More explicitly, the following research question will be in focus: How does the introduction of the graph representation make it possible for elementary teachers to discern and teach the generalization of patterns?

LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

One main concept of algebra involves generalization. Early algebra can be seen as a way to address traditional transition problems from elementary to secondary school. One key aspect of this is working with functions rather than equations, which means addressing the relationship between quantitates and letters as variables rather than unknowns. This invites the possibilities to work with patterns of various sorts, including figural patterns and pattern generalizations.

Dörfler (1991) makes a distinction between theoretical and empirical generalization, where the *empirical generalization* can be seen as a generalization from one situation to another, while *theoretical generalization* is including some form of abstraction. Elements from Dörfler's interpretation of generalization can be found in Radford's concept *algebraic generality*, which is the main theoretical perspective in this paper. Radford (2010) explains algebraic generality in different layers: *factual-, contextual-*, and *symbolic generality*. Factual generality could be described as generality articulated on, for example, numbers, words, and gestures related to the task, for example, when a student talks about a pattern and says, "increases by 3". The layers of contextual and symbolic generality are interpreted as; generality expressed in some linguistically way, for example, through symbols or language. In this study, the difference between the layers of contextual and symbolic generality is interpreted in the following way. In the symbolic generality, the generalization is expressed in variable notation. In contrast, the contextual generality creates opportunities to use language and actions to create meaning of the variable notations, to create what Radford calls *knowledge objectification* (Radford, 2003; 2010). Working with the algebraic generality in different layers, Radford (2010) points to the process of noticing something general and making sense of the general. Radford calls this process of sense-making 'knowledge objectification'. However, from a teacher's perspective, questions arise about how generalization can benefit students' learning in early grades and what generalization is in elementary school. Carraher, Martinez, and Schliemann (2008) and Wilkie (2019) emphasize the importance of both the representation and the reasoning behind the conventional notation when teaching generalization in early grades.

Research, as well as the summary of PME contributions of topics of functions and calculus, indicate the importance of algebraic thinking and using a functional approach in early grades (e.g., Blanton et al., 2015; Hitt & González-Martín, 2016). Working with pattern generalizations is one tool to stimulate algebraic thinking and the idea of generalizations in early grades (Blanton et al., 2015; Wilkie, 2019). The graph representation, along with figural pattern generalizations, could be a pathway to functional thinking and algebraic generalization. Researchers point to the importance of reasoning about the relationship between two (or more) varying quantities when teaching functional thinking (e.g., Radford, 2010; Wilkie, 2019). However, research indicates that students have difficulties identifying covariational relationships, which involves describing how two quantities vary in relation to each other (Wilkie 2019). Similar ideas emerge in (Blanton et al., 2019), where the authors point out teachers' lack of awareness about the functional thinking and how to use the graph representation to visualize covariational relationship and proportionality. Blanton et al. (2019) and Wilkie (2019) stress the importance of further research on how to support elementary teachers in functional thinking in early algebra. Research shows how difficulties in proportional reasoning emerge in the early grades, indicating that students do not use the zero point on the x-axis when working with graph representations (Wilkie, 2019). With the above literature review as background, two general goals for teaching were formulated for a long-term intervention study (see Sterner 2019). In this paper, there is a particular focus on the second theme:

1: The students should be given opportunities to identify a pattern, structure the pattern, and generalize the pattern.

2: The students should be given opportunities to work with algebraic reasoning, including functional thinking and proportional relationship, and determining relations between two or more varying quantities.

METHOD

The intervention in this study is designed as an educational design research, and a continuation of a project, including mathematics teachers from grades 1-6 (Sterner 2015). The author and three mathematics teachers collaborated (one from Grade 1 and two from Grade 6) in three recurring design cycles. The selection of the teachers' teaching groups was done naturally since it was in grades 1 and 6 the teachers' work when the intervention took place. The two goals for teaching, the themes, are seen as Design Principles (DPs) (McKenney & Reeves, 2012) and are used as a theoretical guide for the intervention. Hereafter, these themes are referred to as DP1 and DP2. The background and the content of the DPs are described in more detail in Sterner (2019).

The Swedish Context

There are goals for algebra in the Swedish curriculum materials (National Agency of Education, 2017) for Grades 3–6, but functions and functional thinking are not introduced until Grade 7. However, most teachers, including the participants in the current project, have little experience with functional thinking because of the lack of emphasis on functions in the elementary school. Three mathematics teachers (Clara, Irma, and Jonna) from different schools in Sweden have all of them more than twenty years of experience from teaching in Grades 1–6. Two of the teachers are semi-specialized mathematics teachers and teach in three subjects, mathematics, science, art, or music in Grades 1–6. The third teacher is a general subjects teacher in Grades 1–3.

Empirical Materials in This Paper

The intervention took place over a period of nine months, in the meetings, the teachers and the researcher planned and evaluated the teaching. Excerpts from lessons in Grade 6 are also used to illustrate the challenges that occurred in the teachers' discussions. The teachers' individual reflections about these lessons are also included in the analysis. The empirical data analyzed in this paper included 15 hours of video recordings from the meetings mentioned above where the teachers and the researcher planned and evaluated the teaching, 21 hours of lessons, and 5.5 hours of teachers' individual reflections with the researcher directly after teaching.

Knowledge Objectification and Algebraic Generality as an Analytical Frame

In this study, Radford's knowledge objectification with focus on algebra generality (2003; 2010) is used as a conceptual frame in relation to the empirical data. The algebraic generality (factual-, contextual-, and symbolic generality) is used to explore and exemplify how the graph representation makes it possible for elementary teachers to discern and teach the generalization of patterns. The frame is also used to exemplify how the teachers move within and between the different layers of algebraic generality when using the graph in pattern generalizations. The algebra generality is used as a theoretical frame for the analyses, while the DPs, are seen as a theoretical frame for the intervention and goals for teaching. Through the analysis, the teachers' reflections upon their changed teaching when using graphs were analyzed, i.e., their knowledge objectification.

RESULTS

In the results, transcripts and figures are used to visualize the teachers' process of knowledge objectification of pattern generalizations, and the selection of transcripts will show crucial moments in this process.

The Graph Opened Up for Different Representations

The introduction of the graph revealed that the general formula needs to be visualized in different representations. In the initial process of the intervention, the teachers strongly opposed using the graph as a representation for pattern generalizations. As mentioned, working with graphs as a representation of pattern generalizations was a new challenge for these teachers (Sterner 2019). However, the complexity of understanding, expressing, and making justifications of a general formula becomes visible in the teaching when the teachers challenge their students to explain what the variable notation symbolizes in an equation for example $y = 3x + 5$. This equation represents a pattern generalization that the teachers called *the canoes* (see Fig. 1).

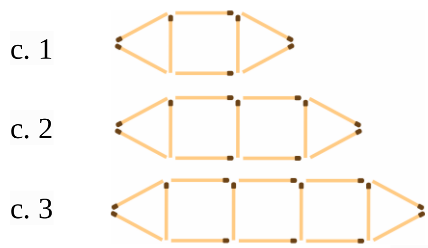

Figure 1: Image of a pattern called 'the canoes.'

This task was a crucial task in the intervention to explore the slope – the change in input and the corresponding change in output. In discussions with each other, the teachers became aware that the general formula itself, do not explains neither the teachers' nor the students' understanding of generalizations. The teachers realized that they had neither the language nor tools to talk about the structure of a general formula. Therefore, I introduced the graph representation as a tool to make a justification for pattern generalizations, in line with Wilkie (2019).

The Graph Visualized the Structure in the General Formula

By introducing the graph, the importance of visualizing the structure in a general formula emerge. The teachers used various examples in their teaching, illustrating linear functions and pattern generalizations. One task exemplifying pattern generalization was 'the canoes' (Fig. 1). Another task illustrating direct proportionality was a pattern concerning a number of dogs and their corresponding number of tails, ears, and legs. The graph made it possible to visualize the slope (m), and the y-intercept (c), ($y = mx + c$). The teachers talk about the 'start-value' when c has the value of zero. The teacher asked the students to work with the figural patterns in various representations, for example, using matches and tables, using the coordinates from the table to make a graph representation, and finding a general formula (see Fig. 2).

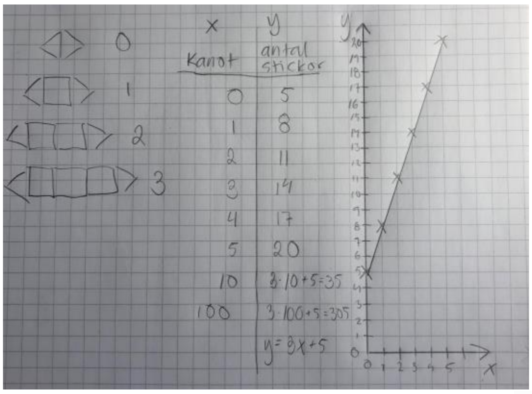

Figure 2: A student's solution of the pattern of 'the canoes.'

The following transcript illustrates a conversation in the whole class discussion when the teacher (Irma) asked two of the students to describe what they had realized when using the graph and the equation to represent the pattern generalization. This conversation illustrates how Irma, in the meeting with her students, comes to realize the potential of using the graph. The transcript indicates that the students (Anna and Kim) used the graph to understand the rate of change and used the graph to understand what happens when the independent value is 0.

Irma:	Kim and Anna, can you tell us what you found when you compared the pattern of matches with the table, the graph, and the general formula?
Kim	Yes…we tried to draw a straight line through the origin, but it didn't work…It didn't end up as a straight line…That made us understand the meaning of the number of 5 in the general formula ($y = 3x + 5$).

Irma: Alright, go on.

Anna: We just realized the meaning of figure 0 in the table. We hadn't thought about it earlier, but now, when we look at the graph, and we didn't manage to get a straight line through the origin…we realized…and saw the 1 step in the right (x-axis) and 3 steps up (y-axis).

Irma: Yes, the graph visualized the start-value in the coordinates (0,5) (Fig. 2) and indicates how we can explain the general formula using the values in the table and the graph for the pattern generalization. [When Irma speaks, she makes it as clear as possible by using various representations]. We have the start-value (0,5) by subtracting the rate of change, 3, from the first entry, 8, in the coordinate (1, 8).

In the whole-class discussion, they talked about what several students have comprehended during the lesson. The teacher then returned to the discussions from the previous day, which is about linear functions and direct proportionality representing dogs and their corresponding tails, ears, and legs. The conversation goes on in the whole-class discussion, and they talk about the relationship between quantitates (x and y). The students talk about what happened when they have "one more seat" in the canoes, and other students talk about what happened when they increase one figure, and a third student looked at the graph and said: "every time we go 1 step on the x-axis, we go 3 steps on the y-axis." The graph helped the teachers to talk about the independent and the dependent value. The graph becomes a representation to go from a specific situation to the general. In the reflection after teaching and in the refining phase, the teachers talked about how they now have realized the importance of understanding the relationship between quantities. The graph was an excellent way to visualize this relation.

Jonna: I've always had difficulties explaining the relationship between x and y, and I didn't know what words I could use to explaining the relationship correctly and simply for the students.

Clara: The graph became a tool that I often came back to, making it possible to visualize the values in the table and visualize and talk about the general formula's content…The graph helped me to talk about what I did not have words for – the relationship between x and y.

The quotation below shows how the teachers use the graph representation as an input to talk about a generalization, through other already known representations.

Clara: Can you believe, I've never connected the graph with the table or the pattern of matches before…I never realized the importance of the relationship between x and y. I don't think I've fully understood what proportional relationship is – sometimes, I feel like the text of the (Swedish) curriculum materials is a bit abstract. I would love for all teachers to be part of something like this.

The teachers described how the graph had been an asset for both their own and the students' understanding of pattern generalization. In the discussion, the teachers stated that they had not previously understood the value of paying attention to what they called 'new small details', for example, what they called the *starting-value* or the relationship between quantitates in a generalized formula or a pattern generalization. The results indicate that the graph representation helped the teacher to talk about the functional relationship as well as the proportional relationship of pattern generalizations.

CONCLUSION AND IMPLICATIONS

The results show that the graph representation became a way of understanding and talking about the structure of a general formula in a pattern generalization. However, the graph representation is not enough. The teachers' discussions show that they had to elaborate at multiple representations, to justify and understand what the teachers called 'new small details'. The small details include, for example, the relationship between quantitates and the slope.

The teachers' discussions changed from interpreting a pattern generalization equal with a general formula – *that*, and nothing else. The teachers' initial interpreting of generalization would be described by Radford (2010) as symbolic generalization. At the end of the intervention process, the teachers interpret and justify the pattern generalization in multiple representations. The graph provided an entrance to justify the pattern generalization. The graph was also used as a tool to understand the structure of a generalization. This is what Radford (2003; 2010) called using different layers of generality to understand the algebraic generality. The contextual generalization (Radford, 2010) became visible when the teachers used the graph with already known representations to find the words and the language to talk about the relationships between variables and the slope. That falls in line with what Blanton et al. (2019) and Carraher et al. (2008) indicate from their study, including students and their use of different representations to support the understanding of variable notations. The graph thus becomes a tool to express the symbolic generality in natural language and creates opportunities to make knowledge objectification for pattern generalization. Working with the graph helped the teachers to understand and talk about proportional relationships and functional thinking, which in line with (Blanton et al., 2015; Blanton et al., 2019; Wilkie, 2019). The graph made it possible to visualize how the generalizations apply not only to a specific situation but also to all cases. This demonstrates to the teachers the importance of both empirical and theoretical generalization, which Dörfler (1990) addresses.

In addition to the results answering the research questions of this study, it is worth considering the methodology used. DPs are used as a theoretical frame for the interventions' content and as goals for the teaching. In design research, the DPs are normally changed, and new conceptual ideas are developed during the process (McKenny & Reeves, 2012). However, in this study, the content of the DPs does not change. Instead, the teachers' *understanding* of the DPs changes, thanks to using the graphs and working with different layers of algebraic generality.

References

Blanton, M., Stephens, A. Knuth, E., Murphy Gardiner, A., Isler, I. & Kim, J. S. (2015). The Development of Children's Algebraic Thinking: The Impact of a Comprehensive Early Algebra Intervention in Third Grade. *Journal for Research in Mathematics Education*, 46, 39–87.

Blanton, M., Stroud, R., Stephens, A., Gardiner, A. M, Stylianou, D. A., Knuth, E., Isler-Baykal, I., & Strachota, S. (2019). Does Early Algebra Matter? The Effectiveness of an Early Algebra Intervention in Grade 3 to 5. *American Educational Research Journal*, 56: 5, 1930-1972.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM, Mathematics Education* 40.3-22.

Dörfler, W. (1991). Forms and means of generalization in mathematics, in A. J. Bishop (ed.), *Mathematical Knowledge: Its Growth through Teaching,* Kluwer Academic Publishers, Dodrecht, 63–85.

Hitt, F. & González-Martín, A. S, (2016). Generalization, Covariation, Functions and Calculus. In Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education*, 3-38.

McKenny, S. & Reeves, T. (2012). Conducting Educational Design Research. London: Routledge.

National Agency for Education (2017). *Curriculum for the primary school, preschool class and leisure time center* 2017. Stockholm: National Agency for Education.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, 5(1), 37-70.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA, 4(2), 37-62.*

Sterner, H. (2015). *Problematisera "görandet", Lärares lärande om kommunikation och resonemang i matematikundervisningen i en organiserad praktikgemenskap*: Licentiate Dissertation, Linnaeus University.

Sterner, H. (2019). Teachers as actors in an educational design research: What is behind the generalized formula? *International Journal on Math, Science and Technology Education* 7(3), 6-27.

Stylianides, G. J. & Silver, E. A. (2009). Reasoning-and-Proving in School Mathematics, The Case of Pattern Identification. In D. A. Stylianou, M. L. Blanton & E. J. Knuth (eds.), *Teaching and learning proof across the grades: A K–16 perspective*. New York, NY: Routledge.

Wilkie, K. J. (2019). Investigating Students' Attention to Covariation Features of their Constructed Graphs in a Figural Pattern Generalization Context. *International Journal of Science and Mathematics Education.* Ministry of Science Technology, Taiwan 2019.

FINNISH AND GERMAN TEACHERS' BELIEFS ON TEACHING PROBLEM SOLVING IN PRIMARY SCHOOL MATHEMATICS: A COMPARATIVE STUDY

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The study of teachers' mathematical beliefs has received much attention in recent years. Yet, the beliefs that teachers hold with respect to specific process areas are limited, though these have been recognized to be central in mathematics classrooms. In this paper, we report on a comparative study on beliefs about teaching problem solving with inservice primary teachers from Finland and Germany. While the results are consistent with the literature in some respects, such as promoting reflection and using good problems regardless of the country, limited teacher input and teaching problem-solving strategies have been viewed differently by the teachers from both countries. Similarities and differences between the two country teachers' beliefs are discussed with regard to their theoretical and practical implications.

INTRODUCTION

Teachers' beliefs and knowledge about mathematics, learners and learning, teaching, subjects or curriculum, interpretation of content, and about self are considered to play a significant role in their teaching practices (e.g., Pajares, 1992). Pajares (1992) explained that "the beliefs teachers hold influence their perceptions and judgments, which, in turn, affect their behavior in the classroom" (p. 307). Research on teachers' beliefs is tremendous ranging from assessing prospective and practicing elementary and secondary teachers' beliefs about mathematics and teaching mathematics to changing beliefs and assessing that change (e.g., Philipp, 2007). Nonetheless, there is little research that focuses on teachers' beliefs about process areas in mathematics.

During the last few decades, mathematics education researchers have called for studies that focus on the teacher in problem-solving instruction (Donaldson, 2011). This is not surprising taken that problem solving has been recognized as the central content in school mathematics (e.g., Reiss & Törner, 2007), and has, therefore, been implemented in international curricula worldwide, such as Finland and Germany. Despite this endorsement, the integration of problem solving into mathematics classes is only present to a limited extent, if at all (e.g., Pehkonen, 2017; Reiss & Törner, 2007). Here, both teachers' beliefs about problem solving, and about teaching problem solving are central. Especially the latter, may influence how problem solving is approached in mathematics lessons, the mathematical opportunities teachers provide their students, and their expectations for students' problem-solving abilities. For instance, Cross

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 121-128. Khon Kaen, Thailand: PME.

(2009) found that beliefs about the nature of mathematics, and beliefs about teaching based on these beliefs are a rather reliable predictor of the instruction in classrooms (Cross, 2009). Thus, understanding teachers' beliefs about teaching problem solving can shed light how the problem-solving standard is implemented in school mathematics. With these assumptions in mind, in this report we focus on Finnish and German elementary teachers' beliefs about teaching problem solving as well as on similarities and differences between the two countries with respect to these.

THEORETICAL PERSPECTIVE

Beliefs refer to "psychologically held understandings, premises or propositions about the world that are thought to be true (Philipp, 2007, p. 259). Unlike knowledge, they are held with varying degrees of conviction and are not consensual (Philipp, 2007). They can be thought of as dispositions toward an action, such as teaching practices. For instance, Cross (2009) reported that teachers' beliefs affect their decisions in teaching problem-solving lessons. Researchers (e.g., Donaldson, 2011; Heinrich, Bruder, & Bauer, 2015; Kilpatrick, 1985) reported on different categories among the many perspectives on how to teach problem solving, noting that teaching problem solving must combine features of several categories. The categories include, but are not limited to: (1) give lot of problems (i.e., initiate many problem-solving activities), (2) give "good" problems (e.g., mathematically rich problems), (3) teach specific or general heuristic strategies, (4) model problem solving, (5) limit teacher input (e.g., by having students work individually or in small groups), (6) reflection (e.g., by asking metacognitive questions), and (7) allow and highlight multiple solutions.

Even though these aspects have been recognized, and advocated already for decades, research shows that teachers hold views that are just partly in accordance with these aspects. For instance, Pehkonen (2017) and Siswono et al. (2019) reported that most primary teachers believed that problem solving is learnt by solving problems. Here, the importance of both teaching materials and tasks used was stressed (Pehkonen, 2017). Additionally, Näveri et al. (2011) reported that, according to teachers, tasks should reflect everyday situations as well as include word problems. Besides that, teaching problem solving in mathematics refers to the use and study of different strategies, which are needed in solving problems (Pehkonen, 2017; Siswono et al., 2019). Here, the teachers emphasized that students need to be able to select and combine proper strategies on the basis of their logical thinking. The studies on teachers' beliefs about their role when teaching problem solving are not conclusive. In Pehkonen's study (2017) with the Finnish primary teachers, the teachers viewed the teacher as a leader when teaching problem solving. Concretely, the teacher should select problems to be dealt with, illustrate and explain own thinking during the solution process to students as well as give them information on different problem-solving strategies. However, in Siswono's et al. (2019) study with Indonesian primary teachers, the teachers viewed the teacher as a facilitator in exploring students' knowledge and skills. Thus, the teacher should give only necessary help to students during problem solving, allowing them to struggle with the problem, and construct some strategies on their own. Lastly, Näveri et al. (2011) reported that the majority of the Finnish primary teacher considered problem solving as teamwork.

RESEARCH PROCESS

For this study, a quantitative research design was chosen. The **participants** were 345 inservice primary teachers ($n_1 = 159$ from Finland, $n_2 = 187$ from Germany) who participated voluntarily in the study. The schools were selected through existing contacts with the researchers' universities and through random enquiries. The questionnaires were returned anonymously to the respective university. The **main source of data** was a questionnaire on teachers' beliefs about problem solving, that was based on an adaptation of the instrument from the work of Pehkonen (1993), and Kuzle (2017). Additionally, new items were developed on the basis of literature on problem solving, teaching problem solving and factors influencing the implementation of problem solving in school mathematics (e.g., Donaldson, 2011; Heinrich et al., 2015; Kilpatrick 1985). The questionnaire consisted of four sections: beliefs (1) about the importance of teaching problem solving, (2) problem solving teaching practices, (3) demands on teaching problem solving, and (4) general conditions for implementation of problem solving in school mathematics. In total 53 closed items were developed that were rated on 5-point Likert scale (1 = strongly disagree, 5 = strongly agree) ($\alpha = .86$). In this paper, we focus on section (2) from the questionnaire. This category was measured with 16 items ($\alpha = .71$).

For the **data analysis** the percentage of agreement was calculated. For this purpose, the original response scale (1–5) was reduced by combining the two response values at the extreme ends of the scale to obtain a scale of: disagree (1 or 2) – neutral (3) – agree (4 or 5). The consensus level as a percentage was defined as suggested by Kuzle (2017) (see Table 1).

Consensus level	Percentage[a]
complete consensus	at least 95%
consensus	85%–94%
near consensus	75%–84%
lack of consensus	60%–74%
no consensus	less than 60%

Table 1: Consensus level as suggested by Kuzle (2017)

Note. [a]The percentages show how many of the test subjects agree with the same extreme end of the scale.

The percentage of consensus was chosen to describe the consensus level of the test subjects' beliefs about the statements, and with it, similarities and differences between the both countries regarding the teachers' beliefs how problem solving should be taught. Additionally, the non-parametric Mann-Whitney test was used to determine whether the differences between countries were significant. The results and effects are only given if they were significant.

FINDINGS

In this section, we present the results concerning the goal of the study, namely Finnish and German elementary teachers' beliefs about teaching problem solving as well as on similarities and differences between the two country teachers' beliefs. The section is structured on the basis of different problem solving-teaching practices which were outlined in the section on theoretical perspective. The results from the questionnaire are presented in Table 2.

The first problem-solving teaching practice, namely *"give lots of problems"* was measured with two items. Both Finnish and German teachers were critical to students' learning problem solving by solving problems only (item 2). While there was no consensus among Finnish teachers (56% of consensus), there was a lack of consensus among German teachers (71% of consensus). The agreement of Finnish teachers (Mdn = 3.0) differs significantly from the agreement of German teachers (Mdn = 3.0), $U = 11513.00$, $z = - 2.55$, $p = .011$, $r = .14$. A significant difference was likewise measured on item 20. While "having enough problems on hand" was important to Finnish teachers (87% of consensus), this was the case with German teachers only to a limited extent (75% of consensus) ($U = 10924.00$, $z = - 3.33$, $p = .001$, $r = .18$).

Three items dealt with the second teaching practice, namely *"give 'good' problems"*. With regard to item 7, teachers from both countries rated the inclusion of real-life problems as particularly important (96% of consensus). There was a lack of consensus on focusing on relevant mathematics problems (item 1, Finland: 60%, Germany: 70%), and problem posing (item 6) with Finland agreeing more strongly on the latter (72%) than Germany (51%). Both differences were significant (item 1: $U = 11180.00$, $z = - 2.42$, $p = .016$, $r = .13$; item 6: $U = 10119.50$, $z = - 4.64$, $p < .001$, $r = .25$).

Three items felt under the scope of the third category, namely *"teach specific or general heuristic strategies"*. With regard to the teaching of heurisms (item 13), there was consensus in Finland and Germany that student representations should be included into teaching of problem solving (Finland: 87%, Germany: 89%). The view that teachers "should provide the students with clear and precise strategies" (item 15) was not shared by teachers in either country (Finland: 59%, Germany: 57%). Regarding the teaching of heuristic strategies (item 5), Finland agreed with 93% (high consensus), whereas German teachers agreed only with 72% (near consensus). The agreement of the Finnish teachers (Mdn = 3.0) was significantly higher than the agreement of the German teachers (Mdn = 3.0), $U = 8079.50$, $z = - 5.71$, $p < .001$, $r = .31$.

The fourth category, namely *"model problem solving"* was evaluated with one item only (item 18). There was no consensus on "students should also practice problem solutions demonstrated by an expert" (item 18). Only 37% of Finnish, and 50% of German teachers considered this problem-solving practice to be important. Three items dealt with the fifth teaching practice, namely *"limit teacher input"*, which can be done by implementing cooperation as well as independent working phases.

	Finland[a]						Germany[b]						Agreement (%)	
					I agree						I agree			
	N/A	1	2	3	4	5	N/A	1	2	3	4	5	F	G
1.	Problem solving should be taught by focusing on problems that occur in the teaching of mathematics.													
	12	2	17	32	87	9	8	3	15	30	101	29	60	70
2.	Problem solving can be learned only by solving problems.													
	5	5	39	21	52	37	8	1	22	23	87	45	56	71
3.	The teaching of problem solving should allow students' creative processes.													
	0	0	0	3	54	102	1	0	3	4	97	81	98	96
5.	The teaching of problem solving should include problem solving strategies.													
	2	0	1	8	43	105	31	0	2	20	80	53	93	72
6.	Students should make up problems themselves, and then solve them.													
	4	0	14	26	85	30	6	5	40	41	77	17	72	51
7.	During problem solving, familiar problems from the students' environment should be used.													
	2	0	0	5	75	77	2	0	1	5	98	80	96	96
9.	It is sufficient that the students find one solution only.[c]													
	9	11	43	28	60	8	6	14	73	35	51	7	34[c]	47[c]
11.	The students should have the possibility to work independently on the problem.													
	4	1	12	31	69	42	1	0	0	3	83	99	70	98
12.	The students should have the possibility to exchange on their problem solving processes with their peers.													
	1	0	0	5	39	114	0	0	0	4	63	119	96	98
13.	The teaching of problem solving should include representations of the students (e.g., drawings, tables, calculations).													
	5	0	3	13	41	97	4	0	2	14	88	78	87	89
15.	The teacher should provide the students with clear and precise strategies for working on problems.													
	9	3	22	32	67	26	4	6	33	38	68	37	59	57
16.	The teacher should motivate the students to solve problems independently.													
	4	1	7	15	77	0	1	0	0	2	81	102	83	98
17.	After problem solving, the students should have time to reflect on their problem solving process.													
	0	0	2	14	51	87	6	0	3	9	87	81	87	90
18.	Students should also practice problem solutions demonstrated by an expert.													
	23	2	18	58	48	10	6	4	40	44	69	23	37	50
19.	Reflecting different solutions can be an obstacle to students' learning.[c]													
	15	69	35	26	13	1	1	31	72	45	33	4	65[c]	55[c]
20.	Enough problems should be on hand.													
	3	7	0	11	0	138	12	10	0	24	0	140	87	75

Table 2: Frequency of responses and level of consensus on the questionnaire

Note. The last column shows the percentage of responses that agreed (response 4 or 5). N/A= not available; 1 = I strongly disagree; 2 = I somewhat disagree; 3 = I neither agree nor disagree; 4 = I somewhat agree; 5 = I strongly agree; F = Finland; G = Germany.
Assignment of the items: give lots of problems: 2, 20; give "good" problems: 1, 6, 7; teach specific or general heuristic strategies: 5, 13, 15; model problem solving: 18; limit teacher input: 11, 12, 16; reflection: 17; allow and highlight multiple solutions: 3, 9, 19.
[a]n = 159. [b]n = 187.
[c]Inverted item, meaning that disagreement was calculated by summarizing response values 1 and 2.

Item 12 measured the relevance of the teachers' commitment during exchange phases. In Finland as well as in Germany there was a complete consensus (Finland: 96%, Germany: 98%). With regard to the role of independent problem solving, the findings were ambivalent. Differences in the degree of agreement were found with regard to giving students the possibility to work on the problem independently (item 11), and that teacher should motivate the students to solve the problem independently (item 16). Whilst in Germany there was a high consensus on both items (98% of consensus), the consensus in Finland was only 70% (item 11) and 83% (item 16), which corresponds to a lack of consensus or no consensus, respectively. In both cases, the agreement of German teachers (*Mdn* = 3.0) was significantly higher than the agreement of Finnish teachers (*Mdn* = 3.0), *U* = 8808.00, *z* = - 6.69, *p* < .001, *r* = .36 (item 11); *U* = 10657.50, *z* = - 4.56, *p* < .001, *r* = .25 (item 16).

In the sixth category, namely *"reflection"* there was a high consensus in both Finland (87% of consensus) and Germany (90% of consensus) that "students should have time to reflect on their problem solving process" (item 17).

In the seventh category, namely *"allow and highlight multiple solutions"*, there was a consensus on item 3 that problem-solving teaching "should allow students' creative processes" (Finland: 98%, Germany: 96%). In contrast, only 55% of Finnish and 65% of German teachers rejected that reflecting different solutions can be an obstacle to students' learning (item 19). The teachers were critical to having students find one solution only (item 9). There was no consensus on this item (Finland: 34%, Germany: 47%). The difference was significant (*U* = 9150.50, *z* = - 6.31, *p* < .001, *r* = .33).

DISCUSSION AND CONCLUSIONS

The results of the study showed both similarities and differences with respect to Finnish and German teachers' beliefs about teaching problem solving. In the category *"give lots of problems"* no clear agreement was reached. This is in line with Kilpatrick (1985) who argued that, even though solving many problems is important, it is not a predictor of becoming a better problem solver. The results in the category *"'good' problems"* showed both similarities and differences between the two countries. Similar to Näveri et al. (2011), both Finnish and German teachers agreed that the tasks should come from the students' environment. However, focusing on relevant mathematics problems was not considered to be as relevant. The often underestimated role of problem posing (Siswono et al., 2019) also underpins the responses of teachers from both countries.

The results in the category *"teach specific or general heuristic strategies"* showed only partly similarities between both countries. Whilst the inclusion of student representations was agreed upon teachers from both countries, conveying problem-solving strategies by the teacher was not shared by either country. That problem-solving teaching should include heuristic strategies was considered important by Finnish teachers significantly more often than by German teachers. This can be explained by unfamiliar wording. The term "problem-solving strategies" was used in Finland and "heuristic strategies" in Germany. The technical vocabulary "heuristic" may not have been familiar to some teachers, which is suggested by 31 "N/A" responses on item 5.

No consensus was reached by teachers of either countries in the category *"model problem solving"*. Most teachers tended to focus on the middle of the Likert scale. This could be caused by the ambiguous meaning of the word "experts". In future questionnaires it would be useful to clarify this and, for example, to list good problem-solving students as experts, as well as to include more items pertaining to this category.

The results with respect to the category *"limit teacher input"* reflected both similarities and differences between the two countries, which also confirmed non-conclusiveness in the literature. On the one side, Finnish teachers viewed themselves as leaders in a problem-solving classroom. Here, student's individual work was significant given more relevance in Germany than in Finland, which confirms the results of Pehkonen (2017). On the other hand, German teachers viewed themselves as facilitators, confirming the results of Siswono et al. (2019) with Indonesian teachers. This might be explained by different cultural or educational system particularities, which could be further investigated in future studies. Nevertheless, there was agreement in both countries regarding the implementation of cooperative work phases, which confirms the results of Näveri et al. (2011) on Finnish teachers' beliefs about teaching problem solving.

A high consensus was reached by teachers from both countries in the category *"reflection"*, confirming the results from the literature (e.g., Donaldson, 2011; Heinrich et al., 2015; Kilpatrick, 1985). Since this category was evaluated on the basis of one item only, it should be taken with reservation. In our future work, including more items with respect this category is of imperative.

The results in the last category do not clearly support the fact that multiple solutions should be allowed and highlighted. Concretely, there was complete consensus in both countries that mathematical problem solving is a creative process, which is consistent with the literature (e.g., Donaldson, 2011). The answers were broadly scattered when it comes to the fact that finding one solution is sufficient and reflecting on several possible solutions hinders student learning. This may be an indication that it is sufficient for primary school teachers if the children find a solution. However, it cannot be excluded that these items have been misunderstood due to the inverse formulation.

It should not be forgotten that the participating teachers only represent the two countries to a limited extent. Also, due to a voluntary participation it may assumed that the teachers were more motivated. Despite these drawbacks, the results of the study not only gave first insights on the Finnish and German inservice primary teachers' beliefs about teaching problem solving, but also shed light how problem solving is implemented in school mathematics.

References

Cross, D. I. (2009). Alignment, cohesion, and change: Examining mathematics teachers' belief structures and their influence on instructional practices. *Journal of Mathematics Teacher Education, 12*, 325–346.

Donaldson, S. E. (2011). *Teaching through problem solving: Practices of four high school mathematics teachers* (Unpublished doctoral dissertation). University of Georgia, Athens, Georgia.

Heinrich, F., Bruder, R., & Bauer, C. (2015). Problemlösen lernen [Learning problem solving]. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 279–301). Berlin, Heidelberg, Germany: Springer Spektrum.

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–15). Hillsdale, NJ: Erlbaum.

Kuzle, A. (2017). German teacher educators' conceptions about teaching problem solving in mathematics classroom – An obstacle to a large-scale dissemination?! *Magistra Iadertina, 12*, 74–93.

Näveri, L., Pehkonen, E., Ahtee, M., Hannula, M. S., Laine, A., & Heinilä, L. (2011). Finnish elementary teachers' espoused beliefs on mathematical problem solving. In B. Roesken & M. Casper (Eds.), *Proceedings of the MAVI-17 Conference* (pp. 161-171). Bochum, Germany: Ruhr-Universität Bochum.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307–332.

Pehkonen, E. (1993). What are Finnish teacher educators' conceptions about the teaching of problem solving in mathematics? *European Journal of Teacher Education*, 16(3), 237–256.

Pehkonen, E. (2017). Finnish elementary teachers' conceptions on problem solving in mathematics teaching. *Matematica e la sua Didattica, 25*(1), 13–27.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 257–315). Charlotte, NC: Information Age.

Reiss, K., & Törner, G. (2007). Problem solving in the mathematics classroom: The German perspective. *ZDM – The International Journal on Mathematics Education, 39*, 431–441.

Siswono, T. Y. E., Kohar, A. W., Hartono, S., Rosyidi, A. H., Kurniasari, I., & Karim, K. (2019). Examining teacher mathematics-related beliefs and problem-solving knowledge for teaching: Evidence from Indonesian primary and secondary teachers. *International Electronic Journal of Elementary Education, 11*(5), 493–506.

MOVING AWAY FROM THE CENTER OF DISCOURSE IN MATHEMATICS PROFESSIONAL DEVELOPMENT

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In this study we examined the use of two moves--wait time and self-removal--in the facilitation practices of three facilitators who led a mathematics professional development program focused on discourse. Our findings show that facilitators promoting higher quality discourse used these moves more often, which suggests that wait time and self-removal can support high-quality discourse among teachers.

INTRODUCTION

The role facilitators play in mathematics professional development (MPD) is key for the quality of participants' experiences. After attending to teachers and teaching during decades that can be conceptualized as the era of teachers (Sfard, 2004), researchers have begun to turn their focus to facilitators and their practice in promoting mathematics teachers' learning. Studies of MPD facilitators and facilitation often build on and extrapolate from what is now known about teachers and teaching, looking to understand and build capacity for the work of MPD leaders (Sztajn, Borko, & Smith, 2017).

An important component of research on facilitators and their professional practices is the understanding that PD settings need to collectively engage teachers as active participants (Desimone, 2009; Darling-Hammond, Hyler, & Gardner, 2017) who are, at the same time, in charge of their own learning. Although facilitating PD is different from teaching in K-12 classrooms (Borko, Koellner, & Jacobs, 2014), teachers still need to experience learning environments in which they engage in practices such as constructing viable arguments and critiquing the arguments of others.

To understand the work of MPD facilitators, in this study we examined facilitator moves in one MPD setting. The MPD focused on the promotion of high-quality discourse in elementary classrooms and, to "walk the walk", facilitators promotion of high-quality discourse in the MPD was critical. Two practices often considered productive for engaging teachers in such high-quality discourse are the main focus of the study: wait time and self-removal.

ATTENDING TO FACILITATION PRACTICE

Research focusing on facilitation practice is identifying productive moves facilitators use in MPD settings. In video based MPD programs, for example, practices such as promoting group collaboration, fostering an inquiry stance, and maintaining a focus on the video and the mathematics have been deemed important, although it is the

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coordination among them that seems to matter most for promoting productive discussions (van Es, Tunney, Goldsmith, & Seago, 2014). Other moves recognized as important for facilitators include creating a climate of respect and establishing collaborative working relationships among participating teachers (Borko, et al., 2014b; Linder, 2011), as well as setting and maintaining explicit social and sociomathematical norms (Elliott, et al., 2009). All these moves speak to the importance of creating a community within the MPD, which positions teachers and facilitators as contributors to the collective discourse. Additional moves such as providing constructive feedback while avoiding judgmental feedback (e.g., Gardiner, 2012; LoCascio, Smeaton, & Waters, 2016), addressing emotional distress (Gardiner, 2012; Linder, 2011; Odell & Ferraro, 1992) or positioning oneself as a colleague (Odell & Ferraro, 1992) also seem to contribute to the productive practice of facilitators in professional development settings.

Several of these moves address the need to build trust in MPD settings (Sztajn, Hackenberg, White & Allexshat-Snider, 2007) creating a climate in which participating teachers discuss ideas, also connecting them to their classroom practices. Facilitators need to make sure teachers are responsible for their MPD discourse, engaging them in professional exchanges with colleagues. To this regard, facilitating discourse in professional development settings, similarly to the classroom, requires the facilitator to encourage teachers to exchange ideas among themselves, asking each other probing and pressing questions. This change in discourse responsibility implicates a move from a “hub and spoke” talk diagram (where all conversations go through the facilitator as the center) to a “star pattern” diagram (Nathan & Knuth, 2003), where teachers are talking to each other.

Two moves are important for this change in discourse pattern that allows teachers to talk to each other: facilitator wait time and self-removal. A key component of the move from the hub-and-spoke to the star-pattern discourse is the facilitator’s intentional withdrawal from consistently being at the center of the conversation. Wait time and self-removal are two productive moves that can quickly support a change in discourse pattern. Wait time refers to the brief pauses or silences between speakers during discourse turns (Ingram & Elliott, 2016). Providing brief uninterrupted periods of time of at least 3-5 seconds for teachers to process information and to consider individual responses has been shown to contribute to improved learning during instruction (Stahl, 1994) and increased response opportunities (Lamella & Tincani, 2012).

Ephratt (2011) distinguished other types of silences that serve as non-verbal communication, including longer interactive silences that signal control, interpersonal attention, or terminate the speech burden through verbal silence, interactive distance, or non-verbal body language. We refer to the type of silence seen through the removal of oneself from the center of conversation or non-verbally communicating one’s removal from the conversation (e.g., breaking eye contact by gazing away or physically repositioning oneself further from others at that time) as self-removal. Discussing silence as part of discourse highlights the relationship between verbal and non-verbal moves as part of communication.

Given the centrality of these two moves in facilitating productive and participatory conversations, our study examined their use by facilitators in an MPD that was, itself, focused on discourse. We addressed the following research questions: how often and in which ways did facilitators use wait time and self-removal to foster productive discussions in a MPD focused on promoting high-quality mathematics discourse?

METHOD

This study includes a retrospective analysis (Cobb, 2000) of data collected within a larger design research experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) that included several cycles of design, implementation, analysis, and revisions of the MPD with the goal of developing theories and designs that can be useful in other learning environments. The analysis presented herein focused on the use of the two moves of interest in one of the early implementations of the program.

Context

Project All Included in Mathematics (AIM) is a highly-specified 40-hour, year-long PD program centered around subtraction. The PD has three main goals within the areas of discourse, content knowledge, and instruction. It attempts to 1) support teachers in promoting high-quality mathematics discourse for all students, 2) develop teachers' mathematics knowledge for teaching needed to implement high-quality discourse, and 3) provide teachers opportunities to practice and use discourse techniques that promote such discourse.

Results from analyses of several implementations of Project AIM have consistently demonstrated teacher growth over the course of the PD program. For example, pre-post analyses revealed increased teacher-reported planning for classroom discourse in mathematics lessons after participation in the PD, and a follow-up quasi-experimental study showed that the difference in growth on this variable between treatment and comparison teachers had a significant effect size of +0.72. The quasi-experimental study also showed a significant effect size of +0.71 for changes in the quality of treatment teachers' practice in relation to comparison teachers, using the *Teaching for Robust Understanding in Mathematics* (TRUMath) rubric system (Schoenfeld, 2013) as a measure of instructional quality.

The PD program is organized into 13 sessions. The first six sessions take place during a summer institute that lasts approximately 20 hours. The remaining sessions take place during the school year in the form of after-school monthly discussions. For this paper, we analyzed videos from the summer institute.

Participants

Eight facilitators worked with 78 teachers during this implementation of Project AIM. Facilitators were all from the same school district as participating teachers and were selected based on their previous mathematics facilitation experiences in the district. The year prior to their implementation of the program, facilitators observed an implementation led by the MPD developers and were, therefore, very familiar with the program's goals and design. Three facilitators were selected for this particular study given the types of discourse they facilitated in their groups according to the four types of discourse represented in Project AIM Discourse Matrix (Sztajn, Heck, Malzhan, & Dick, 2020): correcting, eliciting, probing, and responsive. The first type is based on the initiate-respond-evaluate discourse pattern (Cazden, 1988) and focuses mostly on finding correct answers to facilitator-posed questions. Eliciting discourse is about engaging teachers in the conversation, in an inviting and non-threatening fashion, with the goal of opening up participation. Probing discourse adds the use of probing and pressing questions to deepen the conversation, and responsive discourse shifts discourse responsibility to participants and further pushes to build connections among ideas. Analysis of discourse in Project AIM has suggested that facilitators who engage participants in successful responsive discourse often start with inviting them into the conversation (eliciting), modeling and helping teachers ask and answer probing and pressing questions (probing), and then releasing discourse responsibility to participants (responsive).

Among the eight AIM facilitators in this implementation of Project AIM, 3 led only eliciting, probing, and responsive discourse with their groups, that is, they did not make use of the more traditional correcting discourse: Brenna, Crystal, and Ann. Their facilitation practices were selected for this analysis. In her work with teachers, Brenna often elicited teachers' participation in the conversation (68% eliciting discourse), encouraged the use of probing and pressing questions (14% probing discourse) and allowed teachers to take responsibility for conversations that built connections among important ideas (18% responsive discourse). Among all facilitators, Brenna not only spent no time in correcting discourse but she also led her group in spending the highest amount of time in responsive discourse during the summer institute. Crystal, similar to Brenna, mostly facilitated eliciting discourse in her group (79%). Her use of probing (5%) and responsive (16%) discourse, however, differed from Brenna. Finally, the third facilitator who did not use correcting discourse was Ann. Different from Brenna and Crystal, Ann's group spent most of its time in eliciting discourse (95%) with the remaining of the time spent in probing discourse (5%). Ann did not facilitate any instance of responsive discourse during Project AIM summer institute.

Data collection and analysis

To focus on participants' interaction, data for this study comes from the facilitation of whole group discussions (WGD). All MPD meetings were video recorded during the Summer Institute and we identified all instances in which the groups engaged in WGD, defined as any spoken communication that started with a prompt or a question from the facilitator and engaged teachers, that is, the prompt was not followed by a mini-lesson monologue from the facilitator. When facilitator and MPD participants interacted around a single thread of conversation, with different people contributing at different times (i.e., at least one person outside the facilitator spoke), then we coded that conversation as one instance of a WGD. Because the PD was highly-specified, several of the prompts starting WGD were part of the PD materials provided by the designers. Unplanned prompts that were not in the materials and were added by facilitators were also considered the start of a WGD if they initiated a discussion in which at least one teacher participated.

For the analysis presented herein, we used a WGD as the unit. Two of the authors identified and segmented all WGDs from the entirety of the video data corpus. Given our definition of a WGD, the start of a WGD was identified as the moment when the facilitator initiated a conversation by providing a prompt to participants. The end of the WGD segment came when the conversation thread changed, often due to a new prompt from the facilitator or a call for some other activity. The first two authors segmented the videos independently, setting 80% as the required minimum intercoder agreement.

Once all instances of WGD were identified and segmented, they were coded for the use of wait time and self-removal, defined as follows:

- Wait time: Pause lasting at least 3 seconds once a conversation is initiated, independent of whether the conversation is initiated or followed by facilitator (F) or participants (P): F-P, F-F, P-F, P-P all count as instances of wait time given the facilitator choice to not step in.
- Self-removal: Instances when the facilitator explicitly removes herself from the center of the conversation, which can be signalled verbally with a statement about the intent not to participate, or physically with the repositioning of self in the room, often away from the center or from standing up when all participants are seated. A signalling of self-removal that is not followed by facilitators' choice to remain out of the conversation does not receive the code.

All WGD segments were coded for type of discourse using the Mathematics Discourse Matrix. Again, the two coders worked independently until they achieved 80% intercoder agreement, at which point the third author was engaged to resolve any differences. Once all segments were coded for moves and discourse types, we calculated the average use of the move over time using 10 minutes of WGD as the unit given that most discussions lasted about that long.

RESULTS AND FINDINGS

Results from the counts of WGD, wait time and self-removal, as well as their average use during different types of discourse are presented in Tables 1 and 2.

Facilitator	WGD	Wait time	Self-removal
Brenna	28	131	20
Crystal	19	51	23
Ann	21	18	1

Table 1: Total number of WGD, wait time and self-removal

Facilitator	Eliciting Discourse		Probing or Responsive Discourse	
	Wait time	Self-removal	Wait time	Self-removal
Brenna	7	0	6	2
Crystal	3	1	3	3
Ann	2	0	1	1

Table 2: Average use of moves per 10-minute WGD for different discourse types

Brenna, who promoted the highest amount of responsive discourse, used wait time more and more often than others. The increase in self-removal during probing or responsive discourse, compared to eliciting discourse, was observed for both Brenna and Crystal, indicating they were explicit in signalling to participants the intent to be silent and not a part of the conversation to foster discussions among them. Ann, whose discourse facilitation focuses mostly on getting teachers talking and engaged (95% eliciting), barely used the two moves of interest. This pattern shows that although Ann wanted to get participants engaged and sharing, she remained at the center of the group discourse throughout the summer institute.

Further analysis of the videos to provide additional information regarding the use of the moves showed that facilitators used wait time in qualitatively different ways. For example, Brenna and Crystal used wait time consistently, during almost every WGD. Brenna started using wait time in the first WGD in which her group engaged and continued to do so throughout the Summer Institute, at high rates. Crystal, after reading the WGD prompt, typically waited for responses and sometimes initiated participating teacher think time to give them more time to prepare for sharing before she elicited their responses. When necessary, Crystal used quite long wait time periods to push teachers to talk, sometimes waiting as long as 50 seconds for a response to come from teachers. In contrast, Ann inconsistently employed wait time. For example, she once utilized wait time four times in a single whole-group discussion, but then across six other WGDs she used wait time once only. After reading a prompt, Ann often repeated the prompt multiple times until a participating teacher responded, eliminating “awkward” silences that can be part of wait time.

DISCUSSION AND CONCLUSION

Our findings suggest that similarly to the classroom, wait time can be an important move for MPD facilitators interested in promoting high-quality discourse in which teachers participate and take responsibility. Self-removal can also contribute to this type of discourse. We conjecture that facilitators interested in changing discourse in their MPD from a “hub and spoke” to a “star pattern” can begin to do so by learning to use these two moves. Their simplicity and ease to learn make them quick facilitation tools that, if properly used, can enhance the quality of discourse. The use of these moves, however, does not warrant high-quality discourse: it is important to attend to what teachers are discussing beyond the format of the conversation, making sure the discourse moves from eliciting, to probing, to responsive.

Guskey (2020) suggested changes in MPD script to first support changes in teachers' practices, which latter support changes in teachers' knowledge and belief. We suggest that changing the script in the preparation of MPD facilitators can also be a viable approach to learning to lead. Facilitators who learn to quickly change discourse patterns by using the two simple moves examined in this study can begin to see the power of teacher talk in star-patterned conversations. By creating opportunities to listen to participating teachers, facilitators can begin to understand them better. Thus, allowing teachers to talk to each other through wait time and self-removal can be a productive start for facilitation of high-quality discourse in MPD.

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References

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *The Journal of Mathematical Behavior*, 33, 149–167.

Cazden, C. (1988). *Classroom discourse: The language of teaching and learning*. Portsmouth, NH: Heinemann.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.

Council of Chief State School Officers (CCSSO). (2010). *C*ommon core state standards initiative-Standards for mathematical practices. Washington, DC: Author. Retrieved from http://www.corestandards.org/Math/Practice/

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective Teacher Professional Development*. Palo Alto, CA: Learning Policy Institute.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, 38(3), 181-199.

Elliott, R., Kazemi, E., Lesseig, K., Mumme, J., Carroll, C., & Kelley-Peterson, M. (2009). Conceptualizing the work of leading mathematical tasks in professional development. *Journal of Teacher Education*, 60, 364–379.

Ephratt, M. (2011). Linguistic, paralinguistic and extralinguistic speech and silence. *Journal of Pragmatics*, *43*(9), 2286-2307.

Gardiner, W. (2012). Coaches' and new urban teachers' perceptions of induction coaching: Time, trust, and accelerated learning curves. *The Teacher Educator*, *47*(3), 195-215.

Guskey, T. R. (2020). Flip the script on change. *The Learning Professional, 41*(2), 18-22.

Ingram J., & Elliott V. (2016) A critical analysis of the role of wait time in classroom interactions and the effects on student and teacher interactional behaviours. *Cambridge Journal of Education*, 46(1), 37-53.

Lamella, L., & Tincani, M. (2012). Brief wait time to increase response opportunity and correct responding of children with autism spectrum disorder who display challenging behavior. *Journal of Developmental and Physical Disabilities*, 24(6), 559–573.

Linder, S. M. (2011). The facilitator's role in elementary mathematics professional development. *Mathematics Teacher Education and Development*,13(2), 44–66.

LoCascio, S. J., Smeaton, P. S., & Waters, F. H. (2016). How induction programs affect the decision of alternate route urban teachers to remain teaching. *Education and Urban Society*, 48(2), 103–125.

Nathan, M. J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction*, 21(2), 175-207.

Odell, S. J., & Ferraro, D. P. (1992). Teacher mentoring and teacher retention. *Journal of Teacher Education*, 43(3), 200-204.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM*, 607-621.

Sfard, A. (2004). What could be more practical than good research? On mutual relations between research and practice of mathematics education. In M. Niss (Ed.), *Proceedings of the 10th International Congress on Mathematical Education* (pp. 76–91).

Stahl, R. (1994). Using "Think Time" and "Wait Time" Skillfully in the Classroom. *ERIC Digest*, ED370885, pp. 1-6.

Sztajn, P., Borko, H., & Smith, T. (2017). Research on mathematics professional development. In J. Cai (Ed.) *Compendium for Research in Mathematics Education*, (pp. 793-823). Reston: VA: National Council of Teachers of Mathematics.

Sztajn, P., Hackenberg, A., White, D. Y., & Allexshat-Snider, M. (2007). Mathematics professional development for elementary teachers: Building trust within a school-based mathematics education community. *Teaching and Teacher Education, 23* (6), 970-984.

Sztajn, P., Heck, D. J., Malzahn, K. A., & Dick, L. K. (2020). Decomposing practice in teacher professional development: Examining sequences of learning activities. *Teaching and Teacher Education*, 91, doi:10.1016/j.tate.2020.103039

van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of videos. *Journal of Teacher Education*, 65, 340–356.

HIGH OR LOW SCAFFOLDING? UNDERSTANDING TEACHERS SELECTION OF TOOL-BASED TASKS

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Appropriate tasks are regarded as an important factor for realizing the potentials of teaching with technology but little is known about criteria that teachers apply for task design and task selection. The present study investigated pre-service teachers' decision making when choosing from different versions of a dynamic geometry task that differed with respect to the level of scaffolding. N=29 pre-service teachers in Germany were asked to choose from three versions of a dynamic geometry tasks which had different levels of co-ordination of mathematical depth and technological action. Analysing the written justifications of the pre-service teachers for their task choice, we observed that teachers varied considerably in how they justified their task choice. In addition, we identified three categories that underly teachers' selection of tool-based tasks and which provide a lens for possible tensions that arise during selection and design of tool-based tasks.

INTRODUCTION

Research in the last decade has shown, that teaching with technology like function plotters, geometry packages and computer algebra systems (so called "Mathematics Analysis Software") can support the learning of mathematics in many different ways (e.g. Drijvers et al., 2016; Pierce & Stacey, 2010). For example, technology can support the development of mathematical concepts by allowing students to explore, discover, and develop mathematical concepts on their own. However, simply adding technology to the classroom is not enough. An important aspect of technology integration is the design and selection of appropriate tasks which realize the potentials of technology. Even though there is no overarching or unified theory for tool-based task design, many different heuristics and frameworks have been proposed that can inform the design of tool-based tasks (Leung & Bolite-Frant, 2015; Clark-Wilson & Timotheus, 2013; Leung, 2011; Trocki & Hollebrands, 2018). While these frameworks give some normative guidelines, researchers have noted a great variability of teachers' task design and task selection and little is known about the criteria underlying teachers decision making with respect to task design and selection (Smith et al., 2017a). In this study we draw on the "Dynamic Geometry Task Analysis Framework" by Trocki & Hollebrands (2018) and investigate the reasons underlying pre-service teachers' choices among different types of dynamic geometry tasks, that varied with respect to the level of scaffolding through prompts that co-ordinate mathematical depth and technological action.

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THEORETICAL BACKGROUND

Technology in the Mathematics Classroom

The use of digital technology in the mathematics classroom can comprise a plethora of different technologies. These range from general technology that can be used across different subjects (e.g. word processing software like MS-Word) to subject-specific technology like digital learning environments, function plotters, dynamic geometry systems (DGS) and computer algebra systems (CAS) that are specifically used in mathematics education (Pierce & Stacey, 2010). Research shows, that these technologies can support student learning in many different ways (Drijvers et al., 2016; Pierce & Stacey, 2010). For example, dynamic geometry systems allow for an easy construction of geometric objects and can facilitate constructivist teaching approaches by giving pupils the opportunity to explore mathematical links on their own. In particular, a DGS allows to dynamically interact with geometric objects. Students can drag vertices or line segments of geometrical objects and can observe how properties of the object change or remain invariant. In addition, a DGS can also support students in testing and explaining mathematical conjectures (Laborde, 2001; Mariotti, 2012; Azarello et al., 2002).

Tool-based Tasks for Teaching with Technology

However, the aforementioned potentials will not unfold on their own and the challenge is “to design tasks that can make use of the technology so as to improve mathematical learning” (Hitt & Kieran, 2009, p. 122). In this context the notion of a tool-based task has emerged:

> “A tool-based task is seen as a teacher/researcher design aiming to be a thing to do or act on in order for students to activate an interactive tool-based environment where teacher, students, and resources mutually enrich each other in producing mathematical experiences. In this connection, this type of task design rests heavily on a complex relationship between tool mediation, teaching and learning, and mathematical knowledge.” (Leung & Bolite-Frant 2015, p. 192)

Researchers have described several frameworks and principles that can guide the design of tool-based tasks (Leung & Bolite Frank, 2015; Leung & Baccaglini-Frank, 2017; Leung, 2011; McLain, 2016). For example, Leung & Bollite-Frant (2015) highlight that epistemological and mathematical considerations, tool-representational considerations, pedagogical considerations and discursive consideration are of major importance. Trocki and Hollebrands (2018) proposed the “Dynamic Geometry Task Analysis Framework” that specifically addresses dynamic geometry tools. The authors focus on how prompts (defined as written question or direction related that requires a verbal or written response) may enhance student’s mathematical activity and student argumentation in order to achieve particular learning goals. They suggest that tasks which contain a collection of prompts that co-ordinate technological actions with mathematical depth can increase student learning. Prompts that allow for mathematical depth are prompts that require student to explain mathematical concepts or that require students to make generalizations. Prompts for technological actions are prompts that demand the construction, measuring, dragging or manipulation within geometrical objects. Scaffolding student learning by co-ordination of these prompts may assist in guiding students through the tool-based task in order to develop mathematical knowledge.

However, these guidelines and design principles are normative principles derived by researchers in order to analyse and construct suitable tasks. Little is known about teacher's decision-making when designing and selecting tool-based tasks (Smith et al., 2017a; 2017b) which holds particularly true with respect to selection or design of different versions of the same task. Which criteria do teachers apply for the design and selection of tool-based tasks for a given learning goal? What are the underlying reasons for variations in teachers design and selection of tool-based tasks for a given learning goal?

RESEARCH QUESTION AND METHODOLOGY

The present study investigated what criteria pre-service teachers apply when choosing among tool-based tasks that had different levels of co-ordination of mathematical depth and technological action. For this, pre-service teachers were given three different tasks versions taken from the study of Trocki and Hollebrands (2018) which all address the same two learning goals but varied with respect to the co-ordination of mathematical depth and technological action. The two learning goals were: 1) justify that opposite angles of parallelograms are congruent, 2) justify that the diagonals of parallelograms bisect each other. The three task versions are depicted in Figure 1. Task version A has a low co-ordination of mathematical depth and technological action, task version B has a medium co-ordination of mathematical depth and technological action and task version C has a high co-ordination of mathematical depth and technological action (see Trocki and Hollebrands (2018) for more details). Data was gathered in 2019 from n=29 lower secondary school pre-service teachers. Each pre-service teacher was asked to justify which of the three task versions they think is most suited to accomplish the two learning goals (Figure 1). The pre-service teachers were in their final year before entering in-service teacher training and had worked with dynamic geometry programs throughout several courses.

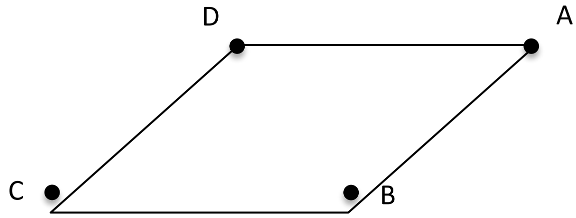
Data was analysed with respect to arguments that referred to prompts of mathematical depth or technological action. In addition, the emergence of categories was pursued that described how pre-service teachers argued with respect to the implications of these prompts. The categories were derived out of the data and were not decided upon prior to data analysis. However special attention was given to known design elements of tasks that researchers have identified. This comprise for example context, language, structure, distribution (openness) and levels of interaction between teacher and students (Barbosa & de Oliveira, 2013).

Task for pre-service teachers:

In the following you find three task versions that all refer to the same DGS-Sketch. Each task version is meant to pursue the following two learning goals: 1) justify that opposite angles of parallelograms are congruent, 2) justify that the diagonals of parallelograms bisect each other.

Which task version (A, B, C) do you regard as most suited to achieve these learning goals? Give a ranking from most suited to less suited: ___________________(e.g. C,B,A would mean C is most suited to achieve the learning goals)

Please justify your ranking and provide arguments why you regard your top choice as most suited to achieve the learning goals. Why are the other task versions less suited?



Parallelogram Task A

1) Describe what a parallelogram looks like.
2) Can you determine a relationship among the angle measures? Measure each angle. What do you notice about their angle measures?
3) Try dragging the vertices. Do your assumptions hold true?
4) Construct diagonals. Mark the point of intersection and label it E. Measure AE, BE, CE, and DE. What do you notice?
5) What is the relationship between diagonals of a parallelogram?

