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Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

VOLUME 2

Research Reports (A – H)

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RESEARCH REPORTS

A to H

PROBING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF VISUAL REPRESENTATIONS OF FUNCTION TRANSFORMATIONS: A MULIPLE SCRIPTING TASK APPROACH

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In this paper, we use multiple scripting tasks as a research tool to investigate prospective secondary mathematics teachers' (PSMTs') mathematical knowledge of function transformations and their inclination to connect multiple representations of functions. Mathematically similar scripting tasks focused on visual representations of function transformations were given at three intervals during a 15-week semester in an undergraduate mathematics course on functions for PSMTs in the United States. Participant responses to these scripting tasks were analysed, and four prevalent themes were identified that reveal initial tendencies to disregard visual observations posed by students in the scripting tasks and limited use of their mathematical knowledge to connect multiple representations of functions.

INTRODUCTION

Prospective secondary mathematics teachers (PSMTs) will be expected to teach mathematics for which the concept of function is a fundamental component. However, Ponte and Chapman (2008) identify "lack of a good understanding of functions" as a consistent issue with the knowledge of PSMTs highlighted in the research literature (p. 227). For example, Even (1993) found that a limited conception of function influenced PSMTs pedagogical reasoning. Also, Hitt (1998) links a group of practicing secondary mathematics teachers' conceptual knowledge to difficulties in passing from one representation of function to another. With the prevalence of graphing technology, visual representations of functions can be easily generated and used in the classroom. However, this may expose ways in which PSMTs' limited understandings of the connections between representations may deter future teachers' capacity to address student understandings and leverage their own understandings.

In this study, we used three mathematically similar scripting tasks to explore any changes in the PSMTs' understanding that may have been influenced by inquiry-based lessons focused on functions and function patterns. Our research questions are: (1) To what aspect of the mathematics in the scripting task do PSMTs choose to attend? In particular, how do they connect different representations or attempt to make mathematical connections to resolve the student's question? (2) How do PSMTs choices to resolve the student's question incorporate or validate the student's mathematical thinking?

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BACKGROUND AND THEORETICAL PERSPECTIVE

Script writing in the context of a mathematics course for preservice teachers can be a useful research tool to investigate mathematical knowledge and understanding for teachers (Zazkis & Zazkis, 2014). A scripting task typically begins with a hypothetical conversation between a teacher and a student, or between multiple students, which is then continued by the PSMT in a written dialogue. Script writing tasks provide PSMTs an opportunity to prepare a well-considered reply to a student, rather than an on-the-fly, in-the-moment response. Scripting tasks allow researchers a written window into the mathematical thinking of the PSMT, together with a view of how the PSMT chooses to address a cognitive conflict as expressed by a student, and their pedagogical sensitivity in assisting students (Kontorovich & Zazkis, 2016).

In a student-centred mathematics classroom, researchers have supported models of effective mathematics instruction in which a teacher fosters students' ability to consider various mathematical solutions (Hiebert et al. 1997). To do this, a teacher must use their own mathematics knowledge flexibly to draw out the important representations, ideas, and conceptions embedded in students' mathematical thinking. Teachers who lack this flexible knowledge of mathematics and student thinking may be more inclined toward ritualized "show-and-tell" (Silver et al., 2005). Ball's (1990) study exhibits this inclination when she probed ten elementary and nine secondary prospective teachers' understanding of division and found that the prospective teachers at both levels tended to search for the particular rules rather than focusing on underlying meanings. "They seemed to assume that stating a rule was tantamount to settling a mathematical question" (p. 141). In 2008, Ball et al. further categorized mathematical knowledge unique to the work of teaching or mathematical knowledge for teaching (MKT). The domains of MKT proposed by Ball et al. (2008) map to two categories – subject matter knowledge and pedagogical content knowledge. In the context of this study, subject matter knowledge is at the core of the sequence of scripting tasks completed by the PSMTs.

Developing a deep understanding of function transformations at the secondary level can require the learner to reconcile multiple representations of function, including graphical, tabular, and symbolic representations (c.f. Eisenberg & Dreyfus, 1994). Oehrtman et al. (2008) "recommend that school curricula and instruction provide more opportunities for students to experience diverse function types emphasizing multiple representations of the same functions" (p. 153). Dynamic visualization software can be a robust tool for students to make these connections as they explore the effect of different transformations (Villarreal, 2000). However, visual information can sometimes negatively influence misconceptions held by the learner (Aspinwall et al., 1997). For example, Álvarez et al. (2020) described a task for practicing teachers in which the teachers struggled to explain an apparent discrepancy between the dynamic visual representation of a vertical dilation of the linear function $f(x) = x$ and a rotation of the graph of $y = x$ about the origin. In addition, Moore and Thompson

(2015) use the study of shape thinking "to offer a new perspective on multiple representations by enabling researchers to be clearer about what a graph represents *to a student*, and thus what students understand multiple representations to be representations of" (p. 788).

METHODS

This study took place at a large, public university in the southwestern United States with an enrolment of over 42,000 students. The university is recognized as one of the most diverse national universities in the United States. Participants in this study consisted of 27 PSMTs enrolled in 2018 fall semester, second-year mathematics content course for PSMTs with a second-semester calculus prerequisite. Twelve participants self-identified as male and 15 self-identified as female.

The mathematics course implemented a unit developed by the *Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers* Project, *Explorations on Functions and Equations* (*EFE*). The *EFE* materials consist of 11 research-based lessons with an objective of deepening and broadening PSMTs function-related mathematical content knowledge from school algebra to calculus by exploring relevant topics in an inquiry-based learning environment. In the 15-week fall 2018 semester, the *EFE* materials spanned the first 10 weeks of the course approximately.

This study centres around three scripting tasks related to two lessons within the *EFE* materials: "Functions Arising from Patterns" and "Indistinguishable Function Transformations and Function Patterns." Zazkis and Zazkis (2014) advocate that scripting tasks "serve as a window for researchers to investigate participant's understanding of mathematics" (p. 68). The scripting tasks in this study are intended to reveal PSMTs' MKT. In particular, MKT related to connections between function transformations and their visual representations. During the eighth week of implementation of the *EFE*, students completed Scripting Task 1 (ST1). This served as a baseline for evaluating participants' MKT, and it was completed outside of class before the lesson on "Functions Arising from Patterns." ST1 (see **¡Error! No se encuentra el origen de la referencia.**) provides a fictional interaction between a teacher and a student, Grace, in which Grace questions the teacher about the horizontal compression she perceives in the transformation $g(x) = a \cdot f(x)$ as a varies dynamically where $f(x) = x^3$ and $a > 1$.

Over the next three 80-minute class meetings, students engaged in the lesson "Functions Arising from Patterns" and then "Indistinguishable Function Transformations and Function Patterns." For the former, PSMTs investigated patterns in the domain of given data sets and resulting patterns in the corresponding range data sets. Specifically, students explored the domain-range patterns within data sets arising from linear, quadratic, power, exponential, and logarithmic functions. They identified patterns such as an addition-product pattern for logarithmic functions by noticing that adding c to subsequent domain values results in a pattern of multiplying the corresponding range values by a constant k (that depends on c).

In the second part of the "Functions Arising from Patterns" lesson, students work with general forms of the functions to verify algebraically that the identified domain-range patterns apply to certain transformations on functions of the same type. For example, they verify the product-addition pattern for logarithmic functions.

You are given the beginning of an interaction between a teacher and a student. Your task is to continue this imaginary conversation in the form of a dialogue between the teacher and student (or several students).

T:	Today we will continue our exploration of transformations of functions. Consider for example $g(x) = a \cdot f(x)$ where $f(x) = x^3$ and $a > 1$. How does the graph of g compare to the graph of f? Yes, Grace?
S:	So you said yesterday that multiplying by a is a vertical stretch.
T:	That is correct as long as $a > 1$.
S:	But my graph of g looks like a horizontal compression of f.
T:	Ok.
S:	So a on the outside means it's both a vertical stretch and a horizontal compression? Or what?
T:	...

Figure 1: ST1

The "Indistinguishable Function Transformations and Function Patterns" lesson examines function patterns that may seemingly produce a dynamic process that defies the algebraic rules previously learned about transformations of functions. PSMTs encounter four scenarios in which a particular transformation represented algebraically simultaneously appears also to correspond to a different type of transformation. They are invited to use appropriate technology in their exploration. The following is an excerpt of one of the scenarios:

> For a given function f, we define a new function $g(x) = f(x + c)$ where $c > 0$. The graph of the new function g is a horizontal translation (shift) of the graph of f, but it also appears to be a vertical translation (shift) of the graph of f. In order to observe this, which function pattern must f have? Explain your reasoning. Identify the function type for which this observation would apply.

Directly after completing these explorations on function patterns and transformations, students were given Scripting Task 2 (ST2) to be completed outside of class. This task is then intended to elicit participants' MKT after they have had the opportunity to delve into these ideas within the *EFE* lessons. Like ST1, ST2 presents a fictional conversation between a student, Isaac, and his teacher. In this conversation though, Isaac asks why he sees a vertical stretch in the transformation $g(x) = f(x + c)$, where $f(x) = 3^x$. Students were asked to carry out this dialogue between the teacher and Isaac. Finally, Scripting Task 3 (ST3) was presented to students as a part of their end-of-course final exam. This task asks students complete another discussion between a teacher and the student, Isaac, where Isaac asks about the vertical stretch he perceives in the transformation $g(x) = f(cx)$, where $f(x) = x^3$ and $c > 0$. Between ST2 and

ST3, students completed explorations outside the *EFE* lessons that centred on ideas of statistical regression, the polar coordinate system, and complex numbers. Thus, ST3 is intended to reveal the MKT that persisted over time.

Following the completion of all three scripting tasks, participant responses were de-identified and linked to a participant number. We then coded responses to identify the ways in which PSMTs leveraged their understandings of function transformations and representations to attend to student questions posed in the scripting tasks. Separately, we each generated initial codes for all the scripting task data. All initial codes were then reviewed and triangulated by the research team, organized into common reactions, defined, and named. We then examined the prevalence of these reactions.

RESULTS

Each scripting task posed ended with a scripting-task-student (STS) question arising from the situation such as "S: So does adding inside a function give you both a horizontal shift and a vertical stretch? Or what?" from ST2. Four dominant reactions were identified when examining participant responses to STS questions across all three scripting tasks. These reactions were applying form-dependent reasoning, directing visual observations, comparing representations, and focusing on algebraic equivalence.

PSMTs' use of form-dependent reasoning involved directly appealing to a rule to redirect the STS claim. For example, on ST3, one PSMT explains, "In this case, it appears as if it is a vertical stretch, but it is not. When the 'c' is larger or smaller, it will appear to look more like a horizontal stretch. Just remember the rules because looks can be deceiving." On ST2, another PSMT also says, "No, it may be perceived that way, but when we have a constant added inside the function than you will always get a vertical or horizontal shift." On ST1, 45% of the PSMTs appealed to the rule or form only whereas 25% and 30% did so on ST2 and ST3, respectively.

The reaction of directing visual observations was identified when PSMTs' explanations directed students to attend only to the changes related to the form of the expression such as the following participant answer on ST1.

T: Well, what exactly is a vertical stretch?

S: It's when the y-values in the graph are bigger than the y-values of the parent function's graph?

T: So, it has nothing to do with the x-values?

S: No, the x-values stay the same.

T: Then, if your x-values are the same, but your y-values are bigger, what does the graph look like?

S: Tall and skinny.

T: Exactly. It looks tall and skinny because the y-values changed, but there is not actually a horizontal compression.

S: Oh, that makes sense. It's the scale of my x-axis that makes it look like a horizontal compression.

On Scripting Tasks 1 and 2, 31% of the PSMTs directed STSs in this way whereas on ST3 only 4% did this.

PSMTs used comparing representations most on ST1 (32%), but then this dropped to 6% on ST2 and increased again to 27% on ST3. For example, on ST2 a PSMT draws a student's attention to a tabular representation to illustrate the transformation but does not validate *why* the student is observing the apparent contradiction to the learned rule.

Focusing on the algebraic representation as a way to explain the apparent contradiction in the STS claim or question only appeared in less than 5% of the responses on ST1, 19% of the responses in ST2, and 38% of the responses on ST3. These responses involved the PSMT showing how the algebraic representation may help illuminate why there is an apparent contradiction to the learned rules.

In addition, we noted that PSMTs were much more likely to validate the STS claims on the final scripting task when compared to the previous tasks. That is, 46% of the participants validated the STS claim on the ST3 versus 19% on ST2 and 14% on ST1. Validating a STS claim did not preclude a participant from then evoking form dependent reasoning, directing visual observations, comparing representations, or appealing to algebraic equivalence in attempts to complete the scripting task. Thus, in most instances, participants were not attending to *why* the student was seeing what they were seeing, but only addressing how they *should be* seeing it. Their responses would continue with "this is why it is not…"

DISCUSSION

To address our research questions, we employed repeated use of related scripting tasks as a research tool. The codes identified suggest that our PSMT participants held views similar to Ball's (1990) prospective elementary and secondary mathematics teachers that "stating a rule was tantamount to settling a mathematical question" (p. 141).

Although the lessons attend to multiple representations of function, PSMTs displayed uneven abilities to connect different representations and use this knowledge to attend to student thinking. Although reliance on "rule following" decreased from 45% to 25% from ST1 to ST2, the persistence of "rule following" indicates further revisions to the lessons or refinements to the facilitation of the tasks is warranted.

The tendency for PSMTs to have the teacher in the script direct the scripting task student's attention to the transformation that they "should see," decreased dramatically to only 4% on ST3. This may have been due to group discussions and review before the final exam in which PSMTs viewed animations directing their attention to seeing these simultaneous transformations does occur and that simply redirecting attention to the "correct" transformation does not help a student understand why they see what they see. This relates Moore and Thompson's (2015) idea that we may not clearly understand what the dynamic situations represent to the PSMT and how PSMTs' understanding enables them to make connections among different representations. PSMTs may question their understanding of function transformations when confronted with conflicting visual information causing visual imagery that interferes with their understanding as seen in Aspinwall, et al. (1997). Development of the MKT to untangle this conflicting visual information was not present in our participants.

Our findings related to PSMTs validating student thinking, but then explaining "why it's not…" give some insight into how PSMTs may have an underdeveloped understanding of representations. The use of multiple scripting tasks to track PSMTs understanding in this way reveals that while the PSMTs overwhelmingly validated student thinking on ST3 more work is needed to help them attend to answering the student's "why" question and not only superficially acknowledge their thinking to move to a standard explanation. We continue to investigate how scripting tasks, used in this manner, can inform curriculum development as well as provide formative assessment on appropriate mathematical concepts.

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SUPPORTING NOTICING OF STUDENTS' MATHMATICAL THINKING THROUGH 360 VIDEO AND PROMPT SCAFFOLDING

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360 video records a complete, spherical view of a scenario and allows the viewer to manipulate what is viewable in each frame. We incorporated 360 video into a teaching mathematics course and used prompts that directed prospective teachers' attention to students' mathematical thinking. Results indicated that prospective teacher noticing was more specific when they responded to prompts about students' thinking as compared to more general prompts. With focused prompts, prospective teachers had increased attention to students' mathematical thinking and were more likely to make interpretations about students' mathematical thinking. The findings show promise for the combination of 360 video and student-focused prompts to support prospective teacher noticing.

INTRODUCTION

During the last twenty years, the research on teacher noticing has spanned contexts and continents as mathematics education researchers and teacher educators have focused efforts on how teachers attend to and interpret students' thinking (Dindyal et al., 2021). Drawing from Mason (2011) and van Es and Sherin (2002), noticing refers to the process of sensitizing oneself to act intentionally in situations, without habit, with the purpose of making sense of how students reason. Teachers who sufficiently notice are more likely to implement teaching practices considerate of students' thinking, a process Jacobs and Spangler (2017) consider a core teaching practice. Dindyal and colleagues (2021) recently outlined the current state of teacher noticing, with a focus on how noticing is conceptualized, studied, and with emphasis on the contexts within which studies of teacher noticing are situated. They conclude that using records of teaching to support the development of noticing is common practice in many teacher education contexts (e.g., Jacobs et al., 2010; Schack et al., 2013; van Es et al., 2017). Despite the focus on noticing, and identification of ways noticing is supported, learning to notice is challenging for prospective teachers (e.g., Ivars et al., 2018; Llinares & Valls, 2010; Roth McDuffie et al., 2014). And consequently, researchers and teacher educators have focused on means to support prospective teacher noticing (Schack et al., 2013).

Given the challenges to support prospective teachers to notice, many teacher educators have implemented instructional practices in teacher education courses to support the development of noticing (Amador et al., 2021). Video is one common tool used in teacher education courses to show a representation of practice and support noticing (Gaudin and Chaliès 2015; Santagata et al., 2021). In a recent review of international

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 11-18). PME.

studies, Santagata et al. (2021) found that many researchers call for an increased use of technologies to support noticing. Consequently, knowing that ‘learning to notice’ is often challenging (van Es, 2011), we designed modules in teacher education courses that would capitalize on recent video technology and aim to scaffold prospective teacher noticing.

In our teacher education courses, we incorporated a recent technological advance in video, that of *360 Video*, which records a complete, spherical view of a scenario, and allows the viewer to manipulate what is viewable in each frame by “dragging” the screen or moving their head when wearing an appropriate headset (Amador et al., 2021; Roche & Gal-Petitfaux, 2017). Researchers have found that the prompts that are used to elicit noticing and promote learning to notice matter for prospective teacher development (Estapa & Amador 2021; Stockero et al. 2017; Weston & Amador, 2021). Therefore, we paired the 360 video clips of mathematics lessons with prompts containing an intentional focus on students’ thinking to support the development of noticing. Santagata et al. (2021) wrote, “the nature of the prompts matters and is consequential for teacher learning (p. 128).” Given that noticing is a core practice, yet difficult to learn, and knowing that video is a tool to support noticing and that the prompts given matter, we designed and implemented a multi-part learning process for prospective teachers as part of mathematics pedagogy. We were interested to know whether or not providing the 360 video support and purposeful prompts resulted in more advanced prospective teacher noticing (van Es, 2011). We answered the research question: *What* and *how* do prospective teachers notice when supported with 360 video and prompts that direct attention to students’ mathematical thinking?

THEORETICAL FRAMING

Noticing is central to the work of teaching (Mason, 2011) and encompasses attending to, interpreting, and making decisions about how to respond, based on students’ thinking (Jacobs et al., 2010). Attending means an ability to pay attention to how students’ think and reason about particulars of mathematics content. Interpreting refers to one’s ability to make sense of what has been attended to and then to draw conclusions about the meaning of the foci in ways that make sense of students’ thinking. Therefore, we consider noticing as a skillset, but as also a way to conceptualize higher order thinking of teachers that is important for effective mathematics teaching. Specific to mathematics, prospective teachers need to learn to notice students’ mathematical thinking and mathematics teacher educators need to purposefully select tasks in their pedagogy courses to support this learning (Roth McDuffie et al., 2014). Increased attention on teacher noticing has resulted in attempts to improve prospective noticing utilizing a variety of platforms within methods courses. Researchers have found that viewing videos can improve prospective teacher noticing (Jacobs et al., 2010). However, the content of video matters for learning to notice (Superfine & Bragelman, 2018).

RELATED LITERATURE

Video is a useful tool to support teacher learning because it allows users to slow down the process of teaching, and closely examine aspects of teaching and learning that may be missed during live observation (Santagata et al., 2021). However, what is viewable in traditional video is often dictated by the person managing the camera, leaving other aspects of the classroom and student learning offscreen. Teacher educators are beginning to use “360 video,” wherein a prospective teacher viewing a 360 video may adjust the perspective to focus on a small group of students to the left, view the students to the right, etc. Prospective teachers who view 360 videos report a greater sense of immersion (Roche & Gal-Petitfaux, 2017), and attend to more specific aspects of mathematics pedagogy (Kosko et al., 2021). Kosko et al. (2021) recently found that prospective teachers who watched 360 video attended to more student actions than peers who watched traditional video. Weston and Amador (2021) demonstrated that the use of 360 video plus prompts can elicit and support professional noticing. However, research on 360 video viewing and noticing is only beginning to emerge, and researchers call for increased studies on how noticing may be supported with the use of 360 video. Given that novice teachers attend to less specific aspects when viewing videos of teaching than more experienced educators (Stockero et al., 2017), and are in the process of learning to notice (van Es, 2011), the use of 360 video in teacher education holds significant promise.

Video is a valuable tool in teacher education; however, how teacher educators use video also affects the learning opportunities for prospective teachers. Estapa and Amador (2021) conducted a qualitative meta-synthesis of the prompts that teacher educators use when eliciting noticing and found that the level of specificity of prompts can influence response to prompts. They noticed that when teacher educators use specific prompts along with video, noticing can be developed. Likewise, Sherin and Russ (2014) note that prompts moderate the learning opportunities that accompany videos. In a close example of prompts, Roth McDuffie et al. (2014) found that the prompts used alongside video supported an increased depth of noticing and prospective teachers were able to attend to students’ thinking and make interpretations based on their thinking, aspects indicative of more advanced noticing. Weston and Amador (2021) demonstrated that the combined use of 360 video and prompts revealed growth in or presence of advanced prospective teacher noticing. Therefore, we were interested in understanding the outcome of the intersection of 360 video and purposeful specific prompts—both of which researchers have identified as supporting noticing (Kosko et al., 2021; Roth McDuffie et al., 2014; Weston & Amador, 2021).

Method

Data were collected from students (n = 173) enrolled in one of two university undergraduate mathematics pedagogy courses (one course had multiple sections). Two of the authors taught one of the courses within an education program at their U.S.-based institution, where prospective teachers worked towards initial licensure to teach. All data were collected during the 2019-2020 academic year (August 2019 through May 2020). One course took place in both Fall 2019 and Spring 2020 (with multiple sections each semester) and focused on PreK through grade 3 (ages 3 to 9). The second course took place in Spring 2020 and had a K-6 (ages 5 to 12) focus.

All prospective teachers were first-time users of 360 video and were provided with the same tutorial for how to watch 360 video, which was a one-and-a-half-minute 360 video the three authors made. The data-collection task, which was about multiplication, took place before the prospective teachers read or learned about that topic. Participants were asked to watch a seven-minute 360 video of a grade 3 (ages 8-9 years) class. In the video, students used Cuisenaire rods to explore the commutative property of multiplication.

Although prospective teachers all watched the same video, by virtue of the 360 feature they were able to observe students at more than one location in the classroom by pivoting their field of view from the camera placement. This meant multiple third-grade students' actions were observable throughout the recorded classroom episode, and likewise many student verbal comments were audible while students worked to complete the task. After watching the 360 video, prospective teachers were asked two questions about the device they used to watch the 360 video. They were next asked two questions about their noticing: Prompt 1: "What did you notice about teaching and learning?" and Prompt 2: "Describe an important student action or statement in the video. Why was that important?" In both cases, prospective teachers responded in writing, using a blank text box with no length limit. The remainder of the questions and prompts in the assignment were about their use of 360 video and are not the focus of this report. Responses were collected using either Google Forms or Qualtrics (depending on the course), with identical wording used in both platforms.

Analysis & Results

We conducted a convergent mixed-methods analysis in which qualitative analysis was conducted to examine prospective teachers' written noticings and then themes were quantitized for statistical analysis (Creswell & Plano Clark, 2018). To begin, the first two authors used van Es (2011) framework for learning to notice student mathematical thinking to independently code a subset of data for both noticing prompts about what and how prospective teachers noticed. The two researchers then met to reconcile codes and further discuss code application, before independently coding the entire data set. The following are examples from the data based on the framework. (see van Es, 2011 for framework)

Code	Example
	What was Noticed
Level 1 Baseline	The classroom setup encouraged collaboration and the teacher was physically moving around the class to observe how the students were

	working with the rods for the math problem.
Level 2 Mixed	I noticed that the teacher was asking thoughtful questions and expanding upon the ideas of the children…I also noticed that many of the children who didn't originally understand the concept were able to get it after using the manipulatives and being able to see it visually.
Level 3 Focused	… Children showed their mathematical thinking with manipulatives (colored rods)…One child used eight rods of seven (black rods). Another child used seven rods of eight (brown rods). The children were encouraged to put the rod on top of each other to see if they fit…They were asked how does it fit? Then they were asked why they fit? This encouraged the children to think about multiplication, and explain their thinking.
Level 4 Focused	No example
	How PTs Noticed
Level 1 Baseline	Teaching was very interactive, the teacher left many things up to the students. They were able to figure things out for themselves by testing their ideas with the rods and with each other.
Level 2 Mixed	An important statement in the video was when the student made the connection between the rods and the numbers. He connected how changing the position of the rods made them the same size and the numbers 8 and 7, which the rods represented, can change position and they are still the same.
Level 3 Focused	One thing that interested me that a student said was towards the end of the video when the teacher was talking about the different rods. One child said, "the numbers are the same, but one is on the other side so you just have to flip it to the other side." This was when he noticed that the numbers are the same in each problem…
Level 4 Focused	I noticed that some students were taking the rods out of their rectangular groups and trying to create a different set up of groups. [The teacher] then had to facilitate and give them more specific directions. Then they were able to see that the rods of each group would exactly fit. This is important because it shows the different thinking processes going on. Some of the students were on the right track, others were taking a different approach and trying to rearrange them. When students struggle it is okay, but if it becomes an unproductive struggle it is important for a teacher to recognize this and step in…

Table 1: Examples of coding

Table 1 shows excerpts from different prospective teachers for Prompts 1 and 2. Many prospective teachers had higher levels of noticing when answering Prompt 2 as compared to Prompt 1; their level of noticing for what they noticed were also sometimes connected with their level of noticing for how they noticed. The following is one example, coded at a Level 1 for both what and how they noticed:

> I took interest in the ending of the video when [the teacher] was letting students share their ideas and thoughts on the question. [The teacher] would ask them to further their thinking and this showed great benefits. I think it is vital to allow ample time for students to work with manipulatives like this.

This was coded as Level 1 for what was noticed because, despite being asked about an important student action or statement, the prospective teacher focused on the teacher and the whole class of students, describing them as a general group. This response was also coded as Level 1 for how the prospective teacher noticed, because there was a general description without any specific instances.

Following qualitative analysis, codings were quantitized as ordinal variables to determine whether the prospective teachers' level of noticing differed between type of prompt (see Table 2). We used a Wilcoxon Signed Ranks test to examine the difference in level of what prospective teachers attended to when provided each prompt. The Wilcoxon Signed Ranks test is a nonparametric statistic used to calculate the magnitude of differences between two paired ordinal variables (Siegel & Castellan, 1988). Results indicated a statistically significant difference ($W = 9.100$, $p < .001$) with prospective teachers demonstrating higher ranks, on average, on the second prompt than the first. Table 2 illustrates the difference in distribution. Notably, when prospective teachers were asked to describe what they noticed "about teaching and learning," responses were overwhelmingly general. When the prompt instead asked for "an important student action or statement," the level of specificity in their noticing increased dramatically.

	Level 1	Level 2	Level 3	Level 4
Prompt #1	91.3% *n* = 158	7.5% *n* = 13	0.0% *n* = 0	1.2% *n* = 2
Prompt #2	52.3% *n* = 90	32.6% *n* = 56	2.9% *n* = 5	12.2% *n* = 21

Table 2: Distribution of what PTs' level of responses by prompt.

Following the comparison with the Wilcoxon Signed Ranks test, we sought to understand how the degree of specificity for what prospective teachers attended to corresponded to how they interpreted what they noticed. We focused our analysis only on the second prompt, as the first prompt was heavily skewed to a Level 1 noticing (91.3%). The Gamma statistic was ideal for this comparison since it "is appropriate for measuring the relation between two ordinally scaled variables" (Siegel & Castellan, 1988, p. 291). Results indicated a statistically significant relationship between what and how prospective teachers attended to mathematics pedagogy in the 360 video ($\gamma =$

.499, $p < .001$). Thus, the ordinal relationship between the prospective teachers' descriptions of what and how they attended were 49.9% more likely to agree than to disagree, meaning there was a positive association between what and how they noticed.

Discussion

Findings indicate that when prospective teachers used 360 video and then responded to prompts to elicit their noticing, levels of noticing were higher for both *what* and *how* they noticed when the prompt was specific to students' thinking. Data indicate that from watching the 360 video, prospective teachers were able to focus on students' mathematical thinking. Although we do not make claims that the 360 video (as compared to standard video) is the reason for the higher levels of noticing, we note that using a combination of 360 video and prompts focused on students' mathematical thinking resulted in advanced levels of noticing from prospective teachers, which is uncommon for novice educators.

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RELEARNING: A UNIFIED CONCEPTUALIZATION ACROSS COGNITIVE PSYCHOLOGY AND MATHEMATICS EDUCATION

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We propose a unifying conceptualization of "relearning", a construct that has a long history in the field of cognitive psychology and has recently been reconceptualized in the mathematics education with respect to teacher training. We argue that existing accounts of relearning are versions of the same phenomenon subjected to different motivations for relearning and intended relearning outcomes. Utilizing the existing theoretical rigor behind existing conceptualizations of relearning, we demonstrate the utility of the unified conceptualization in using findings from one section to suggest new avenues for others, and in addressing issues posed by a lack of theoretical framing in the studies of remedial mathematics education and repeated mathematics courses.

In this report we argue for the utility of a conceptualization of "relearning" in mathematics education, or the experience of learning about mathematical content one has tried to learn about before. Global pushes for widespread access to higher education combined with the hierarchical structure of mathematics has resulted in an increased number of relearning experiences for college mathematics students. In the United States, this can be seen in increasing enrollment in remedial mathematics courses (Chen, 2016) in particular. While such remedial courses are less common outside of the United States, they have begun to gain popularity more globally (Rienties et al., 2008; Brants & Struyven, 2009). Measures concerning their effectiveness remain limited due to the lack of research describing student experiences with teaching and learning in such courses (Grubb, 2001; Cox & Dougherty, 2019). Despite calls for research that investigates the relationship between students and mathematical content in remedial courses (Sitomer et al., 2012; Mesa, Wladis & Watkins, 2014), there exists no theoretical perspective useful for structuring investigations into the phenomenon of relearning in this context, or in college contexts more broadly. We argue that such a perspective may be built by combining and expanding on two similar lines of inquiry: the study of memory in cognitive psychology, and relearning in content courses for future mathematics teachers.

EXISTING CONCEPTUALIZATIONS OF RELEARNING

In cognitive psychology, the term *relearning* is attributed to psychologist Hermann Ebbinghaus' studies of memory and retention. In 1885, Ebbinghaus documented the number of verbal rehearsals necessary for him to memorize strings of randomly-ordered nonsensical syllables as the lengths of the strings varied. He then recorded the number of rehearsals necessary to recite the strings of syllables again from memory after varying intervals of time. Ebbinghaus labeled his experience of

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trying to memorize the same strings of syllables through verbal rehearsal a second time as “relearning”. Of particular significance was his ‘savings in relearning’ result (Nelson, 1985; Murre & Dros, 2015), or the observed inverse logarithmic relationship between the amount of time elapsed from the first learning trial to the relearning trial and the number of rehearsals required in the relearning trial for the individual to reproduce the material perfectly from memory. Ebbinghaus hypothesized that this change in time was proportional to the amount of the syllable string stored in one’s memory. Thus, by studying the amount of time “saved” in each relearning trial, one could estimate the rate at which content held in memory was forgotten.

Such an estimation is undoubtedly valuable in educational contexts, and has led to the adoption of the technique of successive relearning (spaced relearning that anticipates Ebbinghaus’ retention curve; comparable to other types of techniques to promote retention such as self-explanation, spaced practice, or mnemonics) in psychological studies of college students (Dunlosky & Rawson, 2015). However, as noted by more contemporary critiques (Bahrick, 1979), interventions in authentic educational contexts centered around Ebbinghaus’ conceptualization of relearning remain limited in applicability. Namely, because the intended learning outcome under this conceptualization is a successful reproduction of content from memory in as little time as possible, application to contexts involving more complex systems of knowledge such as the structure of a language (Hansen, Umeda, & McKinney, 2002), or mathematics (Rawson, Dunlosky & Janes, 2020) is more limited.

To this discussion, we add that while relearning motivated only by *retention* may have limited applicability in mathematics, relearning motivated by insufficient understanding of mathematical content previously learned is very common. In fact, this traditional tie to memory us on memorization may explain why an entirely separate theory of relearning has recently been developed by Zazkis (2011) in the field of content courses for preservice mathematics teachers. While relearning as a term was used to describe the learning experience of preservice mathematics teachers colloquially prior to Zazkis (2011) (e.g. Nicol, 2006), her work marked the first acknowledgement of relearning as a phenomenon of theoretical significance in undergraduate mathematics education. Zazkis argued that “contemporary” understandings of how people learn mathematics such as constructivism or situated cognition were insufficient in this context, "since prior cognitive structures have been constructed in the learner's mind some time ago, the reconstruction and reorganization processes involved [in relearning] are more challenging for the learner as well as for the instructor" (p. 13). Zazkis’ notion of relearning may be distinguished from relearning as it is conceptualized in studies of memory in cognitive psychology by two features: the intended learning outcome, and the motivation behind relearning.

Under Zazkis’ conceptualization, the intended outcome of relearning in teacher education is “restructuring knowledge,” or revisiting previously-held knowledge in order to reorganize it in a particular way seen as better-suited for the purposes of teaching. This reconstruction is motivated by an insufficient understanding of mathematical content from K-12 experiences, either due to "prior misleading learning" that resulted in misconceptions on the part of the student, or a K-12 experience in which the content was presented with limited depth (Zazkis & Rouleau, 2018). In this way, relearning in cognitive psychology and teacher education have very different proficiency criteria within their motivations. While relearning in cognitive psychology requires that the content be "learned" (memorized) successfully when first introduced in order for relearning to occur in the second encounter, relearning under Zazkis' conceptualization in teacher education *requires* a previously-insufficient content understanding to take place. Furthermore, while agree that Zazkis' definition of relearning is more useful for researchers of preservice teacher mathematics education because it expands learning beyond the notion of retention, we see potential for a more expanded conceptualization. Namely, while Zazkis' conceptualization is useful for describing the *intended* outcome of content courses for future teachers, it has limited utility in describing the phenomenon as it actually occurs for preservice teachers. Student experiences with relearning have been noted to be fraught with resistance from preservice teachers (e.g. Nicol, 2006; Barlow et al., 2018) given that they have seen the material before and may be more comfortable with their previous understandings. Thus, outcomes other than restructuring are not only possible in such courses, but a common point of concern for teacher educators. We contend that a theory meant to describe student experiences learning about content seen before in this context would benefit from the inclusion of such outcomes.

Despite their surface differences, we argue that the inherent phenomenon being described as 'relearning' across the aforementioned fields is inherently the same. Their ostensible dissimilarity comes from the fact that they both describe different types of relearning subject to restrictions that are relevant to the foci of their respective fields. However, by viewing them as separate instantiations of the same general phenomenon, we contend both fields would increase the likelihood of theoretical advancements for mathematics educators. Divorcing the term *relearning* from the norms of a particular context allows for the focus to shift from answering the question: 'what outcome *should* students get as a result of this experience?' to 'what outcomes *are* occurring and how do the circumstances of this particular context determine which outcomes are possible?' Furthermore, considering these areas of research to be contributing to the same overarching field of study means that researchers have access to a wider range of perspectives with which to consider issues of interest.

PROPOSAL OF UNIFIED CONCEPTUALIZATION

At the most basic level, we contend that relearning requires three things: some (mathematical) content, a "time 1" (T1) representing a past occurrence in which one has tried to learn about that content, and a "time 2" (T2) representing the most recent time one has tried to learn about that same content again. Although the name re*learning* appears to suggest some degree of mastery of content at T1, we make no such assumption in our treatment of this construct. That is, T1 learning need not cross any threshold or meet any criteria for relearning to be said to occur at T2. This is not to say that different levels of proficiency do not matter, but instead that a particular level of proficiency at T1 is not *required* for the phenomenon to take place. Furthermore, while the content at T1 and T2 need not be identical, it does need to cross a particular threshold of similarity such that the content learning goals at T2 are essentially the same as those at T1. For some studies of memory in cognitive psychology this criterion is more clearly filled as the materials to be memorized are completely identical at T1 and T2. In the field of mathematics teacher education, the issue of determining content similarity is more complex because mathematics content courses for future teachers often have additional learning goals related to pedagogy that would not be considered in the K-12 context. However, the focus of the mathematical content remains the same.

If these components are considered sufficient to defined a relearning experience, then several other common college mathematics experiences would fall under this category such as retaken college mathematics courses and remedial math courses. In the United States, remedial math courses are either semester-long courses or corequisite sections of courses in college whose content mirrors that of algebra courses offered in the middle and high school settings. This similarity to content learned at a time T1 is often noted as a point of concern for semester-long prerequisite remedial math courses (Stigler, Givvin, and Thompson, 2010) which are sometimes referred to as “high school all over again,” (Ngo, 2020). By placing additional restrictions on the basic components in terms of *motivation* and *intended learning outcome,* we can recognize and compare sub-types of relearning as they are currently conceptualized (Table 1).

Context	Motivation for Relearning	Intended Learning Outcome
Cognitive Psychology	Reduce likelihood of forgetting previously-memorized content.	Content is successfully reproduced from memory in as little time as possible.
Mathematics Teacher Education (Zazkis, 2011)	Mathematical knowledge previously demonstrated to be insufficient for teaching	Restructuring: address misconceptions from T1 and widen “domain of applicability” of content.
Traditional Remedial Mathematics Education	Mathematical knowledge previously demonstrated to be insufficient for subsequent course in mathematics.	Acquire ideal understanding from K-12 experiences; impact on understanding brought to T2 undefined.
Corequisite Remedial	Mathematical knowledge previously demonstrated to	Acquire understanding of only K-12 content related

Mathematics Education	be insufficient for current math/statistics course.	to credit-bearing course; impact on understanding brought to T2 undefined.

Figure 1: Constraints Among Relearning Contexts

By *motivation* we mean the main rationale that justifies the beginning of the relearning experience for the individual. Importantly, this question is asked of the *relearning context* rather than of the individual. For instance, an individual required to participate in a psych study of memory for course credit and an individual required to take a math content course for future teachers might both list 'academic requirement' as their motivation for beginning the relearning experience. The motivation behind the inclusion of relearning in the two scenarios, however, is very different. Historically, contexts involving relearning have used both proficiency-based motivations and memory-based motivations.

By *intended learning outcome* we mean the intended impact on the understanding of material gained at T1 by the end of a relearning experience. This is *not* a grade or an indication of passing/failing. For a scenario in which one is learning for the first time, we ask what content was learned. This may, more or less, be determined by examining a student's answers to a well-designed exam. The same is not true for a relearning scenario. In asking about intended learning outcome, we mean to answer the question: what was the intended additional value of learning about the material this time around? The answer to this question requires one to reference the understanding of content that was developed at T1 as well as to define the impact of the relearning experience on that understanding. Unlike Zazkis' theorizing of relearning within teacher education, relearning has yet to be theoretically investigated within remedial mathematics courses. While there is a general sense that students should reach a level of competency with material that was desired at T1, there is no consensus as to what the impact should be on the understanding of content that the students begins with at T2. However, as we will discuss in the implications, there is existing literature on student understanding in these courses that may serve as starting points for such a theorization.

Comparison to Alternative Conceptualizations

Due the hierarchical structure of mathematics, one could argue that you would be hard-pressed to find any college mathematics course that didn't include learning about at least *some* content that a student had seen before. Thus, one might argue that instances of relearning are really simply special cases of students building on prior knowledge. Recall that in order for a scenario to be labelled as relearning, the content learning goals at T2 are essentially the same as the content learning goals at T1. This would exclude cases, for instance, in which calculus instructors reference common algebraic errors when teaching students how to find critical values of functions whose derivatives involve fractions. The content learning goals are focused on the novel

Calculus concepts of derivatives and local maxima and minima, not the algebra that might be involved in solving a problem related to these concepts.

While it would be possible to view relearning scenarios through the theoretical lens of prior knowledge, we contend that this would be less advantageous for understanding student experiences. Consider the comparison between the above examples from calculus with the educational scenarios described in Cox (2015). In her analysis of instructional activities across six remedial mathematics courses, Cox describes different strategies for teaching students about fraction representations. For instance, one strategy involved positioning the idea of fraction division within a larger domain of part-whole relationships between numbers by asking students to produce problems whose solutions would be represented by various fractions rather than prioritizing simplification of an expression like 3/.25. In evaluating the effectiveness of the strategies, one could conceptualize the phenomenon taking place in this classroom as students building on prior knowledge to produce a new type of understanding of what previously may have been only a mathematical "rule". However, considering this to be a task of helping students *relearn* algebra allows one to shift the focus from the content covered to the relationship a student would build with their already-established understanding of that content in the current context. The primary area of focus would not be that another representation of 3/.25 was learned, but rather how it was learned by students relative to their previous learning experiences. We would argue that the relearning lens is more useful in this context because it attends to the defining features of the classrooms Cox observed (i.e. the situation of learning about the same content again), whereas the use of prior knowledge would work equally-well for analyzing an instructional strategy for learning about algebra for the first time.

Relearning may also be distinguished from McGowen and Tall's notion of a *met-before* (McGowen & Tall, 2010). A met-before is defined as "a mental structure that we have now as a result of experiences we have met-before," (p. 171). McGowen and Tall use met-befores to construct mental models of students' understanding of content by considering how students employ mental structures formed by previous experiences with mathematical content to learn new things. The notion of a met-before is not incompatible with the notion of relearning, but the two terms represent different types of entities. Met-befores are mental structures containing previously seen content, whereas re-learning is an experience that takes place when a student is learning about the same content at a different timepoint. However, met-befores may be a useful concept when examining how a relearning context restricts the kinds of learning outcomes that are possible for students given that they are capable of being both supportive and unsupportive according to the context in which they are encountered.

IMPLICATIONS FOR RELEARNING FIELDS

We have proposed a unified conceptualization of relearning in mathematics education along with the constructs of *motivation* and *intended learning outcomes* that have traditionally been used to define relearning within various sub-disciplines. In doing so, we hope to broaden opportunities within each sub-discipline in two ways. First, in the realm of cognitive psychology and teacher education, we encourage researchers to move beyond the learning outcomes that are intended or desirable within their particular context in order to explore the realm of *possible* learning outcomes that students may encounter. For instance, while the restructuring outcome addressed by Zazkis earlier is the intended learning outcome of a content course for future teachers, it will not always be achieved depending on student engagement with the material. In under-theorized sub-disciplines such as remedial mathematics, looking to the learning outcomes that are possible in other sub-disciplines may serve as a starting point by which to begin to examine student experiences. It may be the case that Zazkis' notion of reconstruction would fit the intended learning outcome for some types of remedial courses, whereas student-generated descriptors of remedial mathematics courses as "refreshers" (Cox & Dougherty, 2019) of their memory, may point to connections to cognitive psychology's treatment of relearning instead. It may also be the case that multiple learning outcomes could exist simultaneously for one individual such that he or she may be reconstructing their understanding of some mathematical topics while achieving different outcomes for others. Determining the range of outcomes that exist in a relearning experience and comparing it to the desired or range of desirable outcomes would be one of the first ways in which one could begin to determine which contextual elements are or are not supporting students in meeting course expectations.

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FROM WHAT WORKS TO SCALING UP: IMPROVING MENTAL STRATEGIES IN SOUTH AFRICAN GRADE 3 CLASSES

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This paper shares results from a national 'familiarisation trial' of a mental mathematics intervention focused on assessing and encouraging strategic calculation methods with Grade 3 students in South Africa. Successful smaller pilots refined the intervention into 6 foci and this paper draws on assessment results from the four provinces that trialled one focus: adding and subtracting using jump strategies. Findings from pre- and post- test results of 1379 students show statistically significant gains in both the fluencies underlying calculating strategically and in items assessing strategic competency. The results indicate that scaling up this model into national implementation is feasible, and that the intervention package can support improvements in mental mathematics learning outcomes.

INTRODUCTION AND CONTEXT

There is a body of evidence on the importance of the interaction between procedural fluency and strategic competence (Mulligan and Mitchelmore (2009), with some researchers demonstrating that attainment in reasoning about number relations in primary school is a better predictor than arithmetical (procedural) fluency of later mathematical attainment (Nunes, Bryant, Sylva, & Barros 2009). Despite such arguments, in some educational systems, including South Africa where we work, teaching continues to prioritise developing students' fluency in mathematical procedures (with a main emphasis, in primary schools, on algorithms for multi-digit calculations). One argument for the continued importance of teaching procedures is that fluency in these leads to structural understanding (a core aspect of strategic competence) since algorithms are rooted in the base-ten system.

Even if it were the case that a procedural-fluency-first approach does lead to understanding structure (a claim that we, the authors, think is debatable), the situation in South Africa is compounded by the emphasis on working procedurally not balanced by an equal emphasis on developing algorithmic fluency. A wealth of research has shown that when working on multi-digit calculations South African students (particularly those in historically disadvantaged schools) reliance on unit counting approaches continues well beyond when counting is appropriate (Schollar, 2008). The students thus do not engage with the structural aspects of the number system. This lack of structural understanding is regarded a critical reason for the continued low standards of attainment in South Africa for many students (Spaull, & Kotze, 2015).

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For over ten years two South African Numeracy Research and Development Chair projects have been exploring ways to change this situation. In the first five years of each initiative, the main emphasis was on ‘what works’; developing didactic approaches and professional development programmes that address teaching for structural understanding whilst fitting with curriculum and inspection constraints and adapting to the dominant, largely teacher centred, pedagogies. The challenge for the second five years of these initiatives was to explore how approaches developed that had been shown to work on a small scale could be scaled up nationally. One such project developed as a collaboration between the two Chairs – the Mental Starters Assessment Project (MSAP). In this paper we report on how this project is being scaled up nationally through collaboration with South Africa’s Department of Basic Education (DBE) and the results of a national ‘familiarisation trial’ that built on early pilot testing and provides a bridge into national adoption.

THEORETICAL BACKGROUND

A focus of the Numeracy Chair initiatives has been on mental calculation skills, chosen not only because this is a required emphasis in the SA curriculum but also because we deemed it a way to wean students off relying on unit counting. The curriculum notes the role of mental processes to “enhance logical and critical thinking, accuracy and problem solving” (DBE 2011, 8–9), with examples of such mental processes including strategies like *bridging through ten* (36 + 9 = 36 + 4 + 5) or *compensation* (36 – 9 = 36 – 10 + 1). To be effective and efficient, such strategies for adding and subtracting mentally require a structural understanding of part-part-whole relations (for example, that 9 can comprise parts of 4 and 5, not simply a collection of 9 single units).

As Askew (2009) notes, a strategy like *bridging through ten*, while drawing on part-part-whole understanding, is only strategic when supported by fluency in number bonds: to efficiently calculate the answer to 36 + 9, rapid recall that 4 is the missing part in 40 = 36 + [ ] and coordinating that with knowing, again rapidly, that 4 and 5 comprise 9, underpin carrying out the strategy. Thus, as well as attending to strategic calculating, our work with teachers needed to focus on students’ rapid recall of number bonds for single digit and multiples of ten addition and subtraction. In addition to rapid recall and strategic calculating we were interested in *strategic reasoning* - reasoning about structural relations between numbers that does not rely on finding specific answers to calculations. Strategic competence is thus a blend of fluency, strategic calculating and strategic reasoning. We chose to work with Grade 3 students as is a year when the move from counting to strategies is needed to ground going forward.

To design a teaching intervention supporting moves into strategic competence we drew on the stream of research demonstrating the importance of using representations that mirror the desired underlying mathematical structure, such as part-whole bar models and empty number lines (see, for example, Van den Heuvel-Panhuizen, 2008). For the instructional part of the lesson starter, the final model comprised teacher led working on fluency in underpinning number bonds and then working strategically through two calculations and then student individual working on a set of three examples.

The intervention overall covered six different strategic 'foci': bridging through ten, jump strategy, doubling and halving, re-ordering, compensation and understanding the relationship between addition and subtraction. These six titles were taken from the Curriculum and Assessment Policy Statement (CAPS) (DBE 2011), the main curriculum document from which teachers plan. We thus expected teachers would recognise these strategies as part of what they were expected to be teaching. Pragmatically, six foci allowed teachers to work on two foci in each of the three terms in the teaching year.

Here we focus on the jump strategy, that is, for a calculation like 36 + 28 only partitioning the 28 into 20 + 8 and adding 20 to 36 and then 8 to 56 (using bridging through 10), initially with the support of an empty number line. This strategy was not widely used in our schools where the dominant approach was to partition both 36 and 28 and add the tens, add the ones and then add the two answers. Not only is the jump strategy a little more efficient, it also transfers more easily to subtraction.

MODEL OF INTERVENTION AT SCALE

An intervention that could effectively be scaled required teachers to perceive the materials as both easy to manage and fitting with curriculum requirements. Such fit with existing circumstances and constraints would be central to any intervention's success, given the evidence of lack of take-up of many previous initiatives as a result of expectations being too far from the 'ground' of South African schools. Here we outline the final intervention model, with brief reasons for the design decisions; for a fuller account of the origins of the model see Graven & Venkat (2021).

CAPS sets out an expectation that each lesson should begin with 10 minutes of oral and mental work, so the materials were designed to fit within those 'lesson starters'. There is a national mathematics workbook that the vast majority of teachers work through with their classes in the part of the lesson following the 10-minute starter: we knew our work in schools that the intervention would fail if we expected teachers either to replace workbook time with other tasks, or to add in extra materials.

Each of the six units was designed to be a three-week cycle comprising a pre-intervention assessment, guidance and materials for eight lesson 'starters' and a post-intervention assessment. While in theory, the two assessments and eight starters could be completed in two weeks, the provision of three weeks meant teachers could extend or revisit any of the starter ideas if they thought their students needed more work on these.

Pre- and post-unit assessments were designed to be easily administered to classes in a time-limited form. Given the evidence that the standard measure of progress in mathematics in many South African classrooms is recording a correct answer to a calculation, irrespective of the means of arriving at that answer (inefficient or copied), we used the low-stakes time-limited format to develop the teachers' awareness of the importance of fluency of basic mathematical facts, and use of time efficient strategies.

The final assessments developed comprised two pages. A first page had 20 rapid recall fluency items, to be completed in two minutes: simple, core number bonds that we expected should be well within the capabilities of being quickly answered by most Grade 3 learners. For the jump-strategy items, typical items included:

57 – 10 = [ ]   79 – 40 [ ]

The second page (to be completed in three minutes) had 10 questions that were a combination of strategic calculating and strategic reasoning items, each of which could be reasonably easily and quickly answered if students had some awareness of mathematical structure. For example, strategic calculating items drawing on jump strategies included:

57 + 26 = [ ]   83 – 24 = [ ]

We expected the strategic reasoning items to be a 'stretch' for both teachers and students as questions not leading to a closed numerical answer are rare in our context, but included these to raise participants' awareness of the importance of structural thinking. Typical jump strategy items included:

61 – 32 = 61 – [ ] – 2   74 – [ ] = 74 – 20 – 5.

Booklets provided to each gave guidance on running for each of eight ten-minute lesson starters, and also included all support materials. The materials were translated and made available in all 11 of South Africa's official languages. As well as print materials, each starter outline had a link, via a QR code, to a short, two to three-minute video demonstrating how to model the strategy, using the empty number line in the case of jump strategies. As noted, each starter had the common format of brief practice of rapid recall items, for example adding a small multiple of ten to a two-digit number (23 + 30; 23 + 50..), then the teacher using the focal strategy to model, with representations, solving two calculations, such as 47 + 21 43 + 24 in case of jump strategies, with students then. individually working on at three similar examples.

In summary, the run of each cycle, over three weeks comprised

- The pre-assessment
- Ten-minute lesson starter teaching aimed at developing fluencies and strategies across the 2–3 weeks following the pre-assessment
- Re-assessment providing feedback on learning.

RESEARCH METHODS

The basic model of the intervention was refined over three phases. The first, design phase comprised a small scale pilot involving three classes across two provinces. In this phase, members of the research and development team worked closely with the teachers to support the enactment of the intervention and to refine the model.

The second phase was a collaborative scaling-up with the national Department of Basic Education (DBE) in a trial that worked across three provinces – nationally, a high socio-economic status (SES) province, a mid-level SES province and a low SES province. In this second trial the teachers' support was 'once-removed' from that provided in the first trial: rather than support from the research and development team, local subject advisors were trained in the use of the materials and they then supported the teachers. Thus, the research question driving this first level of scaling-up was to understand the extent to which the intervention, now mediated by district subject advisers (supported in turn by the two research teams), could produce pre- to post-test gains on two units. The data from the pre- and post-assessments of in this trial showed that such a model could produce good learning gains (see Graven & Venkat, 2021). These with the outcomes pointed to a proof of possibility – that the materials could be used by teachers in the system at large to produce learning gains.

The third phase of development, a *national familiarisation trial*, provided the data reported on here. In this phase, the intervention involved all the early grades' subject advisers in all nine South African provinces, each advisor working with one or two Grade 3 teachers in a school in their own district. As in the second phase, the advisers supported teachers with the rollout of the intervention, including pre- and post-test administration and collating test responses. Now we turn to look at the results from the four of the provinces providing data on jump strategies.

FINDINGS

Figure 1 is a box and whisker plot for the percentage scores on all items pre- and post-test across all four provinces doing jump strategies (matched learners, n= 1379). As Figure 1 shows, the median post-test score was 43%, which was only in the top quartile for the pre-test. The first quartile cut off on the post-test was 20%, which was the same as the median score of 20% on the pre-test. In the pre-test, a quarter of learners scored less than 10% - that cut off point improved to 20% in the post-test. So, on the post-test over half of the learners performed at a level that fewer than a quarter of the learners attained on the pre-test. Also, three-quarters of the learners on the post-test performed at a level that only a half of learners performed at on the pre-test. Post-test, the mean score was 44% compared with 28% in the pre-test.

Table 1 sets out the two-tailed *t*-test results of changes in student scores from the pre-test to the post-test. The provinces are ordered in terms of SES from the one with the highest SES (P1) to that with the lowest (P4). In each case the calculated *t*-score (*t*-cal) is statistically significantly above the t-critical (*t*-crit) score. As can be seen, across all four provinces the gains made were statistically significant.

Table 2 disaggregates the pre- to post-test gains across parts one (fluency) and part 2 (strategic calculating/reasoning) of the assessment. There is no clear pattern of whether gains were more due to improvement on either part: in two provinces (P1 and P3) gains were lower on part 2 of the test, with this pattern reversed in the other two provinces

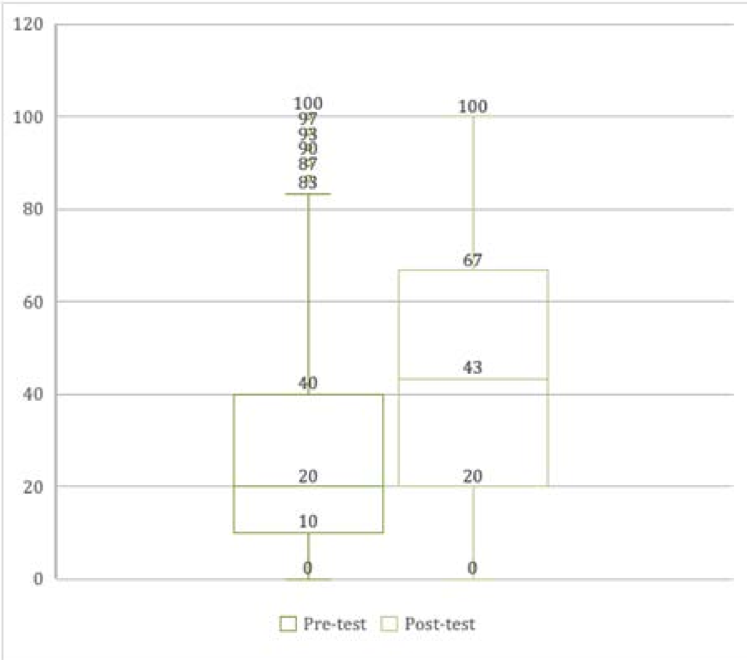

Figure 1. Box and whisker comparing pre- and post-test scores (%) n=1379

Table 1 sets out the two-tailed *t*-test results of changes in student scores from the pre-test to the post-test. The provinces are ordered in terms of SES from the one with the highest SES (P1) to that with the lowest (P4). In each case the calculated *t*-score (*t*-cal) is statistically significantly above the t-critical (*t*-crit) score. As can be seen, across all four provinces the gains made were statistically significant.

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Table 3 presents the effect sizes for each province, and ‘levelled’ on the commonly used interpretation of effect sizes as small (d = 0.2), medium (d = 0.5), and large (d = 0.8). With an effect size above 0.4 deemed worthy of consideration, the effect sizes in each of the provinces easily meet that criterion.

Province	N	Mean gain	SD	*t*-cal	*t*-crit	df	p
P1	443	15.12	20.52	15.51	1.97	442	<0.001.
P2	46	25.80	21.52	8.13	2.01	45	<0.001
P3	366	12.65	18.84	13.09	1.97	365	<0.001
P4	524	19.45	20.63	21.58	1.96	523	<0.001

Table 1. *t*-test Pre- and Post-Test Gains (%) Jump Strategies

Province	N		Mean gain	SD	*t*-cal	*t*-crit	df	p
P1	443	Part 1	16.95	20.53	17.38	1.97	442	<0.001
		Part 2	11.47	26.31	9.17	1.97	442	<0.001
P2	46	Part 1	22.61	25.60	5.99	2.01	45	<0.001
		Part 2	32.17	26.83	8.13	2.01	45	<0.001
P3	366	Part 1	16.02	22.23	13.78	1.97	365	<0.001
		Part 2	5.90	19.28	5.85	1.97	365	<0.001
P4	524	Part 1	19.40	22.30	19.91	1.96	523	<0.001
		Part 2	19.54	27.81	16.09	1.96	523	<0.001

Table 2. *t*-test Part 1 & 2 Pre- and Post-Test Gains (%) Jump Strategies

Province	N	Pre-test Mean %	Post-test Mean %	Cohen's D	Level
P1	443	18	33	1.2	Large
P2	46	34	59	1.07	Large
P3	366	26	39	0.47	Medium
P4	524	37	57	0.78	Medium

Table 3. Effect sizes for each province

DISCUSSION

The statistical significance of the pre- and post-test gains made on the assessments, together with the effect sizes, indicates that the jump strategy intervention raised attainment above what might be expected to have come about simply from the usual teaching that may have taken place over that time. We note that mean gain of 12 percentage points from the pre- to post- amounts to students answering about 4 more questions (out of 30) correctly. But it is also worth noting that the total time teaching underpinning this gain was only around 80 minutes (on the assumption that the teachers carried out eight 10-minute mental starters). Differences in patterns of gains across the four provinces are worthy of further investigations. For example, we might have expected that, given its high SES, gains in P1 would have been substantially higher than in the other provinces. The reasons why two of the provinces show strong gains in the Part 2 strategic competence items are also worth of further inquiry.

In short, across these four provinces, notwithstanding the differences in gains across provinces, there is evidence that the intervention was successful in supporting improved learner performance on adding and subtracting two-digit numbers. Given the many Covid challenges and disruptions that teachers and learner faced we have been especially pleased to note these improvements.

CONCLUSION

It is beyond the scope of this paper to report on the data from other strategies and other provinces though similar gains were noted. National rollout by the DBE, with the support of the Chair teams, is planned for 2022. While continued disruptions to schooling continue and there is a focus on 'catch-up' in relation to weak curriculum coverage over the past two years we have been pleased to hear general buy-in from teachers and provincial co-ordinators and advisors in relation to both the value of the intervention and the quality and ease of use of the support materials. We expect that some of the success of the intervention, despite many challenges relating to curriculum coverage and 'catch-up' concerns is that the intervention has not 'interfered' with the main body of the teaching time of lessons and has focused on supporting more strategic use of the ten-minute warm up session at the start of lessons to promote number sense and structural reasoning around number.

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ASPECTS OF MATHEMATICAL KNOWLEDGE FOR TEACHING: A QUALITATIVE STUDY

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We present a qualitative study aiming at investigating secondary school teachers' Mathematical Knowledge for Teaching regarding the dense ordering of rational numbers. Fifteen secondary math teachers were asked to evaluate the responses of hypothetical students, explain students' thinking, and give feedback. The accuracy of the evaluation, the quality of the explanation, and the use of counterexamples were examined. The results showed shortcomings in various categories of Mathematical Knowledge for Teaching, such as Common Content Knowledge and Specialized Content Knowledge, Knowledge of Content and Students and Knowledge of Content and Teaching.

THEORETICAL FRAMEWORK

One of the major concerns in educational research is the knowledge required for teaching. Ball and her colleagues (Ball, Thames & Phelps, 2008) outlined certain components of Mathematical Knowledge for Teaching that have been a reference point for mathematics education researchers. In this paper, we adopted Ball and colleague's theoretical framework and we studied aspects of the Mathematical Knowledge for Teaching in secondary school math teachers.

One of the aspects of Mathematical Knowledge for Teaching that we focused on is Common Content Knowledge, which is knowledge about the mathematical content that is useful for teaching, albeit not exclusively. The second aspect of Mathematical Knowledge for Teaching we are interested in is Knowledge of Content and Students, in terms of teachers' ability to explain students' thinking, especially when they give incorrect answers. We focused on the use of counterexamples, which relates to Specialized Content Knowledge (i.e., knowledge which is useful exclusively for teaching) and Knowledge of Content and Teaching. Indeed, the appropriate selection and use of counterexamples in teaching is a very important, non-trivial process (Zaslavsky, 2010). The fundamental purpose of a counterexample is to refute a claim. In teaching, however, appropriate selection and use of counterexamples is required to make visible to students the reasons why a claim is false and to create conditions for generalization, beyond the particular claim; these are found to be challenging for teachers (Pele & Zaslavksy, 1997; Zaslavsky, 2010).

The study presented in this paper is part of a larger one investigating secondary math teachers' Mathematical Knowledge for Teaching about rational and real numbers. Here we focus on Mathematical Knowledge for Teaching about the dense ordering of rational numbers. It is amply documented that this property is difficult for students at

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all levels of education, even for tertiary students studying mathematics. Students often argue that between two rational numbers there is a finite, possibly zero number of numbers, as it happens in the set of national numbers. Moreover, even students who describe the number of intermediate numbers as "infinite" often refer to a very large number (e.g., "a billion", "as many as the grains of sand in the desert") that it is finite (Vamvakoussi & Vosniadou, 2012).

Our research questions were: a) Do teachers evaluate correctly students' answers about the number of numbers in an interval? (Common Content Knowledge) b) Are teachers able to explain the students' way of thinking? (Knowledge of Content and Students) and c) What are the characteristics of the feedback they give to the wrong answers? Specifically, for (c), we examined teachers' selection and use of counterexamples (Specialized Content Knowledge, Knowledge of Content and Teaching).

METHOD

Participants

The participants were 15 secondary math teachers (10 women, 5 men) with 1 to 7 years of teaching experience. In Greece, secondary math teachers necessarily have a degree in mathematics. The majority of our participants were either in the process of obtaining or had already obtained a master's degree. One of them had a master's degree in Mathematics Education.

Research tool

To explore teachers' Mathematical Knowledge for Teaching about the dense ordering of rational numbers we used tasks in the form of (hypothetical) classroom scenarios, deemed suitable for such purposes (Biza, Nardi & Zachariades, 2007). Due to space limitations, in this paper we will examine only one of them in detail.

According to this classroom scenario, a hypothetical teacher asks a class of 9th graders how many numbers there are between 1.1 and 1.3; three different responses (A, B, C) by the students are presented: A) "One, 1.2"; B) "19: 1.11, 1.12, 1.13, 1.14, 1.15, .., 1.19, 1.20, 1.21, .., 1.29"; and C) "They are infinite… lots of them… over a billion. Only a computer could find them all." The three hypothetical responses A, B and C correspond to different levels of understanding of the number of intermediate numbers in an interval (Vamvakoussi & Vosniadou, 2012). In A, the student treats the given numbers as natural numbers. In B, the hypothetical student performs the first step of a potentially repeatable process by adding a decimal digit to the given numbers (1.10 and 1.30) but then treats them similarly to student A. Finally, answer C corresponds to the interpretation of the expression "infinity numbers" as "a very large, but finite, number of numbers".

The participants were asked the following questions: a) Is any of these answers, correct? If so, which one? If not, which is the correct answer?, b) Can you explain each student's thinking?, and c) How would you deal with this situation, if you were the teacher of this class? How would you give feedback to these students?

Procedure

Teachers participated individually in semi-structured interviews, which were conducted via Skype. All the interviews were recorded and transcribed.

RESULTS

Evaluating and explaining students’ answers

We first examined whether the teachers evaluated correctly the three responses (A, B, C). Nine of the fifteen teachers correctly evaluated all three answers. While all of them correctly judged that answers A and B were incorrect, six teachers (T2, T3, T6, T7, T11, T13) considered answer C to be correct. For example:

> T7: I agree that a computer could find them all. If it were programmed by a mathematician, the computer would run the algorithm and find them all.

Explanations of the student’s thinking were examined in cases where the assessment was correct and were categorized into 3 categories. The first category (“No Explanation”) included responses in which participants either explicitly said that they were unable to explain; or avoided giving an explanation. The second category (“Trivial explanation”) included all explanations that repeated the student’s answer, or attributed the error to general factors such as the student's background in mathematics (“strong”/ “weak” student), or carelessness. The following extracts present examples of the first (T2) and second (T1) category of explanations:

> T2: Now, how did he come up with it? I don’t know how he thought of it.

> T1: It can be due to a number of factors. This student might be weak, or careless.

Finally, the category “Relevant Explanation” included the explanations that provided a substantial rationale for the hypothetical students’ thinking. For A and B, teachers who gave relevant explanations appeared to recognize that the students’ reasoning was based on natural number knowledge (see T15 in the excerpt below). For C, teachers who provided relevant explanations appeared to acknowledge that the hypothetical student, while using the term “infinity”, was actually referring to a very large, albeit finite, number of intermediate numbers (see T15 in the following excerpt).

> T5: Well, the first one thought that after 1.1… in the decimal part, after 1 there’s 2 and then 3. So, between 1.1 and 1.3 there is 1.2.

> T15: The third one says there are infinitely many, but the fact that he says there are over a billion, he puts a barrier, he is, like, counting them. I don’t think he understands what infinity is.

Table 1 shows the frequency of each category of explanation by hypothetical answer. Less than half of the explanations were found in the “Relevant Explanation” category, with the majority of them for A. We should note that only three teachers gave relevant explanations for all three hypothetical answers (T4, T8, T15). In addition, 5 participants didn’t provide relevant explanations for any of the responses that they had assessed correctly, including two who had assessed all three correctly (T1, T14).

Explanation Category	Hypothetical Answers			Total
	A	B	C	
Relevant Explanation	9	4	5	18
Trivial Explanation	5	7	2	14
No Explanation	1	4	2	7
Total	15	15	9	39

Table 1: Frequency of each category of explanation by hypothetical answer.

It is also interesting to note that 5 of the teachers who gave a substantial explanation for A didn't recognize that B was a similar case (Table 1). One such example is T10:

> T10: The second answer is a bit strange, it's weird. Uh… (pause). I don't know. I don't know where this answer comes from, I really can't imagine.

Feedback: The use of counterexamples

We analyzed the teachers' feedback to the hypothetical students only for the responses they had (correctly) assessed to be incorrect. We note that in many cases the teachers addressed more than one response simultaneously. In the relevant texts, we searched for references to counterexamples, initially individually. We found that there were direct and indirect such references, so the texts were reviewed, the findings were compared and the (few) differences were resolved by discussion.