Parallelogram Task B

1) Drag parallelogram ABCD's vertices. Write a conjecture about the relationship between the measures of opposite angles of this parallelogram.
2) Measure the four angles of parallelogram ABCD. Drag its vertices to make many different size parallelograms. Is your conjecture from #1 true? Explain.
3) Construct diagonals. Mark the point of intersection and label it E. Write a conjecture about the relationships among the line segments AE, BE, CE and DE for parallelograms.
4) Is your conjecture from #3 true? Explain?
5) Drag vertices of the parallelogram. Make a statement describing the relationship between diagonals of parallelograms.

Parallelogram Task C

1) Drag parallelogram ABCD's vertices. Write a conjecture about the relations between the measures of opposite angles of this parallelogram.
2) Measure the four angles of parallelogram ABCD. Drag its vertices to make different size parallelograms. Is your conjecture from #1 true? Explain.
3) Construct diagonals. Mark the point of intersection and label it E. Drag vertices parallelogram ABCD. Write a conjecture about the relationships among the line segments AE, BE, CE, and DE for parallelograms.
4) Measure line segments AE, BE, CE, and DE. Drag the vertices of parallelograms ABCD to make any size parallelogram. Is your conjecture from #3 true? Explain.

5) Based on your work and conjectures in prompts #3 and #4, make a statement describing the relationship between diagonals of parallelograms.

Figure 2: Task for pre-service teachers with the three parallelogram tasks taken from Trocki & Hollebrands (2018).

RESULTS

Out of the 29 pre-service teachers 11 teachers chose task version A (low co-ordination) as the most suited, 8 pre-service teachers chose parallelogram task version B (medium co-ordination) and 9 pre-service teachers chose parallelogram task C (high co-ordination). Hence the distribution among the three task versions was almost balanced. This indicates that there was no consensus among the pre-service teachers which task version is best suited to achieve the two learning goals.

From the analysis of the written statements of the pre-service teachers three central categories emerged:

<u>Discovery learning</u>: This category describes that the co-ordination of mathematical and technological action will have an impact on student discovery learning.

<u>Affective states:</u> This category describes that the co-ordination of mathematical and technological action will have an impact on affective states like student motivation or boredom.

<u>Clarity:</u> This category describes that the co-ordination of mathematical and technological action will have an impact on clarity of the task.

In the following we will give a summary of the analysis of the pre-service teacher arguments that choose task version A (low co-ordination) and task version C (high co-ordination) and illustrate pre-service teachers arguments by examples of excerpts from pre-service teachers' writings.

Task A Preference Group

Pre-service teachers which chose task version A (low co-ordination) highlighted that a high co-ordination of mathematical depth and technological action would be *detrimental* for discovery learning, affective states and clarity.

<u>Discovery learning</u>: Pre-service teachers perceived a high co-ordination offered by the prompts as too strong. They argued that a high co-ordination of mathematical depth and technological action will reduce the openness of the task and will not give students enough opportunity to make discoveries on their own. *"Task version A is best suited since students have much opportunities to discovery the properties of the parallelogram on their own without following a detailed prescribed instruction. Task version C is much to closed."*

<u>Affective states:</u> Pre-service teachers argued that a high level of co-ordination of mathematical depth and technological action will decrease motivation and increase boredom. *"Task version A looks much more interesting and motivating. There is so much text in task version C. This will be demotivating and students will lose interest."*

Clarity: Pre-service teachers argued that a high co-ordination of mathematical depth and technological action decreases clarity of the task: *"All these prompts in task version C will confuse the students. Task version A is clear and precise does only contain the necessary information."*

Task C Preference Group

Pre-service teachers that chose task version C (high co-ordination) highlighted that the prompts would be *beneficial* for discovery learning, affective states and clarity.

Discovery learning: Pre-service teachers perceived the high co-ordination of mathematical depth and technological action is as a prerequisite that discovery learning can take place. *"The prompt to move the vertices of the parallelogram will support discovery learning since students can explore the relationship of the angles and can observe more than only one example of a parallelogram."*

Affective states: Pre-service teachers argued that the high co-ordination of mathematical depth and technological action allows students to maintain or develop positive affective states. *"Students will be motivated by the explicit prompts to alter the parallelogram. By moving the vertices, they can see that something is happening and this will be much more interesting and less boring."*

Clarity: Pre-service teachers argued that the high co-ordination of mathematical depth and technological action increases clarity of the task since it is clear for students what to do: *"The request to drag the vertices and to explain the findings makes clear what students are expected to do. Ambiguity is reduced which is important for a task."*

SUMMARY & DISCUSSION

The study aimed to scrutinize how pre-service teachers choose among tasks which have different levels of scaffolding through prompts that co-ordinate technological action and mathematical depth. The results show that pre-service teachers differed considerably in their task choices. One third of the pre-service teachers chose the task version with high level of co-ordination of technological action and mathematical depth and argued that this task version will support discovery learning, affective states and task clarity. In contrast one third of the pre-service teachers chose the task version with low level of co-ordination of technological action and mathematical depth and argued that this task version would enhance discovery learning, affective states and task clarity. Hence the present study provides some explanation for the large variability in teachers design and selection of tool-based tasks. Discovery learning, affective states and clarity were important goals for all teachers in this study. However, teachers differed largely in how they thought that co-ordination of mathematical depth and technological action might support these goals. Hence variability in teachers task design and task selection is not necessarily grounded in limited knowledge, different criteria or different goals but can be traced back to different perceptions on how task characteristics support these goals. In particular the analysis uncovers potential tensions that may be present when teachers design and select tool-based tasks (Figure 2).

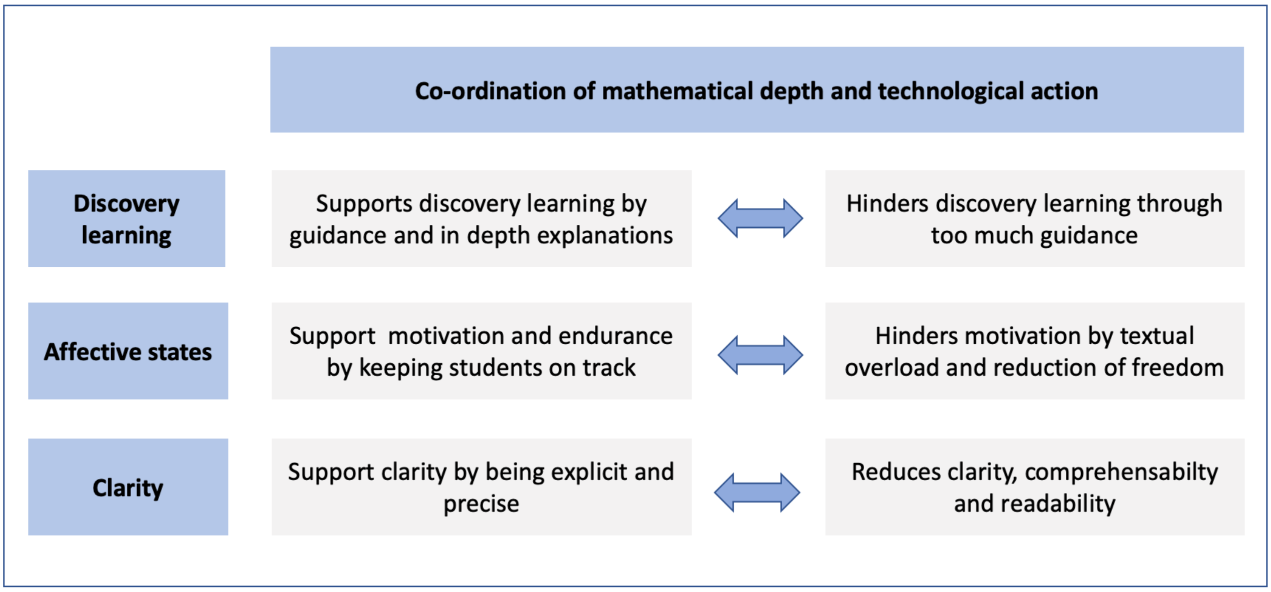


Figure 2: Tensions in the design of tool-based tasks

Therefore, simply providing teachers with research-based frameworks that guide tool-based task design may not be enough. Rather the tensions identified in this study can guide professional development efforts by providing a lens for developing teachers' ability to design and select appropriate tool-based tasks. This is particularly important since professional development programs are often lacking the desired outcomes (Thurm & Barzel, 2020). However, the research described in this paper is still ongoing. More thorough analysis of teachers' tool-based task choices is needed for example by means of cognitive interviews and for different tasks and mathematical topics.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM: The International Journal on Mathematics Education*, *34*(3), 66–72.

Ball, L., Drijvers, P., Ladel, S., Siller, H. S., Tabach, M., & Vale, C. (Eds.). (2018). *Uses of technology in primary and secondary mathematics education: Tools, topics and trends*. Cham, Switzerland: Springer.

Barbosa, J. C., & de Oliveira, A. M. (2013). Collaborative groups and their conflicts in designing tasks. In C. Margolinas (Ed.), *Task design in mathematics education* (Proceedings of the International Commission on Mathematical Instruction Study 22, pp. 541–548), Oxford, UK. Available from http://hal. archives-ouvertes.fr/hal-00834054

Clark-Wilson, A., & Timotheus, J. (2013). Designing tasks within a multi-representational techno- logical environment: An emerging rubric. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI Study 22* (pp. 45–52), Oxford, UK. Available from http://hal. archives-ouvertes.fr/hal-00834054

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Switzerland: Springer.

Hitt, F., & Kieran, C. (2009). Constructing knowledge via a peer interaction in a CAS environment with tasks designed from a task–technique–theory perspective. *International Journal of Computers for Mathematical Learning*, 14(2), 121–152.

Hoyles, C., & Lagrange, J. B. (Eds.). (2010). *Mathematics education and technology: Rethinking the terrain*. Berlin: Springer.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM*, *43*(3), 325–336.

Leung, A., & Baccaglini-Frank, A. (2017). *Digital Technologies in Designing Mathematical Tasks – Potential and Pitfalls.*

Leung, A., & Bolite-Frant, J. (2015). Designing mathematics tasks: The role of tools. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education* (pp. 191–225). Cham, Switzerland: Springer:

Mariotti, M. (2012). Proof and proving in the classroom: Dynamic geometry systems as tools of semiotic mediation. *Research in Mathematics Education*, *14*(2), 163–185.

McLain, C. J. (2016). *Supporting Teachers' Selection of Dynamic Mathematics Environment Tasks*. North Carolina State University, PhD thesis.

Pierce, R., & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*, *15*(1), 1–20.

Smith, R., Shin, D., & Kim, S. (2017a). A Framework for Examining Teachers' Noticing Of Mathematical Cognitive Technologies. *Journal of Computers in Mathematics and Science Teaching*, *36*(1), 41–63.

Smith, R. C., Shin, D., & Kim, S. (2017b). Prospective and current secondary mathematics teachers' criteria for evaluating mathematical cognitive technologies. *International Journal of Mathematical Education in Science and Technology*, *48*(5), 659–681.

Thurm, D., & Barzel, B. (2020). Effects of a professional development program for teaching mathematics with technology on teachers' beliefs, self-efficacy and practices. *ZDM*. https://doi.org/10.1007/s11858-020-01158-6

Trocki, A., & Hollebrands, K. (2018). The development of a framework for assessing dynamic geometry task quality. *Digital Experiences in Mathematics Education*, *4*(2–3), 110–138.

ARE SOME LANGUAGES MORE MATHEMATICALLY PREFERRED FOR COUNTING PURPOSES?

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Language complexity and mathematics performance have been linked together in earlier studies researching effective pedagogical practices for teaching early number sense to young children. Though the choice of language is usually dependent on that of a child's parents or caregivers, little is known about whether there might be any advantage when it comes to learning numbers and numerals for those who are given a choice of language. This study examines the presence of mathematical preferences of adults in their evaluation of invented languages. The results reveal that languages with a simpler, predictable numeral system structure are considered mathematically preferable to the extent that each number word can be easily associated with a distinct symbol and place value in the corresponding numeral system.

INTRODUCTION

Persistent challenges among students from certain countries to rise above their longitudinally below-average standings in international mathematics assessments such as the Trends in International Mathematics and Science Study (TIMSS) have been well documented (Mullis et al., 2016). This has led researchers to explore explanatory factors ranging from national curricula to affective qualities (Kaiser, Luna, & Huntley, 1999; Leung, 2014; Schmidt et al., 2001; Stigler & Hiebert, 2004). In recent years, researchers have examined language differences to explain variations in mathematics performance, albeit inconclusively (Miller et al., 2005; Tjoe, 2017). Some advocate incorporating base-ten blocks into introductory counting lessons for kindergarteners in order to account for the base-ten structure transparency of East Asian languages which has been credited for the consistently high performing mathematics achievement of those countries (Miller et al., 1995).

Nevertheless, it was not clear from these studies whether learning any language might be beneficial in acquiring elementary understanding of a numeral system, especially if students have a choice (Civil & Planas, 2004). Instead of measuring the effectiveness of language choice on mathematics competence in young students, the present research aimed to evaluate such an effect on adults. We approached our goal by analyzing subjects' problem-solving ability in re-constructing a numeral system associated with a list of invented number words simulating four language patterns. Researcher asked subjects about the extent to which one language might be considered more mathematically preferable than others.

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 145-152. Khon Kaen, Thailand: PME.

THEORETICAL FRAMEWORK

Students in East Asian countries have long been perceived to hold a natural linguistic advantage over those in western countries as far as their performance in international mathematics assessments (Geary et al., 1993; Stevenson et al., 1990). Chinese-speaking countries (e.g., Hong Kong–CHN and Chinese Taipei–CHN) specifically have for five consecutive quadrennial periods between 1995 and 2015 been steadily among the world's top five in TIMSS Grades 4 and 8 mathematics (Mullis et al., 2016).

Compared with other languages, Chinese was to a certain extent viewed as better communicating a one-to-one correspondence between Hindu-Arabic numeral symbols and their corresponding number words, as well as a consistent and transparent base-ten structure and positional place value (Miller & Paredes, 1996).

For instance, to master enumeration skills from 1 to 99, Chinese-speaking students needed to learn only ten distinct number words for numerals 1 to 10, because any number words for numerals 11 to 99 could be constructed by combining the first ten distinct number words with a natural base-ten structure and place value in mind (e.g., "yī," "shí-yī," "shí-jiǔ," "èr-shí-yī," and "jiǔ-shí-yī" might be viewed as "1," "10 + 1," "10 + 9," "2 × 10 + 1," and "9 × 10 + 1" for 1, 11, 19, 21, and 91, respectively) (Fuson, Richards, & Briars, 1982). Although Korean-speaking students needed to learn nine additional number words for multiples of ten in addition to ten distinct number words for numerals 1 to 10, a distinct rule was apparent in the separation of place values and their positions (e.g., "hanna," "yoll-hanna," "yoll-ahop," "sumul-hanna," and "ahun hanna" might be viewed as "1," "10 + 1," "10 + 9," "20 + 1," and "90 + 1" for 1, 11, 19, 21, and 91, respectively) (Miller & Paredes, 1996). English-speaking students needed to learn more than ten distinct number words for numerals 1 to 99, with a potential misconstruction of number words given an occasionally reversed place value structure (e.g., "one," "eleven," "nineteen," "twenty-one," and "ninety-one" might be viewed as "1," "11," "9 + 10," "2 × 10 + 1," and "9 × 10 + 1" for 1, 11, 19, 21, and 91, respectively) (Ho & Fuson, 1998). French-speaking students also needed to learn more than ten distinct number words for numerals 1 to 99, as well as the idea of some number words being a compound multiplicity where numbers could be viewed as a multiple of a multiple of ten (e.g., "un," "onze," "dix-neuf," "vingt-et-un," and "quatre-vingt-onze" might be viewed as "1," "11," "10 + 9," "20 + 1," and "4 × 20 + 11" for 1, 11, 19, 21, and 91, respectively) (Miller & Paredes, 1996).

Regard for the efficacy of Chinese prompted some researchers to adapt its transparent base-ten structure to languages with less transparent base-ten structures in the hope of facilitating stronger association between number words and number concepts (Wynn, 1992). Beginners' perspectives on language choice in learning mathematics concepts as early as counting skills remain unclear.

The research of choice in mathematics problem solving itself pointed to a variety of motivations (Nesher et al., 2003; Presmeg, 1986; Silver et al., 1995; Star & Rittle-Johnson, 2008). Expert mathematicians, for their part, reflected on aesthetic values when choosing a preferred solution method among many (Dreyfus & Eisenberg, 1986; Silver & Metzger, 1989; Tjoe, 2015). It was because of the breadth and depth of their mathematical comprehensions that these experts became more disposed toward—and thus appreciative of—the assessment of mathematical beauty (Sinclair, 2001).

Earlier studies have suggested that—implicitly, in the absence of language choice—the regurgitation, if not formulation, of number words influences beginning counters' ability to transfer their early study of numeral symbols into later mathematics achievement (Miller et al., 2005). To put it differently, the preferred language allows beginning counters to recognize the complete harmony—as Sinclair (2004) highlighted in her analysis of mathematical beauty—of the underlying correlation between linguistic and mathematical structures. To this extent, using a different methodology, the present study might be valuable in helping elementary classroom teachers further weigh the pedagogical benefits of integrating a concrete base-ten structure in their early number and operation instructions, as suggested by past studies (Miller et al., 1995).

METHODOLOGY

A total of 120 undergraduate science and engineering students from a large university in the Northeastern region of the United States volunteered for the study, in which a proportionate stratified random sampling technique was used. Most of them were international students. Of the 120 subjects, an equal number of 30 spoke Chinese, Korean, English, and French as their first languages, respectively. Each of the 30 subjects spoke English, but no other languages besides their first language. Prior to participating in the research, subjects were asked about their familiarity with non-decimal numeration systems. All subjects reported that they had taken mathematics courses that surveyed non-decimal numeration systems.

Four languages—namely, Chinese, Korean, English, and French—were selected to reflect four major different levels of transparency of base-ten structure into which most languages around the world are classified (Tjoe, 2017). From these four languages, the authors derived four invented languages (ILs) simulating the actual four languages but reflecting a base-five numeral system instead of the standard base-ten numeral system.

IL1, IL2, IL3, and IL4 were based on Chinese, Korean, English, and French, respectively (see Table 1). For numerals 1 to 44 in base five, all 24 invented number words (except their variations) were kept to single syllables to avoid preferences for the least number of syllables involved in the number words.

(Hindu- Arabic) numeral symbols in base five	IL1 (Chinese-based)	IL2 (Korean-based)	IL3 (English-based)	IL4 (French-based)
1	wie	oak	eor	sap
2	aur	baes	lang	tou
3	piir	ryuu	ngge	miin
4	soat	zar	uos	goo
10	yoap	vis	non	ber
11	yoap-wie	vis-oak	jaen	bappin
12	yoap-aur	vis-baes	samj	touin
13	yoap-piir	vis-ryuu	nggi-nunn	ber-miin
14	yoap-soat	vis-zar	uor-nunn	ber-goo
20	aur-yoap	boir	leng-nann	meng
21	aur-yoap-wie	boir-oak	leng-nann-eor	meng-sap
22	aur-yoap-aur	boir-baes	leng-nann-lang	meng-tou
23	aur-yoap-piir	boir-ryuu	leng-nann-ngge	meng-miin
24	aur-yoap-soat	boir-zar	leng-nann-uos	meng-goo
30	piir-yoap	hia	nggi-nann	meng
31	piir-yoap-wie	hia-oak	nggi-nann-eor	meng-bappin
32	piir-yoap-aur	hia-baes	nggi-nann-lang	meng-touin
33	piir-yoap-piir	hia-ryuu	nggi-nann-ngge	meng-ber-miin
34	piir-yoap-soat	hia-zar	nggi-nann-uos	meng-ber-goo
40	soat-yoap	pott	uor-nann	tou-meng
41	soat-yoap-wie	pott-oak	uor-nann-eor	tou-meng-sap
42	soat-yoap-aur	pott-baes	uor-nann-lang	tou-meng-tou
43	soat-yoap-piir	pott-ryuu	uor-nann-ngge	tou-meng-miin
44	soat-yoap-soat	pott-zar	uor-nann-uos	tou-meng-goo

Table 1: Base-five numerals and number words in four ILs that simulated Chinese, Korean, English, and French languages

All subjects met with the researcher in one classroom at the same time. They were not informed that most subjects did not speak English as their first language. They were provided with a printed set of four ILs comprised of only ten consecutive number words for the numerals 1 to 20 in base five, but without any information on the corresponding (Hindu-Arabic) numeral symbols. Subjects were informed that the number words did not correspond to the base-ten numeral system, but were not informed of the number base to which all number words corresponded, or of whether the number base was the same across the four ILs.

In addition to the printed set of the four ILs for numerals 1 to 20 in base five, blank paper, a pencil, and at least 325 counters were provided to each subject. Without time limit, subjects were asked to: (a) determine the next 14 number words for the subsequent numerals in each of the four ILs; (b) determine the number base to which all number words corresponded, and (c) visualize using the counters provided the corresponding grouping and/or mathematical equivalence of each number word. At the time of the study, subjects who asked whether they needed to figure out corresponding (Hindu-Arabic) numeral symbols for each number word were welcomed to do so.

After completing the three tasks earlier, subjects were immediately provided with a complete list of number words and informed that the 24 number words for the four ILs corresponded to a base-five numeral system. Subjects were asked to: (a) determine with explanations one of the four ILs which they might consider learning if they were to use it for counting objects in the corresponding base system; (b) determine with explanations the rank order of the four ILs beginning from the most to the least preferred languages; (c) determine to which of the four ILs might their own first language be similar, and (d) again visualize using the counters provided the corresponding grouping and/or mathematical equivalence of each number word.

ANALYSIS AND RESULTS

All subjects completed the first three tasks in less than one hour. Their written responses revealed that they were able to complete the next 14 number words for the subsequent numerals in IL1 with higher accuracy than for the other ILs. (Many subjects left blank or answered incorrectly the next 14 number words for IL2, IL3, and IL4.)

Of the 120 subjects, 103 were able to provide precise predictions of the next 14 or more number words for the subsequent numerals in IL1. They were also able to figure out, in a relatively short amount of time, that number words in IL1 corresponded to a base-five numeral system. Of these 103 subjects, all were able to successfully identify mathematical equivalences for the 24 number words in IL1 either by drawing, by using counters, or by writing. Apparently, it was these 103 subjects who took the time to figure out the correct corresponding (Hindu-Arabic) numeral symbols for each number word. (During the study, one subject asked out aloud whether he or she was required to figure out the corresponding (Hindu-Arabic) numeral symbols for each number word. After the researchers answered, “You may or may not choose to do so,” the researchers announced this response to all 120 subjects to avoid partiality. It was obvious that most subjects took this announcement as a pointer.)

In contrast, none of the 120 subjects were able to determine the next 14 number words in IL2, IL3, or IL4. (A few were able to determine only the next 4 number words in IL2 and/or IL3 with multiple errors for the following 10 number words.) Consequently, IL2, IL3, and IL4 were the most challenging for all subjects to determine to which number base their number words corresponded as well as the visualization of these number words.

Upon the completion of the first three tasks, the subjects seemed curious about the complete list of number words provided and about the base-five numeral system to which these four ILs corresponded. (Some subjects studied them for nearly 15 minutes before continuing to proceed with the last four tasks.) Most subjects were surprised to learn that number words in IL3 and IL4 (but not IL1 and IL2) corresponded to a base-five numeral system.

All the 120 subjects reported that they preferred learning IL1, assuming they were to count objects in a base-five numeral system. Their justifications included: "the pattern [of number words] is predictable just like the matching symbols;" "it seems short, simple, and clean unlike other languages;" "pretty neat because the words coordinate well with the symbols;" "I don't have to second guess myself whether my answers (to the next 14 number words) are correct or not;" and "I wish our (number words) were easy like this."

While all 120 subjects considered IL1 the most preferable, 105 considered IL2 the second most preferable. Subjects wrote: "[IL2] has a similar taste as [IL1] … you just need to be careful not to get ahead of yourself after every 5 (number words) spelled out;" "everything seems bundled up in [IL2];" and "seeing a new (number word) is a good sign that we are going back into a new cycle."

Of the 120 subjects, 96 and 101 considered IL3 and IL4 the third and fourth most preferable, respectively. Subjects reported: "[IL3] is not the best but more predictable than [IL4];" "[IL4] is like putting [IL2] and [IL3] together;" "you need to think a lot for (number words in IL4) because it's not a straightforward translation (between number words and numeral symbols)", and (number words in IL4) were repeated in the pattern and even after knowing it is base five, I still need to make sense why they got repeated before and after certain (number words)."

DISCUSSION AND CONCLUSION

Previous studies indicated that no definite agreement exists among researchers using international assessments in mathematics such as TIMSS regarding the effect of languages with transparent base-ten structures (such as Chinese) on the mathematics performance of students from countries speaking such languages. When one had a choice to learn a language associated with a new, non-decimal numeral system, the present research partially demonstrated the existence of a preference for a language with a transparent numeral system. Despite their predictive factor in mathematics achievements four or eight years subsequent, languages more mathematically preferred for counting purposes might be best suited for kindergarteners to learn the conceptualization of number sense, as well as to appreciate the construction of our decimal numeral system.

While the present research did not attempt to predict which language might produce more favorable mathematics achievement by its learners, it suggests, to a certain degree, alternative empirical evidence that early number sense—especially in counting and cardinality—might be comprehended more effectively if the numeral system is chosen to discernibly coincide with numeral symbols. That is, because our numeral symbols are based on the decimal system, the task of transposing between these numeral symbols and the language of the word names, if not the language itself, should also be based as closely as possible on the decimal system. Regarding recognizing a certain language as more mathematically beautiful than others, aesthetic considerations in mathematics classroom learning might be attainable as early as kindergarten (Tjoe, 2016). Further studies are needed to investigate links between language preference and student success in arithmetic operations.

References

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: Does every student have a voice? *For the Learning of Mathematics*, *24*(1), 7–12.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetic of mathematical thought. *For the Learning of Mathematics*, *6*(1), 2–10.

Fuson, K. C., Richards, J., & Briars, D. (1982). The acquisition and elaboration of the number word sequence. In C. Brainerd (Ed.), *Progress in cognitive development: Children's logical and mathematical cognition* (Vol. 1). New York, United States: Springer-Verlag.

Geary, D. C, Bow-Thomas, C. C, Fan, L., & Siegler, R. S. (1993). Even before formal instruction, Chinese children outperform American children in mental addition. *Cognitive Development*, *8*, 517–529.

Ho, C. S-H., & Fuson, K. C. (1998). Children's knowledge of teen quantities as tens and ones: Comparisons of Chinese, British, and American kindergartners. *Journal of Educational Psychology*, *90*, 536–544.

Kaiser, G., Luna, E., & Huntley, I. (1999). *International comparisons in mathematics education*. London, United Kingdom: Falmer Press.

Leung, F. K. S. (2014). What can and should we learn from international studies of mathematics achievement? *Mathematics Education Research Journal*, *26*(3), 579–605.

Miller, K. F., Kelly, M. K., & Zhou, X. (2005). Learning mathematics in China and the United States: Cross-cultural insights into the nature and course of mathematical development. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 163–178). New York, United States: Psychology Press.

Miller, K. F, & Paredes, D. R. (1996). On the shoulders of giants: Cultural tools and mathematical development. In R. J. Sternberg & T Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 83–117). New Jersey, United States: Lawrence Erlbaum.

Miller, K. F., Smith, C. M., Zhu, J., & Zhang, H. (1995). Preschool origins of cross-national differences in mathematical competence: The role of number-naming systems. *Psychological Science*, *6*, 56–60.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics*. Massachusetts, United States: TIMSS & PIRLS International Study Center.

Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Educational Studies in Mathematics*, *52*(2), 151–176.

Presmeg, N. C. (1986). Visualization and mathematical giftedness. *Educational Studies in Mathematics*, *17*(3), 297–311.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H. C., Wiley, D. E., Cogan, L. S., & Wolfe, R. G. (2001). *Why schools matter: A cross-national comparison of curriculum and learning*. California, United States: Jossey-Bass.

Silver, E. A., Leung, S. S., & Cai, J. (1995). Generating multiple solutions for a problem: A comparison of the responses of U.S. and Japanese students. *Educational Studies in Mathematics*, *28*(1), 35–54.

Silver, E. A., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem-solving. In D. McLeod, & V. Adams (Eds.), *Affect and mathematical problem-solving* (pp. 59–74). New York, United States: Springer-Verlag.

Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics*, *21*, 25–33.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*(3), 261–284.

Stigler, J. W., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership*, *61*(5), 12–16.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem-solving: The case of equation solving. *Learning and Instruction*, *18*(6), 565–579.

Stevenson, H. W., Lee, S.-Y., Chen, C., Lummis, M., Stigler, J., Fan, L., & Ge, F. (1990). Mathematics achievement of children in China and the United States. *Child Development*, *61*(4), 1053–1066.

Tjoe, H. (2015). Giftedness and aesthetics: Perspectives of expert mathematicians and mathematically gifted students. *Gifted Child Quarterly*, *59*(3), 165–176.

Tjoe, H. (2016). Aesthetics in school mathematics: A potential model and a possible lesson. *The Mathematics Enthusiast*, *13*(3), 279–302.

Tjoe, H. (2017). The relationship between language complexity and mathematics performance. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy. (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273–280). Singapore: PME.

Wynn, K. (1992). Children’s acquisition of the number words and the counting system. *Cognitive Psychology*, *24*, 220-251.

MEASURING AGENCY IN MATHEMATICS COLLABORATIVE PROBLEM SOLVING

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Collaborative problem solving (CPS) has received much attention recently. However, little is known about how people work in CPS and how to support students working productively in the setting. CPS requires both social and cognitive aspects. This paper focused on the social aspect to examine how students act on their agency in CPS. Using data from a laboratory classroom with a social setting, we analysed the positions that students take when solving an open-ended mathematics problem and their illocutionary force. The results show that students reveal dynamic positions with a combination of different forces. Also, some interactions are more productive in CPS than others.

INTRODUCTION

Collaboration problem solving (CPS), a critical 21st-century skill, has received much attention recently (Graesser et al., 2018). However, less is known about how people work in CPS, especially how to support students to work well in the setting. CPS requires both social and cognitive aspects (ibid.). Concerning the social aspect, how students act on their agency when working with each other needs attention. That is, further research is needed to clarify what position they take and the illocutionary force they have when communicating with each other to solve the problem. Research has indicated that one type of unsuccessful group includes too many leaders (Miller et al., 2013). However, we need to concern about different positions for productive CPS. A productive group needs to utilise dynamic expertise in the group in forming a solution that leads to agreement and makes every member feels inclusive. Drawing on a networking theories approach, we, for an illustrative purpose, examine how the agency is acted on during CPS. We address the research question: *How do students act on their agency when engaging in mathematics collaborative problem-solving?* We advance the literature by developing an analytical lens at a micro-level of talk turns about the agency. In addition, we seek to highlight the association between multiple social aspects, linking between positionings and dialogic talk.

LITERATURE REVIEW

To deepen our understanding of a social learning process, we draw on three different components: interaction and learning, agency, and positioning.

Interaction and Learning

Interaction is a cornerstone component of learning, yet not all are equally productive for learning (Sfard & Kieran, 2001). Sfard and Kieran argue that *intention* is inherent in the act of communication. We cannot fully understand any act of communication

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without thinking about the intention of the participants in that situation. "Intention is the property of utterance that allows it to be followed by some kinds of responses, but not by others." Hence, "… communication is effective if it fulfils its communicative purpose, that is, the different utterances of the interlocutors evoke responses that are in tune with the speakers' meta-discourse expectations." (Sfard & Kieran, p. 49) Therefore, a productive interaction should lead to long term impact on students' future participation in mathematical discourse.

Drawing on the dialogic learning theory of communicative speech acts, Díez-Palomar and Cabré (2015) analysed illocutionary force of utterances used by participants to to highlight students' intention when communication. The intention of someone participating in an episode of interaction might be (a) reaching a consensus towards a particular concept, (b) imposing his/her point of view on that matter, (c) adopting a neutral position, or (d) manifesting expressiveness. Dialogic talk refers to the use of language to reach consensus. Consensus arises when the participants agree on the explanations, arguments, claims they use to justify their position. When participants use dialogic talk, it is more likely for learning to happen, because participants engage in educationally productive interactions (Díez-Palomar & Cabré, 2015). During a dialogic talk, all of them can verify the truthfulness of those explanations, arguments or claims by themselves, using validity claims. Communication becomes effective, only when this process happens. Hence, educational productivity is subjected to the use of validity claims rather than power claims.

Agency

Current research in education recommends promoting agency among students to support their learning. There are different definitions for the agency (Chateris & Smardon, 2018). Cognitivism theories claim that agency is intrinsic to individuals. Authors embracing socio-cultural theories regard that agency emerges in spaces where individuals interact with each other. It is a dynamic process that resides in social situations. Other authors define agency as a "quality of learner engagement with temporal-relational contexts-for-action." (ibid., p. 56) This is what Chateris and Smardon call "new material Theory Assemblage theory" in which agency is defined flexiblely, involving both human and non-human (objects), because sometimes even objects can "make things happen."

Positioning

Positioning might help us understand how cognition evolves throughout the process of interaction. Assuming that learning is a social process, based on the interaction of participants, then a relevant question is to consider if the position that they play within the group matters (or not) in terms of how they learn or construct meanings associated to the mathematical objects being discussed.

In a conversation, not all the participants play the same "role" all the time. Participants are acting as *leaders*, being the ones initiating the dialogues; others prefer to act as *followers*, agreeing, rejecting or resisting the leader(s)' propositions. Positioning has been defined as a dynamic concept (instead of "role"), because the same person might take several positions during the same episode of interaction, even contradictory ones (e.g., powerful or powerless, dominant or submissive, authorised or unauthorised). van Langenhove and Harré (1999) defined positioning as “the discursive construction of personal stories that make a person’s actions intelligible and relatively determinate as social acts and within which the members of the conversation have specific locations.” (p. 16) Drawing on Austin (1962), we can use the illocutionary force of a speech act in order to understand the position that s/he is taking within the group. Illocutionary force is mediated by the speaker’s position within the group. The meaning of their utterances is also mediated by the position they occupy within the group. Therefore, positioning matters in order to interpret the meaning of participants’ acts in a conversation. We draw on positioning and illocutionary force to analyse how students act on their agency during CPS in this study.

METHODOLOGY

Setting and Data

Part of the Social Unit of Learning project (Chan & Clarke, 2017) examined individual, dyadic, small group and whole-class learning and problem-solving in mathematics, the data for the study came from a Year 7 class accompanied by their teacher in a laboratory classroom. The classroom resembled a natural setting but was equipped with advanced video technology that permitted recording of the dialogue and social interactions between students in small groups. The project collected student-written products, video and audio recording of each student and the teacher. The data for the present study came from a group of four students Pandit, Anna, Arman, and John solving an open-ended mathematics problem. The problem was presented on a card to each group.

Fred’s apartment has five rooms. The total area is 60 square metres.

1. *Draw a plan of Fred’s apartment.*
2. *Label each room and show the dimensions (length and width) of all rooms.*

This task can be solved in several ways. It is typical of mathematics problems that allow students multiple entry points and facilitate their interaction and display of collaborative behaviours (Chan & Clarke, 2017).

The group was provided with writing materials that they used freely while solving the problem. They were allocated 20 minutes. Their regular teacher supervised the group and provided no direct instructions or feedback for task completion.

Data Analysis

Transcripts were the primary data source and videos were referred to occasionally when needed for clarification. First, we chose frameworks that reflect the social aspects of collaboration that focus on interrelation agency. When referring to an agency, we regarded the nature of talk (dialogic or not) and positions unfold the relationships. In turn, it explains how knowledge was co-constructed, taken and developed. The dialogic talk framework (Garcia-Carrión & Díez-Palomar, 2015) was used to examine the quality of interaction. We particularly coded for the illocutionary force (neutral, coercion, expressiveness, or consensus). When coding for the position, we adopted Barner’s (2004) 13 positions students have when engaging in CPS (Figure 1), which were grouped into on-task and off-task positions.

When coding, we found some positions were not evident in the data and added a new position of *follower*, who engages in the task by responding to or revoicing others without adding extra information and ideas to the discussion. Figure 1 summarises the coding framework developed.

Position:
· On task: Manager Helper Critic In Need of Help *Follower Expert Outside Expert Facilitator
· Off task: Humourist Entertainer Audience Networker Outsider
Illocutionary force: Expressive Neutral Consensus Coercion

Figure 1: Coding framework

Agency was defined as a result of the position that the speaker plays within the group, also considered the illocutionary force of the speech acts in which s/he participates.

PRELIMINARY RESULTS

We will use exemplars to illustrate the coordination between the positions the students take and their related illocutionary force. We then describe some interactions of positions as students work together on the task.

Coordination between the Position a Student Takes and Illocutionary Force

Eight positions were coded. Except for the *outsider* position, the off-tasks positions were not found. Also, we have not coded for any instance that a student shows the position of an *expert* or *outside expert*. This might show the collective expertise in this group.

In the following excerpt, Pandit (Lines 75, 77) showed the position of a *manager*, who asked for the dimensions of the house with a *consensus* force. Arman (Line 80) took the position of an *outsider*, who did not actively engage in solving the task and gave no sign of seeking to participate in the solution. His interlocutory force was *neutral*. Anna took the position as a *helper* (Line 78), who carried out routine tasks when asked to do so by another group member or acted in a subordinate position, under Pandit's direction with the *neutral* force. Pandit then played the position as a *follower* (Line 79), with the *neutral* force, who revoiced Anna and kept the conversation going without adding more information and ideas to the solution.

75	Pandit:	Wait. It's going to be like - wait …	Manager	Con
76	Anna:	No, guys.		
77	Pandit:	… how big do you want the house to be?	Manager	Con
78	Anna:	It's 60 metres square.	Helper	N
79	Pandit:	Sixty square metres.	Follower	N
80	Arman:	It's… Measure your hand.	Outsider	N

83	Pandit:	How big do you want? Like the length and width?	Manager	Con
84	Anna:	Okay. I'll decide.	Manager	Coe
85	Pandit:	Guys, listen.	Facilitator	N
88	Pandit:	My God, what are you doing?	Critic	Con / E
89	Anna:	What?...	Follower	E
90	Pandit:	Why are you changing it	Critic	Con
91	Anna:	It's a square.	Manager	Coe / N
92	John:	Pick a new one.		
93	Arman:	Why, why, why?	Follower	
94	Pandit:	Don't do square. Why make it a square house?	Critic	Con

When proposing a rectangle as a shape for the apartment, Pandit initiated work and ideas and sought consensus about the dimensions of the rectangle (Line 83). She asked her peers with the position of a *manager* who sought for *consensus* to continue the solution. In this instance, her peer Anna also took a *manager* position, but with a *coercion* force (Line 84). It seems that when Pandit felt to be accountable for the group work, Anna was more for herself. Pandit then (Line 88) took a *critic* position, who sought explanations, looked for alternative methods, or disputed other people's assertions. This happened when Anna tried to change the scale without discussion and gave an explanation for her action. Pandit's force was *consensus* and *expressive* in the first instance (Line 88) and then to *coercion* and *consensus* later (Line 94).

We also found that sometimes, a student played the position of *facilitator*, who acted to keep the group functioning smoothly, gave social support, ensured that nobody is ignored, tried to avoid or resolve conflict. Line 85 provided an example of the position when Pandit tried to bring all the members together. Her force was *neutral*.

The last position we found was *in need of help*, who either claimed not to understand, and explicitly or implicitly asked for help, or accepted an offer of help from another and attended to the explanation. John, a non-native English speaker, sought help for ways to write a division symbol, as he attempted to find out the area of each of the rooms in the apartment.

127	John:	It's huge. Where is… Division is like this or like this. Which one is division? Which one is division?	In need of help	N

Interaction among Positions by Students

Students engaging interaction episodes may navigate between different positions. They alternate leading the action or following others in the group when sharing thoughts in order to solve the task. We have identified four main types of interaction in which participants play different positions: manager-helper, manage-manager, and manager-critic, and manager – follower.

Lines 77-78 showed the interaction of a manager and a helper. When Pandit gave direction, Anna followed the direction and pointed out the information given in the task that was related to the direction Pandit asked for.

A few turns after that, in Lines 83 and 84, we found a certain degree of rivalry among Pandit and Anna. We characterised the interaction among positions as manager-manager. Pandit and Anna took positions of a manager next to each other. Both of them tried to lead the interaction to find an answer to the task. However, they came with different illocutionary forces: consensus for Pandit and coercion for Anna. Whereas Pandit is asking the members of the group about the size of Fred's apartment, Anna just claimed that she "will decide" (how big Fred's apartment would be). This could yield conflict and create a barrier for a collective solution. Next (Line 85), Pandit adopted a position as a facilitator, when she asked the members of the group to listen (to Anna's words). That re-situated the attention back to the task, bringing people together in seeking for a possible answer.

As the discussion was evolving, we identified other types of interaction among positions by students. In some cases, a student gave a direction, who acted as a manager. Another student has then criticised the direction, to seek an explanation or dispute this direction. In different occasions, this led to another direction of further explanation. Lines 114-116 exemplified this type of interaction (manager-critic). Pandit played the critic position to question why Anna changed the scale and choose the shape of the apartment. Anna then reacted and clarified that she just doubles it.

114	Anna:	Let's make two centimetre - guys, let's make the two centimetre square one metre square in this, okay?	Manager	Coe
115	Pandit:	Don't – don't. It's so confusing.	Critic	Coe
116	Anna:	Why not? How is it confusing? You just double it.	Manager	Con

Later on, during that interactive episode, we found other situations in which a student did not act according to the direction of the manager but responded to the manager without adding more information to the conversation. The following except showed this example. This illustrated the manager – follower interaction.

134	Anna:	Okay. Can I just leave this?	Manager	Con
135	Pandit:	Yes.	Follower	Con

DISCUSSION

The preliminary results showed that the students took dynamic positions when solving the problem collaboratively. However, not all dynamic is as good. When a student always takes the position of a helper, in need of help, or follower, a teacher can question this dynamic interaction and has an action to support this student. Also, when a student consistently takes one of the off-task positions, the grouping might need consideration as the student does not contribute nor benefit from solving the problem with the group. A facilitator position is unique in bringing collective efforts, even though it is not necessary to solve the task at hand. We argue that this position might be challenging for young students.

We have not found all the positions suggested in the coding framework. This might be due to the task itself did not offer such opportunities for the position to be acted (e.g., expert, who needs to confirm the mathematical correctness). Whereas, we supplemented with a new position of follower, who might not actively add to the cognitive aspect of CPS, yet to the social aspect that keeps the conversation going.

The coordination of position and force is helpful to clarify the interactions and explain the mechanism for the effectiveness of collaboration. A manager position with an illocutionary force of consensus or neutral may have a different impact on CPS compared with that with coercion. Except for the manager who is also an expert in the group, a coercion force might create tension in the group, which in turn impacts the group cohesion. Future studies can examine the impact of this in a combination of types of tasks. Likewise, a critic with a force of consensus or neutral might be productive in helping others to revise their ideas. In contrast, the coercion force might create hardship and prohibit collaboration. Interestingly, we have not seen a manager with an expressive force. Are these two mutually exclusive, which can be further explored?

Some interactions are more effective than others in collaboration (Sfard & Kieran, 2001). At one instance, if one student takes a position of a manager and others play a position as a helper to proceed with the direction or a follower to give feedback to the manager without significant contribution, the group can move on to a direction. In another case, after the manager initiates an idea, another act as a critic to seek explanation or point out flaws, this is effective for the group as a whole in both solving the task and group cohesion. In contrast, when too many managers work at the same time without listening to others, collaboration is not beneficial as it shows to be more individual instead of collective thoughts. This confirms previous studies about the negativity when there are many leaders in a group (cf. Miller et al., 2013). Another case is when one acts like a manager or critic and others take more of off-task positions, this can prohibit the collaboration. Lastly, when we have more than two people in a group, a facilitator might be crucial to keep the conversation going. Also, it might be helpful to see the relationship between positions of manager, critic and expert and outside expert.

The results were illustrative. We propose conditions for productive CPS and more research is needed for statistical significance. However, the detailed analysis conducted suggests interesting aspects characterising interaction, that researchers would like to consider in further research. This paper contributes to extending previous studies (Díez-Palomar & Cabré, 2015; Sfard & Kieran, 2001), in clarifying how students act on their agency when working in mathematics CPS.

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References

Austin, J. L. (1962). *How to do things with words*. Oxford: Oxford University Press.

Barnes, M. (2004, November). The use of positioning theory in studying student participation in collaborative learning activities. In *Annual Meeting of the Australian Association for Research in Education, Melbourne* (Vol. 28).

Chan, M. C. E., & Clarke, D. (2017). Structured affordances in the use of open-ended tasks to facilitate collaborative problem solving. *ZDM*, 49(6), 951-963.

Charteris, J., & Smardon, D. (2018). A typology of agency in new generation learning environments: emerging relational, ecological and new material considerations. *Pedagogy, Culture & Society*, 26(1), 51-68.

Díez-Palomar, J., & Cabré, J. (2015). Using dialogic talk to teach mathematics: The case of interactive groups. *ZDM*, 47(7), 1299–1312.

García-Carrión, R., & Díez-Palomar, J. (2015). Learning communities: Pathways for educational success and social transformation through interactive groups in mathematics. *European Educational Research Journal*, 14(2), 151-166.

Graesser, A. C., Fiore, S. M., Greiff, S., Andrews-Todd, J., Foltz, P. W., & Hesse, F. W. (2018). Advancing the science of collaborative problem solving. *Psychological Science in the Public Interest*, 19(2), 59-92. doi:10.1177/1529100618808244

Miller, B., Sun, J., Wu, X., & Anderson, R. C. (2013). Child leaders in collaborative groups. In C. E., Hmelo-Silver, C. A., Chinn, C., Chan, & A. M. O'Donnell, (Eds.), *The international handbook of collaborative learning* (pp. 268-279). New York: Routledge.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8(1), 42-76.

Van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. Van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 14-31). Oxford, UK: Blackwell.

THE CONSTRUCTION OF CLASSROOM MATHEMATICAL PRACTICES THAT SUPPORT STUDENT LEARNING OF MULTI-DIGIT MULTIPLICATION

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We report on findings from a study designed to explore the social mathematical practices that emerged in two Year 5 classrooms (5-11 years of age) that supported the development of students' understanding of multi-digit multiplication. During the teaching experiment phase, data were collected via student work samples, classroom video and field notes and analysed from a social constructivist perspective and using an interpretive framework to identify and characterise four key mathematical practices that emerged over the course of the instructional sequence. Theoretical and practical implications of the findings are presented.

INTRODUCTION

Multiplicative thinking has been shown to be an area of weakness for many students (Siemon, 2013). While there is substantial research on the development of multiplicative thinking, research in the field of multi-digit multiplication remains scarce (Hickendorff, Torbeyns & Verschaffel, 2019). Fluency with multi-digit multiplication is an important skill for students which develops from their ability to think and work multiplicatively.

There are two aspects informing the investigation reported in this paper. The first is prior research specific to primary-aged students' conceptual understanding of and strategies for multi-digit multiplication, and the second relates to *how* young students learn mathematics within the social context of the classroom. We briefly introduce each of these aspects before reporting findings from a larger study that sought to document a domain-specific instructional theory for multi-digit multiplication.

LITERATURE

Mathematical understanding is the result of creating connections between mathematical ideas with the depth of students' mathematical understanding proportionate to the quantity, strength and organisation of these connections (Hiebert & Carpenter, 1992). Larsson (2016) describes understanding in multiplication as the result of connections between three elements: the arithmetic properties of commutativity, distributivity and associativity; models of multiplication including the array and area; and strategies for solving multiplicative calculation. While there is not a developmental hierarchy evident in arithmetic properties or models, Larsson (2016) reports an observable progression in students' solution strategies for multi-digit multiplication, developing from addition-based strategies such as repeated

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doubling, through to strategies that draw on multiplication thinking such as decomposition.

Students' difficulty with multiplicative reasoning means that many rely on additive strategies for extended periods of time. (Ambrose, Baek, & Carpenter, 2003; Barmby, Harries, Higgins, & Suggate, 2009; Izsak, 2004). Evidence suggests there is scope to extend these strategies to more efficient ways of computing using multiplicative reasoning. For example, students' intuitive use of repeated doubling to solve multi-digit multiplication problems has been used to pave the way for more complex strategies (Ambrose et al., 2003; Baek, 2005). On the flip side, there is a danger that an overgeneralisation of addition strategies can impede students' conceptualisation of the binary nature of multiplication (Larsson, 2016).

The use of effective multiplicative strategies demonstrates a shift in students' understanding of multiplication (Larsson, 2016) and efficiency in performing calculations (Hickendorf et al., 2019). These strategies for multi-digit multiplication draw on the associative and distributive properties. Analysis of students' invented strategies indicate that some students possess an intuitive understanding of these properties (Ambrose et al., 2003; Barmby et al., 2009). Prior research surrounding the array (Baek, 2005; Larsson, 2016) and discourse explicitly focused on these multiplicative properties (Barmby et al., 2009), indicate that they are both important tools in building students' understanding and fluency in the operation.

Further research is required on the development of understanding and fluency in multi-digit multiplication (Hickendorf et al., 2019; Larsson, 2016) and, at the time of writing this paper, we were unable to locate work on the social development of understanding in the domain. This paper addresses this gap by exploring the research question: What social mathematical practices emerge in the classroom through the implementation of an instructional sequence focused on developing students' understandings in multi-digit multiplication?

THEORETICAL PERSPECTIVE

Learning is both an individual and collective pursuit. From a social constructivist perspective, or *emergent perspective* (Cobb & Yackel, 1995), a cooperative relationship exists between the constructions of the individual and the social culture of learning in the classroom. Individuals construct new knowledge and understandings through mathematical activity while participating in the social role of learning in the classroom. In turn, students' interactions and contributions influence the evolving learning culture and normative ways of reasoning and participating in the classroom. Analysing and documenting learning from an individual and collective perspective allows for rich description of how learning develops within that domain (Stephan & Rasmussen, 2002).

In this paper, we report on four socially constructed practices that emerged during an instructional sequence on multi-digit multiplication implemented in two classrooms. Evidence supporting the emergence of these practices is presented through the mathematical constructions and reasoning from individual students from each class.

METHODS

Design Research methods were employed (as described by Cobb & Gravemeijer, 2008) that allowed the researcher to observe students' thinking and reasoning of multi-digit multiplication first-hand. Three research phases were enacted: a preparatory phase, teaching experiments and a retrospective analysis.

In the preparatory phase, a detailed analysis of multi-digit multiplication literature was conducted as the basis for anticipatory thought experiments. The thought experiments clarified the learning goals, documented the starting points for instruction and from this, a predicted learning pathway or instructional sequence was developed.

The teaching experiment phase was conducted in two different Year 5 (9–11-year-olds) classes from Sydney, Australia; 23 students in Class 1 and 22 in Class 2, creating a sample size of 55 students. The same instructional sequence was taught in both classes. The sequence was implemented over a two-week period and comprised four teaching episodes. Each episode spanned two or three one-hour lessons and was characterised by a focus on a distinct mathematical concept. This research phase was a cyclic process of data analysis, designing and then testing and refining the instructional sequence in the classroom setting. The researcher adopted the role of teacher in each experiment with the regular class teacher present to help facilitate student activity. Data collected included student work samples, classroom video recordings and field notes.

Interpretations and assumptions made in the analysis of data were justified using an interpretative framework based on the previously described emergent perspective (Cobb et al., 1995). Events of the classroom and participatory regularities that were observed were interpreted according to the framework and, from this, conjectures made about the route of learning. Analysis of the social learning were documented as *mathematical practices* (Cobb et al., 1995) and the cognitive learning was informed by individual students' reasoning and their participation in, and contribution to, the emergence of the mathematical practices.

In the retrospective analysis phase a final analysis of data was conducted and generalisations made beyond the specific experiments conducted, forming a grounded theory on the process of students' learning.

RESULTS

Four socially constructed mathematical practices emerged over the course of the instructional sequence, one practice from each teaching episode (Table 1).

Each practice was linked to *conceptual events* rather than specific actions. Events were considered conceptual when a shift in the collective reasoning of the class was observed. Conceptual events were considered significant when they occurred in both iterations of the experiment. The following section briefly describes each mathematical practice (MP1-4) and then presents a detailed description of the conceptual events associated with the third mathematical practice (MP3).

Teaching Episode	Mathematical Practices
1	The array as a tool for sense-making: Partitioning the array
2	The array as a tool for sense-making: Manipulating the array
3	Ways of working mathematically: Thinking multiplicatively
4	Ways of working mathematically: Using friendly numbers

Table 1 – The four mathematical practices

The array as a tool for sense-making – MP1 and MP2

The instructional sequence followed the narrative of a cupcake bakery. In the first two teaching episodes students were asked to calculate the total number of cupcakes in two situations: the number of cakes baked on a given day and the number of cakes sold in boxes of 12. In each situation the cakes were presented to students as an array image. Students could select to use the array to support their calculations if helpful.

MP1 and MP2 (Table 1) centred around the way in which students used the array to make sense of computations and to reason mathematically. The first practice centred students partitioning of the array and recognition that the place value properties of a number could be used to distribute complete rows and columns in the array. The second mathematical practice linked to the associative property of multiplication. This practice saw the array shift from a static to dynamic tool as students utilised common factors to divide and then rearrange the array to aid computation.

Ways of working mathematically – MP3 and MP4

MP3 and MP4 (Table 1) centred on what it means to work multiplicatively. Although students were effectively using strategies based on the associative and distributive properties, misconceptions were evident. The third mathematical practice centred on a structural understanding of associativity and distributivity and is described in the following section. Based on a more complete understanding of associativity and distributivity, students looked for ‘friendly numbers’ and based their strategy selection on the numbers involved in the calculation. This formed the fourth and final practice in the sequence.

MP3 - Ways of working mathematically: Thinking multiplicatively

The third teaching episode continued the narrative of a cupcake baker: *Charlie the baker receives an order of boxes that hold 12 cupcakes. The boxes are a different design from the previous boxes he used. These boxes have the cakes in a ‘skewed array’. He wonders why this might be?*

Students were shown the trays from inside the different cupcake boxes and discussed why one array was skewed and the other was not (Figure 1). Similar suggestions were offered as to why one array may be skewed, one suggestion being that the skewed array took up less space than the regular array. Therefore, its box would not use as much material and consequently be cheaper to make.

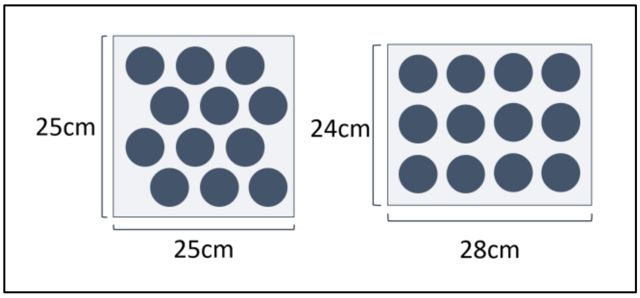

Figure 1 – Two trays from cupcake boxes

The students were asked to predict if it was true that a tray with dimensions 28cm x 24cm was larger than a tray with dimensions 25cm x 25cm. The students' responses in both classes were interesting and unexpected:

Mia: 28 x 24 is like 26 x 26…if you take 2 off the 28 that makes 26, and then you just put that 2 with the 24, then it is 26 too. So, it is 26 and 26 and you just times them together and it's obvious…26 and 26 will make more than 25 and 25.

Ashley: Well, both of them are just 20 multiplied by 20. Then… well, with that one [indicating the 25cm x 25cm array] you do 5 times 5 and then with that one (indicating the other array) you do 4 times 8 and I think…um…wait, it's, yeah, it's 32 and the other one is 25, and 32 is bigger than 25.

These misconceptions became the focus of investigation and the resulting activity and discussions were recorded as significant conceptual events in the emergence and acceptance of the third mathematical practice.

Conceptual Event 1—Recognising and adding all partial products

The first investigation focused on the validity of the comment that 28 x 24 was bigger than 25 x 25 because 8 x 4 has a greater total than 5 x 5. To begin, the students were asked to solve two questions: Which is bigger—28 x 16 or 26 x 19? Which is bigger—26 x 18 or 29 x 16?

Students instinctively reverted to an array as a means of solving the calculations. They partitioned the array into place value parts and found that, when the array for a 2- x 2-digit multiplication was distributed based on place value parts, four partial products were formed. In their initial calculations the students had only acknowledged two of these parts. This learning was the focus of a whole-class conference.

Ellena and Taylor presented their findings from the investigation to the class and stated that the problem with Ellena's initial reasoning was that she didn't use the array.

Taylor: I think that if Ellena used the array, she would have got it right.
Ellena: Yeah, I agree. I think I would have got it right. I just needed the array to see all the parts.
Teacher: What do you mean by that Ellena—that you needed the array to see all the parts?
Ellena: Well, I forgot these two (pointing to the work sample) and so I didn't add them. Well actually, I didn't really know they were there.

The students were invited to use Ellena and Taylor's modified strategy and the students observed that the number of parts in the distributed array corresponded to the number of digits in the problem. Students then realised that each part of both numbers was multiplied together.

The recognition of all partial products formed in a multi-digit multiplication was significant in the negotiation of the third mathematical practice. Students recognised that the two-dimensional structure of multiplication created multiple sections when the array was distributed. As a result of this recognition, the students appreciated the importance of multiplying and adding together all the partial products formed.

Conceptual Event 2—Additive versus multiplicative compensation

The second investigation focused on whether 28 x 24 had the same value as 26 x 26. The students had reasoned that two could be taken from the 28 and added onto the 24 to create 26 x 26. Students began their investigation by using a calculator to check if the two multiplications had the same value. Students realised the two equations were not equal and students selected to use an array to make sense of what was happening.

Two key strategies were observed. The first focused on comparing the common areas of the arrays. Students constructed two arrays, the first measuring 28 x 24 and the second 26 x 26. They proceeded to overlay the two arrays and measured the size of the overhanging sections. The students reasoned that 28 x 24 was not the same as 26 x 26 as the size of the overhangs were not equal (Figure 2). Using the same strategy, they showed that the area of the 28 x 24 array was bigger than the area of the 25 x 25 array.

In a second strategy, students constructed an array measuring 28 x 24, then physically cut two columns off the 28 and added these to make two new rows below the existing 24 rows (Figure 2). These students observed that a gap of four squares had been revealed, and so concluded that the product of 28 x 24 must be 4 less than the product of 26 x 26.

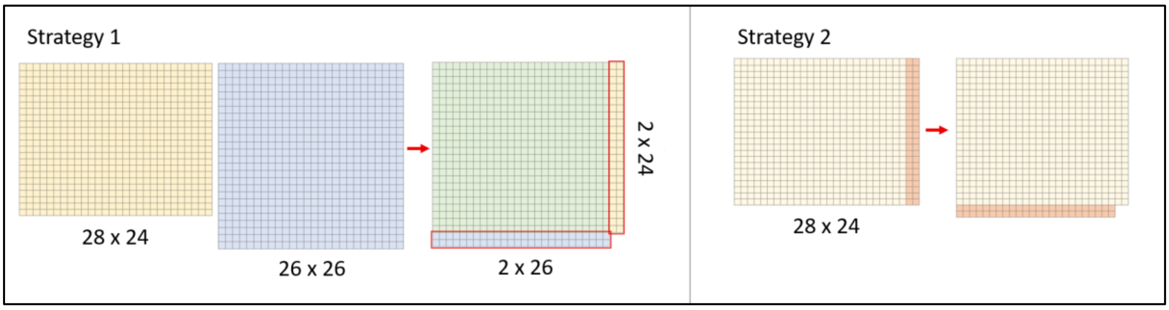

Figure 2 – Two strategies comparing 24 x 28 and 26 x 26

In the ensuing class discussion, both strategies were shared. Of particular significance to the emergence of the third mathematical practice was the conversation relating to the second strategy. Amelie shared how she had used the array to show why 28 x 24 did not equal 26 x 26.

Amelie: I made an array that was 28 across and 24 down and then I cut off two rows from the 28 and I stuck it onto the bottom of the 24. But what I noticed was that the rows are different. These ones are 24 long and the rows this way are 28 long and so they don't match up…these ones are shorter and so it won't work.

Teacher: So, you can't just take two off the 28 and put it on the 24 then?

Amelie: No, it won't work because the rows are different. You can see it here on my array.

Once again, the students were given the time to explore the strategy for themselves and were asked to consider why you can use such a strategy in addition and not for multiplication. In her reasoning, Samar referred back to an incident in the previous episode when students manipulated the array to explore the associative property.