In total, 14 references to counterexamples were identified. Counterexamples were mentioned explicitly (as specific numbers) or descriptively (e.g., "decimals with many decimal digits"); they were also implied via referring to the responses of other hypothetical students or to a modified form of the problem in which more intermediate numbers were considered to be visible to students (e.g., after adding one or more zeros to the decimal part of the given numbers). Texts containing references to counterexamples were first examined as to whether counterexamples are used merely to refute a particular claim or whether their use afforded possibilities for generalization. Two initial categories were created.

The first category ("Claim Refutation", N=4) included cases in which the teacher referred to one or more intermediate numbers, with the intention of refuting the claim that "there are no other intermediate numbers". For example:

> T15: I would ask them, is—let's say—1.135 between these numbers? I think that all the students would say "yes, this is in between". Then I would ask, is 1.1355 between? It is. That's how they would understand that they were wrong.

The second category ("Potential Generalization", N=7) included the cases that referred to a method of generating counterexamples that potentially leads to the infinity of the intermediate numbers in an interval. However, differences were found in the adequacy of the description of the method, and two subcategories were formed. In all cases of the first category ("Potential Generalization – Inadequate description, N= 4), teachers were limited to mentioning the possibility of more/fewer decimal digits in a number, similar to T3 in the extract below:

> T3: I would explain to them that after the decimal point, we can put infinitely many digits, that's a number too. The number 1.113758239 is smaller than 1.2.

In the second subcategory ["Potential Generalization – Adequate Description", N=3], we included cases in which teachers explicitly described a generalizable, repeatable process of generating intermediate numbers, either in a purely numerical context (T1, T10), or in the context of the number line (T12). For example:

> T10: At first, I would pay attention to the first two answers (*A and B*). I would say to the students: the first step is simply to say 1.2. But then, as the other student said, we can take a second step and add two decimal digits, 1.12, 1.13, 1.29. I would say to them, if we got one decimal digit the first time and two decimal digits the second time, why don't we continue to three decimal digits? I would then say that what I'm telling you now, we can do for 4 decimal digits as well. So, it's a process that we can keep doing for any number of decimal digits. Since we can do this for any number of digits, we begin to understand that there are infinitely many numbers in between.

> T12: I will tell them to pick any two numbers on the number line. I'm going to take the point in the middle of the line segment. So, here is a number in between. Then, I will pick one of the two (*endpoints*) and I will do the same. We can zoom in again and again and find infinitely many numbers.

We note that T1 and T12 also expressed a clear intention to address the infinity of intermediate numbers in any interval.

Finally, in the "Other" category (N= 3), three cases of feedback using counterexamples were included, which were judged, for different reasons, as inappropriate (see excerpts below). More specifically, T13 relied on an invalid argument, claiming that since all real numbers are infinitely many, there are infinitely many numbers in any interval. T7's feedback had two parts. In the first, she stated that there are infinitely many numbers in the interval, referring to infinity as an "unending process". In the second, she described vaguely the intermediate numbers as "numbers with "lots of decimal digits". No obvious connection between the two parts was made; and it is unclear how the intermediate numbers are generated, and whether "lots of" is also used to refer to "infinitely many". Finally, T14 based all the counterexamples as well as their generation method, on the sequence of natural numbers, referring to the first four terms of the corresponding sequence of decimal numbers with one decimal digit. It is not clear which number follows 0.9 in his sequence. Assuming that it is 1, then the subsequent terms are not between 0 and 1. Assuming that it is 0.10, then this sequence is presented with the misleading ordering of natural numbers, consistent with the well-known misconception that "longer decimals are larger"

> T13: Because these numbers can be placed on the real line, and because there are infinitely many real numbers, it is obvious that between two numbers there isn't just one, two etc., there are lots of numbers, which are not easy to find.

Usually, in our everyday life and when we teach, we use "easier" numbers such as 1.13, 1.14 etc.

T7: I would tell them that there are infinitely many numbers between 1.1 and 1.3 and we can't say precisely how much "infinitely many" is, infinity means you keep going. Anyway, there will be numbers with lots of decimal digits that approach 1.3.

T14: The answer is the same for all, so I would tell them that the numbers, as we know them, are infinitely many. (…). When I go from 0 to 1, there are also infinitely many numbers. You see, when I count 0, 1, 2, 3, 4, 5, I can continue to infinity. If I want to go from 0 to go to 1, there is the number 0.1. So, I can go on, 0.2, 0.3 up to infinity, just like before. Just like I reached infinity the first time, I also reach infinity by 0.1, 0.2, 0.3, 0.4, there are infinitely many numbers up to 1.

Finally, we would like to highlight another aspect of teachers' feedback which we had not anticipated and emerged through their descriptions. As can be noticed from the preceding excerpts, the students are hardly taken into account. We reviewed the texts for indications of intention to include students in the process in a meaningful way (i.e., intention to ask students to explain their answers, to compare answers, to explore non-trivial questions, etc.). Only two teachers (T1, T6) expressed the intention to engage the students in the process. For example:

T1: I would start with the second student and the rationale of the third so that we can come to the conclusion that no matter what interval we end up taking, we will always find an intermediate number (….) I generally prefer in such cases to ask the children to explain their peers' mistakes. In this way we eventually end up with the correct answer.

DISCUSSION

In this paper we examined aspects of Mathematical Knowledge for Teaching (Ball et al., 2008) of secondary school teachers. We focused on participants' responses to hypothetical students' answers to a question about the number of intermediate numbers in a given interval. The first finding, which relates to Common Content Knowledge (Ball et al., 2008), was that only 9 out of 15 teachers evaluated correctly all three hypothetical responses. This is remarkable given the mathematical background of the particular participants. The remaining 6 teachers agreed with the claim that "a computer can find all the intermediate numbers in a given interval". This reflects a conception of "infinitely many" as "a very large, albeit finite number", similar to the conception documented for primary and secondary school students (Vamvakoussi & Vosniadou, 2012). Regarding Common Content Knowledge, we should also note the invalid argument presented by one participant in the attempt to give feedback to the hypothetical students ("there are infinitely many real numbers, therefore there are infinitely many intermediates in the given interval").

Explaining the hypothetical students' thinking (Knowledge of Content and Students; Ball et al., 2008) also proved challenging. A meaningful explanation, acknowledging implicitly or explicitly that decimal numbers were treated by the first two hypothetical students like natural numbers, was provided by the majority of participants in the simpler case (i.e., “only 1.2 between 1.1 and 1.3”), but by fewer in a similar one (i.e., “only 1.11, 1.12, …1.29 between 1.1 and 1.3”). Only five teachers recognized the misinterpretation of the expression “infinitely many” by the third student. In the majority of cases, no explanation or a trivial explanation was given, attributing the error to factors such as carelessness, or the general mathematical background of the student, which is not conducive to meaningful instructional support for students. We note that accurate evaluations did not necessarily imply relevant explanations, which highlights the fact that Common Content Knowledge is distinct from Knowledge of Content and Students.

Finally, in terms of feedback, we focused on the use of counterexamples, which relates to Specialized Content Knowledge as well as to Knowledge of Content and Teaching (Ball et al., 2008). The particular classroom scenario afforded the use of counterexamples and the participants indeed used them; they also had methods of producing counterexamples at their disposal. Only three teachers, however, placed the counterexamples in an explanatory context that could support students to understand the underlying method and reach the conclusion that there are infinitely many numbers in any interval. This is consistent with findings showing that the appropriate use of counterexamples is challenging for teachers (Zaslavsky, 2010).

Finally, with only two exceptions, the teachers did not explicitly express the intention to engage the students in any productive activity during the feedback process. They typically described a situation where the teacher presents and explains the correct answer; and the students’ participation is minimal, if not trivial.

The findings on feedback, although they give some (alarming) indications regarding the teachers’ ways of dealing with similar situations in the classroom, should be treated with caution. Indeed, it is possible that in real classroom settings the teachers would have engaged in a more meaningful interaction with students or presented more elaborate explanations.

To sum up, the results of this study indicated shortcomings in the aspects of Mathematical Knowledge for Teaching that were investigated. The findings can’t be generalized, due to the small sample size, but can be used as a starting point for deeper exploration of these issues in the future.

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TRANSFORMING CRITICAL EVENTS THROUGH SCRIPT WRITING IN MATHEMATICS TEACHER EDUCATION

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This paper proposes a teacher education strategy based on a combination of critical events and scripting dialogues. This strategy was used in two teacher education contexts in Israel and Greece with thirty-four prospective teachers (PTs). The PTs identified critical events in the context of their field experiences and transformed them into scripting dialogues with the aim to handle students' difficulties. The analysis focuses on the adopted pedagogical actions in the critical events and the scripting dialogues. The PTs used general and mathematics-specific actions to address students' difficulties promoting their conceptual understanding.

INTRODUCTION

In recent decades, there has been a growing interest in the use of critical events in teacher education (Stockero et al., 2020). In particular, critical events serve as a tool for teacher educators to develop PTs' *noticing* of students' mathematical thinking, considered as an important aspect of a teacher's expertise. Noticing entails attending to students' thinking, interpreting it and responding to promote further mathematical understanding (Jacobs et al., 2010). Research identifies a need for studies that focus on the *responding* dimension of noticing, that is, the potential teaching action (Santagata et al., 2021). Scripting dialogues is suggested as a new way of representing teaching practice in an imaginary way (Zazkis et al., 2009). In this paper, we propose a teacher education strategy that uses both critical events and scripting dialogues. PTs are engaged in transforming critical events from real classroom to scripting dialogues so that they become aware of different aspects of mathematics teaching and learning. Similar to other studies (Sun & van Es, 2015), in our study PTs selected critical events related to students' difficulties and attempted to address them in their scripting dialogues through specific teaching actions. We aim to explore the responding dimension of PTs' noticing of critical events related to students' difficulties while engaging in the transformation process.

THEORETICAL BACKGROUND

We take a socio-cultural perspective based on the role of tools and resources in mathematics teacher education facilitating teacher noticing, in particular, the nature of script writing and critical events as tools mediating PTs' noticing. Critical events are moments in which students' mathematical thinking becomes apparent and thus can provide teachers opportunities to delve more deeply into the mathematics discussed in the lesson (Stockero et al., 2020). Research indicates that PTs struggle to respond or

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provide teaching alternatives in ways that are built on students' thinking to promote learning (Sun & van Es, 2015) and that there is a need for studies getting a deeper insight on PTs' suggestions of alternative teaching actions and on the process of formulating them (Santagata et al., 2021). Research has suggested several structures for facilitating PTs' noticing. Often, PTs receive researcher-selected artefacts of practice, such as classroom video clips or narrative cases accompanied with prompts that focus attention on the details of student thinking and learning (Jacobs et al., 2010). In other cases, the PTs are asked to take part in producing the scenarios. For example, Zazkis et al. (2009) proposed the 'lesson play', where participants are presented with a beginning of a scripting dialogue between a teacher and students and are asked to continue the dialogue to resolve an issue they perceive as problematic. This provides insights into PTs' ways of understanding the mathematics in the situation and their images of the potential pedagogical considerations related with its learning. Still, other researchers invite PTs to analyze critical events selected by them in their field-work so as to promote reflection on their classroom experiences (Goodel, 2006).

Research indicates that in responding to students' difficulties, PTs tend to instruct the students what to do or correct occurring errors. These actions potentially remove the opportunity for students to make sense of errors and run counter to research recommendations for using errors as learning opportunities to foster understanding (Borasi, 1996). To classify the nature of PTs' responses to students' errors, Son (2013) suggested the term of Forms of address, distinguishing between two types: show-tell (teaching actions such as telling the definitions, explaining a procedure) and give-ask (e.g., pursuing mathematical explorations, inviting students to evaluate students' arguments). She further proposed three types of use of student error: (i) Active use, when using the student's error as a major tool for instruction, e.g., providing the students opportunities to explore why an error does not work. (ii) Intermediate use, when an error is addressed briefly as a stepping-stone to correct the student. (iii) Rare use, when not addressing the error at all or only remarking that a solution is wrong. In our study, we draw on Son's distinction among types of teachers' use of errors in their instruction and analyze the pedagogical actions in responding to students' errors identified in the critical events and the corresponding scripting dialogues written by the PTs. We use PTs' suggested rationales of developing the scripting dialogues as well reflections on the challenges they faced in the process and the learning outcomes, to better understand the PTs' engagement in the process. We aim to address the questions: (1) *What are the shifts in teaching actions that the PTs followed or suggested to handle students' difficulties from the initial critical event and the scripting one?* (2) *What is the PTs' rationale of developing the scripting dialogue? What are the challenges they faced in this process and what are the learning outcomes?* In this paper, our analysis addresses mainly RQ1. Through two case studies we provide some initial evidence in response to RQ2.

METHODOLOGY

The research took place in the context of two mathematics education undergraduate courses during the same semester in two universities: one in Israel and the other in Greece. The aim of the courses was to engage PTs in critical consideration of aspects of mathematics teaching as they emerge from the complexity of teaching practice in schools. Decisions concerning the design of the study and the data analysis were taken collaboratively through regular online meetings between the researchers. In the Greek context, 9 PTs participated in the study while in the Israeli context, 25 PTs. Enrolling the courses, the PTs had already attended a number of courses on mathematics education. In both contexts, the PTs experienced both University lessons and classroom observations. The structure of the intervention involved: identifying a critical event during classroom observations and illustrating it through a dialogue; providing the rationale of the selection (Why is it critical?); interpreting students' and teacher's actions and providing evidence; suggesting alternative teaching actions and developing a scripting dialogue between them (as teachers) and the students; providing rationale of the scripting dialogue; reflecting on the process (decisions, challenges, learning outcomes). PTs were asked to: address the above tasks in a written report; present briefly their work in the meetings; and discuss their scripting dialogues and the process of developing them in work-out sessions (3-4 PTs). The data consisted of: PTs' written reports; video recordings of the meetings including the work-out sessions; and designed resources (e.g., lesson plans, worksheets).

In this paper we analyze the data from the written reports in relation to one cycle of classroom observations. Initially, we used grounded-theory methods to analyze: the nature of critical events and their interpretation by PTs in relation to the teacher's actions and the students' thinking; the PTs' pedagogical actions in the scripting dialogues; the process of developing the scripting dialogue (rationale, decisions, challenges, learning outcomes). Then, we decided to focus on critical events related to students' difficulties as this was a dominant theme. Next, by combining grounded analysis and theoretical ideas related to teachers' response to students' difficulties we refined our coding of teacher pedagogical actions both in the critical event and the corresponding scripting dialogue to address the first research question. Finally, we analyzed specific cases of PTs to highlight the occurrence of the pedagogical actions as well as dimensions of the transformation process from the critical event to the scripting dialogue.

RESULTS

Most PTs (26 out of 34) selected critical events related to students' difficulties and attempted to address them in their scripting dialogues through specific teaching actions. Our analysis reveals a multiplicity of teaching actions indicating an active use of students' difficulties in the whole classroom as well as at the level of the individual students who faced the difficulty. In Table 1, we present the emerging categories of pedagogical actions identified in the critical events and the corresponding scripting dialogues. We distinguished two main categories of actions: *general* and *mathematics-specific*. The total number of pedagogical actions in the scripting dialogues in both categories is increased in relation to the ones identified in the critical events. Part of the general actions concerned the teacher's interaction with individual students who had the difficulty (i.e. prompting students to explain and justify their solutions, and enabling the student to find her/his own error) while the rest of actions addressed the whole class. The latter indicates that the student's difficulty becomes a central point of discussion with the whole class (active use of students' difficulty) while the former shows a response to the individual student (rare or intermediate use of students' difficulty).

Prompting students either to evaluate other students' statements ("What do you think about Hala's idea?") or to explain and justify their solutions ("Justify your response") were dominant *general* actions both in the critical event (12 and 11 respectively) and the scripting dialogue (15 and 13 respectively). The general actions that appear to be more frequent in the scripting dialogues in relation to the critical events were: making a student's solution visible ("Come on the board to write your solution"); encouraging students to help their peers to overcome their errors ("try to explain Edi what his mistake is."); verifying understanding ("Is there something misunderstood before we continue?"); and enabling the student to find her/his own error (see case 2 below).

Pedagogical actions	**In critical events**	**In scripting dialogues**
General actions		
Prompting students to evaluate other students' statement	12	15
Making a student's solution visible	3	7
Prompting students to explain and justify their solutions	11	13
Prompting diverse solutions	3	5
Encouraging students to help their peers to overcome their errors	2	4
Verifying understanding	5	8
Providing time/homework to think about the error	3	4
Giving hints	6	6
Enabling the student to find her/his own error	6	10
Evaluating a student's answer	9	10
Total number of generic actions	70	82
Mathematics-specific actions		
Using representations and examples	4	8

Guided questioning to engage in a mathematical process	13	22
Modifying the task	1	5
Pointing to big mathematical ideas	5	6
Providing the correct solution	5	1
Providing explanation/exposition	7	9
Total number of math specific actions	35	41

Concerning the *mathematics-specific* actions, the dominant action was guided questioning to engage students in a specific mathematical process (see case 1 and case 2 below) both in the critical event and the scripting dialogue (13 and 22 respectively). The mathematics-specific actions that appear to be more frequent in the scripting dialogues in relation to the critical events were: using representations and examples (e.g., a drawing illustrating how many intersections can be between a circle and straight line) and task modification (see case 1 below). Also, in the scripting dialogues the PTs seem to avoid providing the correct solution. In most cases, transformation in the pedagogical actions is indicated through shifts from intermediate use of student's difficulty in the original event (i.e. student error is addressed briefly as a stepping-stone to correct it) to intermediate or active use of student's difficulty in the scripting dialogues (i.e. using student's difficulty as a major tool of their instruction, provide students' opportunities to discuss and test why a method doesn't work). PTs often brought to the scripting dialogues alternative teaching actions to those identified in the original events in most of the above categories.

THE PROCESS OF TRANSFORMATION – TWO CASES

We illustrate below how PTs modified the initial event to the scripting one through two cases of PTs (Elisavet and Nina) who made an active use of students' errors in their scripting dialogues introducing a large range of pedagogical actions as we have mentioned above. Elisavet and Nina offered 9 and 5 different actions respectively (Elisavet: 5 general, 4 mathematics-specific, Nina: 1 general, 4 mathematics-specific). Through the cases we also illustrate these pedagogical actions.

The case of Elisavet. Elisavet's initial event was taken from her observation of a lesson in Grade 11. The students had already worked on tasks to identify the center of circles of radius R situated either at O(0,0) or in another point of the plane K (x_0, y_0) using the circle equation in two forms. Elisavet considers this event as critical because of the students' difficulty to use the taught content in identifying the center of a circle in textbook tasks. In the initial dialogue, one student (S3) cannot provide a response. The teacher asks another student (S1) and through some guidance (i.e. he writes the formula as $x^2+y^2=5^2$) he recognizes the radius of the circle but not its center. S1 seems to be confused with the different forms of the circle equation. Then the teacher comments that "this is the simplest form of circle, we have referred to it many times" and he asks another student (S2). S2 replies correctly. In the scripting dialogue, Elisavet adopts *general* pedagogical actions (e.g., *making a student's solution visible, enabling the student to find her/his own error*) and *mathematics-specific* ones such as reminding the relevant theory (*exposition*) and asking guiding questions (*guided questioning to engage in a specific mathematical process*). Below, we provide an extract from her dialogue with S2 and S3.

T: We know that the general form of the equation is $(x\text{-}x0)^2+(y\text{-}y0)^2 = r^2$. How can we find the center and the radius of the circle?

S3: The center, let K, have coordinates (x_0, y_0) and the radius will be r.

T: Fine. Now, in the special case where the origin of the axes is the center, what will be the point (x_0, y_0)?

S2: I will have $x_0 = 0$ and $y_0 = 0$.

T: Right. So, what is the equation of the circle in this case?

S2: $x^2 + y^2 = r^2$.

T: Fine. And what will be the center and the radius of the circle in this case?

S2: The center is the origin (0,0) and the radius is r.

The scripting dialogue differs substantially from the dialogue of the original event. Elisavet attributes to the teacher a more active role in using students' difficulties. The construction of the scripting dialogue was not an easy task for Elisavet. In her first attempt, she tended to talk more and then she made the dialogue more interactive. The crucial decisions she made concerned how the teacher could build on students' responses and direct them to develop their own way of thinking: "I think that the teacher should enable students to understand from the beginning the purpose of the task and from there to help them build step by step the final solution". Another difficulty that she experienced was to find appropriate explanations to students: "I had a hard time finding a way to explain to the student why it is wrong that the center passes through the origin". As regards her learning from this experience Elisavet seemed to have developed some awareness of the importance of exploring and developing students' understanding as well as the challenge that a teacher faces to take on-the-moment decisions.

The case of Nina. Nina's initial event was taken from her observation of a lesson in Grade 12 focusing on division of two complex numbers. The teacher solved a few examples on the board, and then asked a volunteer to solve the expression $\frac{1+i}{1-i}$. Amir approached and wrote $\frac{1+i}{1-i} = \frac{(1+i)(1-i)}{(1-i)(1-i)}$. The teacher asked him: "Did we learn to do it this way?" Another student responded: "We learned to multiply by the conjugate of the denominator". Amir responded: "It doesn't matter whether we multiply by the conjugate of the nominator or by the conjugate of the denominator". The teacher responded: "We multiply by the conjugate of the denominator". Nina considered this event as critical because of Amir's response, which reflects an incomplete mathematical thinking. She suggested that Amir cannot "see" his mistake because he does not understand the purpose of multiplying the fraction by the conjugant of the denominator. In the scripting dialogue, Nina suggested alternative teacher actions (e.g., *guided questioning to engage in a specific mathematical process*, *making a student's solution visible, pointing to big mathematical ideas*).

> T: Come to the board. Try to multiply the fraction first by the conjugate of the nominator and then by the conjugate of the denominator…
> [Amir approaches the board and follows the teacher's instructions.]
> Amir: Wow, I was wrong. Multiplying by the conjugate of the nominator does not lead to anything meaningful… multiplying by the conjugate of the denominator resulted in a complex number in the standard form because the multiplication gave me a real number in the denominator.
> T: In conclusion, pay attention that we multiply by the conjugate of the denominator as an effective step for solving the task. It is not a random action. We choose this particular technique so as to receive a real number in the denominator and not to be left with a complex number in the denominator. Any questions before we continue?

The scripting dialogue written by Nina differs from the dialogue of the original event. Nina attributed to the teacher a more active role in using Amir's statement as a tool for her instruction. In contrast to the teacher in the original event, who addressed Amir's suggestion briefly as a stepping-stone to correct him, Nina provided a space for Amir to understand why his initial method is ineffective. Moreover, by inviting Amir to the board and by emphasizing the main ideas in her summary, Nina made the student's solution visible. Nina reported that it was crucial for her to enable Amir to understand the rationale for using a different method. She was uncertain about what would be a good way to do it and whether inviting Amir to the board would embarrass him. Eventually she decided that it would allow the other students to take part in the activity and support a norm that making mistakes is a constructive part of learning. Nina reported that the experience contributed to her developing some awareness of the multiple roles that a teacher fulfils simultaneously, in particular, building on student thinking and enabling students' participation in the learning process.

DISCUSSION

The pedagogical actions that PTs adopted to address students' difficulties in the scripting dialogues indicate an active use of these difficulties. Specifically, PTs attempted to engage all students in the class in discussing the difficulties through general and mathematics-specific actions. The general actions address mainly social norms and issues of participation while the mathematics-specific ones concern the development of students' conceptual understanding. The cyclic transformation process of critical events to scripting dialogues seems to have supported PTs in developing alternative teaching actions that effectively addressed students' difficulties. This finding is rather promising as the existing literature in the field speaks of the dominance of procedural ways by which PTs address students' difficulties tending to 'correct' the error (e.g., Son, 2013). Scrutinizing into the transformation process in the cases of Elisavet and Nina, we see that (a) student's difficulty becomes a central point of discussion with the whole class and (b) students' difficulty is handled in conceptual and social terms. Both of them attempt to address social and mathematical aspects of classroom communication through common and different pedagogical actions, something that is challenging for the teachers (Skott, 2019). The analysis of the two cases suggests that the process of writing a scripting dialogue facilitated further PTs' sensitivity to students' difficulty: not only they focused on the difficulties, but also attempted to understand their sources and develop awareness of critical issues of mathematics teaching and learning. Further analysis of data including also the discussions among PTs in the two courses is expected to allow us getting a deeper understanding of the transformation process from the critical event to the scripting dialogue and its potential for the development of PTs' noticing of students' difficulties.

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MAKING SENSE(S) OF MULTIPLICATION

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This paper examines drawings created by third-grade students during a mathematics lesson using the multi-touch iPad application TouchTimes*, which provides a way of engaging directly with multiplication through the user's fingertips. Using a theoretical perspective that recognises the materiality of mathematical concepts, we study the students' conceptualisations of multiplication (which are imbricated with their fingers and the application) by analysing the multiple sensory meanings expressed in their drawings. We show how these sensory meanings relate to characteristics of multiplication discussed in the literature—through actions we call multi-plying and unitising. We also discuss additional features of the drawings, such as the inclusion of fingers and hands, the order of multiplier and multiplicand and limited use of colour.*

INTRODUCTION

How would you describe multiplication? We asked friends, family and colleagues this question and received answers such as: faster than addition, groups of, patterned, double numbers, eight bags of six apples, fast combinations of numbers, something that rabbits do very quickly and a reference to Lunney Borden et al.'s (2021) 'sets of, rows of, jumps of' ideas. While there is some diversity to these responses, we found a strong emphasis on the arithmetic interpretation of multiplication as repeated addition, with very few multiplicative images evoked (aside from the rabbits).

Indeed, a common approach to introducing multiplication focuses on additive reasoning. However, Askew (2018) suggests that presenting multiplicative reasoning in primary schools by focusing on functional relations would create a better foundation for higher mathematics. Through the use of a multi-touch application called *TouchTimes* (Jackiw & Sinclair, 2019), we wanted to introduce students to multiplication through *both* functional and relational experiences. *TouchTimes* (hereafter TT) allows children to create tactile, pictorial and symbolic representations of multiplicative situations that adjust simultaneously in response to their fingertips on the screen. This paper explores the meanings students express in diagrams they were asked to make about a multiplicative problem.

MEANINGS OF MULTIPLICATION

Multiplication is a concept that consists of several dimensions. Vergnaud's (1988) conceptual field brings many of these dimensions together. A conceptual field consists of, "a set of situations that make the concept meaningful, (…) a set of invariants (objects, properties and relationships) that can be recognized and used by subjects to analyze and master these situations, and a set of symbolic representations" (p. 141).

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In terms of the situations that make multiplication meaningful, Greer (1992) identified ten: equal groups, equal measures, rate, measure conversion, multiplicative comparison, part/whole, multiplicative change, cartesian product, rectangular area and product of measures. The objects of multiplication consist of intensive quantities (Kaput, 1985) and two types of unit counts (Clark & Kamii, 1996). Davydov (1992) distinguished these unit counts, with one being smaller and the other larger. While the former are included among themselves in one level, they are also included in the larger unit counts in another level, thereby creating a one-to-many and many-to-one correspondence between the two distinct unit counts (Clark & Kamii, 1996). This correspondence allows for intensive quantities, which emerge from the ratio between two quantities measured by these two distinct unit counts (Davydov, 1992; Kaput, 1985; Vergnaud, 1988). Multiplication also involves two actions. One can be found in Davydov's (1992) description of multiplication as measuring a quantity indirectly by transferring the unit count from the smaller to the larger one, creating a unit of units. The other can be found in Confrey's (1994) description of multiplication as splitting, or "creating simultaneously multiple versions of an original" (p. 293).

In an equal group situation such as "3 children each having 4 oranges. How many oranges do they have all together?" (Greer, 1992, p. 280), oranges and children correspond to the smaller and the larger unit counts, respectively. Each child corresponds to 4 oranges. Instead of measuring the total amount of oranges additively by counting oranges one by one, the amount can be calculated multiplicatively by counting the children and transferring the amount to the oranges based on the 4:1 ratio between the number of children and the oranges.

MAKING SENSE(S) THROUGH DIAGRAMMING (THEORY)

We assume that mathematical concepts are not abstracted by humans from encounters with the material world, but are inseparable from tools, techniques and bodies (de Freitas & Sinclair, 2017; Rotman, 2008). Concepts become intelligible, "not by a *reductionist abstraction* or by a 'subtraction of determinations' (Aristotle's approach to abstraction), but by the actions (moving, excising, cutting through diagrams and gestures) that awaken potential multiplicities that are always implicit in any material" (de Freitas & Sinclair, 2017, p. 78). Therefore, in studying learners' conceptualising of multiplication, we are particularly interested in the actions they experience (in this case, with TT) and the drawings and gestures they make in response to these experiences. We see drawings not solely as representations of existing conceptualisations, but as material sites for conceptual creation.

Much of the research on children's understanding of multiplication focuses on their ability to solve particular types of multiplication tasks. With Cheeseman et al. (2020), we are interested in exploring the visual meanings of multiplication that students can develop, since these can provide significant conceptual support (Davis et al., 2015), particularly when these visual meanings have dynamic elements (McGarvey et al., 2015). Diagramming can be an effective way for learners to express, create and represent mathematical concepts, particularly spatial and temporal aspects that might be more difficult to verbalise (de Freitas & Sinclair, 2017). Given the novel gestural/visual experiences that TT affords, we ask: what conceptions of multiplication do students' diagrams produce, and how do these relate to aspects of multiplication discussed in the literature?

METHODS

After providing a short description of *TouchTimes*, we will then describe the research context and our approach to analysing the children's drawings.

How *TouchTimes* works

Though TT has two microworlds, the focus of this paper is the Grasplify world. A vertical line divides the Grasplify screen in half, but it is otherwise blank (Figure 1a) until a user places their fingers on the screen. Grasplify is symmetric, so on whichever side of the screen that is touched first, coloured discs (which are called 'pips') will appear (and remain) below the user's fingers (Figure 1b) for as long as screen contact is maintained. Created by finger taps on the other half of the screen, 'pods' are encased groups of pips (seen on the right side of Figure 1c) that replicate the shape and colour of the original pip formation being maintained on the opposite side of the screen (seen on the left side of Figure 1c). All pods are then encircled by a 'lasso' to form the product (seen on the right side in Figure 1c). The composition of the pips within the pods (number, colour and shape) change instantly to reflect the pip-creating half of the screen, as does the numerical expression at the top of the screen (Figure 1d).

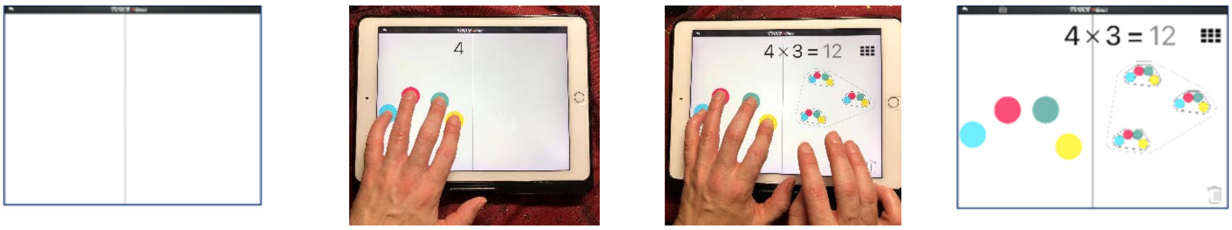

Figure 1: (a) Initial screen (b) Pips (c) Pod creation (d) Two composite units

The design of Grasplify reflects an embodied co-ordination of units approach to multiplication based in measurement (Davydov, 1992), which requires a unit of measure (the multiplicand, or 'pips' in Grasplify) to be established first, followed by the unit quantity (the multiplier, or 'pods'). The first unitising (i.e., making composite units) occurs in the making of the pips (in Figure 1b, there are 4 pips), and the second unitising occurs in the making of pods (three of which appear in Figure 1c) and a third in the encircling of all the pods, which is the product (in Figure 1c, there are 3 pods encircled). The pods (multiplier) are a quantity generated from the original pips (multiplicand) and serve as a measurement unit for the product. Thought of this way, multiplication involves "a count of a [larger] unit for which a relationship to another, smaller unit, is already established" (p. 12).

Study context

This study took place in a French-immersion third-grade classroom (8–9-year-olds) in Metro Vancouver, Canada in October–November 2019. This class was composed of a multicultural group of 18 students, most of whom speak English as their primary language, as does the teacher. The teacher had more than 20 years teaching experience and had volunteered to be part of this multi-phase research project to collaboratively develop tasks for use with TT and to implement TT as a resource for learning multiplication. The research team (who are the authors of this paper) was invited by the teacher to observe the use of Grasplify in her classroom. Three 60-minute lessons were observed and video-recorded. The drawings analysed in this paper were part of the day two activities, which included: approximately 15 minutes of free exploration time using Grasplify; a task requiring that students double the product of $1 \times 3 = 3$ in different ways using Grasplify; an exploration of how to make a 5 with their pod-finger and then to count by fives to 25. The teacher brought the whole class together after each task and had individual students share their answers using an iPad projected on a screen for all could see. In the final activity, which took approximately 10 minutes to complete, students were given the following problem: *How would you use* TouchTimes *to solve this problem? A bunch of buttons fell on the floor. Nick gathered them in heaps of 8 buttons. He made 5 heaps. How many buttons are there*? Students had access to their iPads and were given a sheet of paper on which to make their drawings. While students worked, the teacher, as well as the research team, interacted with students to answer questions and assist them, as necessary.

Analysing the drawings

Drawing on both the literature and the design of TT, we identified the two actions of multiplication described above, namely: unitising and multi-plying. The unit is central to Davydov's (1992) model of multiplication, and in TT, the production of a pod is a unit of the pips on the screen and the product is the set of pods. Visually, the unit of units is the 'lasso' around all of the pods. Multi-plying is central to Confrey's (1994) idea of multiplication as splitting, which emphasises the copying of a unit. In TT, this copying is shown visually through the spatial arrangement and the number and colour of the pips within each pod, which are reflective of the original pip-formation created by the user. We opted for multi-plying (instead of splitting) because splitting refers to the partitioning of the original into identical quantities, while multi-plying refers to the creating of multiple replicas of the original. The emphasis is less on the product (unit of unit of units) than on the pods (unit of units). In analysing the diagrams, we looked for evidence of each aspect, while remaining open to other meanings that could emerge in the diagramming process.

WHAT THE DRAWINGS SHOW (RESULTS)

The students produced a wide variety of drawings, some of them very faithful to elements of the TT screen, and some seemingly unrelated. Multi-plying was the most prominent meaning expressed, found in 15 of the 18 drawings, and unitising in 5 of the 18 drawings. We provide examples of each type of drawing and then consider some additional features of the drawings that were noteworthy in terms of the senses of multiplication being expressed.

Multi-plying

In each of the three drawings shown in Figure 2, the spatial arrangement of the pips within each of the five pods is drawn in a consistent way, which expresses the sense of multiplication as multi-plying. In both Figures 2a and 2b, the pip-creation itself is shown on the left screen (LS). The copying of the original pip-unit on the right screen (RS) is demonstrated by each student's attention to the number and configuration of pips within each pod. Though Figure 2c does not include the original pip creation, the consistency of the pods is apparent. Note the differences in terms of the use of symbols: while the product appears in all three drawings, only Figure 2b includes the equation.

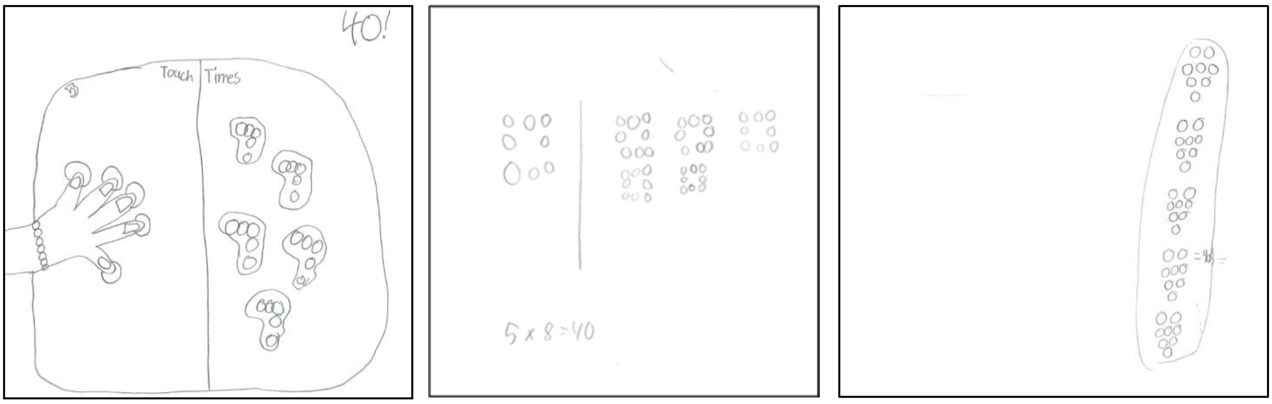

Figure 2: (a) Hand(s)-on multi-plying of 5 (b) Domino-like multi-plying of 8 (c) Pods-only multi-plying of 8

Unitising

Clark and Kamii's (1996) two types of unit counts and Davydov's (1992) unit of units are visible in the Figure 3 drawings. Without emphasising the multi-plying of the Figures 2 drawings, these Figure 3 drawings all express unitising, primarily through an encircling. Both Figures 3a and 3b drawings have included the 'lasso' that visually creates the product of 8 × 5 (or conversely 5 × 8). In addition to showing pips and pods, each drawing includes the mathematical notation that corresponds with the double unitising visible in the drawings. Note that all of the drawings in Figure 3 include equations, with only that of Figure 3a being in the order implied by the button problem.

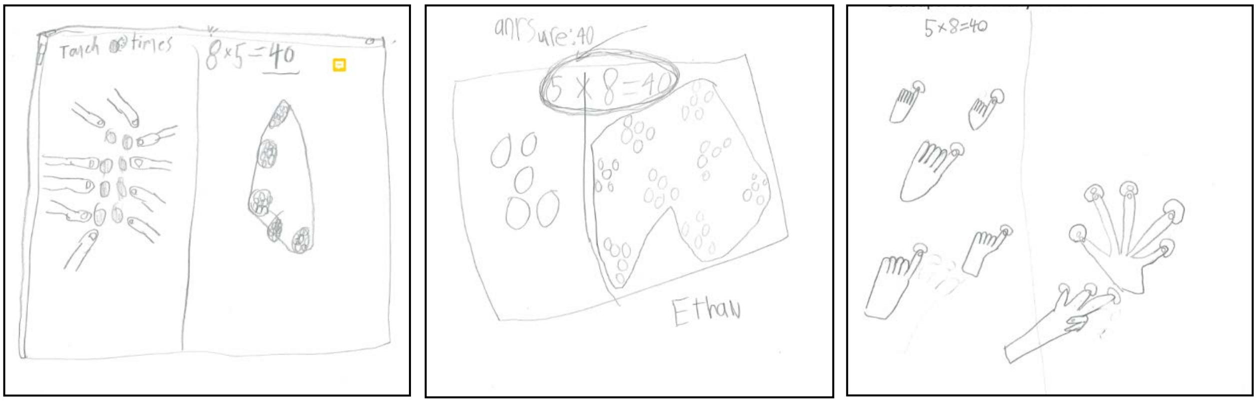

Figure 3: (a) Unitising 8 × 5 (b) Unitising 5 × 8 = 40 (c) Hands-on unitising

Other features

We use the three following examples to discuss other features of the drawings that are relevant to the students' sense of multiplication.

Figure 4a shows unitising as well as some sense of multi-plying, with the pods all looking similarly trapezoidal. However, in expressing 5 × 8 = 40, this drawing does not model the button question. This was true in many drawings (Figure 2a, 3b, 3c). While circulating in the classroom, it was evident that some students understood 5 × 8 and 8 × 5 to be the same; and, indeed, Figure 4b shows this well. The prevalence to using 5 pips may be related to the ease with which one can make 5 pips in *TouchTimes*, which requires only one hand. It may also be related to the 'groups of' thinking common in repeated addition, wherein '5 groups of 8' places the 5 first (and to the left), as opposed to '8, five times'.

In Figure 4c, though the spatial orientation of the pips within the pods is not reflective of the pip-creation on the left screen, each coloured pip on the left has been replicated in the appropriate colour within each of the pods on the right side of the screen. We see this use of colour as demonstrating the 'spreading' effect of each pip-unit across the larger pod-units, which highlights the one-to-many multiplicative relation.

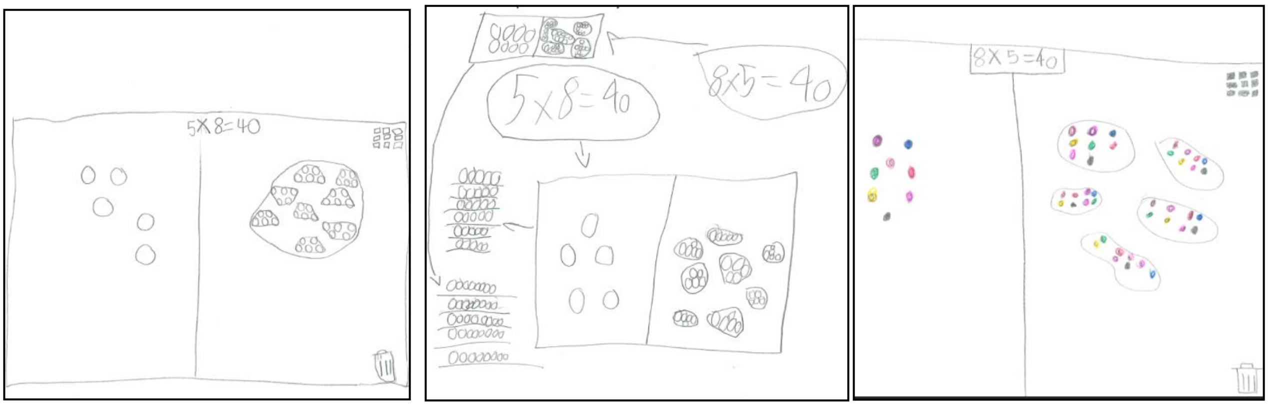

Figure 4: (a) Trapezoidal pods (b) 5 × 8 = 40 and 8 × 5 = 40; pips/pods and arrays (c) The multi-plying of colour

We were surprised to find only one drawing making use of colour, particularly since colour is so important to the TT design as a way of showing the one-to-many relation of multiplication. Perhaps the colour was not deemed important to the students, which would suggest the need for an explicit pedagogical intervention. It may also be that students come to think that colour does not matter in mathematics learning, since it is not frequently used as a significant mathematical sign.

In Figures 2a, 3a and 3c, we see the presence of hands and fingers. These may be re-enactments of touching, but we see them as fundamental components of how students conceptualise their participation in multiplication. In the phrasing of the question: *How would you use* TouchTimes *to solve this problem?* their drawings express a sense of agency—the mathematical action is not just mental, but physical. In Figure 3c, for example, the diagram conveys the actions taken by the student, but in a way that highlights their embodied performance. The diagram shows what the student is paying attention to. On the right side of the diagram, the student draws their hands palm up, showing 8 fingers. This reflects the contemplative and reflective nature of what the student recognised to be significant in their drawing, the unit of 8. The action conveyed in this instance is not one that involved TT directly but is a negotiation of their own awareness before they turn their fingers to touch the screen. On the left side of the diagram, the student is touching once, then twice, and sequentially up to 5 times. Multiplication is shown as a temporal multi-plying of the unit. In Figure 2a, the hand is an essential contribution to multiplication as well. When a pip creating hand is lifted, there is no multiplicative operation since the pips will be gone. Finally, in Figure 3a, ten fingers are drawn with only eight pips, a clear break in a 1-1 correspondence. The student might have lost count, or may simply be expressing the sense of ‘lots’, or the potential of using all ten fingers on the screen.

CONCLUSION

The drawings analysed in this paper highlight the sense of multiplication that these third-grade students experienced through their interactions with TT and in the course of their drawing process. Several aspects of multiplication, such as multi-plying, unitising, and in one instance, spreading, are visible within these drawings. This suggests that TT can provoke different ways of thinking about multiplication, rather than simply focus on a single meaning (or a single model). The unitising and multi-plying multiplicative relations that can be seen in these drawings are very different from common ways of introducing primary students to multiplication that are additive in nature. The inclusion of fingers and hands was significant for some students to express how TT was used to understand the button problem, which shows how the actions they used on the screen affect ways of thinking in their diagrams. These drawings included not only the actions and objects of multiplication, but also the associated symbols, with most (14 of the 18) drawings showing a symbolic multiplicative expression.

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HOW DO FIRST-GRADE STUDENTS RECOGNIZE PATTERNS? AN EYE-TRACKING STUDY

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Recognizing patterns is an important skill in early mathematics learning. Yet only few studies have investigated how first-grade students recognize patterns. These studies mainly analyzed students' expressions and drawings in individual interviews. The study presented in this paper used eye tracking in order to explore the processes of 22 first-grade students while they were trying to recognize repeating patterns. In our study, we used numerical and color pattern tasks with three different repeating patterns (i.e., repeating unit is AB, ABC, or AABB). For each repeating pattern task, students were asked to say the following object of the given pattern. For these patterns, we identified four different processes in recognizing repeating patterns. In addition, we report differences in the observed processes between the patterns used in the tasks.

INTRODUCTION

Mathematics can be described as the science of patterns (Steen, 1988). In early mathematical learning, patterns play a decisive role in the development of algebraic thinking (Carraher & Schliemann, 2007; Clemens & Samara, 2007). Being aware of patterns provides primary students with a mindset that is useful in the later study of algebra (Schoenfeld, 2007). In addition, pattern recognition is a central content of mathematics education in primary school (NCTM, 2000), which makes it a significant topic for mathematics education.

Yet, pattern recognition poses challenges to many students. For example, Clarke et al. (2006) found that for repeating patterns (e.g., ABABAB) first graders are successful in recognizing and extending them in only 31% of the tasks. This calls for support of young children in their ability to recognize patterns. For being able to foster children's pattern-recognition ability, it is crucial to investigate and understand their pattern-recognition processes (Lüken, 2018; Papic et al., 2011). Empirical studies mostly explore pattern-recognition processes based on children's expressions and drawings. The present pilot study in the context of the Erasmus+ project DIDUNAS investigates students' pattern-recognition processes using eye tracking, the recording of eye movements. Eye tracking has shown to be useful for investigating processes of children on early primary level before (Schindler et al., 2020; Sprenger & Benz, 2020). Our study uses eye movement video analysis to explore pattern-recognition processes.

In this pilot study, the aim is to investigate (1) what processes first-graders use in recognizing patterns and (2) whether there are differences in students' use of pattern-recognition processes between different kinds of patterns.

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THEORETICAL BACKGROUND

Pattern recognition in early mathematics learning

One goal of early mathematical learning is the development of algebraic thinking (NCTM, 2000). Algebraic thinking in early grades includes noticing structures (Kieran, 2004). Fostering students in their ability to notice structures and recognize patterns can therefore contribute to their development of algebraic thinking. (Carraher & Schliemann, 2007). In the first grades of primary school, repeating patterns can be used to foster students' ability to recognize patterns (Clemens & Samara, 2007). Repeating patterns refer to patterns that have one unit (e.g., AB) that repeats multiple times (ABABAB). Patterns differ from each other if the repeating unit is of different length or the elements of a repeating unit are arranged differently. For example, pattern ABABAB is different from pattern ABCABCABC, or pattern AABCAABC is different from ABACABAC.

For pattern recognition, it is of particular importance to recognize the recurring unit of a pattern (Papic et al., 2011). The repeating unit can be represented, for example, symbolically (e.g., AB), numerically (e.g., 2 5), or by colors (e.g., ● ●). The present study uses numerical and color patterns. Common pattern tasks for preschoolers and first graders consist of identifying and extending given repeating patterns (Rittle-Johnson et al., 2015). For example, students are given a color pattern (e.g., ● ● ● ● ● ● ●) and asked to say what color the next dot should be. In this paper, student approaches to extend repeating patterns, which are anticipated to entail identification of the repeating unit, are referred to as pattern-recognition processes.

Research on pattern recognition at the beginning of primary school

Some studies investigate the abilities of preschoolers or first graders in the context of repeating or growing patterns. Clarke et al. (2006) found that 76% of the students at the beginning of first grade can copy a repeating pattern, but only 31% can extend it. Rittle-Johnson et al. (2015) found similar results in a study with 64 four-year-old preschoolers. Further studies have used children's expressions and drawings in individual interviews to investigate processes of preschoolers and first graders in recognizing patterns (Lüken, 2018, Papic et al., 2011). For example, Lüken (2018) found that three- to five-year-old children use a process of *comparison* to compare the beginning of the repeating pattern with the part that has to be extended. Lüken also found that the repeating unit was identified and used by the students. Papic et al. (2011) identified similar pattern-recognition processes.

In this study, we were interested in pattern-recognition processes of first-grade students. In contrast to previous studies on this topic, we use eye tracking videos (not student utterances or drawings) to explore student pattern-recognition processes, since eye tracking has proven itself valuable to identify student processes in mathematics (e.g., Schindler et al., 2019, 2020). We ask the following research questions.

1. What processes do first-grade students use in recognizing repeating patterns?
2. Are there differences in students' use of pattern-recognition processes between different kinds of repeating patterns?

METHODS

Participants, procedure, and tasks

The study was conducted with 22 first-grade students (age: *M* = 6.80 years; *SD* = 0.24 years) from a primary school in Cyprus. Fourteen (~63.6%) of the students had Greek as their mother tongue, the others Arabic (*n*=7, ~31.8%) and Bulgarian (*n*=1, ~4.5%).

In addition to the eye-tracking study (see below), we conducted the standardized mathematics test ZAREKI-K for assessing students' mathematical performance level at the transition from kindergarten to primary school (von Aster et al., 2009). We used the adapted version by Walter (2020). The test indicates that twelve of the 22 students (~54.5%) are not at risk for developing math difficulties, but ten are at risk. Thus, the sample has a good spectrum in terms of performance levels.

In the main part of the study, the students worked individually on eight pattern tasks on a computer screen (see Figure 1). Each task consisted of at least three repetitions of a unit followed by a white blob. The students were asked to name the number or color of the object that was hidden behind the white blob (e.g., 1, yellow). Before the first numerical and the first color pattern task, the students worked on a sample task, to ensure that the students understood the task correctly. The following three repeating units were used in the pattern tasks: (1) AB (four tasks), (2) ABC (two tasks), and (3) AABB (two tasks). The students answered by saying aloud the number or color they thought was behind the white blob. The students did not receive feedback and incorrect answers were not corrected. Four tasks had numerical repeating units in form of digits (e.g., 4 1), four in the form of colored dots (e.g., ● ●).

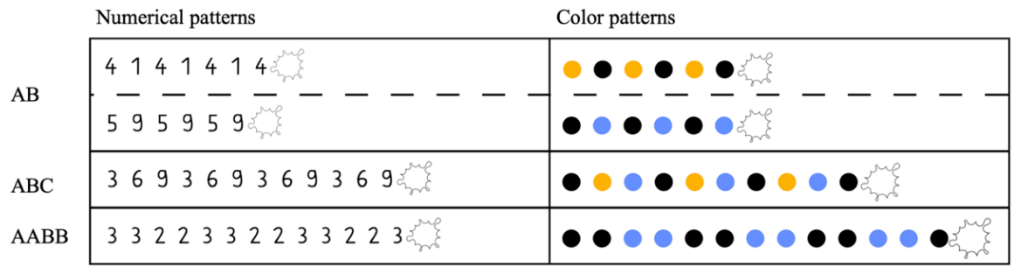

Figure 1: Numerical and color repeating pattern tasks used in the study.

Eye tracking

Students' eye movements were recorded with the screen-based eye tracker Tobii Pro X3-120 (infrared, binocular), with a sampling rate of 120 Hz. The tasks were presented on a 24" monitor. The students' heads were about 60–65 cm away from the monitor. The eye-tracking data showed an average accuracy of 1.37°, which corresponded to an error of about 1.44–1.55 cm on the screen at a head distance of 60–65 cm. The centers of the digits and dots in our tasks were on average 4.16 cm apart from one another on the display on the monitor (3.2–5.5 cm), which means that the eye-tracking accuracy was sufficient to reliably determine what element the students looked at.

Analysis of eye-tracking data

Raw gaze-overlaid videos provided by Tobii Pro Lab software were used to analyze students' pattern recognition processes. In addition, notes were taken during the data collection describing student actions (e.g., when students pointed to the monitor). Although pattern-recognition processes have already been elaborated in research, an inductive approach was chosen for this study. We performed a qualitative content analysis through a data-driven inductive category development (Mayring, 2000), similar to Schindler et al. (2019, 2020), in four stages: *Stage one*: A randomly selected half of the gaze-overlaid videos were viewed and for each video, the gazes were described. *Stage two*: Similar descriptions of the gaze-overlaid videos were subsumed into one category, while categories in this study refer to pattern-recognition processes. A first general description of the respective pattern-recognition process was formulated. *Stage three*: The second half of the gaze-overlaid videos were viewed and coded using the pattern-recognition processes elaborated in stage two. During this coding, existing descriptions of pattern-recognition processes were revised and specified, and new processes were added when existing processes did not seem suitable for describing the gaze-overlaid videos. *Stage four*: Finally, with the complete category system, the first half of the gaze-overlaid videos were re-coded to check the fit of the revised process descriptions and to assess the emergence.

All gaze-overlaid videos were coded by the first author. 22.7% of the videos were analyzed independently by the last author. The interrater agreement was calculated using Cohen's Kappa (Cohen, 1960). With $\kappa = 0.87$, the inter-rater agreement is almost perfect (Landis & Koch, 1977).

Statistical analysis

To determine differences between the pattern-recognition processes and the different patterns, a two-tailed Fisher–Freeman–Halton exact test for $r{\times}c$ contingency tables was performed (Freeman & Halton, 1951) using SPSS 28. This test is an extension of chi-square test and is especially suited for small sample sizes for which the chi-square approximation does not hold (Fagerland et al. 2017). For this analysis, we have grouped the different patterns according to their structure (see Figure 1). For example, the pattern tasks with repeating unit 4 1 and ● ● in Figure 1 fall into group AB.

RESULTS

Pattern-recognizing processes

In the following, we describe the pattern-recognition processes found through the analysis of gaze-overlaid videos of the first-grade students. We use gaze plots to visualize the processes for this paper, even though the analysis was based on the videos. Figure 2 shows idealized gazeplots of these processes as illustrations. These idealized gazeplots are not actual gazeplots of children, but idealized illustrations for the identified processes.

(1) Identifying one repeating unit of the pattern

The gazes go to one repeating unit (sometimes multiple times)—mostly the repeating unit before the white blob. The gazes partially also touch one dot/number before the repeating unit.

(2) Identifying one repeating unit and validating/applying it

(a) Identifying and validation: The gazes go to the repeating unit before the blob (sometimes multiple times) and then go to another repeating unit in the pattern (sometimes multiple times). Afterwards, the pattern is extended by continuing the repeating unit before the blob.

(b) Identifying and application: The gazes go over a repeating unit in the beginning or the middle of the pattern (sometimes multiple times). Afterwards, the pattern is extended by continuing the repeating unit before the blob.

(3) Looking at each element

The gazes go to each dot/number of the pattern individually usually from left to right (sometimes multiple times). Up to three dots/numbers are skipped from the beginning.

(4) Unsystematic jumping over the pattern

The gazes jump fast over the pattern. Often the blob is not looked at. There is no systematic process recognizable.

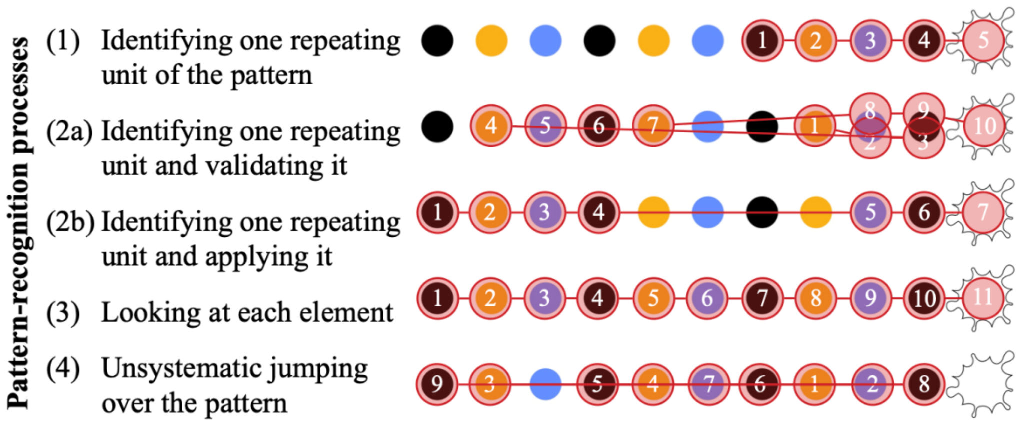

Figure 2: Idealized gazeplots of the pattern-recognition processes with numbers indicating the order in which the students looked at each dot.

Differences in pattern-recognition processes between different patterns

To determine differences between the identified pattern-recognition processes (see Figure 2) and the different kinds of patterns (see Figure 1), Fisher–Freeman–Halton exact test for $r \times c$ contingency tables was performed (see Table 1). The test revealed that the pattern-recognition processes used by the students differed significantly between the three kinds of patterns (p = 0.031). Cramér's V = .20 indicates a moderate relationship between the kind of pattern and pattern-recognition processes.

		Pattern-recognition process				
		(1)	**(2)**	**(3)**	**(4)**	Total
Repeating unit	**AB**	35	24	23	0	82
	ABC	12	22	5	1	40
	AABB	15	16	47	2	40
		63	61	35	3	162

Table 1: Observed number of pattern-recognition processes for the different patterns.

Figure 3 shows the distribution of the pattern-recognition processes over the three kinds of pattern tasks based on the absolute values in Table 1. Figure 3 illustrates that for patterns of kind AB, students used process (1) identifying one repeating unit, the most with 42.68%. Process (2) identifying one repeating unit and validating it, and (3) looking at each element, were equally distributed in the pattern tasks with the repeating unit AB. Process (4) unsystematic jumping over the pattern, in contrast, did not occur. In the pattern tasks with repeating unit ABC, process (2) was identified most often with 55%. The other processes occurred less often. Processes (3) and (4) together were identified only half as often as process (1). For patterns of kind AABB, processes (1) and (2) were identified with almost equal frequency, 37.5% and 40%, respectively. Processes (3) and (4) were identified slightly more often than in patterns of kind ABC. Category (4) appeared exclusively together with wrong answers.

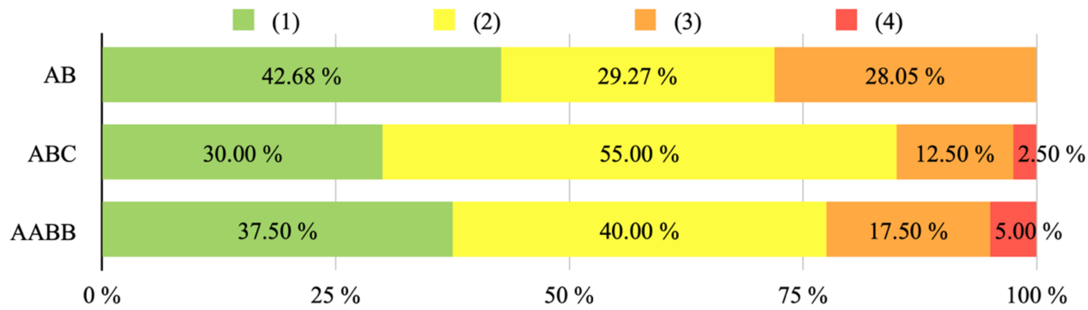

Figure 3: Distribution of pattern-recognition processes over the kinds of patterns.

DISCUSSION

This pilot study aimed to investigate what processes first-grade students use in recognizing repeating patterns and whether there are differences in students' use of pattern-recognition processes between different kinds of repeating patterns. The study was conducted with 22 first-grade students and eight repeating pattern tasks in which students were to name the number or color of the next object of a given pattern. The results of our study show that the first-grade students used four different pattern-recognition processes in numerical and color repeating pattern tasks: (1)

Identifying one repeating unit of the pattern, (2) identifying one repeating unit and validating/applying it, (3) looking at each element, and (4) unsystematic jumping over the pattern. Our results connect to some of the pattern-recognition processes identified in previous studies (Lüken, 2018; Papic et al., 2011) and extend them. Furthermore, we found that these processes were used differently between different kinds of patterns (see Figure 3).

These results should be interpreted considering the following limitations: With 22 students, a relatively small sample was available. It cannot be discounted that with a larger sample, additional pattern-recognition processes could be identified. Also, in future studies, more than eight pattern tasks should and will be conducted. In particular, other patterns than those used in this pilot study (i.e., AB, ABC, AABB) need to be investigated (e.g., AAB, AABC, ABAC).

The results of this study hinted at the value of using eye tracking to explore students' pattern-recognition processes. With regard to the overall purpose of supporting children in pattern recognition processes, the study has shown that eye tracking can inform about student strategies and that these insights can help to support students adaptively and individually in developing further their pattern-recognition processes. In line with this, one aim of the DIDUNAS project is to develop teacher materials that serve to support students, for example, in pattern recognition. The results of this pilot study provide initial insights for the development of such teacher materials.

Acknowledgment

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TEACHERS' JUDGEMENT ACCURACY OF WORD PROBLEMS AND INFLUENCING TASK FEATURES

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The ability to judge accurately the difficulty of mathematical tasks is considered as a central facet of the diagnostic competence of mathematics teachers. An underlying reason is that the accurate judgement of task difficulty is the basis for achieving an optimal level of instruction for the learning group. Although a lot of studies have already investigated the judgement accuracy and the influence of additional factors, like teacher knowledge, there is a lack of a detailed look at the task features as possible influencing factors. Therefore, in the present study, we first investigated the judgement accuracy of word problems with fractions. Afterwards, by means of theoretical varied task features and an empirical study with 153 6th graders as well as 64 prospective teachers, we explored differences in the tasks regarding the judgement accuracy.

THEORETICAL BACKGROUND

Accurate diagnostic judgments are considered to be crucial for adaptive teaching (e.g., Hardy, Decristan, & Klieme, 2019). In mathematics teaching, the task diagnoses play a key role for shaping teaching and influencing learning processes (e.g., Sullivan, Clarke, & Clarke, 2013). In this context, to estimate the task difficulty is one method of adapting the teaching to a learning group (e.g., Hammer, 2016) and for achieving an optimal level of challenge for the learning group (e.g., Urhane, & Wijnia, 2020). Anders et al. (2010) were able to show in a study that students are more cognitively stimulated during instructional activities when a teacher can make an adequate judgement of task difficulty. Thus, accurate judgements of task difficulty have also been identified as one of the core tasks of mathematics teachers.

In a large part of the empirical studies on teachers' diagnostic competence conducted so far, the quality of diagnostic judgments is regarded as judgment accuracy. There is knowledge from over 40 years of research on the accuracy of teacher judgement (e.g., Urhane, & Wijnia, 2020). Judgment accuracy describes the degree to which teachers' judgments on task solution agree with empirically collected solution rates (Hoge, & Coladarci, 2016; Südkamp, Kaiser, & Möller, 2012). It is important to distinguish more precisely between person-related, task-related and person-specific teacher judgement accuracy (e.g., McElvany et al., 2009). In this contribution, we focus on the task-related judgement accuracy. Empirical studies on the judgement of task difficulty could show that the task difficulties are mostly underestimated and are judged with a too low of variance (e.g., Urhane, & Wijnia, 2020). Although the rank component is judged well on average, it shows considerable variance. Furthermore, teacher overestimate the level of their classes up to 45.5 % (e.g., Hosenfeld, Helmke, &

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 67-74). PME.

Schrader, 2002). Only a few studies considered the three components of judgment accuracy in their analysis of teachers' diagnostic competence. However, these few studies reported low positive associations between the three components of accuracy (e.g., Schrader & Praetorius, 2018). Teacher judgment accuracy in a given content area differs across and within studies and show high inter-individual variances in the teacher judgment accuracy (e.g., Südkamp, Kaiser, & Möller, 2012; Urhane, & Wijnia, 2020). Research has identified several moderators that can determine the degree of judgment accuracy (e.g., Südkamp, Kaiser, & Möller, 2012). The accuracy can be influenced by teacher characteristics, judgment characteristics, student characteristics, class-level characteristics as well as test and task characteristics. Test and task characteristics refer to features of the tests or the tasks that have been used to measure student achievement. For example, Südkamp et al. (2012) examined the role of subject matter and the domain specificity of the achievement test as moderators of the judgement accuracy. But, none of these moderators affected teacher accuracy of judging and similary, Machts et al. (2016) found no evidence that test standardization moderated the judgement accuracy.

Despite the long period of research, we can look back on, and despite the numerous factors that have been investigated as influencing factors on the judgment accuracy, there has been little research on the association between teachers' judgment accuracy, the empirical solution frequency and the features of a task.

OBJECTIVE

According to the need for research pointed out in the previous section, we investigate first, whether the judgement accuracy and the interindividual variances regarding the judgment accuracy of student solutions reported in meta-analyses and task difficulty could also be shown in the judgement accuracy of mathematical word problems with fractions. Afterwards, as the focus of the study, we explicitly look more closely at the difficulty-generating task features with regard to the judgement accuracy and to the empirical solution frequency with the aim to analyze task features as influencing factors.

SAMPLE AND METHODS

The tasks that we focus on in this contribution is part of a larger study in which the influence of stress on the cognitive processes underlying diagnostic judgements on tasks (Becker et al., 2020) and the resulting judgement accuracy was examined. The difficulty of the tasks, eight mathematical word problems with fractions, were theoretically determined and empirically verified. For this reason, we designed fraction word problems with varied task features based on tasks frequently found in mathematics textbooks. The difficulty of word problems and fraction tasks has already been investigated in a number of studies. The task features chosen in the previous study were deduced from a review of those studies. In the present study, we considered two mathematical as well as two linguistic task features. First, we differed the relationship between the denominators (like or unlike fractions) (Padberg, & Wartha, 2017). It has been shown that like fractions have a lower requirement for the solution of a task in comparison to arithmetic tasks with unlike fractions and that tasks containing like fractions are easier to solve by students because of the analogy to the familiar natural numbers (e.g., Padberg, & Warta, 2017). Second, we distinguished between the number of calculation steps that have to be executed until the task is solved (one or two steps) (e.g., Jordan et al., 2006). It has been shown that tasks including one calculation step based on one mental model of operation and are therefore easier to solve by students than tasks that require two steps, because they include a further mental model of operation (Jordan et al., 2006). Furthermore, in word problems, the mathematical operation is part of the semantic structure of the text, which can also influence the difficulty of tasks (e.g., Verschaffel et al., 2020). For example, passive constructions can cause a change of the subject and the object of a sentence and can therefore be a further difficulty for students (e.g., Wessel, Büchter, & Prediger, 2018). Therefore, we varied as the third difficultly, the sentence structure of the tasks by using a passive construction in the task or not. Fourth, we distinguish between the use of words that can be considered as unfamiliar to 6th graders and the abandonment of those words, because it has been repeatedly shown that the use of those words can influence the solution of tasks and therefore the difficulty (e.g., Gürsoy et al., 2013). The number of the four difficulty-generating task features determines the theoretical difficulty in the present study.