Samar: Last time we halved and doubled. When you halved, the two parts were the same and so you could move the parts of the array. But when you take just two off, the parts are not the same and so they won't match. So, I think that you can divide but not subtract.

Teacher: What do you mean that you can 'divide'?

Samar: You can divide by 2 and then double or divide by 3 or divide by 4…and you can keep going. But you can't just subtract 2 or 3 or something. You need the parts [of the array] to be the same.

The students were given some time to explore Samar's reasoning and came to the consensus that there was a difference between multiplicative compensation and what was termed additive compensation.

DISCUSSION AND CONCLUSION

This paper presented four socially constructed mathematical practices that emerged through the course of an instructional sequence as students developed increasingly sophisticated understandings in multi-digit multiplication. These practices are of theoretical and practical significance.

Theoretically, the four practices present a viable, generalisable theory for collective learning in multi-digit multiplication. While no instructional sequence would produce the same results in multiple classrooms, the four mathematical practices provide a generalised instructional theory to guide instruction and further research in multi-digit multiplication (Cobb et al., 2008). Our findings confirm the significance of the array and mathematical discourse in developing students' understanding of the multiplicative structure and associated properties (Barmby et al., ref; Larsson, 2016). Students used the array as a tool to support calculations and make sense of the multiplicative structure. The array was also significant in the mathematical discourse of the classroom. Students used the structure to represent and communicate their thinking and reasoning to the class community in discussion focused on making sense of multiplicative properties.

Practically, the mathematical practices presented in this paper provide a substantiated theory that can be used and adapted by educators in the design of instructional sequences. The practices can be used to inform instructional decisions including the selection of tasks, the focus of classroom discourse and the representations and tools used to support learning (Stephan et al., 2002).

References

Ambrose, R., Baek, J. M., & Carpenter, T. (2003). Children's invention of multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *Studies in Mathematical Thinking and Learning. The Development of Arithmetic Concepts and Skills: Constructive Adaptive Expertise* (pp. 305–336). Mahwah: NJ: Erlbaum.

Baek, J. M. (2005). Children's mathematical understanding and invented strategies for multidigit multiplication. *Teaching Children Mathematics*, *12*(5), 242–247.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, *70*(3), 217–241.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A. Kelly, R. Lesh, & J. Baek (Eds.), *Handbook of Design Research Methods in Education: Innovations in Science, Technology, Engineering, and Mathematics Learning and Teaching* (pp. 68–95). New York: Routledge.

Cobb, P., & Yackel, E. (1995). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3–4), 175–190.

Hickendorff, M., Torbeyns J. & Verschaffel L. (2019), Multi-digit Addition, Subtraction, Multiplication, and Division Strategies. In: Fritz A., Haase V.G., Räsänen P. (Eds.) International Handbook of Mathematical Learning Difficulties. Switzerland: Springer.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65–97). New York: Macmillian.

Izsak, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning*, *6*(1), 37–79. Larsson, K. (2016). *Students' Understandings of Multiplication*. Stockholm University.

Siemon, D. (2013). Launching mathematical futures: The key role of multiplicative thinking. In S. Herbert, J. Tilley and T. Spencer (Eds.), *Proceedings of the 24th Biennial Conference of The Australian Association of Mathematics Teachers Inc.* (pp. 36–52), Adelaide: Australian Association of Mathematics Teachers.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behaviour*, *21*(4), 459–490.

THE AD LIB MUSIC SESSION AS A METAPHOR FOR MATHEMATICS CLASSROOM ACTIVITIES IN THE THEORY OF OBJECTIFICATION: A PHONETIC ANALYSIS OF LAUGHTER

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The article aims at articulating the potential impact of Radford's theory of objectification on technical research on mathematics teaching. First, instead of Radford's metaphor of mathematical activities as an orchestra, we proposed an alternative metaphor—mathematical activities as an ad lib music session. With that in mind, we focused on laughter as affective expression in class and conducted phonetic analysis of the laughter in a Japanese high-school mathematics classroom. The analysis revealed that a fun atmosphere that included laughter transformed students' treatment of a mathematical model in the lesson and that the proposed metaphor is more suitable than the original. We, thus, conclude that affective factors might determine the quality of mathematics learning in an ad lib manner.

INTRODUCTION

The theory of objectification (TO), proposed by Radford (2016b, 2018), is a promising theory for holistically capturing the endeavor of mathematics education. We have gradually become aware that teaching and learning mathematics is a complex process that can be neither simply psychological nor simply epistemological (Radford, 2016a, 2018). In the TO, Radford elaborates the concept of joint labor as "an historically produced aesthetic form of life where matter, body, movement, action, rhythm, passion, and sensation come to the fore" (Radford, 2016b, p. 200). From this point of view, teaching-and-learning is regarded "not as two separate activities, but as a single and same activity" (Radford, 2016b, pp. 200–201). Through joint labor, teachers and students engage collaboratively in producing the common work that Radford calls "the sensuous appearance of knowledge" (Radford, 2016a, p. 5). All the participants in joint labor "not only create and re-create knowledge but they also co-produce themselves as subjects" (Radford, 2016b, p. 201). This dual process comprises objectification of knowledge and subjectification of the self (Radford, 2015, 2016b). This theory is, thus, characterized as:

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 169-176. Khon Kaen, Thailand: PME.

> an attempt to understand learning not as the result of the individual student's deeds (as in individualist accounts of learning) but as a cultural-historical situated process, and to offer accounts of the entailed processes of knowing and becoming (Radford, 2015, p. 553).

Empirical episodes collected over several years support the validity of the TO (e.g., Radford, 2015, 2016b). We now know that we should view classroom activity holistically as joint labor, not separating the intellectual and emotional aspects of individual thinking, the teacher and students' engagement, or objectification and subjectification. However, we should be aware that the TO does not explicitly show the extent to which the psychological and epistemological perspectives are narrow. Meaning, we may take a simplified psychological or epistemological view of classroom activities for a particular educational purpose. We do not challenge the generalizability of the empirical episodes in the TO. Rather, our argument relates to the technical–political divide that problematizes the gap between technical research aiming at improving local implementation of teaching and socio-political research aiming at global social justice (Ernest, 2016). If the TO wants to say that technical research on intentionally designed classroom activities for an educational purpose—from a psychological or an epistemological perspective—tends to oversimplify classroom activities, then it must show that a theoretical perspective of the theory can contribute at least to the achievement of the original purposes of technical research. This paper aims at articulating such a potential by presenting episodes of laughter in a Japanese high-school mathematics classroom.

THEORETICAL PERSPECTIVE

In this section, to describe the basis of the TO accurately, we elaborate some theoretical concepts. First, according to the TO, *knowledge* is not an object but a process (Radford, 2013). Knowledge is a cultural codification of a potential way of practice. *Knowing*, thus, corresponds to the actualization of knowledge (Radford, 2013). Through this actualization process, learners gradually become aware of a difference in a chaotic situation in between something that they are the *objects* of and something they are the *subjects* of. The former is called *objectification* of knowledge and the latter is called *subjectification* of the self (Radford, 2013). In any situation, a learner experiences objectification and subjectification. Even if practice is repeated, actualization is not completely stable; this instability engenders new learning.

Next, we carefully reconceptualize the concept of *common work* as a product of *joint labor* in a mathematics classroom. This is our main theoretical proposal. In this regard, Radford (2016a) proposes the *orchestra* as a noteworthy metaphor for common work:

> *Common work* is the bearer of dialectic tensions because of the emotional and conceptual contradictions of which it is made. Through it, knowledge appears sensuously in the classroom (through action, perception, symbols, artifacts, gestures, language), much in the same way and, with similar aesthetic force, that

> music appears aurally in a concert hall through the common work of the members of the orchestra. (p. 5, italics in the original)

However, based on our understanding of the TO, we propose that an *ad lib music session* is a more suitable metaphor for common work in a mathematics classroom. The orchestra metaphor provides three images: (1) The members of the orchestra share a goal pregiven by a score; (2) they consciously follow their conductor to achieve it; and (3) their motivation for achieving the goal comes from the existence of the concert audience to an extent. The corresponding images of a mathematics lesson are: (1) The participants do not share a pregiven goal; (2) students do not necessarily purposefully follow their teacher; and (3) there is no external observer in many cases.

A mathematics lesson is, rather, like an ad lib session. In an ad lib session, once the first player introduces an initial phrase, the other participants freely play the subsequent phrases. Their main purpose may be to enjoy playing itself. Although the audience may evaluate the quality of the music, observers are not always present, and players are not necessarily conscious of any such observers. The same holds true in a mathematical activity. Once a teacher provides an initial mathematical task or topic, all participants freely discuss it. Their main purpose may be to learn mathematics together. Although an observer may evaluate the lessons' quality, such observers are not always present and the participants are not necessarily conscious of such observers when they are present. A mathematics lesson must be a different kind of common work than an orchestra, if students do not behave in a prescribed manner. Note that by term "ad lib," we do not mean ill-planned. Rather, we emphasize the possibility that unanticipated improvisational collaboration produces valuable mathematics learning.

The TO has an interest in the dynamic nature of a mathematics lesson *from an observer point of view*. It focuses on products in the public rather than private domains. Observers and participants may feel differently about what roles the participants play in a lesson. For example, although students may be embarrassed to err in solving a mathematical problem, their mistakes may lead to deeper understanding of the problem for themselves and their peers. Additionally, when considering learning mathematics in a public domain, both cognitive and affective elements must be taken into account. As Roth & Walshaw (2019) argue that we should not regard effect as the sole driving force toward cognitive development. Rather, as with an ad lib music session, a mathematics lesson brings observable cognitive and affective changes in participants engaged in common work in a public domain.

In this paper, we particularly focus on laughter as a kind of affective expressions in a mathematics lesson. Although laughter is known to play a crucial role for interwoven cognitive and affective development (Roth et al., 2011; Roth & Walshaw, 2019), to our best knowledge there is no research on how laughter emerges in a mathematics classroom or on the role it plays when it does emerge.

RESEARCH QUESTION

Based on the abovementioned understanding of the TO, we ask: *What role does laughter potentially play for mutually dependent cognitive and affective development in a mathematics classroom?* As Cobb (2007) argues, an insight is a key criterion for choosing theoretical perspectives. The answer to this question offers good reasons for technical researchers to refer to the TO. In addition, this inquiry is worthwhile because it provides a concrete way to fill the gap between technical and socio-political research.

METHOD

In order to locate the potential roles of laughter, we conducted a phonetic analysis of a classroom discussion.

Participants and data collection

We video recorded a tenth-grade mathematics lesson at a public high school in the Kanto region in Japan. The teacher was the third author of this paper. Thirteen students (3 males and 10 females) in an IB math class participated in the lesson. One video camera was located in the rear of the classroom. The official language of the class is English, and the mathematics textbook is written in English; all teachers and students use English primarily and their native Japanese supplementarily.

The topic of the lesson was quadratic functions. The teacher introduced the following opening problem written in the textbook: A motorcyclist Marvin attempts to jump his motorcycle a long distance from the take-off ramp; supposing his height is given as $H = -0.009x^2 + x + 6$ meters, will he safely reach the landing ramp? Figure 1 shows a photograph of the problem presentation by the teacher in the lesson.

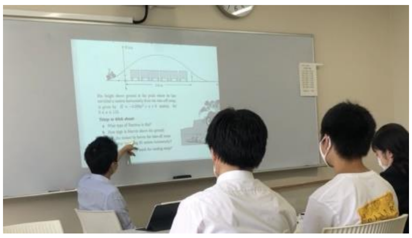

Figure 1: Presentation of the problem in the lesson

Analysis

We processed the video clip of the classroom discussion as follows. First, we transcribed it and attempted to understand what participants talked about. This analysis reveals a cognitive aspect of the discussion. Second, we cropped all but the scenes with participants' laughter. We extracted an audio clip from each scene and graphed the transitions of pitch and intensity using *Praat* speech analysis software (Boersma & Weenink, 2020). The pitch of the laughter indicates whether it was from a female or a male, and the intensity indicates loudness. This analysis reveals an affective aspect of the discussion. In this paper, we report on three scenes of laughter from the lesson.

RESULTS

The original transcript includes English and Japanese. For readability, the Japanese was translated to English and the translations are underlined. Students' names are pseudonyms.

The three scenes of laughter were extracted from a classroom discussion about the opening problem. In the first scene, the teacher drew a picture of a bike and a ramp on the whiteboard and explained the height of the landing ramp, prompting students to imagine the moment of landing.

27	S:	He might get crashed if his height is less than 6 meters.
28	T:	If I magnified this, it would look like this. (Teacher drew the stand in the picture). Thus, he would be here, a little higher than the one I drew.
29	S:	The momentum.
30	T:	Well, it is 6 meters, so here is 6, and 1.1. Like that.
31	S:	Well.
32	T:	I mean, the bike would come like that.
33	SS:	(Laughter) [about six seconds from (1) to (4) in Figure 2]
34	T:	Then, how would it be?
35	Ken:	Isn't the bike you drew too small?
36	T:	Well, it's 6 meters high. 6 meters. This should be fine.

Laughter occurs in #33 because the size of the teacher's bike looked too small from Ken's perspective in #35. However, from the teacher's perspective, the size was valid. When he noticed that the laughing students did not grasp the validity of the scale of the bike, he explained (#36). The laughter seemed to occur because of the cognitive gap between the size of the bike they anticipated and the size of the one the teacher drew.

Figure 2 shows the pitch and intensity of the classroom laughter in #33. A brief episode of laughter began at (1) in Figure 2. The following episode increased in intensity from (2) to (3). From (3) to (4), the intensity decreased. The laughter lasted until the teacher finished drawing the motorcycle. Although the intensity between (2) and (3) vacillates, we can see that it increases overall, representing the swell of the laughter. The better the students grasped the entire picture drawn by the teacher, the louder their laughter become. In addition, because it was the female students who were primarily laughing, the pitch of the laughter is higher than the male teacher's speech in #32 before (1) and in #34 after (4).

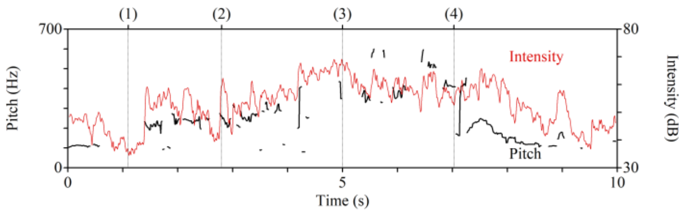

Figure 2: Pitch and intensity of the classroom laughter in the transcript #33

After the long wave of laughter occurred in #33, Ken answered the teacher's question in #34.

37 Nao: By the momentum of the bike.
38 T: By the momentum of the bike.
39 Ken: Because he flew.
40 SS: (Laughter) [about four seconds from (1) to (4) in Figure 3[a]]

Ken thought that the safety of the motorcyclist depended on how vigorously the bike jumped. The nuance of his response in #39 in Japanese manifested a comical image of a bike passing over the landing ramp, probably causing the students' laughter in #40.

Figure 3[a] shows the pitch and intensity of the classroom laughter in #40. Listening to the response from the student, the teacher's lower pitched laughter continued from (1) to (2), and the higher pitched laughter of the female students began from (2). One female student continued laughing from (3) to (4). The intensity of the classroom laughter decreased from (2) to (3). The duration of this laugher was instantaneous.

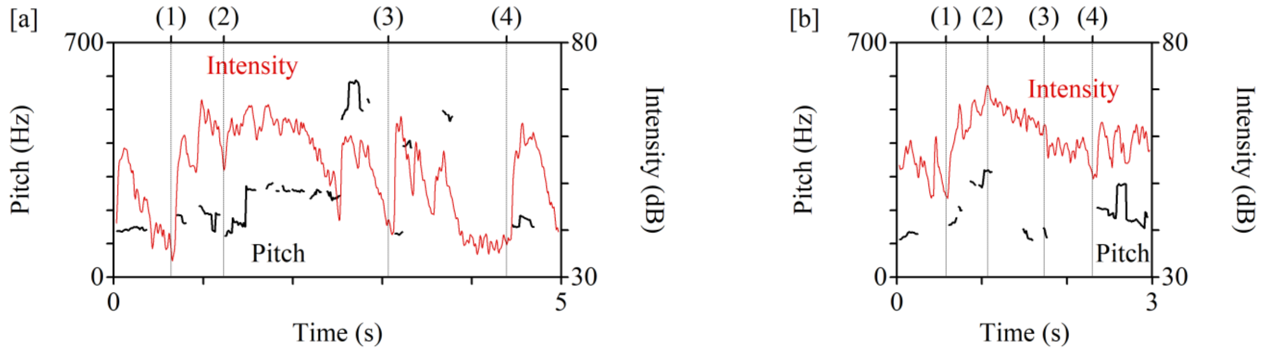

Figure 3: The pitches and the intensities of the classroom laughter in the transcripts #40 [a] and #46 [b]

Following the abovementioned second scene of laughter, the final scene of laughter occurred when the teacher and the students discussed the height of the jump.

41 T: Dangerous? Safe? What do you think?
42 Nao: It seems safe.
43 T: It seems same. Why?
44 Nao: Well, I reckon he might get crashed into the ramp if his speed was less than 6 meters.
45 T: Well, right, if it was less than 6 meters, it is out of the question.
46 SS: (Laughter) [about two seconds from (1) to (4) in Figure 3[b]]
47 Nao: Well, if it was more than 6 meters and about 1 meter (higher than 6 meters), it should be okay.

In #44, Nao argued that the motorcyclist would be safe because the formula H given in the problem indicated that he would not crash into the ramp. The teacher agreed with her in #45, and many students laughed in #46.

Figure 3[b] shows the pitch and the intensity of the classroom laughter in #46. The laughter rapidly swelled from (1) to (2). Since many students kept laughing in turns, the laughter neither vacillated nor rapidly decreased from (2) to (3). It gradually decreased from (2) to (4). The comical image of the bike crash seemed to link with a particular value of the given quadratic function.

DISCUSSION AND CONCLUSION

Figures 2 and 3 corroborate the interwoven relationships between cognition and affect. First, Figure 2 shows that the students took a while to get in a laughing mood. A cognitive factor, misunderstanding the scale, is related to an affective impact on the classroom. Second, the flying bike image provided by Ken in the second scene critically influenced how the students understood Nao's claim in the final scene. As argued in the previous section, Figures 3[a] and [b] indicate that more students were laughing in the final scene than in the second one. Since the image of the bike was referred to twice, more students might have clearly imagined it crashing by the final scene. In addition, Ken's claim characterized Nao's claim as *a necessary condition* for the motorcyclist's safety. Ken suggested that the motorcyclist might have been in danger even if he had jumped sufficiently high. If Ken had not made any claim and the students only discussed how high the motorcyclist jumped, Nao's claim might have been treated as *a sufficient condition* for the safety; students might have implicitly assumed that the motorcyclist had a reliable ability to land on the ramp. This means that the bike image prompted the students to make sense of the mathematical model differently. Our classroom episode, therefore, suggests an affective factor, that is, Ken's funny claim may have contributed to the production of an essentially different mathematical conclusion in the classroom.

Based on the TO, we regarded Ken and Nao's claims as common work in the lesson. Each claim appeared as an intermediate product of mutual engagement in the joint labor. The appearance of these claims could not have been conjectured before the lesson from a solely psychological or solely epistemological perspective. Thus, according to our definition, the classroom activity consisted of *ad lib* collaboration between the teacher and the students.

While *knowledge* of quadratic functions in the sense of the TO provided one possible way of modeling the height of the bike when jumping, the process of *knowing* was not limited to that possibility. Rather, it included dual aspects, objectification and subjectification. As an object, the model was characterized as a necessary condition for safety in practice. As a subject, each student found it socially accepted to discuss mathematical problems with humor. Although we did not capture the changes in the private domains of the students' minds, we did reveal how mutually dependent cognitive and affective elements were in a lesson as a public domain.

In conclusion, the ad lib music metaphor can be more suitable for the mathematics classroom activity than the orchestra one. Laughter in the first scene was an indicator of the students' cognitive understanding, and fun atmosphere with laughter in the second scene influenced the treatment of the mathematical model in the final scene. Therefore, as a potential answer to our research question, we argue that affective factors might determine the quality of mathematics learning in an ad lib manner. As our theoretical contribution to the TO, we also argue that (i) both cognitive and affective development should be regarded as an interwoven achievement of a mathematics lesson and, thus, that (ii) mathematics teachers need to plan their lessons more holistically for better ad lib collaboration. In this regard, using the metaphor of ad lib music as a theoretical framework, future research can seek out what impacts ad lib collaborations have on cognitive and affective development in mathematics classrooms.

In this study, our consideration is limited to laughter and does not focus on the influence of foreign and native languages on cognitive and affective development. A variety of affective and cognitive elements should be investigated in future research.

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References

Boersma, P., & Weenink, D. (2020). *Praat: Doing Phonetics by Computer* (Ver. 6.1.36). https://www.fon.hum.uva.nl/praat/

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspective. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (Vol. 1, pp. 3–38). Information Age Publishing.

Ernest, P. (2016). The unit of analysis in mathematics education: Bridging the political-technical divide? *Educational Studies in Mathematics*, *92*(1), 37–58.

Radford, L. (2013). Three Key Concepts of the Theory of Objectification: Knowledge, Knowing, and Learning. *Journal of Research in Mathematics Education*, 2(1), 7–44.

Radford, L. (2015). Methodological aspects of the theory of objectification. *Perspectivas Da Educação Matemática*, *8*(18), 547–567.

Radford, L. (2016a). Mathematics Education as a Matter of Labor. In M. A. Peters (Ed.), *Encyclopedia of Educational Philosophy and Theory* (pp. 1–6). Springer Singapore.

Radford, L. (2016b). The Theory of Objectification and Its Place among Sociocultural Research in Mathematics Education. *The RIPEM - International Journal for Research in Mathematics Education, 6*(2), 187–206.

Radford, L. (2018). A Cultural-Historical Approach to Teaching and Learning: The Theory of Objectification. In F.-J. Hsieh (Ed.), *Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education* (Vol. 1, pp. 137–147).

Roth, W.-M., Ritchie, S. M., Hudson, P., & Mergard, V. (2011). A study of laughter in science lessons. *Journal of Research in Science Teaching*, *48*(5), 437–458.

Roth, W.-M., & Walshaw, M. (2019). Affect and emotions in mathematics education: Toward a holistic psychology of mathematics education. *Educational Studies in Mathematics*, *102*(1), 111–125.

THE PREDICTIVE ROLE OF DOMAIN-SPECIFIC VOCABULARY IN EARLY PROPORTIONAL REASONING

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Proportional reasoning is a pivotal concept in elementary mathematics. More insight into early predictors of later proportional reasoning might be useful to guide early interventions. Research indicates that language is of major importance for mathematical thinking and learning. The present study investigated the longitudinal association between vocabulary that is specific to proportional situations and children's ability to reason about proportional situations. Results showed that each of the five concepts measured was indeed significantly correlated with proportional reasoning ability, and this relation persisted after controlling for children's SES.

THEORETICAL AND EMPIRICAL BACKGROUND

The role of language in mathematics learning and teaching has been the object of research for many years (e.g. Pimm, 1991; Lefevre et al., 2010, Staples & Truxaw, 2012; Zhang et al., 2017). After all, children learn mathematics in a language-based instructional setting. In this respect, Durkin (1991, p. 3) stated that: "Mathematics education begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language."

For these reasons, it is not surprising that many empirical studies show that language abilities predict math skills (e.g., Abedi & Lord, 2001; Kleemans et al., 2011; LeFevre et al., 2010; McClelland et al., 2007; Purpura & Reid, 2016; Seethaler et al., 2011; van der Walt, 2009; Zhang et al., 2017). For example, LeFevre and colleagues (2010) found that a linguistic pathway, together with a quantitative and spatial attention pathway, contributed independently to early numeracy skills during preschool and kindergarten and was related differentially to performance on a variety of mathematical outcomes two years later.

In most research that focuses on the link between language and mathematics, mathematical performance is considered either as a general ability, or the focus is on the curricular subdomain of whole number and arithmetic. An important question, however, is what role language plays in more advanced domains, such as proportionality.

Proportional reasoning is an important ability for the further development of mathematical understanding, yet it is not achieved easily. Traditionally, proportional reasoning ability is assumed to be acquired towards the end of primary school, i.e. in the formal operational stage (Inhelder & Piaget, 1958) (see for instance the abundance of studies showing that primary school children make additive errors in

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proportional situations). Still, recent research suggests that the development of proportional reasoning starts much earlier. Resnick and Singer (1993), for instance, showed that 5- to 7-year olds give proportionally larger amounts of food to larger fish. Vanluydt et al. (2019) also found that often 5- and 6-year olds can already make sense of one-to-many correspondences, and that some can already handle many-to-many correspondences. Still, at this young age there are large interindividual differences. Some children already demonstrate rather advanced forms of reasoning, while others do not show any evidence of making sense of proportional relations at all.

At this young age, children also show large differences in their language abilities. The individual differences in proportional reasoning that Vanluydt et al. (2019) found in young children may therefore be explained at least partly by their language abilities. Language has been related to proportional reasoning before; Cirino et al. (2016) found an association between general vocabulary and proportional reasoning in sixth graders. We could not find studies on this association at a younger age. Moreover, Cirino et al. (2016) used a *general* vocabulary test. Recently, calls have been made to use more specific mathematical language tests (Purpura & Reid, 2016; Purpura et al., 2017) that address the terminology involved in the mathematical domain at stake instead of general vocabulary tests.

Doing so for proportional reasoning would imply a test that heavily relies on the language of comparison (Lamon, 2006). According to Staples and Truxaw (2012), this includes (1) expressing a relative comparison between two quantities, (2) expressing a comparison between two quantities which are themselves relative comparisons of two quantities, (3) using language that distinguishes a comparison of a proportion from a comparison of absolute values.

RATIONALE AND RESEARCH QUESTIONS

Although several authors (Lamon, 2006; MacGregor, 2002; Staples & Truxaw, 2012) have reflected on the language that children need to reason about proportional situations, we are not aware of empirical evidence linking people’s knowledge of language (and specifically math language) related to proportionality and their proportional reasoning ability. The present study aims to address this gap, particularly in young children. We examined how proportionality-related vocabulary in the first grade of elementary school predicts proportional reasoning ability in the second year of elementary school.

Since several studies show that the home situation of children (and particularly their socio-economic status, SES) has a significant influence on their vocabulary (e.g., Abedi & Lord, 2001; Purpura, 2019), we also investigated whether the predictive relation between proportionality-related language and proportional reasoning ability was still present when SES was included as a control variable.

METHOD

Participants and Design

The study is part of a larger longitudinal research project that focuses on the development of a number of early key mathematical competencies. A cohort of 410 children is followed from the age of 4 until 9. The sample comes from 17 schools (31 classrooms) and is representative for the range of socio-economic backgrounds in Flanders, Belgium. For this study, complete data could be collected from 343 children, of which 172 boys. Informed consents from parents were obtained for all participants, and the study was approved by the ethical committee of KU Leuven.

Proportionality related vocabulary data were collected in the first year of primary school, while proportional reasoning ability was assessed in the second year. All tasks were individually administered in a quiet room in the children's schools.

Instruments

For measuring *proportionality-related vocabulary,* we constructed a task that aimed at measuring the receptive vocabulary knowledge that is important for proportional reasoning. Five concepts were addressed: (1) "fair" (because the proportional context of the ability task involved a fair assignment of food to a number of children – see below), (2) "double", (3) "half", and (4) "three times more" (which verbalize a multiplicative relation) and (5) "three more" (to assess the difference between expressions on an additive relation –three more– and a multiplicative relation –three times more). All items were kept as simple as possible from a calculation point of view, so that the focus would be on children's understanding of the vocabulary involved. For each item, a statement involving one of the five concepts was read aloud by the interviewer, while a picture was shown. Children had to assess whether the statement was true or false when looking at the picture. Each concept was measured by two items: one with a true statement and one with a false statement, leading to a total of ten items. Figure 1 gives examples of the two items for the concepts "half" and "three times more". Given the true/false nature of the task and the fact that we expected children to recognize when a given concept was applicable *and* when it was not applicable, we applied the following scoring rule: Only if children correctly responded to *both* items of a concept, they received a score of one for that concept, otherwise (i.e., only one item of a concept correct or none of the items of a concept correct) they received a score of zero for the concept.

To measure *proportional reasoning ability,* children completed a task consisting of eight items involving two discrete quantities and eight items involving a discrete and a continuous quantity. All items were missing-value problems, involving the assignment of food (discrete: a number of grapes, continuous: a chocolate bar of a certain length) to a number of children (represented by puppets). Children had to construct a set B equivalent to a set A by putting the elements in set B in the same ratio as the elements in set A. Figure 2 shows an example item for both item types. See Vanluydt et al. (2019) for a more detailed description of the proportional reasoning tasks, and for information on the reliability and validity of this instrument. Given the young age of the children, the items were offered orally, supported by concrete materials that children could manipulate, and the need for complex calculations was avoided.

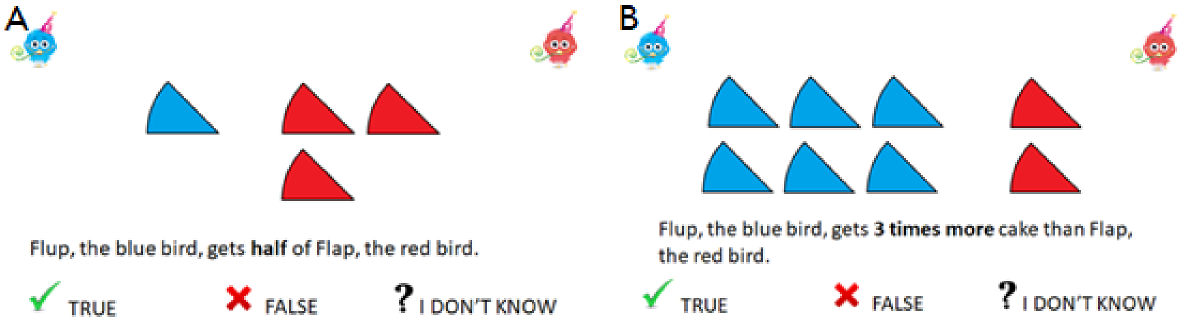

Figure 1. Example items for vocabulary task
(A: incorrect statement about "half", B: correct statement about "three times more")

"All puppets are equally hungry. If I give four grapes to these puppets, how many grapes do you have to give to these puppets for it to be fair?"

"All puppets are equally hungry. If I give this chocolate bar to these puppets, which chocolate bar do you have to give to these puppets for it to be fair?"

Figure 2. Example items for proportional reasoning ability task (left: involving two discrete quantities, right: involving a discrete and a continuous quantity)
(Adapted from Vanluydt et al., 2019)

Socio-economic status (SES) data was collected by means of a parent questionnaire. In line with other research (e.g., Aaro et al., 2009), it was based on the education level of the mother, which is considered a good indicator for the construct at stake: (1) no education, elementary education or lower secondary education; (2) higher secondary education; (3) professional bachelor; (4) academic bachelor, master or PhD.

RESULTS

Table 1 gives an overview of the scores on the vocabulary task. The notion "Fair" in the sharing situation seemed to be well understood by most children. The notion "Double" and the (additive) notion "Three more" were understood by a substantial number of children (but less than half), and "Half" and "Three times more" seemed still quite difficult for more than 4/5 of the children. Still, for each of the notions, there were children who showed an understanding of the notions. Overall, about one third of the children got a score of 3 or more out of 5, while the other two thirds got scores of 2 out of 5 or less.

Fair	***Double***	***Half***	***Three times more***	***Three more***
81.9%	42.0%	18.7%	14.0%	37.3%

Table 1. Percentage of 1-scores answers on each concept of the vocabulary task

On average, children solved 8.65 (SD=3.53, range = 1-16) out of 16 items correctly on the proportional reasoning ability task. As shown in Figure 3, which provides the distribution of the scores on this task, some children already obtained really high scores – solving all or nearly all items correctly – whereas others managed to do well on only a few.

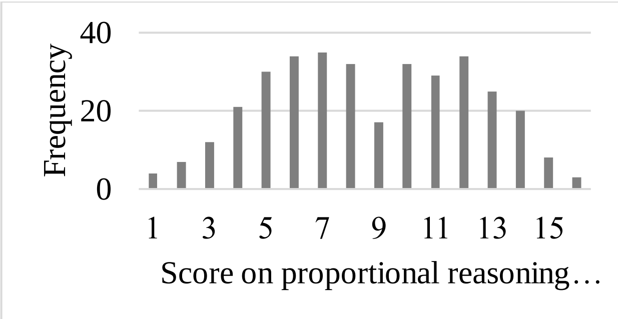

Figure 3. Distribution of scores on proportional reasoning ability task

Given these large interindividual differences on both instruments, it was meaningful to relate the vocabulary scores to the proportional reasoning scores. Table 2 provides a correlation matrix, involving scores on each of the notions tested in the vocabulary task and the score on the proportional reasoning ability task.

This correlation matrix indicates that the five concepts in the vocabulary task are interrelated to some extent. More importantly, each of the concepts correlates significantly with performance on the proportional reasoning ability task, with the strongest correlations being observed for the notions “Double” and “Three more”.

The question then remains whether these correlations with the scores on the proportional reasoning ability task would persist when controlled for children’s SES. Table 3 gives an overview of the partial correlations for each concept, controlling for SES.

	1	2	3	4	5	6
1.Proportional reasoning	--	.183**	.365**	.235**	.110*	.336**
2. Fair		--	.139*	.108*	.037	.080
3. Double			--	.320**	.168**	.272**
4. Half				--	.066	.141**
5. Three times more					--	-.068
6. Three more						--

* *Note* * $p < 0.05$, ** $p < 0.01$.

Table 2. Correlation matrix for the vocabulary task and the proportional reasoning ability task

Fair	***Double***	***Half***	***Three times more***	***Three more***
.117**	.314**	.190**	.087	.280**

* *Note* * $p < 0.05$, ** $p < 0.01$.

Table 3. Partial correlations between the score on the proportional reasoning ability task and the scores on each concept of the vocabulary task, taking into account SES

Although all partial correlations are lower than the correlations reported in Table 2, indicating that children's SES indeed plays a role in explaining the link between proportionality-related vocabulary and proportional reasoning ability, all notions – except for "Three times more" – still significantly correlate with proportional reasoning ability.

CONCLUSIONS AND DISCUSSION

Proportional reasoning is a pivotal concept in elementary mathematics, crucial for more advanced mathematical skills, but hard to apprehend for children (Resnick & Singer, 1993). More insight into early predictors of later proportional reasoning might be useful to guide early interventions.

Theoretical and empirical research indicates that language - general as well as specifically related to the mathematical concept at stake - is of major importance for mathematical thinking and learning (e.g., Abedi & Lord, 2001; Kleemans et al., 2011; LeFevre et al., 2010; McClelland et al., 2007; Purpura & Reid, 2016; Seethaler et al., 2011; van der Walt, 2009; Zhang et al., 2017). The present study, was the first to explicitly investigate the association between vocabulary that is specific to proportional situations, on the one hand, and children's ability to reason about proportional situations, on the other hand. The study was conducted in a large sample, and investigated this relationship longitudinally, by measuring the understanding of the vocabulary in the first grade of elementary school and the proportional reasoning ability in the second year.

We observed that each of the five concepts that was involved in the vocabulary task was indeed significantly correlated with proportional reasoning ability. We additionally tested whether this correlation would persist after controlling for children's SES, which was indeed the case for 4 out of 5 concepts. In line with previous research (e.g., Abedi & Lord, 2001; Purpura, 2019), we found confirmation of the fact that SES is involved in the relation between children's proportionality-related vocabulary and reasoning ability, but even when controlling for this variable a significant correlation persisted.

In contrast to previous research, our study did not involve a general language or vocabulary test, but focused on the specific vocabulary that is involved in the mathematical task at hand. This was done in line with recent calls in the literature (Purpura & Reid, 2016; Purpura et al., 2017). While general language ability is of course important in children's mathematical development, our study may provide more concrete departure points for intervention. It points at the central notions that children need to understand if one wants to involve them in reasoning about proportional situations at a young age. Of course, care must be taken in drawing conclusions about the causal relation between proportionality-related vocabulary and reasoning ability, given that our study was only correlational in nature. Further research may look at the impact of interventions paying particular attention to the relevant vocabulary.

References

Aaro, L., Flisher, A. J., Kaaya, S., Onya, H.,, Namisi, F., & Wubs, A. (2009). Parental education as an indicator of socioeconomic status. *Scandinavial Journal of Public Health, 37, 16-27.*

Abedi, J., & Lord, C. (2001). The language factor in mathematics tests. *Applied Measurement in Education, 14,* 219-234. https://doi.org/10.1207/S15324818AME1403_2

Cirino, P. T., Tolar, T.D., Fuchs, L.S., & Huston-Warren, E. (2016). Cognitive and numerosity predictors of mathematical skills in middle school. *Journal of Experimental Child Psychology, 145,* 95-119. https://doi.org/10.1016/j.jecp.2015.12.010

Durkin, K. (1991). Language in mathematical education: An introduction. In K. Durkin and B. Shire, Eds, *Language in mathematical education. Research and practice* (pp. 3-16). Milton Keynes/ Philadelphia: Open University Press.

Inhelder, B., & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. London: Routledge.

Kleemans, T., Seger, E., & Verhoeven, L. (2011). Cognitive and linguistic precursors to numeracy in kindergarten: Evidence from first and second language learners. *Learning and Individual Differences, 21,* 555-561. https://doi.org/10.1016/j.lindif.2011.07.008

Lamon, S. J. (2006). *Teaching fractions and ratios for understanding: essential content knowledge and instructional strategies for teachers*. Hillsdale: Lawrence Erlbaum.

LeFevre, J., Fast, L., Skwarchuk, S., Smith-Chant, B. L., Bisanz, J., Kamawar, D., & Penner-Wilger, M. (2010). Pathways to mathematics: Longitudinal predictors of performance. *Child Development, 81,* 1753-1767. https://doi.org/10.1111/j.1467-8624.2010.01508.x

MacGregor, M. (2002). Using words to explain mathematical ideas. *Australian Journal of Language and Literacy, 25*(1), 78–88

McClelland, M. M., Cameron, C. E., Connor, C. M., Farris, C. L., Jewkes, A. M., & Morrison, F. J. (2007). Links between behavioral regulation and preschoolers' literacy, vocabulary, and math skills. *Developmental Psychology, 43*(947). https://doi.org/10.1037/0012-1649.43.4.947

Pimm, D. (1991). Communicating mathematically. In K. Durkin & B. Shire, Eds, *Language in mathematical education. Research and practice* (pp. 18-23). Milton Keynes/ Philadelphia: Open University Press.

Purpura, D. J. (2019). Language clearly matters; methods matter too. *Child Development, 90*, 1839-1846. https://doi.org/10.1111/cdev.13327

Purpura, D. J., & Reid, E. E. (2016). Mathematics and language: Individual and group differences in mathematical language skills in young children. *Early Childhood Research Quarterly, 26*, 259-268. https://doi.org/10.1016/j.ecresq.2015.12.020

Purpura, D. J., Logan, J. A. R., Hassinger-Das, B., & Napoli, A. R. (2017). Why do early mathematics skills predict later reading? The role of mathematical language. *Developmental Psychology, 53*, 1633-1642. http://doi.org/10.1037/dev0000375

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107– 130). Hillsdale: Lawrence Erlbaum.

Seethaler, P. M., Fuchs, L. S., Star, J. R., & Bryant, J. (2011). The cognitive predictors of computational skill with whole versus rational numbers: An exploratory study. *Learning and Individual Differences, 21*, 536-542. https://doi.org/10.1016/lindif.2011.05.002

Staples, M. E., & Truxaw, M. P. (2012). An initial framework for the language of higher-order thinking in mathematics practices. *Mathematics Education Research Journal, 24*, 257-281. https://doi.org/10.1007/s13394-012-0038-3

Van der Walt, M. (2009) Study orientation and basic vocabulary in mathematics in primary school. *South African Journal of Science and Technology, 28*, 378-392. https://doi.org/10.4102/satnt.v28i4.73

Vanluydt, E., Verschaffel, L., & Van Dooren, W. (2019). Early stages of proportional reasoning: a cross-sectional study with 5-to 9-year olds. *European Journal of Psychology of Education.* https:/doi.org/10.1007/s10212-019-00434-8

Zhang, J., Fan, X., Cheung, S.K., Meng, Y., Cai, Z., & Hu, Y.H. (2017). The role of early language abilities on math skills among Chinese children. *PLOS One, 12*(7_/ E0181074. https://doi.org./10.1371/Journal.pone.0181074

DERIVATIVE TASKS PROPOSED BY TEACHERS IN TRAINING

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The tasks are fundamental elements in the process of teaching and learning of Mathematics and consequently, the design and selection of these. The present work focuses on characterizing a set of tasks proposed by teachers in training and focused on the derivative of a function at one point, as part of a broader investigation in which we address the meaning that teachers attribute to this topic. We employ the method of content analysis provided with a system of categories that allow us to analyze revealed elements of the meaning of the derivative as well as the cognitive demand encouraged for each of the tasks. We noticed a clear predominance of tasks with low cognitive demand in which the most involved content is the calculation of maxima and minima of a function.

INTRODUCTION

The relationship between the type of homework tasks that students do and the mathematics they learn has been a topic of research for many years (Breen & O'Shea, 2010). Several studies claim that what students learn is highly determined by the tasks assigned by teachers (Sullivan, Clarke, & Clarke, 2013). Concretely, tasks transmit messages about what mathematics is and what it involves to know them, that is, its meaning. Moreover, it is considered that it is through the tasks that students are really given learning opportunities (Anthony & Walshaw, 2009). Some authors even think that posing tasks that invite the student to think for himself is the main stimulus for learning, above any other action in the classroom, (Sullivan, Clarke and Clarke, 2013). Hence, the design and selection of tasks are essential for effective teaching (Watson et al., 2014).

Due to its relevance, in recent years, there has been an increasing interest in addressing investigations about school tasks (e.g. Lithner, 2017; Bobis, et al., 2019). Other aspects also show its relevance: for example, in 2003, at the annual meeting of the international group for the Psychology of Mathematics Education (PME), the design and use of tasks were identified as the main topics of the research reports. As well, in 2008, the International Congress of Mathematical Education (ICME) organized a Topic Study Group (TSG), named Research and development in task design and analysis, and even some journals, such as the Journal of Mathematics Teacher Education (JMTE), have devoted a special issue to this topic.

Therefore, we believe that teachers should be able to pose tasks that promote appropriate learning of their students (Lee, Lee, & Park, 2016). Both the design and

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 185-192. Khon Kaen, Thailand: PME.

the selection of tasks are influenced by the teacher's goals, as well as by his knowledge and beliefs about mathematics (Sullivan, Clarke, & Clarke, 2013). Therefore, as part of a broader investigation in which we address the meaning of the derivative expressed by math teachers, we focus the present work on characterizing the derivative tasks proposed by teachers in training. In this way, not only we approach the future teachers' perception of the derivative, but also, we can also analyze the relevance of these tasks.

BACKGROUND

We understand ‘task’ in terms of Watson et al. (2014) “to mean a wider range of ‘things to do’ than this, and include repetitive exercises, constructing objects, exemplifying definitions, solving single-stage and multi-stage problems, deciding between two possibilities, or carrying out an experiment or investigation” (p. 9-10). Indeed, a task is anything that a teacher uses to ask students to do something.

Different models and approaches have been used for the analysis of mathematics tasks. One widely used has to do with the level of cognitive demand of the task, proposed by Stein, Grover, and Henningsen (1996), and used in many investigations (e.g., Cai, Moyer, Nie, & Wang, 2009; Tekkumru-Kisa, Stein, & Schunn, 2015). In our work we also consider the four levels of cognitive demand raised there. However, since our goal is also related to the meaning of the derivative for math teachers, we extend the analysis of the tasks with a set of categories proposed by Moreno and Ramírez (2016) which have been already used in Vargas, Fernández-Plaza, and Ruiz-Hidalgo (2018). These categories can be organized in two groups:

- Mathematical content and its meaning: considering the theoretical framework based on the meaning of a school mathematical concept developed by Rico (2013), we analyze some elements that make up the meaning of the derivative: the content, the representation systems, the transformation of representation systems that are requested, the context, the situation and the type of function involved.
- Learning or cognitive aspects: regarding this aspect we analyze (a) the demand (Stein et al., 1996); and (b) the mathematical ability fostered by the task.

METHOD

We conduct qualitative research of descriptive nature, which was carried out with 55 Mathematics teachers in training in Spain. At the time of data collection, they were studying for the University Master's Degree in Secondary Education at the University of Granada, intended for graduate students with different academic backgrounds (Mathematics, Engineering, Physics, among others) who wish to access to secondary teacher career. In this way, every considered future teacher has passed at least two Calculus courses in their training.

A survey composed of three questions was used for the data collection. From the questions, we present here the one in which teachers in training were asked to propose a task that was resolved involving the derivative. Through a content analysis, we proceeded to study each of the proposed questions according to each of the following categories.

System of Categories

In the group of Mathematics meanings, we considered the following categories:

1. *Content*: the mathematical content addressed in each task.
2. *Representation systems*: we pay attention to the different representation systems that appear in the statement of the task. These can be: verbal, graphic, numerical, symbolic and / or tabular.
3. *Types of transformations*: under the line of representation systems and following Duval (2006), we analyze whether the proposed task encompasses in its resolution transformations within the same system (treatments) or requires translations from one system to another (conversion).
4. *Situation*: we identify the PISA situation (OECD, 2016) in which the proposed tasks are presented: personal, occupational, societal, or scientific.
5. *Context*: based on our theoretical framework, we take into account the different mathematical contexts or functions to which the derivative attends in each one of the tasks.

From the group of cognitive aspects, we consider:

1. Cognitive demand: To analyze this aspect we use the taxonomy of Stein et al. (1996), in which four types of tasks are considered, according to cognitive demand. The characterization of these can be seen in Table 1.

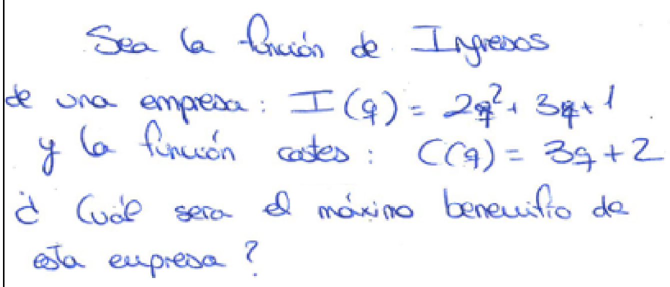
Cognitive demand	Description
Memorization	Regarding those tasks that ask the student to remember facts, rules or definitions. The answers imply an exact and memorized reproduction. No type of procedure is used.
Procedures without connections	The purpose of this kind of task is to apply some algorithm to solve a problem. It is more about applying than understanding. These tasks are characterized by not requiring explanations as well as there is no ambiguity about what to do and how to do it.
Procedures with connections	Although these tasks have a procedure to be solved, their intention goes beyond the process itself, trying to develop deeper levels of understanding about mathematical concepts and ideas. Its main feature is that they are not tasks that can be solved only by knowing the algorithm, but they require some extra effort.
Doing Mathematics	These are the tasks with the highest cognitive demand, since they require non-algorithmic thinking and the solution path is not predetermined. They require a true understanding of the concepts, processes, properties and the establishment of relationships among mathematics concepts.

Table 1: Taxonomy of Stein et al. (1996)

2. Mathematical capability: We adopt the seven Fundamental Mathematical Capabilities from PISA framework (OECD, 2016): communication; mathematising; representation; reasoning and argument; devising strategies for solving problems; using symbolic, formal and technical language and operations; and using mathematical tools.

RESULTS

One out of the 55 participants did not write any task, other six teachers in training drew surfaces in which derivatives should be used, but they did not actually pose a task. Although they were only asked to write a task that was solved through the derivative, some of them posed two or three, so a total of 52 tasks were analyzed.
At first glance, we detected that nine out of the 52 tasks have no solution as they were written, since either the necessary data were not presented, or the data were not consistent. Despite this, these nine tasks took part of the analysis, considering the intention with which they were posed.
To exemplify, we show the analysis we carry out for the task proposed in Figure 1.


Sea la función de Ingresos de una empresa: $I(q) = 2q^2 + 3q + 1$ y la función costes: $C(q) = 3q + 2$ ¿Cuál será el máximo beneficio de esta empresa?

Translation:

The income of a Company is given by the function $I(q) = 2q^2 + 3q + 1$ and its cost by the function $C(q) = 3q + 2$. What will be the maximum benefit of this company?

Figure 1: Example of task posed

The first step of the analysis was to identify optimization as the content that is addressed in the task (Figure 1). This task is contextualized in an occupational situation, in a context of applications of the derivative. Regarding the representation system, both verbal and symbolic are used, where the management required is a symbolic treatment of the given function.

Regarding the demand of the task, the resolution of a plain problem is requested, in which there is already defined the function that models the situation and could even be solved without using the notion of derivative (notice the vertex of a quadratic function). Therefore, it could be solved using procedures without connections. In fact, students usually learn to solve automatically these types of tasks, without really needing the context to find the correct answer. Finally, the capacity that it fosters is classified as the use of operations and symbolic language.

In the following, we show the results obtained after the analysis of the 52 tasks, for each of the categories considered.

Regarding the *mathematical content* addressed in the tasks, problem solving predominated (26), mainly optimization, and particle speed and acceleration; followed by tasks in which it is determined the extremes values of a function and analyze its monotony (16).Other tasks also addressed: derivation rules (2), differential equations (1), calculation of limits (2), tangent and normal lines (3); and others (2). We have included the category 'others' for those tasks that did not really address any content of the derivative, as is the case of a participant who stated: "Indicate the intervals of increasing and decreasing for the function shown [...]". The solution does not require the notion of derivative. In this regard, seven out of the 52 tasks that address relative extremes and monotony do by using quadratic functions, for instance $f(x) = x^2$, so that knowing the sketch of such a graph could determine the extrema without involving the notion of derivative.

A plain function was involved in most tasks (in 33 tasks), mainly polynomials of grade 2 or 3. Four of the tasks slightly suggest more complex functions, and no specific function is proposed in the remaining 15 tasks, but in some cases the solver is who may determine the function modelling the situation to answer the task.

The *representation system* used in the statement and the *types of transformations* that are requested of these can be seen in Table 2. The only representation systems that emerged were verbal and symbolic, or both. The management of representation systems mainly deals with a symbolic treatment. In the case of conversion, it refers in all cases to the translation from the verbal to the symbolic system.

Representation system	Types of transformations	Frequency
Symbolic	Treatments	20
Verbal	Conversion	16
Verbal y symbolic	Treatments	16

Table 2: Representation systems used in tasks and types of transformations

Scientific *situations* predominate, specifically intra-mathematical situations. A half of the tasks analyzed were contextualized in a strictly mathematical situation (26), the second half were categorized in occupational situations (12), mainly issues of business benefits), physics (9, speed and acceleration of bodies), and personal (5). The *context* within which the tasks were proposed was mainly applied (29), followed by geometric (16) and a few within an algebraic-numerical context (7).

An interesting aspect of the tasks is the *cognitive demand* of each of them. Table 3 shows what is requested in the tasks and the related demand.

Cognitive demand		Frequency
Procedures without connections	Direct Calculation	6
	Indirect Calculation	19
	Problem solving	15
Procedures with connections	Identifying	1
	Problem solving	11

Table 3: Cognitive demand of the proposed tasks

We notice that the tasks mainly require procedures without connections to be solved. The only task of identification is not really about derivation. In the same way, the most of the 11 problems considered to have a higher level of demand (procedures with connections) correspond to tasks that had no solution. However, according to the intention of the proposal and the amount of data that would be required for its solution, the demand is different from others problems such as that shown in Figure 1 that can be solved more mechanically. In addition, in these 11 problems the situation plays an important role, since the task must be interpreted in order to find a modelling function.

In a similar way to the cognitive demand of each task, it is possible to analyze the mathematical capacity that each one of them promotes. The most encouraged capacity has to do with calculations and symbolic language (in 50 tasks), regardless of whether tasks are found in different contexts and situations. Since there are so many tasks in the form of contextualized problems, we can also say that the design of strategies to solve problems is promoted and in some of them, mathematising (11).

DISCUSSION

The goal of this work was to characterize tasks posed by teachers in training, for the topic of derivative. The analysis showed that the tasks proposed mainly addressed the content of finding maxima and minima, as well as problem solving. Although many future teachers submitted task in an applied context, the most of them were placed in mathematical situations. We also identified that the tasks were formulated using mainly verbal and symbolic representation systems, and what is requested is essentially a procedure without connections that only requires symbolic transformations (treatments).

This is a noteworthy result since it has been found that tasks should lead to more rigorous ways of thinking. In fact, it has been determined that students learn best when they attend lesson in which they maintain a high level of cognitive demand (Kessler, Stein, & Shunn, 2015), i.e., the tasks proposed must demand a procedure with connections or doing Mathematics. However, we realize that teachers in training essentially propose tasks that promote the handling of quick procedures. Thus, according to Sangwin (2003), it is clear that routine tasks that solve without the use of superior skills predominate. We believe that, regardless of the context, for a task to be worthwhile, it must be interesting and provide a level of challenge that invites reflection and hard work (Cai, Moyer, Nie and Wang, 2009). Even though students prefer simple tasks, they consider that they learn more with demanding tasks (Sullivan, Clarke and Clarke, 2013).

The Stein et al. 's (1996) claim about the tendency of the teacher to reduce the level of potential demand of the task is related to the result of this paper. Although an effort is made to contextualize the task, they are finally very simple problems that are solved by applying a routine procedure. Charalambous (2008) argues that a factor in this is the teacher knowledge. Also, we think the posing of tasks is also related to the content is understood, i.e. to the meaning given to the notion of derivative.

A synthesis of the tasks analyzed shows that the derivative is perceived as an algebraic tool to determine the extremes of a function. Certainly, this is a fairly limited view of what this concept encompasses. We believe that the results obtained can be used as input in the training of teachers in order to enrich the meaning of the concept of derivative, and that in this way teachers can in the future select and design varied tasks that enrich the meaning of this concept in their students.

Acknowledgments

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References

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics*. Brussels, Belgium: International Academy of Education.

Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., Russo, J., & Sullivan, P. (2019). Changing teacher practices while teaching with challenging tasks. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 105-112). Pretoria, South Africa: PME.

Breen, S., & O'Shea, A. (2010). Mathematical Thinking and Task Design. *Irish Mathematical Society Bulletin*, *66*, 39-49.

Cai, J., Moyer, J. C., Nie, B., & Wang, N. (2009). Learning mathematics from classroom instruction using standards-based and traditional curricula: An analysis of instructional tasks. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 692-699). Atlanta, GA: Georgia State University.

Charalambous, C. Y. (2008). Mathematical knowledge for teaching and the unfolding of tasks in mathematics lessons: Integrating two lines of research. In O. Figuras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281-288). Morelia, Mexico: PME.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Kessler, A. M., Stein, M. K., & Schunn, C. D. (2015). Cognitive Demand of Model Tracing Tutor Tasks: Conceptualizing and Predicting How Deeply Students Engage. *Technology, Knowledge and Learning, online*. https://doi.org/10.1007/s10758-015-9248-6

Lee, K. H, Lee E. J., & Park, M. S. (2016) Task Modification and Knowledge Utilization by Korean Prospective Mathematics Teachers, *Pedagogical Research*, *1*(2), 54-66.

Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM Mathematics Education*, *49*, 937-949.

Moreno, A., & Ramírez, R. (2016). Variables y funciones de las tareas matemáticas. In L. Rico & A. Moreno (Eds.), *Elementos de didáctica de la matemática para el profesor de Secundaria* (pp. 243-257). Madrid, Spain: Pirámide.

OECD. (2016). *PISA 2015 assessment and analytical framework: Science, reading, mathematics and financial literacy.* Paris, France: OECD Publishing.

Rico, L. (2013). El método del Análisis Didáctico. *UNIÓN, Revista Iberoamericana de Educación Matemática*, *33*, 11-27.

Sangwin, C. (2003). New opportunities for encouraging higher level mathematical learning by creative use of emerging computer assessment. *Journal of Mathematical Education in Science and Technology*, *34*(6), 813-829.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 255-288.

Sullivan, P., Clarke, D., & Clarke, B. (Eds.). (2013). *Teaching with Tasks for Effective Mathematics Learning*. New York, NY: Springer.

Tekkumru-Kisa, M., Stein, M. K., & Schunn, C. (2015). A Framework for Analyzing Cognitive Demand and Content-Practices Integration: Task Analysis Guide in Science. *Journal of Research in Science Teaching*, *52*(5), 659-685.

Vargas, M. F., Fernández-Plaza, J. A. y Ruiz-Hidalgo, J. F. (2018). Tareas propuestas por los libros de texto de 1º de bachillerato para el tema de la derivada. In L. J. Rodríguez-Muñiz, L. Muñiz-Rodríguez, A. Aguilar-González, P. Alonso, F. J. García-García y A. Bruno (Eds.), *Investigación en Educación Matemática XXII* (pp. 594-603). Gijon, Spain: SEIEM.

Watson, A., Ohtani, M., Ainley, J., Bolite, J., Doorman, M., Kieran, C., Leung, A., Margolinas, C., Sullivan, P., Thompson, D., & Yang, Y. (2014). Introduction. In C. Margolinas (Ed.), *Task Design in Mathematics Education. In Proceedings of ICMI Study 22.* Oxford, United Kingdom: ICMI.

INDIRECT LEARNING PROCESSES AS KEY VARIABLE IN EARLY MATHEMATICS LEARNING

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For children, the first mathematical learning does not only take place when they enter school. It is rather the preschool or the family context in which children have their first encounter with mathematics. The following contribution focuses on such early mathematics learning processes in German preschools (Kindergärten) and highlights especially the role so-called indirect learning, whereby mathematical content is not explicitly negotiated but implicitly involved within adult-child-interactions and a kind of 'learning over time' takes place. It is pointed out that these indirect leaning processes are essential for fundamental mathematical learning. Therefore, two paradigmatic examples of situations with preschool teachers form the erStMaL-study (early Steps in Mathematics Learning) are interpretatively analyzed.

INTRODUCTION

There is no doubt that the development of mathematical thinking already takes place in early childhood and has many sensitive phases during this time (Sarama & Clements, 2009). However, it remains much discussed how, especially at this early age, the best possible support for first mathematical learning can be found. In Germany, in the centre of this discussion is above all the first institutional learning in kindergarten and the support of this learning by preschool teachers. Besides the early learning of mathematics at home with parents and siblings, the preschool is the first place where children encounter mathematics. Although Germany has established different 'curricula' for "Kindergärten" (meaning: preschool) and "Kindertagesstätten" (meaning: day care centres) for children 0 to 10 years old in each of its states since early 2000s, concerning mathematics, there is more or less concrete advice for preschool teachers regarding how to 'teach' different mathematical contents. Only co-constructive learning is picked out as a central theme for all contents in all these curricula. As in classroom learning, the question arises how direct and explicit support by preschool teachers should be, especially in mathematics. On the basis of a (co-)constructivist perspective on early mathematical learning in interaction, the following article takes a closer look at this question and highlights the role of so-called *indirect learning processes* (Bauersfeld, 1995, p. 281) as a possible 'key variable' of early mathematic learning and as a pivot element in the discussion presented above.

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THEORETICAL BACKGROUND

Early Mathematics Learning in Adult-child-interactions within Preschool

Several studies could prove, that preschool improves children's school readiness especially in Mathematics (Melhuish, Quinn, Sylva, Sammons, Siraj-Blatchford, & Taggart, 2010). While learning at home with parents and siblings is surely a formative learning context, learning in kindergarten is also an important factor for children's learning biographies: it is the first time of institutional learning for young children and it, therefore, builds a crucial basis for future schooling (Claesens & Engel, 2013). From a (co-)constructivist perspective on early learning mathematics, the interactions between children and 'more competent others' (Chaiklin, 2003, p. 41; Vygotsky, 1978), such as preschool teachers, are an important parameter. In such interactions (initial) learning processes take place when the children - with the support of the preschool teachers - participate increasingly autonomously in negotiating processes of the mathematical discourse. According to Bauersfeld (1988), for such an increase in autonomy it is necessary that existing "Domains of Subjective Experiences" (in the following called: DSE) of the child, which represent a context-bound knowledge that is activated to cope with situations, are changed or newly formed through processes of negotiation of meaning with other interactants or through interacting with materials. Increasing autonomy in the participation in mathematical discourse is thus conditioned by a (formally) mathematically oriented DSE-formation. Concerning the learning of mathematics, the children should, therefore, participate in interactions that can be characterized as 'mathematically rich' to develop mathematical meaning that is full of relations and outlives the situational context.