The theoretically defined difficulty of the tasks has been proven in an empirical study with $N = 153$ 6th graders at various secondary schools in Germany. For this purpose, the students edited the word problems during their lessons in a randomized order to prevent sequence effects. Correctly solved tasks were subsequently coded with 1, incorrectly solved or unsolved tasks with 0. The students had sufficient time to solve the tasks.

Based on the solution frequency of each task, an empirical difficulty was determined by assigning a corresponding difficulty to the task on a ten-point scale (e.g.: 100 % - 90.1 % solution frequency corresponds to difficulty level 1, 90 - 80.1 % solution frequency corresponds to difficulty level 2, etc.; see task difficulty – students in table 1, 2 and 3). Furthermore, based on the empirical solution frequency, a ranking of the tasks was created.

Afterwards, in the main study, $N = 64$ prospective teachers of the educational university of Heidelberg judged the difficulty of the mathematical word problems in fractions for 6th graders on a ten-point scale. In a previous questionnaire, the semesters of the participants and whether any courses regarding the difficulty of fraction tasks had already been attended, but could be excluded as influencing factors in subsequent calculations. The mean of the participants' judgements is referred to as task difficulty – prospective teacher in the tables below (see task difficulty – p. teachers; table 1, 2 and

3). In the divisions of the means, the values were rounded down when the non-whole number is less than .5, higher than that the values were rounded up.

RESULTS

In all three components of judgment accuracy, the teachers' judgements deviated from the empirically determined difficulties of the tasks. On average, the task difficulty and the variance of task difficulty was underestimated. The rank component showed low positive correlations between prospective teachers' judgments and the empirically solution frequency. Furthermore, the results indicated high inter-individual variances in the teachers' judgments. No correlations were found between the individual components of judgment accuracy (between -0.001 and -0.180; averaged correlation is 0.000).

In view of the aim to identify task features that could influence the judgement accuracy of tasks, in the following, the varied task features of the eight word problems, the judgements of the prospective teachers and the empirical solution frequency of the 6th graders are examined in more detail with regard to each word problem.

The prospective teachers estimated the tasks mostly accurately, that are solved correctly by the students to a large extent (close to 50 % or more) (see table 1). This includes task 1, that is correctly solved by 89 % of the 6th graders, task 5, that is correctly solved by 42 % of the 6th graders, and task 7, that is correctly solved by 48 % of the 6th graders. Taking a closer look at the tasks, that are mostly judged accurately by the prospective teachers and are correctly solved by the 6th graders at a rate of almost 50 %, it is noticeable that task 1, 5 and 7 include only mathematical difficulty-generating task features. The lower the theoretical difficulty of the task, the more often the task is solved correctly. The theoretical task difficulty of task 1 is two, of task 5 it is five and of task 7 it is four.

	task difficulty		task features			
task	p. teachers	students	fraction	steps	lexicology	syntax
1	2	2 (89 %)	1	1	0	0
5	5	6 (42 %)	3	2	0	0
7	5	6 (48 %)	2	2	0	0

Table 1: Task difficulty of task 1, 5 and 7, judged by prospective teachers and derived from the solution frequency of the empirical survey, and task features

Task 2 and task 6 don’t fit into the previously recognized structure, although both include exclusively mathematical difficulty-generating task features and would therefore have to be assigned to table 1. The theoretical task difficulty of task 2 is four and of task 6 it is also four.

	task difficulty		task features			
task	p. teachers	students	fraction	steps	lexicology	syntax
2	5	4 (25 %)	2	2	0	0
6	6	4 (35 %)	2	2	0	0

Table 2: Task difficulty of task 2 and 6, judged by prospective teachers and derived from the solution frequency of the empirical survey, and task features

But the teachers’ judgements are not accurate and the empirical solution frequency is in the lower third. If we take a closer look at task 6, it is noticeable that this is not a classic fraction calculation task. The solution is already given in the word problem. The word problem was therefore less about mathematical calculation and more about understanding the text of the task. Analyzing the verbal protocols of the participants, it is noticeable that some participants noticed this and therefore classified it as easy and other participants classified it as difficult for 6th graders. Some participants did not recognize the given solution in the task and analyzed the mathematical calculation with regard to the difficulty for the 6th graders. No verbal protocols are available for the solutions of the 6th graders. But the solutions of the 6th graders showed that some pupils recognized and noted the solution in the text of the task, other pupils tried to solve the word problem by calculating.

Finally, we will take a closer look at those tasks that are mostly judged accurately by the prospective teachers with regard to the theoretical task difficulty, but that are not accompanied by the empirical solution frequency and thus the empirical difficulty of the tasks (see table 3). Task 3, 4 and 8 include mathematical difficulty-generating task features as well as semantic and linguistic difficulty-generating task-features. The theoretical task difficulty of task 3 is five, of task 4 it is also five and of task 8 it is six.

	task difficulty		task features			
task	p. teachers	students	fraction	steps	lexicology	syntax
3	5	8 (30 %)	2	1	1	1
4	5	9 (18 %)	2	2	1	0
8	6	9 (12 %)	2	2	1	1

Table 3: Task difficulty of task 3, 4 and 8, judged by prospective teachers and derived from the solution frequency of the empirical survey, and task features

DISCUSSION AND CONCLUSION

The present study investigated the judgement accuracy of task difficulty by prospective teachers. The results of the present study for the domain of word problems with fractions are consistent with the research findings reported in the literature presented.

Because these results were consistent with our assumptions, we investigated the task features with regard to the teachers' judgement accuracy and the empirical solution frequency. It is noticeable that in particular such tasks are accurately judged, that include only mathematical difficulty-generating task features and that are solved correctly by a large part of the students (task 1, 5 and 7). The theoretical difficulty, the teachers' judgement and the empirical solution frequency largely coincide for these three tasks. This is consistent with previous research showing that teachers can accurately judge those tasks in particular, that are easier to solve for students in particular (e.g., Urhane, & Wijnia, 2020). Tasks that contain linguistic difficulty-generating task features in addition, may be accurately judged by the prospective teachers with regard to the theoretical task difficulty (task 3, 4 and 8). However, the theoretical difficulty and the judgement do not concur with the empirical solution frequency. Students seem to find the linguistic difficulties more challenging than judged by the teachers. Two tasks were included in the test, where the empirical solution frequency largely correspond with the theoretical difficulty (task 2 and 6). However, it seems that it was difficult for teachers to judge these tasks accurately. The reason could be, for example in task 6, that the solution was already obtained and the task was, insofar as one recognized this as a student, very easy. This was sometimes not recognized by the teachers or was listed as a point of discussion.

Before discussing possible implication for international research on teachers' judgement accuracy, we would like to recall the limitations of this research, which suggest interpreting the evidence with care. First, it must be pointed out that prospective teacher may not yet be familiar with judging task difficulty for students. However, in order to exclude further influencing factors, such as experience, we first conducted the study with prospective teacher. An important further research approach would therefore be to replicate the results also through studies with in-service teachers. Furthermore, the results of this research report are based on only eight tasks, precisely word problems with fractions. It would be crucial to transfer the results to other content areas and task frameworks. Moreover, further research should complement these findings by means of different methodological approaches, especially quantitative data.

However, the findings of the present study provide a first, explorative insight into the influence of task features as influencing factors on teachers' judgment accuracy. Since the interindividual variances of teachers' judgments have still not been satisfactorily elucidated, despite over 40 years of research, the study offers a starting point for further investigation of the influence of task features on the teachers' judgement accuracy.

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"FRACTIONS MY WAY": HOW AN ADAPTIVE LEARNING ENVIRONMENT AFFECTS AND MOTIVATES STUDENTS

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An adaptive learning environment entitled "Fractions My Way" was developed and introduced to 206 fourth- and fifth-grade students who used it to study fractions over one academic year. Follow-up questionnaires and interviews (with students and teachers) revealed that the method enhanced their sense of ability, their responsibility toward their own learning, and their enjoyment in learning (leading to higher motivation). They claimed they learnt and understood the material better. Post-course assessment indicated an overall improvement in knowledge. Two drawbacks were mentioned: the stress associated with knowing the teacher was constantly monitoring performance and the sense of competition between peers.

INTRODUCTION

Attitudes to learning and how they affect student performance

Students' attitudes toward a subject can profoundly influence their achievements in that subjects (Awofala et al., 2014). This is especially true with mathematics given the widespread perception that mathematics is a difficult subject, the psychological fear often associated with it, and a profusion of poor teaching methods (Sezgin Memnum & Akkaya, 2012).

Socio-emotional learning refers to how learners regulate their emotions regarding thinking and self-management to achieve success in school (and in life) (Paolini, 2020). Recent research has shown that good socio-emotional skills lead students to better performance and can predict success in academia or careers (Gehlbach & Hough, 2018) and lead to consistent achievement in studies (Kanopka et al., 2020). As a result, many educational systems are looking for ways to improve students' socio-emotional learning.

The traditional teaching method does not always increase student motivation and achievement, especially when one teacher must stand in front of a heterogeneous class, relays the same material to all, tends to use the same teaching approach, and sets the pace of progress the same for all. Given the limits of human ability and time, it is difficult for that one teacher to take into account the personal needs of every student.

Multimedia can bring concepts to life through sight and sound. Students can receive instant feedback during their experience. A study examining the impact of multimedia showed that students exposed to multimedia during the learning process were more likely to be independent in their learning (Chipangura & Aldridge, 2019) and its use in

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mathematics can improve student engagement and motivation (Chapman & Wang, 2015).

Learning in the technological age: personalized teaching

There is currently an increase in the use of a digital tools and approaches based on artificial intelligence (AI) that can support the learner's experience based on personal needs, learning preferences, and abilities. Learning systems are becoming more developed and gaining momentum due to their ability to adapt to the individual needs and goals of students (Kabudi et al., 2021; Sampayo-Vargas et al., 2013). They are based on a model of the learning process from the students' point of view and analysis of data from previous users allows it to adapt itself and provide well-suited, high-quality learning material to the learner (Kurilovas et al., 2015). They simulate the knowledge and experience of a teacher to instantly judge situations and provide each student with personally tailored support or guidance (Xiao & Yi, 2020).

AI-enabled learning systems offer many advantages: an improved learning experience, flexibility time-wise and in managing the learning experience, and tailored progress (Hwang et al., 2020). They provide personalized automated support (Jadhav & Patil, 2021) and they offer content and questions, assign tasks, and provide clues, making learning smoother and more enjoyable. This will naturally lead to an improvement in the students' attitudes towards learning mathematics (Ma et al., 2014).

The adaptive learning environment: Fractions My Way

One such system is the Adaptive Learning Environment (ALE). This system creates a personal learning track adapted for the user, the aim being to monitor and maintain a state of maximum development. It can provide tasks based on each student's abilities and interests (Walkington, 2013). Research on such in the subject of mathematics indicates that it can greatly improve student performance (Cordova & Lepper, 1996).

The system is based on an computerized algorithmic “engine” model that uses statistical functions to adapt itself for each student and offer them a personal learning track appropriate to their knowledge, pace of learning, and abilities. In the case reported herein, an ALE entitled “Fractions My Way” (FMY) was developed in collaboration with Microsoft for the purpose of teaching fractions to fourth- and fifth-graders based on the requirements of the Ministry of Education.

Each student works on their own at a computer terminal, learning and practicing the subject matter through videos, exercises, enrichment tasks, quizzes, online assistance, and a digital “lab” for personal exploration. The engine determines the sequence of tasks needed to keep the pace of learning challenging and even what tasks can be “skipped.”

While the students are at the terminals, the teacher has access to a “dashboard” by which he or she can see a "performance breakdown" overview of the status of the class or each individual: the number of students working in each unit, their levels of success in real time, overall class progress, what tasks seem to be difficult for some, quiz scores, etc.

Studies show that the adaptive learning system can significantly improve learning efficiency and student performance (Chen et al. 2020; Xie et al., 2019), and that they are effective in adapting to the knowledge and learning needs of individual students, thereby developing higher-order thinking skills in ways that even the most skilled teachers cannot (Wang et al., 2020; Voskoglou & Salem, 2020).

Study's purpose

The report presented herein is part of a broad, ongoing study to track the development and implementation of an ALE for teaching primary school mathematics. It specifically explores the attitudes of fourth- and fifth-graders learning fractions via the FMY ALE.

METHOD

Research question: How do fourth- and fifth-graders perceive the contribution that working in an ALE makes for them with respect to their ability in and enjoyment for learning fractions and mathematics?

Study population: 51 fourth- and 106 fifth-graders studying in the FMY ALE. Students spent approximately 1.5 hours a week (60 hours total over the year) in the ALE. The study also included the class mathematics teachers of each class (20) and a supervising teacher (ST).

Research and data analysis

Stage one: Observations. The supervising teacher (ST) carried out a total of 48 observations in classrooms in which the ALE was integrated and recorded the conduct of the lesson and their impressions in a journal: teacher's decisions based on observation of the dashboard, sitting alongside students working independently, responding to questions raised during the lesson, etc. In parallel, the teachers also observed and recorded their views of the lesson (in their journals).

Stage two: Questionnaires. The observations from the teachers' and ST's journals were used to produce an online "attitudes" questionnaire to be answered by all the participants. It included 17 closed questions (satisfaction in working in the ALE, differences between learning in the ALE and the traditional classroom, responsibility for learning, quality of learning, sense of ability, etc.) and two open questions ("What do you like?" "What would you change/add?"). The questionnaire was embedded in the task sequence in the ALE.

Stage three: In-depth interviews. These were conducted with two groups of three students each. The interviewees were chosen based on their progress such that each group comprised one "very," one "moderately," and one "poorly" successful student. Some of the questions repeated those in the questionnaire to enhance understanding of the answers. The answers were recorded and transcribed for analysis.

Analysis: Systematic content analysis was used for the answers to the open questions (from questionnaires and interviews). Answers were divided into "units of meaning" and sorted into “themes.” The number of students who indicated each theme in their answers was noted. In addition, the ST’s and teachers’ journals were scanned for statements supporting each theme.

FINDINGS

Three major themes emerged: a sense of ability, pleasure in learning, and responsibility for learning. Table 1 shows the number of students who mentioned these themes in answer to the open questions/interviews or who indicated their agreement in the closed questions to a great or very great extent. Table 2 presents some representative statements that emerged from the ST’s journals, teachers’ reports, and student interviews for each theme.

Table 1. Percentage of students who agreed to a “great” or “very great” extent to closed question statements (n=157).

	Statement	%
Sense of ability	The FMW ALE allowed me to learn and understand more about fractions than textbook learning. I worked better.	73
	I understand fractions better this way.	69
	I do better learning with FMW than with a textbook.	69
Respon-si bility	It is important to work in FMW according to the guidelines received from the teacher	78
	It is very important for me to try to answer the exercises on my own when working with the computer.	78
	I needed help when solving the exercises on the computer.*	12
in learning	The animation in FMW is engaging and fun.	80
	The possibility of skipping questions on the computer makes learning more enjoyable.	62
	I enjoyed learning about fractions with FMW than with the textbook.	83
	Computer learning is simpler, easier, and more convenient.	80

* During the in-depth interviews with the students, they noted that the help given to them in the FMY ALE was varied, significant, important, and helpful.

Table 2: Statements given by the participants for each theme.

Theme	Statements
Sense of ability	**ST's Journal** • I feel it is important to note that as we progress in the material, the students need less and less mediation and intervention. • Students who had difficulty sitting in class were able to sit in front of the adaptive for a long time and progress. **Teacher's reports** • I was able to instill in the students the ability to follow instructions. The path of minimal mediation proves itself. • The system provides a solution for all students, both advanced and those with difficulty. Every student works at their own pace and it's great. **Student interviews** • It's fun. It teaches more about fractions than the teacher. • On the computer it is also more convenient because if you are in class and you do not listen then you have no other way, in the "Fractions My Way" you have a video and if you have to, you can go back and watch it again.
Responsibility	**ST's Journal** • Many students use "test," not to try but for understanding. • I feel like it affects learning, most children are not embarrassed to ask for help and that may be why I feel progress in this class. **Teacher's reports** • This system has given them some kind of peace and they manage to manage on their own in the system. • Using a computer makes learning independent. They move at their own pace. **Student interviews** • If there is a video at that point I go and watch it again and then there is also some hints at the side that help. • In class (traditional learning) there is the teacher. If I can't or don't understand something, I can ask her, but on this computer it's me and the computer and that's it. The teacher is not always available and I try even if I do not know and not do "reveal the answer" (a possibility that exists after two attempts).
Enjoyment in learning	**ST's Journal** • I was very happy to see that the children love the system very much, and they are in a different and independent experience. • Students who have difficulty sitting still in class were able to sit with the adaptive environment for a long time and progress. **Teacher's reports** • The visual stimuli facilitated learning and contributed to the students and applied to all the children in the class, no matter what their level. • When I had to postpone a lesson, one of the students expressed disappointment that the "most loved and fun" class had been postponed. **Student interviews** • I love having videos and it's fun when the ghost jumps out [the feedback] that says, "well done you've succeeded in the mission." • It's really good [to learn with the adaptive environment], because if the teacher lets you work on the page you can't go to the next page until you finish it and here it also takes you in stages.

Drawbacks. Two main drawbacks emerged. The first was that some students felt stressed knowing that the teacher was tracking their actions. The second was the sense of competition that emerged between students that were demonstrated during breaks or after school regarding how “quickly” they were progressing through the ALE. However, this could also have a positive impact: "Competing with a friend about who gets to a milestone first, helped motivate me to work harder.”

Although some students still felt the need for teacher support, the teachers claimed that this decreased as students became accustomed to studying in the system and became increasingly independent.

Overall, students reported that FMW ALE improved their understanding of fractions, made learning more enjoyable, gave them an increased sense of worth, improved their capabilities, and increased their sense of personal responsibility for their own learning. They continuously expressed that the method was “fun,” which led to anticipation for the FMW classes. The pass rates of students on the tests and their grades increased.

DISCUSSION AND CONCLUSIONS

Increased interest and motivation are the cornerstones of effective personalized teaching (Potvin & Hasni, 2014). The ALE instituted here contributed greatly to increasing motivation by stimulating students who have difficulty learning traditionally. This corroborates studies that found the use of multimedia for learning mathematics purposes improves student engagement, interest, and motivation (Chapman & Wang, 2015).

The “skipping” effected by the ALE proved to be another advantage. This was based on each student's personal learning data, enabling each to learn according to their specific abilities. Students experienced the “skips” as positive feedback regarding their abilities, which encouraged learning. In fact, they make an effort to get them.

Regarding the concern students felt regarding the consistent monitoring of their actions by the teachers, it might be prudent to explain to them that the information transmitted to the teacher assists the process. Regarding competition, it would be important to address the issue of social responsibility and explain how competition may adversely affect students who are not so proficient in the subject (Kanopka et al., 2020).

In conclusion most students perceived learning in the ALE as instructive, efficient, understandable, enjoyable, and motivating. It promoted personal responsibility while adapting to their needs and seems to be a positive addition to the curriculum.

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A CASE STUDY ON STUDENTS' APPROACH TO EUCLIDEAN PROOF IN THE RATIONALITY PERSPECTIVE

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A Rational Mathematical Template (RMT) is the couple consisting of a mathematical entity (definition, proof, etc.) and a rational (according to Habermas) process aimed at producing an instance of that entity. In this report we develop research on RMTs by a teaching experiment on the RMT of proof in two 10th grade classes. The design of the teaching experiment and the analysis of one student's productions were occasions to focus on the relationships between the rational process and its product and on the role of awareness as condition for the mediating role of RMTs in the classroom.

INTRODUCTION

In the last three decades, attention has been addressed in different disciplines to routines, particularly in the sciences of administration, organization and labor (see Feldman & Pentland, 2003). In Mathematics Education, Lavie, Steiner and Sfard (2019) move from "the thesis that repetition is the gist of learning" to consider routine "as the basic unit of analysis in the study of learning" (p. 153) and to the definition of task and procedure, and of routine as a "task-procedure pair": "a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task" (p. 161). They further elaborate the notion of discursive routines by distinguishing between 'process-oriented discursive routines' (called rituals), and 'product-oriented discursive routines' (called 'explorations'). They claim that discursive routines (guided by the question "How do I proceed?") are expected to undergo gradual de-ritualization until they become explorations (guided by the question "What is it that I want to get?").

In spite of the common interest for the invariant aspects of the activity in similar task situations, the discourse developed in Boero & Turiano (2020), Boero (2022) and in this paper on the RMT construct develops according to motifs that differentiate it from Lavie, Steiner and Sfard' construct. The original motif of the elaboration of our construct was to characterize, in an educational perspective, the components of the "intersubjectively shared lifeworld" in the ideal characterization of communicative rationality proposed by Habermas:

> Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 83-90). PME.

By interpreting Habermas' text in an educational perspective, our research problem was: what may allow students to share problems and solutions when moving, at the individual and collective level, from what they have already experienced to new challenges on a given subject, at the same time nurturing and developing their "intersubjectively shared lifeworld"? RMT, defined as a couple (mathematical entity; rational process aimed at producing one instance of the entity), was conceived as a possible solution for this problem.

Since the beginning, the RMTs were intended as dynamic-evolving objects of teaching and learning (with an ideal reference to the mathematical culture witnessed by the teacher) in order to meet two needs inherent in this expected individual and collective evolution: the need for common references in each phase of the classroom work, suitable to inform the individual and collective activities of production and reflection on the product; and the need for mediators between the students, the students and the teacher, and the students and the culture (see Boero & Turiano, 2020), in the perspective of progressive evolution of the mastery of the entities and of the related processes towards the learning goals of the teacher.

The initial elaboration on the RMTs needed and still needs further developments in order to become an effective tool for designing and analysing teaching in the perspective of rationality. A first contribution was offered by the analysis of the progressive construction (mediated by the teacher) of the RMT of definition (Boero & Turiano, 2020): the RMT tool worked as analytical tool to analyse the progressive evolution of the mastery of definitions by 8th-grade students through classroom discussions "orchestrated" by the teacher. In Boero (2022) the reported study concerns the RMT of counter-example. Focus is on the evolution of the three components of the rational process in two classroom discussions and their contribution to the development of students' rationality. Attention is paid to the conditions that allowed such construction: general and specific knowledge, and the already existing, positive relationships between students and with the teacher.

The case study reported in this paper had the ambitious aim to answer the following research questions: Is it possible to exploit the RMT of proof as a tool to design and analyse the progressive development in the classroom of the mastery of proof (as a mathematical entity) and of proving (as a rational process)? What about the aspects of the RMT of proof, which may allow it to play the role of mediator between the students, the students and the teacher, and the students and the culture on proof? And what about the aspects of the RMT of proof, which may allow it to play a cultural role in order to develop students' (and teacher's) well-being in the classroom?.

THEORETICAL FRAMEWORK

Habermas' construct of rationality

Habermas' construct of rationality (Habermas, 1998) concerns discursive practices that satisfy epistemic, teleological and communicative requirements: conscious checking of the truth of statements and the validity of reasonings according to shared criteria in a given cultural context (*epistemic rationality*); evaluation of strategies developed to attain the aim of the activity, in the perspective of possibly adopting them in similar, future circumstances (*teleological rationality*); choice of suitable communication tools to reach the others in a given social context (*communicative rationality*), the three components being strictly interconnected.

One salient aspect of Habermas' elaboration on rationality is the fact that the three components of rationality are described as ideal characteristics of discursive practices, while human behaviours are considered rational even in the case that they are only purposefully oriented towards that ideal horizon (see Boero & Planas, 2014). This remark looks important in order to adapt Habermas' elaboration on rationality in mathematics education for both analyzing and comparing rationalities inherent in the different domains of mathematics, and designing and analyzing students' and teachers' activities. In particular, since 2006 some researchers in our group and outside it tried to adapt Habermas' construct in mathematics teacher education (one of the studies is reported in Guala & Boero, 2017) and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014 for a general account and a presentation of five studies).

The Rational Mathematical Template of proof

RMT of proof is characterized by specific epistemic, teleological and communicative aspects: the process is aimed at producing a text with the specific logical and communicative requirements of proof, according to the different methods of proof (direct, by contradiction, by contraposition, by induction…). The process of proving may be considered "rational" when its different phases (exploration, construction of the reasoning, writing the proof text – not necessarily in this linear order) are consciously developed and evaluated according to the aim of the activity, attention being paid to epistemic and communicative requirements inherent in the product.

THE TEACHING EXPERIMENT

We will consider a teaching experiment on Euclidean proof, which involved two 10^{th} grade classes of scientific and technological oriented high school, with 19 and 25 students each. The activities were performed in the period November, 17, 2017 - May, 11, 2018, with two hours each week, for most of the school weeks in the period, for a total of 36 hours, in parallel with other activities on algebra, analytic geometry and probability. The activities were preceded (in grade IX, with the same teacher, and at the beginning of grade X) by some preliminary activities in plane geometry on the nature of definitions, and on some statements of theorems already met by students in comprehensive school, with a few easy proofs utilizing them. We will focus on a situation of conjecturing and proving (and related activities) and on the productions of a student that we will name Mario, which took place at the beginning of March, 2018. We have chosen Mario's productions due to the fact that Mario was one of the students who moved from a low level of performances at the beginning of the sequence (and in Mathematics in general), to an over the average level at the end.

The general design of the sequence of activities on the approach to Euclidean proof in Geometry took into account the fact that geometric constructions (with related theoretical justifications) and theorems alternate in Euclid's Elements. This choice allowed a smooth approach to generality and precision of the discourse on geometric figures (through comparisons of construction texts produced by students for "construction tasks") and to proving (through theoretical justifications of constructions). Students' acquired familiarity with geometric constructions allowed them to produce suitable geometric drawings for conjecturing and for proving tasks.

The classroom activities (a couple of tasks for each two hours) included, from the beginning, tasks of individual geometric construction, with related verbal description. They concerned the bisector of a given angle (with related theoretical justification), a circle tangent to two assigned straight lines, a circle of given radius tangent to two intersecting straight lines, the circles inscribed in, and circumscribed to, a given triangle. Each of them was followed by oral (through a classroom discussion) or written individual revision of constructions produced by some schoolfellows and selected by the teacher. Revisions included checking the generality of the construction and the identification of lacking details and erroneous verbal expressions. Tasks of theoretical, written individual justifications of the construction (based on known statements) were proposed for each construction. They were followed by individual comparison and/or individual revision and/or classroom discussion of theoretical justifications produced by some schoolmates. Concerning theorems, conjecturing and proving activities related to geometric figures, and then proving activities of statements proposed by the teacher, started at the beginning of March, 2018 (Mario's proof text reported below concerns the first activity of this kind). Like for the other activities, systematic individual and/or classroom revisions, comparisons, discussions of proof texts followed each individual proving activity, attention being paid to the key elements of the produced statements and proofs (particularly as concerns the expression of the hypothesis and the thesis, and the necessity of a complete and not redundant proof text). Other activities were proposed, starting from January, 2018: individual cloze activities (followed by a classroom discussion) to complete a theoretical justification of a construction, which was provided by the teacher, by choosing the kind of justification of some steps (by construction; by hypothesis; by definition of…; by theorem…); identification of the proof strategy in the proof text of a schoolmate, with search for possible lacks and mistakes and of theorems and definitions needed to get the proof according to that strategy. The alternation of individual productions (or revisions) and classroom comparisons and discussions was aimed at implementing the RMT of proof as a mediator between the students, the students and the teacher, and the students and the culture (see Boero & Turiano, 2020, p. 145).

Mario’s texts and their analysis

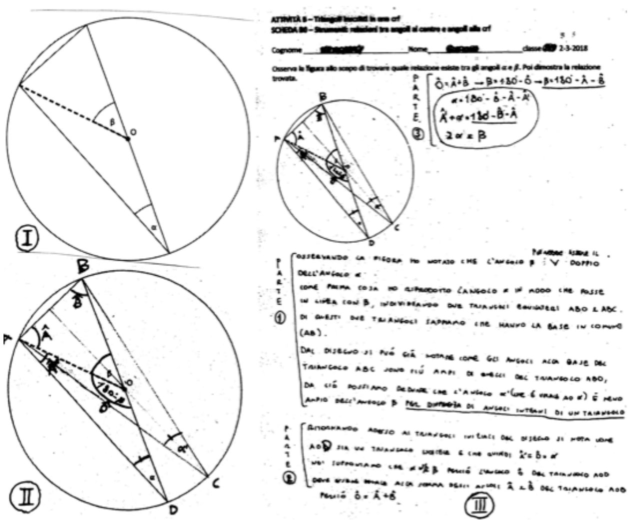

Fig 1: Original figure (I); Mario’s figure (II) and text (III)

(PART 1) By observing the figure, I noticed that angle β might be the double of angle α. As first thing, I reproduced the angle α in such a way that it was aligned with β, by finding two equilateral triangles ABO and ABC. These two triangles have their base in common (AB). From the drawing, we may already notice how the angles adjacent to the base of the triangle ABC are wider than those of the triangle ABO, from which we may deduce that the angle α' (that is equal to α) is less wide that the angle β by difference of internal angles of a triangle.

(PART 2) Now, by coming back to the initial triangles of the figure, we notice how AOD is an isosceles triangle and then $\hat{A}'=\widehat{D}=\alpha$. We suppose that $\alpha=½\beta$ thus the angle $\hat{O}$ of the triangle AOD must be equal to the sum of the angles $\hat{A}$ and $\widehat{B}$ of the triangle AOB, hence $\hat{O}=\hat{A}+\widehat{B}$.

(PART 3)

$\hat{O}=\hat{A}+\widehat{B} \rightarrow \beta=180°-\hat{O} \rightarrow \underline{\beta=180°-\hat{A}-\widehat{B}}$

$$\alpha=180°-\widehat{B}-\hat{A}-\hat{A}'$$
$$\hat{A}'+\alpha=\underline{180°-\widehat{B}-\hat{A}}$$
$$2\alpha=\beta$$

From the teleological point of view, Mario looks aware of the different phases of his conjecturing and proving process (the spatial organization of the text and their labels PARTE 1, PARTE 2, PARTE 3 shows three distinguished steps; within the third step Mario puts the core of the proof into evidence, like in the above quote). Moreover, also his revision of his proof text confirms a high level of awareness:

> In this revision I realized that this worksheet well represents my way of reasoning. A gradual reasoning in which, first, I observe the figure and I notice some possible conjectures, then I try to develop the first thoughts, like that of aligning the triangles ABO and ABC. Thanks to this idea I succeeded in finding the basis of my reasoning (…).

In the following analysis of Mario’s text some weaknesses on the epistemic and communicative ground will be put into evidence by the use of *italic*.

Mario moves from an initial, possible conjecture ("the angle β might be the double of the angle α"; the initial writing was "the angle β is the double of the angle α") to an exploration of the situation. We may notice a *communication mistake* ("equilateral triangles" instead of "isosceles triangles") and the *lack of justification* of *isosceles triangles*. Then Mario exploits the familiarity with geometric constructions to get a suitable figure, and finally he gets the justification of a weaker statement ($\alpha < \beta$) through visual evidence, a theoretical justification ("by difference of internal angles of a triangle") *implicitly* based on the theorem that the sum of the internal angles is the same for any triangle, and an *unjustified claim* ($\alpha' = \alpha$).

In the second part of his reasoning, by exploring the original figure of the worksheet, Mario notices that the angle α is equal to the angle $\hat{A}'$ (by a theoretical, explicit reason related to the fact that the triangle AOD is *isosceles*; however, the *theoretical justification of it is lacking* – only visual evidence is put on the fore. At that point he foresees how to get the proof: he comes back to the initial possible conjecture, that now is expressed as a *hypothesis to derive what follows*, but probably plays the role of a *hypothesis* to be verified, which results in an *abduction*. This is the starting point of a piece of text of difficult interpretation (at the end of part 2 and at the beginning of part 3), in particular it is not clear the meaning of the two arrows. Mario seems to feel the need to work on the angle $\hat{O}$ of the triangle AOD, which *must be* equal to the sum of the angles $\hat{A}$ and $\hat{B}$ in order to find some relationships that are needed to get the proof. It is clear that Mario works on already considered properties of the triangles (the sum of the internal angles, and the congruence of the angles of isosceles triangles) but *explicit justifications are lacking*. This phase seems to play a heuristic role to get the underlined formula: $\underline{\beta = 180° - \hat{A} - \hat{B}}$. At that point Mario starts a sequence of algebraic expressions that bring to the conclusion. From the surrounding line it is clear that Mario considers what is inside as the proof. The *lack of verbal comments and of some intermediate algebraic expressions* (e.g. the recall of $\underline{\beta = 180° - \hat{A} - \hat{B}}$ and of $\hat{A}' = \alpha$) do not prevent the reader from interpreting Mario's reasoning, also thanks to the spatial disposition of the lines.

Mario's text represents an intermediate step in his approach to the RMT of proof; in the classroom, it looks as a (relatively) high level performance, as concerns the mastery of the whole process (from exploration to proof construction), in comparison with most of his mates' productions. However, Mario's text also reveals some weaknesses (which were rather common in the classroom, at that stage of the construction of the RMT of proof), as we have put into evidence in the above analysis. For all these reasons, Mario's text has been proposed to the class as an object of an individual revision task: "Why this proof has been considered in a positive way by the teacher, in spite of lacks and mistakes in part 2 and part 3? How to correct and improve it?", in the perspective of a discussion to share and discuss what students had discovered, and thus to focus on crucial aspects of the proving process and the proof text. The activity helped Mario to identify an important mistake. In his revision he writes:

> Thanks to comparisons with my schoolmates I realized that at the end of Part 2 and at the beginning of Part 3 my reasoning starts with $\beta = 2\alpha$, which is not a hypothesis but the thesis to be proven, while the hypothesis is that the triangle in inscribed in a circle with one side as a diameter (…).

In another individual activity on the same conjecturing and proving task, students were required to correct, complete and re-write the proofs after identifying and maintaining the authors' reasoning, and to put hypotheses, thesis, and theorems and definitions into evidence. This excerpt from Mario's text under this task well represents the high level of awareness already developed by a consistent number of students (about one half of them), and (as the previous excerpt) the climate created through the need of analysing and improving the schoolmates' texts according to the shared rationality criteria:

> In the first solution I realized that there was an unusual reasoning, different from those we had considered in the discussion, but it is correct. However, some points should be improved: the fact that $\alpha = \partial$ does not result from the definition of an isosceles triangle, but from a theorem. There is an important lacking point: the proof that BE is parallel to DO. It is needed to use the theorems on alternate angles.

CONCLUSIONS AND DISCUSSION

Through the description of the sequence of activities and the analysis of Mario's productions we have tried to put into evidence how the use of the RMT of proof may serve the planning and the analysis of classroom activities aimed at student's approach to proving and proof in grade X. As an analytical tool related to Habermas' rationality, the RMT of proof was used to identify weak points of Mario's proof text. They needed (and allowed) interventions (through revision tasks and related discussions) to develop awareness, in particular, of crucial epistemic and communicative aspects of proof.

We may observe how in the planning of the teaching experiment awareness (of the requirements of the product of the process and of the organization of the process) played a crucial role through several specific tasks; this looks necessary to ensure the rationality of the process and the epistemic and communicative quality of its product. The analysis of Mario's productions shows how the role of awareness in the planning of the teaching experiment results directly in the mastery of his personal process and in the revision of the epistemic aspects of his schoolmate's proof text, and indirectly in the climate of the work in the classroom, through the acknowledgment of the contributions of his schoolmates to overcome an important weakness in his text, and the mature, constructive approach to his schoolmate's production. Mario's productions are representative examples of what happened in the two classrooms during the teaching experiment. In particular, the systematic work on students' awareness of the requirements of rationality through the revision, cloze and identification tasks seems to have a supportive, double function on the cultural ground: for the development of a collaborative style of work in the classroom (thus contributing to the well-being of all the involved people), and to ensure the role of mediation that the RMT of proof plays in the long term development of students' proving. This double function looks to be not limited to the case of the RMT of proof and should be elaborated in general, thanks to other teaching experiments on complex, demanding RMTs (like that of analytical model of physical phenomena, or that of probabilistic model of stochastic phenomena).

From the theoretical point of view, the content of the previous Section puts into evidence the distance between the RMT construct and the construct by Lavie, Steiner & Sfard, 2019 beyond what concerns the motifs of the constructs (see Introduction). In particular, in the classroom long term construction of the RMT of proof that we have described it is not possible to distinguish a ritual phase from an exploration phase. Indeed, for intrinsic reasons due to the necessity of developing awareness (a crucial requirement of Habermas' rationality), since the very beginning students are engaged in both productive and reflective activities on accessible tasks, which gradually evolve trough a conscious mastery of more and more complex situations.

However, the definition of RMT still needs an in-depth work, if we want to move from an extensive definition (i. e. a definition concerning a set of assigned "entities", with specifications for the components of the rational process aimed at the production of "instances" of those individual "entities"), to an intensive definition (i.e. a definition based on a characteristic, common property of the "mathematical entities", with specification of the general aspects of the rational process that result from the entity).

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DESIGNING PROBLEMS INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION IN AN INQURY-BASED SETTING

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Research literature argues for the benefits of inquiry-based approaches to provide opportunities for in-depth understanding of mathematics. This paper studies the design of mathematical problems for the purpose of introducing the concept of numerical integration in an inquiry-based setting. We present a series of six developmental stages (represent, refocus, area, accumulate, approximate, and refine) indicating a natural trajectory for students to follow when inquiring on the concept of numerical integration before any formal introduction to the topic. Further, we present a sequence of three problems illustrating how the developmental stages can be applied in problem design.

INTRODUCTION

The work presented in this paper concerns the design of problems in an inquiry-based setting in calculus, and is part of a larger research project focusing on how inquiry approaches in calculus in a Norwegian continuing professional development (CPD) program can support teachers', students' and teacher educators' development of mathematical competencies and inquiring habits of mind. In addition to the mastering of procedural skills, it is important to provide calculus students opportunities to develop deep understanding of context and connections between concepts (Hall, 2010; Sofronas et al., 2011). Mathematics education literature strongly speaks in favour of inquiry-based approaches for in-depth mathematical learning and critical application of knowledge, and such approaches have been promoted in educational policy and mathematics curriculum documents across the world (Artigue & Blomhøj, 2013; Dorier & Maass, 2014). Inquiry nurtures the critical, creative, and reflective mind, and encourages students to engage in mathematics in ways that mathematicians do (Artigue & Blomhøj, 2013; Dorier & Maass, 2014) by using mathematical key ideas to wonder, explore, discuss, justify, interpret, and collaborate with others on mathematical problems. Hence, knowledge on how problems given to students are designed, is crucial for facilitating inquiry-based learning (Artigue & Blomhøj, 2013; Cai, 2010), and is the focus of this paper.

One important element to consider in problem design is how central mathematical properties must be understood and used to solve the problem (Lithner, 2017), and the mathematical topic of this paper is numerical integration. Students' understanding of integration should be given special attention, as the topic is central in calculus and has a broad area of use in the real world (Jones, 2013; Sofronas et al., 2011). Calculus students are expected to make sense of integrals as limits of Riemann sums, develop

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and implement numerical algorithms to calculate integrals, interpret the meaning of integrals in different situations, and use integration to solve problems. By informally approaching the concept of numerical integration through inquiry, students are provided opportunities to develop an in-depth understanding of the topic. This paper asks the following question: *How can problems be designed for the purpose of introducing the concept of numerical integration in an inquiry-based setting*?

INQUIRY-BASED PROBLEMS IN MATHEMATICS

Inquiry-based problems encourage exploration, discussion, the posing of questions, and evaluation. Inquiry is built on the idea of exploring something unknown or challenging, but requires that this can be approached through building on existing knowledge (Artigue & Blomhøj, 2013). Inquiry-based problems must therefore have a delicate balance between creating challenges for the students and enabling them to make sense of the challenges by accumulating their knowledge. A skewness towards the known can contribute to degrade the problem to uncritical rote learning and repetition of known procedures and algorithms. Lithner (2017) suggests not providing pre-decided strategies or procedures in the problem text. The argument is that if a solution method is given, or already known by the students, they often uncritically apply it. Similar arguments given by Schoenfeld (1985) and Cai (2010) suggest that reducing the number of specific algorithms and techniques increases the potential for exploration. On the other side, too many unknown elements may hinder new learning and meaning making as the students do not have the prerequisite knowledge to approach the problem. The importance of providing students the opportunity to solve the problems and justify their solutions through building on what they know, is emphasized in the literature (Artigue & Blomhøj, 2013; Lithner, 2017; Schoenfeld, 1985).

Mathematical inquiry is therefore closely linked to problem solving, emphasising multiple solutions (Cai, 2010). Such problems invite students to develop an ownership to their solution methods (Cai, 2010; Schoenfeld, 2012). They should facilitate reflection beyond concrete situations (Goldin, 2010; Schoenfeld, 2012), for example on what happens if we change some problem criteria or introduce higher number situations. Such reflection can be approached through designing “the most elementary, generic example” (Goldin, 2010, p. 248) or through sequences of problems with increasing complexity (Schoenfeld, 2012). From an inquiry perspective, these approaches enable the students to accumulate their knowledge (Artigue & Blomhøj, 2013).

INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION

Introducing integration in a way that helps students develop in-depth understanding of the concept, is challenging (Orton, 1983). Research suggests emphasizing a variety of representations and interpretations and the connections between them (Hall, 2010; Orton, 1983; Sofronas et al., 2011) to facilitate deeper understanding of the topic.

Using different representations might also help students trace their own solution processes and ideas when working with mathematical problems (Goldin, 2010).

Orton (1983) suggests introducing integration through active use of visual *representations*. For a real valued function of a single variable, the visual representation of an integral is the area between the function and the x-axis in a given interval. Thus, *moving focus* away from the function itself, and to the area under the graph as well as the interpretation of this area, can be considered a key "discovery" when approaching numerical integration through inquiring on visual representations. Calculating this *area* is a rather simple process if the function is linear, and can be considered what Goldin (2010) calls the most elementary example. If the function is a curve or a piecewise function, one might have to split the area into smaller subareas and sum up these areas – introducing *accumulation* to the process and providing the increasing complexity that Schoenfeld (2012) suggests. Approaching integration as an area under a curve or as accumulation of "bits" (Jones, 2013; Sofronas et al., 2011) requires an understanding of covariation between x and $f(x)$, the ability to imagine or visualize the "bits", and an understanding of why an area can give information on another quantity (Thompson & Silverman, 2008). Such understanding, and moreover understanding the connection between area and accumulation, can build deep understanding of the concept of numerical integration and for critical application of this understanding.

To encourage reflection beyond one concrete situation (cf. Goldin, 2010; Schoenfeld, 2012), problems for introducing the concept of numerical integration should stimulate wondering on how to approach situations where the areas cannot be calculated exactly (for example if there are no "simple" geometrical shapes that can be used or if there are too many different shapes). This introduces the idea of *approximation*. Combining this with inquiry on higher number situations (Goldin, 2010), increasing the number of accumulations, can stimulate reflection on how the interval breadth affect the accuracy and efficiency of the approximation, *refining* the approach.

DESIGNING PROBLEMS FOR INTRODUCING THE CONCEPT OF NUMERICAL INTEGRATION IN AN INQUIRY-BASED SETTING

When the aim is to introduce a new concept, the problem does not have to be very difficult (Lithner, 2017). One idea is to create a sequence of problems with increasing complexity (Schoenfeld, 2012) to balance what is known with what is unknown. This approach helps the students develop ways to approach general "find the area under the graph"-problems, reflect on when these approaches become inefficient, and propose ways to tackle this new obstacle. The goal of the problem should be to facilitate students' construction of some aspects of the concept (Lithner, 2017) and reflection beyond one concrete example (Goldin, 2010; Schoenfeld, 2012). This calls for problems that separate from strict prescriptions and encourage informal discoveries through inquiry.

Based on the presented theories, and empirical observations from introducing numerical integration in a CPD calculus course, we suggest that the following developmental stages should be emphasized, in this order, when designing problems for an inquiry-based introduction to the concept of numerical integration:

Stage	Description
Represent	Actively using and combining representations of the problem. This includes extracting information from the problem, translating from text to graphical, geometrical, or algebraic representations and moving between representations.
Refocus	Move focus from the function to the geometrical shape between the function and the x-axis in a given interval.
Area	Discovering that the area of the geometrical shape can be calculated to solve the problem. Inquiring on *why* it is interesting to calculate the area.
Accumulate	Finding ways to calculate the area through dividing it into subareas and accumulating the areas of these “bits”.
Approximate	Reflecting on how to tackle the problem if the area cannot be calculated exactly. Inquiring on effective ways to approximate the area.
Refine	Reflecting on how to improve the approximation.

Table 1: Developmental stages for an inquiry-based introduction to the concept of numerical integration.

These developmental stages follow a natural progression, balancing the known and unknown, from expected knowledge (drawing a graph, understand what a function is) to the mathematical objective (develop an understanding of the concept of numerical integration through accumulation of areas of repeated geometrical shapes with small interval breadth, and inquire on why this can provide useful information on a quantity other than area).

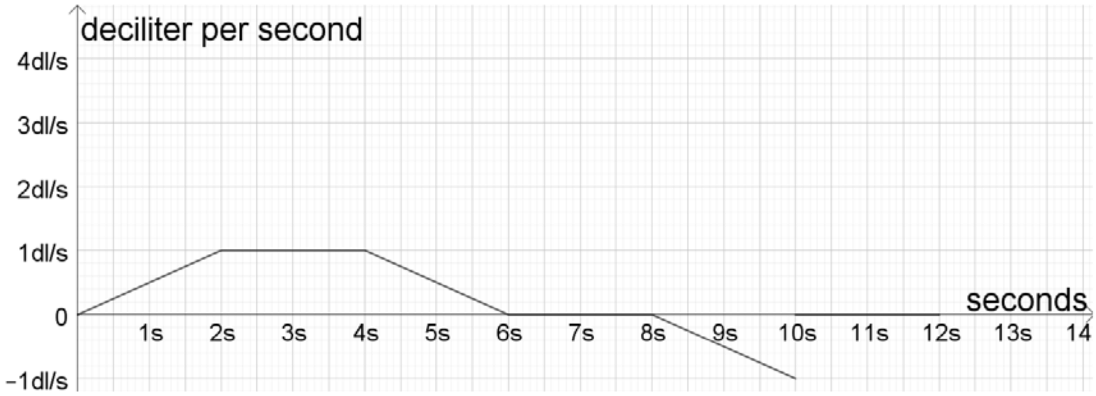
EXAMPLE: A SERIES OF THREE PROBLEMS

The following sequence of three problems for introducing the concept of numerical integration is designed based on the developmental stages and inquiry-based criteria presented above. Naalsund, Bråtalien, & Skogholt (2022) present a short transcript and analysis based on one student group’s collaboration when working with problem 2.

Problem 1 (A car trip)
A family is on a car trip driving with a constant velocity of 50 km/h.

- Choose some time between 0 and 5 hours. Calculate the distance travelled by the family at this time.
- Plot the velocity as a function of time for t between 0 and 5 hours. Discuss if you could have used this graph to find the distance you calculated above.

Problem 2 (Filling a bottle with water)
Anne wants to fill an empty bottle with exactly 3 dl of water. The figure below shows how much water is entering or leaving the bottle at time t. Discuss if Anne is successful.



Problem 3 (Pedal to the metal)
Kåre is driving a car and steps hard on the gas pedal at the time *t=0*. The acceleration of the car is shown as a function of time in the graph below. Estimate the velocity Kåre is driving at after 4 seconds. Write done the assumptions you make if any. Discuss how your estimate can be improved.

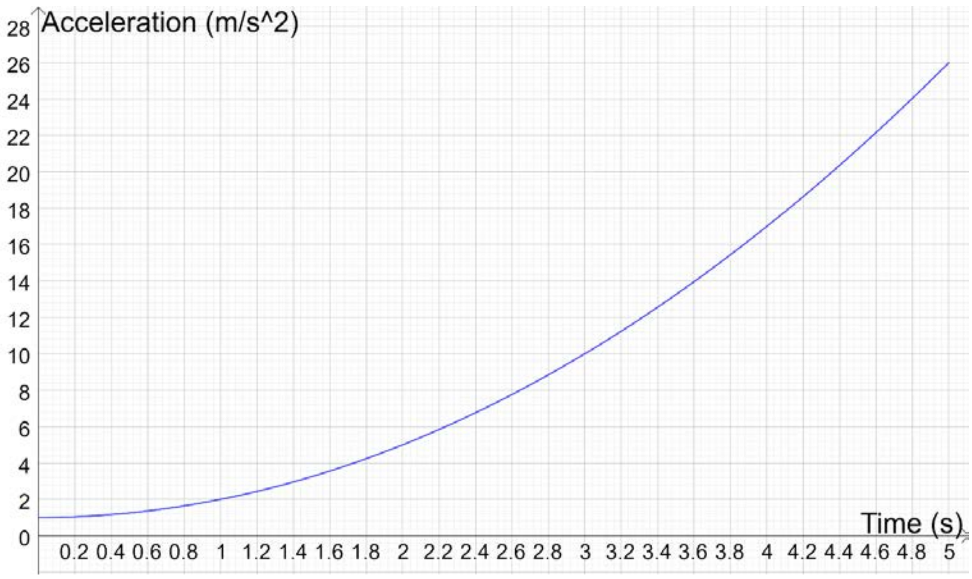


Figure 1: The sequence of problems.

Understanding the concept of integration consists of a network of smaller units (Hall, 2010; Jones, 2013) such as, but not limited to, ideas of area, limits, functions, algebraic operations, geometry, accumulation, and covariation. Problem 1 asks the students to calculate a distance using both algebraic and graphical representations. Asking the students to consider if the distance could be calculated from a velocity-time graph encourages them to combine graphical, algebraic (distance = time · velocity), and geometrical (area = length · breadth) representations to refocus from the graph itself and discover that the area under a graph can have a physical interpretation and hence be used to solve problems involving other quantities than area. This process includes the three first developmental stages of represent, refocus, and area. As the area of interest is a rectangle, the problem can be considered an example of the most elementary example (Goldin 2010).

Problem 2 also encourages the students to refocus and consider the area and its physical interpretation. The graph is piecewise linear, which naturally probes the students to split the area into some combination of triangles, rectangles, and trapezoids. This adds the developmental stage accumulate to the students' inquiry. The combination of positive and negative areas, as well as the graph having segments with increasing and decreasing (but positive) rates of change encourage reflection on the physical interpretation of both the graph and the area.

Problem 3 introduces the developmental stages approximate and refine. The function was chosen so that the area could not be computed exactly, and hence stimulating reflection on approximations to the area. In contrast with problem 2, problem 3 omits any mention of an initial condition to prompt a discussion of the interpretation of the definite integrals as the net change in the quantity considered. A variety of approaches are possible (Cai, 2010), and in our experience the students will consider approximations similar to the trapezoidal rule, the midpoint rule, as well as lower and upper sums with rectangles, even if none of these formal approaches to numerical integration are mentioned or have been formally introduced. The problem therefore invites the students to develop an ownership to their solution methods (Cai, 2010; Schoenfeld, 2012) and a sound foundation for learning about formal approaches. By also asking the students to discuss how their solution can be improved, the problem includes reflection beyond the concrete example (Goldin, 2010; Schoenfeld, 2012), i.e., on higher number cases (Goldin, 2010), encouraging the students to reflect on their solution and introducing the developmental stage refine. The students might discover that it is in principle simple to refine the approximation (by subdividing the x-axis into smaller intervals), but that the computation of the areas becomes an issue. This can be used as motivation for introducing formal integration procedures and even programming.

The three problems presented in this paper include few prescriptions, increasing the potential for exploration (Cai, 2010; Schoenfeld, 1985), but they also provide necessary support for the students to follow the trajectory of the developmental stages presented. Opening the problem by not providing a prescription makes the solution method itself a part of the unknown, engaging the student to wonder, explore and reflect in the process of constructing both strategies and solutions. Such inquiry will include several ideas being what Artigue & Blomhøj (2013) refer to as unknowns to the students. Students might struggle understanding integration as accumulation (Orton, 1983; Thompson & Silverman, 2008), as deep understanding of this requires the ability to visualize the “bits” that should be accumulated and a complex understanding of the area as representing something other than an area (Thompson & Silverman, 2008). To help students approach these unknowns, the problems use units on the axes and rates of change that are considered to be known for the students.

In this paper, we have presented and argued for the benefits of a progression through six developmental stages when designing problems for introducing the concept of numerical integration in an inquiry-based setting. We argue that an informal approach such as inquiry, holding back formal notation, symbols, prescriptions, and methods, allows the students to discover these stages themselves and reflect on obstacles they meet in their inquiry. The three problems we have presented follow the developmental stages, and these problems together with questioning, exploration, discussion, and evaluation that inquiry entails, can provide opportunities for connecting representations and interpretations (Hall, 2010; Orton, 1983; Sofronas et al., 2011), and reflection on the central questions of how and why an accumulation of areas can be used to represent another quantity than an area (Thompson & Silverman, 2010). This can prepare the students for formal methods of numerical integration, such as Riemann sums, where the idea of area and accumulation is combined.

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TURNING MOMENTS: THE CROSSROADS OF THE PROSPECTIVE SECONDARY TEACHERS' ATTITUDE TOWARDS MATH

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Relationship with mathematics strongly affects teachers' practices. Specialist mathematics teachers' positive relationship is often taken for granted, although recent studies suggest that this may not be the case. In this paper we present a narrative research aimed at investigating which events do prospective secondary teachers recognise as crucial in the development of their relationship with mathematics. The analysis reveals that experiences of success/failure in mathematics, teacher's opinion, teacher's charisma, and experiences of helping someone with maths are frequent factors influencing these events, impacting on prospective teachers' emotions, perceived competences, and view of mathematics.

INTRODUCTION AND THEORETICAL FRAMEWORK

According to Nias (1996, p. 293) "affectivity is of fundamental importance in teaching and to teachers". Several studies discussed how teachers' relationship with mathematics (in terms of beliefs, identity, emotions, and attitudes towards mathematics) can strongly affect their decisions and their teaching style (De Simone, 2014). How Zembylas (2005) underlines:

> Teacher knowledge is located in 'the lived lives of teachers, in the values, beliefs, and deep convictions enacted in practice […].' These values, beliefs and *emotions* come into play as teachers make decisions, act and reflect on the different purposes, methods and meanings of teaching (p. 467).

Therefore, it appears to be particularly significant to analyse prospective teachers' relationship with mathematics and its development during the school experience. This knowledge is crucial to understanding if and how teacher education programs should be designed to positively affect this relationship. However, research on teacher affectivity in mathematics education has been mainly focused on prospective and in-service primary teachers (Martinez Sierra et al., 2021). These studies have shown that many of them have developed a very bad relationship with mathematics during their school experience (Hannula et al., 2007). For this reason, it appears significant to fill this gap by considering an affective perspective also in the professional development of prospective secondary school mathematics teachers (PT). Unlike future primary teachers, many secondary teachers have a degree in Mathematics: they are therefore specialist mathematics teachers and their passion for mathematics is, in a certain sense, taken for granted. On the other hand, as shown by recent studies, several

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brilliant students live "crisis moments" during their university experience in a mathematics degree, also developing strong negative feelings towards mathematics (Di Martino & Gregorio, 2019).

Referring to the Three-dimensional Model for Attitude (TMA, see Fig. 1) introduced by Di Martino and Zan (2010) and within a larger study, we conducted a narrative study among PTs to identify crucial events in the development of their relationship with mathematics.

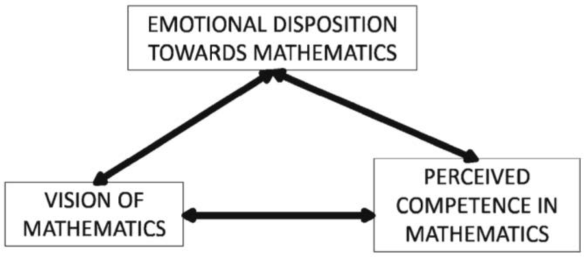

Figure 1: The Three-dimensional Model of Attitude (Di Martino & Zan, 2010).

We were inspired by Bruner's conceptualisation of the idea of *turning points* (1991) in autobiographical accounts. According to Bruner, the marking of a turning point is a narrator's device to signal a rupture inside a habitual and expectable routine. Turning points consist in an inner transformation of the narrator, a change in intentional states, linked to a particular external event or experience:

> By "turning points" I mean those episodes in which, as if to underline the power of the agent's intentional states, the narrator attributes a crucial change or stance in the protagonist's story to a belief, a conviction, a thought (Bruner, 1991, p. 73).

Turning point narratives are characterised by mental verbs indicating internal transformations of which the narrator expresses awareness. We will call *turning moment* an episode, a school period, or a particular experience which PT considers to be determinant in her personal relationship with mathematics and its teaching, since it involved a particular internal change.

Our study was guided by the following research question: *which events do PTs recognise as crucial in the development of their relationship with mathematics?*

METHODOLOGY

The sample and the data collection

The sample of our study consists of 62 students attending the course of Mathematics Education for the master's degree in Mathematics in an Italian university.

In line with well-established methods of narrative data collection (Kaasila, 2007), students were asked to answer the following open prompt in the first session of the course: *Tell us about an episode from your scholastic past career (at any school level, from primary to university) that you consider particularly significant for the development of your relationship with mathematics. Based on what you remember,* *include in your story the details of the situation you experienced, the emotions you felt and finally explain why you consider the episode significant.*

PTs could use as much time as they wanted within the two-hour session and their productions were collected in an anonymous way. According to Connelly and Clandinin (1990):

> Humans are storytelling organisms who, individually and socially, lead storied lives. The study of narrative, therefore, is the study of the ways humans experience the world [...] teachers and learners are storytellers and characters in their own and other's stories (p. 2).

In this frame, how (form) and why (reason) the narrator describes her experiences matter more than a present objectivity of the narrated facts.

Approach to the data

To analyse the collected narratives, we referred to two main independent dimensions, *holistic vs categorical* and *content vs form*:

> The first dimension refers to the unit of analysis, whether an utterance or section abstracted from a complete text or the narrative as a whole. [...] The second dimension, that is, the distinction between the content and form of a story, refers to the traditional dichotomy made in literary reading of texts. (Lieblich et al., 1998, p. 12).

A purely categorical or holistic approach is not practically possible, whereas the combination of the different dimensions allows for a deeper and differentiated understanding of the narratives. In particular, we developed a holistic approach to the data to identify narratives containing events considered crucial by the narrator for her relationship with mathematics and its teaching. Regarding the content/form dimension, the content was the main focus of our analysis to recognize turning moments in the crucial events that participants reported; however, the recurrence of some expressions was considered. After this analysis, we identified recurrent themes characterising the turning moments through a categorical approach, discussing which dimensions of the TMA were involved. This process of analysis was conducted and finally discussed using an investigator triangulation method.

RESULTS

Five PTs did not respond to the given prompt. Four PTs described a stable relationship with mathematics during the school experience. These narratives do not include any event recognized as crucial for the development of the relationship with math, although, in one case, fluctuating emotions are reported (PT11: "over the years it has been a relationship of *odi et amo*"). Thus, the final corpus was composed of 53 narratives of events. Within them, we recognized two types of events:

- Single *episodes* identified as crucial for the relationship with mathematics.
- *Periods* perceived as crucial for the relationship with mathematics. Periods can be short-lived, such as the preparation for the high school diploma, of

medium duration, such as the encounter with a particular teacher, or of longer duration, such as the experience during a whole school level.

The corpus contains a total of 68 narrated events: 39 episodes and 29 periods (some PTs reported more than one event). For both types of events, we determined the school levels in which they occurred (Table 1). It is conceivable that, given the prompt, all the 53 selected narratives include episodes or periods that are crucial for the narrator. However, in this paper, we discuss more in depth those narratives in which the narrator explicitly identifies an episode or a period as a *turning moment* for her relationship with mathematics. 39 turning moments were selected according to the above criteria.

Account type	Primary	Middle	High	University
Episodes	8	3	22	6
Periods	1	4	19	5
Turning moments	5	4	19	11

Table 1: School levels of the accounts, according to the different typologies.

From the point of view of *form*, to turning moments group belong narratives of events in which *mental verbs* appear (Bruner, 1991), indicating the narrator's access to new consciousness, such as, for example, the *emergence of an intention for the future* (PT53: "So once I overcame the initial difficulties due to the new approach to the discipline, I understood that mathematics would be present in my future"), the *achievement of a certain view of mathematics* (PT45: "This made me understand something important: you can always find a solution"), or a *new awareness in the perceived competences*, in mathematics or in its teaching (PT8: "this made me think that maybe I have the gift of a good teacher). In addition, narratives in which there are expressions that indicate a *beginning*, in the context of the relationship with mathematics, also fall into this category: as an example, the adoption of a different method of study (PT49: "Since then I have changed, I started to apply myself more to mathematics") or a new emotional disposition (PT18: "it was from that moment that I started to have fun").

Analysing the *content* of the turning moments' narratives, four main themes emerge as factors influencing the turning moments: experiences of success/failure in mathematics, teacher's opinion about PT's mathematical competence, teacher's charisma, and experiences of helping someone with maths. The related categories of narratives are not disjointed from each other: PTs often refer to more than one factor. We present each of them in detail, also identifying which dimensions of the TMA model are involved.

Experiences of success/failure in mathematics influence about half of the turning moments narrated. In most cases the *success* experience is linked to a school test or a mathematical competition, resulting in a *positive change in the student's perceived competences*. As we could expect, from these accounts *positive emotions* emerge, sometimes very strong (PT56: "that feeling was unforgettable and indescribable"), although often preceded by strongly described moods of *anxiety* (PT41: "panic while waiting [for an evaluation]"; PT56: "heart in the throat while waiting"). Instead, in some cases, the success experiences lead to *contrasting emotional states* (PT18: "this is where my troubled love-hate relationship with mathematics began").

Among the accounts of *failures* only one is from primary school, all the others regard the transition from one school level to another, in particular the transition to high school or university. They are almost always characterised by an *abrupt negative change in the perceived competence*, caused by a bad result in a school test or in a university exam. Nevertheless, although often involving strong negative emotions, such as *sadness*, *anger*, *sorrow*, none results in surrender, but all have a *story of redemption* as consequence (PT19: "I felt somehow encouraged to study mathematics in order to succeed, not to feel inferior to others"). In these stories, PTs claim to have changed their method of study, to have worked hard in a different way, to have felt spurred on.

Teacher's role emerges as decisive in almost half of the turning moment narratives and the main factors influencing them are teacher's opinion about PT's competence in mathematics and/or teacher's charisma.

Teacher's opinion about PT's competence in mathematics is a very recurrent theme in the narrated turning moments. In many cases, PTs refer to *teachers' trust*, which determines the turning moment in different ways. Great trust is narrated as being associated with positive emotions, such as *pride*, or with the *desire not to disappoint* the teacher's expectations (PT26: "Even on the graduation day, my teacher shook my hand and said 'Be sure to enrol in maths'. Some trusted me, I won't let them down"). On the other hand, cases in which the PT feels that she has disappointed the teacher's trust are associated with very negative emotions, such as great *bitterness*, or with a *desire to recover* that trust. Also, the turning moments are narrated as being caused by the teacher's *attribution of innate capacities* to the PT as student, determining student's perceived high competence, or on the contrary by the teacher's perception of their *lack*, always leading to the PT's *will to prove the opposite*, through commitment and determination. Sometimes, PTs consider the *admiration* of the teacher for an outstanding performance as decisive for a turning moment. This situation is described as involving positive emotions, such as a *sense of reassurance*, great *gratification* or even a long-term change in the emotional disposition (PT23: "From that moment on, my teacher too changed her opinion of me and I maybe began to love mathematics a little more").

Teachers are undoubtedly a main actor in PT's narratives. This very particular sample of students – they are enrolled in a math degree – usually judged their school teachers as "good", "excellent", "fascinating", "passionate", "enthusiastic", "open to dialogue with students". The teachers are narrated as able to arouse students' *interest* and *passion* for mathematics, or a feeling of *reassurance*, or to spur a *less algorithmic view* of mathematics. In some cases, a significant *teacher's charisma* is recognized: the feelings induced by the teacher and her acts are described as very intense. This fascination is very often recognized as the main factor for a turning moment (PT29: "his lessons were a joy; they gave me that something new: you know when you fall in love for the first time?!").

The opposite scenario –a negative experience with a math teacher as the main reason for the distaste for mathematics as described in previous studies involving prospective primary teachers (Coppola et al., 2015)– is surely less frequent in PTs' narratives analysed in this work. However, in some narratives it emerges. In these cases, PTs reported a *loss of enthusiasm* or a *worsening of perceived competence* (then overcome). An interesting case is that of a PT who, having become aware of her teacher's lack of inclusiveness, talks about her motivation to seek redemption in her future teaching activity (PT24: "I realised that this teacher had brought forward four or five 'elect' [...] leaving all the others behind. I realised then that I was going to study mathematics, I was going to teach, and I was going to worry about all my students").

Some PTs report an episode or a period in which they experienced *helping someone with maths* as a turning moment that led them to choose teaching mathematics as their *future profession*. In such accounts, PTs report good perceived teaching skills and positive emotions such as *gratification* and *joy* (PT32: "My desire to teach what little I knew, to notice that after I explained, they were able to finish the exercises, aroused great joy in me and slowly my dream grew").

Focusing again on the *form* of the turning moments narratives, it is worth observing how in many cases the narratives are 'teacher-centred' and that the student appears to have a more passive role. In many cases, teachers' exact *words* are quoted (PT40: "At that point, the professor made a weird face and said, 'I doubt you'll be able to do it!'"; PT55: "She looked at me smugly, commented 'my future colleague'") and teachers' emotions as *disappointment* or *satisfaction* are reported. Moreover, many words are spent on describing the teacher as passionate, enthusiastic, "austere at the right point" (PT13) and taken as a model. Sometimes the passive form of verbs is used, almost as if some student's internal changes have been heavily influenced by the teacher's action. Still on the form, in several cases PTs use strong expressions such as "the straightway" or "my way" when reporting the intention to put more effort into mathematics or to choose a degree course in mathematics.

DISCUSSION AND CONCLUSIONS

In this paper we presented a narrative study aimed at investigating which events prospective secondary teachers recognise as crucial in the development of their relationship with mathematics, as a part of a wider study about this topic. The analysis shows that more than half of these events are turning moments (Bruner, 1991), most of them occurring during high school.

We recognized four categories in the narrated turning moments: success/failure experiences, teacher's opinion about PT's mathematical competence, teacher's charisma, and PTs' helping experiences of someone with maths. In undertaking this study, we expected some different results from previous studies on crucial events with prospective primary teachers (Coppola et al., 2015) as our new sample consisted of people who had already earned a bachelor's degree in Mathematics and would be future specialist mathematics teachers. Contrary to what we might expect, conflicting g emotions often emerge in PT's accounts. In some cases, "troubled" or "odi et amo" relationships are narrated and very strong negative emotions, such as anxiety and panic, are often reported alongside positive emotions. As for the future primary teachers, very frequent in the factors influencing the turning moments are the experiences of failure associated, also for PTs, with very strong negative emotions and a sudden lowering in perceived competence. For future primary teachers they very often resulted in life choices aimed at avoiding mathematics. In contrast, for our new sample, these experiences, although causing feelings of crisis, mistrust, and uncertainty, were taken as a *challenge*, either to themselves or to the teacher, which was then won. Many *stories of redemptio*n therefore emerge. The theme of redemption had emerged with future primary teachers too, but only in the form of a desire for the future and to be realised not so much in the study of mathematics but in teaching it. It had been called a "desire for math-redemption" (Di Martino et al., 2013).