Indirect Leaning Processes

Especially in interactions in the kindergarten context, however, mathematics cannot necessarily be experienced and negotiated directly and immediately. On the one hand, this is due to the fact that mathematics "is, as knowledge of abstract relations, not directly accessible" (Steinbring, 2015, p. 281), on the other hand, it is due to the fact that the main focus of the interactants is initially on the effort to maintain the interaction (Krummheuer, 1997) and less on an explicit negotiation of elementary mathematical concepts (Vogler, 2020). Early mathematical negotiations can, therefore, be primarily concerned with more (informal) everyday topics, which are overlapping a mathematical content that can be interpreted - but does not have to be interpreted in order to participate in the interaction (Vogler, 2020). For mathematics, this conclusion seems to be obvious because, in some cases in mathematics, the concrete and every day meaning already contain the general and abstract mathematical meaning. The result is a "double-layer structure" (Vogler, 2020) where learners can participate on both levels of the interaction – the concrete situational and the abstract mathematical meaning. Bauersfeld (1995, p. 281) and Krummheuer (1997, p. 9) characterize these learning processes as "indirect learning". Hence, the concrete meaning superimposes the abstract. However, successful mathematical learning can be regarded as increasing participation on the (latent) *abstract* level of meaning – only, of course, when there is mathematical content contained on this level. According to Krummheuer (1997, p. 9) and Bruner (1990, p. 34), this is not explicitly negotiated but latently involved mathematical content, that underlies the informal interactive process, is called “plot”. While more competent members of a discourse know about the *plot* of a situation, the newcomers become familiar with it, while the participants within interactions that latently involve in this *plot*. So, they learn it actively over time. Whether a *plot* of a situation and its latent meaning is accessible for the learner depends on her or his existing or interactively acquired DSEs. Like Bauersfeld (1995) and Krummheuer (1997) consider this *indirect learning* as a 'key variable' for “fundamental learning” (Miller, 1986, p. 20–21; e.g. Steinbring, 2006, p. 194) in Mathematics. At this point, it remains questionable whether *indirect learning processes* afford all children’s development of (formal) mathematical DSE out of their possibly informal contextualization of the situation. This will be discussed in the following empirically substantiated based on analysed situations with preschool teachers in German kindergartens.

METHODOLOGY AND METHOD

Empirical Data – Mathematical Situations with Preschool Teachers

The analyzed situations come from the empirical data of the erStMaL-study (*early Steps in Mathematics Learning*), whereby early mathematics learning is longitudinally researched. Therefore, 12 German kindergartens consisted of 37 preschool teachers and 144 children were observed over an interval of three years. Besides, situations that are developed and realized by the research team, also self-designed mathematical situations of preschool teachers from the five mathematical domains ‘Numbers and Quantitative Thinking’, ‘Geometry and Spatial Thinking’, ‘Patterns and Algebraic Thinking’, ‘Measurement’ and ‘Data Analysis’ (e.g. Sarama & Clements, 2009) were observed. Two of the 37 mcompiled situations from the mathematical domain of Measurement, which are paradigmatic examples, are presented in the analysis.

Methodology – Reconstruction of Indirect Learning Processes

To analyze the *indirect learning processes* and the latent *plot* of the interactions, a two-step analysis is necessarily implemented. To carve out the *explicit processes of negotiation of meaning* and the included opportunities of the children to participate in this process, (1) the *Analysis of Interaction in Mathematics Education* is used (e.g. Krummheuer, 2002). For the *not explicitly* negotiated but latently involved meanings, an extension of this analysis is needed because interactional analyses mainly take *situational* processes into account which generate “taken as shared meanings” (Krummheuer, 2002; Vogler, 2020). Therefore, (2) the method is enlarged by elements from the objective hermeneutical approach as developed by Oevermann et al. (1979). This approach focuses on the “latent rules of the interactional system” that are characteristics of the *indirect learning processes* on which this paper is focused. Hence, the enlargement also provides the opportunity to even reconstruct meanings that originate from individual “Domains of Subjective Experiences” (Bauersfeld, 1988). By this means, it is possible to reconstruct the ‘hidden’ meaning of the interaction that originates from one of the participants of the interaction as well as the *plot* of a situation. For this purpose, “markers” within the interaction are taken into view. These markers are words, phrases, specific (diagrammatic) actions or gestures that are ‘produced’ by one participant to communicate and used by another to interpret the meaning of communication (e.g. Heller, 2015). In this way, the meanings can also be reconstructed by the researcher.

COMPARATIVE EMPIRICAL ANALYSES

With the help of the briefly described method of analysis, it is possible to confirm the theoretically developed assumption that early learning situations in kindergarten are characterised by interactions in which the learning processes can be described as *indirect*. In these situations, sense structures rich in elementary mathematics emerge initially on a level of meaning that can be described empirical hermeneutically as latent. What is negotiated are rather everyday meanings which overlap latent mathematically rich attributions of meaning. It is remarkable in the comparison of different situations with different preschool teachers that in some situations the previously latent mathematical meanings are manifested by the children in the course of time, while in other situations these manifestations are realized by the teachers.

Situation 1 – “Which are belonging together?”

One of the situations in which the children manifests latent mathematical meanings is described in Vogler (2020). In this situation, four children from a kindergarten in Germany (Hannah (3.3 years), Michael (3.7 years), Bettina (4.7 years) and Martha (5.3 years)) and their preschool teacher Nicola interact with materials on a carpet. The materials which are used include two green paper circles with different diameters (0.5m and 1.0m) and a burlap sack which is filled with ten different yet pairwise similar objects - in each case in two different sizes. The pairwise similar objects are lying on the two paper circles (e.g. Figure 1).

Figure 1: Arrangement of the objects on the paper circles

At the beginning of the scene, the preschool teacher Nicola asks the children to find two things that belong together. She asks: “Which are belonging together?”. After a girl, Bettina, pointed at two building blocks, Nicola continues with her instructions.

456 Nicola:	Take a look [*Bettina put two things together*], here we make a line.
457:	[*…pointing with her finger in a line right beside the paper circles parallel to the edge of the carpet*]
459:	Start right here.
460:	[*pointing at one point near the edge of the carpet]*.

She instructs Bettina to locate the objects on the edge of the carpet where they are separated from the paper circles. After a short interplay, Bettina places the two pins on the place marked by Nicola and Nicola confirms the successful ending of the task and evaluates Bettina's action in the following line #470:

470 Nicola:	Exactly! This way.
471:	[…*adjusts the pins on the carpet the way that they are lying parallel to the edge of the carpet and the heads of the pins are abreast]*.
472:	Who wants to search for two things that belong together now?

In line #471, the teacher corrects the arrangement by putting the nails side by side until the nails are parallel to each other with the carpet's edge. She asks the kids who would like to find the next objects that belong together in #472. In the next scenes, the kids position *all* pairs of objects on the carpet in a line with the first two nails. The next scene follows this 'routine':

583 Nicola:	What else can we do with it? Does anybody have an idea?
584 Martha:	Compare.
585 Nicola:	Compare! How would you do that, Martha?

In that scene, Nicola asks the children about the use of the two 'lines' of objects lying on the carpet. Martha specifies the use as 'comparing' #584. After this scene, an interaction can be observed wherein Nicola asks for the way to compare and the girl astonishingly replies that one can see it because it is 'beautiful'. The last utterances of the children in line #584, as well as the following interaction, can be seen as a manifestation of the latent mathematical *plot* of the situation: Two objects of similar shape and different size are building a pair and should be placed to visualize the exact geometrical difference in size in order to enable a direct mathematical comparison. Especially the 'marker' in line #598, provides evidence that there is this *plot* underlying the following interactional routine of putting all pairs of objects in a line on the carpet. On the surface, the situation 1 deal with is putting nails on a carpet, but, on the latent level of the interaction, the preschool teacher Nicola introduces an early concept of 'size' by directly comparing objects of equal shapes and different sizes (Vogler, 2020).

Situation 2 – "Are these all the same building blocks?"

While in situation 1 the children realize the manifestation of the *plot*, in the following situation (2), the preschool teacher (Doris) explicates the *plot* herself. This situation also takes place in a Germen kindergarten with four children: Nina (4;3 years), Belina (4;8 years), Nuem (4;7 years), and Mario (4;2 years). The four children and Doris are sitting on a table at the beginning of the scene. Doris presents a wooden box with rectangular prisms to the children. The 'blocks' are all similar concerning the material and the color but only pairwise similar in size. The preschool teacher puts the box on the table (e.g. Figure 2).

Figure 2: Illustration of the arrangement in Situation 2

001 Doris:	“What is in it?
002 Nina:	Building blocks
003 Doris:	Building blocks. Have a look!
004:	[*The wooden box with the wooden cuboids light tilting towards Mario and Nuem, then slightly tilting towards Nina]*.

In the following she requested an answer for the question “Are these all the same building blocks?” and the children reply simultaneously.

006 Nina:	Yes
007 Belina:	Yes
008 Doris:	Yes? Have a look, Mario!
010 Doris:	Are these all the same?

The contributions of Doris in line #008 and #010 is considered as an implicit rejection of Nina’s and Belina’s answer in lines #006 and #007. It can be interpreted that Doris focuses on the different sizes of the wooden cuboids, while the children may look at the informal categorization of the wooden cuboids as “building blocks” #002. After none of the children presents the adequate answer (“No”), Doris asks again, while holding a cube and a narrow cuboid in her hand:

029 Doris:	Look at that. What does that look like?
030:	[*Holding a wooden cube a little further up and moving it towards Nina*].
031 Belina:	They do not look the same.
032 Nina:	Square and two corners.

Although Nina's answer contains a distinguishing feature for the building blocks which can be classified as quite mathematically rich in its meaning, Doris also rejects this answer. Only much later the children answer a similarly asked question as in line #010 with “No” and Doris concludes: “They all look different! Various!” (#057). In doing so, the preschool teacher, Doris, is manifesting the latent *plot* of the situation.

Comparison and Analytical Results

In the comparison of the two situations, it can be carved out, that in the Situation 1, an increase in autonomy for most of the children can be observed because the interaction is based on a homogeneous, mathematically rich *plot* that is a kind of "ostinato" (Vogler, 2020) of the manifest negotiation. In addition, at the manifest level of interaction in this situation, a routine of action can be reconstructed in which the latent structures of meaning endure and are in turn present as latent argumentation routines - here, following Krummheuer (1997), one can speak of *double formatting of the interaction*. This structure seems to enable the children to first 'bite down' on the everyday manifest level of interaction, in order to then gradually participate in the latent sense structures on the basis of this (informal-level) participation (Vogler, 2019). In this way, the children seem to succeed in making the transition from an everyday contextualization of the situation to a formal mathematical one. This can be identified based on their contextualization (e.g. line #584). In Situations 2 in which the preschool teacher, in turn, explicates the initially latent mathematical *plot*, this transition does not seem to succeed for the children. Obviously, the informal attribution of the cuboids as building blocks does not change among the children in the course of time. Sometimes this is also because of the contributions that do not correspond to the 'desired' *plots* of the teachers are rejected (see interaction from #029 to #032). In such situation, the patterns of interaction resemble a question-developing (classroom) conversation.

CONCLUSION

The analysis outlined here give a first impression of how central *indirect learning processes* seem to be in the field of early mathematical learning and how important and indispensable they are for *fundamental learning* in their interactional form as 'double formatting'. For, even though it is quite possible that a learning process emerges from interaction patterns such as a question-and-answer conversation through explications, the routinization and the 'indirect' seems to be of particular importance for the 'first' and *fundamental* learning of mathematical contexts. This form of learning seems, from the perspective of the outlined results, to be almost inevitable to enable sustainable mathematical meaning on the part of the children. The analysis suggests that the increasing autonomy of children in mathematical content stems from the fact that the interaction between the existing informal DSEs and the formal mathematical experience of the situation itself is linked through a routine. It is not only the transition does succeed, but also a certain network of DSEs develops. In the comparison of the two situations, it can be assumed that an explication does not necessarily lead to such networking of the child's DSEs if they are realized by the preschool teachers and not by the children themselves. *Indirect learning,* therefore, seems to be a *key variable* for sustainable and networked learning of mathematical content. This also corresponds to the idea of mathematical learning as an enculturation process.

Notes

1. In the transcribed sequence all specialities of the spoken language (mistakes, grammar etc.) are mentioned in the translation of the transcribed sequence. Pauses within the speech are coded by a dot for every second in round brackets.

2. It can be assumed that Nina takes the spatial expansion of the cuboids into account and calls the cuboid with the smaller spatial expansion in one direction a "two corners", while she calls the cube a square (literally translated from German into English: "four corners"). The new word is thus based on the German term for square.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws & T. J. Cooney (Eds.), *Perspectives on* *research on effective mathematics teaching* (pp. 27−46). New Jersey, United States: Lawrence Erlbaum.

Bauersfeld, H. (1995). "Language games" in the mathematics classroom: Their function and their effects. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning. Interaction in classroom cultures* (pp. 230–271). New Jersey, United States: Lawrence Erlbaum.

Bruner, J. (1990). *Acts of meaning*. Massachusetts, United States: Harvard University Press.

Chaiklin, S. (2003). The zone of proximal development in Vygotsky's analysis of learning and instruction. In A. Kozulin, B. Gindis, V. S. Ageyev, & S. M. Miller (Eds.), *Vygotsky's Educational Theory in Cultural Context* (pp. 39–64). Cambridge, United Kingdom: Cambridge University Press.

Claesens, A., & Engel, M. (2013). How important is where you start? Early mathematics knowledge and later school success. *Teachers College Record, 115*(6), 1-29.

Heller, V. (2015). Academic discourse practices in action: Invoking discursive norms in mathematics and language lessons. *Linguistics and Education, 31*, 187-206. doi: 10.1016/j.linged.2014.12.003

Krummheuer, G. (1997). *Narrativität und Lernen. Mikrosoziologische Studien zur sozialen Konstitution schulischen Lernens.* Weinheim: Deutscher Studien Verlag.

Krummheuer, G. (2002). *The comparative analysis in interpretative classroom research in mathematics education.* Paper presented at the Second Conference of the European Society for Research in Mathematics Education, Marianske Laszne.

Melhuish, E., Quinn, L., Sylva, K., Sammons, P., Siraj-Blatchford, I., & Taggart, B. (2010). *Pre-school experience and Key Stage 2 performance in English and Mathematics*. Bangor, Northern Ireland: Department of Education.

Miller, M. (1986). *Kollektive Lernprozesse*. Frankfurt a. M.: Suhrkamp.

Oevermann, U., Allert, T., Konau, E., & Krambeck, J. (1979). Die Methodologie einer objektiven Hermeneutik und ihre allgemeine forschungslogische Bedeutung in den Sozialwissenschaften. In H. G. Soeffner (Eds.), *Interpretative Verfahren in den Sozial- und Textwissenschaften* (pp. 352–434). Stuttgart: J.B. Metzlersche Verlagsbuchhandlung.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research. Learning trajectories for young children*. New York, United States: Routledge.

Steinbring, H. (2006). *The construction of new mathematical knowledge in classroom interaction. An epistemological perspective*. New York, United States: Springer.

Steinbring, H. (2015). Mathematical interaction shaped by communication, epistemological constraints and enactivism. *ZDM: The International Journal on Mathematics Education, 47*, 281–293.

Vogler, A.-M. (2020, accepted). *Chances and obstacles of 'indirect' learning processes in situations with preschool teachers.* Paper presented at the CERME 11, Utrecht, Netherlands.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Massachusetts, United States: Harvard University Press.

SPECIALIZED FRACTIONS DIVISION KNOWLEDGE: A PROPOSED MODEL

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This paper aims to propose a model of specialized fractions division knowledge (SFDK). The model is drawn from specialized content knowledge (SCK) subdomain (Ball et al., 2008) and a synthesis of prior related works which examine the natures of prospective primary teachers' (PTs) specialized knowledge on fractions division. With respect to preceding relevant studies, the proposed model is more comprehensive since it fully considers translating across representations and different conceptualizations of fractions division. Moreover, it has double functions; to examine and develop PTs' SFDK in the teacher education program.

INTRODUCTION

Having a robust understanding of fractions division (FD) is a major challenge for PTs since, unlike other primary mathematics topics, its characteristics are problematic (Prediger, 2006; Ma, 2010). Researches in the last decades (e.g., Simon, 1993; Li & Kulm, 2008; Olanoff, Lo, & Tobias, 2014) reveal that a multitude of PTs have not fulfilled that challenge. For example, when PTs were asked to create a word problem of the division with fractions, most of them either presented fraction multiplication problem or not able at all to come with the answers (Simon, 1993). Studies, with prior (e.g., Jansen & Hohensee, 2016) and after instruction design (e.g., Adu-Gyamfi et al., 2019) unravel quite similar results regarding the low achievement of PTs' specialized knowledge on the topic.

To help prospective teachers develop such kind of knowledge through a well-prepared design of instructions in the teacher education courses, the information on the nature of their knowledge on fractions division is required (Lo & Luo, 2012; Jansen & Hohensee, 2016). Similarly, that information is vital to examine the effectivity of an instructional design which aims to develop PTs' SFDK. Thus, in order to thoroughly understand such knowledge, a model is definitely needed. This paper attempts to address that need by proposing a model of specialized fractions division knowledge.

PRIOR STUDIES ON SFDK

Olanoff, Lo, and Tobias (2014) extensively reviewed a number of studies which focus on mathematical knowledge for teaching fractions from pre-1998 to 2013. Several studies included in the review (e.g., Tirosh & Graeber, 1990; Rizvi, 2004; McAllister & Beaver, 2012; Lo and Luo, 2012) examined the participants' SFDK through tasks which ask them to (1) write words problems from a given number

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sentences or otherwise, and (2) use pictorial representations to solve word problems. However, the researchers were not concerned about the different conceptualizations of fractions division. For instance, Lo and Luo (2012) examined PTs' specialized fractions division knowledge using a task that asks the subjects to write a word problem which represents 8 2/3 ÷ 1/4 = ? and use drawing to solve it. Three representations are involved in solving the problem.

Two related studies are found after Olanoff's et al. (2014) review. Jansen and Hohensee (2016) examined the nature of PTs' conceptions of a partitive division with fractions prior to the instruction. Referring to the notion of productive conceptions (Llyoid & Wilson, 1998), they set two criteria of conceptions, namely *connected* and *flexible*. Translating between representations and being aware that partitive fractions division generate unit rate are the indicators of *connected conception*. *Flexible conception* is defined as becoming aware that division can involve partitioning, iterating, or both. The results of the study reveal that none of the participants has fully connected conceptions and flexible connection. Within the same objectives to examine PTs' content knowledge and different context, the subjects have participated in the related fractions division course, Adu-Gyamfi et al. (2019) presented three tasks to examine PTs' knowledge regarding conceptualizations and connections the subjects made among diagrammatic, verbal, and algebraic representations. The study did not only cover SCK but also knowledge of content and students (Ball, Thames, & Phelps, 2008) since two items of the tasks present example of students' works to be analysed, whether or not it is a correct solution to the first task.

On the whole, aforementioned studies (e.g., Simon, 1993; Li & Kulm, 2008; McAllister & Beaver, 2012; Lo & Luo, 2012) mostly examined how prospective (elementary or middle school) teachers move from one representation to another, for example from symbolic (number sentences) to words or story problem and different interpretations of fractions division was not its concern. Jansen and Hohensee (2016) focus specifically on one conceptualization of fraction division (partitive) but provide a significant tool to examine specialized content knowledge on partitive fraction division. Adu-Gyamfi et al., (2019) studied the PTs' knowledge which refers to the conceptualizations of fraction division, and connections between verbal, diagrammatic, and algebraic representations but the focus were split to another subdomain of MKT. With respect to the researchers, I argue that the PTs' specialized fraction division knowledge needs to be further comprehensively examined with respect to translating across representations used in solving fractions division problems and different conceptualization of fractions division.

THEORETICAL REVIEW

To strengthen the idea of the model, it is imperative to provide a theoretical review about representation systems of fraction division, different conceptualizations of fractions division, and SCK as follows.

Conceptualization of Fractions Division

The literature agrees that fractions division has diverse conceptualizations (Sinicrope, Mick, & Kolb, 2002; Gregg & Gregg, 2007; Lamon, 2012). Sinicrope et al. (2002) conceptualize fractions division into five; measurement, partitive, unit rate, the inverse of an operator multiplication, and the inverse of a Cartesian product. Some authors (e.g., Gregg & Gregg, 2007; Lamon, 2012) include and use unit rate as part of the partitive division with fractions. The model adopts the five categories but includes only three common conceptualizations; measurement, partitive, and unit rate since they are mostly taught in the classrooms and presented in the textbook (Wahyu & Mahfudy, 2018).

Measurement, partitive, and unit rate interpretations of FD have unique features (Gregg & Gregg, 2007; van de Walle, Karp, & Bay-Williams, 2012; Petit et al., 2016; Jansen & Hohensee, 2016; Shin & Lee, 2018) with respect to *components* (dividend and divisor), *typical situation* (e.g., fair-sharing), *solution process* (iterating or partitioning), and *developed algorithm*. I argue that the ability to *differentiate* each conceptualization is definitely necessary for PTs. For example, when $3/4 \div 1/2$ is given, two distinct story problems (measurement or unit rate) could be made, or when given two story problems, they could identify which one is for $2 \div 3/4$ (measurement) and $3/4 \div 2$ (partitive). After all, this process also associates with translating across representations and thus result in a comprehensive SFDK.

Representations of Fractions Division

External representations are visible productions such as diagrams, graphs, manipulatives, formulas and equations, or mathematical expressions which stand for mathematical ideas or relationships (Goldin, 2014). There are three common external representations widely used and referred to fractions division; *concrete* (e.g., fraction bars), *semi-concrete* (e.g., number lines), and *abstract* such as numerical, verbal or symbolic/algebraic (Adu-Gyamfi et al., 2019). These representations have been deploying to develop students' understanding (Gregg & Gregg, 2007; Wahyu, Amin, & Lukito, 2017) and examine (prospective) teachers' FD knowledge (Lo & Luo, 2012). Solving FD problems involves the translation between verbal representation (word problems), pictorial representations (number lines, area model, or sets of objects model), symbolic representation (number sentences), and algebraic representation which are depicted by the proposed model. In Adu-Gyamfi et al. (2019), algorithm algebraic representation is the algebraic representation of procedures to operate fractions division. In this model, the algebraic representation is the procedures and rationales behind it as well.

One of the key aspects of SCK related to fraction divisions is understanding different representations (Ball et al., 2008). For example, when the PTs is asked to solve a story problem on fractions operations (verbal representations), they are certainly demanded to differentiate which operation fits the story problem, what number sentence (symbolic representation) stands for the problem, what appropriate models (number lines or area model) to construct, what algorithm (algebraic representations) to use and how it relates developed models, and finally all of which lead to correct solution. These processes denote *linking across representations*.

Specialized Content Knowledge

SCK is one of the knowledge components under subject matter knowledge that the teachers should possess. It is defined as mathematical knowledge and skill that is peculiar in teaching (Ball et al., 2008). Referring to the description of SCK presented by Ball et al. (2008), the competencies related to fractions division are (1) differentiating the conceptualization of FD, for example, the difference between measurement and partitive interpretation, (2) linking across various representations in solving FD problems, for instance, write a correct number sentence from a FD story problem, and (3) holding decompressed mathematical knowledge such as explaining why invert and multiply or equalize the denominators to divide fractions. The proposed model is built up by these points. The last point is placed in algebraic representation in the model.

THE PROPOSED MODEL

Drawing from aforementioned components of SCK, prior studies (Jansen & Hohensee, 2016; Adu-Gyamfi et al., 2019), and theoretical reviews on conceptualizations and representations of fractions division, this paper proposes a model of SFDK (Figure 1) which can be utilized to examine PTs' SFDK and prepare an instructional design in a mathematics course to develop such knowledge. I introduce the term *connected and flexible SFDK* adapted from Jansen and Hohensee (2016). *Connected SFDK* is PTs' ability to translate across various representations, not only from verbal to pictorial representations or one direction translation. *Flexible SFDK* is PTs' capability of differentiating measurement, partitive, and unit rate interpretation of fractions division which affect their works on the representations.

The model represents two main components of SFDK, i.e., *linking across representations* and *differentiating conceptualizations of fractions division*. There are two parts of Figure 1. Firstly, diagram inside the large rectangle which denote the first component. Secondly, the rectangle itself that indicates the second component that 'guide' the representations. 'Guide' means that each conceptualization uniquely determines the process of moving from one representation to others. I call it unique since measurement FD has distinct features, e.g., repeated subtraction situation, compared to a fair-sharing situation (partitive FD) which affect the approach students or (prospective) teachers used to solve the FD problems. Two-direction arrow denotes the link of various representations meanwhile the one-direction dashed line denotes the process of solving FD problems. The solution process is included since it entails the way PTs develop decompressed mathematical knowledge as part of SCK.

One example is presented to explicate how the model works. Given this word problem of measurement FD: *Ana is making a flower decoration from 2 1/5 metres ribbon. Each decoration requires 3/5 metre ribbon. How many decorations can Ana make?*

Generally, to solve the problem, PTs could begin by either drawing pictorial representations, for example, number lines or determining number sentence. Let us focus on the first starting point. The problem-context (verbal representation) is translated into number lines (pictorial representation). With the number line, PTs can find the number of decorations (model-based solutions) by partitioning and iterating it. The 3 2/3 decorations are the results of counting how many 3/5s are in 2 1/5 (verbal representation ↔ pictorial representation) **[1]**. However, the process does not stop here since teachers will introduce fraction division to the students.

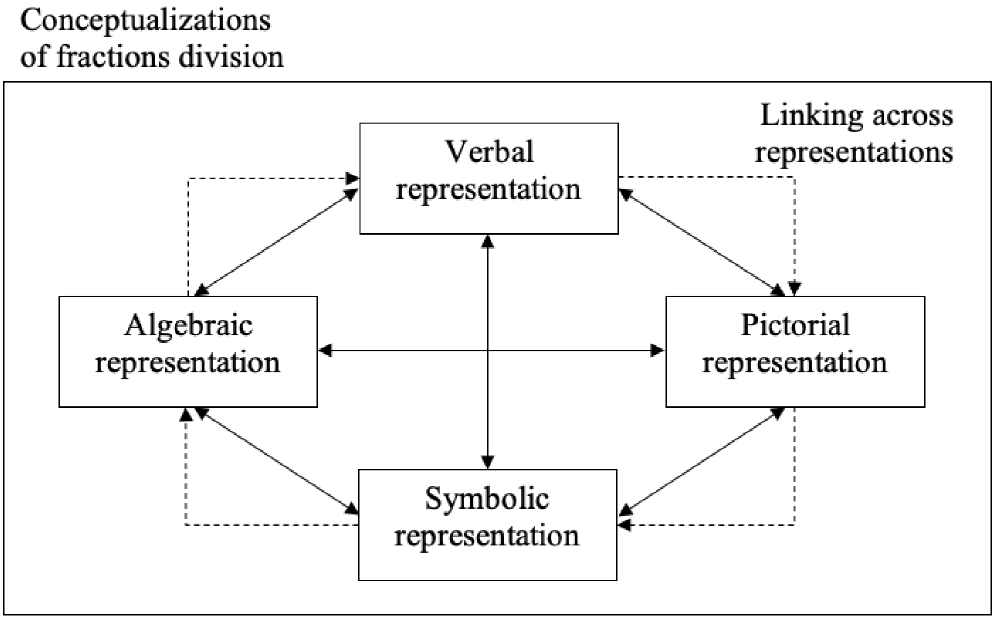

Figure 1: A model of specialized fraction division knowledge

PTs determine number sentence (symbolic representation, 2 1/5÷3/5), or also called as mathematics problem model (verbal representation ↔ symbolic representation) **[2]** which depart from episodic situation comprehension and problem model (Staub & Reusser, 1995). The number sentence is meaningful if PTs could relate it to the number line. Indeed, the number line represents 2 1/5÷3/5 (pictorial representation ↔ symbolic representation) **[3].** The result of 2 1/5÷3/5 could be determined by using the common-denominator algorithm. In this model, the algebraic representation is not only the algorithm or procedure to calculate the quotient as part of common content knowledge (Ball et al., 2008) but also justification why the procedure could be used to divide. The quotient 3 2/3 and the algorithm are meaningful if PTs could link them to the model-based solutions. 2 1/5÷3/5 = 3 2/3 is similar to determining how many 3/5 metres ribbons are in 2 1/5 metres ribbon. It is the reason why 2 1/5 divided by 3/5 results in 3 2/3. The procedure, 2 1/5 ÷ 3/5 = 11/5 ÷ 3/5 = 11 ÷ 3 = 3 2/3, refers to a number of partitions made in the number line for dividend directly divided by the numerator of divisor or number of iterations based on the divisor. It is the argument why common-denominator algorithm could be used (pictorial representation ↔ symbolic representation ↔ algebraic representation ↔ pictorial representation) **[4]**. At last, the process **[1]** to **[4]** is also linked back to verbal representation and otherwise (linking across representations) **[5]**.

These processes, in my perspective, reflect *connected SFDK*. When PTs are able to translate across representations, from **[1]** to **[5]**, no doubt that they will teach FD conceptually. I also argue that linking across the representations depends on the FD conceptualizations. If the problem is partitive FD, the way PTs do **[3]** is different from measurement since both conceptualizations have distinct situation; repeated subtraction and fair-sharing. Likewise, the unit rate conceptualization is not the same as partitive and measurement when PTs do **[4]** since it uses the invert-multiply algorithm. For this reason, moving across representations is not enough, and that is why differentiating each conceptualization *flexible SFDK* is needed.

Using the model, one could design fraction division tasks to reveal PTs' connected and flexible SFDK. Table 1 shows the exemplary tasks which I am using to test the model empirically. The tasks below can also be utilized to develop PTs' SFDK.

No.	Task
1*	Match the following word problems with the given number sentences! You may use one number sentence for more than one word problem. (1) Dwi has 4 kg of flour to be put in a box. One box contains 2/3 kg of flour, how many boxes does she need? (2) … (7) **Number sentences** (a) $\frac{3}{4} \div \frac{1}{2}$; (b) $\frac{2}{3} \div 4$ (c) $\frac{1}{2} \div \frac{3}{4}$; (d) $1\frac{2}{3} \div \frac{1}{4}$; (e) $4 \div \frac{2}{3}$; (f) $\frac{3}{4} \times \frac{1}{2}$; (g) $\frac{1}{4} \div 1\frac{2}{3}$
2	Use only drawings or models to solve the word problems in number 1! Also, determine the quotient of its number sentence through an algorithm!
3	After solving word problems and determine the quotient in number 2, do you get a similar answer for each pair? If NO, which one is correct? If YES, how your models and algorithm are related? Explain your answer!
4	Write a different word problem for $4 \div \frac{2}{3}$; $\frac{2}{3} \div 4$; $\frac{3}{4} \div \frac{1}{2}$; and $1\frac{2}{3} \div \frac{1}{4}$!
5	What is the difference in words problems for $4 \div \frac{2}{3}$ and $\frac{2}{3} \div 4$? *Hint*: Use contextual problems in number 1. You can compare it to your contextual problems in number 4.

*There are seven word problems in number 1

Table 1: Fraction division task to examine and develop PTs' connected and flexible SFDK

CONCLUSION

This paper explicates the proposed model, which can be used to examine PTs' SFDK and entry point for instructional design to develop such knowledge. It represents a connected and flexible SFDK for which PTs should hold in order to teach fractions division conceptually. Prior studies (e.g., Rizvi, 2004; McAllister & Beaver, 2012; Lo and Luo, 2012; Jansen & Hohensee, 2016) which examine PTs' specialized content knowledge on the topic were limited to one direction of translating between representations and had not fully considered the different conceptualizations of fraction divisions. Nevertheless, SCK (Ball et al., 2008) includes three major components, namely (1) differentiating the conceptualization of FD, (2) linking across various representations, and (3) holding decompressed mathematical knowledge. The proposed model extends the foregoing works and covers all the components.

References

Adu-Gyamfi, K., Schwartz, C. S., Sinicrope, R., & Bossé, M. J. (2019). Making sense of fraction division: domain and representation knowledge of preservice elementary teachers on a fraction division task. *Mathematics Education Research Journal*, *31*, 507-528.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 399–407.

Goldin, G. A. (2014). Mathematical representations. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 409-413). Dordrecht: Springer.

Gregg, J., & Gregg, D. U. (2007). Measurement and fair-sharing models for dividing fractions. *Mathematics Teaching in the Middle School, 12*(9), 490-496.

Jansen, A., & Hohensee, C. (2016). Examining and elaborating upon the nature of elementary prospective teachers' conceptions of partitive division with fractions. *Journal of Mathematics Teacher Education*, *19*(6), 503-522.

Lamon, S.J. (2012). *Teaching fractions and ratios for understanding (3rd Edition)*. Routledge: New York.

Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM Mathematics Education, 40,* 833–843.

Lo, J. J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education*, *15*(6), 481-500.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29*(3), 248–274.

Ma, L. (2010). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New York: Routledge.

McAllister, C. J., & Beaver, C. (2012). Identification of error types in preservice teachers' attempts to create fraction story problems for specified operations. *School Science and Mathematics*, *112*(2), 88-98.

Olanoff, D., Lo, J. J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast*, *11*(2), 267-310.

Petit, M. M., Laird, R. E., Marsden, E. L., & Ebby, C. B. (2016). *A focus on fractions: Bringing research to the classroom (2nd Edition)*. New York: Routledge.

Prediger, S. (2006). Continuities and discontinuities for fractions: A proposal for analysing in different levels. In Novotna, J. et al. (Eds.). Proceeding of the 30th PME, Prag 2006, 4-377-384.

Rizvi, N. F. (2004). Prospective teachers' ability to pose word problems. *International Journal for Mathematics Teaching and Learning*. Retrieved from http://www.cimt.org.uk/journal/rizvi.pdf

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*(3), 233–254.

Sinicrope, R., Mick, H. W., & Kolb, J. R. (2002). Interpretations of fraction division. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 153–161). Reston: National Council of Teachers of Mathematics.

Shin, L., & Lee, S. J. (2018). The alignment of student fraction learning with textbooks in Korea and United States. *Journal of Mathematical Behavior*, *51*, 129–149.

Staub, F.C., & Reusser, K. (1995). The role of presentational structures in understanding and solving mathematical word problems. In C.A. Weaver, S. Mannes, & C.R. Fletcher (Eds.), *Discourse comprehension. Essays in honor of Walter Kintsch* (pp. 286-305). Hilsdale, NJ: Lawrence Erlbaum.

Tirosh, D., & Graeber, A. O. (1990). Inconsistencies in preservice elementary teachers' beliefs about multiplication and division. *Focus on Learning Problems in Mathematics*, *12*(3–4), 65–74.

van de Walle, J.A., Karp, K.S., & Bay-Williams, J.M. (2012). *Elementary and middle school mathematics: Teaching developmentally (8th Edition)*. Allyn & Bacon: Boston.

Wahyu, K., Amin, S. M., & Lukito, A. (2017). Motivation cards to support students' understanding on fraction division. *International Journal on Emerging Mathematics Education*, *1*(1), 99-120.

Wahyu, K., & Mahfudy, S. (2018). Opportunity-to-learn fraction division through mathematics tasks in Indonesia's mathematics textbook. Unpublished research report, LP2M UIN Mataram.

MATHEMATICAL ACTIVITY IN COLLABORATIVE LINEAR-ALGEBRA PROBLEM-SOLVING

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Collaborative problem-solving's popularity for encouraging meaningful learning is increasing, both in general, and particularly in mathematics. However, little is known about the processes of collaborative learning in university-level mathematics. Hence, we studied the activity of a pair of students working on a linear-algebra problem. We analysed the mathematical routines they followed and found there was a commognitive conflict that impeded the effectiveness of the collaborative learning activity. We point to further research directions regarding the effectiveness of collaborative learning in university-level mathematics.

THEORETICAL BACKGROUND

In a university setting, collaborative learning, although less researched then in primary and secondary education settings, has been shown to promote positive social and academic outcomes (Cabrera et al., 2002). The benefits of collaborative learning cited in the literature include encouraging discovery, fostering student engagement, promoting student agency, advancing communication and collaboration skills, and fostering appreciation for many solution paths to a correct answer (Barron, 2000).

Yet, along with this long list of potential advantages, some researchers have pointed to the problems that can exist in student collaboration. These include distracting social interactions between members (Barron, 2000) and ineffectual communication (Nilsson & Ryve, 2010; Sfard & Kieran, 2001). Moreover, due to the dearth of studies closely examining mathematical content in collaborative settings, especially in university mathematics, our understanding of the processes that elicit more effective or less effective collaborative learning is very preliminary. In this study, we aim to deepen this understanding of the processes by which student learning advances (or not) through peer interaction. For this, we adopt the commognitive perspective (Sfard, 2008), that has proven beneficial for looking at social interaction and mathematical learning concomitantly (Heyd-Metzuyanim & Sfard, 2012; Sfard & Kieran, 2001).

According to the commognitive theory, mathematical learning is the process whereby learners develop and refine their participation in the mathematical discourse. This includes describing mathematical objects and their properties, manipulating these objects, and preforming routines that result in narratives about said objects, such as finding solutions to equations. Lavie, Steiner and Sfard (2019) define mathematical routines as a task and procedure pair used by a student to achieve a certain goal. These authors differentiate between the *task situation*, which

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 209-216. Khon Kaen, Thailand: PME.

is the way that a task-poser (such as the teacher) defines the task and the *task*, which is the way the task performer (learner) interprets the task.

Mathematical routines can be divided into object-level routines and meta-level routines (Sfard, 2008). Object-level routines deal with mathematical objects and how to manipulate them, such as using scalars to multiply vectors in ways that would cancel them out. Meta-level routines pertain to rules, usually implicit, of how to establish object-level narratives. For example, how to prove, demonstrate or convince another of a certain narrative. Incompatibilities between object-level narratives or routines are usually easily noticed by participants. It is not difficult to observe that someone claiming 2+2=4 is making a different claim than someone claiming 2+2=5. However, differing meta-rules are much more difficult to observe. Thus, when participants abide by different meta-rules, often a commognitive conflict ensues (Sfard, 2008). This is a situation where the differing narratives produced by the participants follow incommensurable rules or assumptions that are not acknowledged.

Commognitive conflicts can hinder learning in a collaborative-learning setting, since the participants do not share criteria for deciding if a narrative should be endorsed (Sfard, 2009). Little is known about the mechanisms of the collaborative-learning process, and what is known is mostly in elementary and secondary schools. Learning university-level mathematics includes many meta-level shifts, due to the numerous new mathematical objects introduced, the rules governing their manipulation, and the meta-rules of formal proof that are unfamiliar to graduates of secondary school (Thoma & Nardi, 2018). Using the conceptual toolset described above, we closely examined the problem-solving process of a pair in a linear-algebra class. We ask:

How could the learning process be described in terms of object-level and meta-level routines? Was its effectiveness impacted by whether the communication was around object-level or meta-level rules?

METHOD

This study is part of a larger project aimed at designing and examining effective learner-centered instructional practices for university linear algebra courses. The project took place in a highly selective, engineering university. Discussion-based workshops were offered to the students (all were majoring in mathematics, computer engineering or electrical engineering). Workshops—including small group activities leading to a classroom discussion—were offered in addition to the regular lectures and tutorials.

This paper focuses on Hadar and Yaniv, a pair that participated in the fourth meeting of the workshop. Preliminary analysis of all the recordings showed that the recording of this pair included a discussion in which some non-canonical narratives (wrong solutions) advanced to canonical (correct) ones. Moreover, there was **no** one partner ostensibly dominating the conversation. This seemingly productive, joint interaction indicated that the collaboration might have been effective for the two students, and so we studied this interaction in more detail to examine the processes involved.

We began the analysis by delineating the mathematical routines used by the pair. We established what task each student was solving from the narratives they offered, and incomplete statements were filled in, using prior and subsequent statements. Once the pair's implementations of the problem-solving routine were established, we compared them to ascertain if they were mathematically aligned, that is, if they were consistent to an expert, external observer.

We also examined the declarations (implicit or explicit) of agreement and disagreement during the pair's discussion. The pair's statements were thus classified by two measures: mathematical alignment and declared agreement. This allowed determining instances of effective communication (where declarations fit the mathematical alignment) and instances of ineffective communication (for example, declared agreement together with misalignment of mathematical narratives).

The workshop that provided the context for this episode dealt with the topic of linear dependence. The workshop began with a reminder of the basic definitions and theorems of linear dependence that were presented in lectures and tutorials. The basic definition was written on the board:

> V is a vector space over F. The set $\{v_1, \ldots, v_n\} \subset V$ is linearly dependent over F if there exist $\alpha_1, \ldots \alpha_n \in F$, not all zero, such that $\Sigma \alpha_i v_i = 0$. Otherwise, the set is linearly independent.

The students were then presented with a task to solve in small groups. The task consisted of assertations that the students were asked to determine whether each one always holds, never holds, or sometimes holds. If it sometimes holds, they were asked to provide an example for which it holds and an example for which it does not hold.

FINDINGS

The pair's discussion about Assertation 2 started with tending to the task, as written on the worksheet. This was the task situation:

> Determine if the set $\{u_1, u_2, u_3, u_4\}$ is linearly dependent or linearly independent, given that $\{u_1, u_2, u_3\}$ is a linearly dependent set. (Canonical answer: This assertation always holds.)

Even though both Hadar and Yaniv read the same words from the same worksheet, our examination revealed that their individually interpreted tasks—that is the task they each assigned to themselves—was different for each of them.

Yaniv's and Hadar's task and procedure

The following excerpt exemplifies Yaniv's initial task and procedure.

27 Yaniv: Yes. It (the assertation) is definitely true.

28 Hadar: A linearly dependent set, u belongs to V, all these together ({u_1, u_2, u_3, u_4}) are linearly dependent…Are you sure it's (the assertation) true?

29 Yaniv: If it ({u_1, u_2, u_3}) is already linearly dependent, and we add another vector, this subset ({u_1, u_2, u_3}) is still linearly dependent.

In this excerpt, Yaniv's task was to prove that the set {u_1, u_2, u_3, u_4} is linearly dependent. He claimed that the assertation is true [27], that is the set {u_1, u_2, u_3, u_4} is linearly dependent and he gave a justification for his claim [29]. This justification hinted at the procedure he used to determine if a set is linearly dependent. This procedure used a theorem proved in the lecture-that a set including a linearly dependent subset is a linearly dependent set.

Hadar initially did not agree that the assertation was true; thus, her initial task differed from Yaniv's. Instead of proving that the assertation was true, her task was to show it was wrong. She chose to do so by proving that there exists a vector u_4 such that the set {u_1, u_2, u_3, u_4} is linearly independent when the set {u_1, u_2, u_3} is linearly dependent. To do this, Hadar suggested the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)} as a counter example to the assertation. To justify her claim, Hadar used an idiosyncratic procedure, where she explored the status of each vector in the set, determining whether it was "linearly dependent" or not. She did so by using the "canceling out" procedure. This was revealed in her statement, "this (the vector (0,1,0,0)), you cannot cancel out if you don't put a zero for it" [46]. This idiosyncratic procedure examined whether the scalar used to "cancel out" the vector (0,1,0,0) is 0, and if so, determined that the vector (0,1,0,0) is a "linearly independent" vector.

Hadar's use of linear independence as a property of a single vector was of course non-canonical. In the canonical mathematical discourse, linear dependence is a property of a set of vectors, not of a single vector. A vector can only be linearly dependent with another vector or with a set of vectors. This meta-rule (that linear dependence is a property of sets of vectors) —as well as Hadar's divergence from it—remained implicit throughout the interaction between the two students and was not exposed or acknowledged by either of them. Each student's tasks were thus different both in the procedures, as well as in their meta-rules.

Commognitive conflict

During the pair's discussion there were instances where they declared agreement, but their mathematical meta-rules were not aligned. This happened, for example, in the following excerpt, where Hadar gave an example of a set that she considered a counter example to the linearly dependent set {u_1, u_2, u_3, u_4}:

36 Hadar: Let's take 3 (vectors) that are dependent with u_1. Let's say here is 2, 3 and 4 (probably meaning {(2,0,0,0), (3,0,0,0), (4,0,0,0)}).

37 Yaniv: So? That's exactly what I am saying. If we add, doesn't matter what we add…these 3 vectors will still be dependent.

38 Hadar: The 3 (vectors) are (linearly dependent). But the fourth isn't. So, the entire set is linearly independent.

39 Yaniv: Why?

In this excerpt Hadar used the procedure of finding a linearly independent vector, utilizing the implied meta-rule of linear dependence as a property of vectors ("The fourth isn't" [38]). Yaniv, in contrast, used the procedure of finding a linearly dependent subset ("doesn't matter what we add" [37]). This misalignment between the pair's mathematical narratives, coupled by Yaniv declaring agreement, ("that's exactly what I am saying" [37]), indicates the existence of a commognitive conflict in their discussion.

This commognitive conflict was also apparent in other parts of the interaction, for example in the excerpt below, where Hadar justified her counter example.

46 Hadar: And this (the linear combination) won't be equal to zero, because this (u_4), you cannot neutralize if you don't put a zero for it.

47 Yaniv: Yes. But it doesn't matter if it will be zero, if all the rest uh, if there is one…

48 Hadar: Then show me how.

49 Yaniv: No, that's what I am saying. If there is at least one…uh…if there is one scalar.

In this excerpt Yaniv and Hadar discussed their disagreement and attempted to resolve their differences. However, their discussion was focused on the object level, dealing with the scalars involved in the procedure for neutralizing vectors. Hadar claimed that the scalar multiplying u_4 has to be zero, "you cannot neutralize (u_4) if you don't put a zero for it" [46]. Yaniv agreed to this, but argued that the set can still be linearly independent in this case, "it doesn't matter if it will be zero" [47]. The crux of their disagreement—the implied meta—rule that linear dependence is a property of sets and not of single vectors-did not emerge.

Whenever Hadar and Yaniv authored contrasting object-level narratives, these were noticed and discussed. For example, when Hadar claimed, "that makes a linearly independent set" [60], Yaniv answered, "No, a dependent set" [61]. Hadar then changed her object-level narrative to agree with Yaniv's and corrected herself with a smile: "linearly dependent (set)" [62]. The disagreement about object-level narratives also compelled the pair to examine their narratives and attempt to justify them, thus advancing their justifications. For instance, when Hadar disagreed with Yaniv's claim that a set was linearly dependent, asking "How is it linearly dependent?" [42] he attempted to justify this by saying, "But it doesn't matter if it (the scalar multiplying u_4) will be zero, if all the rest (aren't)" [47]. This justification challenged Hadar's narrative by suggesting that one, but not all, scalars multiplying vectors in a linearly dependent set can be zero. Hadar built on this object-level clarification by including a zero as a possible scalar in her suggestion, “we should cancel them (u_1, u_2, u_3) in a way that’s not zero, and this (u_4) in a way that it (the coefficient of u_4) is zero” [50].

To conclude, Hadar and Yaniv authored conflicting narratives both at the object level and at the meta level. However, while the conflicting narratives at the object level were noticed and advanced the pair’s discussion, the misaligned, unacknowledged, meta-rules hindered their communication. This had implications for Hadar’s learning, as we show in the next subsection.

The persistence of Hadar’s misaligned meta-rule

In the pair’s discussion of Assertation 2, Hadar ultimately agreed that the set {(1,0,0,0), (2,0,0,0), (3,0,0,0), (0,1,0,0)} is a linearly dependent set, seemingly aligning her narratives with Yaniv’s canonical narratives. However, in the pair’s discussion of the next assertation, she still attributed linear dependency to single vectors. Hadar wondered during this discussion, “If the set is linearly dependent … can each vector be expressed as a linear combination of the others?” [216]. Hadar's wondering seems especially surprising given that she had just, minutes before, co-authored, together with Yaniv, an example of a set where one vector (0,1,0,0) is *not* a linear combination of the others.

Hadar's confusion seems to be based on two notions in linear algebra that are interrelated—linear dependence and linear combinations. If a set, S = {$v_1, \ldots, v_n$}, is a linearly dependent set, then **one** of its elements can be represented as a linear combination of the rest. Hadar questioned if each vector in a linearly dependent set can be written as a linear combination of the others. She was examining properties of single vectors when the property given pertained to a set. After some discussion, during which the non-canonical meta-rule was still not exposed, Hadar summed up by stating, “Bottom line, in a linearly dependent set each vector can be expressed as a linear combination of the others.” [242]

To conclude, the discussion between Hadar and Yaniv was ineffective in dispelling the non-canonical meta-level rule about linear dependence that was repeatedly authored by Hadar. Even though Yaniv protested and consistently followed the canonical meta-rules, he was unable to capture the differences in their meta-rules. The only differences that seemed to be apparent to Hadar and Yaniv were those at the object level. As a result, Hadar’s non-canonical meta-rule did not change, even though she authored, at times, canonical narratives that conflicted with this meta-rule and even though Yaniv challenged some of her narratives.

DISCUSSION

This paper describes an episode of collaborative mathematical learning in a university setting. Former studies have shown that collaborative-learning is not always effective and there can be factors that inhibit learning (Heyd-Metzuyanim & Schwarz, 2017; Sfard & Kieran, 2001). However, much of the obstacles discussed in these studies could be attributed to social issues, or motivation of students to work together and listen to each other. In contrast, the interaction studied here was in a university setting, where the students were mature, motivated, chose to work together and genuinely listened to each other. Ostensibly, the mathematical activity in such a context should have been effective. However, we saw in this case that even in such optimal cases, student learning can be obstructed by commognitive conflicts.

The pair's narratives advanced on the object level, yet the need for a meta-level change in Hadar's discourse was not fulfilled. Meta-level changes in discourse demand access to the new discourse, as well as awareness of the necessity for change (Ben-Zvi & Sfard, 2007). The canonical discourse was available to Hadar through Yaniv's narratives. However, since the commognitive conflict between their narratives was unacknowledged, Hadar had no need to change her meta-rules. In contrast, the object-level conflicts were acknowledged and discussed, and thus Hadar needed to revise her narratives.

The results of this study are limited due to the focus on only one interaction. Nevertheless, such micro-scale examinations point to the need for researching collaborative learning in more detail, in particular in a university setting. While collaborative learning in university should be encouraged, lecturers and tutors should be aware of the difference between object-level and meta-level mathematical rules. A seemingly productive, collaborative discussion, while supporting object-level learning might not be conducive for meta-level learning. Future studies exploring how to expose or minimize these commognitive conflicts could support productive learning on all levels.

References

Barron, B. (2000). Achieving Coordination in Collaborative Problem-Solving Groups. *Journal of the Learning Sciences*, *9*(4), 403–436.

Ben-Zvi, D., & Sfard, A. (2007). Ariadne's thread, daedalus' wings and the learners autonomy. *Education et Didactique*, *1*(3), 117–134.

Cabrera, A. F., Nora, A., Crissman, J. L., Terenzini, P. T., Bernal, E. M., & Pascarella, E. T. (2002). Collaborative learning: Its impact on college students' development and diversity. *Journal of College Student Development*, *43*(1), 20–34.

Heyd-Metzuyanim, E., & Schwarz, B. B. (2017). Conceptual change within dyadic interactions: the dance of conceptual and material agency. *Instructional Science*, *45*(5), 645–677.

Heyd-Metzuyanim, E., & Sfard, A. (2012). Identity struggles in the mathematics classroom : On learning mathematics as an interplay of mathematizing and identifying. *International Journal of Educational Research*, *51–52*, 128–145.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: from ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176.

Nilsson, P., & Ryve, A. (2010). Focal event, contextualization, and effective communication in the mathematics classroom. *Educational Studies in Mathematics*, *74*(3), 241–258.

Sfard, A. (2008). *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2009). Commentary on Baker and Schwarz through the lens of commognition. In *Transformation of knowledge through classroom interaction* (pp. 173–183).

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, *8*(1), 42–76.

Thoma, A., & Nardi, E. (2018). Transition from School to University Mathematics: Manifestations of Unresolved Commognitive Conflict in First Year Students' Examination Scripts. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 161–180.

FACTORS CONTRIBUTING TO PRESERVICE TEACHERS' ENDORSEMENT OF TECHNOLOGY INTEGRATION IN MATHEMATICS CLASSES

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This study investigated the factors contributing to preservice teachers' (PTs') endorsement of technology-integrated mathematics classes. The sample included 91 PTs at two normal universities in Taiwan. Five factors were identified through exploratory factor analysis. From highest to lowest degree of endorsement among the PTs, the factors were innovating instruction, developing interpersonal 21st-century skills, developing self-directed learning skills, developing positive attitudes, and detailing explanations. The results also revealed that the factors' influence on the PTs' endorsements differed according to the teaching context.

INTRODUCTION

In the present digital age, technology inevitably influences the organizational and material resources for mathematics and shapes people's thinking in mathematics training. Researchers have argued that students should acquire mathematical competences relevant to problem-centered approaches through the efficient use of technology in accordance with the prevalence of STEM education and 21st-century skills (Jankvist, Misfeldt, & Aguilar, 2019). In addition, teachers should include technology appropriately when teaching mathematics to aid students' understanding and develop students' technologically instrumented action schemes in mathematical competences (Voogt & Pareja, 2010).

In accordance with international trends, Taiwan launched a new mathematics curriculum in 2019. The curriculum emphasizes teachers' use of technology in class and the development of students' mathematical competences in using technology in mathematics learning and problem solving. However, promoting the use of technology in mathematics in the teacher community is difficult. The emphasis on academic achievement rooted in Confucian heritage culture has resulted in examination-oriented and fast-paced mathematics instruction (Leung, 2001). Many teachers believe that using technology wastes time and jeopardizes examination preparation.

In accordance with the new curriculum, teacher preparation institutions in Taiwan have begun training preservice teachers (PTs) to adapt to new mathematics teaching environments. PTs are crucial for successful educational reform; thus, their willingness to use technology in mathematics classes is critical. Therefore, an investigation of PTs' endorsement of the functions that technology can provide in mathematics classes is beneficial for teacher educators to understand how PTs can

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 217-224. Khon Kaen, Thailand: PME.

be encouraged to adopt technology. The present paper addresses the following research questions:

RQ1. What factors contribute to PTs' endorsement of technology integration in mathematics classes?

RQ2. To what degree do PTs endorse the factors identified in RQ1? Does their degree of endorsement differ among various technology-integrated teaching contexts?

RESEARCH METHOD

Conceptual framework

The framework for exploring PTs' endorsement of technology integration in mathematics class included three dimensions: the help of technology on the cultivation of students' mathematical literacy, the help of technology on pedagogy, and impact of technology (Figure 1). The items in each dimension were selected on the basis of a literature review.

Cultivation of student mathematical literacy

Researchers have referred to mathematical literacy by using various terms, such as mathematical competence or proficiency. Mathematical literacy is considered an essential skill that every future citizen must develop. Niss and Højgaard (2011) proposed a list of mathematical competences characterized by mathematical thinking rather than specific mathematical topics, such as reasoning or representing mathematically. In addition to the thought-oriented competence, content-oriented mathematical competence related to specific mathematical topics, such as factual knowledge, is also included mathematical literacy. Kilpatrick and colleagues (2001) indicated the importance of the affective facet of mathematical competence, such as productive disposition. To address problems in the current digital and rapidly changing age, PISA 2021 mathematics framework includes 21st-century skills (OECD, 2018). Technology integration has been identified as helpful in the cultivation of students' mathematical literacy (Zbiek et al., 2007).

Pedagogy

Traditional mathematics classrooms are led by teachers and based on lectures, during which teachers act as initiators or controllers to convey knowledge to students (Philipsen et al., 2019). Researchers have promoted the transformation of teachers' roles from initiators or controllers to facilitators and that of classroom environments from teacher led to student centered (Bray & Tangney, 2017). Teachers are expected to provide students with opportunities to flexibly inquire and engage in mathematics activities (Mishra & Koehler, 2006). Technology has the potential to increase instructional quality and facilitate students' autonomous investigation, exploration, cooperation, and communication (Hsu, 2008).

Impact of technology

Puentedura (2006) introduced the SAMR hierarchical model to illustrate the effect of technology adoption on teaching and learning. The SAMR model comprises four levels. In the **s**ubstitution level, technology acts as a direct substitute for a traditional method without functional change (e.g., saving time spent writing on a board). In the **a**ugmentation level, technology acts as a substitute for an existing tool that offers functional improvements (e.g., increasing accuracy through the use of GeoGebra). In the **m**odification level, technology facilitates a significant change in task design (e.g., focusing on student exploration). The **r**edefinition level describes a situation in which new tasks that were previously inconceivable can be performed using technology (e.g., connecting different representations naturally).

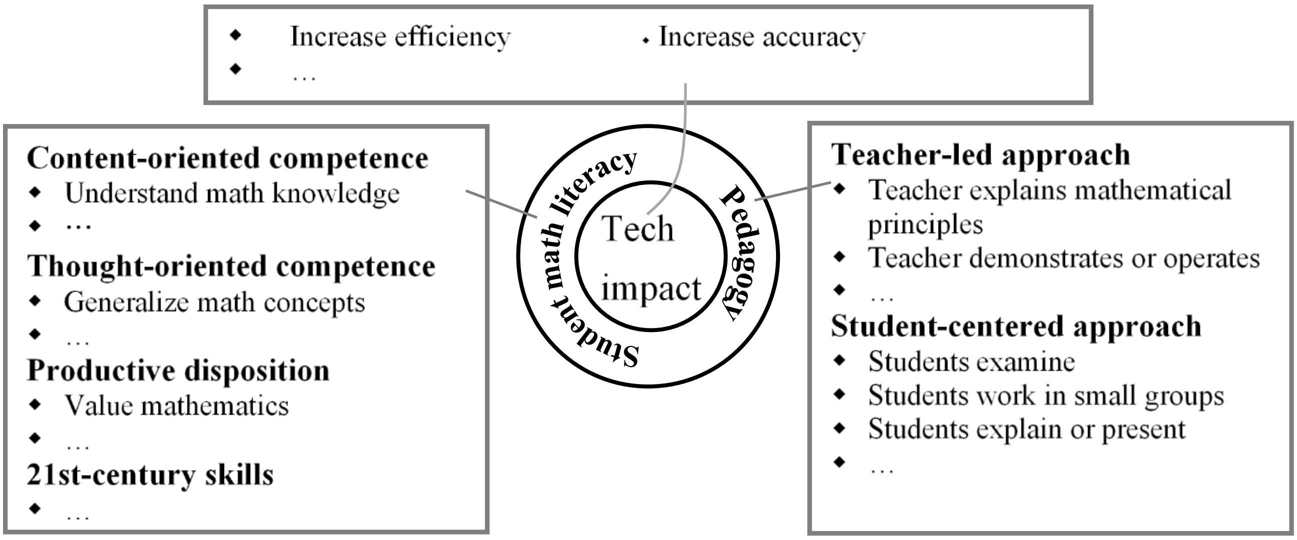

Figure 1: Conceptual framework of this study

Instrument

PTs' endorsement of technology integration in mathematics classes was investigated using a questionnaire. Five vignettes of technology-integrated mathematics classes were created on the basis of content in mathematics textbooks. The vignettes described using calculators to develop students' concepts of logarithms (Teaching context 1), using Desmos/GeoGebra to develop students' understanding of characteristics of graphs of logarithmic functions (Teaching context 2), using Desmos/GeoGebra to help students understand the algorithm $\log_a \mathrm{b} = \frac{\log b}{\log a}$ (Teaching context 3), using Excel to develop students' mathematical competence in modeling with exponential functions (Teaching context 4), and using Desmos/GeoGebra to help students learn to use polar equations and curves to design figures (Teaching context 5). The educational goals of using technology ranged from developing basic ideas to cultivating higher-order mathematical competence. For each vignette, three sets of dichotomous items corresponding to the conceptual framework were designed. The first set, corresponding to the dimension of *mathematics literacy*, contained nine items (ML1–ML9). The prompt was "some teachers believe that the integration of technology helps achieving the following educational goals of mathematics learning; which help is the reason for why you like this technology-integrated math class? Check all that apply." For example, for the item "understanding mathematical knowledge," if a PT likes the technology-integrated mathematics class because the use of technology facilitates the development of student understanding of mathematical knowledge, then he/she can check the item. The second and third sets of items, corresponding to the *pedagogy* and *technology impact* dimensions, consisted of 16 (P1–P16) and 5 (TI1–TI5) items, respectively. Their prompts were similar to that for the first set. This study explored PTs' endorsement of technology integration by using teaching vignettes rather than general questions without context because we aimed to observe the PTs' true perceptions of concrete teaching situations instead of abstract or ideal concepts.

Participants

We surveyed 91 secondary mathematics PTs from two of Taiwan's three normal universities. At the two universities, the teacher preparation programs consisted of two classes of third-year students and two classes of fourth-year students. Four classes of PTs (one class for each year at each university) participated in this study.

Data analysis

For RQ1, exploratory factor analysis (EFA) with oblique rotation was performed using MPlus to determine the factor structures of PTs' endorsement of technology integration in mathematics classes. The exploratory and data-driven approach of EFA was suitable for this study because hypothesized structures were absent. In the analysis, the PTs' responses for the five vignettes were aggregated to present their endorsement of the item content. For example, the degree to which a PT endorsed "understanding mathematical knowledge" was the number of times they check this item among the five vignettes. EFA was performed on the aggregated data. The model fit was evaluated using the comparative fit index (CFI), the Tucker–Lewis Index (TLI), root mean square error of approximation (RMSEA), and the standardized root mean square residual (SRMR). CFI ≥ 0.90, TLI ≥ 0.90, RMSEA ≤ 0.08, and SRMR ≤ 0.05 indicated good fit (Kline, 2011).

For RQ2, the average percentage of checking (POC) for each item and latent factor were computed for each teaching vignette, and the POCs of the five vignettes were averaged to obtain the overall endorsement for each item and latent factor.

RESEARCH FINDINGS

Factor structure (RQ1)

The EFA of the 30 items yielded six factors, one of which included only two items related to teacher questioning in the teacher-led approach. The two items were deleted because Cronbach's α for the factor is not high enough (0.724). EFA was re-performed on the remaining 28 items and five factors were yielded (Table 1). The factors explained 60% of the total variance, and the model fit was good (CFI = 0.917, TLI = 0.907, RMSEA = 0.066, SRMR = 0.043). All the factor loadings were adequate (≥ 0.3).

The first factor, *innovating instruction to develop students' mathematical competence*, represented technology integration to change traditional mathematics classes by arranging student-centered activities, such as conjecture, exploration, and experiments, and providing students with meaningful learning opportunities through teachers' operation of technological tools to help students visualize concepts and connect representations naturally. Students thus efficiently develop mathematical competence in these learning experiences. Unlike the first factor, the second factor, *detailing explanations to deepen students' understanding*, represented technology integration to support traditional teacher-led instruction, including improving their elaboration of mathematical ideas, increasing accuracy of graphs or calculations, and providing instant feedback. The goal of student learning is focused on content-oriented competences. The third factor, *developing students' positive attitudes and valuation of math*, is mainly relevant for the affective facet. This factor represented technology integration to develop students' positive attitudes toward learning mathematics and recognition of the value of mathematics. The remaining two factors concerned the development of students' higher-order competences. The fourth factor, *developing students' self-directed learning skills*, represented technology integration to allow students to inquire problems, examine and provide evidence for solutions, and transfer their experiences to other situations. The fifth factor, *developing students' interpersonal 21st-century skills*, represented technology integration to develop students' communication skills required for the 21st century through discussion, cooperation, and presentation (OECD, 2018). The fourth factor reflected the perspective of radical constructivism, whereas the fifth factor reflected the perspective of social constructivism.

PTs' endorsement of factors of technology integration in math classes (RQ2)

The average POCs of the five factors in descending order were 0.85 for *innovating instruction* (F1), 0.83 for *developing interpersonal 21st-century skills* (F5), 0.81 for *developing self-directed learning skills* (F4), 0.81 for *developing positive attitudes* (F3), and 0.76 for *detailing explanations* (F2). These high averages indicated that all five factors are reasons for PTs to endorse a technology-integrated mathematics class. The most influential factor was F1, indicating that technology transforms traditional classes into innovative student-centered classes. Technology's functions to develop students' higher-order competences (F4 and F5) and positive attitudes (F3) were also appreciated. Technology use to support traditional teacher-led teaching (F2) was least selected as a reason for endorsing a technology-integrated class. Several items (those with POCs > 0.90) were especially endorsed by the PTs. They endorsed using technology to allow students to do hands-on activities (P9), observe (TI3), and explore and experiment (P11). They cared about students' visualization of concepts (ML3) and focus on thinking rather than repeated routine work (TI3), and they emphasized developing students' mathematical competence (ML6) and 21st-century skills (ML7).

Figure 2 presents the PTs' degree of endorsement of each factor in the various teaching contexts. For all teaching contexts, *innovative instruction* (F1) ranked first or second, indicating its strong influence. The most critical reason for the PTs to endorse teaching context 4 was that the use of technology can develop students' communication skills. In the vignette of teaching context 4, a teacher works with students to determine a mathematical model for the growth of the African sacred ibis population. They use Excel to generate graphs and compare the obtained exponential model with actual data; subsequently, they discuss the meaning of mathematical modeling. The students are then asked to work in groups to model the popularity of search queries of the word "mathematics" using Excel. The two tasks are difficult and unfamiliar to the students. For this vignette, the PTs endorsed the ability of technology to promote cooperation and the presentation and exchange of mathematical ideas among the students. Teaching context 3 represented a brand new idea in the textbook. In the vignette, the teacher asks students to use Desmos/GeoGebra to graph the functions $f_1(x) = \log_a x$ and $f_2(x) = \frac{\log x}{\log a}$ to prove that $\log_a x = \frac{\log x}{\log a}$. Compared with the formal and symbolic approach usually used in Taiwan, proving a concept through manipulation, observation, conjecture, and justification with graphs provides students with an opportunity to access knowledge and increases their willingness to engage in the learning process. This might explain why the PTs considered this use of technology to be helpful in developing students' positive attitudes. The case in teaching context 1 is similar.

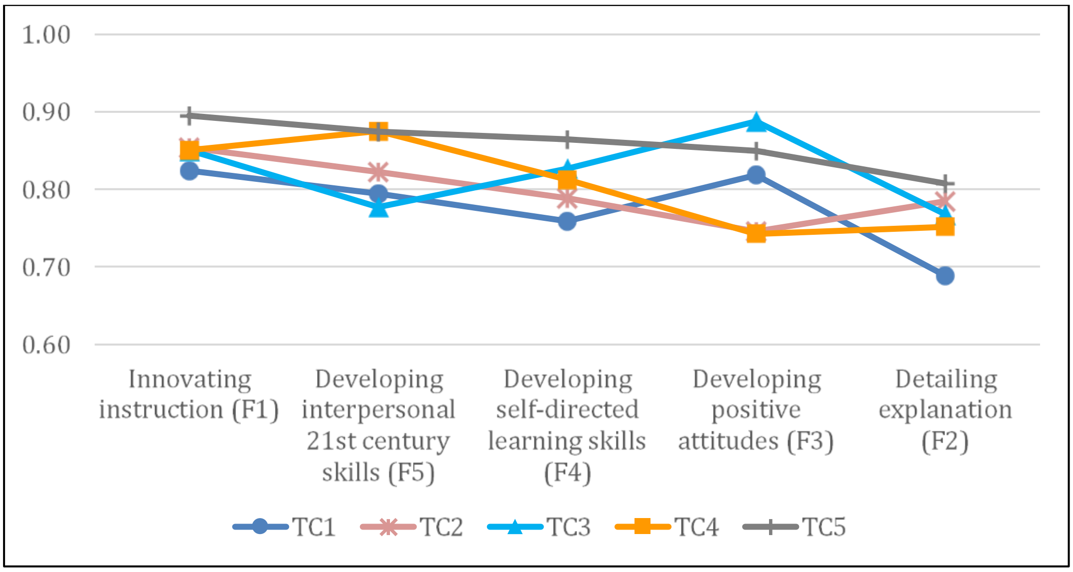

Figure 2: PTs' degree of endorsement of the factors in various teaching contexts. TC = teaching context. Details for each teaching context are presented in the Instrument section.

No.	Item description	Loading	POC
F1: Innovating instruction to develop students' mathematical competence			**0.85**
ML6	Mathematical competence	0.633	0.96
TI3	Students focus on observation and thinking	0.593	0.93
P9	Students use manipulatives	0.843	0.92
ML3	Concretize or visualize mathematical concepts	0.507	0.91

P11	Students explore or experiment	0.802	0.91
TI5	Arrange previously inconceivable activities	0.314	0.85
TI1	Increase efficiency and time effectiveness	0.395	0.84
P10	Students conjecture	0.425	0.84
TI4	Representations connect naturally	0.507	0.83
P4	Teacher demonstrates or operates	0.458	0.73
ML4	Mathematical procedures or skills	0.507	0.69
F2: Detailing explanations to deepen students' understanding			**0.76**
TI2	Increase mathematical accuracy	0.395	0.87
P16	Provide student instant mathematical feedback	0.361	0.78
ML1	Understand mathematical knowledge	0.501	0.74
P2	Teacher elaborates on ideas to deepen student understanding	0.642	0.72
P1	Teacher explains mathematical principles	0.762	0.70
F3: Developing students' positive attitudes and valuation of mathematics			**0.81**
ML8	Positive mathematics learning attitudes	0.484	0.89
ML9	Recognition of the value of mathematics	1.033	0.88
P15	Assess student learning outcomes	0.445	0.75
ML2	Abstract or generalize mathematical concepts	0.314	0.72
F4: Developing students' self-directed learning skills			**0.81**
P13	Students provide evidence	0.979	0.86
P12	Students examine	0.577	0.85
ML5	Students learning transfer	0.301	0.81
P14	Students inquire	0.557	0.72
F5: Developing students' interpersonal 21st-century skills			**0.83**
ML7	21st-century skills	0.300	0.94
P6	Students discuss	0.698	0.85
P8	Students work in small groups	0.526	0.80
P7	Students explain or present	0.421	0.73

Note. Although the factor loading of ML9 is larger than 1, it can be retained in the factor because its residual variance is positive. Loading = factor loading. POC = percentage of checking.

Table 1: EFA loadings and percentages of checking for endorsements

CONCLUSION

Five factors contributed to PTs' endorsement of technology integration in mathematics classes, namely *innovating instruction*, *developing interpersonal 21st-century skills*, *developing self-directed learning skills*, *developing positive attitudes*, and *detailing explanations*. *Innovating instruction* and *detailing explanations* were the most and least influential factors, respectively. In various teaching contexts, the factors influenced the PTs' endorsement to different degrees. For difficult and complicated tasks, the PTs appreciated how technology can help promote student cooperation and discussion. They also endorsed technology as providing an alternative approach to proving mathematical concepts instead of symbolic and formal approaches that increases student engagement.

References

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research: A systematic review of recent trends. *Computers & Education, 114*, 255-273.

Hsu, Y. (2008). Learning about seasons in a technologically enhanced environment: The impact of teacher-guided and student-centered instructional approaches on the process of students' conceptual change. *Science Education, 92*(2), 320-344.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS procedures are objectified? —the case of "solve" and "desolve." *Educational Studies in Mathematics, 101*, 67-81.

Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*(1), 35-51.

Mishra, P. & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning*. Denmark: IMFUFA, Roskilde University.

OECD. (2018). *PISA 2021 mathematics framework draft*. Retrieved from https://pisa2021-maths.oecd.org/files/PISA%202021%20Mathematics%20Framework%20Draft.pdf

Philipsen, B., Tondeur, J., Roblin, N. P., Vanslambrouck, S., & Zhu, C. (2019). Improving teacher professional development for online and blended learning: a systematic meta-aggregative review. *Educational Technology Research and Development*, *67*, 1145-1174.

Puentedura, R. (2006). *Transformation, technology, and education.* Retrieved from http://hippasus.com/resources/tte/

Voogt, J., & Pareja, R. N. (2010). *21st century skills*. Enschede, the Netherlands: Universiteit Twente.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester, Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.1169-1207). Charlotte, NC: Information Age Publishing Inc.

ANSWER PATTERNS OF JAPANESE PRIMARY SCHOOL STUDENTS IN TIMSS 2015 MATHEMATICS SURVEY

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Through secondary analysis of PISA data, the answer patterns of students who have completed compulsory education in Japan have been partially established. However, the answer patterns of Japanese primary school students have not yet been clarified. Therefore, in this study, we conducted a secondary analysis of TIMSS 2015 data on mathematics for fourth-grade primary school students to determine their answer patterns. We performed an international comparative analysis involving 13 countries and areas targeted in previous studies. The results show that Japanese primary school students had a peculiar answer pattern among the 13 countries and areas, and in particular, the calculations were found to be easier than for the students of other countries; the underlying reasons for this must be clarified in future research.

INTRODUCTION

Large-scale international educational assessments such as PISA and TIMSS are influential for mathematics education in Japan (e.g., Nakayasu, 2016; Volante, 2015). PISA and TIMSS show the mathematics achievement level of Japanese students with international comparative scales (e.g., Mullis et al, 2016; OECD, 2016). In addition, they provide data resources useful for conducting secondary analysis to gain new insight into students' mathematics achievements.

For example, Suzukawa et al. (2008) analysed PISA 2003 data to reveal the Japanese students' answer patterns through comparison with data on 13 countries and areas: Australia, Canada, Finland, France, Germany, Hong Kong, Ireland, Italy, Japan, Korea, the Netherlands, New Zealand, and the United States. Their results indicate that Japanese students had peculiar answer patterns and were especially good at solving questions in the 'educational' context of the PISA framework. Conversely, questions in the 'social' context were difficult for them. Following this research, Watanabe (2019, 2020) analysed the data from PISA 2012 and PISA 2015 and identified that Japan still had a peculiar answer pattern overall among the same 13 countries and areas. Further, they identified a partial change: item difficulty in the mathematical content 'uncertainty and data' decreased between PISA 2003 and PISA 2015. It was mentioned that this partial change was due to a 2008 revision in the Japanese mathematics curriculum to include topics related to statistics and probability. In summary, the previous studies reported on the answer pattern of

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Japanese students and implied a relationship between their achievement and the revised mathematics curriculum in Japan.

However, the previous studies analysed the PISA data and examined the answer patterns of 15-year-old students near the end of compulsory education. They did not focus on the answer patterns of students in the middle of compulsory education. Japan participates in the fourth- and eighth-grade mathematics surveys of TIMSS. Hence, TIMSS data (https://timssandpirls.bc.edu/timss2015/international-database/) can be used to examine the answer patterns of the students in the middle of compulsory education, especially focusing on the primary school level.

Thus, this study aimed to reveal the answer patterns of Japanese primary school students through a secondary analysis of TIMSS 2015 fourth-grade mathematics data. The methods used by Suzukawa et al. (2008) and Watanabe (2019, 2020) were adopted to conduct the data analysis. Comparisons with data for the same 13 countries and areas were conducted by applying item response theory (IRT).

METHODS

The present analysis targeted 74,411 fourth-grade students from 13 countries and areas. In TIMSS 2015, 14 different booklets were prepared, and 169 items were given to measure the achievement in mathematics learned in school. One item (item code M061239) was not applicable and hence excluded from scaling at the national level in France (Martin et al, 2016); this item was excluded from this analysis to ensure the precision of comparison. Therefore, 168 items were targeted. Additionally, 9 out of 168 items had partial credit, and answers with partial credit were treated as incorrect to avoid complicating the data analysis. A binary dataset (1 for a correct answer and 0 for an incorrect answer, non-response, or missing answer) was built for this study. Incidentally, items that were not included in the booklets given to the students were regarded as NA (not available) during the statistical analysis that was performed using the statistical data analysis software R version 3.6.1.

This analysis applied the Rasch model of IRT, which is expressed as follows:

$$p_i(\theta) = \frac{1}{1 + \exp(-Da(\theta - b_i))}$$

where θ is the latent trait of ability, p_i denotes the probability whether an answer to item i is correct, b_i denotes the difficulty parameter of item i, and $D = 1.0$ and $a = 1.702$ are constants. The item difficulty parameters b_i (i=1, 2, …,168) of 168 items were estimated for each country by using Rasch model and compared by equating to the scale of Japan with the mean-sigma method. More specifically, let the item difficulties of item i for Japan and country k (k=1, 2, …, 13) be b_{iJPN} and b_{ik}, respectively, and let the mean values of item difficulties be $\overline{b_{JPN}} = \frac{1}{168}\sum_{i=1}^{168} b_{iJPN}$ and $\overline{b_k} = \frac{1}{168}\sum_{i=1}^{168} b_{ik}$, respectively. Then, the item difficulties equated to Japanese scale are defined as $b_{ik}^* = b_{ik} + (\overline{b_{JPN}} - \overline{b_k})$. As $\overline{b_k^*} = \overline{b_{JPN}}$ can be obtained, the mean value of equated item difficulties is combined into $\overline{b_{JPN}}$. Let the mean value of item i in 13 countries be $\overline{b_i^*} = 1/13 \sum_{k=1}^{13} b_{ik}^*$. Assuming a country with the difficulty $\overline{b_i^*}$ for each item, it is possible to set up a country with an average pattern of item difficulty from the 13 countries (hereinafter referred to as 'average country'). Given the difference in item difficulty between each of the 13 countries and the average country, $d_{ik} = b_{ik}^* - \overline{b_i^*}$, we obtain $\overline{d_k} = \overline{d_i} = 0$. In this manner, d_{ik} is obtained as a standardised item difficulty for each country and item and can be used as an indicator of peculiarity of an answer pattern in comparison to the 13 countries. This analysis focuses on d_{ik} to detect the answer pattern of 13 countries. The analysis mainly used the packages 'ltm' and 'plink' in the statistical data analysis software R version 3.6.1 (Rizopoulos, 2018; Weeks, 2017).

RESULTS

Overall features of answer pattern in 13 countries

Let the standard deviation of d_{ik} be $s_k = \sqrt{1/168 \sum_{i=1}^{168} d_{ik}^2}$, where s_k is an indicator of differences in the overall level of item difficulty for the 13 countries. A larger s_k indicates a more divergent answer pattern for the corresponding country. Table 1 lists the value of s_k obtained in the present study and that obtained by Watanabe (2020) for PISA 2015; Figure 1 shows its scatter plot.

Table 1 and Figure 1 indicate that Asian countries, represented by Korea, Japan, and Hong Kong, have divergent answer patterns in PISA 2015. Japan has a high s_k value

Country	TIMSS 2015	PISA 2015
KOR	0.511	0.397
HKG	0.509	0.278
DEU	0.480	0.167
NLD	0.471	0.235
JPN	0.408	0.368
FIN	0.331	0.259
ITA	0.311	0.278
NZL	0.306	0.148
USA	0.298	0.214
FRA	0.285	0.175
IRL	0.281	0.219
AUS	0.276	0.161
CAN	0.215	0.133

Table 1: Values of s_k

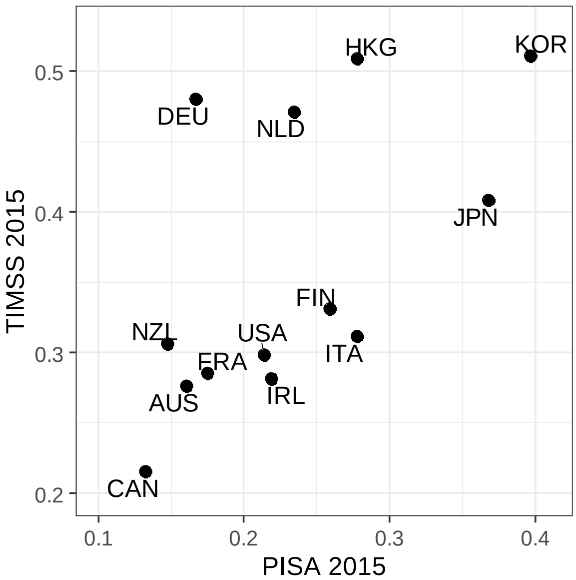


Figure 1: Scatter plot of s_k

in TIMSS 2015, less than those for Korea, Hong Kong, Germany, and the Netherlands. Thus, although not as pronounced as in PISA 2015, Japan can be judged to have a peculiar answer pattern among the 13 countries in TIMSS 2015.

Relationship between Item Difficulty and Item Content

The items in TIMSS 2015 are characterised by two aspects: content domains and cognitive domains. The characteristics of the Japanese answer pattern were examined by focusing on these two aspects. The content domains contain three types of content: numbers (89 items), geometric shapes and measures (56 items), and data display (23 items). The cognitive domains include three content areas: knowing (64 items), applying (71 items), and reasoning (33 items). The details of these definitions are provided in Mullis et al. (2013, pp.13–27). The distribution of d_{ik} for each of these two aspects was checked using a boxplot.

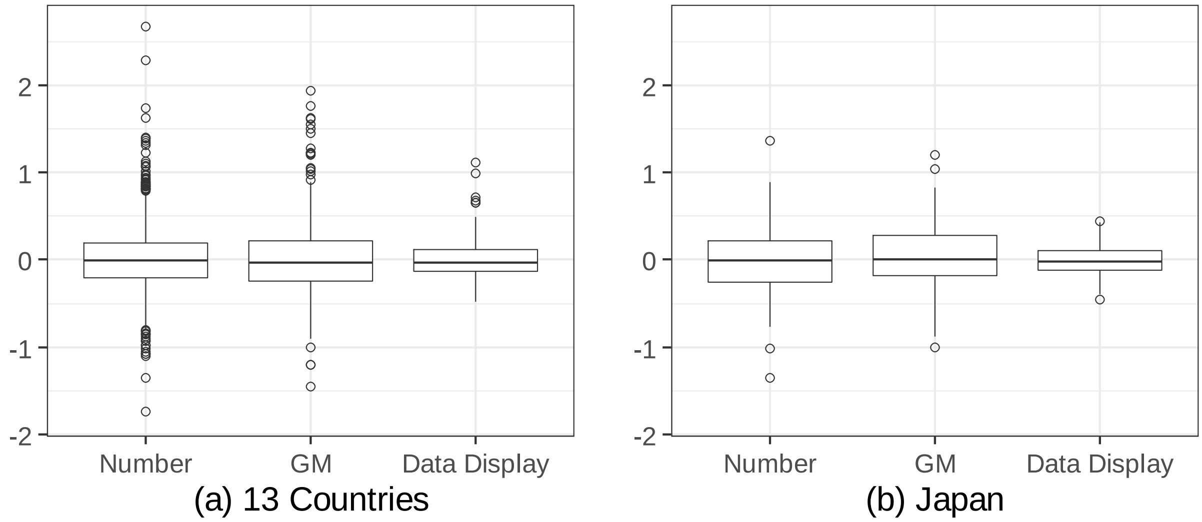


Figure 2: Boxplots for the content domains (GM stands for geometric shapes and measures)

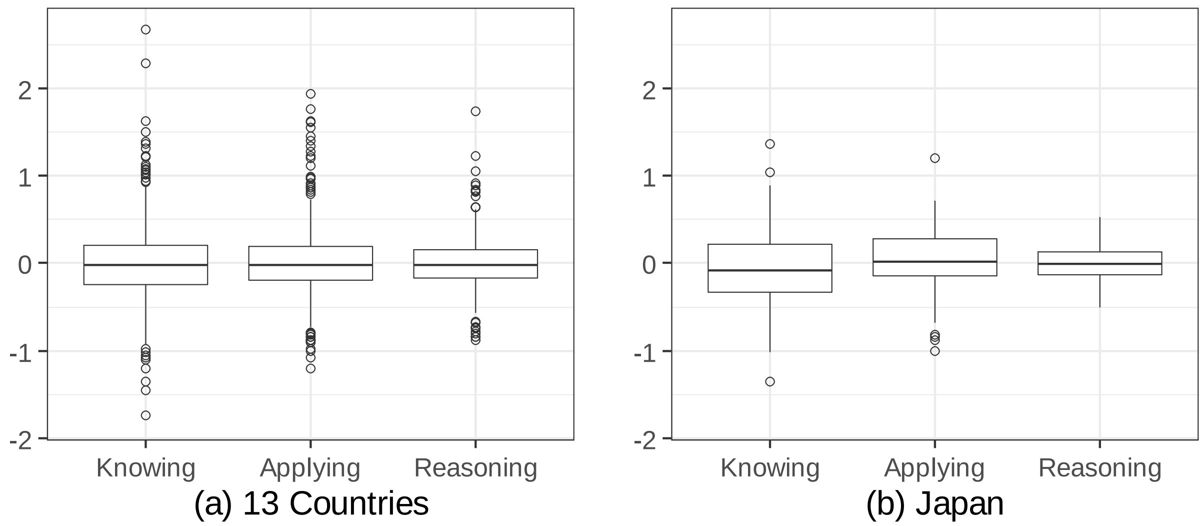


Figure 3: Boxplots for the cognitive domains

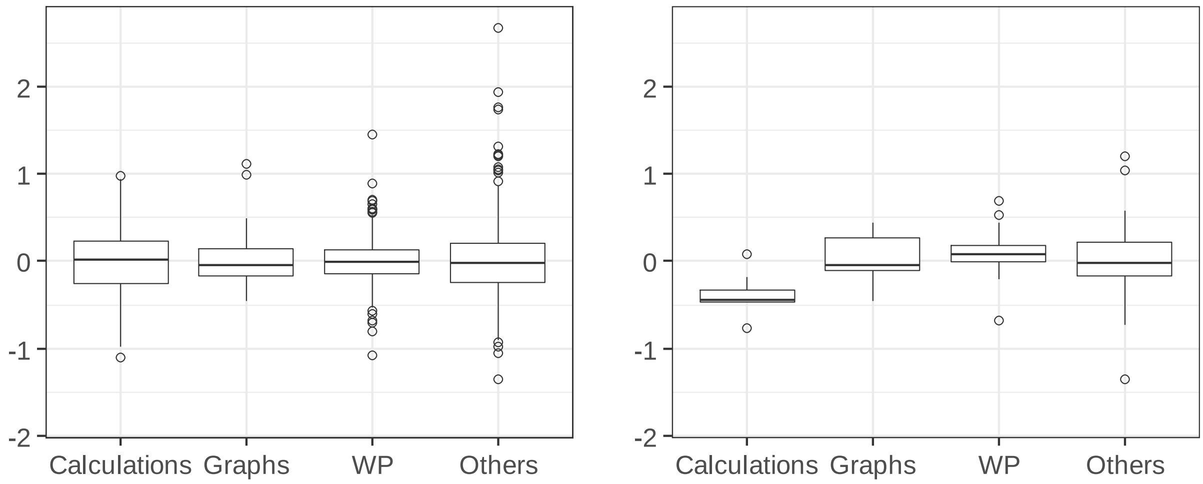
Figures 2 and 3 show the boxplots of the content domains and cognitive domains, respectively. The boxplots in (a) of Figures 2 and 3 depict the distribution for all 13 countries, while those in (b) depict the distribution for Japan. The median values for content and cognitive domains across the 13 countries were close to 0.0. This feature can be observed for Japan, and there is not much difference in the length of the boxes between all 13 countries and Japan. In other words, determining the characteristics of Japanese answer patterns in the TIMSS 2015 framework is challenging.

Analysis focused on released items

In total, 76 out of 168 items were released to the public. The content of the released items was checked, and the 76 items were independently categorised into four types in this analysis: calculations (8 items), graphs (6 items), written problems (24 items), and others (38 items). The items categorised as calculations are listed in Table 2. Thus, the calculations were categorised as addition, subtraction, and division of integers and decimals, as well as solving simple equations.

Item code	Summary of item content
M041087	Addition: $0.36 + 0.77$
M041096	Finding the number that goes into $\square$: $\square - 87 = 23$
M041280	Division: $1362 \div 32$
M041291	Subtraction: $428 - 176$
M051017	Subtraction: $52093 - 4136$
M051205	Subtraction: $4809 - 532$
M061050	Finding the number that goes into $\square$: $6 + 15 = \square + 10$
M061272	Addition: 43+5

Table 2: The items of the type of calculations (Source: NIER, 2017)



(a) 13 Countries (b) Japan

Figure 4: Boxplots for four types of released items (WP stands for written problems)

Figure 4 shows that the distribution of calculations for Japan is clearly lower than that for all 13 countries. This implies that Japanese students solve these items easily. As a characteristic of Japanese answer patterns was found in calculations, we examined the distribution of item difficulty in each country focusing on the items of calculations.

Figure 5 shows the distribution of item difficulty for calculations for each country on a number line. The figure indicates that the distribution of item difficulty for Japan is mostly to the left among the 13 countries. Therefore, the items categorised as calculations are easy for Japanese students, and this is one of the characteristics of the answer pattern of Japanese primary school students.

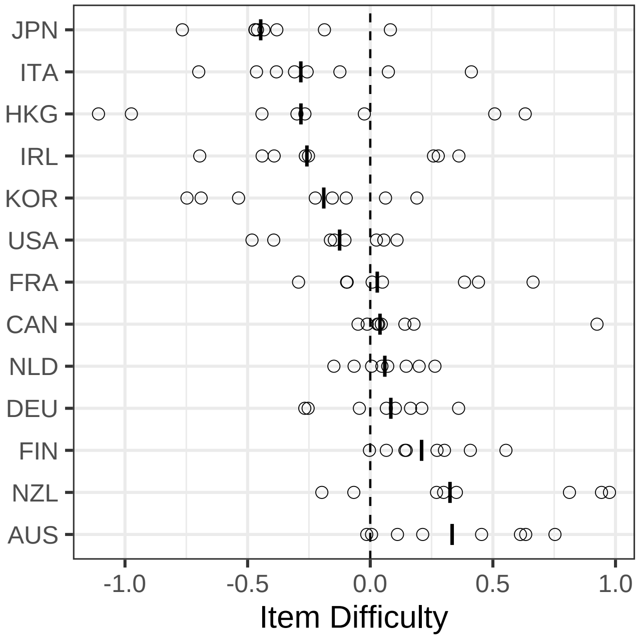


Figure 5: Distribution of item difficulty for calculations (where | is the median value)

DISCUSSION

We conducted a secondary analysis of TIMSS 2015 fourth-grade mathematics data to identify the characteristics of the answer pattern of Japanese primary school students. The standardised item difficulties d_{ik} were calculated, based on which an international comparative analysis was conducted.

In summary, the following two points can be identified as the main findings. First, the overall answer pattern at the fourth-grade primary school level in Japan was found to be peculiar among the 13 countries but not as pronounced as PISA 2015 (Table 1 and Figure 1). Second, although the TIMSS 2015 framework did not capture the details of answer patterns of Japanese students, we developed our own category focusing on the released items and found that the items categorised as calculations, which are listed in Table 2, were particularly easy for Japanese students (Figures 4 and 5). In particular, the second finding is a characteristic of a specific answer pattern at the primary school level in Japan. Conversely, items corresponding to the other types in our category, such as written problems, can be said to be relatively more difficult for Japanese students.

This study was successful to some extent in capturing the answer patterns of Japanese students at the primary school level, similar to the studies performed by Suzukawa at el. (2008) and Watanabe (2019, 2020), who investigated PISA data and showed the answer patterns of Japanese students completing compulsory education.

However, the reason why the calculations were found to be easy and written problems were found to be relatively difficult for Japanese primary school students needs to be investigated in the future.

CONCLUSION

Through this study, we were able to establish the specific answer patterns of Japanese students at the fourth-grade primary school level. For example, calculations were found to be easy for Japanese students. However, this is a result obtained using a rough category that we developed independently based on the contents of the released items of TIMSS 2015. We believe that efforts are needed to refine this category and capture the characteristics of Japanese answer patterns in more detail; for example, it needs to be determined why the Japanese primary school students found the calculations to be easy.

In addition, TIMSS is conducted every 4 years, and TIMSS data are being accumulated. In other words, it is possible to capture changes over time. In particular, the spread of the infectious disease COVID-19 has had a significant impact on schooling, forcing schools to close. It is presumed that this will have an impact on students' actual achievements. To obtain a picture of this impact, we can analyse and contrast the TIMSS data before and after COVID-19. This study is expected to help in drawing a contrast between students' achievements before and after COVID-19. Future work will include a refinement of our category to enhance the contrast with TIMSS 2023, which is scheduled to be implemented in 2023.

Acknowledgment

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References

Martin, M. O., Mullis, I. V. S., & Hooper, M. (2016). *Methods and Procedures in TIMSS 2015*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mullis, I. V. S., & Martin, M. O. (2013). *TIMSS 2015 Assessment Frameworks*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International Results in Mathematics*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Nakayasu, C. (2016). School Curriculum in Japan. *The Curriculum Journal*, 27(1), pp. 134-150.

NIER (National Institute for Educational Policy Research) (2017). *TIMSS2015 sansuu suugaku kyouiku rika kyouiku no kokusai hikaku: kokusai suugaku rika kyouiku doukou chousa no 2015nenn chousa houkokusho [TIMSS 2015 International Comparison of Mathematics and Science Education]*. Tokyo: Akashishobou. (in Japanese).

OECD (2016). *PISA 2015 Results (Volume I). Excellence and Equity in Education*. PISA: OECD Publishing, Paris.

Rizopoulos, D. (2018). Package 'ltm': Latent Trait Models under IRT. Retrieved from https://cran.r-project.org/web/packages/ltm/ltm.pdf.

Suzukawa, Y., Toyoda, H., & Kawahashi, I. (2008). Do Students in Japan Learn How to Use Mathematics in Real-Life Situation? An Analysis of Items in the 2003 Programme for International Student Assessment (PISA 2003). *The Japanese Journal of Educational Psychology*, 56(2), pp. 206-217. (in Japanese).

Volante, L. (2015). The Impact of PISA on Education Governance: Some Insights from Highly Reactive Policy Contexts. *International Studies in Educational Administration*, 43(2), pp. 103-117.

Watanabe, K. (2019). A Study on the Secular Changes of Japanese Students' Mathematical Literacy: Focusing on the Pattern of Item Difficulty Between PISA 2003 and PISA 2012. *Journal of Japan Academic Society of Mathematics Education* (*JASME*)*: Research in Mathematics Education*, 25(1), pp. 1-13. (in Japanese).

Watanabe, K. (2020). A Study on the Changes of Japanese Students' PISA Mathematical Literacy: Focusing on the Content Category of "Uncertainty and Data" on PISA 2003, PISA 2012 and PISA 2015. *Journal of Japan Academic Society of Mathematics Education* (*JASME*)*: Research in Mathematics Education*, 26(1), pp. 1-12. (in Japanese).

Weeks, J. P. (2017). Package 'plink': IRT Separate Calibration Linking Methods. Retrieved from https://cran.r-project.org/web/packages/plink/plink.pdf.

WHAT ARE PARENTAL CONTROLS AND HOW CAN THEY AFFECT CHILDREN'S MATHEMATICS ACHIEVEMENT?

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This paper focuses on the implementation of parental controls as a result of parental involvement in children's mathematics education. The data used were responses from secondary school students (N=128) and their parents (N=85) who live in Melbourne, Australia. The data collection process involved online questionnaires and semi structured face-to-face interviews. A conceptual framework was designed to guide the study. Parental perceptions together with parental controls and students' mathematics outcomes are discussed with respect to culture, gender, and year level.

INTRODUCTION

Parental involvement in their children's education has captivated the attention of the world for some time. The impetus for this research and my motivation to conduct it emerged from readings as well as my life experiences as a student in Sri Lanka – an Asian country, a mathematics teacher, and an immigrant parent in Australia. Different parents may involve themselves differently with their children for numerous reasons. Within the same family, the parenting styles of mother and father may not be the same. Several decades of research have demonstrated that parental controls in children's education are associated with a variety of positive and negative academic and motivational outcomes. Yet, there is no universal pattern of parent controls that results in higher achievement, nor do all forms of involvement enhance learning outcomes (Jeynes, 2011). This paper aims to elucidate the impact of parental involvement in mathematics education resulting in either positive or negative outcomes, which may be perceived by children as support or pressure.

It is argued as well that parental controls may matter more for some children than for others. There are Asian parents, for example, who are often reported to spend time each day in monitoring the academic activities of their children (Fu & Markus, 2014). In her controversial memoir entitled *Battle hymn of the tiger mother*, Chua (2011) depicted a Chinese model of parenting. The term "Tiger mother" self-proclaimed by Chua is sometimes used to describe an authoritarian parenting style in which parents give their children few choices and seldom ask children for opinions (Baumrind, 1967). It is not only Chinese mothers who act as "Tiger mothers", for example, it seems that some non-Chinese parents from other Asian countries such as Korea, Vietnam, India, Bangladesh, and Sri Lanka have similar mindsets. The well-prepared offspring of these "Tiger mothers" seem to be outperforming non-Asian counterparts at schools where both Asian and non-Asian ethnic background students study together (Fu & Markus, 2014).

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 233-240. Khon Kaen, Thailand: PME.

CONCEPTUAL FRAMEWORK

While reviewing research on parental involvement factors and parenting styles, it was found that the theories and concepts employed in previous studies are inter-related and parental involvement factors are directly or indirectly related to the academic achievement of the parents' children. Because of such complexities in the concepts in literature, the methodology of this research involved a conceptual framework and a sequential explanatory mixed methods design to analyse and interpret the data gathered.

In previous research, family rules at home, perceived parental control, and material deprivation were found to be both positive and negative predictors in relation to parental controls (Baxter, Bylund, Imes, & Routsong, 2009). Material deprivation experienced by the child at home is lack of material benefits that are considered to be basic necessities. With regard to parents and children in the study of this digital era, for the appositeness to the study, material deprivation was modified and used as digital deprivation which is defined as inaccessibility to social media and equipment such as computer games, television, and mobile phones due to non-availability within premises or prohibition by parents.

In the conceptual framework, parental attributes such as attitudes, beliefs, expectations, aspirations, values, and academic standards are collectively considered as parental perceptions, which seem to be varied among parents and their cultures. These attributes may influence family rules, perceived parental control, and digital deprivation combined as parental control, and children's perceptions in mathematics achievement. Children's perceptions due to parental control factors may also be divided into positives and negatives depending on how these factors influence children. The possible connections among parental perceptions and control factors together with children's achievement are displayed in the conceptual framework in Figure 1, which shows how these factors may be related.

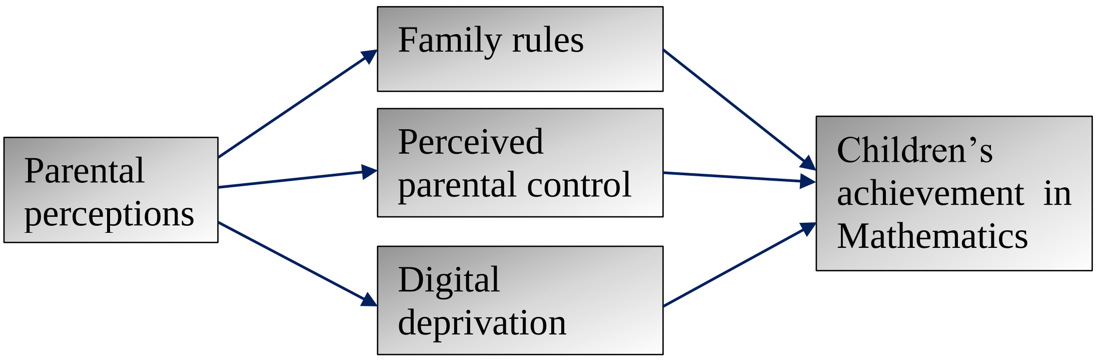

Figure 1: Conceptual framework of parental perceptions, parental controls, and mathematics achievement of children

Even though the conceptual framework guides and shapes the study, initial quantitative results may be inadequate by themselves to describe positive and negative outcomes of students. Therefore, qualitative data are used to explain quantitative results. As a result, the study involves a mixed methods approach. The following are the formulated research questions.

What are the relations among parents' perceptions, parental controls, and children's mathematics achievement in relation to culture, gender, and year level? How do parental controls affect mathematics achievement of children?

METHOD

Questions in both parent and student surveys were based on mathematics education of children. With the permission of education authorities and principals, the consent forms and invitation letters were sent to schools and hard copies of the student questionnaire were distributed to secondary students in three different schools in Melbourne, Australia without being selective regarding their ethnic background, gender, or secondary year level. The schools have a multicultural population of students. There were different groups or clusters of participants in this study. They were both male or female secondary students from Year 7 to Year 12 and their male or female parents from the sets of Asian and European backgrounds who live in Australia. Students were asked to take a copy of the parental questionnaire home and hand it back to their teacher with at least one of the parents' responses. The length of each questionnaire was kept to 20 minutes approximately and they were similar. In addition, it was possible to get permission from school principals to upload the questionnaires and make them available on school websites. This was convenient and enabled students and parents to respond to the questionnaires whenever it suited them.

After the survey using both student and parent questionnaires, from the participants who completed the survey I interviewed a purposefully selected sample of four families (parent and child separately) from each group of European–Australian and Asian–Australian backgrounds. Hence, there were sixteen interviews in total. Interviews were the main means of collecting qualitative data though it was of interest to use responses to the descriptive questions in the questionnaires too. The interviewed participants were also participants in the surveyed sample. This ensured comparison between similar categories of data from both qualitative and quantitative types (Creswell, 2014). Interview questions for parents and children were also similar.

To analyse quantitative data, descriptive and statistical analysis techniques such as correlation, cross-tabulation, independent samples t-test, and analysis of variance (ANOVA) were used. Further, Confirmatory Factor Analysis (CFA) and structural equation modelling (SEM) were involved using the IBM SPSS AMOS Graphics [Version 22.0.0] software package. With the qualitative data, interviews were transcribed and content analysis techniques were employed using the QSR NVivo for Windows [Version 10.0.138.0]. Finally, the results of both quantitative and qualitative analyses were integrated together and interpretations were given to answer research questions. In the analysis, calculations were based on mean values, thus minimising the possible effects on results due to group differences in size. As observed in Year 7 to Year 12 classrooms and experienced with those students, it was evident that the learning needs and methods differed with age. Hence, it was appropriate to split them as junior secondary and senior secondary students to reduce the threats to validity and reliability. As a result, the participants selected for interviews were senior secondary students and their parents.

RESULTS

In quantitative data analysis of parents' data using CFA, final fit indices (n = 85, χ^2 value = 78.66, df = 67, p-value = .16, $RMSEA$ = .05, GFI = .90, RMR = .04, and CFI = .97) have satisfied the requirements of an appropriate CFA model. Composite factor scores found using the model were involved in comparisons in Table 1 show that there is a small positive correlation between parental perceptions and parental controls and a large positive correlation between parental perceptions and children's perceptions in mathematics achievement. Hence, an increase in parental perceptions results in an increase in both parental control and children's perceptions. There is no significant relation between parental controls and children's perceptions at the 0.05 level. Children's data show large, positive correlations between parental perceptions and children's perceptions, parental perceptions and parental controls, and parental controls and children's perceptions. This implies that children felt that they were controlled by their parents more than parents thought they controlled their children.

	M	*SD*	Parental perceptions	Parental controls	Children's perceptions
Parental perceptions	1.924	.497	-		
Parental control	2.332	.555	.236	-	
Children's perceptions	2.028	.542	.740	.108	-

Table 1**:** Correlations among parental perceptions, parental controls, and children's perceptions in mathematics achievement

Independent samples t-tests were used to compare differences between the two ethnic groups in relation to parental perceptions, parental control, and children's perceptions using parents' data. According to the Likert scale used a lower mean value means stronger agreement with the questions asked. Lower mean value in parental control of Asian–Australian parents in Table 2, indicated more control on their children. From the p - values, parental controls showed a significant difference between European–Australian and Asian–Australian participants at the 0.05 level. There was no significant difference between parental perceptions or children's perceptions in the two groups.

	European–Australian (*n* =30)		Asian–Australian (*n* =55)				
	M	SD	M	SD	t(83)	*p*	η^2
Parental perceptions	1.833	.522	1.973	.480	-1.240	.219	.018
Parental control	2.560	.537	2.207	.529	2.918	.005	.093
Children's perceptions	1.886	.558	2.106	.522	-1.818	.073	.038

Table 2: Comparison of ethnic group differences in parental control

The *p* - values in Table 3 for children's data indicate a significant difference in parental control between the two gender groups but related parental perceptions and children's perceptions do not show a significant difference. Further, parents' data do not show a significant difference in any of the three factors. This implies that parents felt that there were no differences in gender when controlling children but their children felt the opposite.

	Male (*n* = 57)		Female (*n* = 67)				
	M	*SD*	*M*	*SD*	*t*(122)	*p*	η^2
Parental perceptions	2.061	.589	1.993	.572	.657	.513	.005
Parental control	2.406	.628	2.660	.732	-2.060	.042	.049
Children's perceptions	2.246	.566	2.206	.617	.372	.711	.002

Table 3: Comparison of gender differences in parental control

Using parents' data, the output from one-way ANOVA across years levels with parental perceptions, parental controls, and children's perceptions show a statistically significant difference among Year 7 to Year 12 groups at the $p < 0.05$ level as shown in Table 4. Although there are some fluctuations in mean values, there was an overall upward trend or an increase in each of the three factors. Likewise, children's data show an upward trend and significant differences in all three factors across year levels. Higher mean values indicate that parental control as well as parents' and children's perceptions of control decreased with the increase in year level.

		Sum of Squares	*Mean Square*	*F*(5, 79)	*p*	η^2
Parental perceptions	Between Groups	3.228	.646	2.910	.018	.156
	Within Groups	17.525	.222			
Parental control	Between Groups	4.367	.873	3.205	.011	.169
	Within Groups	21.530	.273			
Children's perceptions	Between Groups	4.739	.948	3.759	.004	.192
	Within Groups	19.921	.252			

Table 4: One-way ANOVA for parental control among year levels

While some of the families interviewed had rules which were not particularly strict, others reported quite strict rules. The different levels of rules within and among families and the ways of implementing such rules imply that there are more general rules for Asian–Australian children than for their European–Australian counterparts. Furthermore, some families seemed to have unspoken rules, which were expected to be followed by children every day as a standard within the family. By comparing general rules and unspoken rules within families, it was concluded that Asian–Australian families have stricter general rules than European–Australian families. Warnings are another type of parental involvement used to control children. However, Some parent–child dyads did not seem to have rules because they did not require them.

Both parents' and children's data showed that the activities involved in perceptions of control included: assertive involvement, reluctance to grant permission, keeping tracking, limiting socialisation, and punishment. Some parents become too involved in their children's mathematics education, which seems to put pressure on parents and children, creating negative effects on both. Parents' actions could be different when they grant permission to their children. After granting permission, some parents assigned another task for themselves. That is to keep track on the whereabouts of their children. However, keeping track was unnecessary for some students in this study, who were self-disciplined and able to take responsibility for their own actions. Some students gave priority to studies and limited socialising. In addition, some parents punished their children when they neglected their studies. It seems that the type of punishment could be different and varied among families. None of the participants reported hitting with a cane or stick as a punishment.

At present, digital technology has become inseparable from human beings, but sometimes parents find it a disruption to the learning of their children. While the use of digital equipment can enhance teaching and learning mathematics, the addiction to such items seems to be a matter of concern for parents. Data gathered in this study showed that parents confiscate items, cut-off entertainment, or limit entertainment in order to push their children towards academic activities. However, there were some indulgent or permissive parents who did not have concerns about the excessive use of technology. Hence, they did not restrict their children's interests in digital devices. Some parents would limit entertainment to control their children but it seems that further action was regarded as necessary in some cases. Confiscating digital equipment is one such parental action as mentioned by participants.

CONCLUSION

Results indicated a strong and positive espousal of parental controls in both parents' and children's perceptions. Hence, it can be concluded that an increase of parental perceptions such as attitudes, beliefs, expectations, aspirations, values, and academic standards in mathematics education and controlling their children would cause an increase in children's positive or negative perceptions in mathematics achievement. However, when parental aspirations exceeded their children's expectations (Rutherford, 2015) children seemed to have a high level of anxiety, emotional and behavioural difficulties, and lower wellbeing due to perceived pressure, as seen in the interview data. Parental over-aspiration can be detrimental for children's achievement as described by Murayama, Pekrun, Suzuki, Marsh, and Lichtenfeld (2016). Hence, a key implication resulting a positive outcome was the flexibility of parental perceptions, so that parents do not put pressure on children when the latter are seeking to achieve goals. Willingness to change or compromise in parenting was evident in some of the interviews in this study. The findings of this research provide insights on how parental controls can make a difference and why excessive involvement of parents is not always better for children.

Both parents and children reported that the three parental control factors, family rules, perceived parental control, and digital deprivation, differed between the two cultural groups. One of the commonly offered explanations of the differences is based on people from different ethnic backgrounds having different perceptions regarding the parental role in children's education (Wilder, 2014). The process of acculturation over the years, as explained by Sue and Okazaki (1990) can diminish these differences between ethnic groups. Further, it was found that there were no significant differences in the gender of parents when they were involved with children's mathematics education. However, children thought that there were differences in gender when controlling children but parents did not see any difference. Parental perceptions on their involvement in mathematics education of their children and parental control were significantly different across secondary year levels.

The current study, however, affirmed that when children were not doing what they were supposed to do with their studies, parents considered controlling children. As a consequence parents would limit their children's entertainment by reducing TV time, limiting internet usage, or not allowing them to use other digital equipment for a certain period. If there were too many distractions via social media, parents tended to confiscate digital equipment to facilitate their children's engagement in academic activities such as homework. However, some electronic equipment was needed for mathematics and it was impossible to confiscate such items. In this situation some parents implemented ground rules so that children were not able to use digital equipment or the internet after a given time at night. As corporal punishment or hitting with a cane was rare and did not seem to exist anymore the participants reported other types of punishment such as cutting off freedom, grounding, or forcing them to keep studying. Hence, it was clear that if children know their limits in everything they do and obey general rules, parents may not need to place any other restrictions on them or their activities. However, excessively high parental aspirations or unrealistically positive perceptions can lead to over-involvement, resulting in high levels of parental control and excessive pressure on children, which may increase the risk of negative outcomes (Murayama et al., 2016) such as being overwhelmed with worries, depression, and anxiety. As Blondal and Adalbjamardottir (2009) concluded, adolescents who experienced their parents as providers of warmth, trust, and respect while setting fair limits and demanding mature behaviour were more receptive to parental controls. The current study was also in line with the self-determination theory (Deci & Ryan, 1987), which states that children's innate needs for competence, autonomy, and relatedness are undermined when parents are intrusive and controlling. Nevertheless, an appropriate level of parental control is desirable. This study has been able to confirm that children are differentially responsive to how parents become involved and the benefits of such involvement depend on what children themselves bring to their interactions with parents. Importantly, higher parental perceptions can cause higher children's perceptions regarding mathematics achievement and it is a two-way relationship.

References

Baxter, L. A., Bylund, C. L., Imes, R. & Routsong, T. (2009). Parent-Child perceptions of parental behavioral control through rule-setting for risky health choices during adolescence, *Journal of Family Communication*, *9*(4), 251–271. doi: 10.1080/15267430903255920

Baumrind, D. (1967). Child care practices anteceding three patterns of preschool behaviour. *Genetic Psychology Monographs, 75*(1), 43–88.

Blondal, K. S., & Adalbjarnardottir, S. (2009). Parenting practices and school dropout: A longitudinal study. *Adolescence, 44*(176), 729–749.

Chua, A. (2011). *Battle hymn of the tiger mother*. Great Britain: Bloomsbury publishing.

Creswell, J. W. (2014). *Qualitative inquiry and research design* (4th ed.). SAGE Publications, Inc.

Deci, E. L., & Ryan, R. M. (1987). The support of autonomy and the control of behavior. *Journal of Personality and Social Psychology, 53*(6), 1024–1037.

Fu, A. S., & Markus, H. R. (2014). My mother and me: Why tiger mothers motivate Asian Americans but not European Americans. *Personality and Social Psychology Bulletin, 40*(6), 739–749. doi: 10.1177/0146167214524992

Jeynes, W. H. (2011). Aspiration and expectations: Providing pathways to tomorrow. In S. Redding, M. Murphy, & P. Sheley (Eds.), *Handbook on family and community engagement* (pp. 57–59). Lincoln, IL: Academic Development Institute.

Murayama, K., Pekrun, R., Suzuki, M., Marsh, H. W., & Lichtenfeld, S. (2016). Don't aim too high for your kids: Parental overaspiration undermines students' learning in mathematics. *Journal of Personality and Social Psychology, 111*(5), 766–779.

Rutherford, T. (2015). Emotional well-being and discrepancies between child and parent educational expectations and aspirations in middle and high school. *International Journal of Adolescence and Youth, 20*(1), 69–85. doi: 10.1080/02673843.2013.767742

Sue, S., & Okazaki, S. (1990). Asian-American educational experience. *American Psychologist*, *45*(8), 913–920.

Wilder, S. (2014). Effects of parental involvement on academic achievement: A meta-synthesis. *Educational Review*, *66*(3), 377-397. doi: 10.1080/00131911.2013.780009

DISTINGUISHING A FRACTIONAL CONCEPTION OF CHINESE STUDENTS: RESULT OF DIVIDING

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What conceptions, besides the prevalent, part-of-whole conception, may underlie Chinese students' solutions to fractional tasks? To address this question, we draw on a constructivist line of work in western countries that articulated conceptual progressions about fractions as measures (multiplicative relations). We analyze data from two lessons, co-taught by Chinese speaking and English speaking researcher- teachers, which were part of a classroom teaching experiment with 4th graders at a large, urban school in northeast China (N=44). Initially, their reasoning seemed rooted in a conception of fractions we distinguished from part-of-whole, termed "Result of Dividing." Later, they shifted to a conception that involves unit iteration. We discuss implications of these distinctions for theory building and for practice.

INTRODUCTION

Teaching and learning fractions continue to be a great challenge (Beckmann, 2019). Much research has been conducted to study, and promote, students' fractional conceptions (Charalambous & Pitta-Pantazi, 2005; Steffe & Olive, 2010). One prevalent conception, fractions as part(s)-of-whole, has been pointed to as an impediment for deeper understanding and use of fractions (Tzur, 2019). Our work with Chinese students (and teachers) led us to make a subtle, novel distinction of a conception that we could not find in our scrutiny of the literature. In line with the international interest in Chinese students' top-rate outcomes (Tonga et al., 2019), we address the question: What conceptions, besides the prevalent, part-of-whole conception, may underlie Chinese students' solutions to fractional tasks?

To illustrate the newly distinguished conception and issues related with it, we present (Figure 1) the three-phase problem used in a 4th grade class during one of the lessons analyzed in this study. Firstly (Figure 1a), the researcher-teacher presented students with a sequence of actions on the projected computer screen. He drew an unpartitioned (given) whole, then repeated one piece 7 times to show a 7-part whole equal in length to the given whole, colored the first part in blue, pulled out a copy of that blue part, and asked: What fraction is that single blue piece of the given whole? All students answered 1/7, reasoning it is equal to the blue part within the 7-part whole, which is also 1/7 of the given whole. Secondly (Figure 1b), he removed the 7-part whole and asked the same question. This time, students could not figure out what fraction the blue part is of the whole. Thirdly (Figure 1c), he gave students a

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 241-248. Khon Kaen, Thailand: PME.

prompt by copying the blue part (1/7) above the given whole and repeating it only 6 times to produce another unit that is one part (1/7) short of the given whole. Students could still not solve the problem about the blue part below the given whole in spite of the fact it was the piece copied and repeated 6 times. What, we pondered (and invite the reader to ponder with us), could be the conception underlying their reasoning about the "same" unit fraction (1/7) in those three different appearances? And what affordances/limitations it bestows that differ from the part-of-whole or measure (multiplicative relations) conceptions? We address these issues in the Results section.

Figure 1: Screens with different appearances of 1/7 of an unpartitioned whole

CONSTRUCTIVIST CONCEPTUAL FRAMEWORK

This study draws on Steffe's (2010) mental action lens, which articulates partitioning and iterating as inversed operations underlying children's reasoning about whole numbers and fractions. Partitioning refers to a mental action students use to conceive of a whole composed of equal parts while exhausting the whole. It results in equal-size unit fractions (e.g., conceiving the unpartitioned whole in Figure 1a as made of 7 equal parts, 1/7 of that whole). Iterating refers to an action by which a part can be iterated to produce the original whole or parts of that whole (e.g., the iterated whole in Figure 1a). Coordinating both operations into a single way of reasoning, termed splitting (Olive & Steffe, 2010) is a foundation for constructing multiplicative operations on fractions.

Concerning multiplicative operations, researchers stressed that the prevalent part-of-whole conception is deficient, as it hinders students' understanding of fractions and related operations (Simon et al., 2018; Tzur, 1999). Instead, they argued for fostering a conception of fractions as multiplicative relations, or measurement. With such a conception, all a person would need to do for solving any of the tasks in Figure 1 would be to determine the 1-to-7 relation between the blue part and the unpartitioned whole (e.g., reasoning why it would be iterated 7 times to fit within the whole, and thus the whole is 7 times as much as the part). Given a unit fraction (e.g., 1/7), the learner could also produce not only the 7/7 whole but also non-unit fractions (e.g., 6/7, 8/7, 14/7).

Adults might take for granted that a partitioned and an iterated whole are a one and the same entity. For students, however, coordinating those two operations is challenging (Steffe, 2010). In our example (Figure 1), if a child has not seen how the 7-part iterated whole was produced she may only 'see' it as a whole equally partitioned into 7 parts.

METHODS

As part of a larger, cross-culture project[1], we conducted a 6-lesson classroom teaching experiment (Cobb, 2000), two weeks apart, in a grade-4 classroom (N=44) at a mid-size, urban school in northeast China. The larger project used theories and conceptual progressions developed in western countries to study Chinese elementary students' mathematical reasoning. The second author, not speaking Chinese, was the lead teacher-researcher. The first and third authors (Chinese and English) were co-investigators while also serving as real-time translators.

Most fractional tasks were presented/solved using the JavaBars software (Biddlecomb, 1994). This software was developed along with teaching experiments (in the US) that articulated children's fractional conceptions, using physical actions postulated to promote the desired mental actions (e.g., iteration, partitioning). As shown in Figure 1, in JavaBars one could produce bars of various lengths and colors, duplicate (using Copy) and iterate (using Repeat) any such bar as many times as desired, as well as partition any such bar into a desired number of equal pieces (using Parts and Break).

For this paper, we collected and analyzed data from two lessons during the second week of the classroom teaching experiment. During the first week, the researcher-teacher led work on tasks, both with paper strips and with JavaBars, that promoted students' use of iteration to conceptualize unit and non-unit fractions as multiplicative relations (including proper and improper fractions). Two weeks later, we began a lesson with the goal for students to begin partitioning unit fractions. Their inability to solve the 3-phase task (Figures 1a-1b-1c) during the first lesson of the second week led the team to refocus on iteration, this time building on students' available conception, namely, fractions as a result of dividing actions. We thus selected that lesson, and the following one, because they help comparing and contrasting that conception with the more advanced conception in which iteration is used along with partitioning.

We analyzed data in three iterations. First, during the teaching experiment, we conducted ongoing analyses before and after each lesson. In each ongoing session we generated conjectures about students' understanding (e.g., "result of dividing?") and planned the following lesson. Second, the first author observed video records of the two lessons, jotted down logs of main events, and shared those logs with the team. Third, the team identified key indicators of Chinese students' fractional understanding, scrutinized and transcribed relevant video segments, and inferred student conceptions.

RESULTS

We analyze data that help distinguish two conceptions of fractions underlying solutions to fractional tasks by Chinese 4th graders. We believe the first —fraction as a *result of dividing*—is a novel, subtle distinction from part-of-whole. We realized the need to distinguish it during the first lesson, in which students could not solve the last two of the 3-phase problem (Figures 1b-1c). The second conception—fraction as a multiplicative relation—has been a primary goal of our project, and particularly in the second lesson presented here, because our work with Chinese teachers and students indicated it was *novel for them*. Our analysis does not focus on the change process or the teaching that promoted it but rather on its manifestation in the Chinese students.

Conception of Fractions as Results of Dividing.

We begin by analyzing the students' work on the 3-phase problem (Figures 1a-1b-1c). Two key aspects here are that (a) in all three figures the whole is not partitioned and (b) the blue piece in question is *not a part* of that whole. Thus, a person whose conception of fractions is limited to part-of-whole is not likely to successfully solve any of these tasks (see also Tzur, 2019). Our participants could solve the first task (Fig. 1a) but not the second or third (Figs. 1b-1c). We infer their solution to 1a is rooted in the following line of mental actions (reasoning): (a) the 7-part whole was produced through iterating the first piece and is equal to the unpartitioned (given) whole; (b) the given whole could be *imagined* (anticipated) as being partitioned similarly; (c) the first (blue) part on the iterated whole would then be equal to *any of the parts* resulting from the *imaginary (mental) action* of dividing the unpartitioned whole into 7 equal parts and is thus 1/7 of either the given or the iterated wholes; and (d) the pulled-out (blue) part in question is equal to the blue (hence any) part in the iterated whole and, by way of transitivity, is also 1/7 of the given whole. This line of mental actions constitutes a conception of fractions as results of equally partitioning a given whole. Unlike in part-of-whole, where the focus is on an *idle state* with all parts in a partitioned whole showing (and likely highlighting one of them), in fractions as results of dividing the focus is on the dynamic, mental actions yielding that state. While both conceptions involve a part and a whole, the latter supports mental processing leading from an unpartitioned whole to a part that may be embedded in or disembedded from the whole.