Success/failure experiences leading to turning moments are in most cases linked to moments of official assessment (in particular to grading) and only rarely to different moments of mathematical activity. This could be indicative of how much importance is given to assessment in the educational system of our sample of PTs -and how this is often restricted to the attribution of a numerical grade to the student's performance. The teacher appears to play a primary role in determining the turning moments. As recalled above, the teacher's opinion about PT's competence in mathematics and the teachers' charisma are recurrent factors in the turning moments accounts and many narratives are 'teacher-centred' in the form. Moreover, classmates appear only in a few cases in the accounts (and never as peers, but only as learners or almost antagonists of the protagonist). This seems to us indicative of 'traditional school' experiences, in which the control of class activity is almost entirely in the hands of the teacher.

To conclude, we believe that the collected narratives describe a picture of the PT's attitude towards mathematics that is more complex than it might sound. While prospective primary teachers' attitudes towards mathematics are widely negative and strongly marked by negative experiences with school mathematics (Di Martino et al., 2013), our data showed that PT's experiences are not simply the other side of the coin. The variety of the collected narratives and the different phenomena that emerged from our data suggest the need for further studies to describe possible recurrent paths in prospective secondary teachers' attitudes with mathematics. It seems evident that these paths could be strongly affected by socio-cultural issues: therefore, the development of comparative studies between different countries is strongly encouraged.

This description is not an end in itself: as teacher educators, we strongly believe that knowing and understanding prospective teachers' past is crucial for developing effective training programs.

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THE USE OF RATIO AND RATE CONCEPTS BY STUDENTS IN PRIMARY AND SECONDARY SCHOOL

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This paper aims to analyse how primary and secondary school students use the concepts of ratio and rate when solving a ratio comparison problem. 954 primary and secondary school students (11-16 years old) solved a ratio comparison problem that involves four questions designed following the Reflection on the Activity-Effect Relationship mechanism. Students' answers were inductively analysed generating categories in relation to students' use of these concepts and the difficulties they revealed. Results have shown that a large number of students seems not to have the concept of ratio available during and at the end of secondary education, presenting difficulties not only with the identification of the multiplicative relationship between the extensive quantities but also with the norming techniques and with the referent.

THEORETICAL AND EMPIRICAL BACKGROUND

Quantity has been defined as an attribute of an object, which is expressed by an ordered pair, formed by a number and a magnitude unit, for example, two meters. Two types can be distinguished: extensive and intensive quantities. Extensive quantities, such as mass or length, can be measured directly while intensive quantities, such as density or speed, cannot (Schwartz, 1988).

Ratio or internalized ratio is defined as the result of comparing two quantities multiplicatively in a particular situation (Thompson, 1994). For example, in the problem "a car travels 70km in 1h, if it is driven for 5h, how many kilometres has travelled?", the internalized ratio is each iteration "70km in 1h", "140km in 2 hours", ..., so the internalized ratio is the particular ratio for each iteration. When a ratio is conceived beyond a particular situation, a constant ratio is obtained for any situation, called interiorized ratio or rate (Thompson, 1994). In the example, the rate 70km/1h is understood as a new quantity (intensive quantity) that measures the attribute speed, valid for any situation in which the relationship between quantities remains constant. Understanding the concept of rate implies understanding that extensive quantities can vary and still maintain the same relationship. That is, the quantity of kilometres and the quantity of hours (extensive quantities) can vary and the speed (intensive quantity) can stay the same (Simon & Placa, 2012).

Both ratios and rates can be established in ratio comparison problems which are situations where two ratios are given and should be compared. In these problems, students have to identify the multiplicative relationship between quantities that can be equal or unequal, and use norming techniques to favour the comparison between ratios

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(Castillo & Fernández, 2021). Norming describes the process of reconceptualising a system in relation to some fixed unit or standard (Lamon, 1994).

Previous studies have focused on ratio comparison problems showing students' success levels, strategies, misconceptions and the effect of some variables of the problem on students' strategies (Alatorre & Figueras, 2005; Nunes et al., 2003; Yeong et al., 2018). Nunes et al. (2003) showed that primary school students have difficulties solving ratio comparison problems that involve intensive quantities since students have to face two challenges: thinking in terms of proportional relations and understanding the connection between the intensive quantity and the two extensive quantities. Castillo and Fernández (2021) showed that these difficulties persisted also during the secondary education (12-16 years old students). Johnson (2015) conducted a study focused on investigating secondary school students' quantification of ratio and rate as relationships between quantities. She proposed the "change in covarying quantities framework" that shows the operations of comparison (extensive quantities) and coordination (intensive quantities) containing three levels of reasoning each one. This author claimed that the question how students shift from the operation of comparison to the operation of coordination needs further investigation.

As previous studies have shown, primary and secondary school students have difficulties with the concept of rate (intensive quantities). We are developing a cross-sectional study embedded in this line of research. It is focused on examining how primary (6th grade – 11 years old) and secondary school students (from 7th to 10th grade – 12-16 years old) construct the rate concept. For this purpose, we use a characterization of the Reflection on the Activity-Effect Relationship mechanism elaborated from the Reflective Abstraction of Piaget (Simon et al., 2004; Tzur & Simon, 2004).

From this perspective, two stages have been identified in the development of a concept: participatory and anticipatory. The participatory stage starts when a perturbation happens. In this stage, a new concept is abstracted, but it is provisional since it has been built from a single situation. This stage is divided into three phases: projection, reflection type-I and reflection type-II. In the projection phase, students compare what happens when they apply an available concept from a known situation in the proposed one, called generative situation. This comparison leads students to reorganize what they know about both situations (reflection type-I phase) and the new concept (more advanced than the available one) is built, but it is considered only for the generative situation. The reflection type-II phase occurs in the transition between the participatory and anticipatory stages. In this phase, a new situation, different from the generative one but of the same type, is proposed. When students observe that the generative and the new situations are of the same type, the developed concept is rearranged again to add this new situation, making the concept even more complex. Finally, when students can apply the developed concept in situations different from the generative one, they have reached the anticipatory stage, what it means that the concept is no longer provisional.

Three types of tasks related to this process were identified (Tzur, 1999). Initial tasks that involve concepts that students have. Reflective tasks (related to the participatory stage) that seek to cause perturbations to start the construction of the new concept. Anticipatory tasks (related to the anticipatory stage) that students can solve using the new concept that they have developed in the reflective tasks.

This paper is part of the cross-sectional study mentioned before and aims to answer the research question: how do primary and secondary school students use ratio and rate concepts when they solve a ratio comparison problem?

METHOD

Participants and instrument

Participants were 954 primary and secondary school students from 6th grade (n=161), 7th grade (n=188), 8th grade (n=240), 9th grade (n=229) and 10th grade (n=136). There was approximately the same number of boys and girls in each grade, and students were from diverse socio-economic backgrounds.

Participants solved the following problem: Melania's coach tells her that for each 20 meters, she should take 5 seconds to be able to qualify. a) If Melania has covered 250 meters in 60 seconds, has she qualified? b) Melania is competing against Cristina who has covered 300 meters in 70 seconds. What is the speed of each one? Who is faster? c) If Melania runs twice as many meters in twice as many seconds, would her speed change or be the same? Why? If her speed changes, what would this speed be? d) Propose three cases in which the speed would be the same as Cristina's speed (300 meters in 70 seconds). Justify your answer.

This problem was designed taking into account the Reflection on the Activity-Effect Relationship mechanism and the three type of tasks. Question a) is an initial task since the use of the ratio concept is involved and it is considered as an available concept to the students. Question b) is considered a reflective task (reflection type-I) because it implies to identify the ratio as the intensive quantity "speed" (rate). Question c) is considered a reflective task (reflection type-II) because it proposes a different situation from the generative one (question b) but of the same type. In this situation, students can realize that, although the extensive quantities change, the rate (speed) is the same than in question b). At this point, the rate concept should have been built for this particular problem and it should be available. Finally, question d) is considered an anticipatory task because it implies the use of the rate concept in situations different from those in which it was conceived.

Analysis

Three researchers analysed individually a subset of students' answers for the four questions, generating categories. Agreements and disagreements were discussed until an agreement was reached with the final categories. Later, the rest of students' answers were analysed using these categories. If an answer was not fit with the categories generated, it was discussed and a new category was generated.

Four categories emerged in question a): (i) students who did not identify the extensive quantities or they did not identify a multiplicative relationship between them (category A1); (ii) students who identified the extensive quantities and the multiplicative relationship between them, but they had difficulties with the norming techniques to obtain the ratios to be compared (category A2); (iii) students who obtained the ratios correctly using a norming technique but they had difficulties with the referent comparing the ratios (category A3); and (iv) students who obtained and compared the ratios correctly, identifying the inequality of ratios (category A4). The same categories were identified in question b) (categories as B1, B2, B3 and B4, respectively). In question b), a new category was identified: students who compared the ratios correctly, using the speed (ratio m/s) (category B5). In Figure 1, in the category B4, the student compared the ratios 250/60 and 300/70 identifying equivalent fractions. In the category B5, the student compared the same ratios with the quotient, obtaining the speed and specifying the units.

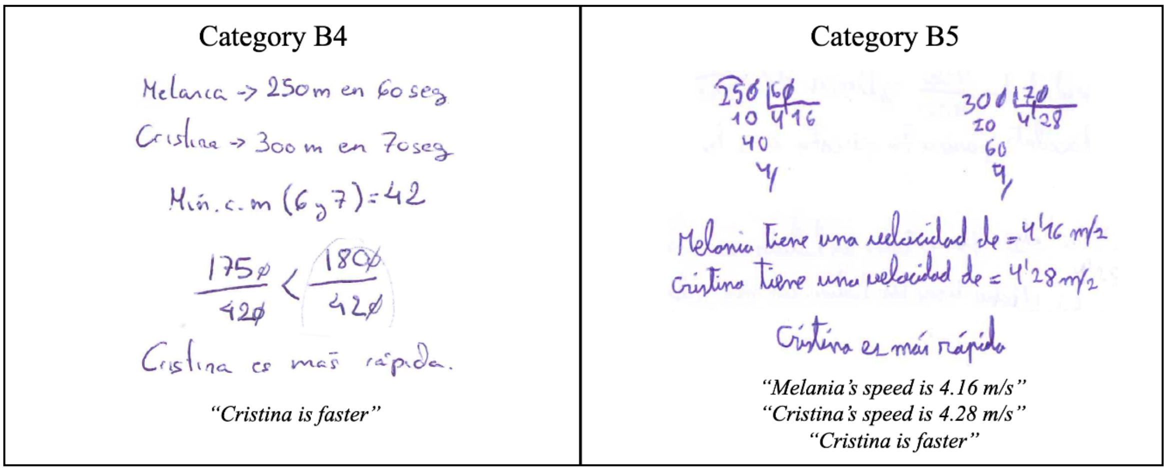

Figure 1: Examples of the categories B4 and B5

In question c), three categories emerged: (i) students who did not identify the variation of the extensive quantities neither the ratio's constancy (category C1); (ii) students who identified the variation but not the constancy (category C2); and (iii) students who identified the variation and the constancy (category C3). In C3, two subcategories were distinguished: students who answered without using numerical relationships (C3A) and students who made operations (C3B). The difference between them is exemplified in Figure 2. In C3A, the student explained that the speed is the same because Melania runs twice as many meters in twice as seconds while in C3B, the student justified the answer multiplying both quantities by 2 and dividing them to check if the speed was equal.

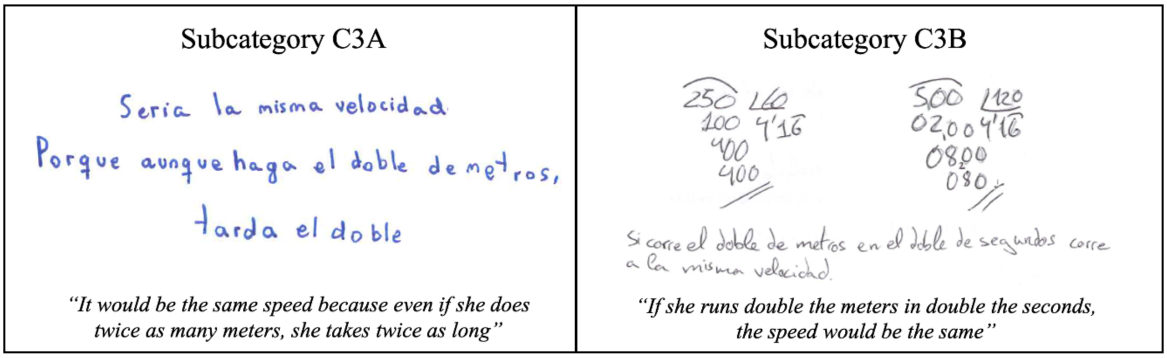

Figure 2: Examples of the subcategories C3A and C3B

In question d), three categories were identified: (i) students who did not identify the ratio's constancy in other situations (category D1); (ii) students who identified the ratio's constancy in other situations multiplying the extensive quantities by the same number (category D2); and students who identified the ratio's constancy in other situations using the speed (ratio m/s) (category D3). Figure 3 shows examples of categories D2 and D3. In the category D2, the student multiplied both meters and seconds of Cristina by 2, 3 and 4, obtaining three situations where her speed is the same. In the category D3, the student multiplied the speed calculated in question b) by three random amounts of seconds, obtaining the respective meters.

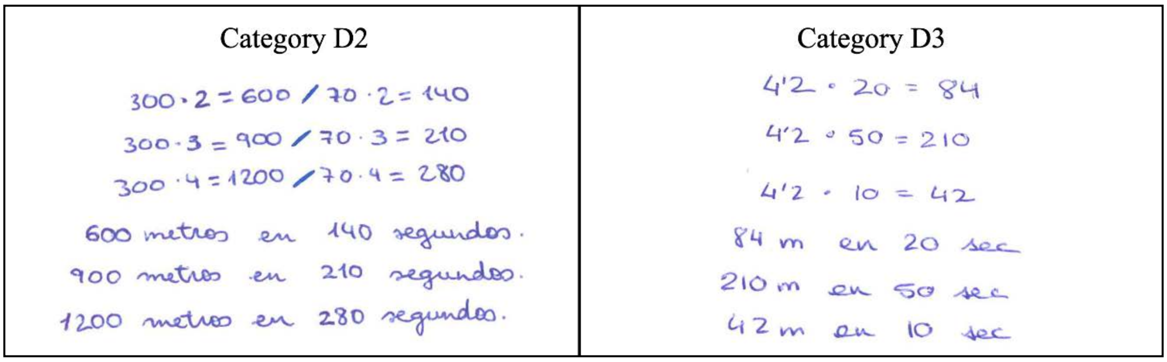

Figure 3: Examples of the categories D2 and D3

RESULTS

Tables 1, 2, 3 and 4 show the percentages of answers in each category by grade in questions a), b), c) and d), respectively.

Category	6th	7th	8th	9th	10th	Total
Blank answers	16.77	6.38	11.67	9.17	7.35	10.27
A1	44.10	51.06	39.58	33.62	43.38	41.72
A2	4.96	8.51	8.75	7.86	6.62	7.55
A3	7.45	8.51	13.75	19.65	15.44	13.31
A4	26.72	25.54	26.25	29.70	27.21	27.15

Table 1: Percentage of answers in each category by grade in question a)

Category	6th	7th	8th	9th	10th	Total
Blank answers	21.74	12.23	20.42	17.47	19.11	18.13
B1	52.18	62.23	46.25	41.05	44.85	48.95
B2	4.97	3.72	6.67	4.80	8.82	5.66
B3	8.07	11.17	6.67	10.48	6.62	8.70
B4	11.80	6.91	7.91	10.48	5.89	8.70
B5	1.24	3.74	12.08	15.72	14.71	9.86

Table 2: Percentage of answers in each category by grade in question b)

In questions a) and b), more than 40% of the students did not identify the extensive quantities or the multiplicative relationship between them (category A1 and B1). Other difficulties were related with the norming techniques or with the referent in the comparison between ratios. Furthermore, less than 30% of the students compared the ratios correctly (categories A4 and B4, B5). Percentages in each category remains similar along the grades. So, a large number of students seems not to have the concept of ratio available along and at the end of secondary education.

Comparing the percentages of correct answers in questions a) (category A4) and b) (categories B4 and B5), students revealed more difficulties in question b) that asks for the intensive quantity (speed) (we added this question as a perturbation). From the group of students who were able to compare the ratios in question b), some of them compared the ratios correctly (B4) but not using the speed (ratio m/s). These students did not observe differences between the known situation (question a) and the generative one (question b), answering equally in both; and others identified the ratio m/s as an intensive quantity (B5). These last students seemed to reorganize what they know about both situations (reflection type-I phase) and the new concept (more advanced than the available one – identifying the speed as a new quantity) is built, but it is considered only for this generative situation.

Category	6th	7th	8th	9th	10th	Total
Blank answers	28.57	23.40	19.58	17.47	28.68	22.63
C1	27.33	22.87	20.83	18.34	12.50	20.55
C2	12.42	15.43	9.58	7.86	11.03	11.01
C3A	13.66	12.23	24.58	24.45	25.00	20.34
C3B	18.02	26.07	25.43	31.88	22.79	25.47

Table 3: Percentage of answers in each category by grade in question c)

Category	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}	Total
Blank answers	56.52	57.98	54.58	53.28	50.74	54.71
D1	22.36	21.28	14.58	14.41	13.97	17.10
D2	21.12	20.21	30.84	31.00	33.82	27.56
D3	0.00	0.53	0.00	1.31	1.47	0.63

Table 4: Percentage of answers in each category by grade in question d)

In question c), 45.81% of the students identified the variation of the extensive quantities and the ratio's (speed) constancy (C3). Therefore, it seems that these students had the rate concept available for this particular problem. Some of them used the concept of rate without using numerical relationships (C3A) and others checked speeds numerically (C3B). However, only 28.19% of the students were able to identify the ratio's constancy in other situations in question d) (D2 and D3; anticipatory task).

DISCUSSION AND CONCLUSIONS

Our study focuses on how primary and secondary school students use the concepts of ratio and rate when solving a ratio comparison problem. Our results have shown that a large number of students seems not to have the concept of ratio available during and at the end of secondary education. These results coincide with those obtained by Nunes et al. (2003) with primary school students and by Castillo and Fernández (2021) with secondary school students. However, for the construction of the interiorized ratio (rate concept), it is fundamental the ratio concept since rate is defined as "reflectively abstracted constant ratio" (Thompson, 1994, p.192).

Yeong et al. (2018) explained that the base of students' misconceptions of ratios is that they do not understand ratio as a relationship between quantities. Our results are in line with this explanation since more than 40% of the students did not identify the extensive quantities or the multiplicative relationship between them (categories A1 and B1). However, other difficulties appeared linked to the norming techniques and the identification of the referent in the comparison. So, it seems that not only the identification of the multiplicative relationship between the extensive quantities is a key issue (Nunes et al., 2003; Thompson, 1994; Yeong et al., 2018) but also other elements such as the norming techniques and the referent in a comparison.

The students who identified the speed (ratio m/s) (category B5) seem to understand this ratio as a new quantity. However, to understand the speed as intensive quantity (rate) it is necessary to identify that the extensive quantities can vary but still maintain the same relationship. A little more of the 40% of students were able to identify it in question c) but few students were able to use the concept of rate in other situations. Therefore, the construction of the rate concept is complex.

Our next step is to identify students' profiles since it could allow us to identify how students move from one stage to another of the Reflection on the Activity-Effect

Relationship mechanism. This identification can also give us information about key elements in the construction of the rate concept.

Acknowledgments

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CYCLES OF EVIDENCE COLLECTION IN THE DEVELOPMENT OF A MEASURE OF TEACHER KNOWLEDGE

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This study highlights some of the tensions that arise during measure development while attending to both Rasch measurement principles and mathematics education's focus on high quality operationalization of complex theoretical constructs. We situate our measure development work within the context of a larger design-based mathematics teacher preparation intervention project focused on improving teacher candidate attentiveness, and illustrate how these tensions have shaped our instrument and item development work over the last four years.

INTRODUCTION

Persistent global concerns regarding the quality and efficacy of mathematics instruction have long influenced mathematics education research agendas (e.g., Council for the Accreditation of Educator Preparation, 2022; Grossman, Hammerness, & McDonald, 2009) and have led to ongoing efforts to (a) articulate the range of constructs related to effective mathematics teaching (Ball, Thames & Phelps, 2008), (b) develop scaled instruments which reliably measure the skills and knowledge associated with each construct (e.g., Mathematical Knowledge for Teaching Measures from the Learning Mathematics for Teaching Project, 2005), and (c) design interventions with the potential to improve teachers' and prospective teachers' position on those scales (Hill, Rowan, & Ball, 2005). This study reports on some of the challenges found at the intersection of measure development, intervention design, and mathematics teacher education program implementation. The focus of this paper is on the development of a measure of teacher attentiveness, the Disciplinary Attentiveness to Student Ideas-Quantitative Reasoning Instrument (DASI-QRI), and items which feature evidence of student quantitative reasoning at the secondary level. Operating under the constraints of mathematics teacher preparation programs and the realities of intervention implementation while also adhering to the charge that "a series of interrelated investigations is required to understand the construct(s) that a measure assesses" (Clark & Watson, 2019, p.1413) has surfaced new complexities associated with measure development for mathematics teacher education. Through a focus on the iterative development of one item in our instrument, we illustrate how multiple cycles of evidence collection and analyses can be used to inform revisions, delineate how these multiple cycles may be necessary to surface a range of different issues, and highlight some of the tensions that must be navigated while designing scalable measures with the potential to yield meaningful data for mathematics education researchers, teacher educators, and professional development providers.

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 115-122). PME.

BACKGROUND

The DASI-QRI was developed to measure *attentiveness* to students' quantitative reasoning. Attentiveness integrates components of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Shulman, 1987), professional noticing (Jacobs et al., 2010), progressive formalization (Freudenthal, 1973; Gravemeijer & van Galen, 2003; Treffers, 1987), and formative assessment (Black & Wiliam, 2009). It is defined as *the ability to analyze and respond to a particular student's mathematical ideas from a progressive formalization perspective* (Carney, Cavey, & Hughes, 2017). Previous work with construct map development for attentiveness (Carney, Totorica, Cavey & Lowenthal, 2019) informs item development for the DASI-QRI.

The instructional intervention associated with the development of the DASI-QRI is designed to increase attentiveness to students' quantitative reasoning and consists of a series of modules with both asynchronous and synchronous components. Each module centers upon a challenging, nontraditional task and features a sequenced collection of curated video and written artifacts of secondary students working on the task. The focus and development of module content has been described elsewhere (e.g., Cavey, Libberton, Totorica, Carney, & Lowenthal, 2020).

The Standards for Educational and Psychological Testing's (AERA, APA, NCME, 2014) argument-based approach to validity encourages conceptualizing development and validation as an ongoing, iterative process. However, certain constraints can lead to more iterations than might typically be expected. For the DASI-QRI, three interrelated, yet distinct factors led to numerous iterations. These factors were:

1. Test development within an instructional intervention development project,
2. Measuring and defining the components of the attentiveness construct, and
3. Use of the Rasch measurement model, which demands consideration of many different test and item indicators, yet also yields a high-quality product.

CYCLES OF EVIDENCE COLLECTION AND ANALYSIS

Annual administrations, each consisting of multiple cycles of evidence collection, has informed the development and revision of DASI-QRI items. The type of evidence collected depended on the status of development for both the instrument and individual items. For example, administration of the DASI-QRI in Year 1 did not include Rasch analysis because the number of participants was limited. Additionally, each selected-response (SR) item was initially developed through analysis of responses to the constructed-response (CR) version and identification of exemplar responses for use in the SR version (Carney, Cavey, & Hughes, 2017). Response process analysis of cognitive interview data examined the degree of match between participant responses to the CR and SR versions of the item and informed SR item revision (Mo, Carney, Cavey, & Totorica, 2021). Once item development/revisions were completed, the DASI-QRI was administered as a pre/post measure in courses using the associated intervention. Rasch analysis techniques were used to examine both individual item functioning and overall item and person statistics for the instrument as a whole. Results of these analyses then prompted additional evidence collection and revision. See Table 1 for a brief overview of the ways in which cycles of evidence collection have impacted the development of the DASI-QRI and one item, in particular, the Truck Intent Item, provided in Figure 1.

		Items			
	n	CR	SR	DASI-QRI Changes	Truck Intent Item Changes
Year 1 Pre	35	15	0	N/A	Development of SR version
Year 2 Pre	89	0	12	added 2 SR items	Revised SR version
Year 3 Pre	127	0	14	added 6 CR items	none
Year 4 Pre	129	6	14	none	none
Year 4 Post	116	6	14	TBD	TBD

Table 1. Cycles of Evidence Collection and Impact

The Truck Intent Item is the first of three questions related to the Algebra I task pictured in Figure 1 (Note: Algebra I refers to the standard first course in algebra for ages 13-14 in the U.S.). Subsequent items include images of secondary student work on the task and prompt candidates to indicate their level of agreement with SR options related to the student's approach and potential teacher responses.

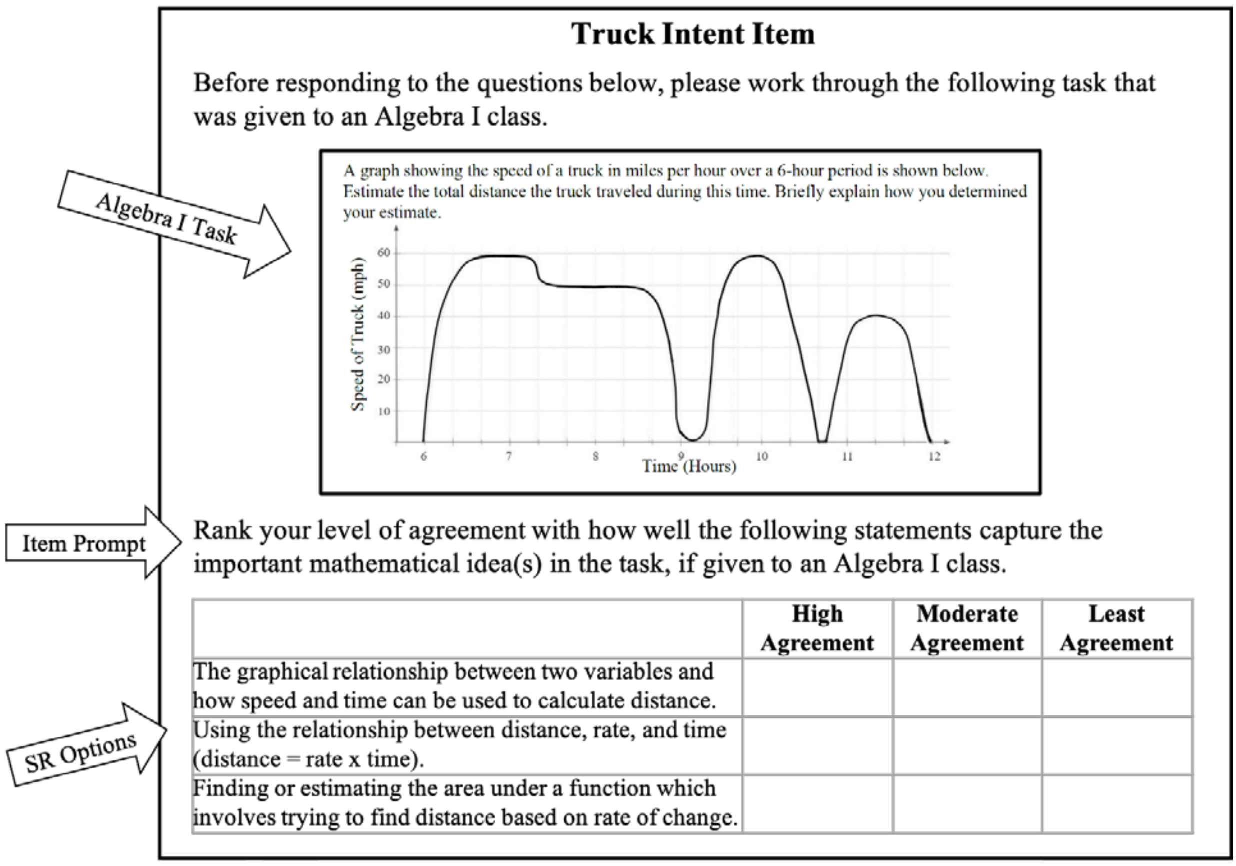

Figure 1. Truck Intent Item

The intended response for the ranked item, listed from high {H} to least {L} agreement, is: {H} The graphical relationship between two variables and how speed and time can be used to calculate distance, {M} Using the relationship between distance, rate, and time (distance = rate × time), and {L} Finding or estimating the area under a function which involves trying to find distance based on rate of change. Given that the stated context is Algebra I, finding the area under a function is not a generally appropriate mathematical focus. Option {H} situates {M} in the context of graphical reasoning, and is thus the more complete and appropriate description of the mathematical focus. The Truck Intent Item is scored based on correctly ranking the {H} (1 point) and the {L} (1 point) (see Table 2).

Ordering of SR Options	Score
{HML}	2
{HLM} or {MHL}	1
{LHM} or {MLH} or {LMH}	0

Table 2. Scoring Scheme for the Truck Intent Item

We focus our results on the Year 4 administration of the DASI-QRI and the Truck Intent Item to illustrate how repeated cycles of evidence collection and analysis are necessary to uncover potential issues. Participants in Year 4 were enrolled in a mathematics course across 13 U.S. universities in which the course instructor implemented the project's intervention. Year 4 analyses of the 20-item instrument, to date, have included Rasch analysis on the pre measure for 129 participants, on the post measure for 116 participants, and on the pre-post paired data for the 62 candidates who appeared to meaningfully engage with the intervention. Qualitative analyses of individual item response trends for the 62 participants and previously collected cognitive interview data for 13 participants were also completed.

With respect to the Rasch analyses, two persons with extreme scores of 0 were dropped from the post measure responses of 116 participants across 20 items. There were no extreme scores on the pre. No extreme scoring categories were dropped from the analyses for either the pre or the post. There were four groups of items based upon the format (SR versus CR) and the number of ranking options for the SR (2, 3, and 4 options). The items within the same grouping share the same partial credit response structure. The JMLE estimation process converged when the maximum logit change was .0041 (.0033 pre).

RESULTS

Rasch Analysis

Overall, the item "test" reliability is .96(.97 pre), which is very high, with a separation index of 5.15(5.81 pre). The sample size allows the item difficulties to be estimated precisely and confirms the item difficulty hierarchy (e.g., high, medium, low item difficulties) of the instrument. The person "test" reliability is .72(.44 pre); the person separation index is 1.62(.89 pre). Thus, the instrument may not be sensitive enough to distinguish between high and low performers or more performance levels in the sample. The raw variance explained by the Rasch measure was 24.09%(24.7% pre). The point-measure correlations (PTMEASUR) were all positive, suggesting that all the items were pointing in the same direction. Except for the Truck Intent Item on the post, the mean-squares (MNSQ) were not excessive, so the misfit was acceptable; the standardized statistics (ZSTD) for both INFIT and OUTFIT were not extreme; thus, we failed to reject the null hypothesis that these data fit the Rasch model. However, the Truck Intent Item had an OUTFIT MNSQ of 1.69 and a ZSTD of 4.64 due to some unexpected responses on the post. This indicated additional analysis may be needed.

For the Truck Intent Item, item category frequency analysis indicates that the average measures advanced with the score categories for the pre but do not advance with the categories on the post; 22 people with a score of 2 had an average measure of -.12, less than the average measure of -.02 of 35 people with a score of 1 (see Table 3).

		Pre		Post	
		Frequency	Mean ability	Frequency	Mean ability
Item Score	0	42 (33%)	-.26	59 (51%)	-.35
	1	44 (34%)	-.15	35 (30%)	-.02
	2	43 (33%)	.07	22 (19%)	-.12*

Table 3. Item Score Frequencies and Mean Ability

The Item Characteristic Curves (ICC) in Figure 2 (pre on the left and post on the right) show the empirical ICC (blue line) of Truck Intent with an unexpected behavior (outside the 95% confidence bands [grey line] around the expected Item Characteristic Curves [red line]). For the pre, there is an unexpected drop in the highest end of the latent variable (i.e., measure), and on the post, there is an unexpected rise at -2.8 and -1.2 and an unexpected drop at 0.5 in the latent variable relative to item difficulty scale.

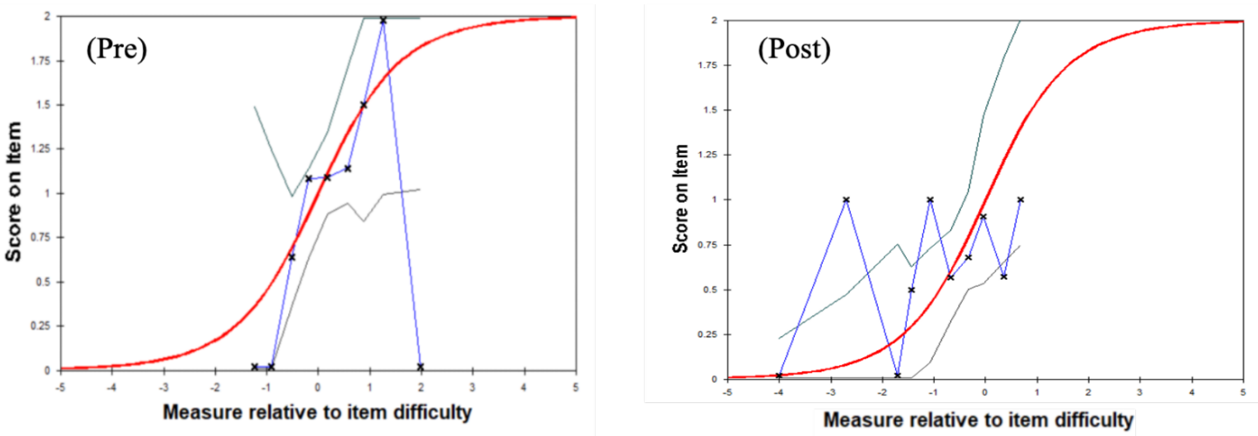

Figure 2. Item Characteristic Curves (ICC) of Truck Intent Item for Pre and Post

Truck Intent Response Analysis

Examination of the raw response data for the subset of 62 participants revealed that 21 (~34%) scored lower on the Truck Intent Item on the post compared to the pre. The highlighted portion of Table 4 shows how participant rankings changed, which includes 12 of the 16 participants (75%) who originally scored a 2 and 10 of the 23 (~43%) participants who originally scored a 1. While there were also participants who scored higher on Truck Intent Item on the post, it was the large percentage of participants with decreased scores that prompted further examination of the data.

		Post-Test Ordering						
		{HML}	{HLM}	{MHL}	{LHM}	{MLH}	{LMH}	Total
Pre-Test Ordering	{HML}	4	1	6	1	1	3	16
	{HLM}	2	1	2	0	0	3	8
	{MHL}	3	1	5	1	2	3	15
	{LHM}	1	0	1	0	1	4	7
	{MLH}	2	3	0	1	4	2	12
	{LMH}	0	0	1	1	1	1	4
	Total	12	6	15	4	9	16	**62**

Table 4. Pre- and Post-Test Rank Ordering of SR Options for the Truck Intent Item

Re-examination of the cognitive interview data, collected in an effort to better understand response process, revealed that of the 10 participants interviewed who mentioned the Algebra 1 context of the Truck Intent Item, all 10 ranked {H} with the highest level of agreement, and 8 responded with the intended ranking {HML}. Interestingly, it did not seem to matter whether or not a participant noticed the potential connection to calculus or the graphical reasoning aspect, though data are limited as there were only 4 different ranking options represented in the sample and only one participant selected {LMH}. In addition, only one participant explicitly compared options {H} and {M}, indicating that further data collection is needed to better understand response processes associated with the Truck Intent Item.

TENSIONS IN MEASURE DEVELOPMENT

Measure development for complex constructs such as attentiveness remains a persistent challenge for the mathematics education community, one that often garners superficial nods to statistics associated with reliability and validity (e.g., Cronbach's alpha) or else is sidestepped altogether in lieu of qualitative assessment. This could be due, in part, to the tensions which arise when attempting to address issues revealed by Rasch analysis while also operating within the context of mathematics education. For example, from the measurement perspective, the DASI-QRI's low person "test" reliability and separation index indicate the need for additional items. Yet from the mathematics education perspective, additional items, especially when considering the cognitive complexity required to elicit evidence of attentiveness, place an undue burden on test-takers in terms of both time and fatigue. Quality instrument development from the measurement development perspective also often depends upon ready availability of large participant pools. In contrast, recruitment of participants within the mathematics education community - especially within the context of a larger instructional intervention project - can be a significant challenge. Furthermore, quality instruments and their items are expected to perform roughly the same across all administrations with an implicit assumption that test-takers complete the assessment with fidelity. However, instrument use in the mathematics education community is often embedded in a mathematics course or professional development as part of an intervention; thus, test-taker motivation and investment in completing the assessment with fidelity can vary depending on the timing of administration and test-taker perceptions of the assessment. Could this be why the Truck Intent Item performed well in some administrations and raised issues of concern in another? Did performance decrease simply because participants missed or ignored the course context when they completed the post? Are variances in Rasch analysis results due to instrument or item failings that can be addressed via revision, or are they due to something else?

FINAL THOUGHTS

We have aimed to highlight the complexity of measure development when meaningfully attending to both Rasch measurement principles and mathematics education's focus on high-quality operationalization of complex theoretical constructs, particularly within the context of developing a measure associated with an instructional intervention. The issues which surface when considering each perspective precipitate different kinds of development work, the outcomes of which can impact the other. This often warrants additional cycles of evidence collection, analysis, and revision, and elicits tensions from the mathematics education side, as we must also consider persistently small sample sizes, the length of the assessment, the time demands on instructors and teacher candidates, and alignment between item design and the cognitive complexity of attentiveness we wish to measure.

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PRE-SERVICE TEACHER`S SPECIALISED KNOWLEDGE ON AREA OF FLAT FIGURES

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This study aims to characterise elements of specialised knowledge of a group of preservice teachers (PST) when solving area tasks. Emphasis is placed on the subdomain of Knowledge of Topics. The written justifications and procedures used in the resolution of one area task are analysed using mixed methods, including qualitative and quantitative analysis. The results indicate that PST who manage to respond to the demand of the task mobilise different registers of representation as well as procedures, justifications, properties, and geometric principles. Results suggest that the use of different representations in the resolution process has an instrumental value that allows other indicators of the subdomain of Knowledge of Topics to be mobilised.

INTRODUCTION

Teachers' knowledge of both content and its didactics has been studied from different approaches (e.g., Ball, Thames & Phelps, 2008; Carrillo et al., 2018). Particularly, we are interested in the content knowledge teachers possess, as it allows them to better understand and justify why they solve mathematical tasks in a certain way. Additionally, possessing content knowledge also allows teachers to know different ways of solving problems and teaching the content to their students (Shulman, 1986). We emphasize the importance of possessing knowledge of area measurement because this content can set the ground to understand other mathematical content in primary education, such as multiplication of natural numbers or fractions (Freudenthal, 1983). Despite the different applications that area measurement may have, numerous investigations conclude that PSTs do not have key content and pedagogical knowledge (Chamberlin and Candelaria, 2018; Simon and Blume, 1994), which has a negative impact on student learning. This study departs from the model of Mathematics Teacher Specialised Knowledge (MTSK) developed by Carrillo et al. (2018) and considers the relevance of the domain of content knowledge on area measurement with the objective to answer the following question: what is the specialised knowledge mobilised by PSTs when facing tasks involving the calculation of area? Thus, our study aims to characterise the Knowledge of Topics (KoT) mobilised by PSTs when solving tasks that require the use of diverse procedures.

THEORETICAL FRAMEWORK

Surfaces' measurement requires understanding and reorganizing the object that is going to be measured, as well as understanding different properties, concepts and procedures involved in measurement processes (Sarama & Clements, 2009).

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 123-130). PME.

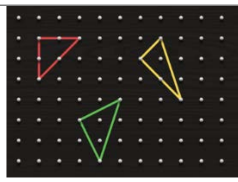
Therefore, it is not surprising that area measurement poses difficulties for PSTs. There are numerous studies that highlight such difficulties (Caviedes, de Gamboa & Badillo, 2021b; Chamberlin & Candelaria, 2018; Simon & Blume, 1994), which are mainly related to poor resolution strategies and limited acquisition of geometric properties. Such difficulties limit the ability of PSTs to propose examples and guide students' wrong answers (Runnalls and Hong, 2019). The tendency that PSTs have towards the use of formulas could be related to difficulties in using and coordinating the different registers of representation (e.g., geometric and symbolic) involved in the resolution of a given task, or else, to the lack of acquisition of geometric properties and principles involved in area measurement processes (Caviedes, de Gamboa, & Badillo,2021b; Hong & Runnalls, 2020; Runnalls & Hong, 2019). Knowledge of such conceptual elements could help PSTs to expand their range of resolution strategies while allowing them to justify what they do and why they do it (Caviedes, de Gamboa & Badillo, 2021b).

In order to understand and develop the different conceptual elements involved in solving area tasks it is necessary to consider the knowledge that PSTs have on such elements. In this sense, we adopt the analytical model of Mathematics Teacher Specialised Knowledge - MTSK (Carrillo et al., 2018), which determines the desirable components that PSTs should know for their future practice (Policastro, Ribeiro, & Fiorentini, 2019; Caviedes, de Gamboa, & Badillo, 2021b). Within the MTSK model, the KoT subdomain describes and makes it possible to distinguish the specific conceptual knowledge that is mobilised in the resolution of area tasks (see Table 3), and their relationships by means of interconceptual connections. Thus, KoT describes what and in what way mathematics teachers (or PSTs) know the content they teach.

METHOD

This study is situated in an interpretative paradigm and is part of a broader research that seeks to characterise the PSTs' specialised knowledge of area measurement. Content analysis (Krippendorff, 2004) is used to make a first interpretation of the PSTs' resolutions using the KoT indicators as analytical categories. In addition, a statistical implicative analysis (Gras & Kuntz, 2008) is conducted to explore relationships between different KoT indicators that PST mobilise in their resolutions. Data collection was carried out in the first term of the 2020-2021 school year. The participants were 147 PSTs enrolled in the third year of the Primary Education Degree at the Universitat Autònoma de Barcelona. The PSTs had had previous instruction on different procedures of area measurement as part of their study programme. A semi-structured open-ended questionnaire (Bailey, 2007) was designed to be completed individually. The PSTs were asked to justify each procedure in writing. To solve the tasks, PSTs could use manipulative materials (cut-outs as an annex to the questionnaire), as well as measuring instruments (except tasks 1, 2 and 3). The questionnaire was structured as follows: three tasks responding to contexts of equal partition, and comparison and reproduction of shapes (Tasks 1, 2 and 3); two measurement tasks (Tasks 4 and 5); one task of classification of statements and one task of the definition of the concept of area (Tasks 6 and 7); finally, one task of analysis of students' responses (Task 8). The PSTs had one week to answer the questionnaire and send it in pdf format. For sake of brevity, we present the analysis of two resolutions of Task 4 (Table 1).

Table 1. Task proposed to the PST group

Formulation	Graphic representation of the Task
Task 4: Look at the triangles constructed on the geoboard. **What is the area of each triangle? Which one has the largest area? Justify your answers using two or three different procedures.**	 (Compiled by authors)

Qualitative and quantitative analysis of PSTs` resolutions

Since we have not found any studies detailing the KoT indicators for area measurement processes, these have been constructed based on the results of a previous study postulating an epistemic configuration of the concept of area (Caviedes, de Gamboa & Badillo, 2021a). From this epistemic configuration, we define the KoT indicators to focus on the analysis of the PSTs responses to the task. Each indicator was adapted to the subcategories that the MTSK model proposes for KoT (phenomenology, representations, procedures, properties and principles, justifications, and intra-conceptual connections) and allowed a deductive coding of the PST responses, with the support of MAXQDA plus software. Table 2 shows the KoT indicators.

Table 2. Categories of specialised knowledge

KoT's categories	Indicators
Representations (R)	(R1) W*ritten:* use of adjectives such as "minor", "major", "double", "half", etc., related to surfaces. (R2) *Manipulative:* use of physical objects or dynamic geometry software. (R3) *Geometric:* use of convenient decompositions or partitions of known figures to calculate the area of unknown figures. (R4) *Symbolic:* use of the R^+ set to compare two or more surfaces, for counting units or adding up areas and-or for the indirect calculation of the area.
Procedures (P)	(P1) Compare two or more surfaces directly by total and-or partial overlapping. (P2) Compare two or more surfaces indirectly by cutting and pasting. (P3) Decompose in a convenient way, graphically or mentally, two

	or more surfaces. (P4) Carry out movements of rotation, translation, and superimposition of figures. (P5) Decompose surfaces into congruent units and/or sub-units to facilitate the process of measuring areas. (P6) Measure areas as an additive process by counting units or sub-units that cover the surface. (P7) Measure linear dimensions and use formulas.
Properties (Pp) and principles (Pr)	(Pp1) Use of conservation. (Pp2) Use of accumulation and additivity. (Pp3) Use of transitivity. (Pr1) Use of the fact that the unit of measurement can be divided into parts to facilitate the process of measuring. (Pr2) Use of the fact that every triangle is equidecomposable from a parallelogram. (Pr3) Use of the fact that the calculation of area is a matter of decomposing the figure into a finite number of parts so these parts can be put back together to form a simpler figure.
Justifications (J)	(J1) The overlapping method to compare two or more surfaces is useful for establishing equivalence or to include relationships. (J2) The mental act of cutting the two-dimensional space into parts of equal area serve as a basis to compare areas. (J3) The change in the shape of a surface does not change the area of the surface, as the figures can be decomposed and reorganised while keeping the same "parts". (J4) The area of the triangle is half of a square or a rectangle with the same base and height that contains it. Therefore, the formula of the triangle is base per height divided by two.

Figure 1 below shows examples of two PSTs (PST 7 and PST 133) that mobilise specialised knowledge. PST 7 uses written (R1), geometric (R3) and symbolic (R4) representations in the resolution process. As we can see, PST 7 mobilises (J2) and (J3) because she decomposes and reorganises triangles A and B into rectangles to later apply the area formula (P7). In addition, PS7 mobilises (J4) as she searches for the square containing triangle C to calculate its area by means of using formulas (P7). Geometric representations (R3) allow PS7 to decompose triangular surfaces by using auxiliary trace. Likewise, they allow PST 7 to use (P4) and (P5) in the case of triangle A, and (P3) and (P4) in the case of triangles B and C. The surface decomposition and reorganization procedures, allow PST 7 to implicitly mobilise (Pp1), (Pp2) and (Pp3) in addition to (Pr1), (Pr2) and (Pr3). This is because PS7 is able to accept that the area of a triangle does not change as its shape changes and that it is possible to simplify a resolution process by decomposing a figure and then rearranging its parts into a new figure. The resolution of PST 133 shows written (R1), geometric (R3) and symbolic (R4) representations. Figure 1 shows that (R3) allows PST 133 to use the auxiliary line tracing and to decompose the area around the triangles (P3) in order to find the legs corresponding to triangles B and C. This procedure allows PST 133 to obtain the length of the sides of triangles B and C by applying the Pythagorean theorem (P10) and (R4). Since the red triangle was located straight on the geoboard, PST 133 calculates its area by means of (P7). The comparison between triangles allows PST 133 to mobilise (Pp3).

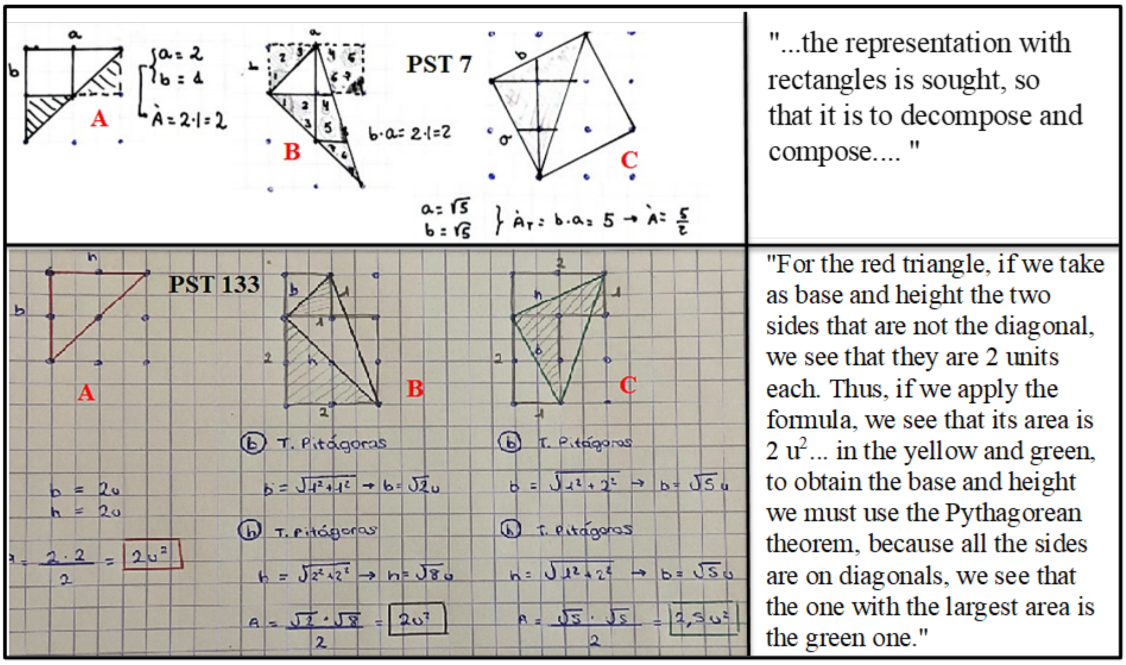


Figure 1. PSTs `resolution for Task 4

With the aim to identify relationships between KoT indicators, we performed a statistical implicative analysis. This analysis makes it possible to identify and organise quasi-implication relationships (implicative relationships between variables with a given probability) by means of a graph with arrows that relates the variables with the strongest implications at different levels and intensities. The quasi-implication between the variables A ➔ B indicates that, if PST respond affirmatively to A, they are likely to respond to B (although a relatively small number of responses may contradict it). That is, A ➔ B is equivalent to the set B not A being almost null (with the understanding that the set of observations A is almost contained in B). In this study, in the implicative graph, we use the arrow ➔ to indicate a quasi-implication according to the meaning described above. The variables considered for the implicative analysis are those arising from the qualitative analysis (presented in Table 3). In order to carry out the analysis, a value of 1 was assigned to each variable mobilised in the PSTs' responses and a value of 0 to each variable that was not mobilised in the PSTs' responses. The package C.H.I.C version 0.27 in the R console version 3.5.2 was used.

RESULTS AND DISCUSSION

Table 3 shows that PTS have a tendency to use symbolic representations (R4) and numerical procedures (P7). PST also struggle to solve Task 4 using different types of procedures. Geometric representations (R4) involving auxiliary line tracing, which allow the use of surface decomposition and reorganization procedures (P3), (P4), (P5), are used by a small number of PSTs. The same happens with the justifications, geometric properties and principles that support the above-mentioned procedures.

Table 3. Categories of specialised knowledge mobilised in Task 4

Code	Frecuency	Code	Frecuency	Code	Frecuency
R1	106	**P5**	38	**J2**	6
R4	141	**P3**	37	**P1**	5
P7	105	**NP**	34	**J1**	4
Pp2	121	**Pr1**	31	**Pp1**	6
R3	63	**J3**	11	**Pr3**	3
J4	44	**P4**	10	**Pp3**	6
P6	39	**Pr2**	7	**R2**	3

The implicative graphs in Figure 2 below (with 98% significance indicated by the red arrows and 95%, indicated by the green arrows) show different relationships between KoT subdomain indicators for those resolutions that make use of different procedures. Graph A (Figure 2) shows that PSTs using procedures related to isometric transformations (P4) make use of geometric representations (R3) by auxiliary line tracing and of procedures that require reorganizing and decomposing surfaces (P3). In turn, PSTs that make use of geometric decompositions (P3) use symbolic representations (R4), indicating a use of different procedures. The symbolic and written representations present a reciprocal relationship (R4➔R1/R1➔R4) due to the fact that a symbolic register is also a written register. Graph B (Figure 2) shows that PSTs simultaneously mobilise the properties of conservation (Pp1) and accumulation and additivity (Pp3). Both properties involve the use of geometric representations (R3), as PSTs decompose triangles by auxiliary line tracing, and subsequently rearrange them into a different figure (rectangle). We also observe that the use of (R3) also implies the use of (R1) and (Pp2), that is, the PSTs justify in written form the decompositions performed and the comparison of the triangles, in order to explain which has the largest area. The use of the transitivity property (Pp2) is also associated with the use of (R4), which indicates that PSTs make comparisons between triangles based on the numerical value of their areas. On the other hand (R1) implies the use of (R4) since both are written registers. Graph C (Figure 2) shows that the use of (J4) implies the use of (R4) and (R1), that is, PSTs justify by writing the relationship that exists between the area of triangles and squares or rectangles. The use of (J2) implies the use of (R3), as PSTs use decompositions to compare and relate the areas of triangles. Graph D (Figure 2) shows that the use of (Pr2) implies the use of (R3), i.e., PSTs use line tracing to decompose figures and identify that a triangle can be transformed into a rectangle. The use of (Pr1) implies the use of (R4) and (R1), which indicates that PSTs justify the decomposition of triangles into congruent units and subunits by means of written and symbolic registers. Again, we observe a reciprocal relationship between symbolic and written representations (R4➔R1/R1➔R4).

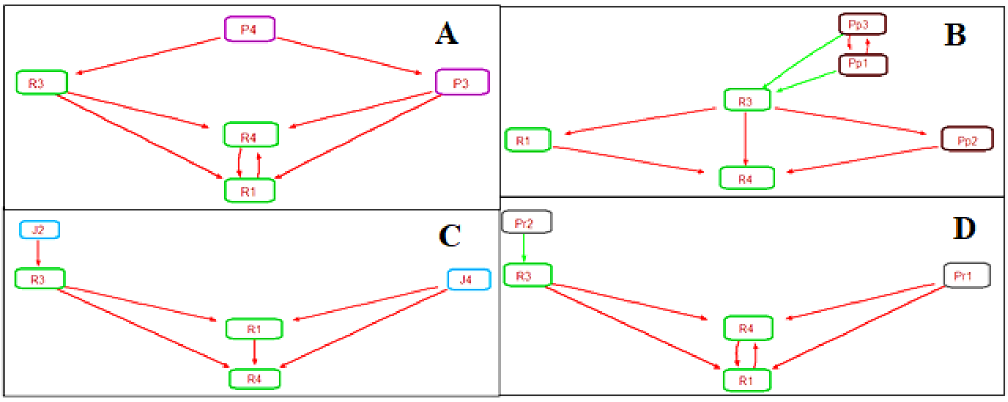


Figure 2. Implicative graph showing relationships between KoT indicators for Task 4.

The results of the qualitative analysis suggest that PSTs tend to associate area with the use of calculations and formulas, through the use of a symbolic register (Caviedes, de Gamboa & Badillo, 2021b; Chamberlin & Candelaria, 2018, Simon & Blume, 1994). Such a tendency explains why PSTs fail to mobilise conceptual elements linked to the measurement of areas, such as properties (Hong & Runnalls, 2020). Broadly speaking, the implicative graphs in Figure 2 show that representations are presented as a key conceptual element within the KoT indicators since they allow PSTs to use diverse resolution procedures. This suggests that representations have an instrumental and organizational value within the KoT subdomain indicators, that is, certain representations allow the use of certain procedures (or justifications, geometric properties, and principles) that would not be possible with the use of other representations. For example, the use of geometric representations allows PSTs to use surface decomposition and reorganization procedures, which would not be possible through the use of symbolic representations. The same geometric register allows the mobilization of properties of conservation and accumulation and additivity, which are not mobilised through the symbolic register. We consider that this instrumental value of representations could have implications for the didactic design of tasks that allow the development of specialised knowledge in PSTs.

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IDEAS OF EARLY DIVISION PRIOR TO FORMAL INSTRUCTION

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Often young children develop ideas of mathematics before they formally meet them at school. Such is the case with early counting concepts. However, little is known about children's early ideas of division. The study reported here investigated the ideas of 114 children (5-6-years old) before they had received any formal instruction about division in their first year at school. A pencil and paper test comprising worded problems with diagrams was read aloud by the teacher. We analysed children's drawings on the diagrams. Results indicate that 74% of children could conceive of at least one division situation prior to any instruction. Some children (20%) could interpret quotitive and partitive division problems. Children drawing on diagrams can provide evidence of their conceptual interpretation of division problems.

INTRODUCTION

Our research interest in children's earliest multiplicative thinking (Cheeseman et al., 2020a), naturally led us to consider how aspects of division interconnect with early multiplication. In particular we were investigating the formation of equal groups from a collection of objects and the enumeration of the group structures. We played dancing games where children made groups of a specified number when the music stopped (Cheeseman et al., 2020b). Whether the thinking in the game was the basis of multiplication or early quotitive division – or both – intrigued us. It is often assumed that young children have no concepts of division before they are formally introduced to division at school. Our earlier research showed that many children achieve early multiplicative reasoning before it is formally taught (Cheeseman et al., 2020b). The research question we sought to answer here was: What concepts of division do young children develop prior to school instruction?

RESEARCH BACKGROUND

Early division

Previous studies have demonstrated that young children (4-5-year-olds) can model division problems using concrete materials before having any formal instruction (e.g., Carpenter et al., 1993), and that children's early understanding of division is underpinned by their experiences of sharing and allocating portions (Correa et al., 1998; Squire & Bryant, 2002). However, Correa et al. argue that an understanding of sharing is not the same as having an understanding of division, as division requires as understanding of the inverse relation between the divisor and the quotient. For example, if 12 sweets are shared between three friends they each receive four sweets, but if 12 sweets are shared between four friends they only receive three sweets each. In

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 131-138). PME.

other words, the more people the fewer sweets each person would receive. This view was confirmed by Squire and Bryant (2002) and Ching and Wu (2021).

Squire and Bryant (2002) explored whether young children (5-8-year-olds) could distinguish between and recognise the role of the divisor and quotient in division problems and whether children found it easier to identify the quotient in a partitive rather than in a quotitive context. Each child was presented with a pictorial representation of the situation and required to provide a verbal response. The same problem (12÷ 4) was used for both experimental contexts. Whilst there was no difference mathematically between the two conditions, as the divisor and quotient were the same in both, the difference between the two conditions was the mental model of division children brought to the problem. The results suggested that children had two different schemas of action each of which was dependent on the nature of the problem context– sharing in partitive division, and forming quotas in quotitive division. The authors argued that providing children with different problem contexts and representations was important to help children to: recognise the dividend, divisor and quotient in a problem; think flexibly in given contexts; and develop a conceptual understanding of the multiplicative relationships. The work of these researchers relates to the present study where we examined children's understanding of different contexts using representations of division.

Recently published work of Ching and Wu (2021) reported that 5-6-year-olds could recognise and reason about multiplicative relationships in partitive and quotitive problems, and that explicit instruction is not a prerequisite for understanding division. Their results also showed that children performed better on partitive situations than quotitive, which resonates with earlier findings (e.g., Correa et al., 1998; Squire & Bryant, 2002). Ching and Wu's findings were particularly relevant to our study.

Matalliotaki (2012) used some similar methodological approaches to the research we report here. She examined 5-6-year-old kindergarten children's capacity to solve quotitive division problem prior to formal instruction. Each child was presented with six problems relating to the context of gloves, socks and football - three presented orally and the same problems were presented pictorially. Matalliotaki found that more children provided a correct response for the pictorial form than the oral (40% compared to 11%). The author found that the pictorial representation enabled the children to keep track of their thinking and collect pairs of objects by focusing on interpreting and coordinating the information in the problem.

Findings of these studies suggest that young children are capable of interpreting and solving partitive and quotitive division problems. Further, that creating mental models or schemas of action for these situations may assist children to recognise the relation between dividend, divisor and quotient – an important underpinning of division.

Assessing young children's mathematical thinking

Pencil-and-paper tests are not commonly used to evaluate young children's mathematical thinking. Such tests involve abstract ideas interpreted through words, diagrams and symbols. It is difficult for children to interpret the questions and to understand how they are required to respond. Written tests are considered inappropriate assessment tools for 5-6-year-old children due to the reading and writing difficulties they present. These difficulties are not confined to young children (White, 2005). For many years mathematics educators have been advocating for more authentic methods of assessing mathematical learning (Ball & Bass, 2000; Clarke & Clarke, 2004). We agree with the sentiments of these authors. However, based on earlier research (Cheeseman & McDonough, 2013) we know that carefully constructed and meaningful pencil-and-paper tests can be used successfully to elicit young children's thinking.

Of course, as an assessment of knowledge and skills the pencil-and-paper test reported here is limited in its scope as a tool to reveal division concepts. Nevertheless, children's responses as shown by their drawings on the test, have provided some interesting data which give insights into children's thinking.

METHOD

The drawn responses to a pencil-and-paper test protocol (Streit-Lehmann, 2019) were analysed to produce the results reported here. Detailed descriptions of the instrument, the sample, and the data analysis follow.

The assessment instrument consisted of two separate forms, each was a page three worded problems. One form had three quotitive division situations where children had to portion objects into equal groups; the other page had three partitive division situations where children had to share all objects equally. Each worded problem on the quotitive division test used matched numbers on the partitive division test ($12 \div 3$, $7 \div 2$, $22 \div 4$) but the problem contexts were different to elicit different thinking. Inclusion of remainders and numbers beyond the children's usual curriculum range (e.g., 22) indicated that the problems became progressively more difficult, yet the language of the test was kept as simple as possible and constructed in short sentences. The protocol required the teacher to read the problems aloud to the children. Diagrams are an element of pencil-and-paper tests known to be difficult for students to interpret (Smith et al., 2011; van den Akker et.al., 2006). As a result, every attempt was made to use simple diagrams recognizable to the children.

An opportunistic sample of students, who were in their first year of school in Australia and had not formally been introduced to division, was obtained by personal links to the second author. From the larger sample of student responses, a randomized sample of 114 students was selected based on: two completed tests for each child, and representation from Government, Catholic and Independent sectors, and multiple classes, representing data across 20 classes from 10 schools.

Categories of responses were proposed for coding and each researcher independently coded a small sample of students' responses. Through discussion, agreement was reached about the interpretation of the children's drawings and the thinking that each type of response indicated. Where there were several acceptable solutions we defined what we considered as "correct". A revised coding system was then applied to the data and results were entered into a spreadsheet. Findings reported here focus on the facility children exhibited with the problems on the test.

FINDINGS AND DISCUSSION

The major finding of the study concerned young children's ability to interpret division contexts. Only about one quarter (26%) of children were unable to provide a correct response to any of the six division worded problems. The corollary is that, 74% (almost three quarters) of children in our sample could provide a correct response and draw their thinking to at least one division worded problem. These children showed some awareness of division, prior to instruction.

Next, we present the percentages of correct solutions for each problem.

		Problem context (calculation)	Correct solution only	Correct solution and representation
Quotitive	Q1	Apples in bags of three (12 ÷3)	67%	43%
	Q2	Socks in pairs (7 ÷2)	51%	25%
	Q3	22 children with 4 at a table (22 ÷4)	41%	32%
Partitive	P1	Candies share between 3 jars (12 ÷3)	64%	41%
	P2	Donuts share between two (7 ÷2)	53%	38%
	P3	4 children sharing 22 cards (22 ÷4)	33%	9%

Table 1: Percentages of correct solutions for each worded problem

Table 1 displays the correct written solutions only and the correct solutions matched to a correct drawn representation of the solution for each diagram. Writing a correct numerical solution only, either as a numeral or text, was easier for the children than drawing their thinking to match their solution. For each question roughly one third of children who could answer numerically could not draw their correct solution (see Table 1). The exceptions to this finding were Q2 *the socks in pairs* (7 ÷2) where half of the children were unable to draw pairs of socks, and P3 *where four children shared 22 cards* (22 ÷4) for which two thirds of children could not share the cards equally. These questions (Q2 and P3) required careful examination to find possible explanations for these results. We noted that drawing the socks in pairs (7 ÷2) was surprisingly difficult for 68% of children.

We hypothesise that the diagram given in the worded problem (Fig. 1) unintentionally made the drawing of solutions more difficult. Perhaps children were trying to match a “left sock” with a “right sock”, or perhaps the illustration of an additional pair of socks was misleading. All we know is that drawing three pairs of socks with one left over was difficult.

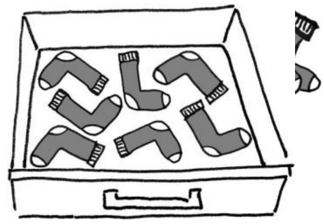

Fig. 1 Socks in a drawer

In Table 2 the numbers and percentages of children who recorded correct solutions matched to correct illustrations to both the quotitive and partitive versions of each division calculation are shown.

Calculation	Quotitive division	Partitive division	n=students (%)
12 ÷3	Q1 Apples in bags of three	P1 Candies shared into 3 jars	23 (20%)
7 ÷2	Q2 Socks in pairs	P2 Donuts share between two	17 (15%)
22 ÷4	Q3 22 children, 4 at a table	P3 4 children share 22 cards	2 (2%)

Table 2: Students correct in both quotitive and partitive worded problem

Twenty-three of 114 children (20%) were able to demonstrate their correct solution to Q1 and P1 (see Fig. 2 for an example).

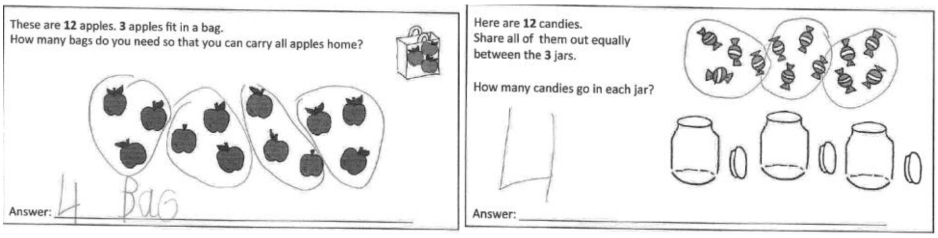

Fig. 2 Child 17’s response to quotitive and partitive problems for 12 ÷3

This child used a grouping strategy for each problem and recorded a correct response. Of note is that she recorded ‘bag’, which shows a connection to the problem. For P1 other children drew lines from each candy to a jar, reflecting one-to-one sharing.

Seventeen children (15%) were able to correctly solve and draw a solution to Q2 and P2 (see Fig. 3 for an example). Child 33 used a grouping strategy for the partitive problem rather than an action of sharing or drawing a line from each donut as might be expected, and also recorded 3 on each plate. Of the 23 children who responded correctly for P2, 13 acknowledged the half, by dividing the remaining donut in two with a drawn line.

Fig. 3 Child 33’s response to quotitive and partitive problems for 7 ÷2

Two children (2%) successfully solved and drew a solution to Q3 and P3 (see Fig 4 for an example).

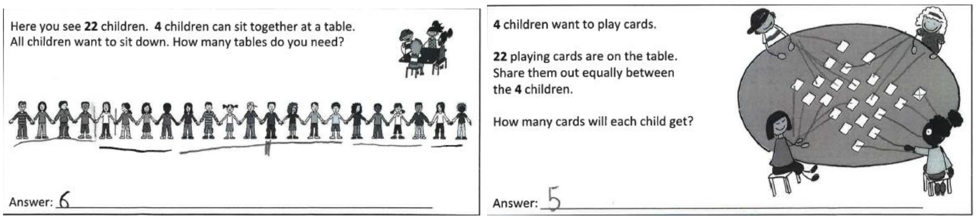

Fig. 4 Child 87’s response to quotitive and partitive problems for 22 ÷4

Unlike the previous examples in which the children used grouping, child 87 used lines to distinguish the groups in Q3 and lines to show the distribution of the cards to each child in P3. The other child who successfully solved both problems used circles to show grouping in each instance. The fact that only two children successfully solved both problems highlights the complexity of these contexts.