Our participants' inability to solve the other tasks indicated a critical limitation of this conception. For them to initiate the sequence of mental actions leading to determining the fractional size of a disembedded part – a partitioned whole equal in size to the given whole seems necessary. In our first task (Figure 1a), that partitioned whole was created by iterating one piece. However, our second task (Figure 1b) included no such partitioned whole (it was removed). The third task (Figure 1c) provided an image of an iterated whole that was not equal to the given whole, apparently requiring a prior mental action of (step) of imagining one more iteration of the same part. Combined, those three tasks highlight the nature of fractions as results of dividing actions: a part does not have to be in the whole, but a completed mental sequence of partitioning actions is needed to make sense of parts (embedded or disembedded) as fractions. Excerpt 1 below, from students' solutions to a problem we presented at the beginning of the second lesson, further illustrates this limitation (S stands for student, R for researcher-teacher; students' responses were translated to English).

<u>Excerpt 1</u>

R: (Draws a bar on the computer screen, partitions is into 5 equal parts, colors each part, copies the partitioned whole and clears the marks, pulls out one (pink) piece from the partitioned whole (Figure 2a), breaks the partitioned whole and relocates each of the (1/5) colored parts on the screen (Figure 2b), erases those 5 parts (Figure 2c), and asks): What fraction of the [purple] unpartitioned whole is the pink part?

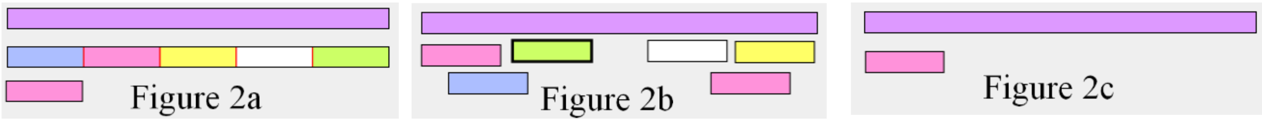

Figure 2: Computer screenshot leading to the first task in the second lesson

S1: The original whole was partitioned into 5 parts. The [purple] bar was not partitioned. However, R copied it from the original [whole]. So [this bar] *could* be partitioned into 5 parts equally. So, [the pink part] is one-fifth of the whole.

R: Anyone who has a different explanation than S1?

S2: Teacher R made an exact same copy of the purple piece. [It] *could* be partitioned into five parts. Four of them were erased, and the pink piece was left. So the pink piece is one-fifth of the purple bar.

R: [Asks the class, after paraphrasing the explanations] Do you all agree with this? Thumbs up if you agree. [All students showed "thumbs up."]

Data in Excerpt 1 provided further evidence to our articulation of the conception of fractions as a result of dividing actions. The students all agreed that the result of quite a complex sequence of actions would be 1/5, because the part in question could have been imagined to be produced through partitioning the given whole into 5 equal parts. Having witnessed the steps of this conception as listed above allowed them to reason about the disembedded (pink) part as a fraction (1/5) of the unpartitioned whole. Again, we emphasize that such a conception affords what a part-of-whole conception would likely not – the pink piece is not part of (in) the given, unpartitioned whole.

Possible Conception: Fractions as Multiplicative Relations.

Having brought forth students' conception of fractions as a result of dividing actions, the researcher-teacher turned to promoting their use of iteration to determine a relation between two units, one being the unpartitioned whole. We remind the reader that, in previous lessons, students have played a game involving iteration for estimating (then confirming) how many times a smaller unit would fit within the given whole. He thus told them to work in pairs, presented a new screen (Figure 3), emphasized the (purple) whole was never partitioned and will not be partitioned, the pink piece is not part of it, and asked them: Is the pink piece a fraction of the whole, and if so – what fraction? The video shows the vast majority of students began, from their position in the room, using their fingers to estimate the number of times the pink piece could be iterated to fit within the given whole (the answer is 9, so 1/9).

Students then shared various answers (1/7, 1/8, 1/9, 1/10, or 1/11), and the researcher-teacher asked for a few explanations.

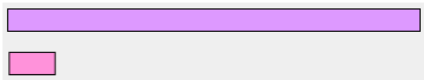

Figure 3: Screenshot of task leading to students' use of iteration

<u>Excerpt 2</u>

R: How did you find one-eighth? Did you do this (moves his fingers)? What did you do?

S3: (Shows the action of moving fingers.)

R: Could you show to all of us on the screen?

S3: (Goes to the screen and uses her thumb and forefinger to measure the length of the pink piece, then, keeping that measure, moves both fingers along the purple bar while counting the number of repeats (ending with 8).

R: Ok. One thing I heard … was a way of thinking. That's what really matters in math. It may not be exactly eight, or nine, or ten. [It is] your FangFa (Chinese for method). How many times does it fit in the whole? I don't need parts; I don't need to divide a whole. I just need to find how many times it fits in the whole. That's what I heard you [S3] saying. It fits 8 times, so 1/8.

R: (to S4) You said it is 1/11, right? How did you get 1/11? What did you do?

S4: (Uses his thumb and forefinger to show a small length) I just counted it one-by-one like this (moving fingers along the purple bar), until I got to the end.

R: (To the class) Do you think what S3 and S4 said is the FangFa? They got different answers ... Thumbs up if you think they did it in the same way. Thumbs up if you did it in the same way. [All students give thumbs up to both questions.]

Excerpt 2 provides a glimpse into students' use of a conception of fractions that includes unit iteration. Based on the purposeful way they engaged in various actions of iteration, we conjecture some students have possibly been using a conception of a fraction as a measure. Being constrained to operate on an unpartitioned whole, *they* independently initiated iteration as a way to determine a possible (anticipated) partition by operating on the given part so they could relate it to the whole. We are careful not to attribute to all (or most students) a conception of a fraction as a multiplicative relation, because they might have slightly extended their conception of fractions as a result of dividing actions by iteration to attain the first step in the dividing actions conception. However, some of the students might have set as a goal to determine a relationship between the unpartitioned whole and the part, in the sense of the whole being *n* times as much as the part. In either of those cases, the crucial inference we make is that these Chinese students began coordinating partitioning with iteration as a means to determine fractional relationships between two given units. Such coordination would serve as a basis for conceptualizing fractions (1/n, m/n) as multiplicative relations, and allow students to solve tasks like those shown in Figures 1b-1c.

DISCUSSION

We articulated a fraction conception that has not yet been documented in research literature. Indeed, teaching practices often pertain to such a conception. This study provides further details of the mental actions that constitute it, and highlights some of its affordances and limitations. We point out two key implications of this distinction. First, for theory building, this study broadens findings of western students' fractional understanding (Charalambous & Pitta-Pantazi, 2005; Simon et al., 2018; Tzur, 2019). Specifically, in the context of fractions, we corroborated Steffe's (2010) argument of levels of fragmenting an unpartitioned whole – a less advanced one based only on partitioning and a more advanced one based also on iteration. In addition, we corroborated the need for students to construct both iterating and partitioning to deal with problem situations in which nothing is being partitioned. Constructing a conception that includes iteration opens the way to establishing more advanced understanding of fractions as multiplicative relations (Norton and Wilkins, 2012).

For practice, this exploratory study demonstrates the importance of viably modeling students' mental actions as a basis for promoting their learning of fractions. Here, we showed that a conception of fractions as a result of dividing, common in China, could serve as a good starting point for shifting to reasoning that involves unit iteration, and ultimately to fractions as multiplicative relations. The tasks we used may serve both as tools for teachers to assess their students' reasoning and thus to also promote their advancement to iteration-based conceptions. Future studies could examine how the conception of fractions as results of dividing actions may evolve and be expounded on.

Acknowledgment

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References

Beckmann, S. (2019). Commentary on Fractions. In A. Norton & M. W. Alibali (Eds.), *Constructing Number: Merging Perspectives from Psychology and Mathematics Education* (pp. 237-248). Cham, Switzerland: Springer International Publishing.

Biddlecomb, B. D. (1994). Theory-based development of computer microworlds. *Journal of Research in Childhood Education*, 8(2), 87–98.

Charalambous, C. Y., & Pitta-Pantazi, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. In Chick, H.L. & Vincent, J. L. (Eds), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol.2, pp. 233–240. Melbourne: PME.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In R. Lesh & A. E. Kelly (Ed.), *Handbook of research design in mathematics and science education* (pp. 307–333).

Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), 557-583.

Olive, J., & Steffe, L. P. (2010). The Partitive, the Iterative, and the Unit Composition Schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 171-223). New York: Springer.

Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, 52, 122–133.

Steffe, L. P. (2010). Articulation of the reorganization hypothesis. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 49–74). New York: Springer.

Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Tonga, F. E., Eryiğit, S., Yalçın, F. A., & Erden, F. T. (2019). Professional development of teachers in PISA achiever countries: Finland, Estonia, Japan, Singapore and China. *Professional Development in Education*, Online First, 1-17.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, 30(4), 390.

Tzur, R. (2019). Developing fractions as multiplicative relations: A model of cognitive reorganization. In A. Norton & M. W. Alibali (Eds.), *Constructing number: Merging perspectives from psychology and mathematics education* (pp. 163-191). Cham, Switzerland: Springer International Publishing.

OPPORTUNITIES FOR EXPLORATIVE PARTICIPATION IN DIFFERENT ACHIEVEMENT GROUPS

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Evidence that grouping students based on their mathematical achievements is detrimental for low achieving students has been accumulating. However, little is known about the opportunities given to students to same different realizations of mathematical objects and to author narratives in different achievement level groups. The purpose of this study is to examine the differences between the opportunities given to students to talk about mathematical objects in different tracks. 34 videotaped mathematics lessons, implementing one identical task (the hexagons task), were analyzed using the Realization Tree Assessment tool (RTA). Results show that the opportunities for explorative participation in low-achievement groups were significantly lower than the opportunities in high-achievement groups.

INTRODUCTION

Tracking, streaming and learning mathematics in achievement groups (AGs) have been some of the most controversial issues in the field of educational research for over 70 years (Gamoran, 2010). Many studies show that there is no significant positive effect on student achievement while learning in AGs (Linchevski & Kutscher, 1998; Slavin, 1990). This may be due to the differences in the curriculum, mathematical content, opportunities to learn and teaching practices between different AGs (Oakes, 1990). Mathematics lessons in low-AGs are characterized by teachers asking closed and simple questions, focus on drill and practice, much time allocated to classroom management, lack of curriculum coverage and diminution of the opportunities to learn (Boaler et al., 2000; Zevenbergen, 2005). However, although these studies refer to mathematics lessons in different AGs, they did so in a general way. They did not focus on the mathematics itself that students are exposed to but rather on more general pedagogical actions such as teacher questioning and classroom management practices. Therefore, it is unclear whether the differences in opportunities to learn (OTLs) are a result of general differences in pedagogical practices and different tasks, or whether, given a potentially rich task, students in different AGs still would receive different OTLs. Therefore, the purpose of this study is to examine the differences in the access given to students from different AGs to engage with mathematical objects, given an identical task.

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 249-256. Khon Kaen, Thailand: PME.

THEORETICAL BACKGROUND

Our conceptualization of opportunities for meaningful participation in mathematical learning rests on the commognitive theory. According to this theory (Lavie et al., 2019; Sfard, 2008) students' explorative participation in mathematics lessons is characterized by routines of participation, where narratives about mathematical objects are produced agentively by students. This, in contrast to ritual participation, in which students mainly reproduce procedures exemplified by an authority (usually the teacher) and their participation in the discourse is for the sake of pleasing others.

A major part of participating exploratively in the mathematical discourse is based on talking about mathematical signifiers as *objects* – as entities that exist in the world regardless of actions, humans and time. In the ritual phase of learning, mathematical objects are often mentioned in the novice learner's utterances as detached signifiers. "Two", "function", "triangle" etc. are all keywords that learners can use without relating them to a mathematical object, but rather as signifiers of only one particular realization. For example, "function" can be signified by $y=3x+5$ but not by the graph of this function. When students become explorative in the discourse on functions, they refer to this object through its multiple different realizations: the algebraic realizations (e.g. $y=5+3x$), the verbal realizations ("I had 5 dollars to begin with, and each day I earned 3 dollars"), the table-of-values realization, the graphic realization, as well as various visual realizations. What makes these different realizations be treated as "the same" object is the fact that every endorsed narrative about one realization can be translated, according to well defined rules, into an endorsed narrative about another realization (Sfard, 2008, p. 154). This process, in which the learner comes to talk about the different realizations of the mathematical object as "the same" is defined by Sfard as "saming".

Advancement in mathematical learning rests on successfully moving from ritual to explorative phases in a hierarchical fashion. When this process fails to occur, students encounter difficulties and low achievements (Ben-Yehuda et al., 2005; Heyd-Metzuyanim, 2013). When tracking is practiced in schools, they may quite quickly find themselves in low-achievement tracks, where they may receive less opportunities to learn.

EXPLORATIVE VS. RITUAL OPPORTUNITIEIS TO LEARN

The term Opportunities-To-Learn (OTLs) includes the teaching practices that afford or constrain student engagement in the process of learning mathematics (Heyd-Metzuyanim, Tabach, & Nachlieli, 2016). Nachlieli & Tabach (2019) theorized the OTLs as types of routines which the students are expected to engage with. They refer to two types of routines: ritual-enabling OTLs which enable students to participate ritually and exploration-requiring OTLs which afford students' explorative participation (ibid, p. 257). In ritual-enabling OTLs, the teachers' questions focus on 'how' to solve the problem and the students are expected to apply a rigid procedure that had been previously demonstrated. In contrast, in exploration-requiring OTLs the students are expected to choose alternative procedures while stating the new narrative produced based on mathematical reasoning (ibid, p. 258).

Previous studies examining differences between AGs point to the possibility that students in low AGs receive restricted opportunities to learn meaningful and interesting mathematics. For example, Boaler and colleagues (2000) found that low-AGs are characterized by low-level work that students find too easy and by much time allocated to coping worked examples off the board rather than solving problems. Similarly, Oakes (1990) found that teaching in low-AGs was characterized by "drill and practice" approaches, whereas the high-AGs included more challenging and open-ended mathematical tasks. Yet the precise differences in the mathematical OTLs that students get exposed to in different AGs are have not yet been sufficiently illuminated.

In the present study, we focus on the differences in mathematical OTLs given to students in high vs. low AGs as evidenced from the Realization Tree Assessment (RTA) tool (Weingarden, Heyd-Metzuyanim, & Nachlieli, 2019). The RTA has been previously shown to be a useful tool for describing different implementations of identical mathematical tasks (Weingarden & Heyd-Metzuyanim, 2019). For example, it showed that lessons can differ widely in the number of realizations of a mathematical object that students get exposed to, the links made between these realizations and who authors these realizations and links – the teacher or the students. However, so far we have not systematically investigated the differences in implementing identical tasks between different AGs. In this study we ask: what are the differences in opportunities to same and link different realizations that students receive in low vs. high AGs?

METHOD

The study was performed as part of the TEAMS (Teaching Exploratively for All Mathematics Students) project. The aim of this professional development program was to expose teachers to various teaching practices that encourage students' explorative participation (Heyd-Metzuyanim, Nachlieli, Weingarden, & Baor, 2018). As part of the professional development program, teachers were asked to implement in their classrooms tasks that can afford explorative participation. These lessons were videotaped by a stationary video camera. Since our goal in the current study was to compare between the implementation of tasks in different AGs (rather than simply the usage of different tasks), we chose to compare only lessons based on one identical task- the hexagons task. This task asks students to describe the perimeter of any train in the hexagons pattern (see Figure 1).

Throughout the TEAMS project, 34 teachers implemented the hexagons task. These teachers taught 7^{th} – 9^{th} grades, in different AGs. Since grouping in middle schools in Israel is very diverse (ranging from no grouping to 5 or even more AGs per grade), the AG category was determined according to the median of all the tracks in that grade. AGs above the median were categorized as 'high-AG' and the others as 'low-

AG'. This produced the following division: low-AGs (N = 13), high-AGs (N = 17) and heterogeneous classes- without grouping (N = 4).

Write a description that could be used to compute the perimeter of any train in the pattern.

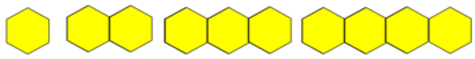

Figure 1 – The hexagons task

The 34 lessons were analysed using the Realization Tree Assessment (RTA) tool (Weingarden et al., 2019) which is presented in Figure 2. The potential mathematical object that can be richly discussed through the hexagons task appears at the top of the tree – "The perimeter of the nth hexagon train". All its various realizations appear below it. For each lesson, the different realizations mentioned in the lesson were shaded according to who articulated them (the students or the teacher) and the links between the different realizations were marked according to who made them (the students or the teacher). Therefore, the RTA examines the extent to which opportunities for explorative participation are given to students by graphically illustrating the different realizations of the mathematical objects mentioned during the lesson, the extent to which links between realizations were made and the extent of student authority.

In addition to the received snapshot image of a lesson, the RTA images were quantified (see Figure 3). This was done by two types of calculations, Saming Realizations (SR) and Students' Authority (SA). The SR measure was calculated by the ratio of the number of realizations and links that appeared in each lesson to the maximum number of links and realizations that appeared in any lesson in the sample. The rationale behind this calculation is that a reasonable assessment of the maximum potential of a task in terms of realizations can be obtain by picking the maximum number of realizations discusses in a wide enough sample of lessons (here- 34). The SA measure was calculated by the ratio of the number of realizations and links authored by students in each lesson to the total number of links and realizations mentioned in the lesson. The rationale here is to quantify the ratio of student authority in relation to the total number of mathematical narratives mentioned in the lesson (by the teacher *and* the students). Analysis of Variances (One-way ANOVA) was conducted to examine whether there were differences between these two measures in the different AGs.

FINDINGS

Our analysis, based on the RTA, revealed that in high-AGs more opportunities were given to students for explorative participation – namely authoring narratives about different realizations and making links between them, than in low-AGs. Before we move to show this quantitatively, let us first illustrate how the difference between lessons in low and high-AG may look like.

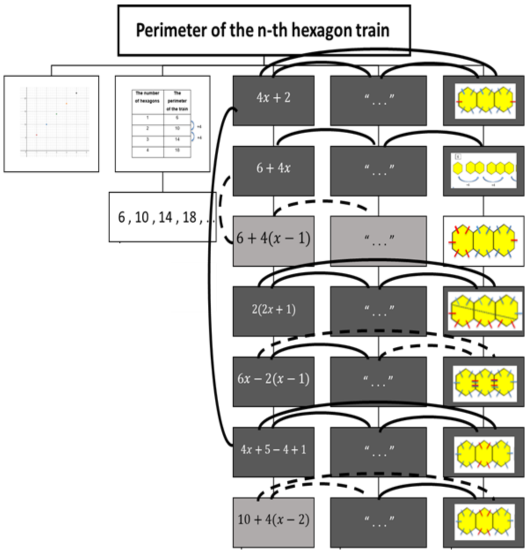

Figure 2.a – RTA image of a lesson in a high-AG

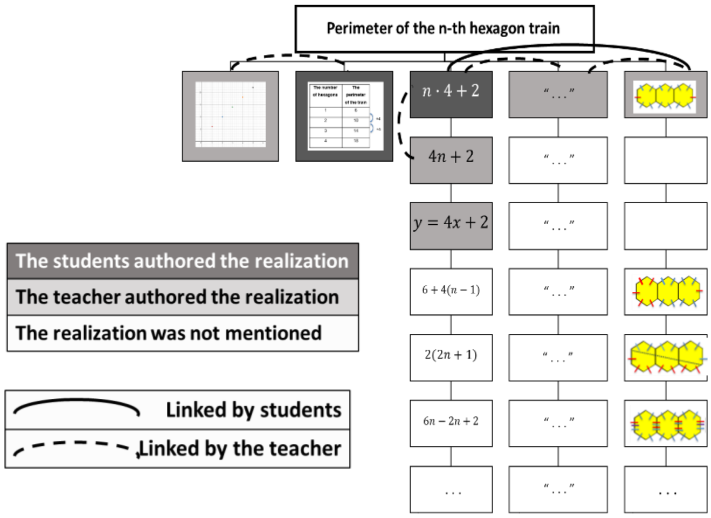

Figure 2.b – RTA image of a lesson in a low-AG

Figure 2 – Comparison between RTA images of lessons in different AGs

Figure 2 depicts the RTA images of two lessons. The teachers in both lessons opened their lesson by launching the hexagon task, explaining the problem briefly and then sending the students to work in groups of 2-4 students. After the group work, they gathered the classroom for discussion, inviting students to present on the overhead board. The RTA images describe the whole-classroom discussion of the two lessons.

The first lesson (Figure 2.a) was videotaped in a high-AG of 8th grade. During the discussion of this lesson, the students with the aid of the teacher, presented their solutions, justified them and linked between different solutions. Through this discussion, multiple algebraic expressions that illustrated the perimeter of the nth hexagon train were produced and justified by linking between the expressions and the visual hexagons' pattern. In addition, some of the algebraic expressions were linked while students produced narratives about the equivalency between the algebraic expressions (for example, that 4x+2 is "the same" as 4x+5-4+1).

In contrast, in the second lesson (Figure 2.b) that took place in a low-AG of 9th grade, the various descriptions that can describe the perimeter of the hexagons train were not mentioned and discussed. The students produced the table-of-values regarding the perimeter of the hexagons train and produced the algebraic expression: n*4+2. The teacher then made the connection between the algebraic expressions n*4+2 and 4n+2, linking them to the different representation of a linear function. In general, only 7 realizations (out of the potential 27 and compared to the 20 of the high-AG lesson) were mentioned, few links were made and most of them were established by the teacher. Therefore, students in the low-AG were getting less access to different realizations and to opportunities for saming these realizations.

Figure 3 exemplifies the quantification of the RTA by showing the calculation of the SR and SA scores of the low-AG lesson.

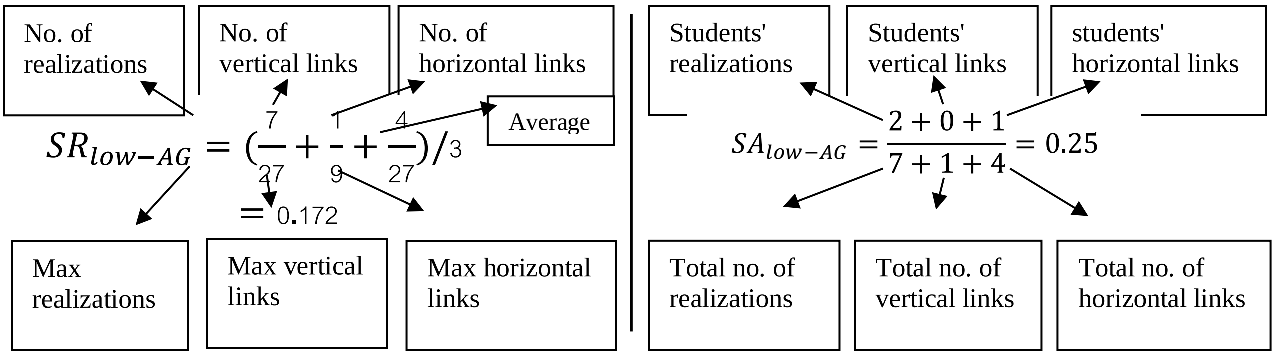

Figure 3 – The calculation of the SR and SA scores

The calculated saming realizations (SR) score of the high-AG lesson was 0.543 (compared to 0.172 of the low-AG) and the calculated students' authority (SA) score for the high-AG lesson was 0.775 (compared to 0.25 of the low-AG). These numbers thus capture (although reductively) the difference between the two lessons in terms of student authority and opportunities for saming realizations.

The above described quantification was performed on all 34 lessons, and comparison of the SR and SA scores over all 34 lessons was examined by a one-way ANOVA test. Results show a significant difference in the opportunities for explorative participation between AGs of different levels, both in the opportunities for saming realizations (SR) ($F(2,31) = 3.456$, $p < 0.05$) and the level of student authority (SA) ($F(2,31) = 16.055$, $p < 0.001$). The results are presented in Table 1. As can be seen in Table 1, in general, students in low-AGs were getting less access to different realizations and links between them than students in high-AGs. These differences were found to be significant according to Tukey HSD post hoc tests.

Achievement group	**N**	**$Mean_{SR}$**	**SD_{SR}**	**$Mean_{SA}$**	**SD_{SA}**
Low-AG	13	0.292	0.189	0.419	0.227
Heterogeneous	4	0.308	0.161	0.785	0.091
High-AG	17	0.469	0.200	0.753	0.125

Table 1 – The SR and SA acores in the different AGs

The situation with the students in the heterogeneous classrooms is somewhat more complex. First, the small number of these lessons (N = 4) limits the statistical power of the analysis. However, despite this limitation, a significant difference involving the heterogeneous classes was found. According to Tukey HSD post hoc tests, the level of students' authority (SA) in the heterogeneous classes *was* significantly different than the level of SA in low-AGs and was found to be similar to the level of students' authority in high-AGs.

DISCUSSION AND CONCLUSIONS

Our goal for this paper was to examine the differences between the opportunities for explorative participation given to students in different AGs, given an identical task. Our findings show that in high-AGs, compared to low-AGs, more realizations were authored, more links were made and most of them were established by students. In addition, the level of student authority in the heterogeneous classes, was found to be significantly higher than the low AGs. These finding are preliminary due to the limited number of classrooms, especially heterogeneous ones. Moreover, we cannot derive any causal relationship between grouping students based on their achievements and the OTLs seen in the RTA. It may be, for example, that the fact that several strong students sit in a heterogeneous class is responsible for the high SA seen in these classrooms. However, given previous works that showed the benefits of learning in heterogeneous classrooms (e.g. Slavin, 1990), our findings may strenghten the understanding of how low-achieving students may benefit from sitting in heterogeneous classrooms, and the price of sitting in low-achievments tracks.

From the difference in the opportunities for saming realizations between low and high AGs we can conclude that for students sitting in a low-AG class, the chances they will hear or construct multiple realizations of the mathematical object and same them are probably lower than if they would sit in a high-AG class, even given an identical task. This conclusion strengthens previous studies that found discrepancies between low and high AGs with regards to the mathematical content and the opportunities to learn (e.g. Boaler et al., 2000). Future studies will be necessary in order to (a) understand whether our findings can be generalized, and (b) better understand how the exposure to mathematical objects plays out differently in low vs. high AGs and heterogeneous classrooms. Given the growing achievement gaps in Israeli mathematics tests (Linchevski & Kutscher, 1998; Razer et al., 2018), and the concern worldwide that marginalized populations may get limited access to mathematical content through grouping practices (Boaler et al., 2000), it is imperative that we better understand the processes of marginalization. This study takes one step forward in this process.

Acknowledgments

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References

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing Wrong with Words: What Bars Students' Access to Arithmetical Discourses. *Journal for Research in Mathematics Education*, *36*(3), 176–247.

Boaler, J., Wiliam, D., & Brown, M. (2000). Students' Experiences of Ability Grouping—disaffection, polarisation and the\nconstruction of failure. *British Educational Research Journal*, *26*(5), 631–648.

Gamoran, A. (2010). Tracking and inequality, New directions for research and practice. In M. W. Apple, S. J. Ball, & L. Armando Gandin (Eds.), *The Routledge International Handbook of the Sociology of Education* (pp. 213–228).

Heyd-Metzuyanim, E. (2013). The co-construction of learning difficulties in mathematics—teacher–student interactions and their role in the development of a disabled mathematical identity. *Educational Studies in Mathematics*, *83*(3), 341–368.

Heyd-Metzuyanim, E., Nachlieli, T., Weingarden, M., & Baor, R. (2018). The "TEAMS" Professional Development Program for Enhancing Explorative Instruction. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 63). Umeå, Sweden: PME.

Heyd-Metzuyanim, E., Tabach, M., & Nachlieli, T. (2016). Opportunities for learning given to prospective mathematics teachers: between ritual and explorative instruction. *Journal of Mathematics Teacher Education*, *19*(6), 547–574.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: from ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176.

Linchevski, L., & Kutscher, B. (1998). Tell Me With Whom You're Learning, and I'll Tell You How Much You've Learned: Mixed-Ability Versus Same-Ability Grouping in Mathematics. *Journal for Research in Mathematics Education*, *29*(5), 533–554.

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*, 253–271.

Oakes, J. (1990). *Multiplying Inequalities: The Effects of Race, Social Class, and Tracking on Opportunities to Learn Mathematics and Science.*

Razer, M., Mittelberg, D., & Ayalon, S. (2018). The ability-track glass ceiling of Israeli schooling: lessons from a comparative analysis of Israeli and Australian PISA 2012 data. International Journal of Inclusive Education, 22(2), 192–214.

Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.

Slavin, R. E. (1990). Achievement effects of ability grouping in secondary schools: A best-evidence synthesis.. *Review of Educational Research*, *60*(3), 471–499.

Weingarden, M., & Heyd-Metzuyanim, E. (2019). Opportunities for Saming Realizations in Different Tasks. *In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). (2019). Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education (Vol 3). Pp. 430-437. Pretoria, South Africa: PME.*

Weingarden, M., Heyd-Metzuyanim, E., & Nachlieli, T. (2019). The realization tree assessment tool – examining explorative participation in mathematics lessons. *Journal of Mathematical Behavior*, 56.

Zevenbergen, R. (2005). The construction of a mathematical habitus: Implications of ability grouping in the middle years. *Journal of Curriculum Studies*, *37*(5), 607–619.

CATEGORISING PERCEIVED ADVANTAGES OF INSTRUCTIONAL MATHEMATICS VIDEOS

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Due to the Corona Pandemic, the importance of digital learning tools got re-emphasised. However, already before, online instructional videos targeting school content had been very popular among students. While teachers in many countries might see the Corona Pandemic as an incentive to engage more with this particular medium, many students already watch these videos regularly out of school. This paper confirms and extends a clustering of the advantages of instructional mathematics videos from upper secondary school students' viewpoint based on data collected in an online survey in 2019 with more than 1000 participants. 17 different categories could be identified and differences based on the students' mathematics grades exist.

INTRODUCTION

In the 21st century, teaching and learning with digital tools is becoming more and more common. One teaching method which has gained widespread attention over the last years is the Flipped Classroom model (Bergmann & Sams, 2012) in which students usually watch videos out of school covering new topics to free up time in school for intensive practice. Apart from such a controlled use of videos for educational purposes initiated by a teacher, many students also watch freely available online instructional videos out of school on their own account (Wolf, 2018; Ratnayake et al., 2019). Opposed to when watching a video assigned to them by their teacher, students decide on their own which video on which topic to watch, when, where and how. This opens up room for many questions: if students watch such videos, and we will focus in this work on videos explaining mathematical content, how does this influence their learning in and understanding of mathematics? A first step to answer these questions is to analyse students' motivation to watch instructional mathematics videos outside of school. For this purpose, we examine the advantages students attribute to this medium.

THEORETICAL FRAMEWORK

Instructional mathematics videos

An instructional video (sometimes also referred to as explanation video or educational video) is a short video explaining a concept or how something is done (Wolf, 2018). Research on instructional videos is still a rather new but growing field (Ratnayake et al., 2019). Many students use instructional videos, especially on YouTube, e.g. to complete their homework (Wolf, 2018). Freely online available, they can be watched on demand at any desired moment granted one has internet access and a computer or mobile device. Instructional videos can also enrich mathematics lessons. With numerous videos available for many different topics, they can be used e.g. as part of a

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 257-264. Khon Kaen, Thailand: PME.

flipped mathematics classroom (Bergmann & Sams, 2012). Due to the need for distance learning during the Corona Pandemic, instructional videos are also used as an easy mean to teach a certain topic from afar (Bersch et al., 2020). However, instructional mathematics videos are often discussed controversially. Acuña-Soto et al. (2018) state that many advantages of the medium only apply when they are not regarded as a standalone but rather a complementary resource. Bersch et al. (2020) argue that students are forced to take on a passive role when watching such videos. They further claim that these videos are often superficial and erroneous. Kulgemeyer and Peters (2016) warn about what they call the "illusion of understanding" when learners have watched an instructional video and think they have understood a topic when in fact, they have not or only partially.

Students' perspective

Apart from the didactical perspective, it is also important to understand how and why students use instructional mathematics videos. There are some studies analysing how university students use and engage with instructional videos (e.g. Guo et al., 2014; Schiltz; 2015) but there is only few data available for school students. Kulgemeyer and Peters (2016) voice the need for research on the criteria based on which students select instructional videos on YouTube and suggest that they include factors such as an "impressive" use of media rather than didactical criteria. Acuña-Soto et al. (2018) emphasise the role of the availability of such videos as they offer students "fast and inexpensive access to educational contents at their own convenience". Bersch et al. (2020) assert that instructional mathematics videos are mainly used by students for repetition purposes or as some sort of free tutoring. Most available research is theory-based, though, with very little empirical data available. One source for empirical data is a representative interview-study conducted in Germany with 818 young people between 12 and 19 years (Rat für Kulturelle Bildung, 2019), of which n = 520 were both, students and YouTube users. These 520 students were asked in which regards YouTube was better than school. The authors clustered the students' free speech answers in 13 categories such as: other explanations, the possibility to repeat a topic, fun content, that they can be watched anywhere and anytime and that they can be watched at one's own pace. The first author of this paper also conducted an exploratory online study in 2019 for her thesis. The goal of the study was to get a better understanding of when, how and why students use instructional mathematics videos outside of school. A total of 2025 school students of all ages filled in an extensive questionnaire that was distributed online, mainly over the YouTube channel *DorFuchs*. Selected results of this study were published in Wetzel and Ludwig (2020) such as two main reasons for watching instructional videos outside of school being the explanations in the videos and the possibility to control the own learning pace. Not all data was evaluated due to the large data set with many different item types. In this paper, we want to thoroughly analyse the answers to one open text question in this questionnaire regarding the advantages of instructional mathematics videos with the goal to arrive at a detailed list as a basis for future research.

RESEARCH QUESTIONS

Understanding students' motivation to use instructional mathematics videos also implicitly helps to learn more about how students learn or at least how they prefer to learn. For this purpose, we want to analyse the advantages students attribute to instructional mathematics videos. Students have individual preferences and needs based on which they choose the videos they watch outside of school. There has been some research in the last years trying to come up with quality criteria for "good" instructional mathematics videos, mostly from a didactical point of view (see e.g. Ratnayake et al., 2019; Bersch et al., 2020). These criteria might guide teachers when creating or choosing videos, however, they do not necessarily affect students' readiness to watch them outside of school, as for example Kulgemeyer and Peters (2016) suggest. To better understand students' motivation, we want to arrive at a detailed set of categories of advantages students see in instructional mathematics videos. Our starting point are the 13 advantages listed in *Rat für kulturelle Bildung* (2019). Using the data and sample from the study we presented in Wetzel and Ludwig (2020), we arrive at our first research question:

- RQ1: Can the categories of advantages of instructional videos identified in *Rat für kulturelle Bildung* (2019) be confirmed for the present sample of students in upper-secondary school for instructional *mathematics* videos in particular?

We decided to focus on upper-secondary students only to work with a more homogeneous sample. What is missing from the data in *Rat für kulturelle Bildung* (2019) is numerical data or a ranking of the advantages. A ranking of the advantages can give an indicator which are the most important factors for students.

- RQ2: Which are the most important advantages of instructional mathematics videos from the perspective of upper-secondary students in this sample?

What is remarkable about the sample from Wetzel and Ludwig (2020) is the high number of participants with very good mathematics grades. This positive-selection of participants aggravates arriving at general conclusions but, on the other hand, opens up room for new questions which leads to our third research question:

- RQ3: Do upper secondary students with the grade "very good" in mathematics see other advantages than other upper-secondary students?

METHODOLOGY

The data we use for our analysis results from an openly-available online explorative study from 2019 (see Wetzel & Ludwig, 2020). The used questionnaire contained quantitative and qualitative items. To answer our research questions, we focus on the open-text question "Which advantages do instructional mathematics videos have compared to mathematics lessons in school?" and the variables mathematics grade (on a scale from 1-6), grade level (on a scale from 1-13), sex (f, m, d) and whether the participants watch instructional mathematics videos outside of school (yes / no). Filtering the sample for grade level, we got a set of n = 1017 students in upper-secondary school (German grade level 10-13) who completed the whole questionnaire, answered "yes" to the question whether they watch instructional mathematics videos outside of school and gave a non-empty answer to our target question. All analyses were performed on this sample (f: 25.7%, m: 73.6%, d: 0.7%). The mathematics grade distribution (where 1 corresponds to “very good” / A; 6 corresponds to “failed” / F) is as follows: 1 – 445 students (43.8%), 2 – 324 students (31.9%), 3 – 167 students (16.4%), 4 – 57 students (5.6%), 5 – 21 students (2.1%) and 6 – 3 students (0.3%).

To come up with a list of advantages, we used a deductive-inductive approach. We started with the 13 advantages listed in *Rat für kulturelle Bildung* (2019). After a first sighting of our data, we added seven further categories. Based on these 20 categories, we created an extensive rating guide. One rater then assigned all 1017 answer to these 20 categories. A second rater assigned 30% of the answers so we could determine the inter-rater reliability (Cohen’s kappa) for each category. For each student answer, at least one category was assigned but the assignment of more than one category was possible and common. We decided to reject categories to which less than 2% of the answers were assigned. We did not set a too high threshold as the sample is not representative to not exclude valid categories. Based on the values for Cohen’s kappa in each category, we determined whether we needed to merge categories. We then ranked the remaining categories in accordance with the number of answers assigned to them by the first rater. To answer RQ3, we formed two groups as disjoint subsets of the original sample based on the participants’ mathematics grade. Group VG consists of 445 students with the German grade 1 (“very good” / A) and group NVG consists of 572 students with other grades (grades between 2 / “good” / B and 6 / “Failed” / F). For both groups, we compared the five most important categories and whether categories were exclusive to one group, again with an acceptance threshold of 2% of the respective group size.

RESULTS AND DISCUSSION

The final 17 categories can be found in Table 1. Two categories were rejected due to the 2%-threshold. The analysis of Cohen’s kappa yielded “substantial agreement” ($\kappa > 0.61$) for all but one category and “almost perfect” agreement ($\kappa > 0.81$) for two categories (Landis & Koch, 1977). The category “Structure” with less than substantial agreement was too similar to the category “Short and concise”. We thus merged both as a union set into the new category “Short and structured” ($\kappa = 0.909$). We can thus confirm 11 out of the original 13 categories for instructional mathematics videos in *Rat für kulturelle Bildung* (2019). The category “Videos are modern” was rejected due to the 2%-threshold and we merged the categories “Short” and “Structured” into a new category. Of the seven additional categories suggested by us, six categories remain (A8, A11, A12, A14, A15, A16) with one being rejected. We can thus confirm nearly all initial categories for instructional mathematics videos, however, we extend the clustering to a total of 17 categories for RQ1.

Category	Frequency	Example quote
A1 Own pace**	400 (39.3%)	“Pausing and rewinding allow for individual learning.”
A2 Short and structured**	261 (25.7%)	“They focus on the most important points.”
A3 Repeating a topic or video*	256 (25.2%)	“You can repeat things that you have not understood.”
A4 Other explanations*	178 (17.5%)	“Offer another perspective apart from the teacher’s.”
A5 Choose topics oneself*	118 (11.6%)	“You can choose yourself what to learn.”
A6 Anywhere or anytime*	108 (10.6%)	“You can watch them whenever you want.”
A7 Interesting or advanced content*	99 (9.7%)	“Topics which are not covered in school. I can live out my passion.”
A8 Instructive*	83 (8.2%)	“They are more comprehensible.”
A9 Visualisations*	79 (7.8%)	“A visual representation is often helpful.”
A10 Many videos to choose from*	56 (5.5%)	“If you do not understand the explanation of one YouTuber, you can choose another.”
A11 Relaxed Atmosphere*	54 (5.3%)	“You are not bullied if you do not understand it at once.”
A12 Concentration*	52 (5.1%)	“No disturbing classmates”
A13 Fun*	33 (3.2%)	“They make math more fun.”
A14 More memorable*	31 (3.0%)	“You can remember a formula better than when it’s just written down on paper”
A15 Emotion towards explainer*	26 (2.6%)	“The content creators radiate positive energy. They motivate more than my teacher.”
A16 Other examples*	23 (2.3%)	“They use examples to which I can relate.”
A17 Detailed*	22 (2.2%)	“More depth.”

Table 1: Overview of 17 categories of advantages of instructional mathematics videos as perceived by upper-secondary students (n = 1017), ordered by frequency. *Cohen’s kappa greater than 0.61; **Cohen’s kappa greater than 0.81.

To answer RQ2 for our sample, we use the ranking of the categories in Table 1. The first category “A1 Own pace” is more than ten percentage points ahead of the next one. Hence, learning at one’s own pace can be considered a very important advantage. This is also emphasised by the variety of answers that were attributed to this category: “if you ask something in class, you delay the whole lesson”, “you can pause if something is unclear” or “no waiting for others who are too slow”. This underlines the importance of individuality and the possibility to actively control the own learning pace. Other advantages in the list can also be linked to the activity of the learner, e.g. “A3 Repeating a topic or video”, “A5 Choose topics yourself” and “A6 Anywhere or anytime”. Other important advantages in the list can be linked to the content of the videos and how the content is depicted such as “A2 Short and structured”, “A4 Other explanations”, “A7 Interesting or advanced content”, “A8 Instructive” and “A9 Visualisations”. It is noticeable, however, that all categories appear rather “general” and not specifically linked to mathematics. Based on our data, it cannot be determined whether there are advantages exclusive to instructional *mathematics* videos.

To answer RQ3, a table similar to Table 1 was computed for the groups VG and NVG. For both groups, 16 categories scored above the threshold. For VG, “A16 Better examples” only scored 1.6% and for NVG, “A 17 Detailed” only scored 1.9%. Table 2 lists the five most mentioned advantages for both groups. Four categories can be found in the top five of both groups. However, “A7 Interesting / advanced content” is the fourth most mentioned advantage in group VG with 18% but has only rank 12 in group NVG with 3.3%. “A6 Anywhere or anytime” is in the top five of NVG but not VG, but got assigned nearly the same percentage of answers (NVG: 10.6%, VG: 10.3%). Category “A3 Repeating a topic” is noticeably more important for group NVG (29.2%) than for group VG (20.0%). It can be stated that some differences between the two groups exist which need to be examined in more depth in future research.

Group VG	Group NVG
A1 Own pace (37.3%)	A1 Own pace (40.9%)
A2 Short and structured (25.4%)	A3 Repeating a topic or video (29.2%)
A3 Repeating a topic or video (20%)	A2 Short and structured (25.9%)
A7 Interesting / advanced content (18%)	A4 Other explanations (18.5%)
A4 Other explanations (16.2%)	A6 Anywhere or anytime (10.6%)

Table 2: Five most mentioned advantages for VG (n = 445) and NVG (n = 572).

In sum, the 17 categories in Table 1 give an interesting insight why student use instructional mathematics videos. Even though we cannot conclude if some of the categories are exclusive to mathematical content, the present data can help gain a better understanding of students’ motivation and learning preferences which can ultimately help improve learning settings in school. Students value the possibility to individually control when (e.g. A1, A6, A11), how (e.g. A1, A4, A9) and what (e.g. A3, A5, A7) they learn. They further view instructional mathematics videos as an additional resource for learning (e.g. A2, A4) catering to different learning types (e.g. A9). Very good students also choose videos with advanced content which interest them (A7).

Instructional mathematics videos can have different types of content and didactic purposes. Videos aiming “to explain why” or to “demonstrate a procedure” (Lim & Wilson, 2018) can fulfil the didactical function of “repetition” (Bersch et al., 2020). This particular purpose seems to be very important for the students, as it corresponds to the advantages “A2 Short and structured” and “A3 Repeating a topic or video”. Other didactical purposes are not as important for students when using instructional videos on their own such as the “introduction” of e.g. an object to model (ibid.) which makes more sense as part of a whole lesson unit in school. When evaluating instructional videos and assessing their quality, it is thus necessary to differentiate whether students watch them on their own or as part of a school assignment and, if the latter, what didactical function this assignment fulfils. As opposed to Bersch et al. (2020) who criticise the allegedly passivity of students when watching instructional mathematics videos, the students in our sample seem to especially value their own active role and the ways they can control their learning. However, this does not necessarily mean that students' learning with instructional mathematics videos is successful. Meaningful scenarios for mathematics lessons integrating instructional videos are equally necessary as supporting students who use instructional mathematics videos on their own. Students need to be informed about different types and purposes of videos, to what they need to pay attention to if they want to recognise a "good" video and how to optimise their learning with this medium. When teachers and students are prepared to purposefully use instructional mathematics videos, this medium can enrich and support the learning of many students all over the world.

Limitations

The most relevant limitation regarding the significance of the presented results is that the sample is not representative. It is even a particular positive selection as a consequence of the self-selection of the participants. Hence, the answers to the research questions must be regarded in the context of this particular sample. Nevertheless, the results can be regarded a hypothesises and it has to be reviewed whether they hold true for other samples as well. Another limiting factor is that the sample only consists of German students. It is unclear, in how far the results hold true for other countries. Interestingly, some students from the present sample indicated to also watch internationally relevant mathematics channels on YouTube such as *3Num1Brown*, *Numberphile* or *KhanAcademy*. The results nevertheless need validation from studies in other countries. The age of the data should also be kept in mind, as the study was conducted in 2019 before Corona. The pandemic has most likely not fundamentally changed the advantages students attribute to the medium, so the results should still be valid. On a positive note, the data was collected when there was not yet an unusual high amount of attention on this medium due to the need for distance learning. The found categories might therefore be more reliable.

CONCLUSION AND OUTLOOK

The exhaustive list of categories demonstrates that there are numerous advantages in the usage of instructional mathematics videos from students' point of view. It is thus essential to not neglect this medium in school: it needs to be discussed how to maximise the benefits from using it but also to underline its limitations and potential risks. The Corona Pandemic has given a boost to this medium, which makes it even more important to engage into more related research. Our results emphasise that reasons to watch instructional mathematics videos can vary greatly. Some students use the videos for repetition, homework or to prepare for a test while others search for videos which fulfil their desire to know more about mathematics. More studies in other countries are needed to confirm our categories. The categories suggest that there are many opportunities to use instructional mathematics videos for individual support and differentiation for both weaker and more skilled students which must be investigated in future research.

References

Acuña-Soto, C. M., Liern, V., & Pérez-Gladish, B. (2019). A VIKOR-based approach for the ranking of mathematical instructional videos. *Management Decision*, 57(2), 501-522.

Bergmann, J., & Sams, A. (2012). *Flip your classroom: Reach every student in every class every day*. Eugene: International society for technology in education.

Bersch, S., Merkel, A., Oldenburg, R., & Weckerle, M. (2020). Erklärvideos: Chancen und Risiken - Zwischen fachlicher Korrektheit und didaktischen Zielen. [Explanation videos: Opportunities and risks – Between technical correctness and didactical goals]. *Mitteilungen der Gesellschaft für Didaktik der Mathematik*, 46(109), 58-63.

Guo, P. J., Kim, J., & Rubin, R. (2014). How video production affects student engagement: An empirical study of MOOC videos. In *Proc. of the first ACM conf. on Learning@scale* (pp. 41-50).

Kulgemeyer, C. & Peters, C. H. (2016). Exploring the explaining quality of physics online explanatory videos. *European Journal of Physics*, 37, 1-14.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics*, 33, 159-174.

Lim, K. H., & Wilson, A. D. (2018). Flipped learning: Embedding questions in videos. *Mathematics Teaching in the Middle School*, 23(7), 378-385.

Rat für kulturelle Bildung (2019). Jugend / YouTube / Kulturelle Bildung: Horizont 2019. [Youth / YouTube / cultural education: horizon 2019]. Retrieved from https://www.rat-kulturelle-bildung.de/fileadmin/user_upload/pdf/Studie_YouTube_Webversion_final.pdf [01/12/2020].

Ratnayake, I., Bruder, R., & Klein, T. (2019). Integrating video tutorials in mathematics teaching: quality criteria to choose good videos. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.), *Proc. 43rd Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, p. 177).

Schiltz, G. (2015). Video Analytics: when and how do students use tutorial videos. In *Proceedings of the 23rd International Conference on Computers in Education ICCE* (pp. 774-776).

Wetzel, S., & Ludwig, M. (2020). Wozu Mathematikunterricht, wenn es auch YouTube gibt? Mathematische Lernvideos unter der Lupe. [Why mathematics lessons, if there is YouTube? A close examination of mathematical explanation videos]. In H.-S. Siller, W. Weigel & J. F. Wörler (Eds.), *Beiträge zum Mathematikunterricht 2020* (pp. 1029-1032). Münster: WTM-Verlag.

Wolf, K. (2018). Video statt Lehrkraft? Erklärvideos als didaktisches Element im Unterricht [Video replacing teachers? Explanation videos as part of instruction]. *Computer+ Unterricht*, 109, 4-7.

THE VIRTUAL PERCENTAGE STRIP – AN INTERVENTION STUDY IN GRADE 6

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Learning percentages is known to be challenging. To support learners, visualisations and (virtual) manipulatives have been proposed, e.g. to build up mental models or initiate the use of visualisation strategies. In the study presented in this contribution, a tablet-based percentage strip and a double number line were used in an intervention study to investigate effects on strategy choice and learning outcomes on the topic of percentage changes. Results indicate that the use of the double number line and the virtual percentage strip alone is not effective to increase in performance. Rather, it seems to be relevant whether the learners adopt the structure of the double number line as their own visualisation strategy or not. Based on these results, we discuss possibilities to encourage learners to adopt graphical representations.

INTRODUCTION

Percentages are a mathematical topic with a long tradition and high relevance. At the same time, many learners have difficulties in solving tasks from this subject area (Parker & Leinhardt, 1995). One way to address these problems is to incorporate visualisations or virtual manipulatives into classroom teaching. For percentages, the pie chart or the percentage bar are suggested. The so-called double number line, which represents the relationships between the quantities given in a situation on a number line, has been discussed but rarely researched systematically. A dynamic variant of the double number line, which can act as a manipulative for learning percentages, is the percentage strip (van den Heuvel-Panhuizen, 2003). This contribution focuses on the question if and how the application of visualisations based on the double number line can indeed contribute to students' learning on percentage change.

THEORETICAL AND EMPIRICAL BACKGROUND

Percentages as proportional variable

From a mathematical point of view, two different approaches to percentages can be distinguished. On the one hand, percentages can be conceived as specific fractions with denominator 100, that are particularly easy to compare, or percentages can be considered as values which are proportional to a given variable (Parker & Leinhardt, 1995). Based on the second conceptualisation, many textbooks use the rule of three as primary solution strategy to solve percentage tasks. According to this strategy, unknown values are calculated from a set of provided values by proportional inference. With the help of the rule of three, a wide range of tasks for percentages can be solved, ranging from basic calculations (two of the three relevant values are given and the third is to be calculated), to tasks involving percentage changes (e.g. "A number is increased by 15

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 265-272. Khon Kaen, Thailand: PME.

%. You get 230. What is the number?”) (Kleine, Jordan, & Harvey, 2005). Beyond identifying the relevant values and their relations in the tasks, applying the rule of three requires to find appropriate common divisors as for intermediate steps. Accordingly, this ability is an essential prerequisite when using this strategy. For example, to calculate 15% of 600, proportional inference over intermediate percentages 1% or 5% makes sense, while inference to 7% is probably of little help (van Galen & van Eerde, 2013).

Visualisations and manipulatives for percentages

In the context of teaching percentages, the percentage bar is often suggested as a useful visualisation (Hoven & Garelick, 2007). Van Galen and van Eerde (2013) see possible advantages in the use of the percentage bar in the representation of the relationship between provided and unknown values, a natural approach to the introduction of calculation strategies, and the possibility that goal-directed intermediate steps in a calculation can be visualised by simply drawing in these steps into the bar (Fig. 1 left). That the percentage bar can actually support the learning process could be shown by Walkington and colleagues (2013). The learners in their study showed better results when the task text was supplemented with the visualisation of a percentage bar than when the task text was presented without a percentage bar.

An alternative visualisation that can support the learning process is the double number line (Küchemann, Hodgen, & Brown, 2011). It represents two proportionally linked variables on two connected number lines and can be used, for example, to represent proportional mappings (e.g., kilometers and miles). In the context of percentages, the two relevant variables are the percentages and the values of an associated variable (van den Heuvel-Panhuizen, 2003). In contrast to the percentage bar, which is strongly linked to the conceptualisation of percentages as fractions, the double number line emphasises the concept of percentages as proportional quantities. The representation of percentages and associated values on an “empty number line” also connects to the learners’ prior experience with basic arithmetic operations, such as the visualisation of a multiplication on the number line with operator arrows (Klein, Beishuizen, & Treffers, 1998). By representating of intermediate steps, solution strategies based on the rule of three can be visualised directly in the double number line representation (Fig. 1 right). A physical implementation of the ordinal structure of the double number line in a manipulative may be achieved with the percentage strip, an elastic rubber band contain a “percentages” scale that can be adjusted dynamically to a scale of associated values (van den Heuvel-Panhuizen, 2003).

The effects of using the double number line has been investigated in case studies and teaching experiments (Küchemann et al., 2011; van den Heuvel-Panhuizen, 2003), showing mixed results. On the one hand, the authors argue that learners are able to use the double number line meaningfully even without prior conceptual knowledge, and that the ordinal structure of the double number line may support estimation and validation of task results. On the other hand, comparing the double number line with the percentage bar, van Galen and van Eerde “prefer the bar because it gives a clear and concrete picture of the relations between the total and its parts" (van Galen & van Eerde, 2013). However, it is not explained in detail how the double number line is not appropriate or why the double number line does not represent a concrete picture of the relations between the total and its parts. Beyond these reports, systematic evidence on the effect of the use of the double number line and related manipulatives is scarce. In particular for the field of percentage changes, the double number line may have advantages, because also percentages above 100% can be represented more easily than in the bar representation (where the full bar usually represents 100%).

How visualisations and manipulatives may support learning

Different mechanisms could explain positive effects of visualisations and manipulatives such as the double number line or the percentage strip on students' learning. Firstly, some authors propose that working with external representations and manipulatives allows students to build up "mental models", i.e. a well-connected mental structure that connects different representations of a concept, which can be used directly and mentally when approaching new problems (Goldin & Kaput, 1996). The structure of the manipulative may influence the structure of the mental model (Schnotz & Bannert, 2003). To support the generation of such well-connected mental structures, some authors have proposed to combine different representations of a concept into one integrated visualisation (Ainsworth, 2006), such as linking of the double number line and the rule of three in an integrated visualisation (Fig. 1 right). It is an open question, however, if and how an ordinal mental model of percentages can be built up using a double number line representation.

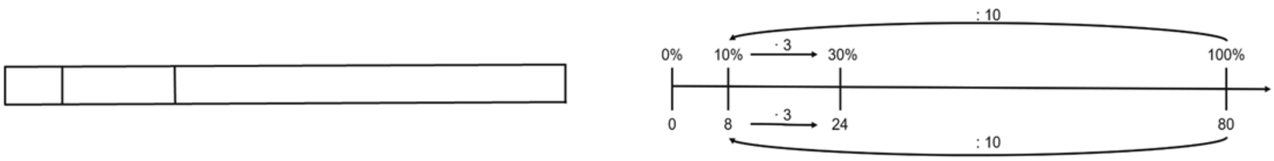

Figure 1: Left: A percentage bar with intermediate steps at 10 % and 30 %, Right: integrated visualisation of the double number line and rule of three strategy.

Secondly, instead of building up directly accessible mental models, students may adopt representations used in the classroom to generate external visualisations (in the sense of drawings) while working on new problems. This mechanism would best be described as adopting a visualisation strategy, which includes selecting the double number line as a suitable basis for self-generated drawings (cf. Heinze, Star, & Verschaffel, 2009). While prior research has found that spontaneous use of drawings can be beneficial, this did not automatically extend to instructed drawing (Cox, 1999). It seems that students need sufficient experience with visualisations to generate sufficiently accurate drawings (Rellensmann, Schukajlow, & Leopold, 2017).

THE CURRENT STUDY AND RESEARCH QUESTIONS

In a preliminary study with German 6th graders ranging over a sequence of 10 lessons, no evidence was found that using the percent strip would support the generation of mental models as proposed by the first mechanism described above (Willms & Ufer, 2018). Thus, the question arises whether the corresponding effects might be strengthened by encouraging students to use the double number line as a visualisation strategy (2nd mechanism), and whether building up such mental models for percentages might require an integration of the percentage strip with a representation of the rule of three strategy (1st mechanism with integrated visualisation). In an experimental intervention study on percentage change, we investigated the two assumed mechanisms that could lead to effects of the percentage strip and the double number line:

1) We compared whether instruction using these two visualisations would have different effects than instruction without these visualisations on student learning. Beyond this, we differentiated between an integrated visualisation, including the rule of three strategy in the percentage strip, and separate, non-integrated use of the visualisations and the rule of three strategy. Based on theoretical assumptions (Ainsworth, 2006; Schnotz & Bannert, 2003), we expect a positive effect of using the percentage strip (H1), which should be more pronounced, for the integrated visualisation (H2).
2) Additionally, we aim to study whether adopting of the double number line as a visualisation strategy could yield a positive effect on the learning outcomes. We investigated whether learners, who adopted the double number line as a visualisation strategy, showed a greater learning gain in percentages than learners who did not adopt such a strategy. Following Rellensmann and colleagues (2017), we expect higher performance when learners decided to adopt the double number line to solve tasks in percentages (H3).

METHODS

To make a contribution to these questions, we conducted an experimental intervention study with 334 grade sixth students at 14 secondary schools (Gymnasium) in a large city in southern Germany. The students took part in the experimental study in a pre-post-test design with 3 experimental groups (no visualisations, separate visualisations, integrated visualisation). Participation was voluntary and based on explicit parents' consent. The study took part in the usual classroom setting, but instruction was done by the first author and a trained assistant. Students within each classroom were allocated randomly to one of the three groups, covering two different groups in separate groups within each classroom. The study followed a sequence of 45 min. pre-test, 15 min. break, 90 min. intervention, 15 min. break, and 45 min. post-test.

For the pre-test, two prior knowledge scales on percentages (3 technical items, $\alpha = .62$, $M = 0.69$, $SD = 0.33$; 3 textual items, $\alpha = .65$, $M = 0.69$, $SD = 0.33$) were developed . Two additional scales (knowledge of divisors, 5 items, $\alpha = .74$, $M = 0.82$, $SD = 0.13$, coding: relative number of correct divisors; numerical estimation via number line tasks (Siegler & Opfer, 2003), 10 items, $\alpha = .57$, $M = 0.06$, $SD = 0.04$) were used. For the post-test, 9 items were developed on the topic "percentage increase and decrease" including 6 technical items ($\alpha = .74$, $M = 0.71$, $SD = 0.24$) and 3 textual items ($\alpha = .78$, $M = 0.55$, $SD = 0.31$). For each solution we coded whether a double number line visualisation was visible in the student solution. Furthermore, an adapted version of the number line tasks for percentages was developed to survey the numerical estimation in percentages (6 items, $\alpha = .53$, $M = 0.14$, $SD = 0.09$). For all number line tasks, the difference between the estimation and the real value divided by the real value was used as a score. Lower values correspond to more exact estimations. All other tasks were coded dichotomously (correct/false).

Similar to its physical counterpart, the tablet-based version of the percentage strip consists of two scales which can be linked proportionally. Each can be scaled by dragging with the finger. Auxiliary lines may support structuring (Figure 2).

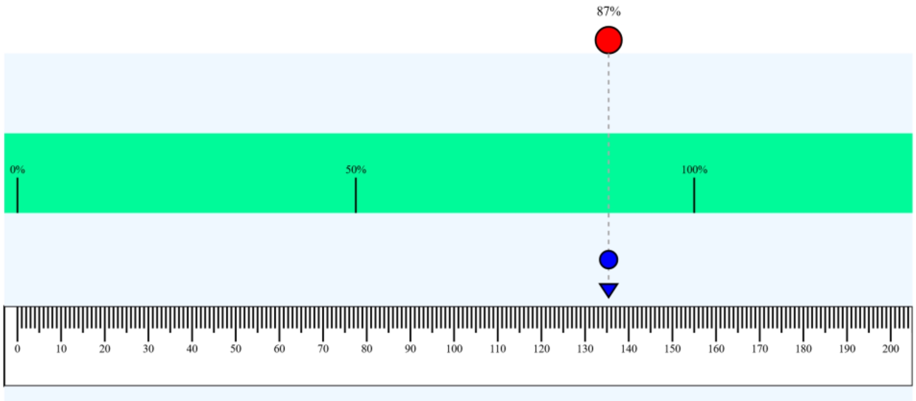

Figure 2: The virtual percentage strip with one inserted auxiliary line

In the control group (CG), applying the rule of three for solving percentage change tasks was discussed along an elaborate teaching script involving group discussions and individual work. When working on the tasks, the learners were guided to follow a structured approach: First, unknown values and relevant information from the task was identified and structured. Then, students searched a suitable intermediate step calculated the target value. The experimental groups (EG1, EG2) followed the same teaching script. In EG 1, the virtual percentage strip was introduced as a manipulative and the double number line as a related visualisation. In EG2, integrated versions of the percentage strip and the double number line, including the rule of three strategy were used. In both experimental groups, adopting a visualisation strategy based on the double number line was explicitly encouraged.