Three major findings result from this study.

- Almost three quarters (74%) of children showed some awareness of division prior to instruction. They could provide a correct response and draw their thinking to a least one worded division problem.
- Twenty percent of young children could interpret both quotitive and partitive division problems. These results extend Matalliotaki’s (2012) findings by focusing on both conceptual forms of division.
- Children’s drawing on diagrams can provide evidence of their conceptual interpretations of division problems.

CONCLUSION

The limitations of the reported empirical study include: its assessment method – i.e., the use of a pencil and paper test to investigate the thinking of young children, the randomised sample of responses was taken from data gathered in a relatively small geographical region, and the results are indicative rather than broadly generalisable.

We posed the research question: What concepts of division do young children develop prior to school instruction? Our findings indicate that some young children (26% of those we analysed) displayed no knowledge of division via a pencil-and- paper test.

This is unsurprising as the children had not formally met division and the test method was unfamiliar to many of them. Of interest, was the fact that 74% of children in this study were able to conceive of either a quotitive or partitive division context for at least one worded problem. We have found no other research evidence that has reported similar findings with 5-6-year-old children. Our results indicated that the context determined the way children solved each worded problem. This finding echoes the work of Squire and Bryant (2002) and Ching and Wu (2021). Our study raises questions about the suitability of particular worded problems researchers select and the diagrams we use with children. For example, the static objects depicted on paper are difficult for many children to use to represent their dynamic thinking about partitive division. We contend that it may be simpler to draw a loop around a quota of objects to divide them into a new unit, than to organise lines to represent sharing objects one by one to a person or target. Therefore, the action of drawing could impact on the results of a study and this finding requires further study.

The results presented here make us keen to continue to investigate young children's conceptual development of division. The main implication from our findings is to question the assumption that young children have no early multiplicative concepts - including rudimentary ideas of division. We are conscious that there is a lot that we are yet to discover about early mathematics concept development in young children. We encourage fellow researcher to investigate such thinking. Our argument is not about arithmetical calculations of division problems but the interpretation of contexts that require either partitive or quotitive thinking, as almost three-quarters of the children whose tests we examined could display some conceptual awareness of division, prior to any formal instruction.

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EXPLORING THE AFFORDANCES OF A WORKED EXAMPLE OFFLOADED FROM A TEXTBOOK

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In designing a set of instructional materials to use in his classroom, a teacher heavily offloaded items (e.g., worked examples, practice questions, exercises) from school-based materials and textbooks. At a cursory level, one may easily dismiss this as a thoughtless lifting of curricular materials. But upon careful analysis – as is detailed in this paper – a different picture emerges. In this paper, we describe and analyse how this teacher adapted one of many worked examples, beyond its typical use, during instruction to develop students' conceptual understanding of proportionality. We argue that he noticed and harnessed multiple affordances in a single item that most teachers may overlook, without the need to modify the example, and propose a notion of "affordance space" as a lens to view teachers' design of instructional materials.

INTRODUCTION

Emerging research on Singapore mathematics teachers as designers of instructional tasks and materials has illustrated the innovative ways that teachers can adapt and improvise tasks, representations, and sequencing to achieve various instructional goals (e.g., Cheng et al., 2021; Leong et al., 2019). However, there are teachers who choose to heavily rely on tasks and procedures from curricular materials for instruction, otherwise known as *offloading* (Brown, 2009). While using an item directly from a textbook may appear to be inherently less complex and involve less "design thinking", Brown (2009) noted that offloading should not be mistaken for being inferior to adapting or improvising, nor does it necessarily imply teachers who offload are negligent or less competent. In a study conducted by Amador (2016), four teachers with 1 to 17 years of teaching experience engaged in offloading, as well as adapting and improvising; two of whom initially offloaded and shifted to adapting during a lesson. Furthermore, as Choy and Dindyal (2021) demonstrated, despite offloading "typical" tasks from past-examination papers, a teacher, Alice, was able to implement them in unexpected and productive ways to develop students' conceptual understanding. They proposed this was due to the teacher's ability to "effectively *notice* and *harness the affordances* of these materials in mathematically productive ways" (p. 196). We build on Gibson's (1986) idea and refer to the set of possibilities for how a task may be used as the affordances of a task. In addition, we follow Choy and Dindyal (2021) in seeing that the affordances of a task are always there, "independent of teachers' ability to perceive them" and "do not vary as teachers' instructional goals or needs change" (p. 198).

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In comparison to the abundance of research on the affordances of "challenging" and "rich" tasks, there is an underrepresentation of research on the affordances of "typical" and "routine" items. Hence, the aim of our study is to examine the affordances of a worked example that was offloaded from a textbook by a secondary mathematics teacher, Peter (pseudonym), but somehow implemented in a non-typical and non-routine way. We hypothesize that Peter engaged in a nuanced form of offloading based on noticing and harnessing *multiple* task affordances—which may not always be immediately obvious—simultaneously. We propose the notion of an *affordance space* to describe the cognitive space in which teachers work with tasks whose dimension is dependent on the number of affordances they perceive. The more affordances a teacher perceives in a task, the great number of ways they can use the task beyond its "typical" procedural use. Further details of the affordance space will be discussed later. Our research questions are: What affordances does a teacher perceive in a typical worked example that influenced their decision to offload? And how do these affordances influence their implementation of the worked example?

METHODS

The data reported is drawn from a larger study on secondary mathematics teachers' design of instructional materials (IMs). Four teachers from two local secondary schools in Singapore engaged in 3 to 6 design cycles involving individual design of IM drafts, one-on-one semi-structured interviews after each draft, and subsequent professional learning community (PLC) discussions with their colleagues. The topics of their IMs were Ratio and Rate, with an underlying emphasis on proportionality. Then, the teachers implemented their IMs and one-on-one semi-structured interviews were conducted after every lesson. The teacher discussed in this paper is Peter (pseudonym). At the time of the study, Peter had over 10 years of mathematics teaching experience, predominantly at upper secondary (Year 11-12), and it was his first-year teaching Year 9 mathematics. He implemented his IMs over four lessons, each lasting 40-70 mins. All interviews, PLC discussions, and lessons were recorded and transcribed. Peter's IM drafts and the curriculum materials he used—a set of school-based worksheets and a textbook—were collected.

To analyse the data, we adopted two grain sizes of analysis. Firstly, at the *item-level* we examined the individual items (e.g., worked examples, practice questions, investigation tasks) within Peter's worksheets to determine: (i) instances of offloading, adapting, or improvising; and (ii) potential task affordances that influenced Peter's offloading, adapting, or improvising. Then, at the *set-level* we examined Peter's tasks as a collective, to determine overarching instructional goals. This dual item-level and set-level analysis was conducted initially on Peter's worksheets and his design interviews, then the implementation and post-lesson interviews were used to triangulate the affordances and goals.

FINDINGS

In this section, we begin by summarising Peter's selection of items for his IMs before we present a vignette of how Peter had used one of the worked examples to develop students' understanding of proportionality. We then highlight two of the affordances inferred from Peter's use of the example. Table 1 summarises the offloads, adaptations, and new items we determined in our first round of item-level analysis. Out of a total 35 items, 27 items were offloaded, suggesting that Peter heavily relied on the school-based worksheet and textbook. Due to length constraints, we will focus our discussion on one item that was offloaded from the textbook in his Ratio worksheet to explore the affordances Peter noticed and harnessed to adapt the task during instruction.

	School-based worksheet		Textbook		Peter's new items	Total items
	Offloaded	*Adapted*	*Offloaded*	*Adapted*		
Ratio	14	2	6	1	1	24
Rates	6	2	1	2	0	11
Total	20	4	7	3	1	35

Table 1: Summary of items in Peter's instructional materials

The worked example Peter offloaded resembled those *typical problems* (Choy & Dindyal, 2021) found in any textbook or examination paper about ratios (Figure 1). It shows how to find a ratio between two quantities that have different units, followed by two short questions for students to ponder. In general, worked examples are used to demonstrate a solution method for students to imitate. Hence, most teachers would typically read these with students, possibly bringing key steps to students' attention, before applying the same method to a similar problem. This is how one would expect Peter to use the worked example, especially given that he directly offloaded it from the textbook into his worksheet and followed it with a similar question ("Andrew and Sueda took 90 seconds and $\frac{21}{3}$ minutes respectively to answer an IQ question. Find the ratio of Andrew's time to Sueda's time.").

Yet, this was not how Peter implemented the item, nor was it his intention to use the worked example as a demonstration for the subsequent question. Instead, Peter used the task to engage the class in a discussion about a fundamental concept of proportionality over a 10-minute episode. He briefly went over the working in four short sentences, and then quickly moved to focus on question (a):

Peter: You're supposed to find the ratio of Bobby's time to Aravin's time. Do take note if you are comparing using the same units. In this case, Aravin's time converted into minutes, that should give you three over two minutes. Then we actually can compare the ratio. Now, question! What if instead of converting Aravin's time from seconds to minutes, what if we compare in seconds?

Students: Times 60!

Peter: Before we even calculate, do you think the ratio would be the same?

Students: No... (some students begin to write)

Peter: Wait, ah! Don't calculate first. Wait, wait, wait! What happens if we compare them in seconds? Who says it will be different? Raise your hand.

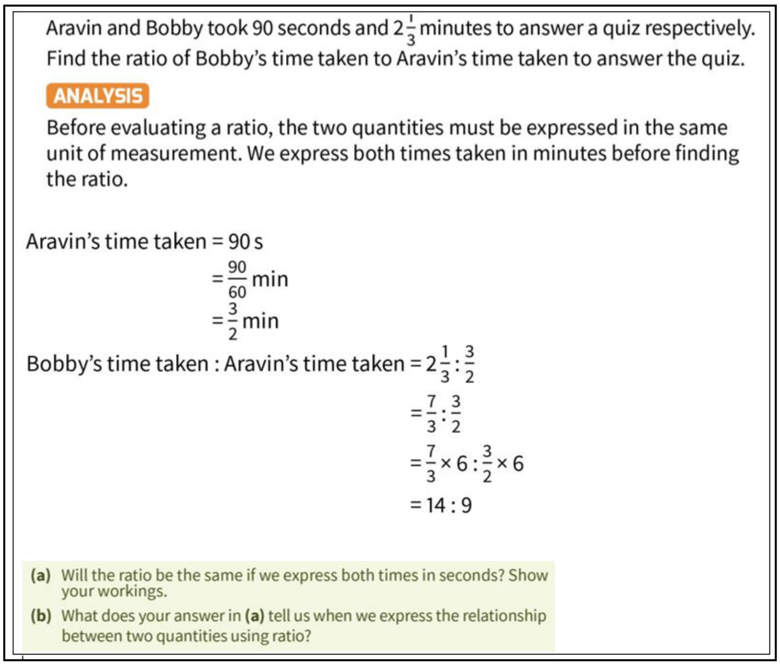
The students looked around the classroom. Those who had initially raised their hands lowered them slowly, and those who still believed it would be different sheepishly kept their hands up as low as possible. Peter asked them again:

Peter: Who thinks the ratio will be different? It's okay. I remember seeing three or four hands, then becomes two hands now? I was pretty sure I heard more than one voice. Who says it will be the same? Raise your hand!

Some students began writing on their worksheets while others continued to look around the classroom. Out of a class of 38 students, eight students raised their hands. It was evident that there was uncertainty amongst the class and clearly the worked example was not useful in resolving this. Peter had fostered curiosity amongst the students, creating the need for the class to investigate this before moving on to the next task. Peter orchestrated a whole-class discussion in which he asked the students to suggest the actual working of the solutions to the same question in a different unit. As he followed their instructions, he drew arrows on the side of each step (Figure 2) and said, "Whatever you do to one side, you do to the other side". When the students shouted out the solution without stating their reasoning, Peter asked them, "How do you know?" Eventually the class arrived at the solution 14 : 9 and numerous students yelled, "They are the same!" One student exclaimed, "They are *equivalent*!".

Affordance 1: Developing conceptual understanding about proportionality

As an experienced teacher, Peter was likely aware that worked examples are commonly used for demonstrating the steps to solving a problem. However, it was not used to ensure students understood the necessity of converting quantities to the same units, nor was it about how to simplify ratios. Instead, Peter's requests for students to think about whether the ratios would be the same or different "before we even calculate" illustrated that his intention was more focused on developing students' conceptual understanding about ratio. In an interview about one of his worksheet drafts, he mentioned that it was "a thought I'd like to plant in their heads". He paid little attention to the solving procedure and utilised the worked example as a foundation for exploring with students the preservation of proportionality (i.e., the ratio will be the same, regardless of the units). In the post-lesson interview, Peter revealed that he deliberately spent more time on the worked example because he didn't "want them to think proportionality questions always [involved] systematically comparing the process. I also want them to think in context as well." Furthermore, this use of worked examples to develop students' conceptual understanding about proportionality was not exclusive to this item. It was observed in another worked example on comparing the rate of fuel consumption of a car using two different units (Figure 3). Evidently, Peter saw the affordance of using worked examples to go beyond demonstration of procedures.



Aravin and Bobby took 90 seconds and $2\frac{1}{3}$ minutes to answer a quiz respectively. Find the ratio of Bobby's time taken to Aravin's time taken to answer the quiz.

ANALYSIS

Before evaluating a ratio, the two quantities must be expressed in the same unit of measurement. We express both times taken in minutes before finding the ratio.

Aravin's time taken = 90 s
$= \frac{90}{60}$ min
$= \frac{3}{2}$ min

Bobby's time taken : Aravin's time taken $= 2\frac{1}{3} : \frac{3}{2}$
$= \frac{7}{3} : \frac{3}{2}$
$= \frac{7}{3} \times 6 : \frac{3}{2} \times 6$
$= 14 : 9$

(a) Will the ratio be the same if we express both times in seconds? Show your workings.

(b) What does your answer in **(a)** tell us when we express the relationship between two quantities using ratio?

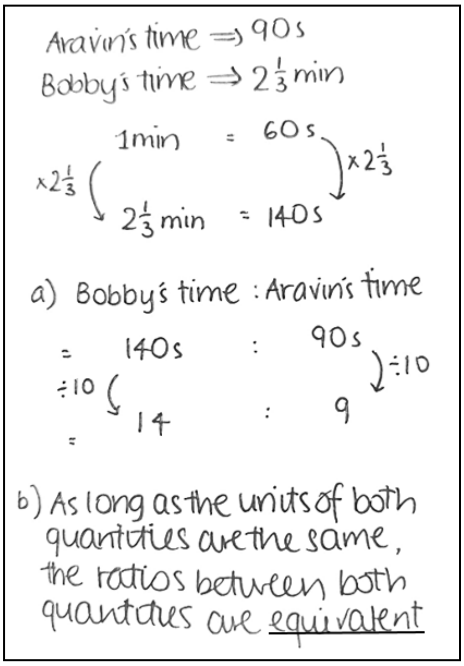


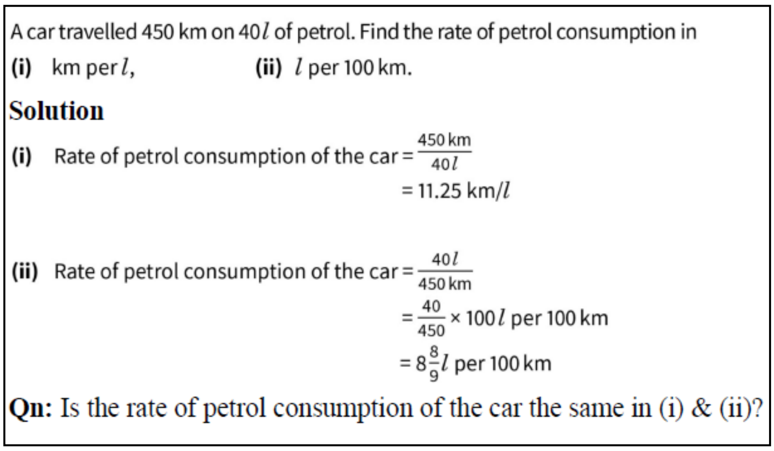
Figure 1: Worked example offloaded from the textbook onto Peter's worksheet

Figure 2: Peter's written working on the whiteboard (rewritten for readability)

Affordance 2: Representations that make proportionality more visible

If Peter had written the students' working on the board in a similar manner to the worked example, he would have still been able to show that the ratios were the same. Yet, he chose to adapt from the worked example and adopt the use of a new representation, the arrows (Figure 2). With the worked example projected onto one side of the whiteboard and Peter's writing on the other, a comparison of the two would show that the underlying proportionality in simplifying ratios is more visible when using the arrows. On top of serving as a reminder to students that simplifying ratios requires treatment to both quantities, it illustrates why proportionality is preserved because of the equal treatment to both quantities. Hence, an affordance of offloading this worked example directly from the textbook was also to be able to demonstrate in contrast to another representation of proportionality that would aid students in making sense of the solving procedure.

There was no clear evidence in the worked example about how Peter came to using arrows. However, when we analysed the implementation of other worked examples, which did not have arrows present on the worksheet, we found Peter had also used this arrow method as an alternative representation (Figure 4). Furthermore, he asked students "I know in the example there isn't an arrow, but can you please write in the arrow in the example just for you to see, so you can follow" on another worked example. Our analysis of the 35 items on the worksheets at the set-level identified only three instances of some form of arrows; one was an improvisation, and two others were adaptations. However, when we zoomed out to examine the 35 items implemented during the lesson, we noticed he had adapted them all by consistently using arrows as a representation of proportionality. This consistent and well-rehearsed use of arrows suggests that although he offloaded most of his worksheet items from the school-based worksheet and textbook, he intended to adapt the implementation all along.


A car travelled 450 km on 40l of petrol. Find the rate of petrol consumption in

(i) km per l, **(ii)** l per 100 km.

Solution

(i) Rate of petrol consumption of the car $= \frac{450 \text{ km}}{40l}$

$= 11.25$ km/l

(ii) Rate of petrol consumption of the car $= \frac{40l}{450 \text{ km}}$

$= \frac{40}{450} \times 100l$ per 100 km

$= 8\frac{8}{9}l$ per 100 km

Qn: Is the rate of petrol consumption of the car the same in (i) & (ii)?

Figure 3: Another offloaded worked example

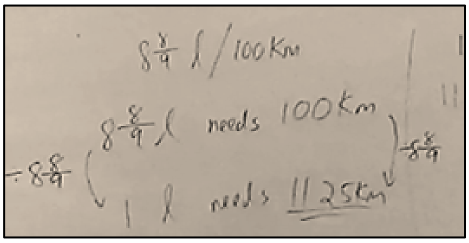


Figure 4: Another instance of Peter's use of arrows

In this 10-minute episode, Peter's implementation of the worked example was noteworthy for two reasons. Firstly, he demonstrated how typical worked examples need not be used for imitating solving procedures but could instead be a catalyst for whole class discussions on fundamental components of a concept. Secondly, although he essentially ignored the procedural elements of the worked example, he was still able to target procedures related to proportionality through his use of arrows to make the reasoning process more visible to students. This dual achievement of both conceptual and procedural developments is an example of how multiple affordances can be noticed and harnessed within a typical item.

DISCUSSION AND CONCLUDING REMARKS

The research questions of our study were: (i) What affordances does a teacher perceive in a typical worked example that influence their decision to offload? And (ii) how do these affordances influence their implementation of the worked example? Instead of using the worked example in the usual manner to demonstrate a solving procedure, Peter perceived a key affordance as being able to facilitate an investigation about the preservation of proportionality when forming ratios involving a unit conversion. Furthermore, he utilised and demonstrated to students how adopting a different representation—the arrows—when simplifying ratios could be useful in making the underlying proportionality in ratio problems more pronounced and easier to follow.

Building on the work of Choy and Dindyal (2021), we propose the notion of an affordance space. On the basis that the potential of a task is dependent on the teacher's ability to notice and harness its affordances, teachers who see a task's sole affordance to facilitate procedural development can be said to be working in a one-dimensional affordance space and therefore less likely to use the task in adaptive or productive ways. However, teachers who notice multiple affordances of a task work in an affordance space of higher dimension and can take the task in various directions beyond procedural development. As Alice in Choy and Dindyal's (2021) study and Peter in this paper demonstrate, research on the affordances of typical task can make clearer the work of teachers, while also demonstrating the complexity of teachers' work in the interesting ways they may use such tasks. Unfortunately, amongst this sea of innovative teachers adopting challenging tasks and adapting and improvising others, teachers like Peter and Alice are easily missed or disregarded.

Lastly, Brown's (2009) definition of offloading does not seem to fully capture the phenomena we observed with Peter. If we adopt the notion that offloading is fundamentally judged on the instructional outcomes, then Peter cannot be said to be offloading at all. But what does that mean for his design of instructional materials which clearly demonstrate the offload of the task from one resource to his worksheet? Furthermore, Amador (2016) noted that Brown's (2009) description of teachers' interactions with curriculum resources implied a static interaction. However, in her study, as well as ours, she documented two teachers who shifted from offloading in lesson design to adapting during instruction. While their shift was triggered by unexpected incidents that meant students would be unable to achieve the instructional goals, interestingly, in the case of Peter, his shift was not triggered during the lesson. His adapting was evidently planned due to the casual and well-rehearsed way he skipped through the solution method to focus on the preservation of proportionality, as well as his consistent use of arrows throughout his implementation.

This brings to question the need to redefine offloading, or at least elaborate and extend on it to encapsulate such instances. In our analysis of Peter's implementation, we wondered if there was possibly no such thing as completely offloading because every teacher brings with them their own unique knowledge and contexts. On a broad-grained scale we might see teachers simply carrying out the task as described in the textbook—or as Brown (2009) gave the example of teachers reading from the curriculum materials—but when we zoom in to the teaching episode, we can likely capture teachers asking additional questions or even very nuanced moments where the teacher provides some alternative scaffolding that was not prescribed in the textbook.

As our findings pertain to a single teacher, and are hence not generalizable, future research should aim to study the various affordances that teachers notice and attempt to *simultaneously* harness in typical tasks to develop the concept of affordance space. In particular, instances where there appears to be a disconnect (or shift) between how teachers interact with tasks during lesson design and implementation would be worthwhile pursuing. To do so, a similar item-level and set-level analysis approach used in this study would help to identify and examine shifts in teachers' interactions at different grain-sizes.

Acknowledgement

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NEGOTIATING MATHEMATICAL GOALS IN COACHING CONVERSATIONS

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Content-focused coaching aims to focus the coach-teacher conversations on the quality of mathematical goals and associated activities while co-planning a lesson. The coach's role is to support the teacher to articulate a mathematical goal that represents an important mathematical idea, to plan activities in support of that goal, and to anticipate how students will respond to the mathematical activities. We use a framework that emphasizes the dialogic and mediated nature of the coach-teacher conversations. We found that coaching conversations focused primarily on instrumental aspects of lesson planning and only less so on deeper articulations of content, and we found differences across coaches' practices. The findings provide nuance to empirical findings of coach-teacher conversations.

SUPPORTING TEACHERS TO ENGAGE IN AMBITIOUS INSTRUCTION

There are ongoing efforts in many countries to engage students in disciplinary practices, which entails greater support for teachers (cf. Andrews, 2013; Li & Ni, 2012). For teachers who are implementing challenging instructional practices, one type of professional support that is growing internationally is coaching (c.f, Kickbusch & Kelly, 2021). Coaching provides teachers feedback (Boston & Candela, 2018) and focuses them on core aspects of instruction (Coburn et al., 2012). Coaching typically involves someone with content and pedagogical expertise who works in a one-to-one setting with a teacher. Many coaching models have a three-part *coaching cycle*, in which the coach and teacher co-plan a lesson, the teacher and /or the coach teaches the lesson, and the coach and teacher then reflect on the lesson together (Campbell & Malkus, 2011; Gibbons & Cobb, 2016; Russel et al., 2020).

There needs to be a greater understanding of how coaching supports teachers' capacity to enact ambitious instructional practices (Gibbons & Cobb, 2016); mathematics educators need a more nuanced understanding of how coaches engage teachers in substantive mathematical and pedagogical discussions. A deeper understanding of the nature and impact of coaching conversations will inform how to engage teachers in the challenges necessary to transform their teaching.

THEORETICAL FRAMEWORK

Our framework emphasizes the dialogic nature of learning (Bakhtin, 1986; Vygotsky, 1986). Vygotsky emphasized the asymmetric nature of the dialogue between two parties in which one is well-versed in the principles of a discipline and the other is conversant in everyday formulations of the content. Vygotsky emphasized that for the learner there is a dynamic interplay between the disciplinary concepts introduced by

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the more expert member and the learner's informal and empirically-based formulations. Moreover, dialogue involves a process of interanimation (Bakhtin, 1986; Gee, 1999) in which multiple voices become intertwined in the thoughts and speech of the interlocutors. That is, dialogue involves learning by both members of the dyad and a shared way of conversing about the content in question. Importantly, the learner begins to incorporate characteristics of disciplinary discourses.

Considerations about the role of *communities of practice* in mathematics specifically has gained attention as a way to consider the situated aspects of learning, including coaching (Voskoglou, 2019). Within the context of coaching, the coach-teacher pair bridge multiple communities of practice as they negotiate principles of mathematics instruction, building from disciplinary and practical knowledge to develop a common language around and vision of instruction. Their work is mediated (e.g., Wertsch, 1995) by a range of contextual factors, such as individual characteristics, professional experiences, and curriculum materials.

Our framework focuses on four factors that mediate the coach-teacher interactions in a coaching cycle: coach characteristics, teacher characteristics, the coach-teacher relationship, and the content of the coaching cycle. A key coach characteristic is their *coaching stance* (Gillespie et al., 2019), of which we identify two basic stances: the reflective and the directive stances. The reflective stance involves inquiry into the teacher's thinking; the directive stance involves direct assistance to the teacher in the forms of evaluation, explanation, and suggestions. Teacher characteristics include their prior practices, beliefs, and knowledge. The coach-teacher relationship involves communication style and trust that evolves over multiple coaching cycles. The content of the coaching cycle pertains to the mathematical goals and tasks identified by the teacher and coach as the focal points of the lesson. See Figure 1 for a visual of our framework.

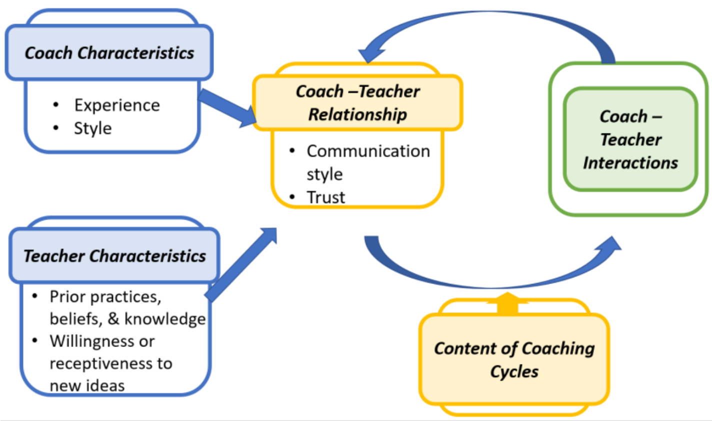

Figure 1: Diagram of Coaching Cycle

CONTENT-FOCUSED COACHING

Content-focused coaching has emerged in the USA as an effective model to help teachers develop productive instructional practices specific to their content area (West & Staub, 2003), and has been shown to have positive effects on teachers' instructional practices and student achievement in the area of literacy (Matsumura, Garnier, & Spybrook, 2012) and mathematics (Campbell & Malkus, 2011; McLaughlin, 2012; Neuberger, 2012). Stein et al. (2021) recently studied content-focused coaching specific to mathematics with 32 coaches and found that one-on-one content-focused coaching around the planning of a lesson led to positive outcomes. Results also indicated that the content-focused coaching model supported a shift from a focus on *what teachers* will do in a lesson to *how students* might think in a lesson, a trend we contend could support a focus on lesson goals.

RESEARCH METHODS

This study emerged from a larger study in which we designed and researched an online three-part professional development model for mathematics teachers in rural contexts (Choppin et al., 2020). We designed the three components as a set of coordinated experiences that took place across two academic years. Part of the model included video-based coaching in which the coach and teacher met via Zoom to plan the lessons, the teacher video-recorded the lesson using Swivl technology, the coach and teacher annotated the video in a Swivl library, and then met afterward to discuss the lesson via Zoom. In this paper, we focus only on the planning meeting conducted in advance of the lesson.

We focused on four cycles of coaching for five coach-teacher pairs involving four coaches and six teachers. We analyzed a total of 20 planning transcripts. Our analysis focused on the discussion around the mathematical goals, using the rubric in Table 1. We parsed the transcripts of the planning conversations into stanzas, which ran anywhere from four to 20 turns of coach-teacher conversation. Using a process that involved pairwise coding and a consensus discussion, we coded the stanzas as having or not having a mention or discussion around mathematical goals. Then all stanzas were coded line by line using the rubric in Table 1.

Rating	Description
1	Discussion focuses on mathematical goals without the connections mentioned in the level 3 rating, such as when the coach presses the teacher to clarify or revise the mathematical goal.
2	Discussion focuses on one or more of the following, without explicit connections made to the mathematical goal: the task, potential student strategies, or students' prior mathematical experiences.

3	Discussion entails one or more of the following connections central to core principles of content-focused coaching: the disciplinary connections between the goal and task; the ways in which the task supports student engagement with the goal; forms of student thinking that represent understanding of the goal; or how the goal represents part of a connected set of mathematical experiences or ideas.

Table 1: Rubric to analyze discussion around mathematical goals

Our research questions were:

1. To what extent did the coaching conversations highlight important connections related to the mathematical goals in the planned lessons?
2. What were the differences between the contributions of the coaches and teachers in the coaching conversations?
3. What were the important differences across the coaches' practices?

RESULTS

We discuss the results in order of the three research questions. In terms of the first question, nearly three-quarters of all turns were coded as focused exclusively on the goals (27%), task (32%), or potential student strategies (15%). By contrast, only 17% of all turns were coded at Level 3. These findings highlight the instrumental nature of the coaching conversations in negotiating the lesson plan elements versus engaging in deeper discussions around how those lesson elements represent connections to broader sets of mathematical experiences or topics. Below, we present samples from the data that represent these findings. In Episode 1, the coach (Reiss) pressed the teacher to clarify the goal.

Reiss: Students will evaluate a situation and determine—to determine and apply whether to use multiplication or division. Students will be able to interpret a remainder? Or—because we want them to be able to interpret what that remainder actually means in the situation—in a context.

Sandoval: Interpret situation in context, if or when, if remainders are part of the answer. Hmm, that's still not clear. No. Students will interpret what to do with remainders? [Laughter] Or how to—I know that over here it says that they have to interpret it. Including problems which remainders must be interpreted. Maybe it's just that simple.

In Episode 2, the coach (Bishop) and teacher worked out details of the task.

Bishop: What would be your next steps? How would you move them towards the symbolic expression, or would you want to wait to do that? What's your thinking around that?

Wise: I would draw a triangle and label the sides, a, b and c. Then have them draw the squares off of it, and see how—could they represent the area of the square. Hopefully they would say, squared and then how would they represent the other square on the leg. There'd be a picture of the right triangle with a squared, b squared and c squared in the square.

Bishop: Oh, I think it's great, yeah.

In Episode 3, the coach and teacher engaged in a substantive discussion (rated a 3) in which the coach and teacher connected the goal to bigger mathematical ideas.

Reiss: Because they might. Even though—I mean, I divided in number one to find the answer, I can see a kid actually working the opposite way and multiplying up to try to get to that answer of twenty-four. In that sense, if we think kids—if we really want them to understand multiplication and division and how to apply it to a situation, that would be different than taking a situation and deciding, "Do I have to use multiplication, or do I have to use division?" They might work in the opposite way.

Sandoval: I see what you're saying, because now that you're saying that, I'm, "Oh, yeah, we just spent a week talking about factors, so—" They might work in an opposite way saying— because today we did review, that they're—when I asked, "How are factors important to the multiplication process?" Then they were, "Oh!" Well, this is why." Yeah, so, [laughter] oh, boy.

In terms of the second question, coaches contributed more frequently (58% of total turns) than the teachers and had 2.6 times as many turns rated at Level 3 than the teachers. By contrast, the teachers had nearly as many turns rated at Level 2 (170 compared to 183) as the coaches, which constituted 64% of the teachers' turns and only 50% of the coaches' turns. This indicates the coaches focused more consistently on the disciplinary connections than the teachers and less on the instrumental aspects of lesson planning, even though just over half of the coach turns were rated at Level 2. Roughly half of the Level 3 codes for the coaches were *the goal and task are connected from a disciplinary perspective*. Episode 4 is an example of the coach providing most of the explanations and connections.

Lowrey: Yeah. In this one—because I know that when they're breaking up the brownies, it's like each brownie has its own area.

Fernandez: Right.

Lowrey: And if we step back, just back a step, I would think that they're really actually practicing fractions within a set. It's just using like, each brownie would have its own area and model, but it's really a set of brownies. So, it's a set of seven brownies that we're going to share among four people.

Fernandez: Right.

Lowrey: So, the idea of division, and what results out of this division is a fraction. Like, what each person's actually going to get, to be able to have that?

Fernandez: Right.

In terms of the third question, there were differences across coaches in the distribution of turns in the rubric categories. Alvarez's turns were evenly distributed across the three categories, while Bishop's turns were primarily concentrated at Level 2 (58%). The other two coaches were similar to Bishop but had more balanced distributions, as seen in Table 2.

Rating	Alvarez	Bishop	Lowrey	Reiss
1	32	31	18	28
2	35	58	46	45
3	33	12	36	27

Table 2: Percentages of coach turns in each category

These differences primarily point to distinctions in the characteristics of the coach; we hope to better understand the impact of teacher characteristics when we analyze the coaching cycles from the next two years of data in which the same coaches worked with different teachers.

DISCUSSION

Our results highlight the instrumental nature of much of the discussion between teachers and coaches (see Episode 2) despite an explicit focus in the coaching model on engaging students in substantive discussions around content. The high number of turns focused on goals (27%) suggests an emphasis on the part of the coaches to identify a goal that is clear and connected to a big mathematical idea, as illustrated in Episode 1. The results also illustrate the coach's role as the expert responsible for directing the discussion, as seen in Episode 4. The coaches had 58% of the overall turns and 72% of all of the turns rated at Level 3. The variation in the patterns of coach moves reveals, unsurprisingly, differences in the coach characteristics that contributed to the dynamics of the discussions. The study provides necessary nuance into coach-teacher discussions that will hopefully serve as a reference for future coaching studies in the US and elsewhere.

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FAIRNESS IN POLITICAL DISTRICTING: EXPLORING MATHEMATICAL REASONING

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For this paper, I explored the informal reasoning of undergraduate social science students in a mathematics class. They were looking into the mathematics of political districting, in particular gerrymandering. Using Sellars' notion of the space of reasons and analytic categories from the socioscientific issues literature, I examined the reasons students gave for the positions they took. I observed the way mathematics played a role in their reasoning and, how, when they addressed a social issue, their reasoning was more holistic. The analytic categories illuminated my data on how mathematics was integrated into the students' informal reasoning.

INTRODUCTION

Reasoning is an essential component of mathematics education, but in a recent article, Kollosche (2021) notes that there is a dearth of research on what reasoning takes place outside the formal kinds, such as deductive. He also states that there is no theoretical framing for reasoning in the mathematics education literature. In this study, I examine students' reasoning in a mathematics activity that engages with a social issue. I am interested in reasoning other than the objective, detached, formal kind. Pursuing this interest, I explored the question of fairness in political districting, a topic seen to be integrally related to mathematics (Staples & Evans, in press). I want to know how student use mathematics in their reasoning and how the mathematics interweaves with their personal values and the social context of the activity.

Research by Byers (2007) shows that connection to an authentic situation improves students' ability to reason. They are not just algorithmically completing a task, but grappling with ambiguity, values, and complexity. Addressing social concerns in mathematics education improves students' ability to conjecture and connect mathematical knowledge to social issues. This helps them becoming socially responsible citizens (Labaree, 1997). Applying mathematics to social issues moves education away from teaching of 'facts' *about* socio-mathematical issues, such as climate change or gerrymandering, to involving the reasoning about such issues.

In this paper, I use the theoretical framework of Wilfred Sellars' (1963) "space of reasons," which posits that knowledge is not accumulated through experience or rational thought but through justifying with reasons. That is, in the giving of reasons when grappling with an ill-structured situation a person becomes aware of their decisions and justifications (Chang & Chiu, 2008). In addition, I also draw on the informal reasoning categories of Sadler et al. (2007), as analytic tools. Sadler et al. are

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based in the socioscientific (SSI) field, which I posit aligns with Sellars' space of reasons. In this study I ask students to engage in a political districting activity. I ask what kinds of reasons are given and how mathematics fits in with those reasons.

THEORETICAL FRAMEWORK

Space of Reasons

The space of reasons (Sellars, 1963) is a philosophical notion that is a move away from mental representation toward increased awareness of the reasons one chooses to justify a claim. Giving reasons is a shift from problem-solving, in which the focus is based in trying to figure things out, to the contemplation of a situation in which contextual factors and emotional stance participate in the decision making of something that may not be able to be figured out. In this way, the space of reasons embraces broad reasoning, one not solely based in rational, objective thought. The purpose for using the space of reasons as a theoretical foundation is that it provides an opportunity to see students become aware of their own values and stance through the reasons they express. Mackrell & Pratt (2017) bring Sellar's space of reasons into mathematics education and suggest that: "human minds develop through an initiation into the space of reasons in that our thoughts and actions are increasingly guided by what there is reason to think about or do" (p. 427). I contend the sort of activity that I gave to the students in this study contributes to students reasoning because the context of a districting activity gives them an increased sense of what they can do.

Informal Reasoning

Sadler et al. (2007) who combine science with social issues, known in science education as SSI, gives currency to the space of reasons. Through their empirical work, they have come up with four kinds of reasoning in the context of working with a social issue: [1] complexity, [2] skepticism, [3] perspective taking, [4] inquiry. Briefly, [1] *complexity* is an unwillingness to commit to a simple solution because of an awareness of the multiple factors inherent in a situation. [2] *Skepticism* is an awareness of potential bias in a situation. [3] *Perspective taking* is looking at a situation from different positions and recognizing that in the social world different people have different priorities. [4] *Inquiry* is an exploration of a situation which may require further investigation.

The affinity between the space of reasons and the reasoning categories can be seen in the following passage:

> a key to interpreting a phenomenon as belonging within the space of reasons is whether the person holding the belief or desire or engaging in the action is aware that the belief, desire or action could be different and can ask the question whether their belief desire or action should be as it is (Mackrel & Pratt, 2017, p. 426).

The question of whether the students have an awareness of fairness in districting and whether their decisions "could be different" is the question that motivates this study. In particular, I ask what kinds of informal mathematical reasons are students giving in response to issues of fairness in a districting activity?

METHODS

In 2021, I taught an undergraduate mathematics education course at a university in western Canada. The course was intended for social science students who required a mathematics course to complete their degree. There were twelve 3-hour classes. There were 34 students registered and attending the class. The course was focused on quantitative reasoning and on topics such as the pigeonhole principle and proof by contradiction. The mathematics curriculum of the province in which the course was taught articulates two areas for mathematics teaching: content and competencies. Mathematical competencies are the ability to use mathematics rather than know the mathematics itself ("content"). According to the curriculum, students develop competencies through "reasoning." The curriculum recommends engaging in "inquiry", shifting "perspectives", and "reflecting" on activities (BCMoE, 2015). These correspond to three of Sadler et al.'s (2007) four categories.

Districting context

One of the topics I introduced this year was the redistricting of voting constituencies. One of the reasons I chose this topic (in addition to its connection to reasoning) was that there was a federal election in Canada occurring three weeks into our course. The winner of the election had fewer overall votes than the loser, not an uncommon result in a first-past-the-post electoral system. In the class we had the day after the election, I asked the students whether they thought the result of the election were fair. Almost all of them thought it was not.

Data

This topic of districting was taught from a mathematical perspective rather than a political one. The mathematics included comparing the general population and the district population and analysing how the proportions between these two depends on district boundaries, compactness metrics (such as the relationship between perimeter and area), and voter counts and its relationship with wasted votes. One mathematics formula introduced was the efficiency gap; it is a measure that compares the difference between each party's total votes minus wasted votes, divided by overall total votes. It was also essential to connect these mathematical ideas with the social issues of voting, taking into consideration concerns such as geographical obstacles, district contiguity, packing, and cracking.

To give a sense of the kinds of activities I presented to my students, I projected a grid like the one seen in Figure 1a and asked them to consider districting in different ways. Figures 1b and 1c show the extreme results of this activity.

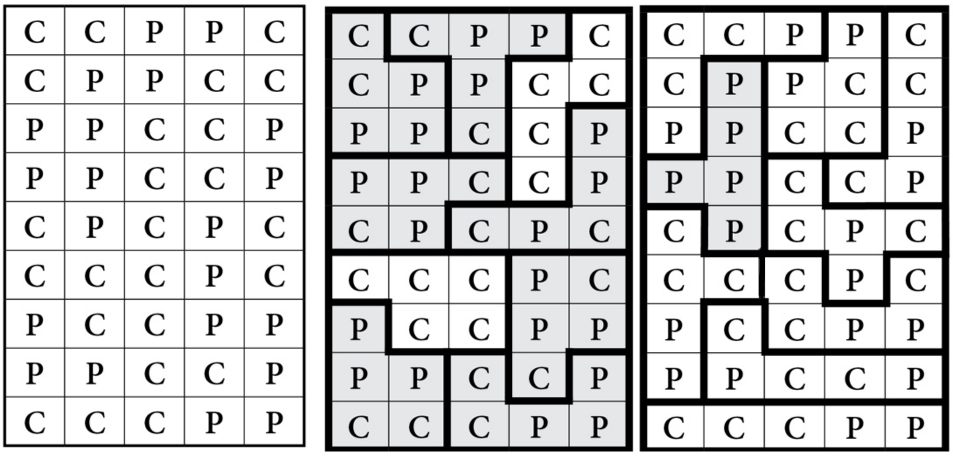

Figure 1: a) grid without any districts; b) districts favouring “P”; “P”s win 7-2; c) districts favouring “C”; “C”s win 8-1

Throughout the course, we investigated the redistricting prompts I gave through class activities, group discussions, and reflections. All data presented below comes from two take-home reflection assignments. The assignments were based on the notion of fairness, with the purpose of eliciting decisions and justifications. Due to the lack of definitive answers, mathematics, in this activity, can be considered a practice of giving reasons.

The data was rich and there was a large set of responses, but due to restrictions of space, I present responses from four students, two from one prompt and two from another. The examples presented below, however, are representative of the overall set of responses. All quotes below are verbatim. I present data and analysis together.

FINDINGS

In the following analysis, I report on two prompts.

Prompt 1

What is fair when districting a population?

Looking at the grids below, what is similar; what is different? Which set of divisions do you prefer and why?

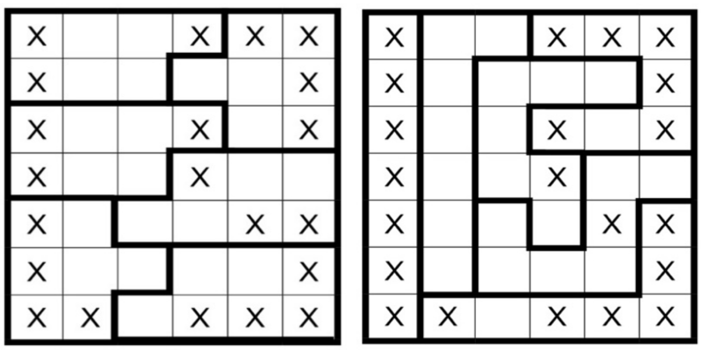

Figure 2: Two grids with the same “X” distribution but with different district boundaries

Inquiry

Xian wrote:

> The grid on the left is much more compact than the grid on the right; so it seems 'technically' better. However, both grids have the same (correct) outcome, showing that either type of division can be used 'fairly'. Having said that, I do think the type of divisions on the right can be used more effectively for gerrymandering ~ I tried to redistrict and gerrymander the left side while maintaining the compact districts and was unable to do so; but then tried the right and could not do so either ~ so really, is one better than the other? No in that regard.

Brown and Walter (1983) articulate a "what if" method as mathematical problem-posing. In Xian's response, we see an example of asking a "what if" question when she implicitly asks "what if" I try to manipulate the boundaries? Xian wondered whether the two grids had different potentials in terms of gerrymandering. Her justification introduced an inquiry of manipulating boundaries to see which grid was closer to the possibility of gerrymandering than the other. But since she found that neither could be easily gerrymandered, she reasoned that neither is better based on that metric. I had not taught anything related to whether grids are closer to or farther from possible gerrymandering. She reasoned with that idea on her own. There is also mathematical legitimacy here in the consideration of "close or not close" to gerrymandering in the mathematics field of "outlier analysis."

Skepticism

Barth wrote the following:

> Both sets of division share the same result of the X's and O's tying. The result of both sides are the same thus I PERSONALLY do not have a preference… But it is to be acknowledged that the diagram on the right can be seen as problematic as most districts are seemingly "packed", which can be an indicator of gerrymandering. Both districts also have the same amount of wasted votes (9) and have an efficiency gap of 0.

Barth's decision is strongly based in calculations; he calculates the winner, the number of wasted votes, and the efficiency gap. He positions these three calculations as central, and by emphasizing "PERSONALLY" he indicates that he feels uncomfortable with making a decision without more information. It is interesting to note that the calculations do not take precedence over his reasoning, in that he notices that there appears to be at least some potential for packing. But he cannot confirm or deny that there was. He notes it only as a possibility. In terms of skepticism, he is aware that there is not enough information and is therefore reluctant to express a preference.

Prompt 2

To simulate a random districting, draw a 5 by 5 district plan in an empty grid. Then flip a coin to propagate Xs and Os in the table as you move systematically through the grid. Is the result of this experiment appear fair for "X"s? If you can change your original district boundaries, what would you change to make it more fair?

Complexity

Helena wrote:

> The Xs occupy 13/25 cells and the Os occupy 12/25 cells. The Xs occupy 52% of the cells but 60% of the districts. I believe if we are basing this off majority/minority it is fair that the Xs won as they occupy the majority of the cells with over 50%. I believe this is fair as it represents an adequate picture of the reality within the cells….Let's say I created just simple linear districts going by columns, the Xs would still win with the same margin of 3/5. If we drew the districts by rows the Os would win, due to the packing of the Xs in the first row. I don't think I would make the change as it represents gerrymandering to allow the minority (O) to win by forcing packing of Xs in row number #1.

Helena's outcome seems fair to her and she is content, but she also evaluates various configurations to see whether her original results are as fair as they could be. She checks what would happen if she uses columns as districts and then rows. In the case of columns, there is no change in outcome; but with rows, the outcome was that the Os would win overall since her cells in the top row were all Xs. She concluded that she would not choose boundaries based on rows because the count for the districts would not match the count for the population. Her comment on the "reality within the cells" addresses proportional comparison. She has a mathematical sense of fairness in that the proportion of the popular vote should align closely with the population of district outcomes. Proportion in pure mathematics is at its root a statement of equality: a/b=c/d, where c/d can be any magnitude bigger (or smaller) than a/b. Helena's dissatisfaction was based on her acceptance of this proportional statement of equivalence.

Perspective Taking

In response to prompt 2, David wrote:

> Even though this experiment did randomly end up giving quite fair results, as there were not many wasted votes at all (4 wasted for X, and 6 wasted for O) even if the results have been more skewed, I don't think you can change them to be more "fair". The districts were created completely randomly, and the grid was filled in that way too, which is essentially the fairest this can be. If you were to change it to try to reduce the amount of wasted votes for one side, it would just become unfair to the other side.

David is willing to accept any grid created randomly. He associates fairness with objectivity. David is relying on mathematics as objective, because it has no intent, it cannot be nefarious. He is using mathematics as a way to establish his stance. If there is human intervention in drawing and/or manipulating of boundaries, that is when problems can occur. He justifies this by recognizing that as soon as you reduce wasted votes for one party, you increase it for the other party. He noticed through a change of perspective that nothing would been gained if he were to interject. He would not solve anything, thus confirming his original stance.

DISCUSSION

This study is about how mathematics is used in student reasoning about social questions. I focused on the mathematical content and competencies that emerge in reasoning about fairness in a political districting activity. I wanted to see how students use mathematics when making decisions.

Returning to the notion of the space of reasons, it appears that students use both the social aspects of districting and mathematical justifications. Each student presented a unique approach to the reasoning in which they engaged. Xian engaged in a mathematical inquiry that paralleled Brown and Walters' "what if" method to explore two arbitrary grids in terms of their mathematical proximity to bias (gerrymandering). This inquiry not only aligns with Sadler et al.'s categories, but is also based on her personal sense of fairness. Barth feels uncomfortable expressing his own opinion and uses three calculations to ground his decision. But he remains skeptical about whether those calculations are satisfactory as there are still social variables that may be unaccounted for.

Helena grounds her stance in proportional equivalence and states her reasons as seeking the closest proportional equivalence between the population and the districts. She thinks that would be the fairest. She redraws district boundaries to determine whether there are better configurations. She, in fact, finds one that is worse. David relies on mathematical randomness as being fair and supports this by noting that in terms of wasted votes, he cannot improve the situation. Taking the perspective of one party and improving their situation will only negatively affect the other party.

In each case, the kind of mathematics is different. Xian, Helena, and David base their reasoning on mathematical competencies, and Barth bases his on mathematical content. It is also evident that the mathematical reasoning cannot be separated from the context or the student's values. This is confirmed by the various decisions made and how each student framed their stance, some confidently in mathematics, others without confidence in wanting to know more. The mathematics emerged from consideration of a social situation. For example, the proportionality between a population and its districts only exists because of the first-past-the-post system.

This study looks to assess how students use mathematics in the reasoning that informs their decisions. It is to understand that students can rearrange, count, calculate, identify, and verify to confirm a posed problem. That is, they develop reasons based on consequences of actions.

We could use more evidence-based classroom studies to advance our understanding of (1) how mathematics teachers can better prepare themselves to use social issues in teaching mathematics and (2) how students can broaden their views of mathematics and see how reasons develop in action.

This study cannot be generalised as only one class was studied, but the study is suggestive that engaging in a social issue in mathematics class may develop a more robust form of reasoning, one that interweaves the mathematics and the social.

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MAKING VISIBLE A TEACHER'S PEDAGOGICAL REASONING: AN ASPECT OF PEDAGOGICAL DOCUMENTATION

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Much of a teacher's practice and professional learning remains unseen despite recent calls to incorporate practice-based and inquiry-based approaches to improve mathematics instruction. Although the idea of pedagogical reasoning and action can provide a way to unpack these unseen aspects of practice, it remains to be seen how a teacher's actions and thinking can be made visible. In this paper, we present a case of how a teacher's pedagogical reasoning is made visible through pedagogical documentation, which suggests the possibility of using documentation to unpack these unseen aspects of a teacher's practices.

INTRODUCTION

Preparing teachers to learn from teaching is a powerful way of thinking about professional learning. Hiebert et al. (2007) proposed that teachers should learn to specify the learning goals, collect evidence of learning from classroom observations, think about the effectiveness of their instructional approaches, and improve their instruction based on the evidence collected. In other words, teachers should have opportunities to examine their understanding of content, curriculum materials, learning and instruction (Sherin, 2002). Despite recent developments in adopting practice-based and inquiry-based approaches to improve mathematics instruction, much of the complexity surrounding teacher learning and the different elements of a teacher's practice remains unseen. Shulman's (1987) proposed model of pedagogical reasoning and action can be seen as "a starting point for unpacking the unseen aspects of practice" (Loughran et al., 2016, p. 388). Yet, whether a teacher has gained new comprehension (Shulman, 1987) from reflection, and how this new learning has taken place still resides in a black box. This paper presents how a teacher's pedagogical reasoning can be made visible using pedagogical documentation.

THEORETICAL CONSIDERATIONS

Teaching "begins with an act of reason" and "continues with a process of reasoning" to culminate in a series of pedagogical decisions (Shulman, 1987, p. 13). In other words, teachers need to learn how to apply their knowledge for teaching to provide justifications for their instructional decisions. Doing this involves taking one's understanding about content and "making it ready for effective instruction" (Shulman, 1987, p. 14), through a cycle of activities involving comprehension, transformation, instruction, evaluation, and reflection, leading to new comprehension. According to Shulman (1987), comprehension refers to how teachers can understand what they teach and relate these ideas to other ideas within and beyond the subject in different ways. A

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teacher then transforms his or her knowledge into “forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (Shulman, 1987, p. 15). Transforming this knowledge involves preparation, representation, instructional selections, adaptations of these representations and tailoring the representations to specific students’ profiles. Although comprehension and transformation can occur at any time during teaching, Shulman (1987, p. 18) sees these two processes as “prospective”, occurring before instruction, an “enactive” performance in the classrooms. Shulman (1987) then highlights evaluation as the process of assessing students’ understanding to provide feedback about the teacher’s instruction. However, it is when a teacher reflects on the instructional experiences that learning from teaching can occur. This new learning in the form of better understanding about teaching and learning will then be part of a teacher’s new comprehension, which becomes the starting point for planning future lessons.

While Shulman’s model may provide a lens to examine a teacher’s instruction, much of a teacher’s pedagogical reasoning remains invisible. How can we document a teacher’s thinking about instruction to make it more visible? For this, we turn to the idea of pedagogical documentation (Dahlberg & Asen, 1994; Lee-Hammond & Bjervås, 2021), which is widely practised in early childhood education settings. The practice of pedagogical documentation involves teachers in collecting written notes, audio and video recordings, photographs, or students’ learning artifacts for describing what and how students learn, which then serve as a basis for reflection and making instructional decisions (Lee-Hammond & Bjervås, 2021). In this way, the documentation is both a product and a process, and has been demonstrated to support teachers in professional learning. However, pedagogical documentation is scarce in mathematics education contexts, and we wonder if this practice could be incorporated as part of a mathematics teacher’s everyday activities to enhance professional learning. In this paper, we present a case of how a teacher’s pedagogical reasoning is made visible through pedagogical documentation, unpacking the unseen aspects of teaching and learning. The key question framing this paper is: What aspects of a teacher’s pedagogical reasoning and action are captured in her pedagogical documentation?

METHODS

The data presented in this paper were collected as part of a larger project, aimed at developing a proof of concept for a sustainable professional learning model for mathematics teachers. Drawing on current theoretical perspectives of teacher noticing (Dindyal et al., 2021; Fernandez & Choy, 2019), we conceptualized professional learning sessions where teachers have opportunities to work and co-learn with us in a community of inquiry (Jaworski, 2006). At the time of this present study, face-to-face sessions with teachers were not feasible due to prevailing Covid-19 restrictions. Hence, we conducted two online professional learning (PL) sessions for six elementary school teachers: In the first session, we elicit teachers’ ideas about ratio and challenges associated with teaching ratio; in the second session, we shared ideas about proportional reasoning and discussed the teaching of ratio for Grade 5 students (age 11). After the PL sessions, we followed up with two of the teachers, who volunteered their lessons for the entire unit on ratio for us to observe. In this paper, we uncover the pedagogical reasoning of one of these experienced teachers, Kathy (pseudonym), as she planned, taught, and reflected on a series of four lessons on ratio.

Data were generated from the voice and video recordings of the lessons, Kathy's lesson plans and instructional materials, and an interview with Kathy at the end of the study. In addition, we leveraged on the idea of pedagogical documentation to capture teachers' thinking about content and their pedagogical reasoning as they reflected on the planning and teaching of the lessons. More specifically, we used Padlet (https://padlet.com/), a digital notice board, as a platform for Kathy to curate her pedagogical documentation. We did not impose any number for the reflections—instead, we asked her to post her reflections, photos, videos, or documents related to any incident that she had found interesting on Padlet—and we left all instructional decisions to Kathy. Our role was to observe what she had learned from our sessions, her considerations for the selection of tasks and the instructional decisions made during her lessons. Findings were developed through identifying and analyzing critical incidents (Goodell, 2006), which are "everyday" events "encountered by a teacher in his or her practice that makes the teacher question the decisions that were made and provides an entry to improving teaching" (p. 224), during her planning and teaching. We analyzed these critical incidents by a "thematic approach" (Bryman, 2016, p. 578) to highlight aspects of concepts related to ratio, students' confusion about ratio, and instructional decisions before we tried to relate these incidents to Shulman's (1987) model of pedagogical reasoning and action.

FINDINGS

For this paper, we present one of these critical incidents, which centred about Kathy's reflections on her selection, modification, and implementation of a colour mixture task (see Figure 3). We begin by highlighting aspects of Kathy's comprehension of the ratio concept and making explicit her thinking about the colour mixture task before and after the task was implemented from her pedagogical documentation.

Kathy's comprehension of the ratio concept

In the first PL session, we asked teachers to share their understanding about ratio and anticipate the possible confusion that their students might have. Referring to Figure 1, we observed that Kathy was aware of some important ideas about ratio. She understood ratio "as a way of comparing 2 or more quantities", without specifying whether the quantities are of the same kind (Lamon, 2012). Kathy also highlighted that working with ratio "involves proportional reasoning" (Tourniaire & Pulos, 1985) and ratios are connected "to other topics like fraction and decimal". Moreover, she was cognizant of students' tendency to "use the additive idea" instead of multiplicative thinking when working with ratios (Clark & Kamii, 1996). Students' inability to apply multiplicative thinking strategies to solve missing-value problems was also highlighted by Kathy with a specific example of "8 : 12 = ? : 15". Lastly, Kathy also surfaced the issue that students might not understand ratio as an ordered comparison.

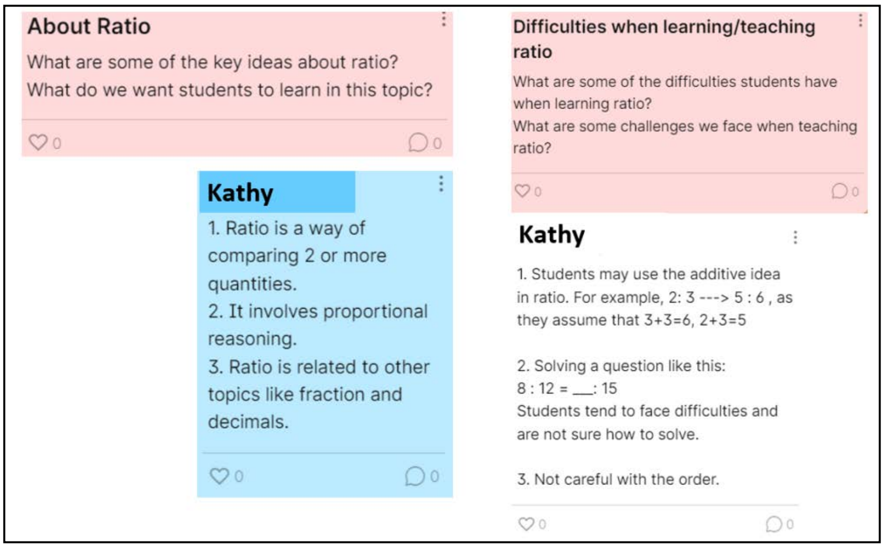

Figure 1: Snapshots of Kathy's responses on padlet.

In the second PL session, we shared other nuanced notions of ratio, emphasizing ideas such as absolute comparisons, relative comparisons, part-part comparisons and part-whole comparisons, as well as making a distinction between ratio, proportion, rate, and proportional reasoning (Yeo, 2019). We then invited Kathy to post her thoughts and reflections whenever something interesting came to her mind during the planning and implementation of her lessons.

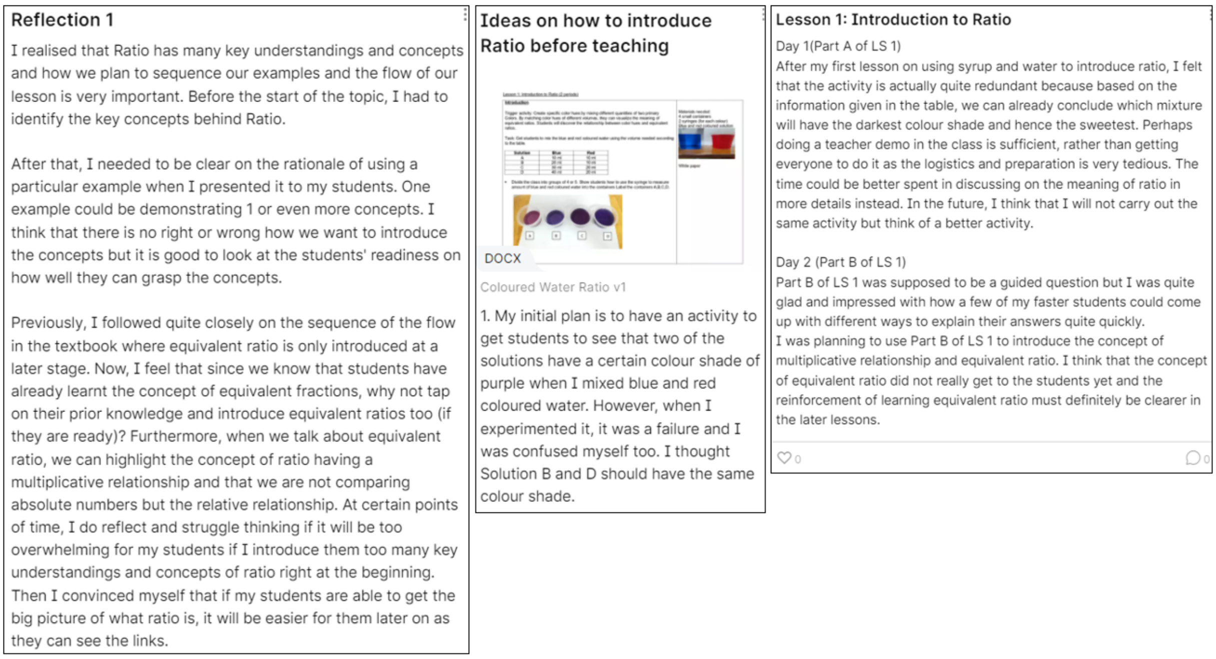

Figure 2: Snapshots of Kathy's pedagogical documentation.

Figure 2 shows a snapshot of her pedagogical documentation at the end of the unit. As seen from her reflections, Kathy became more aware that "ratio has many key understandings and concepts". She was able to highlight how equivalent ratios are premised on "multiplicative relationship" and the difference between "absolute" and "relative" comparisons. Interestingly, Kathy agonized over the ideas and the sequence in which they should be introduced. She was also thinking about the profile of her students and considered the possibility of introducing the various inter-related ideas about ratio instead of presenting them in the sequence as presented in the textbooks. In particular, she entertained the idea that equivalent ratios could be presented to her students earlier even through the concept was introduced much later in the textbook. Her enriched comprehension of the concepts also contributed to the changes in her choice of the initial ratio task.

Kathy's versions of the colour mixture task

In her original lesson plan, she wanted to introduce the concept of ratio through an activity involving students making a dough using different number of cups of flour and water. After the two sessions we had conducted, Kathy began to think about the use of a colour mixture task (Figure 3), where students had opportunities to think about how the different amounts of blue and red dye contributed to the colours of four different solutions. The mixture problem, and its variations, has been used in other studies to develop students' proportional reasoning (Lamon, 2012; Tourniaire & Pulos, 1985). In Kathy's version, she used measurement units instead of non-standard units like cups.

Task: Get students to mix the blue and red coloured water using the volume needed according to the table.

Solution	Blue	Red
A	10 ml	10 ml
B	20 ml	10 ml
C	30 ml	20 ml
D	40 ml	20 ml

Figure 3: Excerpt from Kathy's colour mixture task.

However, as we can see from Kathy's reflection ("Ideas on how to introduce ratio before teaching"), she tried the activity but did not get the expected outcomes ("B and D should be of the same shade"). On one hand, the use of same units may help students to see ratio as comparison of quantities of the same kind and the fact that ratio has no unit (Yeo, 2019). On the other hand, the use of "quantities expressed in the same unit may be more confusing" (Tourniaire & Pulos, 1985, p. 184) as in the case of Kathy. Another possible point of confusion is that the volumes of blue and red dyes are different, and each mixture had a different volume, which may lead to a discussion on rate rather than ratio. This may be difficult for students who are formally learning ratio for the first time. It is possible that Kathy might have taken that into consideration by keeping constant one of the volumes in the second version of the task as shown in Figure 4.

Part A: Task: Mix the syrup and water in a container using the volume given below.

Mixture	Volume of syrup (ml)	Volume of water (ml)
A	5	100
B	10	100
C	20	100

1. What do you notice about the colour shades of the mixtures?

____

2. a) Rank the mixtures from the sweetest to the least sweet. ______, ______, ______
(sweetest) (least sweet)

b) Explain your answer. What makes you say that?

Figure 4: Kathy’s second version of the colour mixture task.

Referring to Figure 4, we see that Kathy had changed the context of the task from comparing colour of a mixture to that of comparing taste (albeit through the colour shades) of the mixture. Kathy had intended her students to mix the syrup with water in class as evidenced in the lesson plan. Although Kathy had “addressed” one issue by keeping the volume of water at 100 ml for all three mixtures, it created another issue of students being able to solve the problem without mixing the syrup and water. Furthermore, the numbers made the solution obvious, which then reduced the demands of the task. While one may argue that Kathy could have caught the problem before the task implementation, it is noteworthy that she noticed the issue after the lesson. To be clear, the lesson went on well and the students were engaged with the task. But as Kathy had noted in her reflection on Lesson 1 (see Figure 2), she realised that the task was “redundant”, and she could have done “a teacher demo” and spent the time “discussing the meaning of ratio” in greater depth. Here, we see Kathy’s reflection of her instruction and assessment during the lesson leading to her new comprehension of how ratio could be approached differently. This new comprehension reinforced the importance of thinking about the first examples as highlighted in her “Reflection 1” (see Figure 2), which could potentially lead to her thinking about a “better activity”.

DISCUSSION

In this paper, we gave an account of what Kathy understood about ratio and how she reflected on her selection, modification, and implementation of a mixture task by making her pedagogical reasoning visible using pedagogical documentation. For example, Kathy’s reflections about the content provide a window into her understanding of ratio, highlighting the aspects of her mathematical knowledge for teaching ratio. More importantly, we could “see” how Kathy transformed her understanding into the design of the task and how she eventually reflected on her instruction to modify her thinking about the lesson design for future lessons. Thus, Kathy had gone beyond documenting her practice—before, during, and after lessons—and used her documentation as a basis for reflection to make instructional decisions (Lee-Hammond & Bjervås, 2021).

The power of pedagogical documentation to make visible a teacher's pedagogical reasoning has important implications. For researchers, the idea of pedagogical documentation can be repurposed to focus on teacher learning and be extended to include teacher artifacts before and after lessons. The use of such documentation helps to pinpoint the areas for intervention and support as mathematics educators work with teachers to improve their instruction. For teachers, documenting their practices provide opportunities for them to scrutinise and negotiate among three aspects of their teacher knowledge: understanding of mathematics, curriculum materials, and knowledge of how students learn (Sherin, 2002), a pre-requisite for teachers to learn from their own teaching. Moreover, a teacher's pedagogical reasoning and action can also be made visible to other teachers as part of their professional learning activities. Discussions around teachers' pedagogical documentation can then form the basis of pedagogical shifts in one's daily teaching activities not just for a teacher, but for the whole community.