ANALYSES AND RESULTS

There were no significant differences between the three groups for all four covariates ($p > .460$). We used linear mixed models with the scales from the pre-test as covariates and the group membership as well as the adoption of a double number line visualisation strategy (0=not used, 1=used at least once) as fixed factors. Classroom differences were controlled by a random factor. The three scales from the post-test were used as dependent variables in separate analyses.

1st mechanism: The results showed no significant relationship between group membership and the technical ($M_{CG} = 0.72$, $M_{EG1} = 0.69$, $M_{EG2} = 0.72$; $F(2{,}262.27) = 0.99$, $p = .371$) and textual skills ($M_{CG} = 0.57$, $M_{EG1} = 0.53$, $M_{EG2} = 0.55$; $F(2{,}314.07) = 1.15$, $p = .318$) in the post-test, nor was there a significant relationship between group membership and the numerical estimation in percentages ($M_{CG} = 0.15$, $M_{EG1} = 0.14$, $M_{EG2} = 0.14$; $F(2,326) = 0.60$, $p = .547$). Contrary to H1 and H2, no significant effect of group membership on the performance in the three post-test measures were found. This indicates that neither the use of the virtual percentage strip and the double number line nor the integrated visualisation with the rule of three strategy had a systematic effect on students' learning.

2nd mechanism: In total, 28 % of the learners were classified as DNL users, i.e. they adopted a double number line based visualisation at least once in the post-test. There were substantially more DNL users in the experimental groups than in the control group (0.01 % in CG, 37 % in EG1, 43 % in EG2), but the difference in the number between the two experimental groups was not significant ($\chi^2 = 0.575$, $df = 1$, $p = .448$).

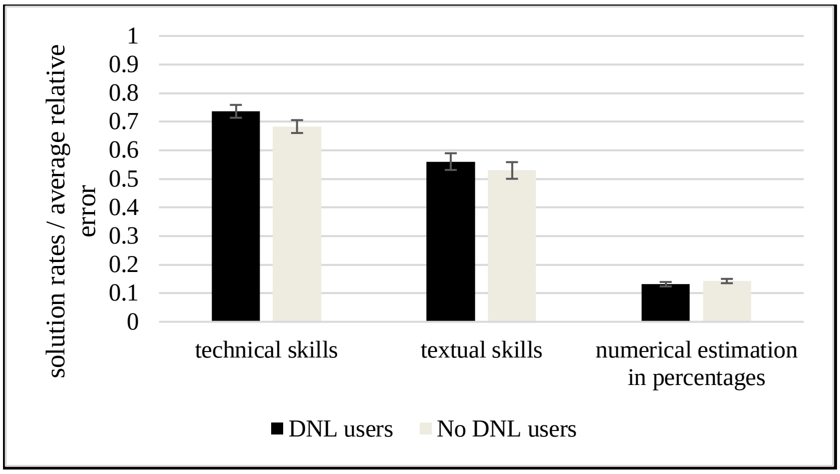

Figure 3: solution rates (technical and textual skills) resp. average relative error (estimation) for the DNL users and the rest of the learners in the experimental groups.

DNL users showed significantly better technical skills in the post-test, controlling for pre-test measures ($M_{\text{DNL users}} = 0.73$, $M_{\text{no DNL}} = 0.70$; $B = 0.07$, $p = .008$, $d = 0.13$). We found a tendency in favour of DNL users for textual skills ($M_{\text{DNL users}} = 0.55$, $M_{\text{no DNL}} = 0.54$; $B = 0.07$, $p = .086$, $d = 0.02$). For numerical estimation in percentages the difference was not significant ($M_{\text{DNL users}} = 0.13$, $M_{\text{no DNL}} = 0.15$; $B = -0.01$, $p = .240$).

For further analysis, the total sample was restricted to the experimental groups. For each of the four pre-test scales, an ANOVA was calculated with the pre-test scale as the dependent variable and DNL use as the independent fixed factor. The results were not significant for all calculations (technical skills: $F(1,225) = 0.58$, $p = .449$; textual skills: $F(1,225) = 0.69$, $p = .408$; knowledge of divisors: $F(1,225) = 0.22$, $p = .639$; numerical estimation: $F(1,225) = 0.11$, p = .737). Overall, it is not possible to explain better performance of DNL users in the post test by better pre-test scores.

DISCUSSION AND OUTLOOK

We presented a study using a virtual percentage strip as a manipulative and a double number line as a visualisation for learning precentages. Similar to a prior study (Willms & Ufer, 2018), the analyses did not provide evidence that the visualization supported learners to generate or extend usable mental models of percentages (Schnotz & Bannert,

2003) or that an integrated presentation of representations (Ainsworth, 2006) would support this effect. However, the adoption of a visualisation strategy based on the virtual percentage strip was accompanied by better technical skills in percentages, and this effect could not be traced back to better prior knowledge. This extends results by Rellensmann et al. (2017), indicating that systematic training of a DNL based visualisation strategy can lead to sufficiently high-quality visualisation strategies to support students performance. From a theoretical point of view, this work substantiates the view that not the sole integration of a virtual manipulative per se is not necessarily conducive to learning. We conclude that the virtual percentage strip, which has been rarely used so far, can provide effective support under certain conditions even within a short time span of 90 minutes. However, it seems crucial that students are repeatedly encouraged to use the double number line as a visualisation strategy. Further research is needed, which should focus on the question under which individual conditions and instructional frameworks learners can be urged to adopt a visualisation structure that is conducive to learning, and also supports estimation skills. It also remains an open question if an interactive virtual manipulative such as the percentage strip is beneficial, or if the double number line as a static visualisation might be sufficiently effective.

References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, *16*(3), 183–198.

Cox, R. (1999). Representation construction, externalised cognition and individual differences. *Learning and Instruction*, *9*(4), 343–363.

Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397–430). Mahwah, NJ: Erlbaum.

Heinze, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM, 41*(5), 535–540.

Hoven, J., & Garelick, B. (2007). Singapore Math: Simple or Complex?. *Educational Leadership*, *65*(3), 28–31.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The Empty Number Line in Dutch Second Grades: Realistic versus Gradual Program Design. *Journal for Research in Mathematics Education*, *29*(4), 443–464.

Kleine, M., Jordan, A., & Harvey, E. (2005). With a focus on `Grundvorstellungen' Part 2: `Grundvorstellungen' as a theoretical and empirical criterion. *ZDM*, *37*(3), 234–239.

Küchemann, D., Hodgen, J., & Brown, M. (2011). Using the Double Number Line To Model Multiplication. In M. Pytlak, T. Rowland, & T. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (CERME7)* (pp. 326–335). Poland: University of Rzesów.

Parker, M. S., & Leinhardt, G. (1995). Percent: A Privileged Proportion. *Review of Educational Research*, *65*(4), 421–481.

Rellensmann, J., Schukajlow, S., & Leopold, C. (2017). Make a drawing. Effects of strategic knowledge, drawing accuracy, and type of drawing on students' mathematical modelling performance. *Educational Studies in Mathematics*, *95*(1), 53–78.

Schnotz, W., & Bannert, M. (2003). Construction and interference in learning from multiple representation. *Learning and Instruction*, *13*(2), 141–156.

Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological Science*, *14*(3), 237–243.

van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1), 9–35.

van Galen, F., & van Eerde, D. (2013). Solving Problems with The Percentage Bar. *Indonesian Mathematical Society Journal on Mathematics Education*, *4*(1), 1–8.

Walkington, C., Cooper, J., & Howell, E. (2013). The effects of visual representations and interest-based personalization on solving percent problems. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of 35th PME-NA* (pp. 533–536). Chicago: Universität von Illinois.

Willms, A., & Ufer, S. (2018). Das Prozentband als Arbeitsmittel im Mathematikunterricht der Sekundarstufe [The percentage strip as a manipulative in secondary mathematics education]. *Beiträge zum Mathematikunterricht 2018* (pp. 1991–1994). Münster: WTM-Verlag.

COMPARING THE PREDICTIVE POWER OF FIRST GRADERS' GENERAL COGNITIVE SKILLS AND PRIOR KNOWLEDGE ON LATER ACHIEVEMENT

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Knowing more about the early predictors of math achievement is a fundamental prerequisite for targeted learning interventions. We compare the extent to which some of the most discussed sets of predictors – (1) working memory capacity and attention (2) personality and (3) prior math knowledge – predict math achievement in grade 2 beyond fluid intelligence. We have collected a rich dataset of about 500 children, including information on various subcomponents of these sets of predictors, thus our regression framework is able to shed light on these subcomponents. Including all sets of predictors within the same regression reveals two verbal working memory tests, the personality trait "openness" and all prior knowledge subtests as significant predictors for math achievement.

INTRODUCTION

Early skills are strong determinants of later school achievement, and later deficit-compensatory programs are much less promising than early interventions (Heckman 2007). Therefore, knowing the main predictors for school success is of great value for developing targeted early childhood interventions. Moreover, since disadvantaged children benefit most from early interventions (Heckman 2007), knowledge of these main predictors can help to identify students at risk and to decrease the educational achievement gap at early stages. Indisputably, fluid intelligence is a strong predictor for later school success. However, for educational research and practice, those predictors that are malleable by teaching are of particular relevance. Among the most discussed predictors for math achievement, students' working memory (WM) capacity, their personality and their prior knowledge are frequently reported (e.g. Watts et al. 2014). However, most of the present studies have a quite small selection of predictors and the results are hardly comparable across studies, because of different data and methodologies used. Thus, we need to include several of the most discussed predictor sets within the framework of one study to be able to compare the explanatory power of the different predictors.

Even though the relationship between working memory capacity and math achievement is well established (e.g. Raghubar et al. 2010, Menon 2010), it is not clear which WM components are generally most predictive of math achievement (Friso-van den Bos et al., 2013). Hence, it is useful to provide more evidence on different WM tasks. Furthermore, there is a strong link between working memory and attention: it has been shown that attention control and working memory capacity

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 273-279. Khon Kaen, Thailand: PME.

are separate, yet correlated, factors and there is only little knowledge about the overlap between them (Raghubar et al. 2010, Winkel et al. 2018).

In another strand of literature, different personality traits have been found to be associated with math achievement at different ages (Laidra et al. 2007), but previous research mainly focused on older children and adolescents. It is not clear, to what extend these results can be transferred to primary school children (see e.g. Poropat, 2009). Among the few existing studies with primary school children, the personality trait "openness" seems to be the strongest predictor (e.g. Allik et al. 2007, Neuenschwander et al. 2013). Neuenschwander and her colleagues demand: "*Thus, more empirical findings are needed within younger age groups in order to integrate these findings into a developmental framework*." (2013, p. 118).

Finally, prior mathematics knowledge is assumed to predict later math achievement (e.g., Hemmings et al. 2011). Even though this relation e.g. between school-entry math skills and later math achievement (Duncan et al. 2007) might seem obvious, it remains to be shown how strong it is compared to the other discussed predictors. First, many studies control for prior achievement when predicting school achievement, but they do not discuss the predictive power of prior knowledge compared to other psychometric measures. Second, in some cases, prior knowledge is assessed by tests and in others, it is assessed by teacher questionnaires or grades. Since these perspectives might complement each other, it might be revealing to include both, precise test measures as well as a more holistic teacher rating.

To be able to compare the predictive power of all predictor sets mentioned above – i. e. WM & attention, personality, and prior math knowledge – in this paper we first regress math achievement on each of these predictor sets and finally we include them all in one regression model. Based on early data from first grade (as predictors) as well as data collected one year later, we investigate the following research question: *To which extent do first graders' working memory capacity, their attention control, their personality and their prior knowledge predict later math achievement (beyond fluid intelligence and further control measures)?*

THEORETICAL BACKGROUND

Since we will combine predictor sets from quite different fields of research, we will give a brief insight to all three predictor sets before we describe our research design.

Working Memory and Attention

According to Baddeley & Hitch (1974), WM is a cognitive system that provides temporary storage and manipulation of information. For example, mental arithmetic problems require students to store several numbers and interim results in mind while manipulating and combining them with prior knowledge. WM capacity is a theoretical construct that is not easy to measure. The storage component characterizes an amount of information that is temporarily in a very accessible state. This memory has proven to be further separable into a verbal and a visuo-spatial component (confirming Baddeley and Hitch's (1974) model of the "phonological loop" and the "visuo-spatial sketchpad"). Tests should account for this complexity, so that both simple as well as more complex tasks are needed to include both the storage and the processing component.

Beside the typical aspect of attention as focusing on relevant information, a second aspect of attention is inhibition control – the mental ability to ignore distracting information or to inhibit unwanted responses. Especially this second component of inhibition is found to be correlated to math achievement as well as to verbal and visuo-spatial WM. Thus, both subcomponents should be considered.

Personality

Another strand of literature assumes students' personality to be predictive of math achievement. The five-factor model, also known as the "Big Five", is a psychological concept that is used to model personality. It assumes that differences in people's personalities can be traced back to five personality traits: openness to experience, conscientiousness, extraversion, agreeableness and neuroticism. Personality is assumed to be related to academic performance in two ways: first, personality is closely linked to motivational constructs that are in turn related to school achievement. Second, personality and academic performance are thought to be associated due to common links between intelligence and the personality trait openness (Poropat, 2009).

Prior mathematics knowledge

When assessing prior knowledge, it is of great importance to have detailed knowledge about the curriculum, the mathematical concepts, specific difficulties and the expected skill levels at the specific times of measurement. At the beginning of the second half of grade 1 the students deal with whole numbers up to 20 and begin to add and subtract them more and more in mind. 15 months later in grade 2, the students add or subtract in the number range up to 100. Yet, they are generally more confident if one of the numbers is single-digit. Beside this computational perspective, the literature reveals evidence for the predictive power of further conceptual numeracy skills specifically for the first school years: the development of number sense (e.g. Jordan et al. 2010). Thus, a test with visual number representations has to be included beside tests about formal computations.

RESEARCH DESIGN

Participants and data collection

In our study, we focus on a sample of more than 500 typically developing primary school children from an own field experimental study in Mainz (Germany). While our main study (Berger et al. 2020) was a randomized controlled trial, the present paper does not analyse the intervention, but controls for it instead. We focus on test data from the first evaluation wave and from a 1-year follow-up as well as on questionnaire data. The attrition rate within this year was very low (7%) and the teacher response rate was 100%. For the selected variables, we obtain a sample of 493 children without any missing data. Mean age at test in first grade was 7.0 years, mean age at test in second grade was 8.3 years; 51% were female. The first graders completed highly standardized digital school achievement tests (see below), using touchscreens, and standardized auditory instructions via headphones. Furthermore, teachers completed a digital questionnaire on students' skills and some background characteristics. All data was collected by a professional data collection service provider on our behalf.

Main variables

Here we just give a brief overview on the main variables. More detailed descriptions and test-specific references are documented in our main study (Berger et al. 2020).

Grade 2 mathematics performance (outcome variable): Mathematics performance was assessed by a number sense task, tasks about addition and subtraction (with auditory and written problems) and a geometry task.

Grade 1 working memory and attention control: We used three subtests in the area of verbal and visuo-spatial working memory. We used a digit span task for a verbal simple span, an object span task for the verbal complex span and a location span task for the visuo-spatial complex span. In the area of attention control we employ a "GoNoGo test" and the "bp-test" to measure inhibition and concentration.

Grade 1 prior mathematics knowledge: The subtests were parallel to our outcome variable but with different items and adapted difficulties. To complement these specific test measures with a more holistic view we asked teachers for an overall rating of their students' mathematical skills.

Personality: Teacher-rated personality questionnaire, covering the "Big Five" personality traits: openness to experience, conscientiousness, extraversion, agreeableness and neuroticism.

Control variables: Age, gender, IQ (subset of Raven's Progressive Matrices, included due to high correlations to WM, attention and achievement), and dummies due to the nested structure of the data (class-fixed effects).

Methods

We use linear regression models (with school-fixed effects, estimated by OLS) and regress grade 2 math achievement on each of the three outlined predictor sets. We begin the analysis with a baseline model including IQ, age, gender, and class fixed effects to control for unobserved differences between the classes and schools. By adding each predictor set of our grade 1 measures individually (i.e. set by set), and finally combining all predictor sets within the framework of one single regression model, we are able to compare effect sizes and changes in the coefficient of determination R^2 relative to the baseline model to assess the relative explanatory power of each predictor. Importantly, to facilitate the comparison of effect sizes, the scores for all measures have been z-standardized to mean of 0 and standard deviation of 1. Beyond the results reported in this regression table, we also checked descriptive statistics, correlations, and more detailed or stepwise regressions with the same variables on the same sample.

RESULTS

Our baseline model (column 0 of Table 1) – including fluid intelligence, age, gender and the other listed control variables – explains already 34.1% of the variance math achievement one year later. In contrast, the same model without fluid intelligence explains 12.5% of the variance.

Model (1) Working memory and attention

By adding a set of three different working memory measures, R^2 increases from 34.1% to 47.3%. All three working memory tests contribute significantly ($p < 0.01$) to this increase of 13.2 percentage points of overall variance. Without the WM measures included in the regression, one of our attention measures has a small, but significant effect size (not shown in the regression table), but in combination, the WM measures dominate the effect of attentional control (column 2 of Table 1).

Model (2) Personality

By including the five personality traits to the baseline model, the explanatory power of the model increases by 18 percentage points of overall variance (compared to the baseline model). The driving force is the personality trait "openness to experiences", whose contribution is highly significant ($p < 0.01$): when openness increases by one standard deviation, second grade math achievement increases by about 0.46 standard deviations. The personality trait "conscientiousness" contributes a bit, but the other three personality traits do not seem to play a major role in our model.

Model (3) Prior mathematics performance

By including our grade 1 measures for prior mathematics knowledge and the more holistic teacher-rated math skills, we can explain an additional 20.6 percentage points relative to our baseline model. Each of the three subtests as well as the overall teacher rating contribute significantly to the increase in R^2 (column 3 of Table 1).

	(0)	(1)	(2)	(3)	(4)
IQ (Control measure)	0.487***	0.309***	0.278***	0.204**	0.161**
WM: Digit Span		0.176***			0.080*
WM: Object Span		0.118*			0.079*
WM: Location Span		0.187***			0.072
Att.: Attention Errors		-0.099			-0.064
Att.: Inhibition Errors		0.036			0.063
Attention: bp-Test Errors		-0.077			0.020
Personality: Openness			0.462***		0.244***
Pers.: Conscientiousness			0.075		0.038
Personality: Extraversion			-0.094		-0.073
Pers.: Agreeableness			-0.052		-0.009
Personality: Neuroticism			-0.059		0.038
Math: Number Sense				0.104*	0.085
Math: Add & Subtract				0.197***	0.082*
Math: Geometry				0.138**	0.094*
Math: Teacher Rating				0.284***	0.124*

(further control measures including age, gender and class fixed effects are not listed here)					
Constant	3.413	2.609	1.684	1.615	1.517
Observations	493	493	493	493	493
R^2	0.341	0.473	0.521	0.547	0.594
Adjusted R^2	0.294	0.428	0.481	0.510	0.550

Table 1: Results from linear regression of grade 2 math achievement on grade 1 measures for (1) WM & attention, (2) personality, (3) prior knowledge, and (4) all three predictor sets (M=0, SD=1, robust standard errors)

Overall model (4)

The rightmost column shows which predictors remain dominant if all predictors are included in the regression: the three documented predictor sets are able to explain an additional 25.3 percentage points of the overall variance beyond IQ and all other control measures. Interestingly none of the three predictor sets dominates another one substantially: Beyond IQ (1) the two verbal WM tasks, (2) the personality trait openness and (3) all prior math tests remain significant in our overall model.

DISCUSSION

Our regressions reveal that beyond IQ, prior math knowledge, openness to experience, as well as verbal working memory are all strong predictors of math performance, also when simultaneously included in the regression.

Despite the large proportion of shared variance with IQ, in the WM model each of our three working memory tests itself has a significant effect size. The two verbal memory tests even remain significant in the last overall model. Since it is not clear from the literature, which WM components are generally most predictive of math achievement (Friso-van den Bos et al., 2013), this result further emphasizes the importance of verbal WM for typical math skills in this age group. Our attention measures do not seem to play a major role beyond WM and IQ.

Also in line with the existing literature, we find the personality trait openness to be a strong predictor in this age group. Openness even remains significant in our overall model. This result adds to findings by Neuenschwander et al. (2013), underlining the relevance of this personality trait in this age group.

Finally, all subtests for prior knowledge are significant predictors for mathematics achievement one year later. This result confirms for example findings from Jordan et al. (2010) and Watts et al. (2014).

Our contribution to the literature – above the already mentioned aspects – is that we compare the magnitude between different predictor sets within the same dataset. This result complements existing findings and helps to complete the complex mosaic of predictors of math achievement. Since these predictors can only be interpreted in a correlational way and since the results always depend on the concrete tests, more studies with diverse predictor sets, similar highly standardized test measures and even longer time spans are needed before clear practical conclusions for school practice can be drawn. Moreover, in the context of future work, we want to study the role of moderators and mediators for the association between the detected early predictors and later math achievement. Nevertheless, even without this further work, the results presented and discussed here clearly emphasize the relevance of early interventions (see Heckman 2007) for improving malleable predictors, such as early math skills and WM capacity.

References

Berger, E., Fehr, E., Hermes, H., Schunk, D. & Winkel, K. (2020). The impact of Working Memory Training on Children's Cognitive and Noncognitive Skills.

Working Paper. Retrieved 14.1.2021, from https://papers.ssrn.com/sol3/papers.

cfm?abstract_id=3622985

Duncan, G. J., Dowsett, C., Claessens, A., Magnuson, K., Huston, A., Klebanov, P., ... & Sexton, H. (2007). School readiness and later achievement. Developmental psychology, 43(6), p. 1428.

Friso-van den Bos, I., van der Ven, S., Kroesbergen, E., & van Luit, J. (2013). Working memory and mathematics in primary school children: A meta-analysis. Educational research review, 10, pp. 29-44.

Heckman, J. (2007). The economics, technology, and neuroscience of human capability formation. Proceedings of the national Academy of Sciences, 104(33), pp. 13250-13255.

Hemmings, B., Grootenboer, P., & Kay, R. (2011). Predicting mathematics achievement: The influence of prior achievement and attitudes. International Journal of Science and Mathematics Education, 9(3), 691-705.

Jordan, N. C., Glutting, J., & Ramineni, C. (2010). The importance of number sense to mathematics achievement in first and third grades. *Learning and individual differences*, 20(2), 82-88.

Laidra, K., Pullmann, H., & Allik, J. (2007). Personality and intelligence as predictors of academic achievement: A cross-sectional study from elementary to secondary school. *Personality and individual differences*, 42(3), 441-451.

Menon, V. (2010). Developmental cognitive neuroscience of arithmetic: implications for learning and education. *ZDM*, 42(6), 515-525.

Neuenschwander, R., Cimeli, P., Röthlisberger, M., & Roebers, C. (2013). Personality factors in elementary school children: Contributions to academic performance over and above executive functions?. *Learning and individual differences*, 25, 118-125.

Poropat, A. (2009). A meta-analysis of the five-factor model of personality and academic performance. Psychological bulletin, 135(2), pp. 322.

Raghubar, K., Barnes, M., & Hecht, S. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and individual differences*, 20(2), 110-122.

Watts, T. et al. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher*, 43(7), pp. 352-360.

Winkel, K., Mueller, H., and Schunk, D. (2018): Mathematics achievement and the role of working memory and attention – Evidence from a large-scale study with first graders. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 443-450). Umeå, Sweden: PME.

THE SPATIAL CONTIGUITY PRINCIPLE IN MATHEMATICS TEXTBOOKS

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The spatial contiguity principle recommends that related pictures and text should be displayed together on a page. This study explored whether diagram and text placement influences textbook-users' perceptions of mathematics explanation quality. We asked students and teachers to compare real-world textbook explanations using a comparative judgement technique. This method enabled us to understand whether we could meaningfully measure explanation quality on a single scale. We found that participants tended to perceive explanations as being higher quality if diagrams were placed closer to related text, rather than apart. These findings support the spatial contiguity principle and also suggest that this principle translates into real-world mathematical applications.

RATIONALE FOR RESEARCH

In this report, we investigate the *spatial contiguity principle*: the idea that related text and pictures should be displayed together on a page (Mayer, 2020). Mayer (2020) developed this principle as a subset of the *split-attention effect*; this effect suggests that we should avoid designing educational materials which cause learners to split their attention between more than one location (Ayres & Sweller, 2005). This is an issue because, using the theoretical rationale of Cognitive Load Theory, splitting attention across locations increases extraneous cognitive load (Ayres & Sweller, 2005). This type of cognitive load is detrimental to learning because it is load imposed by ineffective instructional design rather than load directly related to learning the material (Sweller et al., 1998). If related information is separated then learners have to use cognitive resources to keep concepts in mind as they search the material to make connections (Ayres & Sweller, 2005). If related information is presented closer together then connections between the information are implied and learners can dedicate more cognitive resources to understanding the material.

More recently, Schroeder and Cenkci (2020) gave a slightly different explanation for the mechanisms behind the split-attention effect. Instead of suggesting that split designs impose increased extraneous cognitive load, they proposed that integrated designs help to allocate germane resources. Germane cognitive load is the load imposed by the learning process (Sweller et al., 1998) and, Schroeder and Cenkci suggested that, integrated designs facilitate the integration aspect of cognitive processing. Either way, there is a wealth of evidence for the split-attention effect and the spatial contiguity principle in educational materials (Mayer, 2020).

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In mathematics education, early research on split-attention effects evolved from research using worked examples (Tarmizzi and Sweller, 1988). However, there is limited research using other types of mathematical materials, such as explanations. Indeed, the majority of research on the split-attention principle uses (i) materials focused on the teaching and learning of science and also (ii) materials created or adapted for the purpose of experimentally testing a theory, including adapted materials from existing textbooks or resources (e.g., Chandler & Sweller, 1991; Johnson & Mayer, 2012; Makransky et al., 2019; Mayer, 1989; Moreno & Mayer, 1999; Tindall-Ford et al., 1997). Much less work uses genuine real-world pedagogical materials. We addressed this gap by using unmodified textbook explanations to investigate the spatial contiguity principle in mathematics.

THEORETICAL FRAMEWORK

The spatial contiguity principle was originally developed by Mayer (2005) from his Cognitive Theory of Multimedia Learning. This theory assumes a cognitive model of multimedia learning where multimedia refers to the presentation of material using both static or moving pictures and written or spoken text. Researchers have developed various principles for multimedia learning from this model, including the spatial contiguity principle.

The most relevant aspect of the model here is its adoption of *dual-coding theory*, which identifies two channels that we process information through in memory (Mayer, 2005). Mayer proposed that we process words through an auditory channel and pictures through a visual channel. Each channel has its own processing capacity, so our visual channel could reach capacity even though our auditory channel is not being used, or vice versa. This means that we can build stronger learning outcomes using verbal and pictorial learning models rather than only one or the other. The spatial contiguity principle builds on this by recognising that placement of text and pictures affects their effectiveness: if text and pictures are displayed together then we are more likely to recognise their relationship which enables us to build a more complete understanding using both resources (Mayer, 2020). On the other hand, if information is physically separated then we might struggle to connect the two representations and therefore build a less complete learning outcome, or we could use cognitive resources to search for connections between the two representations so that less resources are available to build the resultant learning outcome.

METHODOLOGY

Our study investigated the spatial contiguity principle in mathematical explanations. We assessed perceived explanation quality using comparative judgement. This method is built on the understanding that humans make better evaluations if they compare two objects rather than if they use specific criteria to evaluate an isolated object (Thurstone, 1927). Comparative judgement has been developed for educational purposes as an innovative assessment tool (e.g., https://www.nomoremarking.com) and it has also been used for various research projects. In particular, education researchers have adopted this tool and have investigated teachers' assessments of students' work in a variety of subjects, including creative writing (Heldsinger & Humphry, 2013) and Chemistry laboratory reports (McMahon & Jones, 2013). More specifically, mathematics education researchers have used comparative judgement to assess students' understanding of *p*-values, derivatives and algebra (Bisson et al., 2016), students' ability to solve maths problems (Jones & Inglis, 2015), and even ambiguous constructs such as which student is 'the better mathematician' (Jones et al., 2016). The common thread between all of these studies is the assumption that teachers and students have an intuitive understanding of these constructs; one which they are not expected to be able to articulate or explain (Pollitt, 2012). In line with their reasoning, explanation quality clearly fits alongside these constructs.

STUDY DESIGN

In this study, we used 16 explanations taken from a mathematics textbook (Jefferson et al., 2017) designed for A Level students (A Levels are post-compulsory mathematics qualifications taken by 16-18 year-olds in England, Wales and Northern Ireland). We used a single textbook so that design and style would be consistent across the explanations; using multiple books would make it harder to isolate which characteristics affected participants' rankings. The selected mathematical explanations were all at a similar level and none directly relied upon material presented elsewhere in the textbook. We omitted topic headings to minimise the inclination for participants to rank explanations by topic preference or difficulty. Nine of the explanations included a diagram; three explanations contained diagrams in the text and the remaining six had diagrams placed away from the text, in the margins. Figures 1 and 2 show two example explanations.

A **quadratic function** can be written in the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$

A **quadratic equation** can be written in the general form $ax^2 + bx + c = 0$

Curves of quadratic functions, $y = ax^2 + bx + c$, have the same general shape. The curve crosses the y-axis when $x = 0$, and the curve crosses the x-axis at any **roots** (or solutions) of the equation $ax^2 + bx + c = 0$

Quadratic curves are symmetrical about their **vertex** (the turning point). For $a > 0$, this vertex is always a **minimum** point, and for $a < 0$ this vertex is always a **maximum** point.

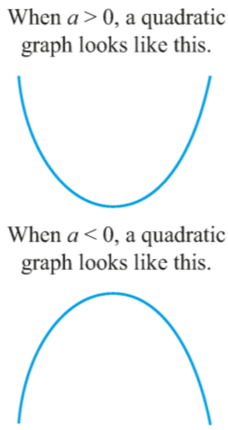


Figure 1: An explanation with a diagram in the margin (Jefferson et al., 2017, p. 16)

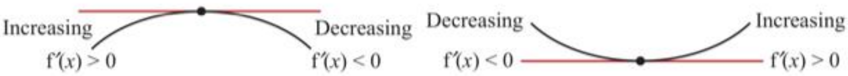
When a curve changes from an increasing function to a decreasing function or vice versa, it passes through a point where it is stationary. This is called a **turning point** or **stationary point**.

Key point

At a turning point, the gradient of the tangent is zero. Therefore, you can work out the coordinates of the turning point by equating the derivative to zero.

A turning point is a stationary point, but a stationary point is not necessarily a turning point. You will learn about other types of stationary point in **Section 15.1**

Here are examples of a **maximum** turning point and a **minimum** turning point.



At a maximum turning point, as x increases, the gradient changes from positive through zero to negative.

At a minimum turning point, as x increases, the gradient changes from negative through zero to positive.

Figure 2: An explanation with a diagram in the main body of the text (Jefferson et al., 2017, p. 106)

We recruited groups of A Level mathematics teachers (N = 77), A Level mathematics students (N = 62), and undergraduate mathematics students (N = 214) to participate. Each participant completed 15 paired comparisons: for each comparison, they read two expository texts and then selected which one they thought was a ‘better’ explanation.

The resulting data were fitted to the Bradley-Terry model (Bradley & Terry, 1952), separately for each group, to produce parameters estimating the perceived quality of each explanation. We calculated reliability coefficients to determine whether or not explanation quality could be meaningfully measured on a single scale. The first reliability measure indicated whether members of each group agreed on explanation quality within groups. This measure is considered analogous to Cronbach’s alpha in comparative judgement research and is known as the Scale Separation Reliability coefficient. The second measure used correlation coefficients to measure whether the same judgements would have been made by a different group of judges from the same population; it is known as the inter-rater reliability (Bisson et al., 2016).

RESULTS

We found reliability was reasonably high for all of the participant groups, as shown in Table 1. The Scale Separation Reliability is high for all three groups which suggests that participants broadly agreed with each other within groups. The correlations for the inter-rater reliability are mostly high too, although the correlation for A Level teachers is slightly lower than the other two groups. This could potentially reflect the diverse levels of experience between teachers. However, overall these measures suggest that participants broadly agreed with each other about the quality of the explanations.

Group	Scale separation Reliability	Inter-rater reliability
A Level teachers	.75	.55
Undergraduates	.92	.82
A Level students	.88	.77

Table 1: Reliability measures for each group

We compared the mean parameter for the explanations that were in line with the spatial contiguity principle with the mean parameter for those not so designed. These data are shown in Figure 3. A 3×2 (group×diagram-placement) Analysis of Variance (ANOVA) found a significant main effect of diagram placement, $F(1,7) = 6.48$, $p = .038$, $\eta_p^2 = .481$. Explanations achieved higher scores if they included diagrams in the text rather than in the margin. In reference to the spatial contiguity principle, we suggest that explanations with diagrams in the margin achieved lower scores because learners needed to use additional cognitive load to integrate the diagrams and text if they were not spatially close together.

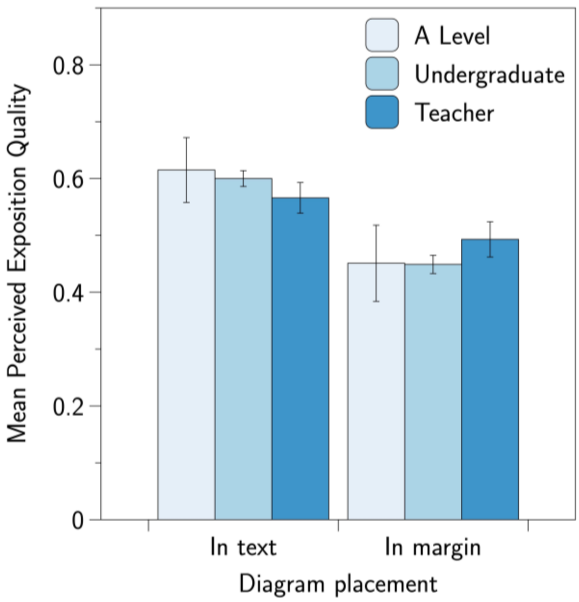


Figure 3: The mean perceived expository quality of the explanations, split by group and diagram location

DISCUSSION

Our research shows that explanation quality can be measured in a meaningful way using comparative judgement, and supports the idea that the spatial contiguity principle is relevant in mathematics textbook design, at least from the perspective of learner experience. It leaves open the question of whether learner experience of explanation quality is systematically related to learning outcomes, but raises this as a question for further study.

The materials we used also raise questions about how design might affect learner behaviour, and therefore suggest another mechanism by which diagram placement might affect perception of explanation quality. In the explanations we used, diagrams placed in the text clearly lead on from that text whereas diagrams in the margin do not clearly fit into the textual explanation. This might affect the extent to which learners examine diagrams: a diagram not in the main body of text might appear inessential and less worthy of attention. This would be consistent with work by Jarodzka et al. (2015) whose eye-tracking study found that learners spent significantly less time looking at pictures presented in a split format rather than integrated into the main body of a resource. Given the robust result that diagrams benefit learning (e.g., Bui & McDaniel, 2015; Lindner et al., 2017) it could be that learners rank explanations with integrated diagrams more highly because they look less at diagrams in split formats and so benefit less from their explanatory power.

CONCLUSIONS

Altogether, these findings give insight into the impact of the spatial contiguity principle on learners' perceptions of explanation quality. Our findings show that diagrams integrated into text are associated with better perceived explanation quality, so this principle does appear relevant for real-world textbook design.

References

Ayres, P., & Sweller, J. (2005). The split-attention principle in multimedia learning. In R. E. Mayer (Ed.), *The Cambridge Handbook of Multimedia Learning* (pp. 135–146). Cambridge University Press. https://doi.org/10.1017/CBO9781139547369.011

Bisson, M. J., Gilmore, C., Inglis, M., & Jones, I. (2016). Measuring Conceptual Understanding Using Comparative Judgement. *International Journal of Research in Undergraduate Mathematics Education, 2*(2), 141–164. https://doi.org/10.1007/s40753-016-0024-3

Bradley, R. A., & Terry, M. E. (1952). Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons. *Biometrika, 39*(3/4), 324. https://doi.org/10.2307/2334029

Bui, D. C., & McDaniel, M. A. (2015). Enhancing learning during lecture note-taking using outlines and illustrative diagrams. *Journal of Applied Research in Memory and Cognition, 4*(2), 129–135. https://doi.org/10.1016/j.jarmac.2015.03.002

Chandler, P., & Sweller, J. (1991). Cognitive Load Theory and the Format of Instruction. *Cognition and Instruction, 8*(4), 293–332. https://doi.org/10.1207/s1532690xci0804_2

Heldsinger, S. A., & Humphry, S. M. (2013). Using calibrated exemplars in the teacher-assessment of writing: an empirical study. *Educational Research, 55*(3), 219–235. https://doi.org/10.1080/00131881.2013.825159

Jarodzka, H., Janssen, N., Kirschner, P. A., & Erkens, G. (2015). Avoiding split attention in computer-based testing: Is neglecting additional information facilitative? *British Journal of Educational Technology, 46*(4), 803–817. https://doi.org/10.1111/bjet.12174

Jefferson, B., Bowles, D., Mullan, E., Wiseman, G., Rayneau, J., Wood, K., Rowland, M., Heylings, M., Williams, P., & Wagner, R. (2017). *AQA A Level Maths: Year 1 + Year 2*. Oxford University Press.

Johnson, C. I., & Mayer, R. E. (2012). An eye movement analysis of the spatial contiguity effect in multimedia learning. *Journal of Experimental Psychology: Applied, 18*(2), 178–191. https://doi.org/10.1037/a0026923

Jones, I., & Inglis, M. (2015). The problem of assessing problem solving: can comparative judgement help? *Educational Studies in Mathematics, 89*(3), 337–355. https://doi.org/10.1007/s10649-015-9607-1

Jones, I., Wheadon, C., Humphries, S., & Inglis, M. (2016). Fifty years of A-level mathematics: Have standards changed? *British Educational Research Journal, 42*(4), 543–560. https://doi.org/10.1002/berj.3224

Lindner, M. A., Eitel, A., Strobel, B., & Köller, O. (2017). Identifying processes underlying the multimedia effect in testing: An eye-movement analysis. *Learning and Instruction, 47*, 91–102. https://doi.org/10.1016/j.learninstruc.2016.10.007

Makransky, G., Terkildsen, T. S., & Mayer, R. E. (2019). Role of subjective and objective measures of cognitive processing during learning in explaining the spatial contiguity effect. *Learning and Instruction, 61*, 23–34. https://doi.org/10.1016/j.learninstruc.2018.12.001

Mayer, R. E. (2005). Cognitive theory of multimedia learning. In R. E. Mayer (Ed.), *The Cambridge Handbook of Multimedia Learning* (pp. 31–48). Cambridge University Press.

Mayer, R. E. (2020). *Multimedia Learning* (3rd ed.). Cambridge University Press. https://doi.org/10.1017/9781316941355

Mayer, R. E. (1989). Systematic thinking fostered by illustrations in scientific text. *Journal of Educational Psychology, 81*(2), 240–246. https://doi.org/10.1037/0022-0663.81.2.240

McMahon, S., & Jones, I. (2015). A comparative judgement approach to teacher assessment. *Assessment in Education: Principles, Policy & Practice, 22*(3), 368–389. https://doi.org/10.1080/0969594X.2014.978839

Moreno, R., & Mayer, R. E. (1999). Cognitive principles of multimedia learning: The role of modality and contiguity. *Journal of Educational Psychology, 91*(2), 358–368. https://doi.org/10.1037/0022-0663.91.2.358

Pollitt, A. (2012). The method of Adaptive Comparative Judgement. *Assessment in Education: Principles, Policy & Practice, 19*(3), 281–300. https://doi.org/10.1080/0969594X.2012.665354

Schroeder, N. L., & Cenkci, A. T. (2020). Do measures of cognitive load explain the spatial split-attention principle in multimedia learning environments? *A systematic review. Journal of Educational Psychology, 112*(2), 254–270. https://doi.org/10.1037/edu0000372

Sweller, J., van Merrienboer, J. J. G., & Paas, F. G. W. C. (1998). Cognitive Architecture and Instructional Design. *Educational Psychology Review, 10*, 251–296. https://doi.org/10.1023/A:1022193728205

Tarmizi, R. A., & Sweller, J. (1988). Guidance during mathematical problem solving. *Journal of Educational Psychology, 80*(4), 424–436. https://doi.org/10.1037/0022-0663.80.4.424

Thurstone, L. L. (1927). A law of comparative judgement. *Psychological Review, 34*(4), 273–286.

Tindall-Ford, S., Chandler, P., & Sweller, J. (1997). When two sensory modes are better than one. *Journal of Experimental Psychology: Applied, 3*(4), 257–287. https://doi.org/10.1037/1076-898X.3.4.257

DECISIONS TOWARDS TRANSLATION EQUIVALENCE: THE CASE OF JAPANESE ELEMENTARY SCHOOL TEXTBOOKS

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In this paper, the research question "Is it possible to classify the decisions taken by translators in their effort to equivalently translate a mathematical textbook?" is examined by using the Model for Translation Equivalence. The Model is based on the four requirements of a translation by Nida (1964), the levels of didactic codetermination by Chevallard (2002), and the extension and integration principle for task sequence. The examination shows that the parameters on the Model allow the codification of the translator's decision to attempt the equivalent translation of Japanese elementary school mathematics textbooks.

INTRODUCTION

Starting on April 2020, new mathematics curriculum standards will be implemented in elementary schools in Japan. As part on the implementation process, textbook companies prepared textbooks that were certified by the Ministry of Education (MEXT) and selected by every school district in 2019. The certified textbooks are distributed by the government for free according to the selection made by each district. On private schools that teach in English, the English edition of those certified textbooks is used. Three textbook companies, Gakko Tosho, Keirinkan and Tokyo Shoseki, have published translations of their products until today. Gakko Tosho has translated every revision since 2005 because its editions were chosen by various countries such as Chile, Indonesia, Marshall Islands, Mexico, Papua New Guinea, and Thailand through bilateral research collaboration.

Gakko Tosho's consistent translation activities have resulted in the accumulation of rich experiences with its international editorial board. This experiences generated the research questions of this report: *Is it possible to classify the decisions taken by translators in their effort to equivalently translate a mathematical textbook? For this effort, what should be considered?*

In this research report, the *Model for Translation Equivalence* is proposed to examine the above questions. This model was designed using the *four basic requirements for a translation* (Nida, 1964), the *levels of didactic codetermination* (Chevallard, 2002), and the *extension and integration principle for task sequence*. The objective of this model is to provide parameters to codify a translator's decision to attempt the equivalent translation of Japanese elementary school mathematics textbooks.

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THEORETICAL BACKGROUND

Four requirements for a translation.

The translation practice requires consistency from the translator. This is based on the premise that a translation is a "creative process, not just a mechanical one, and as an expressive activity, requires choosing words and creating a voice" (Halfon, Dillman, Hahn, & Patt, 2019, p.455). Consequently, this creates difficulties for the translation of mathematical textbooks because "although it may not be deliberate, translation reduces communication by giving more emphasis to language than to mathematics" (Planas, 2014, p.63).

The proposal made by Nida (1964) gives insight into the importance of leaving behind the idea that words have a fixed meaning and advance towards a functional definition of meaning. In this approach, "a word acquires meaning through its context and produces varying responses according to the culture" (Munday, 2001, p.38). Therefore, Nida's (1964) classification of equivalences in translations as formal equivalence and dynamic equivalence became a significant contribution. Summarized by Munday (2001), the first one is referred to accuracy and correctness based on the structure of the source text, while the second one is based on the principle of equivalent effect. Through this principle, the "relationship between the receptor and message should be the same as the one that existed between the original receptors and message" (Nida, 1964, p.159). As a result, it is possible to make adjustments in grammar and cultural references as a tool to achieve the "closest natural equivalent to the source language message" (Nida, 1964, p.166).

Under Nida's (1964) classification of equivalences, the four basic requirements for a translation are: "(1) making sense, (2) conveying the spirit and manner of the original, (3) having a natural and easy form of expression, and (4) producing a similar response" (p.164). In this framework, the numbers (1) to (4) do not have a hierarchical meaning.

Levels of didactic codetermination.

The levels of didactic codetermination (LDC) is a framework that describes the broad context in which didactical and mathematical organizations occur. This theory proposed by Chevallard (2002) shows the mutual interaction of ordered institutional levels that successively condition one another. Described by Artigue & Winslow (2010), the LDC identifies as higher levels the discipline, pedagogy, school, society and civilization by which teaching is conditioned. These teaching conditions are generally not changed by an individual teacher, but "may be further modified by others, such as school principals, curriculum developers, or politicians" (Artigue & Winslow, 2010, p.5). The lower or sub disciplinary levels are the domain, sector, theme, and subject. These levels are linked to the components of the praxeologies (Task, Technique, Technology, and Theory) they determine.

All the different levels that affect the process of studying and teaching are exterior to the teaching practice. Therefore, the construction of knowledge in teaching situations “may vary between students and also may be different from what was intended by the teacher, school, society, and so on” (Artigue & Winslow, 2010, p.6).

Even though the LDC describes the context by which teaching situations are conditioned, the authors adapted this theory to the analysis of textbooks because it provided a broad range of interpretations for translations.

Extension and integration principle for task sequence.

Japanese mathematics educators and teachers have developed theories for curriculum and teaching as a result of hundreds of years of lesson study. Explained by Isoda (2012), one of these theories is the problem solving approach, which is a shared theory of teaching for developing children who learn mathematics by and for themselves.

The Japanese elementary school textbook are designed for the problem solving approach and employ the extension and integration principle as a curriculum principle to develop mathematical thinking. As described by Isoda & Olfos (in printing), this principle is embedded into the textbook as a task sequence based on what the children have learned before. This allows the interpretation of mathematical Japanese textbooks based on existed tasks at previous pages and grades.

MODEL FOR TRANSLATION EQUIVALENCE

The *Model for Translation Equivalence* is a proposal that brings together Translation Studies and Mathematics Education. This model *describes* and *codifies* the decisions taken by translators in their attempt to elaborate the closest natural equivalent (Nida, 1964) of a Japanese mathematics textbook.

The *description* is done through the following three parameters: <u>(A) What is the location of the task related to tasks in previous pages or grades? (B) By which level of didactic codetermination (Chevallard, 2002) is the text conditioned? (C) Given B, which of the four basic requirements for a translation (Nida, 1964) is considered for deciding the equivalent translation of the text?</u> Note that, under parameter B, multiple levels can be chosen. Several combinations between B and C are possible.

The *codification* is done by assigning a code to parameters B and C. Shown in Table 1, the model’s coding matrix is a 4 by 9 matrix. The columns of the matrix contain the levels of didactic codetermination (Chevallard, 2002) namely Civilization (Ci), Society (So), School (Sc), Pedagogy (Pe), Discipline (Di), Domain (Do), Sector (Se), Theme (Th), and Subject (Su). The rows of the matrix contain the four basic requirements of a translation (Nida, 1964) namely making sense (1), conveying the spirit and manner of the original (2), having a natural and easy form of expression (3), and producing a similar response (4).

Parameter A is not codified because it is fixed in the case: The principle of extension and integration for task sequence provides the context for the interpretation of Japanese elementary school textbooks.

	(Ci)	(So)	(Sc)	(Pe)	(Di)	(Do)	(Se)	(Th)	(Su)
(1)	Ci1	So1	Sc1	Pe1	Di1	Do1	Se1	Th1	Su1
(2)	Ci2	So2	Sc2	Pe2	Di2	Do2	Se2	Th2	Su2
(3)	Ci3	So3	Sc3	Pe3	Di3	Do3	Se3	Th3	Su3
(4)	Ci4	So4	Sc4	Pe4	Di4	Do4	Se4	Th4	Su4

Table 1: Coding matrix for parameters B and C

METHOD TO ILLUSTRATE THE MODEL

Based on the above mention of the way to describe and code, we examined the changes of translation from *Study with Your Friends: Mathematics for Elementary School, 10 vol. (2015)* to *Study with Your Friends: Mathematics for Elementary School, 12 vol. (2020)* published by Gakko Tosho in Japanese and English. In total, more than 61 examples were considered in which the translation changed from the 2015 edition to the 2020 edition. After the consideration of various possibilities to change the translation, the description and codification was given. There were various possibilities for coding but the codes were sieved by understandable codes for others.

In the following subtitles of figures, the description Jap15 means Japanese edition (2015) and Eng20 means English edition (2020).

EXAMINATION OF CASES

To illustrate the way of description and coding using the Model for Translation Equivalence three cases were chosen. The first case shows a Japanese word which has various translations in English, the second case shows the simplified Japanese ideograms, and the third case shows the grammatical difference between Japanese and English.

Angles with a size larger than 180 degrees

The extracts shown on Figure 1 illustrate a Task in which the angle 210° must be drawn. The Technique for the Task is interpreted through the word "くふうして" (kufuushite). As shown in the corresponding translation, the translator on the 2015 edition interpreted "くふうして" as "in various ways".

Different from the interpretation shown above, on Figure 2, the word "くふうして" was translated in the 2020 edition as "using learned ideas". The Model for Translation Equivalence describes and codifies the decision taken by this translator as follows:

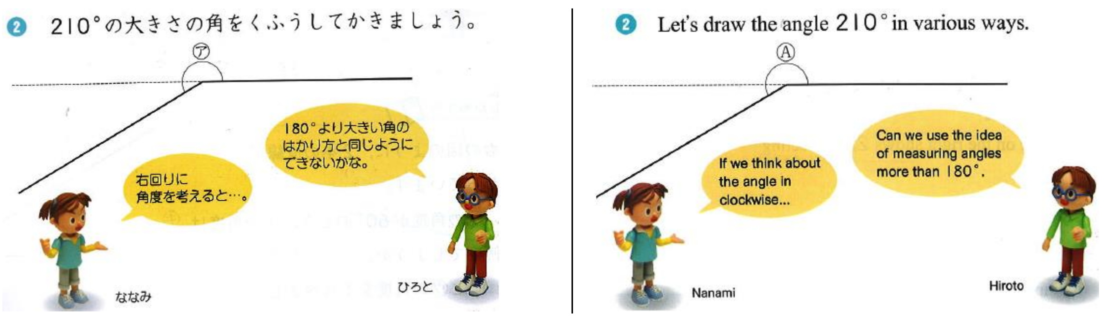

Figure 1: Extracts from Jap15 and Eng15.

A) Location of the Task: This Task is located after the introduction of the size of an angle in 4th grade. As part of this introduction, students learn how to measure angles that are less than 180° by using the protractor. The measurement of angles larger than 180° is achieved by identifying how many degrees more than 180° (addition) or how many degrees less than 360°(subtracting). The next task sequence is to extend from the drawing of angles that are less than 180° towards more than 180°.The Technique for the Task in Figure 2 is based on the addition and subtraction of angles and the measurement of angles larger than 180°.

B) Levels of didactic codetermination: "くふうして" is a word that has been part of the Japanese language for hundreds of years and makes reference to the ability to find/create/consider a new or improved idea. This word cannot be translated into English in a single way, since the word has a cultural attachment (**Ci**vilization). Originally, this word was only used by non-scholars since the word "かんがえて" (kangaete) was reserved for intellectuals who had the responsibility to think/meditate/generate a solution or method as a profession. Therefore, this is a word that is usually used with children to reconsider what has been learned before with previous tasks (**Su**bject).

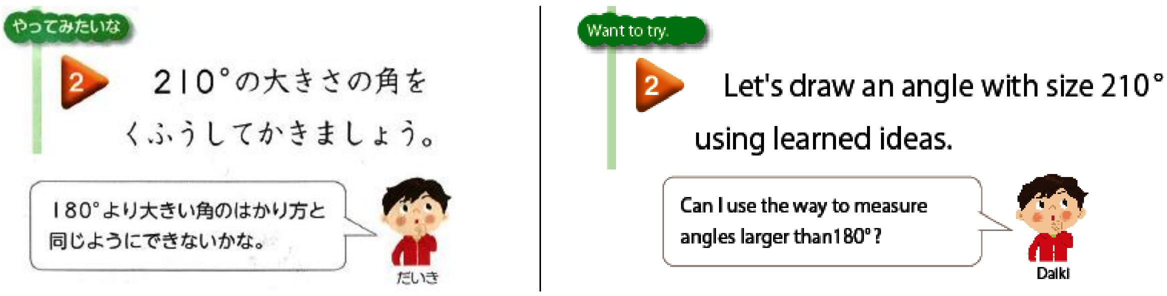

Figure 2: Extracts from Jap20 and Eng20

C) Basic requirements for a translation: Even though the task written in both Japanese editions is the same, the latest English version was modified to get closer to the original intention (2). With the words "learned ideas", the student is expected to associate his/her response (4) to the measurement of angles larger than 180° and the addition and subtraction of angles.

Codification: Cultural/original intention (Ci2) and previous task/response (Su4).

Division algorithm by vertical form.

The extracts shown on Figure 3 illustrate the description of a Technique for a Task that is explained by numbers. The text is a step by step process to characterize the division algorithm by vertical form. On step number two, the expression "九五４５" (ku[9]go[5] 45) is used to represent the number that is closest (without going over) to 48 in the multiplication table of nine. As shown in the corresponding translation, the translator on the 2015 edition interpreted "九五４５" as "9 multiplied by 5 equals 45".

Different from the interpretation shown above, on Figure 4, the expression "九五４５" was translated in the 2020 edition as "9 and 5 is 45". The Model for Translation Equivalence describes and codifies the decision taken by this translator as follows:

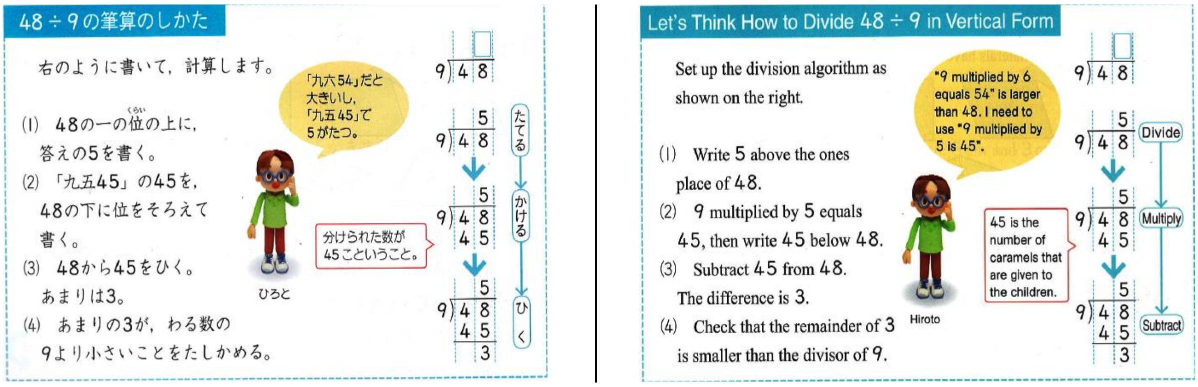

Figure 3: Extracts from Jap15 and Eng15

A) Location of the task: This 4th grade Task is located after Japanese students learned the meaning of division in 3rd grade. Initially, students learned how to divide objects given three terms: total amount, how many equal parts, and amount for every part. These terms are learned by using the associated terms in multiplication. Then, students are able to use multiplication tables for division problems. Still on 3rd grade, the remainder is introduced. In 4th grade, the division of 2-digit numbers by 1-digit numbers is manipulated through blocks. This Task is located previous to Figure 3 and uses the Technique of separation by "sets of 10" and "sets of 1". This is how Figure 3 becomes the summary of this previous Technique.

B) Levels of didactic codetermination: The expression "九五４５" is exclusive in Japanese and Chinese mathematics classrooms, since ideograms make possible the memorization of multiplication tables without using the symbol "×" (**<u>Pe</u>**dagogy). The word "かける" (multiply) is the way of reading the operation "×". This compressed reading is learned in 2nd grade, therefore this 4th grade textbook uses the same manner in which multiplication tables were learned in the past. This representation is a useful tool for efficiency and memorization, therefore is integrated in the solution of tasks (**<u>Su</u>**bject).

C) Basic requirements for a translation: Even though the description of the technique written in both Japanese editions is the same, the latest English edition was modified to <u>keep the summarized notation (3)</u> that is benefited from the use of ideograms. Without translating it with "multiplied by", the expression still <u>makes sense (1)</u>.
Codification: Exclusive use/summarized expression (Pe3) and useful tool/sense (Su1).

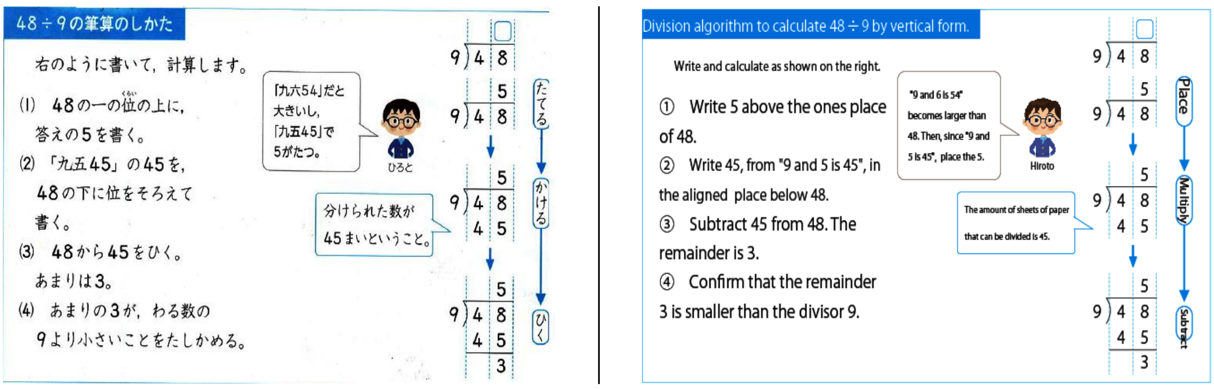

Figure 4: Extracts from Jap20 and Eng20

Changes on the translation depending on users: The case of multiplication.
This example illustrates the difficulty of translation based on the grammatical difference between Japanese and English. The description cannot follow the previous format because the task sequence is beyond several grades and became too long to adapt the same format.

Figure 5 is an example of $2 \times 5 = 10$ in Japanese grammar. This means 5 times 2 $(2 + 2 + 2 + 2 + 2)$: 5 is the multiplier. However in English grammar, the same expression means $5 + 5$: 2 is the multiplier (Isoda & Olfos, in printing). This grammatical difference has an influence (A) to division, some tables and proportional number lines in upper grade textbooks, such as 4th grade. For example, on real situations $ax = y$, a Japanese textbook from 3rd to 6th grade usually draws a table with x in the bottom row and y in the upper row until proportion is defined.
There are two ways for challenging the translation equivalence. The first way is keeping the Japanese grammar for English School users in Japan (Ci4). The English users in foreign countries feel the grammatical contradiction but understand that the English edition is written with the original spirit of Japanese grammar (Ci2). The second way is changing it under English grammar. This adaptation includes changing situations and diagrams. When a translator makes the effort to adapt the Japanese textbook into other country's curriculum, it loses the original spirit, but the adaptation makes sense under the new curriculum (Pe1).

Figure 5: $2 \times 5 = 10$ in 2nd grade (Eng15)

CONCLUSION

This report illustrated the Model for Translation Equivalence. The examination showed that the parameters A, B, and C allow the codification of the translator's attempt an equivalent translation of Japanese elementary school mathematics textbooks.
Although the case studied is a translation from Japanese to English, the model could be applied in other languages. The pre-requisite is to incorporate the curricular principles that set the design of the source textbook. This offers the opportunity to unpack the hidden efforts on the source text based on pre-requested principles, such as the original spirit. This interpretation for unpacking reaffirms that languages include "linguistic features of benefit for the acquisition of mathematical concepts that can be used for mathematical teaching and learning" (Phakeng, 2016, p.14).
Also, the authors are conscious about the criticism around Nida's (1964) work, especially to those that suggest that an equivalent translation can be a purpose, but is not easy to achieve. That is the justification for choosing the words *effort* and *attempt*.