But it is challenging and time-consuming for teachers to document their practices in ways that enhance their pedagogical reasoning and action. As Kathy had said during the final interview:

> I don't really like [documentation] because it takes some time. But it's good, and, you know, you got asked to write all this stuff. It really forces us to think, you know, what are the things that is in our mind? And then we can refer to that [documentation] later, even after a long period of time.

While the benefits of pedagogical documentation may justify the efforts needed to document one's practices, such tensions about effort and benefit should not be ignored if we want to move towards the idea of learning from one's teaching. What else can teachers do to document their practices? How can teachers be supported to document their practices? These are the important questions for future research.

Acknowledgement

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TOWARDS A SOCIO-ECOLOGICAL PERSPECTIVE OF MATHEMATICS EDUCATION

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In this theoretical research report we propose a socio-ecological perspective, as relevant for research in mathematics education that takes account of our complex, precarious present and imagined future, while recognising its historical roots. We discuss briefly work that considers the social, political and ecological, and build from this scholarship. A 'socio-ecological' perspective considers the social and ecological as entangled, and mathematics (education) as both shaping and shaped by these entanglements. This is a mathematics (education) that gains meaning from questions that emerge in socio-ecological relations. We ground our theoretical argument using a project located in a community living in a polluted region of Mexico, where a river is central to the questions motivating community activism and our research.

RATIONALE

Contemporary world events offer stark evidence of the inseparability of social, ecological, health, spatial and political issues such as: climate change effects related to water, heat, biodiversity loss; health pandemics; poverty; inequality; unemployment; migration; totalitarianism and loss of voice. This is a rapidly changing world characterised by complexity, uncertainty, vulnerability, movement, and informality, with the pace of change outstripping our knowledge of this world. These events challenge the mathematics education community to consider, in Latour's (2004) words, "Are we not like those mechanical toys that endlessly make the same gesture when everything else has changed around them?" (p.225). We conceptualise mathematics education as making "gestures" in the form of recontextualised knowledge, curriculum organisations, textbooks, professional development opportunities, and anything that becomes visible in the context of teaching and learning. We are prompted by Latour to ask: What might be the "gesture" of a recontextualised mathematics? We ask this in a context in which a supposedly neutral and universal mathematics, valued for its descriptive, categorical and predictive possibilities, has, in action, in science and technology, come to format the world as 'calculable' (Mbembe, 2021; Skovsmose, 2011). What might be some alternative "gestures" of a mathematics education that is commonly and unquestioningly considered a necessary individual and social 'good'?

In what follows, we propose a socio-ecological perspective as one response to the aforementioned challenges faced by mathematics education, taking the social in 'socio' as inherently political. We do not see the socio-ecological as replacing other perspectives, but rather as complementing and building on them to offer insights that recognise the entanglement of the social and ecological, and the role of mathematics

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(education) therein. To ground our theoretical argument in this report, we use a project situated in Tlaxcala State, Mexico, in which the first and third authors participated and alongside which our thinking about the socio-ecological has developed. We describe this context first, acknowledging that any such description cannot capture its complexity and history. We then discuss briefly existing mathematics education scholarship relevant for our consideration of the social, political and ecological. Finally, we describe our proposed perspective and some possible future directions.

THE ATOYAC RIVER PROJECT

The Atoyac River in Mexico is the third most polluted in the country. From a visit to the region by the first and third authors, it is clear the river no longer supports animal life. Coloured dyes from a textile factory and heavy metals from a car parts factory (both of them internationally owned) are regular discharges into the river. The toxic smell is noticeable over 1km away, in a primary school playground, and the significant negative health effects on the local population, such as child leukemia, are documented. From having a central role in the life of the community and its rituals, the river is now rarely visited. A network was instigated decades ago, by community members living near the river, and including non-governmental organisations, school teachers and academic scientists from a range of disciplines, in order to respond to the pollution issue. The first and third authors of this report were invited (having won a grant from the UK's Global Challenges Research Fund, EP/T003545/1) to bring an education perspective to the network. The initial research question they were challenged with was, how the Mexican primary curriculum, including the mathematics therein, might become "relevant" in such contexts of complexity, vulnerability, and marginalisation. Over the course of an academic year, the primary school children involved in the project engaged in many activities relating to the pollution of the river. Mathematics was not always present; one task where it was involved comparing data, looking at the biodiversity of the region today and comparing this to the biodiversity remembered by the children's parents, grandparents and other elders in the community.

This curriculum project (henceforth the *Atoyac River project*) is productive for thinking about the socio-ecological, for it is *the river* that had been studied for decades, that is central to the context. It is the dramatic changes in the river that provoked changes in lifestyles in the region (e.g., a disappearance of fishing and recreation in the river). It is the river which is at the centre of the community's social activism ("Coordinadora por un Atoyac Con Vida" [Coordinator for a Living Atoyac] and "Centro Fray Julián Garcés Derechos Humanos y Desarrollo Local", [Fray Julián Garcés Human Rights and Local Development centre]). And it is around the river that the network (and the questions it asks) was conceived. We return to this project through the next section in a hypothetical way, to illustrate how it could be approached from different perspectives, and again in a concluding section, where we describe the project's influence on how we have come to think about the socio-ecological and the questions it provokes.

TRENDS IN RESEARCH IN MATHEMATICS EDUCATION RELATING TO SOCIAL, POLITICAL AND ECOLOGICAL CONCERNS

The past two decades have seen a growth in a *socio-political perspective* of mathematics education (following Gutiérrez, 2013; Valero, 2004). Using notions of knowledge, power, and subjectivity, this perspective conceptualises mathematics and mathematics education as historical, social, and political practices. Broadly, it is concerned with understanding how mathematics (education) might (re)produce wider practices and structures of inequality, and with acting towards a more socially just and ethical world. In this section we discuss particular named areas of the work within this perspective, as relevant for our focus on the socio-ecological, noting the constraints on space, and that the definitions of research areas and their relations are contested.

Critical mathematics education (CME), is united by particular concerns, commonly raised from within the dominant Euro-modern knowledge and education structures (Vithal & Skovsmose, 1997). Firstly, how mathematics (re)produces, or 'writes', the world through action in, for example, science, technology, economics. And also how mathematics education (re)produces particular subjectivities and knowledges. Secondly, CME is concerned with mathematics (education) for understanding, or 'reading' the world, aspiring to the possibilities of (re)writing for a more democratic, socially just world (see, for example, the edited volumes: Alrø, Ravn, & Valero, 2010; Andersson & Barwell, 2021). CME's view of the role of mathematics (education) in society informs perspectives variously named as *mathematics for social justice/peace/democracy*. In the Atoyac River project, taking a CME approach would suggest using school mathematics (education) to understand the social injustices of the pollution to the river, in terms of health outcomes, and to provoke action for change.

More recently, CME scholarship has demonstrated the potential of mathematics (education) to write and read the contemporary ecological condition of the world, or 'climate change' (e.g. Hauge & Barwell, 2017; the edited volume by Coles et al., 2013), in what might be called a *mathematics for environmental sustainability*. If following such an approach, the Atoyac River project would focus on a mathematical understanding the ecological health and future of the river itself, again accompanied by action against the ecological injustices.

Socio-critical modelling, described as an "emancipatory perspective" (Kaiser & Sririman, 2006, p.304) of mathematical modelling for a critical understanding of the world, has strong links to CME. Educationally, socio-critical modelling centers "students' ability to be critical modelers and [to] recognize their power", rather than their mathematical understanding and skills (Abassian et al., 2020, p.61). For the Atoyac River project, such an approach would foreground the social action that could follow a critical investigation of the river and its pollution and the modelling of trends.

Other areas of socio-political mathematics education have thought from the 'outside' of the dominant canon and structures (Vithal & Skovsmose, 1997), and have commonly been shaped by the marginalised positions in which they emerge. For example, *ethnomathematics* challenges dominant narratives of the history of mathematics, and identifies what are considered culturally and socially embedded ways of thinking and acting mathematically that are decentred by dominant mathematics (education) (e.g. Powell & Frankenstein, 1997; Rosa et al., 2017). In its naming, *ethno*mathematics specifically focuses on the practices of social groups, with the ecological a context in which human activities are developed, for example, land measurement practices. An ethnomathematics perspective in the Atoyac River project would involve attention to the past and present mathematical knowledge and practices used by the community, located in the context of the river.

Scholarship on *indigenous ways of knowing* commonly foregrounds the ways of knowing, acting, being and using language of variously named indigenous communities that have been traditional marginalised in dominant mathematics (education) (e.g. the edited volume by Nicol et al., 2020). As with ethnomathematics, the very nature of relations in these contexts signals the presence of the ecological. In the Atoyac River project this would mean attending to ways of the community, which could include the very manner in which human-river relations have been enacted.

Learnings from ethnomathematics and indigenous ways of knowing are used to inform what is called *culturally responsive mathematics education*. With their focus on groups marginalised by coloniality, neoliberal globalisation, and so on, all three are presented as ways to 'decolonise' mathematics education. However, *decolonial and antiracist perspectives* of mathematics education specifically draw from traditions of post/decolonial and/or critical race theory to: (a) understand and surface the co-constitution of racial (and related) difference in the entanglement of mathematics (education) in historical processes; and (b) promote an active process of becoming of mathematics knowledges and knowers (e.g. Martin, 2019; Swanson & Chronaki, 2017). In the Atoyac River project, the focus would be the role of mathematics (education) in historical and contemporary processes that (re)produce hierarchical difference and by which the region has come to be vulnerable and marginalised.

TOWARDS A SOCIO-ECOLOGICAL PERSPECTIVE

The brief discussion in the previous section suggests that, within a socio-political perspective of mathematics education, there is scholarship that takes account of the environment, or of ecological issues. But, at the same time, it appears that the ecological may be taken as context in which peoples and mathematics (education) act, rather than being strongly theorised in itself, or in relation to other actors. A move towards greater acknowledgment of the ecological is, however, discernible in the recent work of some scholars who supplement the aforementioned approaches using theoretical traditions, again, 'outside' of mathematics education such as *(eco)feminist, ecocritical, ecojustice, and posthumanist* ideas (e.g. Coles, 2017; Gutiérrez, 2017; Khan, 2020; Wolfmeyer, Lupinacci, & Chesky, 2018).

It is in these recent moves in the field that we locate our argument for a *socio-ecological* perspective of mathematics education. Specifically, we conceptualise the past, present and future world as entanglements and interdependencies between the social and ecological, and consider the role (or not) of multiple forms of mathematical knowing and being therein. Such a perspective recognises multiple actors in the constitution of the world, and hence the presence of the (un)quantifiable within socio-ecological contexts and challenges. Such as perspective may, at times, 'decentre' particular forms of mathematical knowing. Conscious that decentring mathematics might be read as counter-intuitive, in the context of a mathematics education conference, we stress that we are proposing a shift towards thinking about mathematics (education) as gaining meaning through relations between actors, which include the human and non-human (e.g. the environment and mathematics itself). We view the perspective as a move to starting with questions, that is, the perspective is relevant when the questions we are asking, or concerns we carry, are themselves socio-ecological, for instance taking a river and its relations as the starting point of questions and of our research.

Also relevant to a socio-ecological perspective is the emergence of new materialisms (e.g. de Freitas & Sinclair, 2014; Appelby & Pennycook, 2017) which see technology, language and the natural world as actors within and besides the human. Such new materialisms lead, for us, into the kind of entangled view of the world that emerges through the socio-ecological. The socio-ecological allows us to inhabit the intersection of social constructs such as gender, class, language, and race *and* bodies, things, ecologies, space, in semiotic and material assemblages. How might our thinking respect, or engage with, the complexity of what we are thinking about (Bateson, 1972)?

The socio-ecological perspective, emerging from questions in the Atoyac River project that we have started to lay out in this section, has epistemological and ontological implications. We start to explicate these next, before discussing further questions prompted by our thinking about the socio-ecological.

Epistemology and ontology of a socio-ecological perspective

From several theoretical positions, a relational ontology is being proposed (e.g. de Freitas & Sinclair, 2014) and such relationality is important also within a socio-ecological perspective. The entanglement of the social and ecological means, for instance, the entanglement of subject and object. Rather than thinking about an encounter as a meeting of two pre-existing entities, a relational ontology implies the view that it is through encounter that subject and object inhabit an identity (for the duration of that encounter). Rather than asking, e.g., "who acts?" (a question which presupposes an already existing subject) a more relational question would be, "how is it that such a subject is able to act in this way" (Benjamin, 2015, p.87). This second question invites attention to the always-already existing webs of relations that allow action in the first place. And, from a socio-ecological perspective, this web of relations include culture, politics, ecology, history. Subjectivity is an after-effect of the socio-ecological relations that allow its emergence, not a pre-condition of those relations.

In terms of epistemology, a certain humility is required. Others' ways of knowing may be radically different to our own and yet equally valid. Furthermore, there are implications for the kinds of knowing that are significant. From a socio-ecological perspective, what is important is to develop wisdom about the complexity of inter-relationships in which we are enmeshed, and less valued will be instrumental knowledge of apparently linear cause and effect relations. Taking a wider systemic view, all relations end up in loops that cycle and become iterative (Bateson, 1972). Thus, some "gestures" (Latour, 2004, p.225) of mathematics education research from a socio-ecological perspective might include: listening to marginalised actors and the questions they provoke; attending to the ecological precarity of communities and adaptations being made to issues such as pollution or climate change; seeking double or multiple descriptions; paying attention to the different scales at which actions take place; questioning the spatial imagination that constrains thinking about a relationship; questioning the role of mathematics in conceptulisations of the ecological.

Towards new questions

Having articulated some of the philosophical ideas we have been led to, we now reflect on the kinds of questions a socio-ecological perspective might prompt us to ask, if we take seriously a relational ontology and an epistemology that is sensitive to ecologies. We offer a diverse set, starting with the Atoyac River and becoming more general.

How is the river remembered? How do pollution levels vary over time and what is the impact? What is the route back to a healthy river? How are ecological precarities experienced differently and what inequalities does this expose? What is the relationship between climate change and inequality? How might reparations for loss and damage through climate change be calculated fairly? What is the role of disciplinary knowledge and thinking, in relation to inter-disciplinary competencies? What mathematical fields (e.g., systems theory, non-linear dynamics) are relevant to socio-ecological questions? What kinds of curriculum organisation allow a centring of non-human, ecological concerns? What mathematics is done by ecologies?

We offer these questions, conscious they are disparate, as provocations and in the hope that the theoretical work of this report will prompt others to become attuned to possible socio-ecological questions relevant to their own contexts.

THE ATOYAC RIVER AND A SOCIO-ECOLOGICAL PERSPECTIVE

We end this report with a final set of reflections on the Atoyac River project. Our thinking about the socio-ecological has developed alongside this research; at the time of starting the project, it was aligned with a socio-critical perspective on mathematical modelling (in part, drawing on the modelling expertise of the third author). What this meant was that social action was a central concern. A disturbing feature of the Atoyac context (that we believe is repeated in many other places) is a normalisation of illegal pollution. The river has been polluted for so long that primary school children (and some of their teachers) have never known it otherwise and hence it can appear as though an alternative future is not possible. There can seem to be an inevitability to how the river is now, because it seems it has always been this way. There was a hope among many participants in the research that the work would spark action and a belief in the possibilities for different futures. What we have come to recognise is the centrality of the river, as described in section 2, not in any sense at the expense of social or political concerns, but rather as a focus for these concerns.

We are conscious of the possible objection that what we are describing is not properly mathematics education. However, and crucially, we want to argue that a socio-ecological perspective *is* relevant to mathematics education and that mathematics education needs to accommodate a perspective (not exclusively, but as a possibility) which re-imagines mathematics in a position which is not central to the problems it addresses, but is defined by its relationality, and might gain its relevance from questions that emerge in the socio-ecological. Indeed, in the Atoyac River project, the community subscribes to a complex and entangled understanding of social, economic, historical, cultural, biological and human and non-human health and rights issues. As researchers in this project, we have been provoked to reimagine our thinking about the gestures of mathematics education and mathematics in this socio-ecological context.

We are conscious we have only made a start at setting out a socio-ecological perspective of mathematics education; work we hope to continue and encourage others at PME to join. What we have offered in this report is, in part, our own process of finding ways to engage, to listen, in contexts that can sometimes feel overwhelming.

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IDENTIFYING PREVERBAL CHILDREN'S MATHEMATICAL CONCEPTIONS THROUGH BISHOP'S REFRAMED MATHEMATICAL ACTIVITIES

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Preverbal children demonstrate and develop mathematical conceptions as they engage with and investigate their environment. This engagement may involve navigating the environment or self-initiated play, neither of which includes an adult or educator interacting with the child. This paper applies Bishop's (1988, 1991) mathematical activities, as reframed by Cooke and Jay (2021), to identify the mathematical conceptions preverbal children are engaging in during self-initiated play within their environment. Limitations of this approach are considered.

INTRODUCTION

Libertus et al. (2020) state psychological research has indicated very young children engage with mathematical thinking. Preverbal children are found to demonstrate understandings related to numbers and quantity (Libertus et al., 2020); sequencing of actions (Verschoor et al., 2015); time (de Hevia et al., 2020); expectation and anticipation (Ruffman et al., 2005), which relate to prediction (Stapel et al., 2016) and probability (Daum et al., 2016); and recognition of features on objects (Needham et al., 2005). However, the child's engagement with the activities in the research is adult-driven.

A greater focus on the mathematics preverbal children may conceptualise when engaged in their everyday lives is evident in mathematics education literature. This focus involves child-initiated engagement with the world, more reflecting what Björklund (2018) considers 'mathematising' – that is, in terms of children making sense of the world they inhabit. This paper applies Bishop's (1988, 1991) mathematical activities as reframed by Cooke and Jay (2021) to identify preverbal children's mathematical conceptions.

REVIEW OF LITERATURE

Identification of very young preverbal children's mathematical conceptions.

Psychological research into the mathematical cognitions of preverbal children has identified a range of mathematical understandings and thinking that these children engage with. As these children are preverbal, non-language based methodology is used. These incorporate adult-controlled activities, usually in laboratory settings. In their research investigating the capacity of children aged between four and five months to distinguish between collections with different numbers of items, Wynn et al. (2002) used a habituation methodology, where the children were shown collections to the

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 179-186). PME.

point that they are habituated to that display of a specific number. Using this methodology, the children were found to differentiate between collections of items, which Wynn et al. interpreted as the children being "capable of genuine numerical representation” (p. B61). Ruffman et al. (2005) used a different approach, an anticipatory looking task, where the focus is on whether the child demonstrates their anticipation of what will happen by looking towards a specific location. Their research with children aged between two and a half and five months investigated whether, after a training phase to recognise an audio cue, the children could anticipate where to look for a hidden object. They found a significant difference for the children looking to the correct location when there was a two second delay but not for an eight second delay.

Although invaluable in identifying the mathematical conceptions of preverbal children, the methodology is not transferrable to early learning centres or everyday settings. Likewise, the use of adult-initiated and adult-controlled experiences runs counter to an early childhood education emphasis on young children’s opportunities for child-initiated engagement (Department of Education, Employment and Workplace Relations [DEEWR], 2009). As a result, other ways of identifying the mathematical conceptions of preverbal children need to be used.

Very young children’s engagement with their world through mathematising.

Björklund (2018) proposes that preverbal children engage in mathematics to make sense of the situations they encounter while exploring their environment and during play. Garvis and Nislev (2017) found that many activities undertaken in everyday family life show mathematics is a social activity, making mathematics relevant to the child’s life and creating a positive impact on children’s mathematical learning. Björklund and Pramling (2017) propose that careful observation of very young children’s exploration of their everyday environments reveals how they engage with mathematics. Franzén’s (2015) analysis of the mathematics involved when a very young child interacted with a climb-in toy car demonstrates how child-initiated solitary activities provided opportunities for observers to notice mathematical engagement and thinking. Franzen suggests that the starting point is the child; what they know; their interests and ability to express their thinking with their own language and actions, will allow children to explore their world mathematically. That is, everyday interactions, whether planned or child-initiated, provide many opportunities for young children to develop and demonstrate their mathematical conceptions.

THEORETICAL FRAMEWORK

Barad (2007) states that language has been granted much more influence and authority than perhaps it should have been. She develops this further by arguing that language has unfairly been positioned to be powerful and trustworthy over all other elements of the environment. These ideas underpin this paper. In terms of very young children, focusing on their language over their actions when interpreting their understandings is denying the existence of preverbal children’s understandings of their world.

Specifically, this paper investigates how actions undertaken through preverbal children's engagement with their world can be used as a valuable way to identify their mathematical conceptions.

REFRAMING BISHOP'S (1988, 1991) MATHEMATICAL ACTIVITIES

Bishop's (1988, 1991) mathematical activities form a useful structure for observing and categorising mathematical conceptions but as they are based on language, they were problematical for use with preverbal children. Cooke and Jay (2021) found it necessary to reframe Bishop's mathematical activities to use them to identify possible mathematical conceptions in the actions of preverbal children. The section below expands on the reframed activities provided by Cooke and Jay.

Counting Reframed.

While engaging with objects, preverbal children will be unaware of numeric order but will observe and manipulate a variety of sets of objects. This involves identifying how many of the objects can be viewed, touched, held or moved. A child picking up one item in one hand or selecting one item from a collection indicates recognition of discrete quantities (also involving locating). It includes reorganising a collection into categories or patterns, selecting specific items from a collection, or differences in quantities of collections. For example, a child will focus on one item in a collection of many or match one item to one item (such as grasping one toy in one hand), slide beads along an abacus wire either one at a time or in groups, identifying whether items have been added to or removed from a collection that has been of or is currently of interest, or when one collection has a different amount to another collection.

Locating Reframed.

Locating activities for the preverbal child focus on the placement of their body in space and the way in which they move their body to interact with the space around them. This includes successful and unsuccessful attempts to move in various directions. Therefore, the child's positioning of their body in a purposeful way in the environment would involve the 'Locating' activity. A child who shows they cannot reach from their current position through moving to a new position incorporates 'locating' activity (including measuring). Similarly, a child's frequent moving of items or objects to examine them, make them usable, communicate an idea (handing a ball to an adult to initiate a game), or following verbal directions incorporates 'locating'.

Measuring Reframed.

Measuring activities involve exploration of the environment and its objects, addressing textures and surfaces; size and shape; familiar and unfamiliar objects, to consider 'how much' of an attribute a feature might have. Measuring activities will be repeated several times and re-visited on recurring occasions to check the same attributes still apply. During this exploratory activity, children are 'cataloguing' their world to make sense of it and this gives them a point of reference when encountering new objects (also involves designing). Preverbal children shake objects, roll or turn them while carefully examining them from all sides, and heft an object to feel its weight. After constant and regular examination and exploration of objects, children may specifically choose an item with more of a desired feature or attribute. This activity can be observed when a child tries, then chooses to use both hands to move or lift an object due to the size or weight, or selecting an object that has more of a desired feature (such as a fluffier blanket).

Designing Reframed.

Preverbal children may not yet be able to create or design patterns within their world but they are able to recognise pattern, shape and design of familiar objects and then translate that knowledge to new similar objects. A preverbal child who has discovered that a plastic toy with brightly coloured shapes will light up and play songs when the shapes are touched will be seen hitting similar brightly coloured shapes on another toy to achieve the same result, thus being able to recognise specific design features. Children involved in this activity recognise attributes of an object or items in a collection that perform desired functions, such as a toy with wheels on that is rolled (‘driven’) over the floor (also incorporating the activity of playing), and are able to plan for and manipulate objects in the environment to perform a function. Choosing matching shapes on a posting box (or shape sorter) correctly, even if they are unable to accurately place the object through the hole, is an example of designing.

Playing Reframed.

Bishop’s reference to “imagined and hypothetical behaviour” (Bishop, 1991, p. 23) is witnessed in preverbal children’s emerging playing activity. Children use objects symbolically, showing imagination and understanding of their world. A child who picks up a wooden car and rolls it is demonstrating imagination and knowledge of other toys. Preverbal children are often seen operating alone as they investigate their environment. While investigating their environment preverbal children are creating or checking their ‘hypothesising’ of the world. Engaging in the ‘Playing’ activity is often seen as an application of previous encounters with ‘rules’ of objects (which also incorporates the activity of design) and place, such as, water is always wet or sand always changes shape when squeezed between fingers.

Explaining Reframed.

Preverbal children constantly seek ways to explain their environment and the objects within it. They pose and test theories as they engage with people, places and things. As they become increasingly more confident with a tested theory (also incorporating the activity of playing) they are able to organise items to demonstrate similarities or purpose. Purposefully reaching for an object, closing their fingers around the object to grasp it and bringing it to themselves shows that the theory they had about their body’s movement, the accessibility of the object, and movement through space are confirmed (also involving locating). With increasing confidence and agility, they move to new objects and perform more complicated actions within their environment. When young children move an object out of another's reach or position their body to block another's access, they show they have made sense of, or 'explained', their world.

METHOD

Observations are a recognised way of collecting data and there are various types of observations available to researchers (Bryman 2012). Lynch and Stanley (2018) describe young children's behaviours as "self-directed, internally motivated, process-oriented interactions" (p. 61). In terms of the EYLF (DEEWA, 2009), the focus was on child-initiated activities. A 360-degree video camera was securely attached to the ceiling of the room at a point close to the centre and was recording when the children entered the room with the educators and turned off by the researchers at the end of the session after 30 minutes. Researchers did not interact with the children, engaging in what Bryman (2012) terms a simple unstructured observation. The full videos were viewed and then instances that were initiated by the child were analysed. In two videos, eight children with two educators were filmed and in another, four children with one educator. Consent to be videoed was obtained from the parents, educators, and the centre manager. Although educators planned activities that would involve the children engaging in mathematical thinking, the focus of this paper is on preverbal children engaged in child-initiated activities.

Selection and analysis of the video segments.

The video segments in this paper were chosen to fit specific criteria – involving only one child younger than 2 years of age who did not speak during the segment; child-initiated activities with no interaction with an educator; and they were encapsulated and short. Each segment was viewed multiple times individually by the two researchers and notes taken. After the notes were completed, each video was viewed several more times by the researchers together while reviewing the notes to check their veracity, accuracy, and interpretation. All notes were analysed to identify the components of Bishop's (1988, 1991) reframed mathematical activities then the video was viewed again as a final check.

APPLYING BISHOP'S REFRAMED MATHEMATICAL ACTIVITIES

This paper presents two video segments to demonstrate how the children's actions were interpreted in terms of Bishop's (1988, 1991) reframed mathematical activities (which are in [square brackets] after the action). Brief descriptions of each video segment are provided, followed by the analysis (Figure 1).

Segment 1: Jumping on a pillow.

Eight children and two educators were in the room. The child had moved towards pillows decorated and shaped like pieces of a tree. She picked up the pillow shaped like a cross-section and threw it. She then walked to and stepped on the pillow, stayed there for a short period of time, and then stepped forward off the pillow.

Segment 2: Moving a toy truck.

Four children and one educator were in the room. The child goes to the shelf containing a toy truck with a string that can be used to pull the truck along. He tries to use the string to move the truck off the shelf, but finally uses his hand to pick it up. He puts it on the floor and then pulls it along, almost tripping over a toy as he does so.

Segment 1: Jumping on a pillow.	**Segment 2: Moving a toy truck.**
1. The child took seven steps towards the pillows [*locating*, positioning her body and moving towards desired objects; *measuring*, moving forward to be within reach of the objects].	1. The child walks to shelf with truck and reaches out with his left hand to grasp string attached to the toy truck [*locating*, positioning his body by moving towards desired objects; *measuring*, moving forward to be within reach of the objects; *explaining*, reaching for the string as it is connected to the truck].
2. She picks up one pillow with both hands and turns as she throws it onto the mat [*measuring*, determining the distance to move her hands to reach the pillow; *locating*, reaching for and picking up the pillow; *designing*, manipulating the objects in the environment by throwing the pillow; *counting*, selecting one of many].	2. Pulls the string while slightly backing away from shelf but the toy does not move [*designing*, creating a plan to achieve an outcome; *measuring*, determining the distance to move back from the shelf; *locating*, moving his body back from the shelf].
3. She takes two steps towards the pillow, pauses one second, then steps with one foot at a time until both feet are on the pillow [*locating*, positioning her body within the spatial environment; *measuring*, accurately traversing the distance between the original position and the required position; *designing*, planning for and manipulating her environment through her body position].	3. Grabs the string with his right hand, holding it briefly with both hands, and then releases the string from his left hand [*locating*, positioning his hands to grasp the string; *measuring*, accurately judging the distance between the original position of his hands and the required position; *counting*, removing one hand from the string to make available to pick up the truck; *explaining*, trying to move the truck by pulling the string as the string is attached to the truck].
4. She stands for two seconds, wobbling slightly, then moves her right foot slightly (but not lifting it from the pillow) [*explaining*, the relationship between her body position and her standing on the pillow; *locating*, moving her foot in relation to her body].	4. Grabs truck with left hand and removes it from the shelf [*designing*, creating a new plan to move the truck; *locating*, moving his hands with the string and the truck; *measuring*, moving the distance to reach the truck; *playing*, changing his approach to increase the chance that he will move the truck].
5. Stands for three seconds then lifts and replaces her right foot [*locating*, positioning her right foot in the spatial environment; *playing*, staying on the pillow rather than moving off].	5. Turns and takes two steps away from shelf to place truck on the floor, with string still in right hand [*locating*, positioning the truck on the floor; *measuring*, moving the truck and himself away from the shelf so that the truck can be moved].
6. She stands, again wobbling slightly, for four seconds [*locating*, keeping her position on the pillow; *playing*, staying on the pillow rather than moving off].	6. Takes five steps to walk around the truck with string pulled taut [*locating*, moving himself around the truck; *designing*, making the string taut to pull the truck; *playing*, using the string to pull the truck].
7. She then takes four steps with her feet still returning to the pillow but moving slightly forward until her right foot steps partially off and she walks forward off the pillow [*locating*, positioning her body within the spatial environment; *designing*, stepping forward until she is off the pillow].	7. He then turns to look behind at the truck as he walks forward three steps [*locating*, positioning his body within the spatial environment; *playing*, watching the truck as it moves with him as he pulls the string and moves forward].
	8. He almost steps on a toy on the floor – he stops, looks down at the toy, takes two steps to the side and then walks forward again while pulling the truck [*locating*, positioning his body within the spatial environment; *designing*, stepping around the toy on the floor; *playing*, pulling the string to move the truck; *measuring*, estimating how far to walk around the toy on the floor; *counting*, recognising there is something on the floor that requires him to walk around it].

Figure 1: The analysis of the two video segments.

DISCUSSION AND CONCLUSION

Barad (2007, p. 353) states “believing something is true doesn’t make it true”, which emphasises the importance of using evidence based and rigorous processes. This paper uses Cooke and Jay's (2021) reframed version of Bishop's (1988, 1991) mathematical activities to identify preverbal children's potential mathematical conceptions. All of Bishop's reframed mathematical activities were evident in the behaviour of the preverbal children and, multiple mathematical activities occurred concurrently. Bishop's (1991, p. 108) concession that there is overlap between the concepts within some of the mathematical activities was evident in the analysis. The nature of child-initiated play enables the child to explore their environment and, through this, develop an understanding of mathematical concepts and relationships (Björklund & Pramling 2017). Child-initiated play provides opportunities for preverbal children to work through and demonstrate in their behaviours and actions all of Bishop's mathematical activities. The presence of all of Bishop's reframed mathematical activities indicates how powerful this structure is for identifying mathematics in child-initiated play. Further research would need to be conducted to determine whether the bounded nature of child-initiated play consistently demonstrates Bishop's reframed mathematical activities.

There are limitations in this approach. It was the interpretations of the researchers of the preverbal children's that was used that was linked to Bishop's (1988, 1991) mathematical activities. Franzén (2015) stated, the researcher has power in the way they interpret what they see – the goal is for the interpretation to be close to the perspective of the observed child as possible. This, she explains, needs to be carefully done, particularly when the child does not use language, as what the researcher observes and interprets will always be ambiguous and incomplete.

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DIGITAL RESOURCE DESIGN AS A PROBLEM SOLVING ACTIVITY: THE KEY-ROLE OF MONITORING PROCESSES

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In this paper we analyse upper secondary school students' design of digital resources by interpreting digital resource design as a problem solving activity strongly influenced by the process of instrumental genesis. Our research questions concern the monitoring processes activated by students-designers. Through our analysis, we identified different levels of monitoring during digital resource design, highlighting how monitoring processes are influenced by the students' systems of conceptual and procedural knowledge and by the artifact's constraints.

INTRODUCTION AND BACKGROUND

The study presented in this paper sets in the mainstream of research focused on the role played by digital tools in the task-design process (Leung & Baccaglini-Frank, 2017). In different areas of disciplinary education, an increasing interest has emerged in investigating how the design process affects the designers' learning itself, focusing on the role that students could play as designers or co-designers of different kinds of digital resources (Kimber & Wyott-Smith, 2006; Tracy & Jordan, 2012). Few studies have focused on this aspect in the context of mathematics education (see, for instance, Diamantidis, Kynigos & Papadopoulos, 2019; Alessio et al., 2021). With the aim of contributing to this research issue, in this paper we analyse the process of digital resource design (in the following, DR-design) carried out by a group of secondary school students. In order to develop this analysis, we interpret the design process as a problem solving activity, as Shaffer (2007) does in his investigation of the nature of problem solving in architectural design, where design is conceived as a process aimed at resolving an open-ended problem through a series of intermediate solutions.

Most of the frameworks developed to investigate students' problem solving processes have identified specific phases that characterize them (see, for instance, Schoenfeld, 1985; Garofalo & Lester, 1985; Carlson & Bloom, 2005). Identifying these phases enables researchers to parse protocols of students' interactions focused on problem solving into episodes that can be classified according to specific categories (Schoenfeld, 1985). By investigating experts' problem solving processes, Carlson and Bloom (2005) characterise the problem solving process in terms of nested cycles of repeated actions. In particular, they noticed that experts move through four main phases when completing a problem: (a) *orienting*, when a mental image of the problem situation is constructed and the solver attempts to make sense of the question; (b) *planning*, when conjectures are initially devised and the playing-out of possible approaches is imagined; (c) *executing*, when the strategies devised during the planning

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phase are concretely carried out; (d) *checking*, when the focus is on assessing the correctness of the implemented approaches. When the first phase is completed, the planning-executing-checking cycle is repeated throughout the remainder of the solution process.

A key feature, within all the frameworks developed to explain the solvers' ways of dealing with problems, is represented by the importance of engaging in metacognitive behaviours to regulate own processes and make decisions. Schoenfeld (1985), in particular, focuses on *control*, defining it as a category of behaviour that deals with the ways in which individuals use the information at disposal and with the major decisions that they make when they are solving a problem.

Since good problem-solvers are not those that always make the right decisions, but those that can recover from erroneous decisions, Schoenfeld stresses that a major component of effective control consists of the periodic monitoring and assessment of solutions as they evolve. In tune with Schoenfeld's studies, Carlson and Bloom (2005) notice that experts monitor their thought processes and products regularly during all the problem solving phases, with the aim of both making decisions about their solution approaches and reflecting on the effectiveness of their decisions and actions. Here we refer to their definition of *monitoring* as "reflection on and regulation of one's thought processes and products at any point in the solution process" (pp. 54-55).

Carlson and Bloom (2005) highlight that, in the case of expert problem-solvers, the effectiveness of their monitoring is assured by their strong conceptual and procedural knowledge, since they could draw on this knowledge to verify the reasonableness of their results and the correctness of the actions they carry out. On the other hand, unstable systems of conceptual and procedural knowledge generate problems in activating monitoring processes (Schoenfeld, 1985). Lesh (1982) observes that, when the conceptual system is poorly coordinated, students risk to ignore salient features of a problem or to distort the interpretation of the problem situation. In case the procedural system is unstable, students' work is often characterized by rigidity in procedure execution and inability to anticipate the consequence of actions during the execution.

In the case in which the problem solving process under investigation is the design of digital resources for mathematics, both the identification of specific strategies to be implemented and the activation of effective monitoring processes are influenced not only by standard mathematical knowledge, but also by the knowledge about the digital artifact that is used and by the computational transposition of mathematical knowledge that the use of the artifact involves (Artigue, 2002). During the design process the digital artifact is gradually transformed into an instrument, by means of a process of instrumental genesis. This process works in two directions: on one side (instrumentalisation), the artifact is progressively loaded with potentialities and transformed for specific uses (Artigue, 2002); on the other side (instrumentation), constraints and potentialities of an artifact shape the subject's activity (Trouche, 2005), leading to the development of schemes of instrumented actions (Artigue, 2002).

Trouche (2005) distinguishes between three kinds of constraints: internal constraints (physical and electronic), command constraints (linked to the different commands and to the artifact's syntax), organization constraints (linked to the screen organization).

RESEARCH CONTEXT

The study presented in this paper involved 20 upper secondary school students (grades 12-13) who participated in a university STEM literacy program for students in secondary-tertiary transition (Alessio et al., 2021), which took place in the Polytechnic University of Marche in the period between October and December 2020. The part of the program devoted to Mathematics was aimed at giving students the opportunity to deepen their knowledge of specific mathematical topics through the use of GeoGebra as a tool for DR-design. It was articulated into 5 sessions (20 hours in total). During the first session, focused on the presentation of the GeoGebra software, the participants were involved in a guided design process in order to explore the features of the software and to gain confidence with its commands. The two following sessions were devoted to introducing a mathematical topic that participants had not faced at school: the theory of complex numbers. During the last two sessions, the participants worked in small groups (7 groups in total) and were asked to design two (or more) GeoGebra applets to support students' learning of complex numbers. Due to the pandemic emergency, most of the activities (including the working group activities) were developed at distance, by means of the Zoom platform. Three university tutors were always available to support the students during the DR-design process.

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

As mentioned above, the focus of the research documented in this paper is on students as designers of digital resources. The hypothesis on which our study is based is that, particularly in the case of students as designers, the problem solving process that characterizes DR-design is strongly influenced by the parallel process of instrumental genesis that characterizes students' construction of personal schemes in the use of a digital artifact and their appropriation of pre-existing schemes. The instrumentation process, in particular in its first phase, when several techniques and strategies appear to burst, could strongly affect the process of DR-design, due to the role played by the artifact's constraints in shaping the designer's activity.

Considering the key role played by metacognitive behaviours in regulating problem solving processes, in this paper we focus on the monitoring processes in DR-design and on the influence of the artifact's constraints and of students-designers' systems of procedural and conceptual knowledge on this monitoring. Specifically, we address the following research questions: *What kind of monitoring processes are implemented by beginner students-designers when they face the task of DR-design? How are these monitoring processes affected by the artifact's constraints and by the students-designers' systems of procedural and conceptual knowledge?*

In order to investigate these aspects, we video-recorded the working group activities with the aim of highlighting students' spontaneous in-the-moment discussions on their design process. The analysis of the collected video-recordings was performed according to the following steps: (1) transcription of the students' discussions and description of the actions performed by students on the shared screen during the DR-design process; (2) first qualitative analysis of the video-recordings' transcripts aimed at parsing the transcripts into episodes; (3) identification of the episodes during which students activate monitoring processes and analysis of these episodes to highlight their peculiarities and to investigate the effectiveness (or ineffectiveness) of these processes in supporting the DR-design.

During step 2, we classified the identified episodes according to Carlson and Bloom's (2005) four phases. To develop this classification, we interpreted these phases in relation to the DR-design process in the following way: (a) *orienting* corresponds to the initial phase in which the goals of the DR-design process are identified and the general structure to be given to the digital resource is agreed, by identifying its main components; (b) *planning* corresponds to the choice of the techniques to be implemented to create specific components of the digital resource; (c) *executing* corresponds to the implementation of a selected technique; (d) *verifying* corresponds to the activation of monitoring processes to highlight the effectiveness (or ineffectiveness) of the implemented technique in relation to the set goals. The planning-executing-verifying cycle is repeated each time the monitoring process highlights the ineffectiveness of an implemented technique.

DATA ANALYSIS

The results that we present refer to the analysis of the work carried out by the groups of students-designers in their first DR-design process, aimed at creating a GeoGebra applet to support students in the investigation of the different representations (algebraic, trigonometric, graphical) of complex numbers. Since all the students-designers were beginner designers, not having had previous experiences in DR-design, the DR-design process developed by the different groups was characterized by similar dynamics. Moreover, all of the participants had had few experiences in using the GeoGebra software. Due to space limitations, here we focus on the work carried out by one group of students-designers constituted by two girls (S, C) and one boy (V). The main protagonists of the design work are V, who shares his screen and works on the GeoGebra applet, and S, who poses herself as a guide for V. C rarely intervenes.

The *orienting* phase is almost missing in the work of the group. The students-designers immediately start to work on their GeoGebra file without previously discussing the general structure to be given to their applet. The result is that, from the very beginning, they proceed step by step: (a) setting single goals to pursue in the design process (constructing specific components of their applet); (b) identifying strategies to pursue these goals (techniques to adopt to construct each applet's component); (c) implementing these strategies; and (d) assessing their effectiveness. When students-designers think that they have reached each goal, they set a new goal and proceed through the same steps. In this way, the construction of the applet's components is developed without having clear the general structure of the applet and the relations between its different components.

In the following, we present our analysis of three short excerpts from the transcripts of the video-recording of the group's activity. These excerpts were selected with the aim of introducing paradigmatic examples of typical monitoring processes carried out by students-designers to assess the effectiveness of the techniques adopted to pursue specific goals. All the excerpts are focused on the first *goal* that students-designers set in their DR-design process, that is to introduce the algebraic definition of complex numbers and their representation on the Cartesian plane. Each excerpt corresponds to one micro-cycle of planning-executing-verifying and is presented by introducing the *techniques* adopted and implemented to pursue the goal and the *characteristics of the monitoring process* activated by students-designers.

Excerpt 1

The *first technique* that the group adopts to pursue the set goal is to create two sliders a and b and to write $z = a + ib$ in the input bar. GeoGebra recognizes $z = a + ib$ as a surface and the students-designers immediately realize that the representation that appears on the screen is not what they expected. The activation of an incorrect technique testifies students' weaknesses at the procedural level, in managing command constraints. The unexpected feedback from GeoGebra boosts a *monitoring process on the first technique* being implemented, leading students-designers to recognize the incorrectness of the chosen formula. This feedback is unexpected since it is not the result of an intentional process aimed at an aware activation of control strategies. The monitoring process is rapid and it simply consists in highlighting the need of adopting a different technique, without reflecting, at the conceptual level, on the reasons why writing $z = a + ib$ has produced an unexpected representation.

Excerpt 2

Although the monitoring process in excerpt 1 is not associated with deep reflections on the conceptual aspects related to the feedback received from the applet, it makes students-designers progress in their instrumentation process by exploring new formulas to be used. At the beginning of this exploration, V proposes to define a new point whose coordinates are a and b. S agrees and suggests to V to define this point by writing $A = (a; ib)$ in the input bar. This represents the *second technique* implemented by the group *to pursue their first goal*. When V writes $A = (a; ib)$ in the input bar, one tutor intervenes with a question for the group:

Tutor: Why did you write ib as a coordinate of the point?

S: Because… No! …It should be just b, since the *y*-axis is the imaginary axis.

The *monitoring process on the second technique*, activated by S, is boosted by the implicit feedback given by the tutor through his question. Therefore, also this monitoring process is not intentionally activated. Differently from what happens in excerpt 1, S not only proposes a correction of the second technique (to write b instead of ib), but she also explains the reasons behind the correction, showing awareness of the meanings associated with the ways in which the second technique is modified. In explaining these reasons, S effectively refers to her system of conceptual knowledge, recognizing the connections between the formula to be written in the input bar and the underlying mathematical knowledge.

Excerpt 3

Guided by S, V implements a *third technique*, which is simply the correction of the second technique adopted by the group and writes $A = (a; b)$ in the input bar. Then V starts moving the two sliders *a* and *b* and a silent verifying phase begins. Although, in students-designers' intentions, *a* and *b* should represent, respectively, the abscissa and the ordinate of the point, when V leaves the slider *a* fixed and varies the slider *b*, the point moves along a circumference centred in the origin, instead of moving on a line parallel to the y-axis. This problem is due to the fact that, since the coordinates of the point are separated by a semicolon instead of a comma, GeoGebra recognizes *a* and *b* as polar coordinates of the point A (not as cartesian coordinates, as students expected). The *monitoring process on the correction of the second technique* is intentionally activated: students-designers do not limit themselves to interpret the feedback received by GeoGebra or by a tutor, but intentionally interact with their applet to verify the effectiveness of the implemented technique. However, this process is ineffective since the students-designers do not correctly interpret the feedback given by GeoGebra when they interact with their applet, without noticing the problem related to the formula that they write in the input bar. The ineffectiveness of the monitoring process is due to their weaknesses both at the procedural level (they are not able to manage the command constraints) and at the conceptual level (they are not able to correctly interpret, in mathematical terms, the variation of the point when *a* and *b* vary).

DISCUSSION

In this paper we presented the results of the analysis of data collected during a study focused on the role of students as designers of digital resources. We analysed the DR-design process as a particular problem solving activity, by identifying micro cycles of planning-executing-verifying during the whole process. In particular, we focused on the monitoring processes activated by students-designers. The data analysis enabled us to identify different levels of monitoring, according to two main interrelated elements: the intentionality of the monitoring process and the students-designers' awareness in reflecting on their design at both the procedural and conceptual levels.

In excerpt 1, the monitoring process is not intentionally activated: students limit themselves to react to the feedback provided by the GeoGebra applet, recognizing that the representation that appears on their screen is completely different from the expected one. The decision that is taken (identify a different formula to be written in the input bar) is not motivated by explicitly referring to the related systems of procedural (the syntax of GeoGebra's commands) and conceptual knowledge (the mathematical meaning of the representation that appears on the screen).

In excerpt 2, it is the tutor that gives implicit feedback to the students-designers, so, again, the monitoring process is not intentionally activated. However, differently from what happens in excerpt 1, one of the students-designers (S) effectively draws on her systems of conceptual and procedural knowledge to reflect on the reasons behind the ineffectiveness of the second technique that was implemented.

Differently from the previous excerpts, in excerpt 3 the monitoring process is intentionally activated. However, this intentional monitoring process is not associated with an effective interpretation of what the students-designers observe on their screen. Their weaknesses at both the procedural and conceptual level prevent them from being aware of the connections between the ways in which they move the sliders *a* and *b* and the variation of the constructed representation.

The analysis of the three excerpts highlights the role played by the artifact's constraints (in particular command constraints) and by the students-designers' systems of procedural and conceptual knowledge in preventing them from activating effective monitoring processes. Since the command constraints are at the basis of the feedback provided by the applet, they could potentially boost students-designers' activation of monitoring processes that make them realize the need of adopting different techniques (like in excerpt 1). However, the students-designers' lack of awareness at both the procedural and conceptual level prevent them from activating intentional monitoring processes (like in excerpts 1 and 2) or from correctly interpreting the feedback provided by the applet in order to identify the reasons behind the ineffectiveness of an implemented technique (like in excerpt 3).

The fact that all the students involved in the study were beginner designers and had had a little experience in the use of the GeoGebra software clearly affected their DR-design. It implied, for example, that the orienting phase was almost missing in the DR-design of all the groups of students-designers, preventing them from having a clear overview of the structure to be given to their applet. Moreover, it prevented students-designers from carrying out effective monitoring processes by: (a) intentionally act on the digital resource, guided by anticipating thoughts about the expected effects of these actions; (b) correctly interpret the feedback provided by the digital resource to infer about the effectiveness of the implemented techniques; (c) reflecting in an aware way on the reasons underlying the effectiveness (or not) of the implemented techniques. Carrying out these kinds of processes requires a deep awareness of the mathematical knowledge related to the representations constructed through the applet and a stable system of procedural knowledge.

As a further step of our research, we plan to involve students-designers on a long-term program with the aim of investigating how the characteristics of their monitoring processes evolve when students progress in their experience of DR-design. Moreover, we plan to study what are the main factors that influence a positive evolution of students-designers' monitoring toward intentional and aware processes.

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BETWEEN PAST AND FUTURE: STORIES OF PRE-SERVICE MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

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We study the efficiency of a pre-service teachers' education method that is based on a theory-informed analysis of teaching-learning processes, design of tasks for pupils and subsequent creation of fictional classroom discussions focused on the same tasks. A key element of the method is the request, to pre-service teachers, of writing down, after each session of the course in which the method is implemented, accounts of the session, under the guide of suggestive questions. In the contribution we analyse such accounts in order to study the evolution of pre-service teachers' attitude towards mathematics and mathematics teaching and the development of their identity as future teachers.

INTRODUCTION AND THEORETICAL BACKGROUND

In this contribution we focus on a method for primary school pre-service mathematics teacher (in the following, PMT) education, designed with a special focus on affect. This choice is in tune with what Hodgen and Askew (2011) advocate, stressing that teachers' affect plays a crucial role in their professional development, thus it should be considered already in teacher education programs. We rely on the work by Di Martino et al. (2013), who study the link between the past experiences of PMTs as students and their future perspectives of becoming mathematics teachers and describe the phenomenon of the "desire for math redemption", i.e. "*the desire to face the "challenge" of teaching mathematics, starting from a personal reconstruction of the relationship with the discipline*" (p. 226). Such a redemption can be achieved by means of specific interventions, such as the education program for PMTs designed by Morselli and Sabena (2015), which is based on problem solving activities and on narrative reconstruction of PMTs' "affective pathways" during problem solving. They point out that PMT education should act in two ways: "in continuity with respect to the need for redemption […], but also in discontinuity with the widespread procedural view of mathematics" (p. 1232). In this contribution, we present a method for PMT education and we propose the use of two theoretical lenses to discuss the efficiency of this method in fostering such *continuity* and *discontinuity*. The first theoretical lens is the construct of identity, that in the last years has gained increasing interest among researchers in teacher education, as evidenced by the recent overviews and systematic reviews of research in the field (Lutovac & Kaasila, 2018; Graven & Heyd-Metzuyanim, 2019). We refer to the definition of identity by Sfard and Prusak (2005, p.1): "Identity is a set of reifying, significant, endorsable stories about a person". According to the narrator, the referee and the recipient, it is possible to distinguish between AAA stories (told by oneself, about oneself, to oneself), AAC

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stories (told by oneself, about oneself, to another recipient), BAC stories (told by a narrator, about another person, to a third recipient). The second theoretical lens is the construct of attitude, defined by Di Martino and Zan (2014) by means of a model made up of three interrelated dimensions: *emotions* toward mathematics, *vision* of mathematics, *perceived competence* in mathematics. The model, initially theorized for students' attitude, was adapted to study PMTs' attitude towards mathematics and its teaching (Di Martino et al., 2013).

OUR METHOD FOR PMT EDUCATION

We rely on the method introduced by Cusi and Malara (2016), that encompasses the use of theoretical tools for both the design of classroom activities and the a-posteriori analysis of teaching-learning processes. We adapted this method to the case of PMTs by designing a PMT education course characterized by the following kind of activities (see also Cusi & Morselli, 2018): 1) sharing and study of theoretical tools (concerning teaching-learning processes and the roles played by the teacher during classroom discussions); 2) analysis of tasks for students and of videos from classroom activities, by referring to the theoretical tools; 3) design of tasks for pupils and creation of fictional classroom discussions focused on the same tasks, by referring to the theoretical tools); 4) sharing and comparison between the different tasks and fictional classroom discussions created by PMTs; 5) individual reflections, after each session of the course, to be shared with the teacher educator (one of the authors) in the form of a written reports. Individual reflections were guided by some questions suggested by the teacher educator at the end of each session. Suggested questions varied according to the content of the sessions. However, recurrent questions were: "What are the aspects that struck you more? What have you learnt? What have you discovered?".

RESEARCH AIM AND QUESTION

In a previous work (Cusi & Morselli, 2018) we focused on activity 3, showing that the specific activity of creating fictional classroom discussions promoted a change of perspective, from university students to future teachers, and led PMTs to appreciate theoretical lenses as a support for creating discussions, but also as relevant guidelines for their future practice as teachers. In this paper we focus on the following research question: *Is the method adopted during the course efficient in fostering a continuity with respect to the PMTs' need for redemption and a discontinuity with respect to the procedural vision of mathematics?* To address this question, we focus on the data collected through activity 5, that is on the individual reflections that are performed after each session of the course, and analyse them in terms of attitude towards mathematics teaching and learning and development of teacher identity.

RESEARCH METHOD

The course on which this study is focused involved a group of 80 primary school PMTs attending their first university year (5 years totally; practicum starts in their second year), and lasted 32 hours. Totally, we collected 10 accounts for participant. In our analysis, we looked at such written accounts produced during the course as a collection of stories told by PMTs about themselves and directed to the teacher educator as the final recipient (AAC stories). We initially selected parts of the written accounts where the PMTs speak about themselves, that is parts that could represent *reifying* stories, referring to Sfard & Prusak (2005)'s definition. In a second moment, among the reifying stories, we identified sub-stories that are recurrent, i.e. the PMT treats them more than one time in her written accounts. This characteristic makes the selected stories also *significant* for the narrator. Moreover, we focused on sub-stories that are proposed spontaneously by the narrator, without a direct request by the teacher educator, that is on *endorsable* stories. Once selected the sub-stories, we analysed them in terms of attitude towards mathematics and its teaching and learning. Investigating the development of the sub-stories throughout the whole course, we came to reconstruct a story about each PMT, which is a CAD story, since it is our way of narrating about the PMT to a third recipient (the reader of this paper). We outline that the themes of sub-stories may vary from PMT to PMT. Our narration brings to the fore the prevailing themes for one PMT, thus contributing to characterize her identity as a future teacher. In this contribution we confine to two stories of development so as to start our reflection on the efficiency of the method. Our work will be later integrated with the analysis of problematic stories, so as to understand in which cases PMTs' participation in the course does not promote their professional development.

DATA ANALYSIS

We present and discuss the sub-stories of two PMTs, Zelia and Ella, narrating them by means of relevant excerpts and analysing them through our theoretical lenses.

The first prevailing theme in Zelia's sub-stories is "mathematics and its teaching". In the following excerpt from her 3rd written account, she highlights the fact that the classroom activities proposed during the third lesson are designed to support pupils in making their reasoning process explicit. Reflecting on the fact that too often, in mathematics teaching, the product and the application of rules are considered more important than the process, Zelia explains that she is reconsidering her *vision* of mathematics. She asks herself if her love for mathematics is a simple infatuation, associated to an incomplete imagine of this discipline:

> I do not hate math. On the contrary, this subject has become enjoyable year after year. But I think that this course is instilling in me the doubt that I've never really loved it…I ask myself if the mathematics that I thought to love is the real mathematics, or if I simply loved the being able to perform exercises, using rules (Zelia, 3rd written account).

Zelia's awareness that positive emotions toward mathematics could be connected to procedural *visions* of the subject is also proposed in her 4th written account. Discussing on a teaching episode analysed during the fourth session, Zelia reflects on the role played by the teacher, which was very different from the approach that her upper secondary school teacher used to adopt. Zelia declares that she used to prefer traditional lessons, where the teacher explained contents at the blackboard, to lessons focused on problem solving, and calls again her *vision* into question:

> I am trying to discover if my vision of mathematics, under a veil of positivity, is only "mathematics as rules to be applied" (Zelia, 4th written account).

In the same written account, Zelia narrates an exchange between herself and another PMT, concerning her colleague's doubt if mathematics really opens the mind or not:

> I answered her that mathematics could help in reasoning and in fostering understanding. I am quoting this experience because I am asking myself if I am myself expressing a prejudice, that is if I am not reflecting enough about it. Maybe I convinced myself that mathematics opens the mind, while I am continuing using mathematics in a 'mechanical' way (Zelia, 4th written account).

After session 6, Zelia reflects on her way of approaching problems (looking for symbolic expressions to represent relations between variables) and recognizes her difficulty in appreciating alternative solutions; once again, she connects this difficulty to her poor *vision* of mathematics:

> The fact that I am not able to find out other strategies makes me reflect on my way of approaching mathematics and on my fear that my answer is always "not enough mathematical"… I am not used at verbalizing my resolutions. I think it could be due to two possible reasons: my approach has become an automatism (it is a problem if it prevents me from elaborating other strategies), or my vision of mathematics focuses only on the product and does not care of the process…It makes me reflect on the fact that some approaches are so well-established that I need to continuously reflect on them (Zelia, 6th written account).

Referring to the three-dimensional model of *attitude*, we may say that the course promoted Zelia's reflection on her attitude towards mathematics: from the beginning she reports *positive emotional dispositions* towards the discipline, but throughout the course she starts questioning her *vision* of mathematics. When she faces difficulty in performing process-oriented activities, she does not stick in a *low perceived competence*, rather she feels more motivated to work on a more elaborated *vision* of the discipline. We may note that Zelia not only reflects on her *past* as a student, wishing to improve her vision of mathematics; Zelia is also aware of the fact that the *vision* of mathematics could influence the teacher in planning her teaching approach. This leads to the second theme, "roles of the mathematics teacher". The first excerpt refers to session 2, where the discussion with the teacher educator and her mates made Zelia reflect on her role-model teacher and compare her with other kinds of teachers, referred to as "lazy teachers", who contributed to her mates' experiences of failure in mathematics:

> He is among the models of teacher [...] I take when I imagine me as a future teacher, and that at deeper level maybe gave me the desire to teach and to be for my pupils what he was for me. Conversely, in my colleagues' accounts I recognized the lazy teacher, ready to

label students, that can contribute the student's failure, aspect that is not so far from my experience as a primary student. (Zelia, 2nd written account).

In her 5th written account, Zelia reports that the activity of analysis of teaching episodes enabled her to become aware of the importance of teacher's flexibility in managing classroom discussions, and of the necessity of an accurate design of each lesson.

I was struck by the way the teacher was ready to create the metaphor [to make the student's reasoning clearer]. This makes me wonder in which way the teacher had prepared the lesson. I wonder whether the teacher took some time to make hypotheses in the students' difficulties, or the metaphor was already known and she was able to recall and use it at the good moment. (Zelia, 5th written account)

Reflecting on session 8, devoted to the presentation of a theoretical construct aimed at characterizing the roles played by the teacher during classroom discussions, Zelda realizes that some of these roles are fundamental in supporting pupils in overcoming a "static" *vision* of mathematics.

We can highlight the importance of whole classroom discussions, since it is from the crash/encounter of different thoughts, approaches, strategies, that we can grasp the richness of thinking and overcome the traditional static vision of mathematics, according to which, for each problem, there is only one solution and one way to find it (Zelia, 8th written account).

In reference to the second theme ("roles of the mathematics teacher"), Zelia realizes that the teacher has a role in influencing students' *attitude* towards mathematics (2nd written account, reference to her mates' experiences of failure). Afterwards, Zelia recognizes the importance of specific roles concerning planning and managing class activities, thus enriching the three components of her *attitude* towards mathematics teaching in terms of *vision* (she values specific roles such as planning, managing discussions), *emotional disposition* (she is positively struck by the teaching episode!) and *perceived competence* (she realizes that class discussion can help her in proposing students meaningful and "non static" lesson, as she wished).

The second PMT we refer to is Ella. Her first theme concerns "mathematics and its learning". In the account after session 2 (where Ella and her mates were proposed a task of conjecture and proof), Ella recognizes that she never experienced such rich mathematical activities when she was a student, and this lack made her develop a poor *vision* of mathematics, based on procedures rather than on argumentation and reasoning. Moreover, she reports a *negative disposition* towards mathematics (algebra in particular), coupled with a constant fear of making mistakes.

My approach to mathematics, in my experience as a student, led me to underestimate the goals of learning, to internalize isolated concepts; even worse, to provide mechanical solutions, often without conscious argumentations. […] Fear of making mistakes. […] The feeling of having a dangerous relationship with the discipline, always on the edge between the desire to move beyond and the sensation of falling down, step into the wrong calculation and into my limits. (Ella, 2nd written account)

Throughout the course, Ella reflects on the proposed examples of teaching activities and recognizes that such activities are improving her *vision* of mathematics:

> I never thought about the great potentialities of the discipline in such a perspective. Working for the construction of our mathematical thinking, starting from primary school. Learning to "speak mathematically". Experience mathematics as a language. (Ella, 6th written account).

In the final account, Ella recognizes some improvement in her *attitude* towards mathematics, with reference to the *vision* of the discipline. She speaks again about mistakes, but finally she is keen to accept mistakes as an unavoidable part of the process of doing mathematics.

> I bring with me my past as an afraid and insecure student. […] I learnt to hate mathematics, even before looking it in the face. I became able to hide behind an exercise, to circumvent questions, to repeat minimal operations, avoiding the overall vision and the search for meaning. For this reason, the biggest difficulty during this course was to become the protagonist. […] To ask questions I had never dared to share. […] to make mistakes, above all. To learn to make mistakes. To desire and to allow myself to make mistakes. Because in my former non-experience as shy and unsecure student, the mistake was not allowed. (Ella, Final written account).

Referring to the three-dimensional model of *attitude*, we may say that thanks to the course Ella moved from an *attitude* towards mathematics characterized by *negative emotional disposition* (fear, danger) and poor *vision* of the discipline (mechanical procedures, emphasis on the final product without mistakes) to a more positive one, characterized by a new *emotional disposition* (to dare to ask questions and make mistakes) and a new, richer *vision* of the discipline (mathematics as language) characterized by argumentation and search for meaning.

The second theme refers to the "roles of the mathematics teacher". After session 2, Ella reflects on the proposed activity (solving a task of conjecture and proof) and on the crucial role of the teacher educator in the collective discussion on the task. In particular, Ella points out a new way of conceiving mistakes as resources for the teacher:

> […]. I highly appreciated the way the teacher [educator] was able to guide the reasoning of the group without interfering, not judging, using mistakes as resources for the reasoning that was in construction. (Ella, 2nd written account)

After session 3, Ella reflects on the fact that the teacher needs to be flexible and adapt her lesson plan to the students' interventions. Ella expresses the interest for learning how to manage class discussions in a fruitful way.

> It seems to me that, in this sense, the teacher's art is very close to that of the master craftsman. And I would like to experiment and I would like to experience firsthand […] possible techniques of design and presentation to the class. (Ella, 3rd written account)

The 6th written account, where Ella reflects on a session focused on the analysis of classroom activities, contains an interesting reflection on the role of the teacher who asks questions to the pupils, but also to her/himself.

> What strikes me […] is the interactive learning and the cooperative climate that the teacher stimulates through the acquisition of specific roles. Children learn to confront themselves in group, to argue their own reasonings and strategies, to change their minds [..]. The teacher learns: [to be an] equilibrist in search of questions to ask students and to ask herself, bearer of an idea of transparent teaching, which replaces the question, research and thought as nourishment for the individual and the community to the imposition of univocal solutions. (Ella, 6th written account).

In reference to the second theme ("roles of the mathematics teacher"), we may find in Ella's accounts instances of a positive *attitude* towards mathematics teaching. During the course, the *vision* of mathematics teaching becomes richer and richer, with reference to the crucial roles of the teacher in managing class discussions, guiding students' reasoning without imposing a strategy, using mistakes as resources. Ella also expresses a positive *disposition* towards her future role and declares herself ready to learn and experiment how to plan and implement meaningful classroom activities. This good will and optimism concerning her future as a teacher may be interpreted in terms of good *perceived competence*.

PRELIMINARY CONCLUSIONS

This contribution was aimed at discussing the efficiency of a method for PMT education, with a special focus on affect. We adopted a double theoretical lens (identity and attitude) to analyse written reports of two PMTs (Zelia and Ella) throughout all the course. The two reported stories, and their swinging between the *past* and the *future*, show the development of their *identities* as future teachers. The stories of Zelia and Ella reveal differences in the two PMTs' relation with the *past*: Zelia focuses on *teaching*, reflecting on a positive example of teacher she experienced, while Ella reports her own difficulties in *learning* mathematics. Concerning *future*, both PMTs show an increasing awareness and appreciation of the *roles* they'll have to play as teachers. Interestingly, the reflection on the roles of the teachers that is promoted during the course is efficient in bridging the two PMTs *from the past to the future*: Zelia recognizes that by playing such roles she will be able to act as the teacher she had when she was a student; Ella thinks that by playing such roles as a teacher she will be able to help her students learn in a better way in comparison to her experience as a student.

Data analysis shows the evolution of the PMTs' *attitude towards mathematics*, encompassing a more elaborated *vision* of mathematics, thus acting in discontinuity with the *past*. We also found evidences of improvement in *attitude towards mathematics teaching*, both in terms of *vision* of mathematics teaching and in terms of *positive disposition* and *perceived competence*. Such an improvement is linked to a projection towards the *future* work as teachers and takes place in continuity with the need for "mathematical redemption".

Although the method for PMT education is put in action in the whole sequence of activities of the course, the analysis of the PMTs' stories enabled us to identify some activities that seem particularly effective: the analysis of tasks and of videos of classroom activities, the sharing of theoretical tools, the reflections in small groups on the design of specific tasks, and the active participation to laboratory workshops. The written accounts also proved to be effective, because they allowed the creation of a free space to reflect on the course (the *present*) and connect it with their experiences as teacher (their *past*) and their forthcoming role as teachers (their *future*).

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STUDENTS' REFLECTIONS ON THE DESIGN OF DIGITAL RESOURCES TO SCAFFOLD METACOGNITIVE ACTIVITIES

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In this paper we investigate the efficiency of the design of a digital resource aimed at scaffolding students' metacognitive processes during problem solving activities. We develop this investigation by focusing on students' a-posteriori reflections on their interaction with the digital resource. Through the analysis of students' reflections, we highlight the digital meta-scaffolding elements that are relevant for students and their level of awareness about the provided metacognitive support.

INTRODUCTION AND BACKGROUND

The research documented in this paper is set within a wider study that concerns the design and implementation of digital resources to foster individualization processes at university level (Cusi & Telloni, 2020A-B). In particular, we focus on students' reflections on *digital meta-scaffolding elements* (in the following, DMSEs), that is on the elements of scaffolding provided, within digital environments, with the aim of fostering students' metacognitive processes.

Research in mathematics education has widely stressed the key-role played by metacognition in problem solving (Schoenfeld, 1992; Holton & Clarke, 2006). Here we adopt Holton and Clarke's (2006) definition of metacognition as "any thinking act that operates on a cognitive thought in order to assist in the process of learning or the solution of a problem" (p.133). This definition shifts the focus on the idea of "acts" to distinguish them from all the factors that could influence metacognition but are not metacognitive in themselves (such as beliefs, intuition and knowledge). In tune with this idea, we refer to Meijer et al.'s (2006) categorization of metacognitive activities, defined as "the strategic application of metacognitive knowledge to achieve cognitive goals" (p.209). We focus on five categories of metacognitive activities identified by the authors: (1) *orientating*, which involves activities such as activating prior knowledge, establishing task demands, identifying important information, re-reading questions carefully, establish givens, observing; (2) *planning*, which involves activities such as looking for particular information in text, sub-goaling, using external source to get explanation, backward reasoning, formulating action plan; (3) *monitoring*, which involves activities such as error detection and correction, noticing inconsistency, checking plausibility, claiming progress in understanding, giving meaning to symbols or formulae; (4) *evaluation*, which involves activities such as explaining strategies, finding similarities, interpreting, quitting, self-critiquing, verifying; (5) *elaboration*, which involves activities such as inferring, checking representations, commenting on the difficulty of problems, commenting on personal habits.