References

Artigue, M. & Winslow, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherches en didactique des mathématiques, 30*(1), pp.47-82.

Chevallard, Y. (2002). Organiser l´étude. 3. Écologie & regulation. In J-L. Dorier et al (Eds.), *Actes de la XI école d´été de didactique des mathématiques* .Grenoble, France: La Pensée Sauvage.

Halfon, E., Dillman, L., Hahn, D., & Patt, A. (2019). The purest form of writing, the most intimate form of reading. *The Massachusetts Review*, 60(3), pp.448-463.

Hitotsumatsu, S. & Okada, Y. (Eds.). (2015, 2020). *Study with your friends: Mathematics For Elementary School* (in Japanese.). Tokyo, Japan: Gakko Tosho.

Isoda, M. (2012). Introductory chapter: Problem solving approach to develop mathematical thinking. In M. Isoda & S. Katagiri (Eds.), *Mathematical Thinking: How to develop it in the classroom*. (pp.1-28). Singapore: World Scientific.

Isoda, M. (Eds.). (2015, 2020). *Study with your friends: Mathematics For Elementary School* (in English). Tokyo, Japan: Gakko Tosho.

Isoda, M. & Olfos, R. (in Printing). *Teaching Multiplication with Lesson Study*. Switzerland: Springer.

Munday, J. (2001). *Introducing translation studies: Theories and applications* (1st ed.). London, U.K: Routledge.

Nida, E. (1964). *Towards a science of translating* (1st ed.). E. J. Brill.

Phakeng, M. (2016). Mathematics education and language diversity: Past, present, and future. In A. Halai & P. Clarkson (Eds.), *Teaching and learning mathematics in multilingual classrooms* (pp. 11–24). Rotterdam, Netherlands: Sense Publishers.

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics*, 87(1), pp.51-66.

WHAT ARE THE REASONS WHY PRESERVICE MATHEMATICS TEACHERS ENDORSE THE INTEGRATION OF TECHNOLOGY IN MATHEMATICS CLASS？

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This study employed latent profile analysis to explore the reasons why preservice mathematics teachers endorse the integration of technology in mathematics classes. The sample included 91 secondary mathematics preservice teachers, from two normal universities in Taiwan. Four profiles were characterized as active integrated-oriented, practical integrated-oriented, operation emphasized-oriented, and traditional instruction-oriented. The percentages of active integrated-oriented and operation emphasized-oriented profiles were both around 32%, while that of traditional instruction-oriented profile was only 12%. Most preservice teachers appreciated using technology to innovate math classes, however, they are more easily attracted by the apparent functions of integration than the implicit functions inside students' mind.

INTRODUCTION

In the era of rapid development in technology, people are inevitably affected by digitalization, automation and globalization. Society, organizations and individuals in society are increasingly dependent on technology. In this case, mathematics education should prepare students for applying mathematics with the integration of technology in all sorts of future work and everyday-life situations (Gravemeijer et al., 2017). Teachers are expected to use technology to improve teaching quality (Alacaci & McDonald, 2014) and cultivate students' competence of using technology to learn mathematics and solve problems (Zbiek et al., 2007).

Aligning with the international stream, the new curriculum in Taiwan, which was launched in 2019, clearly requested the integration of technology in middle and high school mathematics classes (Ministry of Education, 2018). The mathematics teacher preparation programs should provide corresponding training for preservice teachers. Understanding preservice teachers' endorsement of the various functions provided by the integration of technology will help teacher educators know how to encourage them to use it, thus this research aims to identify the different types of preservice teachers' preference for technology integration. The following research questions were addressed:

1. What are the profiles that portray secondary mathematics preservice teachers' endorsement of technology integration in mathematics classes?
2. What are the commonalities and differences among the profiles identified in the first research question?

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 295-302. Khon Kaen, Thailand: PME.

RESEARCH METHOD

Conceptual framework

The conceptual framework for integrating technology into mathematics class in this study included three dimensions which were the cultivation of student' competence, pedagogy, and technology impact.

Cultivation of students' mathematical competence

Mathematical competences can be distinguished into two types (Hsieh et al, 2012; Niss, 2003). On type is content-oriented mathematics competence and the other is thought-oriented mathematics competence. The former is related to the factual knowledge and skills of specific mathematical contents, whereas the latter is about the ways of mathematical thinking, such as reasoning mathematically or representing mathematically (OECD, 2010). In addition to cultivating students' mathematical competences aforementioned, PISA 2021 further included 21st century skills, such as, research and inquiry, critical thinking, and communication, into their mathematics framework (OECD, 2018). The use of technology has been evidenced to help the cultivation of students' mathematical competences (Alacaci & McDonald, 2014).

Pedagogy

Literatures has discussed teaching behaviors and teachers' role in the mathematics classes. In traditional mathematics classes, the role of teachers is an explainer or lecturer, while the researchers and teacher educators have promoted a transformation of teachers' role to a questioner or facilitator (Ismail et al., 2015; Suffolk, 2007). In traditional classrooms, teachers are responsible to lecture and convey knowledge, and students listen to the lecture and receive what teachers provide in the teacher-led activities. In a student-centered class where teachers are to a questioner or facilitator, students are provided with activities including observation, exploration, or experimentation. With the help of technology, the teacher can arrange these activities in a previously inconceivable manner.

Technology impact

When technology is integrating into mathematics class, the technology impact is then inevitable. Several studies have claimed that the integration of technology can help to increase collaboration, motivation, and bring about more of an emphasis on practical applications of mathematics, through modelling, visualization, manipulation, and supporting the link between students' actions and symbolic representations (Drijvers et al., 2010; Geiger et al., 2010). The improvement of motivation also contributes to the improvement of productive disposition, which is one of the five interrelated strands that together, constitute mathematical proficiency (Kilpatrick, 2001). This proficiency can be seen in the positive mathematics learning attitudes and recognition of mathematics value in this study. Puentedura (2014) proposed a SAMR model to describe four levels of instructional quality regarding the impact of the integration of technology. Corresponding to our study, students may observe the graphics in the substitution level, by using technology to increase mathematical accuracy in the augmentation level. They can also manipulate objects in modification level, and in the redefinition level, they can design experiments to solve the real world problems with technology.

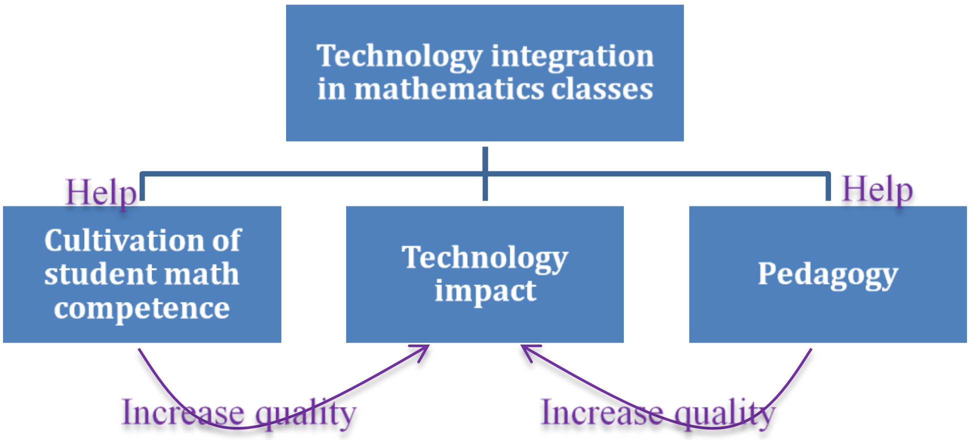

Figure 1: The framework of this study

Instrument

This study was to explore the reasons why preservice mathematics teachers endorse the integration of technology in mathematics classes. A questionnaire with five teaching vignettes was employed to assist preservice teachers with less teaching experiences to think. The vignettes were adapted from contents in textbooks aligning with the new curriculum, including the teaching contexts such as using technology to develop students' understanding of characteristics of graphs of logarithmic functions, or to develop students' mathematical competence in modeling with exponential functions. After the preservice teachers finished reading of each vignette, they were asked to fill out the same three sets of dichotomous items to indicate which help the technology provided would be the reason they endorse the technology-integrated class. The three sets of items were designed according to the three dimensions of the framework through literature review. *Cultivation of student math competences*, *pedagogy*, and *technology impact* consisted of 9, 16, and 5 items respectively.

Participants

The sample included 91 secondary mathematics preservice teachers from two normal universities in Taiwan (45 and 46 in each). They are juniors or seniors in the universities.

Data Analysis

This study conducted latent profile analysis (LPA) on the preservice teachers' responses with M-plus. LPA is a person-centred approach that assumes the existence of an underlying unobserved categorical variable that divides a population into mutually exclusive and exhaustive classes. For each item, the preservice teachers' response for each teaching vignette were summed (check=1, not check=0). Thus, the value for each item ranged from 0 to 5. Through performing LPA on the aggregated data, this study identified groups of preservice teachers with similar endorsement patterns across three dimensions—how technology helps the development of student competences, how it helps pedagogy, and technology impact.

Log likelihood (LL) and adjusted Bayesian information criterion (BIC) statistics were employed to provide information on goodness-of-fit of models with lower values indicating better model fit. Entropy was used to measure the accuracy of classification, with the value above 0.8 for high level. Differences in BIC and LL statistics, Vuong-Lo-Mendall-Rubin (VLMR) test, and Bootstrapped Likelihood Ratio (BLR) test compare each model with the model with one less class to identify whether there is a significant improvement in model fit.

RESEARCH FINDINGS

<u>Profiles that portray preservice teachers' endorsement of technology integration in mathematics classes</u>

Models with more classes indicated better model fit according to the LL and BIC criteria. The entropy values for all five models are approaching 1 indicating clear delineation of classes. However, differences in BIC and LL gradually diminished as the number of classes increased, indicating that improvements in model parsimony decreased. The VLMR and BLR tests suggested that the 5-class model did not fit the data better than the 4-class model (p = .760 & p = .150), but the BLR tests indicated that the 4-class model offered a significantly more adequate fit than the 3-class model (p = .000). The 4-class model was selected due to the consideration of the aforementioned fit statistics.

No. of classes	Log likelihood	Diff(LL)	Adjusted BIC	Difference in BIC	Entropy	VLMR p value	BLR p value
1	-4611.095		9298.045				
2	-4251.465	359.630	8618.067	-679.978	0.972	0.001	0.000
3	-4144.541	106.924	8443.500	-174.567	0.985	0.279	0.000
4	-4095.212	49.329	8384.124	-59.376	0.967	0.776	0.000
5	-4068.305	26.907	8369.592	-14.532	0.970	0.760	0.150

Table 1: Fit statistics for latent class analysis

The items were categorized into five groups for depicting the preservice teachers' profiles. The five groups were generated through exploratory factor analysis (EFA) which were *innovating instruction to develop students' mathematical competence*, *developing students' positive attitudes and valuations toward mathematics*, *detailing explanation to deepen students' understanding*, *developing students self-directed learning skills*, and *developing students' interpersonal 21st century skills* (Wang, manuscript in preparation). As shown in Figure 2, Class 1 (32% of the preservice teachers), comparing to other classes, endorsed all factors to the highest degree. Preservice teachers in this class were the advocators in integrating technology into the mathematical classes. They appreciated the functions the technology provided for innovating the class, developing students' positive attitudes and competences, and increasing the quality of teacher-directed teaching behaviours. The profile this class portrayed was thus characterized as "active integrated-oriented." The preservice teachers in Class 2 (24%) were characterized as "practical integrated-oriented." They relatively endorsed the factors "Innovating instruction to develop students' mathematical competence," "Detailing explanation to deepen students' understanding," and "Developing students' interpersonal 21st century skills" to a higher degree. These teachers appreciated the functions of technology to help the development of student competences through student-centered activities such as inquiry, experimentation, manipulation, examination, and providing evidences, and to help the development of student knowledge through supporting teachers' instruction. They also endorsed how technology helps develop student 21st century interpersonal skills through discussion or presenting.

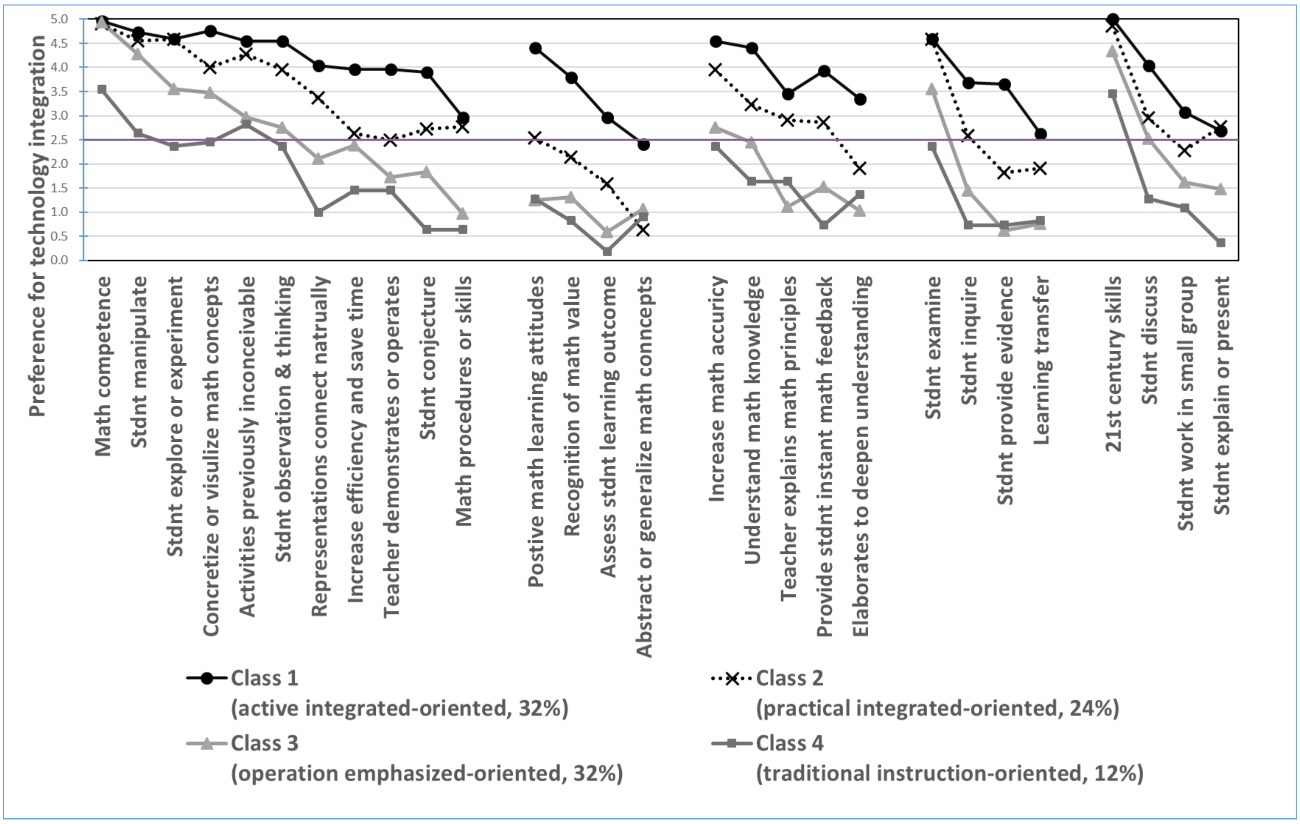

Figure 2: Four profiles of preservice teachers' preference for technology integration

Comparing with Class 2, the preservice teachers in Class 3 (32%) endorsed the functions of technology to provide students opportunities to inquire problems and engage in manipulative activities to develop students' competences, but did not appreciated using technology to support teachers' instruction to develop students' knowledge. That is, they endorsed the integration of technology to support their role as facilitators but not as explainer. In addition, they did not endorse using technology to develop students' positive attitudes and valuations toward mathematics at all. This study thus characterized the profile as "operation emphasized-oriented." In contrast to the first three classes, the last class was an extreme case. They did not endorse any "actual" functions provided by use of technology in mathematics class. They only indicated endorsement on some conceptual descriptions, such as using technology to help the development of students' mathematical competences or the cultivation of students' 21st century skills. We thus characterized the profile as "traditional instruction-oriented" (12%).

The commonalities and differences among the four profiles of preservice teachers' endorsement of technology integration in mathematics classes

All four classes of preservice teachers endorsed the use of technology to develop students' thought-oriented competences and 21st century skills, however, only Class 1 endorsed using technology to develop students' positive attitudes and valuations toward mathematics. It is possible that the teachers in other three classes did not consider the use of technology can help developing positive attitudes or valuations, or that they thought technology can help but it is not the reason why they endorse a technology-integrated mathematics class.

Regarding the factor of *innovating instruction to develop students' mathematical competences*, all classes of preservice teachers, with exception of Class 4, endorsed the benefits of integrating technology into mathematics class to arrange activities previously inconceivable, help students to focus on observation and thinking, explore or experiment, and concretize or visualize mathematical concepts. However, the preservice teachers endorsed using technology to help connecting representations naturally and to promote students' conjecture to a relatively lower degree. For the factor of *developing students' interpersonal 21st century skills*, the three classes endorsed student discussion, and for the factor of *detailing explanation to deepen students' understanding*, they endorsed increase of accuracy. The phenomena indicated that the preservice teachers endorsed the visual and operational effects brought by the integration of technology to a higher degree than the technology's impact on student cognitive development and thinking. That is, they are more easily attracted by the apparent and visible functions provided by the integration of technology than the implicit functions inside students' mind.

CONCLUSION

By using EFA, four profiles of Taiwanese secondary mathematics preservice teachers' endorsement of technology integration in mathematics classes were identified as active integrated-oriented profile (32%), practical integrated-oriented profile (24%), operation emphasized-oriented profile (32%), and traditional instruction-oriented profile (12%).

In the current teaching environment in Taiwan, in-service teachers still have many doubts and resistances about integrating technology into mathematics courses. However, in terms of preservice teachers, the proportion of "active integrated-oriented" (Class 1, 32%), "practical integrated-oriented" (Class 2, 24%), and "operation emphasized-oriented" (Class 3, 32%) are much higher than the proportion of "traditional instruction-oriented " (Class 4, 12%). This shows that under the influence of international trends and the promotion of the new mathematics

curriculum, preservice teachers have gradually developed the tendency to integrate technology into the classroom.
Class 2 with “active integrated-oriented” profile (32%) and Class 3 with “operation emphasized-oriented” profile (32%) are the two largest classes. This indicated two typical perspectives of preservice teachers regarding integrating technology into mathematics classes. The first perspective is that, the aid of technology can be greatly helpful no matter it is for the manipulative activities of students or for the instruction of teachers, both. In addition, the value of technology integration even includes the enhancement of students' learning attitude. The second perspective is that the most valuable function of technology is providing opportunities for students to interact and have hands-on activities, which can make up for the shortcomings of traditional teaching. However, in terms of teacher explanation to help knowledge understanding, the use of technology is not required.

References

Alacaci, Cengiz & McDonald, Gaby. (2014). The Impact of Technology on High School Mathematics Curriculum. *Turkish Journal of Computer and Mathematics Education*. 3. 10.16949/turcomat.66271.

Drijvers, P., Mariotti, M. A., Olive, J., & Sacristan, A. I. (2010). Introduction to section 2. In C. Hoyles, & J. B. Lagrange (Eds.), Mathematics Education and Technology - rethinking the Terrain: *The 17th ICMI Study* (Vol. 13, pp. 81e88). Springer.

Fennema, E., & Franke, M. L. (1992). Teachers’ knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: MacMillan.

Geiger, V., Faragher, R. & Goos, M. Cas-enabled technologies as ‘agents provocateurs’ in teaching and learning mathematical modelling in secondary school classrooms. *Math Ed Res J* **22,** 48–68 (2010). https://doi.org/10.1007/BF03217565

Gravemeijer, K., Stephan, M., Julie, C., Lin, F.-L. & Ohtani, M.. What Mathematics Education May Prepare Students for the Society of the Future? *Int J of Sci and Math Educ* **15,** 105–123 (2017). https://doi.org/10.1007/s10763-017-9814-6

Hsieh, F.-J., Lin, P.-J., & Wang, T.-Y. (2012). Mathematics related teaching competence of Taiwanese Primary Future Teachers: Evidence from the TEDS-M. *ZDM – The International Journal on Mathematics Education, 44*(3), 277-292.

Ismail, S. F. Z. H., Shahrill, M., & Mundia, L. (2015). Factors Contributing to Effective Mathematics Teaching in Secondary Schools in Brunei Darussalam. *Procedia - Social and Behavioral Sciences*, 186, 474–481. https://doi.org/10.1016/j.sbspro.2015.04.169

Kilpatrick, J., Swafford, J., & Findell, B. (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.

Ministry of Education (2018). 十二年國民基本教育課程綱要國民中小學暨普通型高級中等學校 [Grade 1-12 Curriculum Guidelines]. Retrieved from https://cirn.moe.edu.tw/WebNews/details.aspx?sid=11&nid=1402&mid=50

Niss, M. A. (2003). Mathematical competencies and the learning of mathematics: the Danish KOM project. In A. Gagatsis & S. Papastavridis (Eds.), *3rd Mediterranean Conference on Mathematical Education-Athens, Hellas 3-4-5 January 2003* (pp. 116–124). Athens: Hellenic Mathematical Society.

OECD (2010). PISA 2012 Mathematics Framework. OECD, Paris. Retrieved from http://www.oecd.org/pisa/pisaproducts/46961598.pdf

OECD (2018), PISA 2021 Mathematics Framework (Draft). Retrieved from https://www.oecd.org/pisa/sitedocument/PISA-2021-mathematics-framework.pdf

Park. S., & Oliver. J. S. (2008). Revisiting the conceptualization of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education*, 38(3), 261-284.

Puentedura, R. (2014). Building transformation: An introduction to the SAMR model [Blog post]. Retrieved from http://www.hippasus.com/rrpweblog/archives/2014/08/22/BuildingTransformation_AnIntroductionToSAMR.pdf.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester, Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.1169-1207). Charlotte, NC: Information Age Publishing Inc.

TRACING LEARNERS' GROWTH OF MATHEMATICAL UNDERSTANDING IN A DYNAMIC GEOMETRY ENVIRONMENT: NETWORKING MULTIPLE THEORIES

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This research coordinated the Pirie-Kieren theory and instrumental genesis to examine learner's growth of mathematical understanding in a dynamic geometry environment. Data analysis suggested that coordinating the two theoretical approaches provided a productive means to capture learner's growth of geometry understanding in a dynamic geometry environment. By networking the two theoretical approaches, this paper presents a model for studying learner's growth of mathematical understanding in a dynamic learning environment while accounting for interaction with digital tools.

INTRODUCTION

One of the most widely accepted ideas in mathematics education is that learners should understand mathematics. Over the past a few decades, mathematics education researchers have categorized understanding into different types (e.g., instrumental vs. relational, conceptual vs. procedure, concrete vs. symbolic, intuitive vs. formal) and developed theories to capture the process of coming to understand (see Meel (2003) for a brief history of searching for the meaning of "understanding" in mathematics education). Meanwhile, with the emergence of interactive mathematics software in the early 1990s, a large body of research in mathematics education has considered ways that dynamic geometry environments (DGEs) might influence the learning of geometry among learners of all ages. Current research on DGEs has documented ways learners interact with DGEs and the impact of these interactions on their understanding (e.g., Barabash, 2019; Baccaglini-Frank & Mariotti, 2010; González & Herbst, 2009; Olivero & Robutti, 2007). This body of literature provides compelling evidence that features of DGEs contribute to learners' understanding of geometry. Indeed, researchers have argued eloquently that DGEs provide a productive means for allowing students to model both mathematical and real-life situations, to connect multiple representations of mathematical ideas, to formulate conjectures about geometric relationships, to generalize geometric properties, and to justify and explain geometry theorems. Although these technology-mediated activities and processes have been extensively studied, much of the literature concerning DGEs focuses on specific activities and processes that are not linked to form characterization of the overall growth of a learner's mathematical understanding in DGEs. This research aimed to capture the evolution of individual

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learner's growth of mathematical understanding in a dynamic geometry environment through coordinating multiple theoretical approaches.

CONCEPTUAL FRAMEWORK: NETWORKING THE PIRIE-KIEREN THEORY AND INSTRUMENTAL GENESIS

This research coordinated the Pirie-Kieren theory for the growth of mathematical understanding and instrumental genesis to examine learner's growth of geometric understanding in a dynamic geometry environment. This allowed the research to trace learner's growth of geometric understanding while identifying how interactions with technology contribute to such growth.

The Pirie–Kieren Theory

Adopting Glasersfeld's conceptualization of understanding as a continuing process of organizing one's knowledge structures, the Pirie–Kieren theory (1994) perceives learner understanding as a dynamic, leveled but nonlinear, recursive process and describes eight potential levels of actions for mathematical understanding (Figure 1). According to this model, the process of coming to understand starts with *Primitive Knowing*, which includes all the knowledge brought to the learning situation by a learner. At the second level, called *Image Making*, the learner engages in specific physical actions that aim at helping him/her to gain an image of the concept under exploration. Images of the concept developed at this level cannot be separated from the specific actions that produce them. By the level of *Image Having*, images associated with activities are replaced by mental pictures. The learner at this level can imagine a concept unconstrained by the physical processes that produced the image and to carry out specific mathematical actions with a general mental plan. At the level of *Property Noticing*, the learner can reflect on his mental image and recognize attributes and features of it. When *Formalizing*, the learner abstracts a method or common quality from classes of mental images and develops class-like mental objects built from the noticed properties. Description of these class-like mental objects results in the production of mathematical definitions or algorithms. The level of *Observing* entails the ability to observe, structure, and organize personal thought processes and recognize the ramifications of the thought processes. *Structuring* occurs when the learner is aware of how a collection of theorems is connected and seeks justification of statements through logical or meta-mathematical argument. At the outermost level is *Inventing*. A person at this level can break free of structured knowledge and create new questions that go beyond the initial domain of inquiry.

The Pirie-Kieren theory has been used by many researchers to study the growth of mathematical understanding at different grade levels within various mathematical contexts (e.g., Gülkilika, Ugurlu, & Yürük, 2015; Gokalp & Bulut, 2018). Although recognizing the importance of social interactions and tools in learner's growth of mathematical understanding, the theory did not further elaborate on the impact of learner's interaction with tools on the growth of mathematical understanding.

Instrumental Genesis

Drawing on Vygotsky's central concept of object-oriented, tool-mediated activity and Piaget's notion of schema, the theory of instrument (Rabardel & Beguin, 2005) considers situations in which an instrument mediates actions between a subject and an object. An instrument is a mixed entity, consisting of both an artifact and the associated utilization schemes that a user develops to use the artifact for accomplishing specific tasks. An artifact becomes an instrument when its user is aware of how the artifact can extend one's capacities for accomplishing a task and has developed means of using a tool for specific purposes. Instrumental genesis describes the process of appropriation and elaboration of an instrument throughout the interaction between a subject and an artifact. Two sub-processes jointly contribute to instrumental genesis. The instrumentalization process concerns the emergence and evolution of the artifact side of the instrument. It is a process in which a learner enriches the artifact's properties in his interactions with it. Instrumentation is relative to the emergence and evolution of the user's utilization schemes and instrumented actions. A utilization scheme has three main functions: A pragmatic function as it allows the agent to do something, a heuristic function as it enables the agent to anticipate and plan actions, and an epistemic function as it allows the agent to understand something. During instrumentation, techniques for tool usage and insights into concepts interweave and co-evolve in a close relationship. This makes the instrumental approach particularly well adapted for investigating the relationship between tool usage and learning in technology-enriched environments.

GSP contains various tools such as dragging, measuring, locus of points, and various primitive construction commands that enable learners to take actions on mathematical entities and explore the properties of these entities. Through instrumental genesis, a user develops utilization schemes for these tools. Dragging is an essential feature of dynamic geometry software and a conceptual tool for exploring properties and relationships of geometric objects. Baccaglini-Frank and Mariotti (2010) developed a model to explain the cognitive processes behind different types of dragging and differentiated four different types of dragging. *Wandering dragging* is dragging that aims to look for regularities. *Maintaining dragging* is dragging elements of a dynamic diagram so that it maintains certain properties. *Dragging with trace activated* is dragging a point or its parent point with its trace activated. *Dragging test* is dragging elements to test whether certain properties will hold under certain conditions. Measuring is another important feature of GSP. Olivero and Robutti (2007) identified different measuring modalities in a dynamic geometry environment. *Wandering measuring* is measuring some elements of the configuration to identify quantitative relations, invariants, congruencies, etc. *Guided measuring* is measuring to obtain a configuration from a generic diagram that contains free or semi-free elements. *Perceptual measuring* is measuring to check the validity of a perceptual observation. *Validation measuring* is measuring to check a conjecture within a dynamic geometry environment to accept it or refute it. *Proof measuring* is measuring to get a better explanation or understanding of a proof that students have already constructed. These dragging and measuring modalities be utilization scheme users develop while using dragging and measuring tools. When studying construction in dynamic geometry environments, researchers introduced the notion of “soft” and “robust” constructions (Laborde, 2005). A *robust construction* is a construction that passes the dragging test. It is obtained by using geometrical objects and relationships characterizing the construction. In contrast, a *soft construction* is a partial construction in which variation is part of the construction itself, and a mathematical property becomes evident only at the point in which another property is satisfied. Both soft and robust constructions are shaped by utilization schemes developing from invariant use of one or multiple tools within a specific dynamic geometry environment.

METHODOLOGY

Data for this paper came from a research project that aimed to examine the relationship between Geometer’s Sketchpad (GSP) usage and the development of mathematical understanding around various geometry topics (e.g., centers of triangle, quadrilaterals, and geometric transformations) through a series of task-based interviews. Task-based interview was chosen because it allowed the researcher to gain knowledge about learner’s existing and developing mathematical knowledge and problem-solving behaviors. A carefully constructed task is essential to a task-based interview. In this research, researcher selected tasks for which GSP could potentially facilitate exploration and analysis of mathematical relationships, provide alternative approaches for problem-solving, or generate new problems that otherwise could not be posed.

The participants were three undergraduate preservice teachers enrolled in a secondary mathematics education program. The participants were selected based on voluntary participation. All the participants reported that they had not completed any geometry course after high school. None of the participants had been introduced to Geometer’s Sketchpad (GSP) prior to the study. Therefore, a 60-minute tutorial session was held for each participant prior to the interviews. The goal was to get participants familiar with what tools are available in GSP, focusing on tools under the “construct” menu, “transform” menu, and “measure” menu. However, the participants were not taught ways of using these tools.

Each participant spent eight sessions exploring geometry problems around various geometric topics. Each session lasted approximately two hours. Participants’ interactions with GSP were screen-recorded. Interactions between the interviewer and the participant were recorded. The GSP files produced during each interview were collected. Data analysis consisted of several stages. Video of each participant in each interview session was first segmented into episodes according to the transition of mathematical tasks. Each episode was then transcribed, focusing on both what was said and what was done with GSP tools. In the second stage, the participant’s growth of geometric understanding in each episode was analyzed using the Pirie-Kieren theory. It involved associating each understanding activity, as manifested both in words and actions, with a specific level of understanding described in the Pirie-Kieren theory. The analysis at this stage resulted in a diagram that provided a global picture of a participant's growth of understanding in the Pirie-Kieren model. The third stage of analysis involved zooming into each level of understanding to examine participant's interaction with GSP. Specific dragging/measuring modalities (see the instrumental genesis section) and invariant use of one or multiple construction tools were identified. The analysis at this stage enabled me to examine the relationship between the development of utilization schemes and the emergence of new geometric knowledge at each level of understanding in the Pirie-Kieren model. The fourth stage involved synthesizing the analysis in the second and third stages.

RESULTS

Analysis of the participants' GSP-mediated understanding activities suggests that integrating the Pirie-Kieren theory and instrumental genesis provides a productive means to capture learner's growth of mathematical understanding within the dynamic geometry environment. Due to space limitation, this paper shares analysis of only one participant's growth of understanding of inscribing a square of each given square to demonstrate the descriptive power of the integrated framework.

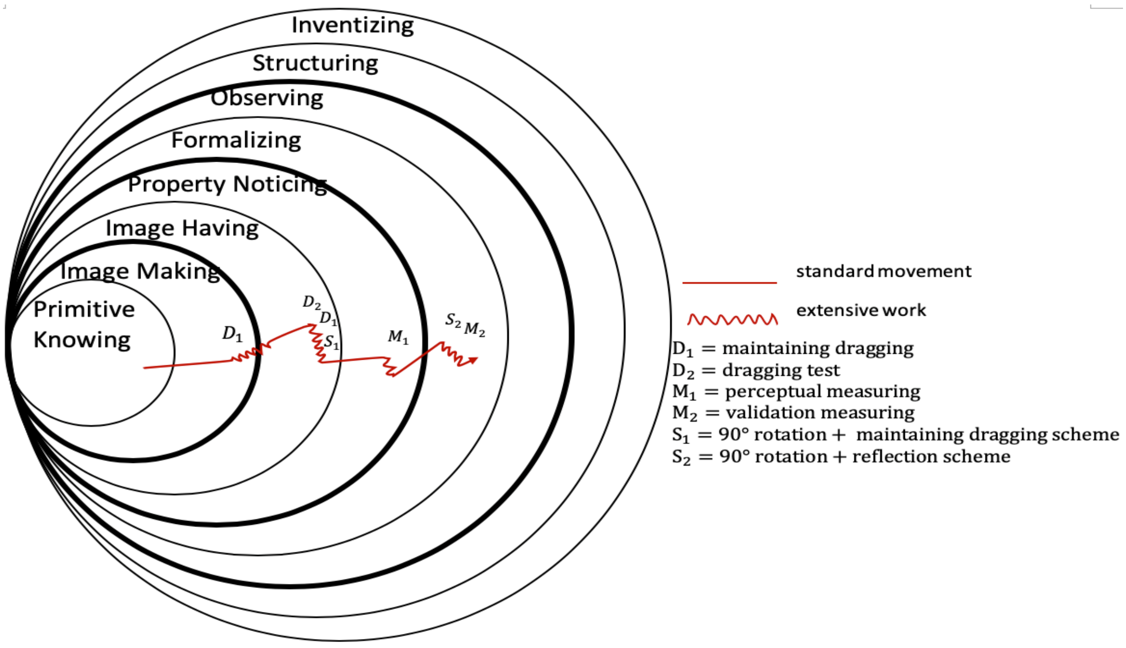

Figure 1: Chen's growth of understanding of inscribing a given square

In one interview, Chen was given a square and asked to inscribe a square such that all its four vertices lie on a side of the given square. Through investigation, Chen developed a formal construction for the inscribed square. Figure 1 represents Chen's process of gaining this understanding. As a start, Chen relied on his *Primitive Knowing* to solve the problem. He used the "midpoint" tool to find the midpoints of four sides of the parent square, connected the four midpoints, and stated that the inscribed quadrilateral was a square (Figure 2a). His justification was the following: Its four sides were congruent since the four triangles at the corners were congruent and all its angles were 90° since the four triangles were 45°-45°-90° triangles. By asking Chen whether that was the only inscribed square, the interviewer moved Chen to *Image Making* since he became uncertain. He drew a line segment $\overline{FE}$ from $\overline{AB}$ to $\overline{AD}$, rotated $\overline{FE}$ 90° about F, then dragged E (*90° rotation + maintaining dragging scheme*) to move E' to $\overline{BC}$ (Figure 2a). Chen created different configurations by changing the location of F and dragging E (*maintaining dragging*) to move E' to $\overline{BC}$. As a result, he observed that for every fixed point on $\overline{AB}$ there was only one location for E that would yield the desired configuration. This indicates that Chen advanced to *Image Having* and started to form a refined mental image about the inscribed square.

Guided by this mental image, Chen started to explore how to create the inscribed square. He deleted E and its dependent elements, left on the screen the parent square, the inscribed midpoint square, and point F. Using the "circle by center + point" tool, he constructed a circle with F on $\overline{AB}$ as the center and G on $\overline{AD}$ as a point on the circle, and labeled H as the point of intersection of the circle and $\overline{BC}$. However, when dragging F (*dragging test*), he realized the circle did not always intersect with $\overline{BC}$ and the two radii were not constructed to be perpendicular. He then rotated $\overline{FG}$ 90° around F and then dragged F (*maintaining dragging*) such that G' and H coincided. After that, Chen reflected $\overline{FG}$ and $\overline{FG'}$ over line GG' to complete the square (Figure 2b). He stated that he could use the above procedure to find an inscribed square for every point on $\overline{AB}$. Meanwhile, Chen was aware that his procedure did not pass the dragging test since he relied on visual image to find where G' and H coincided. When he dragged F (*dragging test*), the construction indeed collapsed. Here, Chen attempted to develop a construction for the inscribed square but did not succeed. He relied on soft construction and maintaining dragging to obtain the inscribed square.

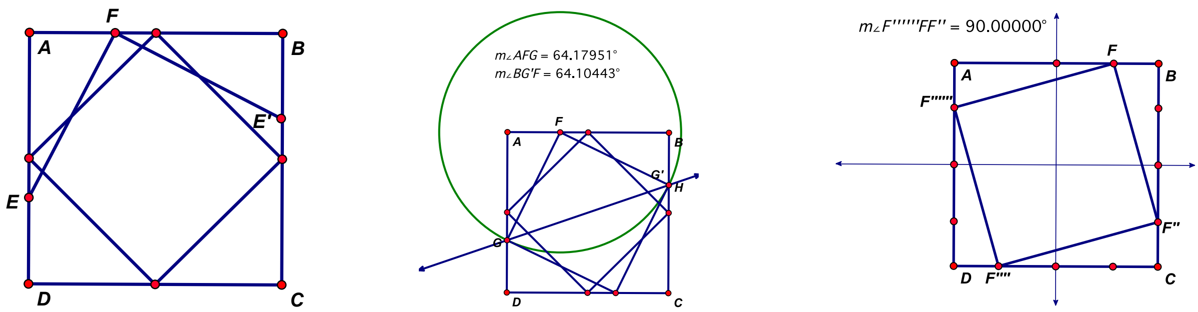

Figure 2: Screenshots of Chen's work

The interviewer then asked Chen what he noticed about the diagram, particularly the triangles at the corners. This moved Chen to *Property Noticing*. Chen stated that the two triangles (ΔAFG and $\Delta BG'F$) were congruent and justified his observation by triangle congruency theorem (i.e., $\overline{FG} = \overline{FG'}, \angle A = \angle B = 90°, \angle AFG = \angle BG'F$, *perceptual measuring* was used to confirm $\angle AFG = \angle BG'F$). He then started to think about how to construct the inscribed square, which led him to move toward *Formalizing*. After about one minute of silence, Chen shared that he was thinking about constructing congruent triangles at each corner through transformations. He achieved this goal by transforming a given point on one side of the square to the other three sides through a sequence of compositions of a 90° rotation followed by a reflection. After deleting his previous work (left only the parent square and point

F), Chen brought back all the midpoints and drew one line passing through the midpoints of $\overline{AB}$ and $\overline{CD}$ and another line passing through midpoints of $\overline{AD}$ and $\overline{BC}$. He rotated F 90° around B and then reflected F' over the line connecting the midpoints of $\overline{AD}$ and $\overline{BC}$ to get F'', rotated F'' 90° about C to get F''' and then reflected F''' across the line connecting midpoints of $\overline{AB}$ and $\overline{DC}$, rotated F'''' 90° around D to get F''''' and reflected F''''' across the line connecting midpoints of $\overline{AD}$ and $\overline{BC}$ to get F'''''' (*90° rotation + reflection scheme*, Figure 2c). After finishing the construction, Chen claimed that the figure $FF''F''''F''''''$ is a square and measured $\angle F''''''FF''$ (*validation measuring*) to confirm his claim. As a result, Chen developed a formal construction for inscribing a given square. Here, the invariant use of a 90° rotation followed by a reflection indicated the formation of a utilization scheme that allowed Chen to develop a formal construction for inscribing a dynamic square of a given square.

DISCUSSION

During the past decade mathematics education researchers have devoted efforts to understand how theories can be connected successfully while recognizing their underlying conceptual and methodological assumptions, a process called "networking theories". Exploring ways of connecting theories may help researchers to better grasp the complexity of learning and teaching processes. Radford (2008) argued that networking theories can happen at the level of principles, at the level of methodologies, at the level of questions, or as combinations of these. Prediger, Bikner-Ahsbahs, and Arzarello (2008) described different strategies for networking multiple theoretical approaches, including making one's own theory understandable, understanding others, comparing, contrasting, combining, coordinating, integrating locally, synthesizing, and unifying globally. The conceptual framework in this research is built by well-fitting elements from the Pirie-Kieren theory and instrumental genesis to capture learner's growth of geometric understanding in a dynamic geometry environment. It provides an example of networking by coordinating principles of two theoretical approaches. More specifically, while the Pirie-Kieren theory served as a conceptual tool for tracing learner's growth of mathematical understanding, instrumental genesis acts as an analytical tool at the microlevel for describing instrument-mediated actions at each level of understanding in the Pirie-Kieren theory.

The coordination enriched the Pirie-Kieren theory. Although the original theory recognizes the importance of tools in learner's growth of mathematical understanding, learner's interaction with tools and its impact on the growth of mathematical understanding was not further elaborated. The coordination makes the Pirie-Kieren a more viable tool to capture learner's growth of mathematical understanding in a dynamic geometry environment while accounting for in detail the impact of technology on such growth. Moreover, the coordination suggested that the study of utilization scheme in a dynamic geometry environment should go beyond the study of various dragging/measuring modalities, an area that has been studied for two decades and accumulated a significant amount of research. The study of geometric understanding requires researchers to attend to the synergy of features of dynamic geometry software because a learner might use multiple digital tools in the process of creating and transforming geometric objects and discovering new geometric properties. This was indeed the case in the episode shared in this paper. The utilization schemes (i.e. 90° rotation+ maintaining dragging and 90° rotation + reflection) for inscribing a given square require the integration of multiple DGE features. Further research is needed to examine the various utilization schemes learners might develop when using multiple tools available in a dynamic geometry environment to solve mathematical problems and their impact on the growth of mathematical understanding.

References

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225-253.

Barabash, M. (2019). Dragging as a geometric construction tool: Continuity considerations inspired by students' attempts. *Digital Experiences in Mathematics Education*, 1-21.

Gokalp, N. D., & Bulut, S. (2018). A new form of understanding maps: Multiple representations with Pirie and Kieren model of understanding. *International Journal of Innovation in Science and Mathematics Education*, *26*(6), 1-21.

González, G., & Herbst, P. G. (2009). Students' conceptions of congruency through the use of dynamic geometry software. *International Journal of Computers for Mathematical Learning*, *14*(2), 153-182.

Gülkilika, H., Ugurlu, H. H., & Yürük, N. (2015). Examining students' mathematical understanding of geometric transformations using the Pirie-Kieren model. *Educational Sciences: Theory and Practice*, *15*(6), 1531-1548.

Laborde, C. (2005). Robust and soft constructions: Two sides of the use of dynamics geometry environments. In *Proceedings of the Tenth Asian Technology Conference in Mathematics* (pp. 22–35). Cheong-Ju, South Korea: Korea National University of Education.

Meel, D. E. (2003). Models and theories of mathematical understanding: Comparing Pirie and Kieren's model of the growth of mathematical understanding and APOS theory. *CBMS Issues in Mathematics Education*, *12*(2), 132-181.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*, *12*(2), 135-156.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26(2-3), 165-190.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM*, *40*(2), 165-178.

Rabardel, P., & Beguin, P. (2005). Instrument mediated activity: From subject development to anthropocentric design. *Theoretical Issues in Ergonomics Science*, *6*(5), 429-461.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM*, *40*(2), 317-327.

EXAMINING MATHEMATICS TEACHERS' PROFESSIONAL KNOWLEDGE BASE DURING THE PANDEMIC CRISIS: THE PERSPECTIVE OF SWOC ANALYSIS

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The study focused on Hong Kong mathematics teachers' personal experiences and challenges in online teaching during the COVID-19 pandemic crisis. A SWOC analysis is adopted. Results showed that they tried to offer e-learning as an alternative mode of teaching to maintain the education service within the pandemic. Hong Kong teachers tended to focus on external factors rather than internal factors in the initial period of online teaching. They agreed that there was a paradigm shift in the role of teaching during the crisis. They were commonly building a sufficient level in using technology, but this was not equivalent to be competent in achieving instructional goals. Mathematics teachers' professional knowledge is influenced by their personal online teaching experiences. Their attitudes and beliefs are crucial factors in the role and effectiveness of using online teaching.

INTRODUCTION

Due to the outbreak of COVID-19, the Education Bureau (EDB) in Hong Kong announced that all students were asked to stay home after the Lunar New Year Break ended in late January 2020. During the period of class suspension, the EDB requested schools to provide student useful learning materials through internets or other effective means. The suggestions that given by the EDB was not a compulsory action, according to the first notice for all schools from EDB, the purpose of school use of e-learning is "students can make good use of their time at home to continue their studies" (The Government of the Hong Kong SAR, 2020). Therefore, schools in Hong Kong were initially slow to embrace online learning at that time. Unfortunately, the outbreak of COVID-19 in Hong Kong was continued, the school suspension has been extended until April after the Easter holidays. The education chief stated that the class suspension was not an extension of the school holidays. Schools are making use of different modes of learning, including e-learning to achieve the goal of "suspending classes without suspending learning" (Hong Kong's Information Services Department, 2020). Schools in Hong Kong started to prepare e-learning materials or online teaching after the outbreak of the second wave. However, this alternative model of teaching brings difficulties for both teachers and students. Most students cannot finish distance learning independently, and their parents expected more interactive support feedback from school during the online learning, at the same time, the related professional training for teachers is needed to

2021. In Inprasitha, M, Changsri, N., & Boonsena, N. (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 311-318. Khon Kaen, Thailand: PME.

strength (Lau & Lee, 2020). All the teachers changed their usual way of teaching and put into action in a short time with new teaching approaches and methods; at the same time, they were also expected to teach as best as possible. This was undoubtedly a very challenging situation for teachers, that brought to the fore their perceptions about teaching. In this situation, the study of analyzing teachers' voice or feedback is imperative for improving students' online learning performance and teachers' professional development. This article focuses on studying teachers' voices of their online learning experiences via utilizing a combination of SWOC analysis (Dhawan, 2020) in teachers' professional knowledge base (Shulman, 1987), and provide support and recommendations for studies about online learning teachers' professional training in the further.

RESEARCH METHOD

Conceptual framework

The SWOC model (Dhawan, 2020) is adopted to be the analytical tool used for analysing the data which amassed from different sources in this study. The research method is descriptive research. The SWOC analysis was conducted to understand various strengths, weaknesses, opportunities, and challenges associated with mathematics teachers' perceptions of their online teaching experiences during the COVID-19 pandemic in Hong Kong.

Research questions

Our intention with this study was to understand further the situation teachers precepted during this pandemic crisis. Moreover, the findings of the qualitative study can provide some suggestions and recommendations for the success of the online mode of learning and teaching. This article focuses on the qualitative part of this study. In order to understand the situation of teachers' online teaching mathematics experiences, this study endeavoured to answer the research questions are two-fold:
RQ1: What is the situation of e-learning and teaching experience of mathematics teachers in Hong Kong?

RQ2: What are the strengths, weaknesses, opportunities and challenges (SWOC) perceived by the mathematics teachers regarding online teaching in Hong Kong?

Participants and instruments

The 13 participants of this qualitative study were selected from the large population of a quantitative study (n=109) regarding by their background and position in schools. The 13 teachers were from primary and secondary schools participated in this study. Among them, 4 were novice teachers (<5 teaching years) and 4 were competent teachers (between 5 to 10 teaching years), the other 5 were expert teachers (≥ 15 years). These teachers teach different levels in schools, including all grades in primary school and all levels in secondary school. Excluding be a mathematics teacher, their primary duties in schools include mathematics panel head, curriculum leader, grade coordinator, STEM coordinator, and academic committee head. All participants were invited by this project coordinator. The participation was voluntary to attend an in-depth online interview to narrate their teaching experiences about e-learning, online teaching, perceived challenges, etc. All interviews were held by an experienced qualitative researcher in the study. Semi-structure questions and guided questions were constructed and the whole interview process was recorded.

Data analysis

The data from the interviews to respond to teachers' experiences in this pandemic period were transcribed and coded with the software Nvivo for analyses. First of all, the grounded coding strategies were used to derive meanings of data collected. Secondly, thematic coding strategy was used to generate a set of main concepts or categories of ideas that were mentioned by the teachers. Through this process, data were organized and sorted into four major themes SWOC: Strength, Weakness, Opportunities and Challenges. During the development of coding categories, it involved an interactive review with related literature that required considering previous research about teachers' professional knowledge (Shulman, 1987). After categorization of every SWOC themes, the basic common themes and categories of interviews data included knowledge of using technology, online teaching methods, classroom management difficulties, special learning need concern, curriculum adjustment, assessment, etc. We explored that these themes were related to how the teacher solve the problems in online teaching by their knowledge. In line with Shulman's ideas on teachers' professional knowledge base, we split each SWOC theme into three categories included content knowledge (CK), pedagogical knowledge (PK) and pedagogical content knowledge (PCK) and combined into a full matrix. Through this process, data were organized into twelve categories: SCK, SPK, SPCK; WCK, WPK, WPCK; OCK, OPK, OPCK; CCK, CPK, CPCK. All data were analyzed. For example, one of teacher A argued in her interview that "*From the beginning of online teaching, teachers among us started to discuss online teaching couldn't replace face-to-face lesson. We all worried about how to teach mathematic distantly. Our teaching includes a lot of hands-on activities, but we could not teach as past now. Teaching time are shortened too*……". A content analysis was conducted in a similar way and identify categories of "Challenges" and "PCK".

The qualitative data were analysed by two researchers. The first researcher conducted the initial analysis, and the second researcher checked the analysis. The interview data were coded independently by the researchers and categorized. Then, the coding analysis was compared, conflicts between the two researchers were discussed, and a consensus with regard to coding the few discrepancies was resolved collaboratively by discussing the nature of the online teaching. Results were summarized in Table 2 and some details are discussed in the next section.

RESULT

In the period of COVID-19, teachers' interviews display positive comments towards embracing online teaching and likely to adopt e-learning during the coronavirus outbreak. As all of the schools were closed in the age of COVID-19, it is necessary for mathematics teachers to explore online teaching. The interview data show that a typology of teaching knowledge that reflects how online teaching experience related to their teacher knowledge. First, results of the analysis of the content of interviews data are presented according to four themes: (a)strength, (b) weakness, (c)opportunities, (d) challenges. Second, the teaching knowledge related to each theme is presented in three categories. Finally, the relationships between SWOC and teachers' knowledge are exemplified in the results.

The smallest theme, weakness, consisted of 7 quotations only but the largest theme is challenges which consisted of 66 quotations (see Table 1). According to the SWOC analysis, strengths and weaknesses are internal factors, while opportunities and challenges are external factors. The data presented that teachers' perception of online teaching was highly impacted by external factors. The data presented a view of how teachers interpreted their perception of online teaching experiences in this period of time. The difficulties and problems that they faced in online teaching what caused by the external factors, for example, technology, resources, teaching schedules, assessment arrangement, etc. In another turn, teachers commonly have a strong belief that their competence in teaching is common strengths in adapting and applying online teaching. In this study showed that teachers' belief that they have strong internal factors to deal with online teaching situation, for example, personal strength, rich teaching experiences, strong subject knowledge and curriculum understanding, etc. However, teachers' interviews focused on challenges were mainly from external factors, for example, school and government policy, school administration, parents support, etc. This result showed a contrast between internal and external impact factors of online teaching.

Table 1 No of quotations in SWOC categories

Themes	CK	PK	PCK	Total
S (internal)	0	13	7	20
W (internal)	0	7	0	7
O (external)	0	8	2	10
C (external)	2	40	24	66

Note. S = Strength; W = weakness; O = opportunities; C = challenges; CK = content knowledge; PK = pedagogical knowledge; PCK = pedagogical content knowledge.

After further analysing the data, we derived a comprehensive picture of the teachers' experiences in online teaching and sorted each SWOC themes into Shulman's professional knowledge-base categories. Several interesting findings were noted. First, the "Strength and content knowledge", "Weakness and CK", "Weakness and PCK" and "Opportunities and CK" categories have none of the quotations. That means teachers' interviews did not contain any narration related to these categories. In the "Strength" categories, 13 quotes were linked to "Strength and pedagogical knowledge categories", such as good competency in IT to shift teaching run online shortly. There were 7 quotes related to “Strength and pedagogical content knowledge” categories, for example, teachers have flexible mathematical teaching methods to deal with online classroom situation. In “Weakness and pedagogical knowledge”, 7 quotes linked to this category, for example, lack of online teaching experiences in the past, bad time management and provided not enough support for individual differences.

The categories of “Opportunities and PK” and “Opportunities and PCK categories” contained 8 and 2 quotes respectively(see Table 2). Teachers commonly agreed that this period of online teaching experiences explored and developed new teaching strategies or reflected the role of school and teacher in the new normal. The “Challenges” were the largest categories consisted of 2 quotes in content knowledge both related to curriculum adjustments, 40 quotes in “PK” and 24 quotes in “PCK”. Teachers mentioned reasons for the challenges were focused on general online teaching difficulties, for example, technical problems, online homework and assessment arrangement, online classroom management problems, shorten teaching times and student low learning motivation etc. which are analysed and organized into six different categories. In the PCK category, teachers concerned how to adapt online teaching and apply ICT in mathematic teaching. Several teachers commented and compared online and traditional mathematic teaching differences such as which concepts and topics were suitable taught through online or not. Teachers also provided examples to illustrate how to deal with those challenges in an online situation. Interestingly, teachers mentioned that the nature of mathematic teaching causes those challenges.

Table 2 Categories of Opportunities

Categories	No.	Examples
Opportunities + Content Knowledge (OCK)	0	—
Opportunities + Pedagogical Knowledge (OPK)	8	In fact, this is the result I want. Because I was also thinking about it (ability grouping). A student came back yesterday and told me that it wasn’t the kind of cross-subject grouping that had been done many years ago. Is that whether the students are of the same age, do they need to go to that grade?....
Opportunities +Pedagogical Content Knowledge (OPCK)	2	I also think that this period of online teaching will not be just one time, it will not be over and nothing will return to the past. When we open a road, the whole world will know that this is a good way, and it will be easy...

The teacher is the most important person in curriculum implementation. Analysis of the interviews attested that the strength of most teachers interviewed in the study are knowledgeable of the current school mathematics curriculum, the practice of mathematical teaching. Moreover, they are responsible for introducing the curriculum in the classroom individually. Thus, teachers expressed their concern in several curriculum issues related to online teaching during the coronavirus outbreak. An example of Teacher N:

Interviewer: Whether you are familiar with the environment and requirements of secondary school?

Teacher N: It's all related, because of the current teaching topic “percentage”, students still have to continue to learn in Form One. And the topic “area” must continue to study like surface area in Secondary One. In Form 2, the area and volume of the cylinder must be studied. If it is not consolidated now, what will happen in the future? You should continue to study the topic “Circle” in Form 5. If the foundation is not good, it will be very troublesome in the future. Because I am very familiar with the teaching content of secondary school, and I am now hesitating whether to consider teaching topics have to teach first.

DISCUSSION

The purpose of this study was to utilize the voices of mathematics teachers in Hong Kong to understand their perspective of facing online teaching. Moreover, this study presented a picture of how teachers deal with difficulties and challenges in term of teacher pedagogical knowledge. Three interesting phenomena were observed in the study.

First, Hong Kong teachers tend to focus on external factors rather than internal factors in the initial period of online teaching. They have omitted personal weakness or strength in facing situation changes. In fact, teachers felt that they had the competencies and confident about their ability to teach mathematic online, although most of the teachers stated that this was the first time to teach online. Few of them likely to explore new tools or possibilities to improve their teachings. Besides of this, teachers also showed strong confidence about their ability in teaching, but seldom mentioned their worries or personal weakness in online teaching. Only a few teachers were likely to improve their teaching methods, including teaching contents or planning. Most of them expressed the classroom management problems as external challenges that they could not control. These are the reactions of Hong Kong teachers when face changes or challenges. Teachers are aware of the external factors rather than internal personal factors in the initial phase of change.

Studies on teacher change usually refer to a situation of professional development and not to a situation of emergency. The pandemic turns all teaching run online, which forced teachers to have to change in a short period of time. This convinced to Peirce’s belief theory that “fixation of belief”. He stated that beliefs are our stability and highly resistant to change. The formation of belief is the base state of cognition through which we make sense of the world (Cunningham, 1998). All past learning and teaching in traditional classroom experiences form teachers' belief (Richardson, 1996). Their beliefs are highly stable and resistant to change. This gives educators and teacher trainers an insight that understanding personal belief can help teachers to move on. Investigating which beliefs can be changed or modified are important to establish a new form of teaching. Liljedahl (2010) argued that change may happen in a rapid and profound way when an existing belief starts to be questioned or even rejected by the teacher; such a change is profound when the teacher finds a new belief to replace the former one. If online teaching becomes an irreversible change, teaching as a personal specific and implicit practice. Assisting teachers to consolidate their personal strength and weakness is an essential process to prepare teachers for the future.

Second, teachers commonly agree that during the pandemic there is a paradigm shift in the role of teaching. There are variations in technology usage in the study, those reflected the differences in teachers' beliefs about the utility of technology in teaching. The teachers' words, as expressed in the interviews, raised the questions of adjustment of teaching contents in mathematics topics and concepts due to temporary closure of school and transition to online teaching: Which topics or concepts should be included? Hong Kong teachers have highly relied on the centralized curriculum for example textbook or government-provided curriculum. In general, Hong Kong teachers were not actively involved in curriculum design or planning although they are knowledgeable (CK) with good teaching practices (PK). They have to consider the adjustment of their teaching schedule and content affected by online teaching.

Third, teachers are commonly building a sufficient level in using technology, but this is not equivalenced to competence in achieving instructional goals by using technology. The problem of effectiveness in online teaching raises within interviews. The competences to achieve effective online teaching shall be the future teacher professional development approach.

CONCLUSION

Teachers' narration provided a whole picture to illustrate the challenges that they were faced. Moreover, their experiences provided rich insights to handle this transition from traditional teaching mode to online teaching in the future. Pandemic kicks off the new era of teaching. This study has provided further evidence that mathematics teachers can also learn via online mode. The study has explored the tension of mathematic class online. Results of the study reveal that online teaching mathematics have differences in clustering. The reason for these differences could be due to teachers' beliefs and attitudes towards the use of technology in learning mathematics. If a teacher lacks the skill and knowledge on how to use technology in teaching mathematics, which may be characterized by a negative attitude thereby recording low motivation in online teaching. Understanding their existing belief is a method to assist teacher adapted and shifted their belief to a new status.

Results also revealed that teachers' online teaching mathematics were very high. One reason for these high scores is that teachers exhibited hood tic-skills to engage in online mathematics teaching and had the necessary technological tools to facilitate their online interactions. Teachers' professional knowledge base could be regarded as one of the core competence of effective teaching and it is important to understand the knowledge base under online teaching. Educators have good computer technological knowledge, subject knowledge, pedagogical knowledge and pedagogical content knowledge, but the effectiveness of their online teaching can be low. This part is waiting for further discussion and studies in the future.

References

Cunningham, D. J. (1998). Cognition as semiosis: The role of inference. *Theory & Psychology*, *8*(6), 827–840.

Dhawan, S. (2020). Online Learning: A Panacea in the Time of COVID-19 Crisis. *Journal of Educational Technology Systems*, *49*(1), 5–22. https://doi.org/10.1177/0047239520934018

The Government of the Hong Kong SAR (2020). SED opening remarks at press conference. Retrieved from https://www.info.gov.hk/gia/general/202002/25/P2020022500760.htm

Hong Kong's Information Services Department (2020). No classes before Mar 16. Retrieved from: https://www.news.gov.hk/eng/2020/02/20200213/20200213_112930_864.html

Lau, E. Y. H., & Lee, K. (2020). Parents' views on young children's distance learning and screen time during COVID-19 class suspension in Hong Kong. *Early Education and Development*, 1-18. https://doi.org/10.1080/10409289.2020.1843925

Liljedahl, P. (2010). Noticing rapid and profound mathematics teacher change. *Journal of Mathematics Teacher Education*, *13*(5), 411–423.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (2nd ed., pp. 102–119). New York: Macmillan.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57(1)*, 1-2

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