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When the focus is on fostering students' metacognitive activities within digital learning environments, the design of DMSEs provided to students deserves special attention. The close interrelation between metacognition and scaffolding has been highlighted by Holton and Clarke (2006), who assert that acts of scaffolding and acts of metacognition could be potentially identified. Moreover, students' effective use of the scaffolding provided to them and their subsequent development of awareness about the role of scaffolding require that they activate themselves at the metacognitive level (Holton & Clarke, 2006). This is in tune with Pea's (2004) reflection on the crucial role played by meta-scaffolding, conceived as the scaffolding for the scaffolding. This is particularly relevant in the context of digital environments, where a good balance between procedural and metacognitive scaffolding is needed (Sharma & Hannafin, 2007). Our previous studies (Cusi & Telloni, 2020A-B) confirmed these reflections, highlighting university students' widespread lack of awareness about the aims of the DMSEs provided to them within specific digital learning environments. In particular, we highlighted how this issue is interrelated with students' lack of awareness about their weaknesses and learning needs and with their lack of metacognitive control in monitoring their problem-solving processes.

THE RESEARCH CONTEXT AND THE DESIGN OF DMSES

The context of this study is a Mathematics course for students enrolled in the "Chemistry and pharmaceutical technologies" degree course at Sapienza University of Rome (Italy). The course, scheduled for the first term of the first year, is aimed at providing students with basic Mathematics notions useful to be applied in the study of pharmaceutical chemistry. The course program covers basic knowledge related to different topics: algebra, analytical geometry, goniometry, probability, statistics, calculus. Within the part of the program devoted to calculus, the topic of differential equations is faced, with a focus on linear equations with constant coefficients and on their use in modelling simple problems. For many students enrolled in the Mathematics course it is the first approach to this content, since it is usually not faced in most upper secondary schools in Italy. The experience of the teacher of the course (one of the authors) during previous academic years has shown widespread students' difficulties with this topic. In particular, the written examinations have highlighted students' blocks in carrying out two fundamental processes: (a) the aware construction of the differential equations that models these kinds of problems (often students construct these equations in an automatic way and are not aware of the meanings of the different terms that appear within them); (b) the effective interpretation of the graph of the function that represents the problem's solution, by connecting specific properties of the graph to the corresponding characteristics of the represented phenomenon.

To support students in overcoming these blocks, we designed a digital resource (a GeoGebra applet) to be used to face a problem that could be modelled through a linear differential equation with constant coefficients. The text of the problem is: "An industry produces mobile phones at a rate of 20% per month. Every month, 150 mobile phones are sold. Suppose that at time t = 0 there are 700 mobile phones ready to be sold. Is the production's rate sufficient to meet market needs?". The digital resource has been designed to include specific DMSEs aimed at fostering metacognitive activities during students' resolution of the problem. Table 1 summarizes the main DMSEs provided to students and the metacognitive activities fostered by each DMSE. The design of the digital resource does not include DMSEs aimed at fostering students' reflections on their difficulties in solving the problem or on their personal habits in doing problem solving. For this reason, the *elaboration* category is missing in Table 1.

Digital meta-scaffolding elements	Fostered metacognitive activities
1) During the whole activity, students are guided to follow the different steps that structure the resolution process. At the beginning, two sub-goals are set: to construct the differential equation that models the problem and to sketch the graph of its solution.	*Planning*, since sub-goals are stated, and students are supported to formulate an action plan.
2) If students fail in the initial construction of the differential equation, they are provided with the general form of the equation they have to construct ($y' = py + q$) and guided, by means of specific questions, to read the problem's text, identifying the information that could help them in determine the coefficients p and q.	*Orientating*, since students are supported in the identification of important information and in establishing given values within the problem's text. *Planning*, since students are guided to look at particular information in the problem's text and in selecting pieces of information useful to achieve the goal of constructing the differential equation that models the problem.
3) After students' construction of the correct differential equation, they are asked to interpret the equation, making the meaning of each term of the equation (y',py,q) explicit.	*Monitoring*, since students are supported in the interpretation of the differential equation in relation to the problem (highlighting, or not, their progress in the understanding of what they are doing) and in giving meaning to the mathematical objects they are working with.
4) Theoretical hints are provided to students if they fail in specific steps of the task. Students can	*Orientating*, since students are guided to focus on important information that could support their work on the task.

autonomously use these hints to be supported in their resolution of the differential equation or if they want to check the correctness of their work.	*Planning*, since students are enabled to use external sources to get more explanations.
5) After students' resolution of the differential equation, they can choose to use GeoGebra to draw the graph of its solution. Afterwards, they are supported in the interpretation of the graph in relation to the problem. Specifically, students are asked to identify, within a list of properties of the graph, the property to which they should refer in order to answer to the problem's question.	*Orientating*, since students are supported in a careful re-reading of the questions and in the interpretation of the constructed graph. *Evaluation*, since the focus is on the interpretation of the result of the employing process, which could also give strength to the explanation of the strategy suggested at the beginning (constructing and solving an equation, then drawing the graph of its solution).
6) During the whole activity, at each step students are provided with reminders about the main results of the previous steps. Students are also asked to make the strategies implemented during the employing process explicit, by selecting the correct strategy within a list of possible strategies.	*Planning*, since students are guided in formulating their action plan. *Monitoring*, since students are supported in referring to the outcomes of the previous steps of the problem's resolution and in keeping track of their work.
7) During the whole activity, error messages are provided, together with partial corrections, that is operative hints aimed at supporting students in detecting their mistakes.	*Monitoring*, since students are guided to detect mistakes and understand possible reasons related to partial failures in the resolution process. *Evaluation*, since students have the opportunity to check their work in progress and to develop a self-critique about the chosen approaches.

Table 1: DMSEs and related metacognitive activities

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

The aim of this study is to investigate the efficiency our design, by focusing on students' a-posteriori reflections on their interaction with the digital resource presented in the previous section. In particular, we address the following research questions: (1)

What DMSEs of the digital resource are relevant for students who interacted with it and why? (2) *What aims related to the design of DMSEs are students aware of*?

To investigate these issues, we developed an exploratory study with a group of 11 students that were attending the course in Autumn 2021. The students, enrolled on voluntary basis, worked in small groups (5 groups of 2 or 3 students) at distance, by means of the Zoom platform. During the groups' work, which lasted from 20 to 30 minutes (no time limit was a-priori set), one student for each group shared his/her screen and directly interacted with the digital resource. Each group's work was video-recorded to keep track of both the students' interaction with the digital resource and the dialogues between students. The choice of making students work in small groups was specifically aimed at fostering their explicitation of cognitive and metacognitive processes while working with the digital resource. Moreover, two researchers (two of the authors) were always present during the groups' work. One of them took notes about the observed interactions, the other played the role of tutor, posing specific questions to the students to make them share their cognitive and metacognitive processes and to support their reflection on DMSEs.

Here we focus on the reflections carried out by the students during a short interview developed by the tutor, immediately after students have completed their work with the digital resource. During the interview, students were asked to provide feedback about the effectiveness of the design of the digital resource in supporting their resolution of the problem and to identify the most supportive elements of this design. We analysed the transcripts of the interviews by highlighting: (a) the DMSEs on which students focused; (b) the ways in which students reflected on these DMSEs; (c) students' metacognitive activities emerging during the interviews. The results of this analysis are presented in the following section.

ANALYSIS

The DMSE on which almost all the groups' reflections are focused (4 groups referred to it) is the first one, that is the choice of structuring the task in different steps. F, a student from group 3, for example, states: "All the steps are in order, so nothing is lost, the procedure that needs to be done is clearer to me". The general idea that students share is that they have assimilated this scaffolding, as testified in this reflection:

> "…[the resolution process] is much more schematized and ordered. The mental order is much easier to be achieved in this way, because it [the digital resource] asks exactly what the procedure is and creates a mental set" (A, group 2).

Although students widely focus on the ordered structure of the digital resource and on the possibility of re-constructing all the passages developed within the resolution steps, DMSE 6 (giving reminders about the results of the previous steps and asking students to identify a correct employing strategy within a list of possible strategies) is more implicit in students' reflections. However, some students propose interesting reflections about the effectiveness of DMSE 6 in making the employing strategies explicit. They stress, in particular, that asking students to identify their strategies among different options enabled them to make the reasons connected to these strategies more explicit to themselves. This idea is evident in V's (group 2) reflection:

> "To answer to the question that asks what we have to do to determine the particular solution [of the differential equation], we must have understood what 0 and 700 represent and why we have to replace x with 0 and y with 700."

The further DMSE on which most of the groups (3 groups) focus is the seventh, that is the error messages that are provided, together with partial corrections. S (group 5) proposes strategic use of DMSE 7 that she carried out when working on the digital resource: "When you try to put an answer and it is wrong, it [the digital resource] gives you some suggestions to get you to the right answer. This is useful". The following reflections highlight, in particular, that students appreciate the immediate feedback provided through DMSE 7, interpreting it as an opportunity to reflect about mistakes:

> "The messages that come out are very useful, because they immediately tell you where you went wrong and they also refer to the theory so you can immediately see your error." (F, group 2)

> "In fact, they give you a second chance, they make you think about the mistake you made. They also tell you if you can go ahead or if you need to review something." (A, group 2)

F's reflection enables us to shift our attention on a DMSE on which only two groups focused, that is the theoretical hints provided within the digital resource (DMSE 4). Besides F (group 4), only one other student, A (group 2), implicitly mentions this element, declaring her awareness about the importance of referring to theoretical tools when facing this kind of tasks: "You have to make reference to the theory you assimilated during the course, then you have to specifically use it to face this problem". This assertion could also be interpreted in terms of an *elaboration* activity emerging during the interview, since A is providing to herself feedback about self-regulation.

The other DMSEs are rarely mentioned in students' reflections. The DMSE 2 (supporting students' reading of the problem's text to identify the information that have to be used to model the problem), in particular, is never mentioned by students. This is certainly due to the fact that one group did not receive this scaffolding since they were able to immediately construct the correct equation and other two groups made minor mistakes in the construction of the equation, so this DMSE was not really necessary for them. As regards the other two groups, we think that they did not mention the DMSE 2 because the difficulty they faced was not related to the identification of the information useful to construct the equation, but to how to use this information.

Although few students spontaneously propose reflections on DMSE 3 (support in the interpretation of the constructed equation in relation to the problem) and DMSE 5 (support in the interpretation of the graph in relation to the problem), our analysis enabled us to show that reflecting on these two DMSEs during the interview fostered the *evaluation* and *elaboration* activities. The following reflection testifies, for example, the awareness about the role of DMSE 5 in guiding students' effective use of the graph within the resolution process:

> "This question is useful to understand how to extrapolate, from the graph, the information we need to solve the problem, therefore how to obtain the needed information from the modeling of the problem" (R, group 1).

DMSE 3 is explicitly mentioned only by V (group 2), who declares that a lack in her approach is that she did not deepen the interpretation of the different components of the differential equation in relation to the problem. Therefore, even if students do not mention DMSE 3 as relevant for them, the reflection on the difficulty faced in interacting with this DMSE boosted their *elaboration* activity during the interview, making them become more aware of their difficulties and develop self-critique. The reflection on DMSE 5 also boosted the development of the *evaluation* and *elaboration* activities during the interview. This is testified by the following reflection, in which a student highlights the interrelation between facing difficulties when interacting with this DMSE and becoming more aware of unclear aspects of the strategy adopted to solve the problem:

> "When we got stuck on this question related to the graph, then understanding what was the right answer, among the three answers, helped us a lot. We were more confident about other things and we got stuck on this one, but then I understood why." (S, group 5)

Our analysis highlighted also other examples of how the students' reflection on the effectiveness of the DMSEs make them develop *evaluation* and *elaboration* activities. It happened especially when students focused on those elements that disoriented them, as testified in the following reflection, developed by V (group 2) when speaking about the confusion they faced in the initial part of their work on the task:

> "In fact, we were wrong because when you write the Cauchy problem, you write like this, not as we did. In my opinion, the problem is not in the formulation of the task and in how it is set up. We were really wrong because we have read the request with little attention".

FINAL DISCUSSION

The analysis presented in the previous section enabled us to show that not all the DMSEs included in our design are relevant for students. Students mostly mention the support provided by the structuring of the task in different steps (DMSE 1), the reminders about the results of the previous steps and the guide in making the employing strategies explicit (DMSE 6), the error messages associated to partial corrections (DMSE 7) and the theoretical hints (DMSE 4). This result shows that students are more aware of those DMSEs that simplify the task by suggesting what are the actions required to reach the solution or of those that keep them in pursuit of specific goals.

The fact that students never or rarely mentioned the DMSEs directly aimed at supporting the phases of construction of a formal representation to solve the problem (DMSE 2) and of interpretation of the representations with which they interact (DMSE

3 and 5) suggests us the need of a re-design of the digital resource with the aim of enriching the meta-level of the provided scaffolding, enabling students to become aware of the role of specific DMSEs within the digital resource. The problems related to the students' lack of this awareness have been also highlighted during the phase of students' work with the digital resources, when, in tune with our previous studies (Cusi & Telloni, 2020B), the tutor played a crucial role in making students effectively exploit the provided scaffolding. The different excerpts that show students' development of *evaluation* and *elaboration* activities when reflecting, during the interviews, on the usefulness of the provided DMSEs, testify the importance of fostering this kind of reflection as a further element of the meta-scaffolding itself.

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ENTAGLEMENTS OF CIRCLES, PHI AND STRINGS

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This study focuses on the entanglements of material discursive agents in a mathematical explorative activity, where learners systematically investigate the properties of the circle. A diffractive methodology is used to explore the emerging intra-activity in-between the teacher, the learners, different teaching materials, the mathematical concepts and formulas, as abstract and concrete aspects of mathematics education are set in motion. The results show that concrete material-discursive agents, such as strings, rulers and bodies are entangled with abstract mathematical concepts such as circumference, formulas and phi.

INTRODUCTION

> The classroom is full of energy. There are about 25 learners in grade 7 and for the fly on the wall there is a constant babble among them. Strings, rulers and large white sheets of paper with printed circles are on the tables. The teacher, Gert, walks calmly around, approaching pairs who need support. He seems to know each of the learners very well. At times, one can see a ruler hanging in the string as a necklace or attached to a hat as a horn. At other occasions, the same learners ask Gert for support to come further in the task. The communication is friendly, focused on central concepts, such as perimeter, diameter and radius.

Do mathematical concepts exist in things perceptible by senses or are they independent from them, and do mathematical concepts exist inside or outside the mind? These questions have long been a central concern within the philosophy of mathematics (Ernest, 2018) and are also reflected in the longstanding controversy over concrete versus abstract aspects in mathematics instruction. There is a widespread assumption, in most learning theories and approaches to pedagogy, that abstract and concrete aspects – abstraction and concretization – are important for the learning of the concepts of mathematics (Coles & Sinclair, 2019).

Abstraction can be seen as ignoring the specific nature of concepts, focusing on general operations or relationships, closely linked to contextualization, whereas concretion is about making concepts considered abstract more concrete and thus, understandable for learners (Dreyfus, 2014). However, controversies regarding if mathematics should be taught from “concrete to abstract” or from “abstract to concrete” have been discussed for a long time. The “concrete to abstract” progression is evident in work building on Piaget and Bruner, such as Ding and Li (2014). They investigated how concrete and abstract representations of concepts may be structured to facilitate learning and identified features that may enable the transition from concrete examples to abstract knowledge, by gradually increasing the abstraction to the higher-level structure of abstract knowledge. Contrariwise, the “abstract-to-abstract”

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development, promoted by Kaminski et al. (2009) in a much-discussed study, claimed that learners benefit more from learning mathematical concepts through abstract, symbolic representations than from concrete examples. This has, in turn, been criticized by De Bock et al. (2011) who conducted a partial replication of the study by Kaminski et al., resulting in questions about what learners actually learned from the abstract concept exemplifications. This implies that a "concrete to abstract" position might be more productive for learners.

Despite the ongoing discussion of whether mathematics instruction should start with concrete or abstract examples, concrete materials or manipulatives, including physical, visual and pictographic objects, are widely used in mathematics classrooms. This is based on the assumption that the "concrete" will make mathematics meaningful to learners which is reinforcing the dichotomy between concrete and abstract. This dichotomy has played an important part in promoting the use of manipulatives in the teaching and learning of mathematics (Coles & Sinclair, 2019). However, questions are raised about the very substance of this dichotomy and the conceptualizing of the concrete/abstract distinction in the learning of mathematical concepts. Adam and Chigeza (2015) question this (and other) binaries with regard to teachers' pedagogical choices and propose a more relational and contextual approach to knowledge.

A different approach taken is work building on new materialist stances (e.g. Barad, 2007) where concrete and abstract aspects are not seen as dichotomized, but as entangled. In Barad's view the materiality of concepts is highlighted and concepts are to be understood, not as abstract mental objects, but as a material arrangement. Furthermore, materiality is not seen as "dead" or passive in relation to the active discursive human being. Instead, materiality – matter and concepts, matter and thought, matter and meaning, matter and discourse – is understood as agential in the co-construction of meaning and learning (Barad, 2007). Hence, new materialist perspectives have the potential to address concerns about how concrete and abstract aspects of mathematics teaching relate to each other. In this study we take departure in Barad´s (2007) agential realism and the concept of entanglement, to explore how the material-discursive aspects of the abstract/concrete dichotomy is enacted in a teaching sequence. We ask the question of how this entanglement can be identified in mathematical explorative activities. In this case concerning the circle and its properties.

AGENTIAL REALISM

From an agential realism stance (Barad, 2007) the focus is set on material-discursive practices. That is, phenomena (or concepts) are constructed by both material and discursive practices in simultaneousness and mutuality. In a Baradian agential realist account, matter and meaning are always co-constituted and entangled as neither discursive nor material practices are prior to each other, reducible to each other or privileged over the other. From this viewpoint of materiality, neither concepts nor things, have determined boundaries, properties or meanings, but emerges through entanglements of material-discursive intra-actions in an ongoing production process. Barad (2007) replaces the notion of interaction with intra-actions and thereby shifts focus from a human-to-human interaction to intra-actions in-between human as well as non-human materialities. Intra-actions as agential are to be understood as enactments, something that is to be done or acted upon (Barad, 2007) rather than as an inherent property of an individual or human to be exercised. Thus, all forms of materiality: bodies, matter, concepts etcetera, are performative agents engaged in entangled relations.

Drawing on posthumanism and new materialism de Freitas and Sinclair (2014) is questioning the, in their meaning, not very fruitful division between the mathematically abstract and the physically concrete. This foregrounds questions about the entanglements of mathematical concepts, material-discursive aspects and how mathematical concepts partake in agential ways (Freitas and Sinclair, 2014). From Barad´s theory they understand concepts as material arrangements of things in intra-actions with each other and apply that on mathematical concepts. Thereof, mathematical objects and concepts take part in an entangled and ongoing process so that abstract thoughts and materiality are entwined. Mathematical concepts are thus considered as performative material agents entangled with other material agents in material-discursive intra-actions (de Freitas & Sinclair, 2014).

METHODOLOGY

This study is part of a larger project (TRACE) on novice mathematics teachers' learning and what they bring from teacher education into the practice of teaching in school. The data used in this study derives from video-recorded lesson observations that were conducted over an extended period, from the last practicum period in teacher education to three years of experience as teacher. In this paper we focus on a lesson that the teacher, Gert, holds at the school where he works, two years after graduation.

The lesson focuses on phi, and the task is to systematically investigate the properties of the circle. Gert's plan is to let the learners work in pairs to determine the radius, diameter and circumference of circles of different sizes. All pairs have a ruler, a string, and a large sheet of paper. On the sheet there are circles of five different sizes and a table in the lower right corner. According to the measurements they have done and noted in the table, the learners are asked to divide the length circumference with the length of the diameter and (hopefully) find that the quotient is close to the same in all cases.

Diffractive analysis

With a material-discursive focus on intra-actions comes methodological issues concerning material aspects of knowledge production (Barad, 2007) opening up for the possibility to unfold complexities when studying the doing of a specific phenomenon. Furthermore, it makes possible to acknowledge and analyze both entanglements and differences emerging in events of encounters of material-discursive intra-actions. This is putting into work what Barad (2007) calls a diffractive methodology.

Diffractive methodology is inspired from the concept diffraction which describes a physical phenomenon and can serve as a counterpoint to reflection (Barad, 2007). Barad put forward a diffractive methodology by contrasting reflection and diffraction. Reflection can hence be read as mirroring and sameness, whereas diffraction can be read as patterns of difference with a purpose of exploring the entangled effects that differences make, and the very nature of entanglements. A diffractive methodology is looking for differences within phenomena, focusing on encounters and entanglements and what these differences might do (Barad, 2007). Hence, a diffractive methodology provides a way to attend to the entanglements of material-discursive agents, such as abstract concepts (e.g., phi) and concrete physical things (e.g., strings and rulers).

The process of analysis is guided by openness to entanglements in each situation, which ,with sensibility and care enables data to take part in the process. As a researcher one has to have trust in the unexpected and be open to the things that will occur in the process (MacLure, 2013). Therefore, the process of analysis is not only about finding themes and patterns but also to make the contradictions, movements, messiness and shifts visible.

Identifying demarcations, delimitations and boundaries in data is, according to Barad (2007), a process of making agential cuts. In the agential cuts empirical data, theoretical concepts, the researcher(s) and previous research intra-act as agents during the analysis. The cuts are guided by questions about what the data can tell us about a complex world in an attempt to unfold these complexities and call to our attention by glowing (MacLure, 2013). The glow can be places or actions in data where things are set in motion. This, in turn, makes the encounters impossible to ignore in the attempt to entangle the relations in-between the different actants involved.

In putting a diffractive analysis in action our focus has been on the emerging intra-activity amongst the teacher, the learners, different teaching materials and the mathematical concepts and formulas. The diffractive methodology allowed us to investigate the entanglements and differences of abstract and concrete aspects enacted in the classroom, and how different materialities co-operate and interact. In the analysis we were engaged in attending and responding to the material-discursive intra-actions in the data, exploring how they seemed to matter. The data cuts were chosen as we identified that abstract and concrete aspects were set in motion, and patterns of entanglements and differences of material-discursive practices were enacted. Analyzing these cuts more closely made possible for abstract and concrete aspects to emerge in new ways. For example, the analysis allowed us to see beyond the teacher instructing the learners, and instead see a circle, a string and the concept of phi co-constructing knowledge. This is one example of how the concept of phi “became” in certain ways within these intra-actions.

RESULTS

When looking at the video-recorded lesson we were absolutely taken by how the physical things, the scissors, strings, rulers and papers, the bodies of the teacher and the learners; the discourses occurring in the classroom, and the mathematical concepts and formulas were intertwined in. Among the first things that struck us watching the recording was that even though the plan of the lesson was to move from the concrete measuring of dimeters, radius and circumference of circles, to the abstract concept of phi, phi and formulas were thrown around by the learners very early on. There was a vivid atmosphere in the classroom, and even though we present the results in three different sections, the entanglements of the material-discursive intra-actions occurred almost simultaneously.

The results focus on the entanglements and differences of material-discursive intra-actions that different materialities in the classroom were engaged in; entanglements of circumference, strings, rulers and bodies, entanglements of circumference, mathematical formulas and calculators and entanglements of diameters, radius and phi. The first entanglement focuses foremost on physical concrete things and bodies while the second and third focus on abstract mathematical concepts.

Entanglements of circumference, strings, rulers, scissors and bodies

Strings were used to measure the circumference of the circles; rulers to measure the diameters; and radius and fingers and other bodily parts were co-acting with them. Physical concrete things and bodies' intra-actions were very much involved in the lesson. Within these intra-actions of measuring circles, different active material-discursive agents such as strings, rulers and papers made themselves known and co-constructed with learners' and teachers' bodies in an entangled way. One example is the teacher Gert, circle, table and string intra-acting in the beginning of the lesson.

Gert: Your task is to measure all circles on the paper. You shall measure the radius; you shall measure the diameter [He holds a string and a circle made by paper in his hands above his head]. You shall note each measure in the table [picks up a string from the table while a learner asks about the table].

Gert: You can start with the smallest, and then we have them all in order - or the largest. Just do it in whatever order you want to. /…/

Gert: And then, as we measure the circumference, you try to put the string around. The two of you help each other. Maybe you can put it little by little, because this is messy.

The fact that it is difficult to measure the circumference of a circle was an important part of the lesson and an example of the string as a performative agent, entangled with other agents. The difficulty was highlighted by the teacher in the beginning of the lesson, saying that it is 'messy' and 'you have to be careful and help each other to sort it out'. The activity evidently called for learners' fingers to co-act with the string and the circles but the difficulties with the string, in putting it around the circle's circumference, was making itself known throughout the lesson. Learners had different solutions to this difficulty. Some thoroughly and methodically put the string around the circle and thereafter measured it with the ruler. Others came up with suggestions of other ways to measure the circumference such as in the example below. Unfortunately, the suggestion that the learner came up with was inaudible, perhaps a measuring tape?

L1:	One can measure with [inaudible]
Gert:	Yes, but today we shall measure with the string.

Intra-actions with the string lead to different outcomes for the learners. While some overcame the difficulty by thoroughness and methodical work, others ignored the string and tried to find out other ways to determine the circumference, for example by calculating it, starting from the measure of the diameter or radius. Gert approaches a pair of learners. One of them holds the string in her hands.

L2:	3 times 3,14…
Gert:	Is it too much for you to measure with the string?
L3:	No, but it's a bit hard… [grabs the string and shows how messy it is].
L2:	But, do we take the radius or diameter?
Gert	[simultaneously]: Have you thought about the results you will get?
L2:	Do we take the radius or diameter?
Gert:	You can try once and see what is most reasonable.
L3:	Okay… The calculator, please [starts calculating on the iPad].

The example above shows that in intra-action, the string and the learners co-create an emerging need for abstract mathematical concepts such as formulas for calculating circumferences and phi (3.14). Thus, formulas and phi became agential in these intra-actions, which is visualized below.

Entanglements of circumference, mathematical formulas and calculators

As indicated above, mathematical formulas to determine the circumference of circles were several times in intra-action with other agents, such as the learners and the string. The string (and the difficulty of measuring with it) in intra-action with the learners seemed to provoke a need for different ways of determining the circumference, namely formulas. Hence, formulas became active material-discursive agents making themselves known and co-constructing with other agents in an entangled way. The abstract mathematical formulas are here understood as material articulations intra-acting with other matter (circumference, strings, rulers and bodies). To use formulas was not the intention of the lesson. However, it was evident that the string shifted the focus for some learners towards formulas determining the circumference. But, since the learners were not sure of the formula (diameter multiplied with phi), questions of how to articulate it arose. They were uncertain whether they should multiply phi with the diameter or the radius: "But, do we take the radius or diameter?". They even took the chance to ask the one who holds the camera: "Circumference, is that the radius times phi or is it the diameter times phi?"

The learners repeatedly asked Gert for the formula, which he did not offer. The entanglements of material-discursive agents in intra-actions, often groundedin the messiness of measuring with the string, evoke a call for formulas as the learners expressed the unneccessity to measure with the string if one knows the formula. The formulas in turn called for intra-actions with calculators as a tool when using the formula, as described below.

Entanglements of diameters, radius and phi

Mathematical concepts of diameters, radius and phi were in different ways actively taking part throughout the lesson. Within the intra-actions of measuring circles, different material-discursive agents, such as diameters, radius and phi made themselves known and co-constructed with the learners' and teacher's bodies in an entangled way. As the example of formulas, the abstract mathematical concepts diameter, radius and phi are seen as material articulations intra-acting with other matters – circumference, formulas, strings, rulers and bodies. Phi played an important role in the lesson, as the intention was that the learners would find that the quotient is a constant, phi. However, entangled intra-actions with strings and other material-discursive agents called for phi to make itself known from very early in the lesson. In these examples, phi is making itself known in connection to formulas as in the following interaction:

Gert: This is how I think. If we measure all around here, there will be some string left over. Let's say we have measured to this point, then I measure this distance with the ruler, like this [holds up a string and a cut-out circle]. How long was it? And then write it down. /…/

L4: Can't we just take the diameter times phi?

Gert: Yes, we could do that [but sticks to the original plan].

Gert eventually takes notice of the important role phi has in the lesson and directs the learners' focus to interesting things about phi, like how it was invented and the number of decimals in a rather long discussion in the end of the lesson.

Discussion/Conclusion

By following Barad (2007) about matter and meaning as inseparable embodied knowledge, and the materiality of mathematical concepts (de Freitas and Sinclair, 2014) knowledge about abstract mathematical concepts come to the fore as inseparable and entangled with physical concrete matter. Knowledge about these mathematical concepts would not exist in this way without the entanglement and intra-actions with the material-discursive agents involved in the lesson. Our result is twisting the question of whether mathematics should be taught from "concrete to abstract" or from "abstract to concrete", and instead focus "how to entangle concrete and abstract aspects".

A diffractive methodology enabled us to engage with the entanglements and differences in the data. The ´glow` of the co-constructing of knowledge amongst agents in the classroom was impossible to ignore and directed our cuts in the data. This made our attempt to unfold and entangle the relations between the agents possible. Thus, we ask ourselves: What emerges if we fix our gaze on material-discursive intra-activities that are taking place and what intra-actions emerge amongst the different agents, regardless of whether these are human or non-human?

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UNIVERSITY STUDENTS' DISCOURSE ABOUT IRREDUCIBLE POLYNOMIALS

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The purpose of this paper is to analyse, following Sfard's theoretical framework, the students' discourse about irreducible polynomials in order to understand their difficulties. Through the students' explanations, given during the interviews conducted after carrying out the questionnaire, it is possible to study which processes were triggered by them to solve the task. In particular, we focused on the concept of the irreducible polynomials and the link between the roots of the polynomial and its reducibility in university students.

INTRODUCTION

The focus of this research is to analyse university students' discourse about irreducible polynomials: the link with the roots of the polynomial, the field being worked on, the idea of algebraic or graphic representations of a polynomial. The data were analysed through the theoretical framework of Anna Sfard's commognitive approach (Sfard, 2008).

Nowadays there are not many studies at university level that use this approach apart from a few works concerning Calculus, Group Theory, and the shift to mathematical proof (Nardi et al., 2014). More generally, the literature about the learning of irreducible polynomials appears scarce (if any) and not study about this topic has been carried from the perspective of the commognitive approach.

Adopting a commognitive perspective, the process of teaching/learning can be described as initiation to a discourse. In the case of irreducible polynomial (in the Italian case) such initiation starts in high school and then narratives about such polynomials are often used in scientific faculties (like Physics, Chemistry, Information Technologies, etc.) without discussing again the rules of the discourse. The discourse of lecturers (to which we refer as academic discourse) may be quite different than the discourse developed by students in high school. For instance, Güçler (2013) shows that when a lecturer shifts the discourse on limits (from limit as a number to limit as a process) without making such shifts explicit to the students, they do not even notice it. On the contrary, lecturers' understanding of the difficulties met by students in taking part to the university mathematical discourse may help them in making explicit – at least – their use of words. This appears as a necessary, even if not sufficient, condition to realize an effective communication between students shifting from high school to university and their lecturers (Stadler, 2011; Nardi et al., 2014).

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THEORETICAL FRAMEWORK

Sfard's theoretical framework is based on the notion that thinking is an interpersonal form of communication and she coins the new word 'commognition' to denote the combination of communication and cognition (Sfard, 2008). According to Sfard, mathematics is a discourse and, as such, it is characterized by four features. First, a discourse is determined by the *word use,* meaning the keywords characterizing a discourse. Word use refers both to the use of mathematical terms and of more colloquial words having a specific meaning within mathematics (such as 'field' or 'roots').

Discourses in general, and the mathematical discourse in particular, have specific *visual mediators*: these are visible objects intervening during the communication process (like gestures, inscriptions, drawings, and so on). In this feature we consider mediators of mathematical meaning (such as symbol and algebraic notation) and material objects useful during teaching of the mathematics.
In this context, it is useful to refer to Zazkis and Liljedahl's studies, who studied the representation of prime and irrational numbers, analysing how students perceive and understand these concepts (Zazkis & Liljedahl, 2004, Zazkis, 2005). They distinguish between transparent representations, which completely shows the meaning of the represented structures, and opaque representations, that highlights some aspects of the structures while hiding others. Recent studies of Zazkis and Lilhedal (2004) show that the difficulty of finding a transparent representation prime numbers or irrationals leads to difficulties in learning the concepts themselves. We can conjecture that the same could apply to polynomials.

A discourse is not characterized only by the objects of the discourse, but also by the rules of production of narratives. In Sfard's framework, *narratives* are texts, written or oral, such as descriptions of the objects and links between them. Narratives are submitted to endorsement or rejection, using the processes and rules accepted by the community (such as axioms, deduction rules, accepted definitions, etc.). Furthermore, the discourse is produced following established *routines*. These are repetitive schemes that characterize a discourse. Sfard distinguishes three types of routines:

- Deeds: a routine is called in this way if there is a physical change in the objects. Deeds may be defined as a set of rules that produce or modify the objects;
- Explorations: a routine is called this way if it helps produce endorsed narratives. You can divide it into constructive explorations, which lead to approvable narratives, justificatory explorations, which help to decide to approve a narrative, and recall narrative, which are processes applied to evoke an endorsed narrative;

- Rituals: when there is a sequence of discursive actions, which are intended more to create and maintain a relationship with people (for instance meeting expectations) rather than exploring within the discourse.

Sfard highlights two types of learning:

- Object-level learning: occurs when there is an expansion of the discourse, expanding vocabulary, building new routines, producing new endorsed narratives;
- Meta-level-learning: causes changes in the discourse metarules (these rules define the models for producing and validating object-level narratives). This change means that some familiar tasks will be performed differently and some familiar words will change their use.

Sfard does not believe that students start a meta change on their own (Sfard, 2008). It is possible, in fact, that this change originates from the direct meeting between student and new discourse. This meeting brings a *commognitive conflict*, that is a situation in which individuals apply different metarules. When students move from high school (when irreducible polynomials are usually introduced in the Italian school context) to university, their discourse about irreducible polynomial meet the discourse of scholar experts, which may be incommensurable, meaning that "they do not share criteria for deciding whether a given narrative should be endorsed" (Sfard, 2008, p. 257). Students and experts can use words differently without being aware of such differences. Characterizing university students' discourse when they enter university appear important to then understand how to help them and their lecturers towards a "gradual mutual adjusting of their discursive ways" (Sfard, 2008, p.145).

METHODOLOGY

Inspired by the research of Zazkis and Liljedahl (2004), we created a questionnaire to characterize students' discourse on the irreducibility of polynomials. This research was conducted at the level of university students in science faculties. They had different backgrounds due to different courses they followed and their previous studies. In particular, all of them studied the concept of irreducibility of polynomials at the high school level and not in the university, therefore their answers to the questionnaire were based on previous studies.

In a similar fashion to what Park (2013) did for the concept of derivative, we investigate students' discourse about irreducible polynomials in terms of object-level learning.

The subjects of this study are 14 students from Chemistry, Physics, and Computer Science courses. The questionnaire consists of five open-ended questions preceded by a definition of irreducible polynomials and an example of reducible polynomial. The subjects responded individually to the questionnaire in 30 minutes, and they had to take note of each personal consideration and to explain their answers. Afterwards, we conducted an unstructured interview individually to better understand their explanations. During the interview, we asked them to explicit the processes implemented.

In this paper we focus on two questions from the questionnaire:

- Is the polynomial $x^3 + 6x^2 + 7$ reducible or irreducible?
- Is the polynomial $(x^2 + 1)(x^2 + 2)$ reducible or irreducible?

The example given before the questions is the following:

The polynomial $x^2 – 1 = (x + 1) (x – 1)$ is reducible.

ANALYSIS OF THE COLLECTED DATA AND DISCUSSION

Use of the words

A first feature characterizing the discourse is the use of the words. Analysing the students' productions and their interviews, we noticed an ambiguity in the use of the words roots and solutions. For instance, a student of the degree course in Physics, answering to first question, tried to decompose the polynomial by collecting x^2, but he did not know whether this decomposition was valid. Furthermore, he did not remember how to use Ruffini's rule. He stated:

> "I always forget Ruffini's rule, I have not memorized the process".

From this statement, we conducted the interview to find out what the student thought about the relationship between *reducibility* of polynomial and *roots* of polynomial. The student stated:

> "I do not remember if there is a relationship between the roots of the polynomial and its reducibility. If the polynomial is reduced, this leaves it easier to see the solutions, but if I do not see the solution, that does not mean the polynomial does not have them".

In this second statement, we can see the use of the terms roots and solutions as synonyms. This inappropriate use may be a consequence of the use of the words *equation* and *polynomial* as synonyms.

It is possible to observe that an incorrect use of the word guided the actions of the students. Indeed, another student, answering to second question, stated:

> "It does not say: "is the reduced polynomial reducible?" but only "the polynomial". That is why I did not think that the question was about the initial polynomial."

From this statement we can observe that the use of the term *polynomial* without the word 'reduced' confused the student. This underlines that student did not recognise the product written as a polynomial. This difficulty also crops up in others, in fact, there are some students who solved the product and then adopted known ***routines*** to decompose the obtained polynomial. Many students interpreted the second question in this way: "is the polynomial further decomposable?", this can be seen from their answers, in fact, they answered "not further decomposable" or "irreducible, because it is already reduced".

Visual mediators

We can than explain this behaviour as a misunderstanding in the task (because of the word use), but we could also interpret it as a lack of transparency (for the students) of the provided visual mediator. This representation is not transparent to them, as they interpreted the product as something to be calculated and not as a polynomial.

Referring to visual mediators, we also noticed that no one referred to graphical representation to address the questions. Some students stated that they had difficulty in understanding the presence of roots from given graph.

One student of the degree course in Chemistry answered to the first question:

> "I have no tools to say that"

During the interview, we reflected with him on techniques for determining the reducibility of third-degree polynomials. We explored his knowledge about graphical representation of the polynomial.

> Interviewer: […] It may be useful to represent it graphically. Have you thought about this possibility?
>
> Student: No, we calculate at most the derivative… but to take it to the graph and understand whether the polynomial is reducible or not, I am not able to.
>
> Interviewer: Are you able to say if there are roots by having the graph?
>
> Student: No… I do not think.

Therefore, some students do not have established routines for using the graphical representation of these mathematical concepts, or they do not consider the production of a graphical representation as an endorsed narrative. A consequence of this is that students are not even able to interpret provided graphs.

Endorsed narrative

We can see the role of endorsed narrative in the discourse also when students try to remember the link between the presence of roots and the reducibility of the polynomial (or the reducibility of the polynomial in the different fields) for example:

> "[…] If the polynomial is reduced, this leaves it easier to see the solutions, but if I do not see the solution, that does not mean the polynomial does not have them".

Another student of the degree course in Physics stated:

> "Irreducible in real field, it can be reducible in complex field".

We investigated the motivation that led him to this statement. He explained that he tried to use Ruffini's rule, but he did not find any suitable constant to use this technique, therefore he concluded that the polynomial was irreducible in real field. He thought it was possible to extend Ruffini's rule to searche roots in the complex field too.

Routines

The use of rote-learned theorems or deduction rules are the most common routines to perform the tasks. The students applied different procedures to solve the questions: Ruffini's rule, decomposition with factoring trinomial, total or partial collecting and finding the solutions of the equation associated to the polynomial. As noted above, for the second question, students needed to solve the product and then applied the chosen routine. One student of the degree course in Chemistry answered, for example:

> "This is the result of the reducible polynomial $x^4 + 3x^2 + 2$, through decomposition with factoring trinomial".

Thanks to this argument, we can deduce that student was able to recognise the reducibility of this polynomial after calculating the product and decomposing it with factoring trinomial. Moreover, he said that he was not able to decompose more this polynomial with the knowledge at his disposal.

This underlines the difficulty of the students in identifying the polynomial written as a product. Moreover, we can observe that some of them had problems with the equality operator. In fact, one student of the same course, after solving the product, stated:

> Student: Reducible because this is given by the decomposition of $x^4 + 3x^2 + 2$?
>
> Interviewer: What are your doubts about this solving process?
>
> Student: I do not know if this decomposition is right. If this decomposition is right, the polynomial is reducible.

This highlights the difficulty of the students in interpreting the equal symbol as an equivalence, but they see it as a one-way procedural operator.

Many of the students' routines can be classified as "deeds" because they appear to be acting on the algebraic symbolism more than on the involved mathematical objects.

Some students stated that they did not remember the Ruffini's rule:

> "I always forget Ruffini's rule, I have not memorized the process".

From this statement it can be observed that students sometimes apply repetitive procedures, in a ritualistic way, apparently without understanding the meaning behind them. In fact, some of them tried to use Ruffini's rule to find the roots of the polynomial in the real field and in some cases in the complex field.

> "Irreducible in real field, it can be reducible in complex field".

These activities underline how students interpret the Ruffini's rule like a decomposition technique and not a procedure to find the roots of the polynomial. These routines can be classified as "rituals" because they applied the repetitive sequences to solve the task without questioning when this procedure might be applied. Most likely, this procedure was introduced by their high school teachers (considered as the "ultimate substantiators" of their narratives, Sfard, 2008, p. 234) and applied in a ritualistic way during high school.

CONCLUSIONS

This study focused on university students' discourse about the concept of polynomial reducibility according to Sfard's theoretical framework.

Some of students' arguments are inconsistent with the academic discourse about reducible polynomials, this highlights the importance of their motivations to understand their discourse and then their difficulties in approaching the academic discourse about this concept.

To the first question almost all participants stated that $x^3 + 6x^2 + 7$ is irreducible, but they used different motivations to justify their answer.

This question was used to investigate the students' ability to apply techniques different than Ruffini's rule. Analysing the arguments, it can be seen that many students used Ruffini's rule in a ritualistic way and some students were confused about the relationship between roots of the polynomial and the reducibility of the polynomial. We explained this difficulty as an ambiguity in their use of words as 'polynomial' and 'equation'.

Taking into account the second task, most of the students answered that $(x^2 + 1)(x^2 + 2)$ is reducible, but there were also few of them saying that the polynomial is not reducible or that it depends on the field. This question was designed to investigate the students' ability to work with fourth-degree polynomials and their ability to recognise properties highlighted by (what we considered as) a transparent visual mediator. This is the why we gave them a decomposed polynomial. However, many students had problem with the meaning of the question, in fact, they interpreted the question in this way: "is the polynomial further decomposable?". This may depend on the lack of transparency (for them) of the visual mediator, or on the different metarules applied to the discourse. Apparently, for some of them, factorization routines must be applied to a polynomial that is not already expressed as a product of polynomials.

Thanks to interviews conducted, we were able to characterize the discourse they use to justify their solutions. The difference between their discourse and the academic discourse explain why they are not able to reason in unfamiliar context or have difficulties with questions posed in different way than what they saw in high school. As pointed out by Sfard (2001) "one has no chance to modify one's discursive habits on her own. In order to change them, one has to be led outside her own discourse by others. Only then can the conflict necessary to create the learning-engendering experience of incomprehension eventually arise" (p. 47). The conducted analysis suggests that, for narrowing the distance between students' discourse and the academic one, several lines of intervention could be adopted by university lecturers. It would be important to expand their discourse both by discussing word use, visual mediators, routines, and endorsed narratives. Results from this study provide significant hints for designing teaching experiments with such goal.

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FOSTERING A CONCEPT OF FUNCTION WITH COMBINED EXPERIMENTS IN DISTANCE AND IN-CLASS LEARNING

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Hands-on experiments and simulations foster functional thinking (FT) in different ways. Both benefits can be combined effectively, when a focus is set on the difficult aspect of covariation through a qualitative approach. Self-directed learning in such settings produces significantly higher gains in FT that rather numeric consideration of experiments (Digel and Roth, 2021). Both settings were implemented as in-classroom (N=219) and distance learning environments (N=113) respectively, within the given constraints due to COVID-19. The results for distance learning Hammerstein et al. (2021) report in their meta-study are inconsistent, but with clear negative tendency. In the study reported here both learning modes show comparable results and the overall differences between covariational and numeric setting persist in both modes as well.

DISTANCE LEARING DURING THE PANDEMIC SITUATION

The restrictions due to COVID-19 set a focus on digital teaching strategies and revealed deficits in the school system concerning this topic. Despite intensive efforts of many schools, the average increase in learning during the first lockdown in Germany was comparable to that during the summer holidays, i.e. without school operations (Hammerstein et al., 2021). With regard to mathematics, the picture in Germany is not entirely consistent at the first glance. In their study on digital learning environments (bettermarks), Spitzer and Musslick (2021) found performance increases in the cohort with lessons under corona conditions comparable to the previous year's cohort. In an annual school performance study in Baden-Württemberg, there were significant learning deficits compared to previous years, especially in operational, mathematical skills, while performance in arithmetic (calculation-related) skills was at the level of previous years (Schult et al., 2021). The authors interpret these findings as indicators for an arithmetic focus in mathematics distance learning. Considering that the digital learning environments of bettermarks also strongly emphasise arithmetic, lower gains for operational and conceptual learning can be assumed in this case as well.

Distance Learning of weak learners

Related to pre-COVID performance, distance learning also increased the differences in learning achievement. In particular, low achievers showed lower gains compared to previous years (Schult et al., 2021), which could also be due to the significantly shorter learning time for this group. In addition, learners with low SES background had significantly poorer conditions for distance learning (Hofer et al., 2022). Our study does not replicate these negative findings. The learning environments for FT based on

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experiments with hands-on and digital material showed comparable learning gains in both grammar and comprehensive schools during distance and in-class learning.

DEVELOPING A CONCEPT OF FUNCTION

The concept of functions is a major concept and at the same time a major hurdle in mathematics at school. Hence a considerable amount of research has been dedicated to the teaching and learning of functions. For the learning environments used in this study we try to bring together several branches of evidence to a coherent approach to the concept of functions. Breidenbach et al. (1992) used the Action-Process-Object-Scheme (APOS) theory for a developmental perspective on students' conceptualization of functions. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky & Wilson, 2013).

Aspects of functional thinking

The developmental stages of APOS are in line with key elements of a function concept, that are described as aspects of functional thinking (FT) by Vollrath (1989) as follows: the correspondence of an element of the definition set to exactly one element of the set of values; the covariation of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an object. Although with the APOS perspective one might deduce a teaching sequence with an initial focus on correspondence, then covariation and finally object, current research advocates for a major role of covariation. Thompson and Carlson (2017) argue that the correspondence aspect alone does not evoke an intellectual need for the new concept function and difficulties with functional relationships are mainly rooted in lacking ability and opportunity to reason covariationally. Johnson (2015) points out that correspondence induces a static view on a functional relationship, while a dynamic perspective is a prerequisite for covariation and a process concept. These arguments lead to the call for a qualitative approach to functional relationships in school.

Experimenting fosters functional thinking

Learning environments with experimentation activities have proven to be beneficial for functional thinking (Lichti & Roth, 2018). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view. Another benefit of student experiment is the inherit constructivist learning approach that leads to higher learning gains in combination with digital technologies (Drijvers, 2019). Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and analysis (conclusions) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (ibid.), but a closer look reveals disparities that can be explained with the instrumental genesis.

Hands-on experiments and simulations in the light of instrumental genesis

The instrumental approach (Rabardel, 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers, 2019). When using simulations, schemes that develop are dynamic and concerned with variation as well as transition and hence support the covariation aspect (Lichti, 2019). Measurement procedures of the hands-on material induce static schemes for values and conditions, fostering the correspondence aspect (ibd.). While hands-on material stimulates basic modelling schemes, relating the situation to mathematical description, a simulation already contains a model of the situation. When used as multi-representational systems, the simulation illustrates connections between model and mathematical representations (e.g. graph and table) that evoke schemes for these representations and their transfer. The study presented here attempts to make use of both beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

Fostering the conceptual development

To foster FT we combine hands-on experiments and simulations with the premise of a productive instrumental genesis as follows: hands-on material at the beginning initiates modelling schemes. Subsequently, simulations facilitate the representational transfer (table – graph; situation/animation – graph), enable dynamic exploration of the relationship and systematic variation, thus fostering an understanding of covariation. Finally, measurements with hands-on material convey the correspondence aspect. The two different settings developed for this study are outlined as a scientific experimentation process with the three phases hypotheses, experimentation, analyses.

The numeric setting follows the APOS steps sequentially and gives the measurement procedure a dominant role in the experimentation phase. This sets a focus on the correspondence aspect. In the analyses phase the learners access the covariation aspect with simulations, that connect an accordingly designed animation with the dynamic representations of the relationship in a table and a graph.The second, covariational setting consequently fosters a dynamic view on the relationship and the related variables. It is implemented with two shorter scientific experimentation processes. After initial hypotheses with hands-on material, the experimentation phase with simulations immediately sets the focus on (co-)variation through. The analyses phase complements the animation with a dynamic representation of the relationship in a graph. Only after this phase, measurement data is generated with hands-on material and fed into the simulation to test the results on the relationships drawn so far.

Both settings use a story of two friends preparing to build a treehouse and contain identical contexts, hands-on material and simulations. The tasks of each setting are similar, but adapted to the numerical and covariational focus respectively. Both settings can be accessed in digital classrooms (www.geogebra.org/classroom numerical Setting: HQX7 UZRQ and covariational Setting: D3XM DDSB).

STUDY DESIGN

A comparative intervention study (pre-post design) is implemented both in distance and in-classroom learning mode with seventh and eighth graders at grammar and comprehensive schools. It contrasts the covariational and numerical settings and includes an additional control group with the simulation only implementation of Lichti and Roth (see above). The intervention is designed for six lessons (split into three sessions). It is preceded and followed by a short test on functional thinking (FT-short, online version: www.geogebra.org/m/undht8rb, Rasch-scalable, 27 items, see Digel & Roth, 2020), to compare the learning outcomes in both settings. Students work in teams of two pairs. A pilot study (ibid.) verified the comparability of the covariational and numerical setting in terms of processing time and difficulty. In this paper we present results with a focus on school form and learning mode:

RQ 1: Which setting is most beneficial for FT in the different school forms?

Hypothesis Grammar > Comprehensive: Large studies on student assessment regularly show lower competence levels in comprehensive schools to grammar schools (OECD, 2019), a gap that is getting wider (Guill et al., 2017). Regarding the different settings, the focus on the difficult covariation aspect in the covariational setting could overburden lower competence levels and thus increase the competence gap. Dubinsky and Wilson (2013) in contrast foster low achievers on all APOS levels of the concept of function successfully.

RQ 2: Does the learning mode (in-class/distance) have an impact on the learning gains in the compared settings?

Hypothesis In-Class > Distance: All three settings focus on conceptual competences, while arithmetic competences are secondary. According to the discussion in the first section in this paper, lower learning gains can be expected in distance learning, especially in comprehensive schools.

METHOD

Data analysis was conducted according to Item Response Theory. The dichotomous one-dimensional Rasch model and a virtual persons approach were used to estimate item difficulties for FT-short. The person ability was then estimated with fixed item difficulties. We applied mixed ANOVAs (between factors: setting, school form, learning mode; within factor: time) after controlling data for normal distribution and homogeneity of variance. Pairwise t-tests were used to investigate differences of the settings. A statistical power analysis (3 groups, 2 measurements, power .9, α =.05) for a medium effect (η_p^2 = .06) in a mixed ANOVA gave a desired sample size of 204.

RESULTS

Here we present quantitative results of the main study (N = 332, 121 female, 187 male, age M = 13.0, SD = 4.8). The distribution of the sample over the settings and constraints is shown in table 1. The estimation of the Rasch-model, used to determine the person abilities for the total sample, showed good reliabilities in the pre- and post-test: $EAP\text{-}Rel_{pre}$ = .86 and $EAP\text{-}Rel_{post}$ = .80 as well as $WLE\text{-}Rel_{pre}$ = .85 and $WLE\text{-}Rel_{post}$ =.80.

Table 1: Sample sizes and effect sizes Cohens d (pre/post) of subgroups

	Covariational Setting		Numerical Setting		Control Group		Total
	N	d	N	d	N	d	N
Total	114	.51***	125	.25***	93	.27***	332
Comprehensive/ Grammar	39 75	.63*** .48***	52 73	.32*** .27***	66 26	.34*** .28***	157 175
Distance In-Class	36 78	.48*** .56***	39 86	.33** .28***	38 55	.36** .30**	113 219

Comparisons of the settings under constraints

Regarding the **school form** (see Figure 1 left) the mixed ANOVA showed a significant main effect for time (F(1, 326) = 197.34, p <.001, η_p^2 =.38) and a significant effect of school form (F(1, 326) = 87.82, p <.001, η_p^2 =.21). Above, there are two significant interaction effects: between time and setting (F(2, 326) = 5.92, p <.005, η_p^2 =.018) and between time and school form (F(2, 326) = 9.57, p <.005, η_p^2 =.029). The grammar school students outperformed the comprehensive school students in the pretest significantly (t(174) = 8.09, p <.001, d = .61), but for both school forms students' ability increased significantly with a small to medium effect (grammar: t(425) = 7.08, p <.001, d = .34; comprehensive: t(216) = 5.84, p <.001, d = .40). In both school forms students in the covariational settings showed the highest learning gains (see Table 1).

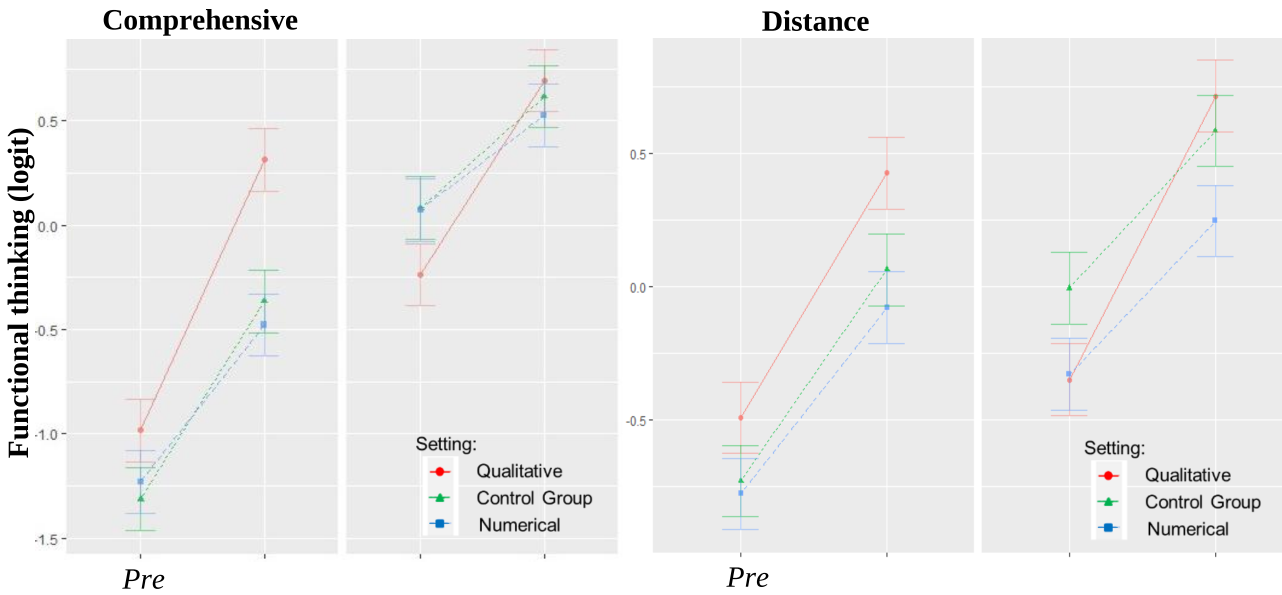

Figure 1: Increase in FT pre/post by setting, school form (left) / learning mode (right)

The mixed ANOVA for **learning mode** (see Figure 1 right) resulted in one significant main effect for time ($F(1, 326) = 170.88$, $p <.001$, $\eta_p^2 =.34$), one for learning mode ($F(1, 326) = 10.85$, $p < .001$, $\eta_p^2 = .03$) and a significant interaction effect of time and setting ($F(2, 326) = 3.63$, $p < .005$, $\eta_p^2 = .02$). In both learning modes the covariational setting shows the highest learning gains (see table 1). The students with in-class learning showed slightly higher results in the pretest ($t(153) = 2.19$, $p < .05$, $d = .18$). The overall learning gains in distance learning ($t(150) = 3.48$, $p < .001$, $d = .28$) are comparable to those in-class ($t(149) = 4.57$, $p < .001$, $d = .38$).

DISCUSSION

First of all, the results are not generalizable without reservation, since they depend on the concrete settings developed in the study. Another restriction is the disbalance of subgroups, caused by altering pandemic restrictions in participating schools. Nonetheless, results of the total sample show that both settings foster FT, while the covariational setting is significantly more beneficial for FT than the numerical setting, but the learning effects in the latter do not differ significantly from those in the control group (see Digel & Roth, 2021). Two characteristics of the covariational setting seem most influential: first, the early focus on the dynamics of the observed variables provides opportunities to reason variationally and to develop a dynamic view on functions. Second, replacing early measurement with investigation and observation of the relationship initiates practice in covariational reasoning.

The significant advantages of the covariational settings also appear in both school forms (RQ1). Significant difference in pretest between grammar and comprehensive schools in FTshort ($d = 0.61$) are as expected, but these disparate competence levels are not reinforced by the intervention, in contrary, learning gains in the comprehensive school sample outperform those of the grammar school sample. FT seems to be accessible in the three settings to learners on all competence levels and the covariational focus is also beneficial to lower levels of FT and not restricted to high achievers, which replicates Dubinsky and Wilson (2013).

The results regarding RQ2 are limited through a possibly lower level of engagement and focus in the pre- and post-tests in distance learning. Nevertheless, we can conclude from the results that all three learning environments promote functional thinking in distance and in-class to a comparable extent. This is contrary to previous studies on the effectiveness of distance learning, especially in the case of conceptual skills, such as FT here. There are three different explanations for this: On the one hand, motivational influences may have favoured the learning process in distance, since hands-on experiments set in everyday contexts and group work with individual coaches stand out positively. Secondly, inquiry-based learning with open tasks contrasts distance learning which is rather dominated by arithmetic and initiates intensive interaction with the concept as well as higher cognitive activation. A continuous interaction with partners/teams enables co-construction processes and mathematical communication about ideas, hypotheses, approaches and thus intensifies interaction with the content.

To sum up, the covariational approach to functions with experiments (1) attains higher learning gains across competence levels, (2) successfully transfers in-class activities to distance learning with comparable learning gains, (3) makes the covariational aspect accessible for high and low achievers and (4) benefits from the combination of hands-on material and simulations. In classroom practice (distance or in-class), an approach to functions designed accordingly has the potential to enhance learning gains.

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NETWORKED MULTIPLE APPROACHES TO DEVELOPING FUNTIONAL THINKING IN ELEMENTARY MATHEMATICS TEXTBOOKS: A CASE STUDY IN CHINA

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By focusing on functional thinking (FT), the core component of algebraic thinking, this study aimed to explore the features of developing FT in textbooks from grades 1 to 6 by examining a popular reform-oriented mathematics textbook series in China. A framework of FT developed by Pittalis and colleagues (2020) was used to examine the FT related tasks in the textbook series. Based on a fine-grained coding analysis, it was found that multiple modes of FT are intended to be developed since the very beginning of elementary school. Multi-modes of FT have been developed and evolved simultaneously and progressively as grades increase, serving as an enhancement for arithmetic learning. Different types of FT tasks provide various opportunities for students to explore these multi-modes of FT while learning and consolidating arithmetic across grades.

INTRODUCTION

Researchers identified that the traditional sequence of "arithmetic-then-algebra" is the hurdle for students in learning algebra (Stigler, et al., 1999), and equation-entry toward learning algebra may limit students' learning of advanced mathematics (Thompson & Carlson, 2017). As one of four big ideas of algebraic thinking (Blanton & Kaput, 2011), functional thinking (FT) has been recommended as a better organizing concept for teaching and learning algebra than the concepts typically used (e.g., expressions and equations) (Stephen, et al., 2017). Although researchers have shown that elementary students are capable of engaging in generalizing and representing functional relationships (e.g., Blanton, 2008), the topic of FT has not been addressed purposefully and systemically in the early mathematics curriculum (Carraher, Schliemann & Schwartz, 2008), or neglected altogether (Stephen et al., 2017). Consequently, it remains largely unknown how FT can be systematically and effectively introduced and developed in elementary schools in tandem with numeration and arithmetic, as well as how to prepare students for formal algebraic learning.

Studies on Chinese mathematics textbooks have drawn international attention because Chinese 15-year-old students have consistently outperformed their counterparts in the Western countries on PISA (OECD, 2020). In the traditional Chinese mathematics textbooks, the function concept is not introduced formally, but function ideas are explored implicitly (Cai, Ng & Moyer, 2011). Since the implementation of the new curriculum in China in 2011 (MoE, 2011), which has adopted numerous research findings and innovative ideas from Western literature (Xu, 2013), the officially

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endorsed textbooks have been developed and refined over the years. Thus, a systematic examination of the most popular reform-oriented textbooks in the elementary school with a focus on FT could provide insight for international readers regarding curriculum and teaching materials related to developing early algebra thinking, and FT in particular.

ANALYTICAL FRAMEWORK

A three-dimensional framework was created for analyzing learning opportunities intended in the textbooks. It includes the modes of FT, types of function tasks, and grade levels (as shown in Fig.1). The function tasks are used as the smallest unit in the textbook analysis (Stylianides, 2009), and their types are categorized. Each FT task is determined whether it is intended to develop any mode of FT. Finally, we use the dimension of grade levels (lower grades (1-2), middle grades (3-4) and upper grades (5-6)) to analyze how the textbooks arrange different kinds of function tasks in sequence to develop different modes of FT.

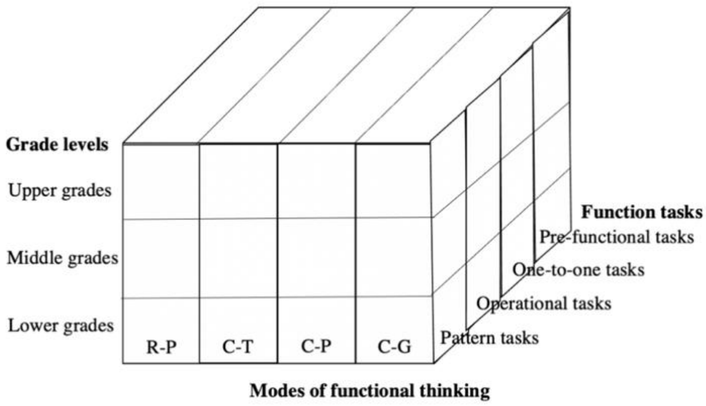


Figure 1: Analytical framework for the development of functional thinking

Modes of functional thinking

In this study, we adopted Pittalis et al.'s four modes of FT: recursive patterning (R-P), covariational thinking (C-T), correspondence relationships-particular (C-P), and correspondence relationships-general (C-G). Students exhibiting *R-P* focus only on one variable, finding variation within a sequence of values (Blanton & Kaput, 2011). *Covariation relationship* describes how two quantities co-vary simultaneously and students who hold this view keep that change as an explicit and dynamic part of a function's description (Blanton & Kaput, 2011), e.g., "if one can describe how x_1 changes to x_2, and how y_1 changes to y_2, he has described a functional relationship between x and y" (Confrey & Smith, 1991, p.57), which is *C-T*. Students exhibiting *C-P* means that they could notice the correspondence relations between corresponding pairs of values, e.g., one can complete a table involving two related quantities, while *C-G* means that students could identify and express the general relations between quantities or variables in word or symbols (Pittalis et al., 2020).

Function tasks

In this study if a task in the textbooks implicitly or explicitly reflects any mode of FT, it is called “a function task.” Through coding and comparing the tasks in the textbooks, we combined similar types of tasks and grouped them into four categories: *pattern tasks, arithmetic operation set tasks*, *one-to-one corresponding tasks* and *pre-functional tasks*, which were described as below:

- *Pattern tasks* are problems which require one to seek patterns in numerical and geometric sequences;
- *Arithmetic operation set tasks* (Abbr. operational tasks) include function machine tasks and well-ordered arithmetic calculation items;
- *One-to-one corresponding tasks* (Abbr. one-to-one tasks) include comparing tasks in grade 1, and the corresponding tasks between numbers (or number pairs) and points on the number line and the Cartesian coordinate system.
- *Pre-functional tasks* are mainly the real-life problems which include two varying quantities, for example, using letters to represent varying quantities, and tabular problems which include proportional or linear relationships.

Different from Demosthenous and Stylianides (2018), we define the explicitness of a task regarding its relationship to modes of FT. If in answering one question in the task, students have to use one specific mode of FT, we consider this question to be “explicitly” developing this mode of FT; if students may or may not use one mode of FT, we call it “implicit.”

The purpose of this study is to explore how FT evolves across grades in the selected reform-oriented mathematical textbook series in China (e.g., PEP textbooks). To achieve this goal this study seeks answers to the three research questions: how are function tasks arranged in the PEP textbooks to develop different modes of FT, how different modes of FT are embedded in the PEP textbooks and how do they evolve throughout the textbooks and if there are any particular routes for developing FT in the PEP textbooks.

METHODOLOGY

We selected Chinese PEP (People’s Education Press) mathematics textbooks (Lu & Yang, 2012) for two reasons. The first is the reputation of the publisher, with this publisher being the only publisher to produce textbooks in China before 2001. The second is there are six sets of officially endorsed elementary mathematics textbooks in Mainland China, but the textbooks published by PEP are used by 63% of students there. All of the grade 1-6 student textbooks (12 volumes) and corresponding teacher guidebooks were selected.

In general, a content analysis was used to code and analyze the curriculum materials (Fan, 2013). There were two rounds of coding, first coding of function tasks, and second, coding of modes and explicit levels of FT. Four research assistants and the first author developed the coding system.

The first round of coding identified and categorized function tasks in the PEP textbooks. The third author read through the whole series of textbooks and identified function tasks based on the definitions of each of the four modes of FT (R-P, C-T, C-P and C-G). A research assistant read the teachers' guidebooks and marked all the points that declared that the design of the worked examples or practice problems intended to develop FT. Then a group meeting (including all the authors and the research assistants) was arranged to discuss the classification of the collection of function tasks into the four categories. Then all group members re-checked the functional tasks in the PEP textbooks, and the interrater reliability was checked using Cohen's kappa (usually kappa should be 0.7; Leech, Barrett & Morgan, 2008). The average kappa of 0.90 indicated good agreement between coders. Finally, we had a whole group discussion and all team members agreed with all the coding results.

Each identified function task included one or more questions, with the possibility of each question developing different modes of FT. So, there are two steps in coding the modes of FT reflected by each function task. Firstly, we counted the number of questions included in each task; and secondly, we determined the modes and explicit levels of FT developed by the question.

RESULTS

An overall distribution of tasks and relevant targeting modes of FT is shown in Fig.3. The figure reveals three salient features related to our research questions. (1) Multiple modes of FT are embedded in the math textbooks in all grades simultaneously;(2) Two routes could be identified for developing FT. These features are illustrated in the sections that follow.

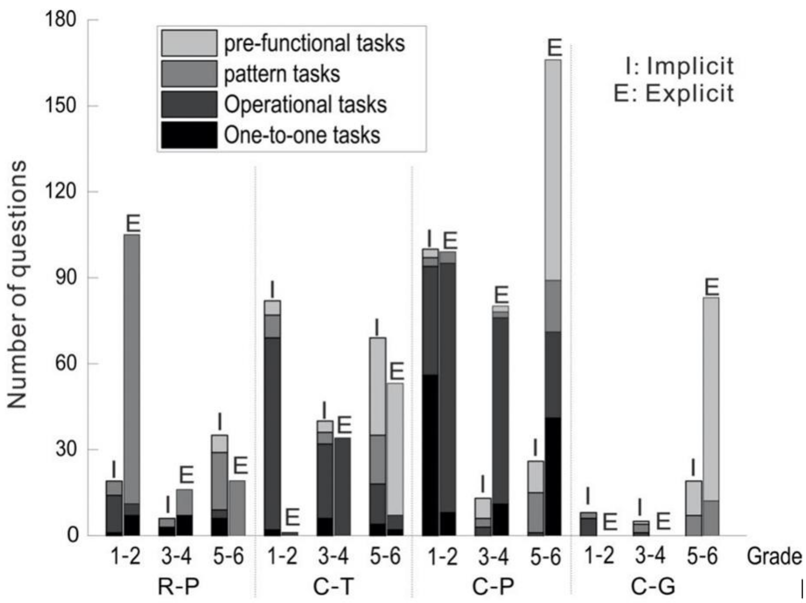


Figure 2: Distribution of function tasks that develop different modes of FT

Multiple modes of FT are developed simultaneously

Figure 2 illustrates that three modes of FT (R-P, C-T, and C-P) are developed simultaneously across three grade levels. In lower grades, two modes of FT (R-P and C-P) are developed explicitly through pattern tasks and operational tasks correspondingly. C-T is developed implicitly through operational tasks, while one-to-one tasks implicitly are used to develop C-P. In middle grades, C-T and C-P are emphasized simultaneously through the operational tasks. In upper grades, pre-functional tasks are used to develop C-T, C-P and C-G at the same time, and the pattern tasks, operational tasks and one-to-one tasks are still useful for developing C-P and C-T.

Grades	One-to-one tasks		Operational tasks		Pattern tasks		Pre-functional tasks		Total
Lower grades	30	20%	67	44%	55	36%	1	0%	153
Middle grades	9	16%	34	61%	6	11%	7	13%	56
Upper grades	26	24%	16	15%	22	20%	45	41%	109
Total	65	20%	117	37%	83	26%	54	17%	318

Table 1: Distribution of different types of function tasks across grades

Generally, there are 318 function tasks and 656 sub-questions in 12 textbooks, with the number of operational tasks being the most prevalent (117, 37%), and the number of pre-function tasks being the least (54,17%). The total number of function tasks in lower grades is the most (153), the number in upper grades is second (109), while the numbers in middle grades is the least (56).

Two routes are identified to develop functional thinking progressively

From Figure 2 and the above analysis, one can see two explicit and continuous routes in which FT is developed across grades. One explicit route is mainly using the pattern tasks, from R-P to C-G through C-P explicitly and C-T implicitly (Figure 3). The other is primarily utilizing the operational tasks and pre-functional tasks, from C-P to C-G, through C-T (Figure 4).

Route A is from R-P to C-G via C-P or C-T. The geometric patterns appear as repeated patterns in lower grades, simple growing patterns (additive or multiplicative) in middle grades, and complicated growing patterns (linear, non-linear) in upper grades. For the number pattern tasks, PEP textbooks combined number sense learning with the growing pattern. In grade 1 number patterns grow by 1, 3, 5, etc., which helps students understand the number sequence. In grade 2 the patterns might grow by 100s, 10s, or 1s, and in middle grades they might grow by 0.1s, 0.01s, 0.001s, etc., which is highly correlated with the knowledge of place value base 10.

In upper grades the PEP textbooks usually present number and geometric patterns together, and the figures help students to generalize the rules. In this way it is easier for students to match the figure and number (C-P), observe the change of both the figures and values (C-P, even C-T), and finally find out the rule (C-G). For some complicated patterns, the textbooks encourage students to observe the geometric figures and find the near-generalization term, which gives students opportunities to experience the non-linear relations which prepares them for the future leaning of functions.

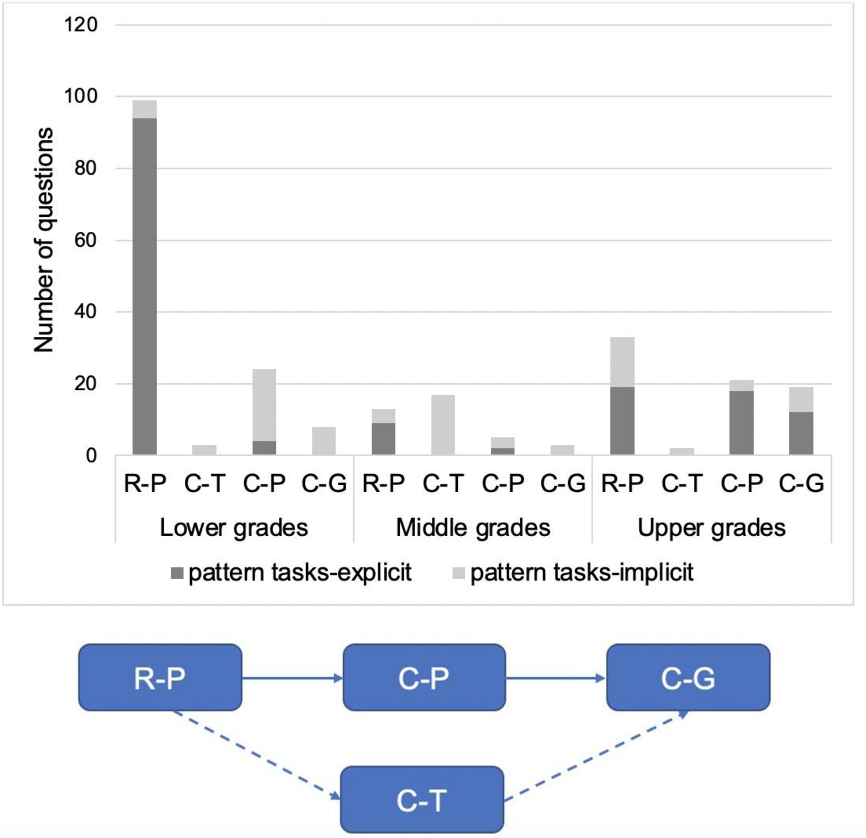
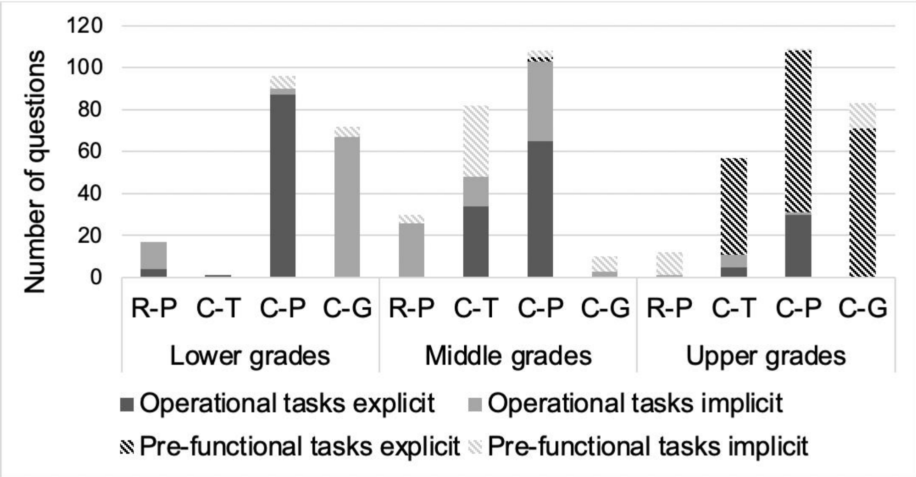


Figure 3: Route A develop FT from R-P to C-G

Route B is mainly from C-P to C-G through C-T. In the lower grades the textbooks usually used operational tasks and comparing tasks, explicitly developing C-P. Through utilizing the organizing table tasks, the textbooks begin to develop C-T implicitly. In middle grades, there are also function machine tasks which include fractions and decimals (explicit C-P), and some pre-functional table tasks, which frequently present the quantities in real-life situations, such as distance and time, price and cost, etc. These pre-functional tasks not only require students to fill in tables based on calculating (C-P), but also help students to experience the co-varying of two quantities. In upper grades, the pre-functional tasks usually involve direct or reverse proportional relationships, and they require students to generate the rules or judge the relationships (C-G) and also explicitly describe the co-variational relationship between the quantities. Many tasks develop C-P in upper grades, but some are by-products of C-G and C-T (e.g., find the corresponding values according to the rules they generate,), and some are preparation for future study (such as, non-linear functions).



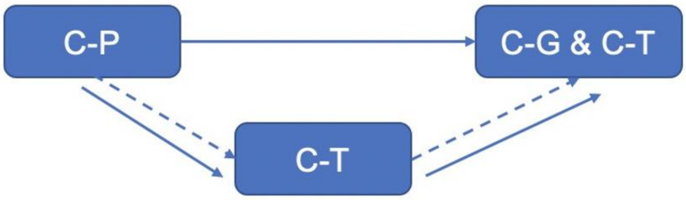


Figure 4: Route B develop FT from C-P to C-G & C-T

CONCLUDING REMARKS

This study provides several implications for developing FT in elementary mathematics. First development of FT should be embedded in learning arithmetic as an enhancement rather than an addition to the crowded existing content. Secondly, the 3D framework which integrates tasks, modes of FT and grade levels provides a useful analytical tool for examining textbooks regarding the development of functional thinking. Finally, we revealed two main pathways for the development of FT, which are aligned with the learning progression and pathways as described by other studies (Stephens, et al., 2017; Pitallis, et al., 2020). Thus, the ways of developing FT in elementary textbooks in China may provide insight for textbook development in other countries.

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PROSPECTIVE TEACHERS' COMPETENCE OF FOSTERING STUDENTS' UNDERSTANDING IN SCRIPT WRITING TASK

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In different research traditions teachers' diagnostic competence has always been characterised as being interwoven with fostering students in order to enhance their understanding. While the first one has been investigated thoroughly there is only a limited empirical access to the second one so far, especially in a content-specific way meaning focussing on the specificity of mathematical content areas. As prospective teachers have been shown to struggle with formulating adequate diagnostic judgements and fostering students, we especially investigate their practices of fostering students' understanding identified in script writing tasks analysed with the epistemic matrix. The results indicate that there are three typical impulse pathways in the matrix.

THEORETICAL BACKGROUND

Prospective teachers' competence of fostering students' understanding

For teaching that is centred around students' understanding, teachers' diagnostic competence has been found to be important (Empson & Jacobs, 2008). This competence has already been defined and conceptualized in different frameworks, which often do not focus on the mathematical content specifically. Therefore, the authors of this paper follow a content-specific approach on conceptual and procedural knowledge elements of the current learning content (here: conditional probabilities) or the prior mathematical content as described and explained in Dröse, Griese & Wessel (accepted). Due to space limitations this explanation cannot be presented in detail here.

While diagnostic competence is well conceptualized, there is not yet a definition of teachers' competence of fostering students' understanding applicable. Ball, Thames & Phelps (2010) describe 'knowledge of content and students' as well as 'knowledge of content and teaching' as important facets of teachers' knowledge. 'Knowledge of content and teaching' particularly comprises that teachers need to make instructional decisions, meaning to know "when to pause for more clarification, (…) when to ask a new question or pose a new task" (p. 401). But, the competence of fostering students' understanding can be seen as relating more aspects than knowledge facets alone:

(1) Teacher student interaction is often based on a task that is linked to the learning trajectories intended by the teacher, and the learning goal(s) set for the individual student. These aspects fall within the teacher's subject matter and pedagogical content knowledge (Ball et al. 2010).

(2) When students solve the given task, their individual thinking and learning pro-

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 243-250). PME.

cesses need to be perceived and interpreted by the teacher in order to guide the teacher's decision-making within the learning process. These aspects are described as cognitive diagnostic thinking processes by Loibl et al. (2020).

(3) What is more in order to react adequately and enhance students' understanding are communicational skills building up on the made decisions, taking the students' thinking processes as well as the task or the learning goal into consideration and guiding the students' learning process through the teacher.

Current studies, e.g. Prediger & Buró (2021) identified teachers' competence of 'Enhancing students understanding' to be a central job for teachers' expertise when working with students at-risk next to the jobs of 'Specifying learning contents' and 'Monitoring students' learning process'. As prospective teachers have been shown to struggle with identifying content-specific aspects in their diagnostic judgements (Jansen & Spitzer, 2009) and therefore maybe also in fostering students' understanding, the authors limit their research to prospective teachers. For them, an important practice to become proficient in is, amongst others, the practice of explaining. It is needed in discursive situations of fostering students in one-on-one or classroom discussions. For this paper, the authors rely on earlier research on explaining practices because of its identified potential when extending ideas to the competence of fostering students in discursive settings.

The epistemic matrix for characterising explaining practices

For capturing the different modes of the practice of explaining, Erath and Prediger (2014) developed further the epistemic matrix (cf. Fig. 1). The epistemic matrix distinguishes epistemic modes and logical levels of an explaining practice: Logical levels referring to the content of explaining (explanandum) are unfolded in conceptual and procedural levels and their epistemic modes are characterised as follows:

- *Labelling & naming*: mode that expresses names and labels, e.g. in one word
- *Explicit formulation*: more elaborated mode, including e.g. definitions of concepts or formulation of procedures
- *Exemplification*: mode of expressing examples and counterexamples
- *Meaning & connection*: mode of expressing connections between conceptual and procedural knowledge or to other concepts, pre-knowledge or graphical representation
- *Purpose*: mode of describing the "inner mathematical or everyday functions" (Erath & Prediger, 2014, p. 18) of concepts and procedures
- *Evaluation*: mode that "appears in the context of presenting solutions in class" (Erath & Prediger, 2014, p. 18)

We can rely on the matrix with its epistemic modes as well as logical levels for systematizing epistemic fields of an explanation in a content-specific approach. Since our study focuses on the current mathematical content of conditional probabilities, we exemplify the epistemic fields for this content in the next Section. As the epistemic modes of “purpose” and “evaluation” are connected to rather general classroom discussions, we focus only on four modes for the content of conditional probabilities.

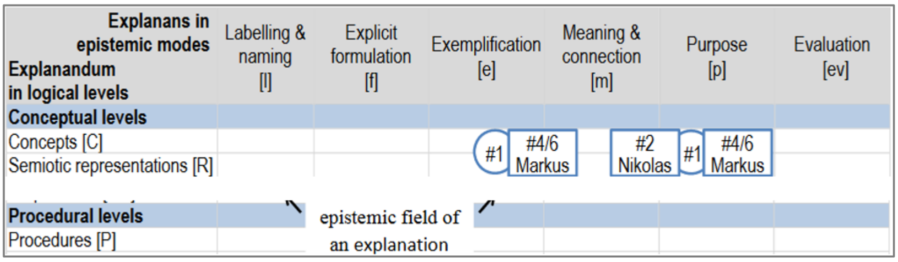


Figure 1: Excerpt of an example of epistemic matrix (Erath & Prediger, 2014)

The epistemic matrix for explaining conditional probabilities

In order to follow the content-specific focus on conceptual and procedural knowledge elements, the epistemic matrix is now exemplarily filled out for the current mathematical content of conditional probabilities (Fig. 2). We differentiate current mathematical content from prior mathematical content because focussing on the content - and with it on students’ prior knowledge in contrast to the current content - as separate and constituent element of diagnostic judgments and fostering students’ understanding has only rarely been investigated (Dröse, Griese & Wessel, accepted). The following explanations regard Fig. 2 for the conceptual and the procedural level:

On the *conceptual level*, concepts concerning stochastic (in)dependence are knowledge elements of the current mathematical content (cf. row --CC--) (Hoffrage et al., 2015), while they build upon conceptual knowledge elements from prior mathematical content (cf. row --CP--), e.g. the part-whole or part-of-part relationship as concepts of fractions and the multiplication of fractions (Post & Prediger, 2020; Prediger & Schink, 2009). The epistemic modes can be described as follows. Here examples are given:

- *Naming:* “conditional probability”, --CC--, |L|
- *Explicit formulation:* “the definition of conditional probabilities”, -CC--, |F|
- *Exemplification:* “distinguish joint and conditional probabilities”, --CC--, |E|
- *Meaning & connection:* “visualize conditional probabilities”, --CC--, |M|

On the *procedural level* different procedures can be focused either in the current mathematical content of conditional probabilities (cf. row --PC--) (see overview in Binder et al., 2020) or in prior mathematical contexts as routine calculations on fractions (cf. row --PP--) (Prediger & Schink, 2009). Again, different epistemic modes for these procedures can be distinguished (examples given):

- *Naming:* “rule of Bayes”, row --PC--, column |L|
- *Explicit formulation:* “formulate the formula of Bayes”, --PC--, |F|
- *Exemplification:* “express conditions of applying the formula”, --PC--, |E|
- *Meaning & connection:* “explaining the formula”, --PC--, |M|

Epistemic modes / Logical levels	Labeling & naming |L|	Explicit formulation |F|	Exemplification |E|	Meaning & connection |M|
Conceptual levels				
Current mathe-matical content --CC--	Term: Conditional probability	Definition of conditional probabilities	Distinguishing joint and conditional probabilities	Concept or visuali-zation of conditional probabilities
Prior mathe-matical content --CP--	Term: fraction	Definition of fractions, parts etc.	Examples of fractions and parts	Unit square as visualization
Procedural levels				
Current mathe-matical content --PC--	Rule of Bayes	Formulating the formula of Bayes	Expressing conditions of application of the formula	Explaining the formula
Prior mathe-matical content --PP--	Formula of multiplying fraction	Procedure of multiplying fractions	Expressing conditions of application of the formula	Explaining the formula part-of-part relation

Figure 2: Epistemic matrix exemplarily specified for conditional probabilities (adapted from Erath & Prediger, 2014)

Research questions

For pursuing the described research interest, we investigate this research questions:

(RQ1) How can teachers' competence of fostering students' understanding be investigated with the epistemic matrix?

(RQ2) Which pathways through the epistemic matrix can be identified in prospective teachers' moves for fostering a student (hereafter abbreviated as PTMF)?

METHODS

Data collection

The data was collected in a university mathematics education course with n=26 prospective secondary school teachers in Germany. The sample can be characterized as follows: 81% of the prospective teachers' study for upper secondary school and 19% study for vocational schools as well as 69% of the prospective teachers attend the course in their sixth semester and 31% attend the course in their eighth semester and all in the last year of their bachelor programme. The course covers among other topics content knowledge and pedagogical content knowledge on conditional probabilities.

For assessing the prospective teachers' competences, a vignette displayed in Fig. 3 is used as an already established instrument in mathematics education research (cf. overview in Buchbinder & Kuntze, 2018). The vignette, consisting of a written student solution and a following transcript, based on a real dialogue (in Post & Prediger, 2020), that give insights into the student's understanding and obstacles concerning conditional probabilities and fractions as the underlying prior mathematical content. For assessing the prospective teachers' competence of fostering students' understanding a script wri-

ting task inspired by lesson plays (Zazki, Liljedahl, & Sinclair, 2009) is integrated.

Transcript vignette

For the following task you should put yourself in the teachers' position and react appropriately within the situation.

Background information for the following scene: The class in a German upper secondary school has covered conditional probabilities and their calculation.

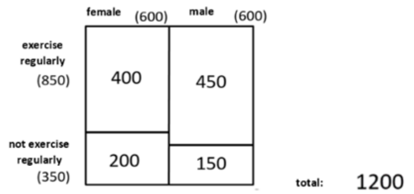
In the following you will read two transcripts displaying excerpts from conversations between two students (Ole and Nazan) and their teacher, regarding the task displayed on the right.

Task: Exercising Teenagers

In a survey, 1200 teenagers were asked if they exercise regularly. 600 out of the 1200 teenagers are female.
$\frac{1}{8}$ of the female teenagers do not exercise regularly. $\frac{3}{8}$ of the teenagers are male and exercise on a regular basis.

What is the probability that a random male person exercises regularly?

(JIM study 2018)



Part 1: Ole solves the task. He writes down the following solution.

$\frac{3}{8}$ = 0.375= 0.375%, the probability is 0.375%.

The following interaction with the teacher evolves [not printed here].

Task: Analyze the two transcripts:

Formulate an impulse for continuing the dialogue. Express how you would react to foster Ole. Write a continuation for the transcript.

Figure 3: Transcript vignette with task for prospective teachers
(completely printed in Dröse, Griese & Wessel, accepted)

Data analysis

The 26 written documents containing PTMF were coded in two steps.

(1) Two raters coded the written documents containing PTMF for logical levels with the codes: --conceptual level of current mathematical content--, --conceptual level of prior mathematical content--, --procedural level of current mathematical content-- and --procedural level of prior mathematical content-- with an interrater reliability of Cohen's $\kappa = 0.89$.

(2) In a second step, for each knowledge element the epistemic mode has been coded and double-checked by the second rater.

(3) The codes have been displayed in the epistemic matrix and the pathways through the epistemic matrix have been categorized into different types.

EMPIRICAL FINDINGS ON PROSPECTIVE TEACHERS' COMPETENCE OF FOSTERING STUDENTS' UNDERSTANDING

The application of the data analysis method to the written documents enabled us to access which epistemic fields (epistemic mode + logical level) have been addressed in the dialogue containing PTMF by the prospective teachers in which order. With this identification of addressed epistemic fields, we suggest a tool for investigating a facet of teachers' competence of fostering students' understanding (RQ1). These analyses provide us with three main types of moves for fostering Ole in the given data set. For assessing RQ2, the three main types are now presented and illustrated by examples.

The first type is characterised by starting with an |exemplification| (|E|) on the --procedural level of the prior mathematical content-- (--PP--) of shortening fractions. After that, the moves continue addressing the |meaning and connections| (|M|) also in the --prior mathematical content but on a conceptual level-- (--CP--) of fractions. This type was found among 5 out of 17 dialogues containing PTMF. The following extract shows only the teachers' moves, as they would have been set by the prospective teachers. The example pathways through the epistemic matrix as displayed in Fig. 4.

1	T:	So then try out, if it is the same.
3	T:	Okay. And what is the meaning of 3/8? Perhaps it is helpful if you read the text again carefully and include the unit square.
5	T:	Right. And the 3/4? What is this part? And what has been searched for in the task?

The second type found among 4 out of 17 dialogues containing PTMF has the same starting point but continues by addressing different epistemic modes in the --conceptual level of the current mathematical content--(--CC--). The example pathway that addresses the |exemplification| (|E|) of the current mathematical content is displayed in Fig. 4.

1	T:	Have a look at the two fractions again, if the two numbers can be the same.
3	T:	3/8 is the probability, that out of all teenagers a random chosen person is male and exercise. Can you see the difference [to 450/600]?

The third type found by 8 out of 17 prospective teachers also has the same starting point and continues on the --conceptual level of the prior mathematical content-- (--CP--) as well as afterwards on the --conceptual level of the current mathematical content-- (--CC--). The pathway through the epistemic matrix for the following teacher moves is again displayed in Fig. 4.

1	T:	We can calculate, if it is the same. With which number can 450/600 be shortened?
3	T:	Very good. And is this the same part as the one you have had before?
5	T:	Ok, lets have a look at the task again. Which part had to be calculated?
7	T:	Very good, so we have calculated a conditional probability. What has been the condition in this task?

Fig. 4 displays all three types found for the continuation of the dialogue with teachers' move in order to foster Ole. T1-T7 describe the turns of the teacher moves.

Contrasting the three types, we see that all prospective teachers start with addressing the same epistemic field. After that, the types take different routes through the epistemic matrix. The first type addresses also the --conceptual level of the prior mathematical content-- while it does not reach the current mathematical content of the task. The second type reaches the current mathematical content, but might miss potential students' obstacles in the --conceptual level of the prior mathematical content-- belonging to the --procedural level of the prior mathematical content--. The third type addresses the --conceptual level of the prior mathematical content-- as well as the --conceptual level of the current mathematical content-- and therefore unites impulses of type one and two.

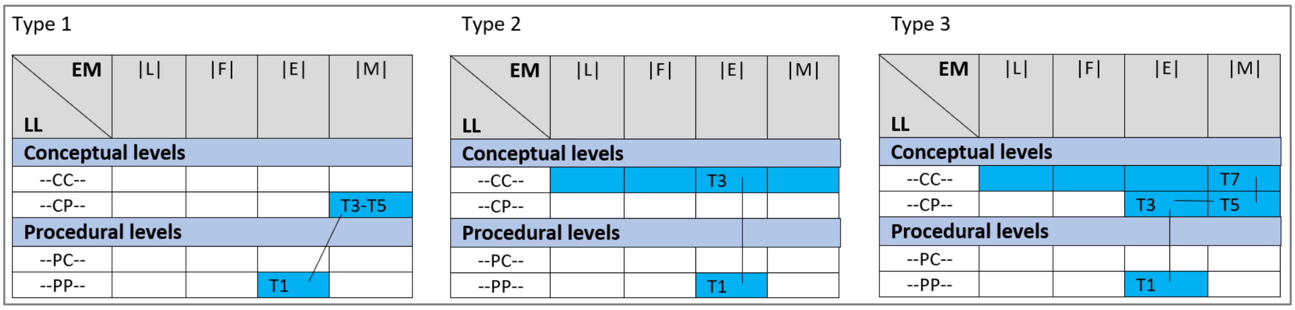


Figure 4: Three types of pathways through the epistemic matrix identified for Ole

DISCUSSION AND OUTLOOK

Concerning the first research question, we see that applying the epistemic matrix for explaining (Erath & Prediger, 2014) to the written prospective teachers' moves for fostering Ole provides different types of teacher moves and therefore might give deeper insights into what is more in teachers' competence for fostering students under-standing than the already identified facets (Ball et al. 2008; Loibl et al., 2020).

The second research question aims at investigating which pathways through the epistemic matrix prospective teachers take in their written moves. This led to the afore mentioned three types of teacher moves for fostering Ole. Those types cover different logical levels and epistemic modes that are of theoretical importance as they enrich the already existing research on prospective teachers' obstacles (Jansen & Spitzer, 2009) with deeper knowledge on prospective teachers' moves for a specific vignette. In addition, this is especially important for teacher educators as knowing prospective teachers' moves makes it possible to adjust teaching-learning arrangements to the prospective teachers' competencies.

Meanwhile, our research is limited due to the small sample size of only 26 prospective teachers and the specific content and given transcript vignette. Future research has to extend on the one hand the sample size and provide further insights for other mathematical content areas and vignette formats (Buchbinder & Kuntze, 2018).

Acknowledgements.

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REPLICATING A STUDY WITH TASKS ASSOCIATED WITH THE EQUALS SIGN IN AN ONLINE ENVIRONMENT

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This paper presents a case study of a conceptual replication study. We replicated the famous and widely cited task presented in Falkner et al. (1999), 8+4=__+5. In contrast to the original study, we administered the task with the same age group (Grade 6) in a different system (Denmark) and via a large-scale online learning environment (OLE), with a larger sample and two decades later. Our replication indicates that the Danish students performed very significantly better than the students in the original study. We discuss why this is the case and argue that OLEs such as the one we used provide an important opportunity to replicate, and thus better understand, similar results.

INTRODUCTION

There is an increasing interest in replication studies in mathematics education at PME (e.g. Inglis et al., 2018) and beyond (e.g. Jankvist et al., 2021). This interest stems from the replication crisis in psychology research, which has highlighted a large proportion of false-positive results (e.g., Open Science Collaboration, 2015). In part, this may be due to the high degree of flexibility in quantitative and experimental researchers' analytic and design choices (Simmons et al., 2011). The imperative for replication studies in mathematics education is, however, broader than this. Aguilar's (2020) literature review highlights the majority of studies published even in respected mathematics education journals are small-scale and hence influenced by contextual factors that are poorly understood. Hence, replication can perform a crucial function in deepening and extending the validity of findings, because "[t]hrough variations to studies, we can delineate the bounds of the original study's findings" (Melhuish & Thanheiser, 2018, p. 106). Jankvist et al. (2021) emphasises that replication studies are important in the mathematics education community because they enable a more deeply understanding of the phenomena and results. Replication studies can help clarify under which conditions a particular finding is true or not and replication whether the results are stable over time, across different educational systems or different populations (e.g. Cai et al., 2018). Aguilar (2020) concludes that knowing more about the conditions that make it possible for a research finding to take place, and the boundaries of where it remains true, advances our research field as it allows us to broaden our understanding of the contextual variables under which the research finding occurs. This in turn has direct implications for the implementation of research findings in practice.

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In this paper, we present the data of a conceptual replication study (Aguilar, 2020) of the study presented in Falkner et al. (1999) and Carpenter et al. (2003), reporting findings from the use of their famous task $8 + 4 = \_ + 5$. This result that is widely cited in the literature on equivalence (e.g. Knuth et al., 2006). From the study presented in Falkner et al. (1999) we learn that an entire range of 145 sixth grade students provided 12 and/or 17 as the number that should go in the empty space. Students argue that 12 is the answer, because the numbers on the left together makes 12, neglecting the meaning of the +5 on the right side and reflecting an operational, rather than a relational, understanding of the equal sign (Knuth et al., 2006). Others argue that what goes in the empty space is the value of all the numbers added resulting in 17. In the two original studies, we are presented with the following data;

Percent of children offering various solutions to 8 + 4 = ☐ + 5

Grade	Answers Given					Number of Children
	7	12	17	12 and 17	Other	
1	0	79	7	0	14	42
1 and 2	6	54	20	0	20	84
2	6	55	10	14	15	174
3	10	60	20	5	5	208
4	7	9	44	30	11	57
5	7	48	45	0	0	42
6	0	84	14	2	0	145

Table 1: Data from answer provided to $8 + 4 = \_\_ + 5$ (Falkner et al., 1999, p. 223)

We have recreated the above task with two additional variations $4 + \_ = 7 + 5$ and $6 + \_ = 4 + 5$, and implemented them in a Danish OLE called *matematikfessor.dk*. The variations are made in order to investigate the bounds of Falkner et al.'s (1999) findings. The first variation uses the same format as the original task but the empty space has been moved to the left side of the equals sign. This is done in order to investigate how willing students are to put the number 3, completing the sum $4 + 3 = 7$, ignoring the number 5 at the end, similarly to the original task. We did however not expect the students to be willing to put in 16 (the total sum of the numbers present) but were curious whether the students would put 12 completing the sum on the right side ($12 = 7 + 5$). The third variation also features the empty space on the left side of the equals sign. In this variation we wanted to investigate what numbers students were willing to put in when the number completing the sum disregarding the last number, should be a negative number. We expected this encourage students to view the equation as more of a whole, thereby including the +5 at the end, because negative numbers might be an unacceptable answer or option (Vlassis, 2002).

The context: Matematikfessor.dk an online learning environment for mathematics

In Denmark, as in many other systems, teachers and students increasingly use OLEs. *Matematikfessor.dk*, the environment discussed in this paper, has been running for over 10 years. More than 500,000 students in primary and lower secondary schools have access to the environment and, on a typical day, 45,000 unique students use the variety of tasks offered by the site, and collectively answer 1,500,000 tasks. OLEs like *matematikfessor.dk* therefore have access to a large amount of data and can quickly host replications of tasks such as the ones presented in the sections above in order to generate large amounts of responses. This leads us to the following research question;

What similarities and differences do we see more than 20 years after the original study when implementing the task presented in Falkner et al. (1999) *in an OLE?*

THEORETICAL BACKGROUND

In this section, we collect research about students' conception of the equals sign and comments on the difficulties that emerge from these conceptions. Rittle-Johnson et al. (2011) gives four levels of interpretations of or four meanings to apply to the equals sign in given situations (see table 2).

Level	Description	Core equation structures
Level 4: Comparative relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing that performing the same operations on both sides maintains equivalence. Recognize relational definition of equal sign as the best definition.	Operations on both sides with multidigit numbers or multiple instances of a variable
Level 3: Basic relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recognize *and generate* a relational definition of the equal sign.	Operations on both sides, e.g.: $a + b = c + d$ $a + b - c = d + e$
Level 2: Flexible operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right: $c = a + b$ or No operations: $a = a$
Level 1: Rigid operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.	Operations on left: $a + b = c$ (including when blank is before the equal sign)

Table 2. 'Construct Map for Mathematical Equivalence Knowledge' (Rittle-Johnson et al., 2011, p. 3).

One of the central difficulties that students encounter in the transition from an arithmetic thought process to an algebraic one is that they continue to view the equals sign as a 'do something" signal' (Kieran, 1981), or they maintain an urge to 'calculate', out of habit (Alibali et al., 2007). In the context of the task chosen for this study, children do need to be able to consider the right side of an expression involving an equals sign as an expression in its own right. In the words of Rittle-Johnson et al. (2011) an operational view or meaning attached to the equals sign. The main purpose of the task ($8 + 4 = \_ + 7$) is to determine what interpretation of the equals sign a student would apply. In the earlier years in school mathematics students might perceive the equals sign as indication for that calculations has to be made and that the operations on the left side results in a single number on the right side of the equals sign (Alibali et al., 2007; Kieran, 1981).

METHODOLOGICAL CONSIDERATIONS

In August 2020 we implemented the task from Falkner et al. (1999) in the OLE *matematikfessor.dk* as parts of three sets of formative tasks, with a total of 49 unique items about linear equations. The sets were only available for teachers to assign to their students, not for students to find on their own within the environment. A promotion campaign was established in order to notify the teachers subscribing to the services of *matematikfessor.dk* of the formative sets existence and applications. The data (in the form of unique answers) was extracted from *matematikfessor.dk's* database on the 4th of November 2021.

DATA RESULTS

The original task 8+4=__+5

In total 2345 answers were given to the original task presented in Falkner et al. (1999) when we implemented out version in the OLE. In a review of these, we found that only 92 of these answers were from students solving the task multiple times. In table 3 is an overview of the answers the students provided. (64 total answers were omitted. These answers were belonged to a range of 16 additional groups of answers that were less than 1% of the answer total)

Answer	Freq	%
7	1546	65.9
17	363	15.5
12	343	14.6
3	29	1.2

Table 3: Overview of the answers to the task 8+4=__+5

Answer	Freq	%
7	501	62.9
17	143	17.9
12	119	14.9
3	8	1.0

Table 4: Overview of the answers to the task 8+4=__+5 (age 12 and 13) We examined how 12 to 13 year olds (6th graders) from Denmark answered the task in order to be able to compare with the same age group from the original study. In total 797 students from this age group answered the implementation of the original task. The results can be seen in table 4.

The amount of 12 year olds that gave the answer 7 is 57.3% where the 13 year olds sum up to 64.0%. The average age of the children represented in the data for the original task is 13.97 years, slightly lower than the total average age of 14.08 years of the children represented in all three tasks. See age distribution in figure 1.

The first variation 4+__=7+5

For the second task (the first variation), we received a total of 1203 answers. In a review of these, we found that only 40 of these answers were from students solving the task multiple times. In table 5 is an overview of the answers the students provided. (45 total answers were omitted. These answers were belonged to a range of 14 additional groups of answers that were less than 1% of the answer total)

Answer	Freq	%
8	996	82.7
3	99	8.2
9	35	2.9
12	27	2.2

Table 5: Overview of the answers to the task $4 + \_ = 7 + 5$

The second variation 6+__=4+5

For the third task (the second variation), we received a total of 824 answers. In a review of these, we found that only 43 of these answers were from students solving the task multiple times. In table 6 is an overview of the answers the students provided. (27 total answers were omitted. These answers were belonged to a range of 13 additional groups of answers that were less than 1% of the answer total)

Answer	Freq	%
3	751	94.9
4	14	1.8
-2	11	1.4
9	11	1.4
2	10	1.3

Table 6: Overview of the answers to the task $6 + \_ = 4 + 5$

Additional results

A total of 351 students have provided answers to all three items. Based on these data the facility of the original task is 69.5%. The facility of the first variation is 83.3% and the facility of the second variation is 93.3%. These students actually represent the overall data very well. Only 32 students have provided two answers to one or more of the items where one of the answers were wrong. We thought it might be interesting to know the exact number of students who either got it wrong first and then right and vice versa. Twenty of the students that provided answers to the task $8 + 4 = \_ + 7$ provided two answers, where the first answer was wrong and the second answer correct. Most of these cases were a situation where either 12 or 17 was the first answer and 7 the second. Five students did in fact provide a correct answer as the first and a wrong answer the second time around.

DISCUSSION

In this section, we discuss the similarities and differences in data results compared to the original studies. In addition, we discuss what possible influence the OLE have on with the similarities and differences. If we compare the data from the original study presented in Falkner et al. (1999) we immediately notice the striking difference in facility among 6th grade students. In the original study, every 6th grade student gave the wrong answer to the task. A later publication (Carpenter et al., 2003) provides additional information about the performance of the task and interpretations made by the authors. The author's comment that the data show that older students are more likely to get the task wrong than younger students are and the author hint at that maybe students get a progressively more operational interpretation of the equals sign based on the teaching at this point in time. Knuth et al. (2006) emphasises that poor performance on measures of understanding the equals sign should not be surprising given the lack of explicit focus in American middle school curricula, although we note that a recent study indicates that American students may have a better understanding of equivalence more generally than some European countries (Simsek et al., 2021). McNeil (2007) finds that performance on equivalence problems such as the ones discussed in this paper decreases with American students from age 7-9 before it increases again from age 9-11. Hence, performance on this item may be particularly influenced by pedagogic and curricular choices. Nonetheless, the data from our study show that students in 6th grade (12-13 year olds) give a correct answer 63% of the time and matches the overall distribution very well. We acknowledge that the original study does not specifically intend to provide information on how 6th grade students perform on a task such as $8 + 4 = \_ + 7$. Rather they intend to provide teachers with a reminder that students' interpretation of the equals sign is of great importance and does not need to be corrected at an older age rather than classroom discussions about the meaning (definition) of the equals sign at lower grades are particular meaningful (Carpenter et al., 2003).

We do get the same wrong answers in our study as in the original. This to some extent prove that the task is not performing in a significantly different way i.e. producing different answers than 20 years ago. We do however wonder why we see the huge difference in the distribution of the answers. 20 years ago in the original study, less than 10% of the participants at every class level gave the answer 7. Now we see a rate of approximately 65%. Granted our data stems from 12-17 years old. With most of the participant being 13-15 (83%). Falkner et al. (1999) mentions that the task was originally carried out by a teacher in a single classroom. When this teacher realized that every student in that classroom provided a wrong answer, she asked her colleagues to use the task with their students resulting in the data in table 1. This means that the observations all stem from the same school. In our study, the data stems from at least 197 schools due to the task being implemented in an OLE. We are however not certain that none of the students in our study received help solving the task. This fact might skew the correct answer percentage towards a higher number. However, it seems unlikely that this should leave us with 60+% correct answers compared to none or almost none. Another obvious difference is nationality of the populations observed in the original study we have American students and in our study the observations stem from Danish students. According to PISA 2018 (https://factsmaps.com/pisa-2018-worldwide-ranking-average-score-of-mathematics-science-reading/) the overall difference in the performance of students in the United States and Denmark is not statistically (or indeed practically) significant. Of course, the task was presented to the American grade 6 students more than 20 years ago and it may be that teachers are now more aware of student's understandings of, and misconceptions about, the equals sign, because of the curricular changes made as a result of the introduction of mathematical competencies in Denmark in 2002. The data collected on the variations of the original task suggests that a similar operational view of the equals sign is being applied even though the empty space is moved to the left side of the equation. This was to be expected, as it is still possible to apply the same operational procedure as the original problem with the empty space on the right side. With the last task, we see an even better performance. The last variation is as expected not similar to the second variation because -2 is not as frequent as the number 3 was in the second variation. This to some extent proves that the choice of numbers matter when designing tasks such as the original task even though the empty space is on the left side of the equals sign. This choice of numbers indicate that students might be more likely to apply a relational interpretation of the equals sign to avoid negative numbers or simply because negative numbers are not accepted in a situation such as this.

CONCLUSION

Based on the differences in the data we believe that, although this task from Falkner et al. (1999), in our opinion is a very good task, the data presented by the authors is not representative of how difficult the task is for 6th grade students. Our data show that the majority of the wrong answers was identical to the ones observed in the original study. This does in our opinion encapsulate one side of the importance of replications studies in mathematics education. On the other hand, our data show a huge deviation from the facility scores of the original study. This is also an important finding for the sake of replication studies in mathematics education. Even though the point of the task presented in Falkner et al. (1999) is not primarily to indicate how difficult it is and present quantitative scores, it is nonetheless important to observe that the scores presented in the original study is an extreme case compared to data collected from a large collection of schools in Denmark 20 years later. With all that said using OLEs to replicate studies such as the performance of the famous task from Falkner et al. (1999) can be great and efficient platforms for achieving additional and in some cases updated information and knowledge.

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META-SCIENTIFIC REFLECTION OF UNDERGRADUATE STUDENTS: IS MATHEMATICS A NATURAL SCIENCE?

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Reflecting on the nature of mathematics is an important activity for undergraduate students. To analyse students' reflection, we address the questions how students categorize mathematics in the system of scientific disciplines and what arguments they use to support their decision, in particular. In an online-survey, we implemented two open-ended items to gather information about the meta-scientific reflection of 296 undergraduate students enrolled in a mathematics-related study program. By analysing students' answers, we identified nine subthemes that can be grouped in three themes: (1) the content, (2) the method, and (3) the purpose of mathematics. Most of the students concentrated on only one of the three themes. Based on these results, we discuss in which way prompts can support students' meta-scientific reflection.

INTRODUCTION

Citizens of modern societies are regularly confronted with problems that are somehow related to scientific findings: whether in their personal daily lives, or in their social responsibility. Those problems may be solved or caused by scientific findings and disciplines, respectively. One recent prominent example is an adequate dealing with information about the COVID-19 pandemic. A sufficient understanding, a reasonable dealing with scientific findings and a reflection of scientific ideas subsumed under the term "meta-scientific reflection", are key components of citizens' participation in their personal, social and public life. This reflection is mentioned e.g. in German school curricula under the term "scientific propaedeutics" ("Wissenschaftspropädeutik") which is an aim of upper secondary schools (KMK, 1972/2021).

So far, only few studies have examined whether students are able to reflect on a meta-scientific level or not (Oschatz et al., 2018). One possible reason for this absence of studies involving meta-scientific reflection is the lack of validated instruments for measuring meta-scientific reflection (Dettmers et al., 2010), especially on mathematics as a scientific discipline. Therefore, we developed an instrument to gain information about students' meta-scientific reflection on mathematics; in particular, to answer the questions how undergraduate students categorize mathematics in the wide system of scientific disciplines and what arguments they use to support their categorization.

THEORETICAL BACKGROUND

The reflection on scientific findings is important for the preparation of a university study program as well as being part of the modern scientific-orientated society. We call the concept *meta-scientific reflection* that covers reflection on an epistemological level:

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> Speaking about mathematics from an epistemological perspective, for example, about the characteristic distinctions between mathematics and other sciences, about the nature and the origin of mathematical knowledge, and so on (Neubrand, 2000, p. 256).

This quote emphasises the importance of reflecting on the differences between scientific disciplines and getting an orientation in the system of scientific disciplines. This perspective on meta-scientific reflection focuses on the development of skills that are helpful for understanding and evaluating scientific findings based on the used methods, conditions, and limits. To be able to reflect on scientific disciplines meta-scientifically, one has to have enough meta-scientific knowledge and an appropriate methodological awareness. Therefore, meta-scientific reflection is most difficult to achieve and maybe even not achievable by every high school student (Klafki, 1984).
The existing empirical research literature focuses strongly on meta-scientific reflection regarding natural sciences (e.g., Oschatz et al., 2018) or just scientific thinking and working in general (e.g., Dettmers et al., 2010). To the best of our knowledge, there are no validated instruments that measure meta-scientific reflection on mathematics. Literature addressing close-related concepts such as epistemological knowledge about mathematics can be found (e.g., Hoffmann & Even, 2021; Zazkis & Leikin, 2010). These projects investigated knowledge rather than reflection and mostly focus on teachers and not on high school graduates. Contrary to Rott and Leuders (2017), we are not interested in students' beliefs concerning mathematics, e.g., if mathematical knowledge is certain or not, but more general what mathematics look like. Our study on meta-scientific reflection of undergraduate students addresses this research gap.

Presenting the whole nature of mathematics, is not possible due to page restrictions. Thus, we use the following statement to illustrate the dual nature of mathematics:

> […] mathematics can be best understood as a framework for studying concrete real-world phenomena in terms of underlying abstract mathematical models. (Hansen, 2008, p. 1).

Hansen (2008) differentiates between an abstract and a concrete site of mathematics. Whereas abstract mathematics focuses on the analysis of formal structures and abstract ideas, concrete mathematics is related to the application of the abstract patterns and structures in real-life situations.

To further characterize scientific disciplines like mathematics, it is sensible to look at the various disciplines from different perspectives, namely in regard to (1) its contents (objects, structures and theories), (2) its methods and processes and (3) its purpose and goals etc. (Niss, 2014). These three aspects can be used to distinguish scientific disciplines from others.

The study in this contribution is a follow-up study of a study in which we investigated reasoning patterns that students used to argue whether mathematics is a natural science or not (Fesser & Rach, 2020). The findings indicated that more than half of the participating students think that mathematics is a natural science. Students' reasons for their decision referred to nine subthemes that can be grouped into three themes inspired by Niss (2014): (1) the content of mathematics, dealing with the question: What does mathematics contain? (2) the method of mathematical studies, which deals with the question: How are mathematical findings obtained? (3) the purpose of mathematics, dealing with the question: What are the goals of mathematics? (see Figure 1).

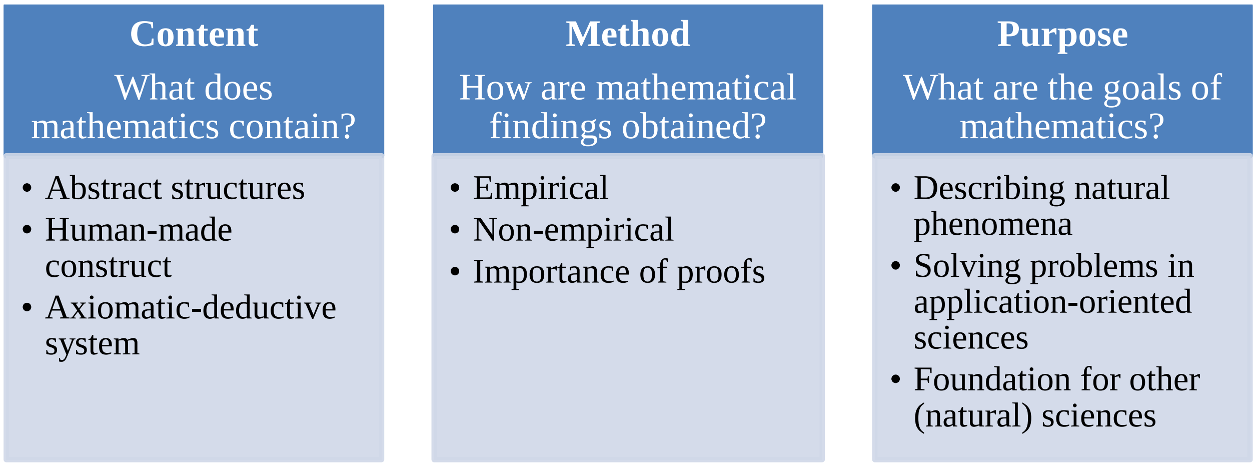

Figure 1: Framework for students' reflection on the nature of mathematics.

In the previous study (Fesser & Rach, 2020), we identified which themes students mentioned when reflecting on the question if mathematics is a natural science. Now, we analyse which themes and subthemes students use to agree or disagree to this statement to gain a deeper insight into students' meta-scientific reflection.

RESEARCH QUESTIONS

In the current study, we focus on undergraduate students who were enrolled in a Bachelor's Degree Program in which students have to participate in mathematics courses. Our aim is to investigate whether undergraduate students are able to reflect on the nature of mathematics. The research questions for this study are as follows:

- In which way do undergraduate students categorize mathematics as a scientific discipline?
- What arguments do undergraduate students use to support their categorization?

METHODS

Sample

The study was conducted at a middle-large university in Germany at the beginning of the winter term 2020/21. The target group of this study were first-year students who were enrolled in Bachelor's Degree Programs that are somehow related to mathematics (e.g., STEM subjects, economics, …). For this study, we focused on first-year students because we want to learn more about meta-scientific reflection of undergraduate students. The sample was collected via convenience sampling and participating was voluntary and anonymous. Out of 313 students who participated in the study, 296 participants (94.6%) answered the open-ended items. Those 296 participants (56.1% female, 82.4% younger than 22 years) were enrolled in the following study programs:

49.7% in economics-related, 21.3% in teacher training, 14.2% in technical-related, 7.4% in computer science, 4.1% in mathematics, and 3.3% in other study programs.

Data collection and data analysis

Due to the pandemic situation and other pragmatic reasons, we conducted the study via the online survey tool *SoSci Survey*. The main data source included answers to two open-ended items, which were implemented in a questionnaire. The two items to collect data about students' meta-scientific reflection were given in German and can be translated as follows:

> The term natural sciences includes e.g. sciences like biology, chemistry, and physics. Take position on the following statement "Mathematics is a natural science.".
>
> - What is in your opinion the strongest argument that (a) supports and (b) contradicts the given statement?
> - Compare both arguments and come to a decision whether mathematics is a natural science or not. Explain your answer.

At the beginning of this questionnaire, participants were instructed to think of mathematics as a scientific discipline. The participating students were asked to answer the questions within 2-4 whole sentences. In this contribution, we focus our analysis on the second item.

The qualitative data was gathered and then analysed, applying the summarizing qualitative content analysis (Mayring, 2015). In the first step (initial read-through), we read all the given answers and gained an overview about the whole data set. After that, we marked all sentences that dealt with a categorization of mathematics as a scientific discipline and possible arguments supporting ones positioning. Then we used the framework in Figure 1 and a peer-validated category system (Fesser & Rach, 2020) for the coding of the answers, enabling room for new categories if needed. Thus, the analysis was deductive and inductive, gathering also insight about adjacent themes.

FINDINGS

Students' answers on the categorization of mathematics as a scientific discipline and the related arguments to support the positions were associated with the three themes and the nine subthemes. A further analysis indicated that no more themes and subthemes emerged from the data. Firstly, we describe students' categorization of mathematics as a scientific discipline and secondly the used argumentation patterns. Thirdly, we give an insight about further findings and a new perspective mentioned by the participants.

Categorization of mathematics as a scientific discipline

The first research question deals with undergraduate students' categorization of mathematics as a scientific discipline: 137 participants categorized mathematics as a natural science (46.3%), 140 disagreed with the statement "Mathematics is a natural science" (47.3%) and 19 has not clearly positioned themselves (6.4%). Therefore, we have a rather equal distribution of students who agree and students who disagree with the statement that mathematics is a natural science (see Figure 2).

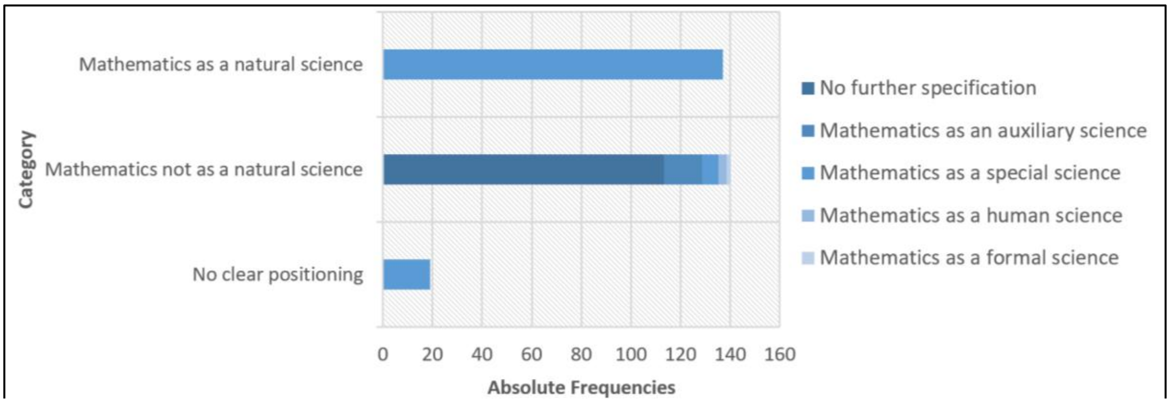

Figure 2: Students' categorization of mathematics as a scientific discipline.

Out of the participants who disagreed with the statement that mathematics is a natural science, 34 students took a further approach to specify mathematics as a scientific discipline. The students labelled mathematics mostly as an "auxiliary science" (55.9%), followed by "special science" (23.5%), "human science" (11.8%) and "formal science" (5.9%). One student who disagreed with the statement that mathematics is a natural science, also stated that mathematics is not a scientific discipline at all:

> As mathematics is a tool for explaining natural phenomena, I would say that mathematics is not a scientific discipline, rather than just a tool for natural sciences so that they [natural sciences] can keep working and researching (A170_2).

Argumentation patterns

Besides the categorization of mathematics itself, we also analysed the arguments that students used to support their categorization. 222 students gave a reason for their positioning. Out of those 222 participants, 195 students (87.8%) gave a sensible answer that we could further analyse. Whereas each of the three themes was addressed in the answers, the number of mentions varied substantially between and within the themes.

Content of mathematics

48 (24.6%) out of the total 195 students are associated with this theme, dealing with the questions: What does mathematics contain? Most of the students within this theme referred to abstract structures as the specific research objects of mathematics (56.3%), for example: "The research objects of mathematics are not natural phenomena, but the structure of formal objects" (A29_2). Fewer students referred to mathematics as a mental construct that is made by humans (29.2%); respectively mentioning that mathematics is characterized by its logical axiomatic-deductive structure (14.6%). Out of the 48 students that mentioned this theme, only one student supported the statement that mathematics is a natural science. It seems like referring to this theme relates to disagreeing with the statement that mathematics is a natural science.

Method of mathematics

The second theme could be emerged from 17 (8.7 %) students' answers, dealing with the question: How are mathematical findings obtained? Similar to the first theme only one out of the 17 students argued that mathematics is a natural science. Most of the students within this theme associated with mathematics being a non-empirical science (76.5%). These students often referred to differences between mathematics and the natural sciences, e.g., as mathematics is not an experience-based science, it does not generate findings based on experiments and observations. Further "it is not common to collect empirical data" (A85_2) for doing mathematics. Three students focused on the importance of proofs as the main criteria of evidence in mathematical research: Mathematics is characterized by "proving specific logical statements and theorems" (A91_2). Students use this subtheme to make clear that mathematical findings has to be proven prior to be used by other sciences.

Purpose of mathematics

Most of the students associated with the third theme (64.6%). Within this theme, we could find the following distribution to the subthemes: 62.2% mathematics as a foundation for the natural sciences, 26.0% mathematics as a tool for describing natural phenomena, and 11.8% mathematics for solving problems in scientific or daily life situations. Concerning the first subtheme, students reported that mathematics is a key component of the scientific working in the natural sciences. Mathematical methods are needed to predict developments and to represent natural phenomena via various diagrams. The first subtheme "mathematics as a foundation" was used to support (50.6%) and to oppose the statement that mathematics is a natural science (49.4%). On the one hand, students argued that mathematics is a foundation and therefore a part of the system of natural sciences and on the other hand, students argued that as mathematics is a foundation for natural sciences, it cannot be a natural science itself. That means that this subtheme is evenly distributed among the positions.

Further findings

Three students' answers could not be associated with any of the formulated themes. Those answers had in common that they are referring to their school experiences with mathematics, for example:

> I think mathematics is a natural science because that it is how I was taught in school. For me, that is like a fact. (A221_2).

Those argumentation patterns does not give any hint of a reflecting process, but are only based on prior experiences and knowledge (knowing that mathematics is a natural science) that was accumulated in school. Therefore, we does not expand our category system (see figure 1) with a new category or theme.

Apart from the findings above, we could also generate some interesting results concerning students' opinions and meta-scientific reflection on mathematics: Some students reported that it was their first time thinking about the place of mathematics in the wide system of sciences. One student even reported that reflecting about this question changed her/his former view on mathematics as a scientific discipline:

> For me, mathematics belonged to the natural sciences because that is how it was always portrayed to me. However, based on the arguments I found, I would say that mathematics is not a natural science but a scientific discipline that revolves around logic (A221_2).

We did not expect that students in higher secondary schools were taught that mathematics is a natural science. Likewise, we were surprised about students reporting that they had not reflected on the nature of mathematics before.

DISCUSSION

The conceptual framework based on Niss (2014) and the developed category system in a prior study (Fesser & Rach, 2020), was useful to examine how undergraduate students categorize mathematics as a scientific discipline and what arguments they use to support their decision. The analysis of students' answers showed that students mentioned all nine subthemes. For arguing that mathematics is a natural science, students referred to the purpose of mathematics: arguing against, they used the content and the method of mathematics. Thus, the purpose of mathematics as an abstract discipline seems not to be clear to students and therefore should be more explicated in mathematical classes to gain a more holistic understanding of mathematics. Besides referring to the representation of mathematics as a natural science in school, no additional themes were associated by the students.

As this study was implemented in a written survey, we were limited when analysing the given word material. Therefore, we were not able to ask students to explain deeply arguments they put forward, e.g., that mathematics was taught in high school. Future projects may collect data on meta-scientific reflection in interviews to get an insight about the quality of argumentations as to which Rott et al. (2014) provided results.

Our findings suggest that most of the students are able to categorize mathematics as a scientific discipline giving reasons. Even though most students gave reasons, the referred themes differ between students and many students ignore the abstract site of mathematics (Hansen, 2008). To support student in reflection, it seems not to be sufficient using reflection prompts only in surveys, but it has to be implemented in regular lessons (see Liebendörfer & Schukaljow, 2020). To gain a deeper insight into meta-scientific reflection, more research is needed to consider (1) whether the categorization is related to students' characteristics, (2) understand the differences between students' reasoning and (3) investigate whether mathematics is portrayed as a natural science in school or not.

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CONTRIBUTION OF FLEXIBILITY IN DEALING WITH MATHEMATICAL SITUATIONS TO WORD-PROBLEM SOLVING BEYOND ESTABLISHED PREDICTORS

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To solve mathematical word problems, students need to build appropriate models of the described situations, which they can describe with mathematical operations. Various studies have confirmed the importance of general cognitive skills, basic arithmetic skills, and language skills for word-problem solving. Beyond these, we investigate flexibility in dealing with mathematical situations, a new construct that describes the skill to re-interpret everyday situations from various perspectives. In a study with N = 113 second graders, an instrument to measure this flexibility construct has been developed and investigated. We find that the construct explains word-problem solving skills beyond the established predictors. Being able to flexibly re-interpret everyday situations may be beneficial for word-problem solving.

Students' skills to solve word problems diverge strongly. It has been well investigated particularly for additive one-step word problems, which predictors explain these differences. Additive one-step word problems are mathematical problems embedded in a verbally described everyday situation that can be solved with a single arithmetic operation (addition or subtraction) and do not contain irrelevant information (Verschaffel & De Corte, 1997). Recently, a new skill construct, *flexibility in dealing with mathematical situations*, has been proposed to support learning regarding additive (one-step) word problems (e.g., Gabler & Ufer, 2021). However, the role of this skill among other well-established predictors is unclear yet. This paper aims to fill this gap.

CURRENT STATE OF RESEARCH

Solving additive one-step word problems

Common theories on word-problem solving (e.g., Kintsch & Greeno, 1985) assume that learners construct two models when solving word problems: a situation model and a mathematical model. When learners encounter the text base (the verbal description of the mathematical situation), they construct a situation model based on this information. The situation model is the learner's internal, mental presentation of the given situation (Czocher, 2018). Learners then connect their situation model to mathematical concepts and transform it into a mathematical model. In the context of additive one-step word problems, students rely on conceptual knowledge on addition and subtraction, which needs to be available and activated to find an adequate mathematical model. For example, some word problems may refer to subtraction as the idea of "taking something away", while others may relate to a difference between two sets, making a connection to subtraction less salient. In literature, these different situations connected

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to addition and subtraction have often been classified into four different types ("semantic structures"; Riley, Greeno, & Heller, 1983): change, combination, comparison, or equalization of sets. Once the mathematical model is successfully constructed, learners can proceed with solving the word problem.

Individual predictors for solving additive one-step word problems

During this solution process, a number of individual predictors influence the students' performance when solving word problems (Daroczy et al., 2015).

For example, domain-general skills are discussed to predict students' word-problem solving skills. Solving word problems successfully depends on *general cognitive skills* (e.g., Jõgi & Kikas, 2016; Renkl & Stern, 1994), which may help with handling new, unfamiliar challenges (Warner et al., 2003). It is assumed that other domain-specific skills mediate the effects of general cognitive skills at least to some extent (Zheng, Swanson, & Marcoulides, 2011).

In addition, students' *language skills* play a role in word-problem solving (Daroczy et al., 2015). In particular, reading comprehension skills are considered crucial to decode the text base and derive an accurate situation model from this text (Vilenius-Tuohimaa, Aunola, & Nurmi, 2008). Indeed, studies have repeatedly identified reading comprehension skills as significant predictors of word-problem solving skills (e.g., Beal, Adams, & Cohen, 2010; Muth, 1984; Vilenius-Tuohimaa et al., 2008).

Besides such domain-general skills, students also need certain subject-specific, *basic arithmetic skills* for word-problem solving (Daroczy et al., 2015). In the context of additive one-step word problems, not only technical skills to solve additive equations are considered necessary, but also knowledge on number concepts (e.g., part-whole relationships, addition and subtraction as complementary operations; Renkl & Stern, 1994). This was confirmed by several studies, which report higher word-problem solving skills for students with higher basic arithmetic skills (e.g., Bjork & Bowyer-Crane, 2013 for grade 2; Muth, 1984 for grade 6).

Beyond these well-established predictors, it may play a role for students during word-problem solving, if they can deal flexibly with the given mathematical situation. This idea has first been suggested in the eighties and nineties (e.g., by Greeno, 1980; Stern, 1993) and conceptualized as a new skill construct within this project. Some learners struggle with constructing and mathematizing their situation model. In this case, it may help them to be able to add alternative perspectives to their situation model and further, to find mathematical operations that describe their situation model. In this sense, *flexibility in dealing with mathematical situations (FDMS)* can be defined as the skill to enrich their individual situation models of additive one-step word problems with further information, which is not verbalized in the text base. For example, learners could reinterpret compare problems as equalize problems: Additionally to the given description (e.g., "Susi has 2 marbles less than Max."), learners could imagine an equalization of Max's set: "If Max gets 2 more marbles, he has as many marbles as

Susi has.” (similarly suggested by Greeno, 1980). Another idea is to change the perspective on the situation: Instead of Susi’s perspective on the relation (“Susi has 2 marbles less than Max.”), learners could also add the perspective of Max: “Max has 2 marbles less than Susi.” (as suggested by Stern, 1993). Learners could integrate these different descriptions of the situation into a network of linked perspectives (Scheibling-Sève, Pasquinelli, & Sander, 2020). One basic assumption of this idea is that this skill complements the learners’ conceptual knowledge in word-problem solving. Learners with a high FDMS could then draw on the perspective that seems most helpful for them to find an adequate mathematical operation.

The suggested construct may be connected with other predictors. Handling new, unfamiliar challenges such as having to re-interpret a word problem (Warner et al., 2003) seems to be connected with general cognitive skills. Imagining different descriptions of mathematical situations is likely to be influenced by language skills and conceptual arithmetic knowledge. It is an open question, if FDMS can be operationalized and measured, and if this construct contributes to word-problem solving skills beyond the other mentioned predictors.

AIMS AND RESEARCH QUESTIONS

Although the idea behind FDMS has been suggested quite early, it has only recently been proposed as a skill construct. We investigated the following research questions:

RQ1: Is it possible to measure FDMS with sufficient reliability?

RQ2: How do general cognitive skills, basic arithmetic skills, and language skills explain inter-individual differences regarding FDMS?

RQ3: How does FDMS explain inter-individual differences in word-problem solving skills beyond general cognitive skills, basic arithmetic skills, and language skills?

Based on prior research, we expected general cognitive skills, basic arithmetic skills, and language skills to predict word-problem solving skills. Due to the reported theoretical foundations, we assumed FDMS to have a direct effect on word-problem solving skills beyond the other predictors.

METHOD

To answer the research questions, paper-and-pencil based tests were used in a cross-sectional study with second graders from ten classrooms in Germany ($N = 113$, 56 female, 57 male). The average age of the participating students was 7.7 years. There were 47% of students with German as their only family language, 19% with only non-German family language(s), and 34% of students with mixed family languages (at least German and another language). The study spans over two measurement times, between 6 and 21 days apart. On the first day, we measured the students’ language skills, their general cognitive skills, and their basic arithmetic skills. On the second day, we collected data on the students’ word-problem solving skills and their FDMS.

Instruments

Language skills were measured using the ELFE II reading comprehension test (Lenhard & Schneider, 2018). This test provides the opportunity to assess language skills based on reading fluency and accuracy with a larger sample. On average, the students achieved M = 45.03 raw points out of 111 total points with a standard deviation of SD = 15.5, which is in line with the average performance of the norm sample of the test. The reliability was excellent (α = .97).

General cognitive skills were measured by using the subscales "Similarities", "Classifications", and "Matrices" of the Culture Fair Intelligence Test "CFT 1-R" (Weiß & Osterland, 2013), which measure characteristics of general cognitive skills in a culturally fair, language-free setting. The reliability of the three subscales was acceptable (subscale "Similarities": α = .66; "Classifications": α = .73; "Matrices": α = .80). The three subscales were combined into one joint indicator. On average, the students scored M = 30.41 points out of 45 total points, with a standard deviation of SD = 5.98.

Basic arithmetic skills were measured with a test, which was developed for third graders within the LaMa project (Bochnik, 2017) and adapted for second graders in this study. Some of the tasks relate to technical skills in adding and subtracting numbers ranging until 100. Further tasks required conceptual knowledge, for example on the relationship between addition and subtraction (e.g., by asking for all four calculations that can be conducted with the numbers 7, 8, and 15). The reliability is satisfying (α = .82). On average, the students scored M = 7.49 points out of 16 total points with a standard deviation of SD = 3.80.

Word-problem solving skills were measured with a newly developed test ("word problem test"). This test was implemented in a multi-matrix-design: learners solved ten different word problems from a pool of 20 word problems based on the work of Stern (1993). The tasks systematically varied typical features (e.g., semantic structure), so that the whole range of possible types of additive one-step word problems was covered. The arithmetic and linguistic complexity of all 20 items was at a similar level. The data were scaled with a one-dimensional Rasch model. The WLE reliability of the instrument was .68. The average item difficulty was -1.04, indicating a relatively low difficulty of the test instrument.

Flexibility in dealing with mathematical situations (FDMS) was also measured with a test, which was newly developed within this project ("flexibility test"). The 20 items measuring FDMS were embedded into a story about twins, who tell the learners about a birthday party they visited. The learners were asked to decide, if the statements of the twins are equivalent or not (see Figure 1). The items emphasize different perspectives on mathematical situations in line with the ideas of Greeno (1980) and Stern (1993). For example, learners contrasted different perspectives on relations (as in Figure 1) or on actions (e.g., "Ben gave Alma 4 cards." vs. "Alma got 4 cards from Ben."). There were also items, in which two different semantic structures were contrasted (e.g., comparison: “There are 3 children more than adults at the party.” vs. equalization: “If 3 children leave, there are as many children as adults at the party.”). This facilitates the assessment of situational understanding and the skill to deal flexibly with such mathematical situations without the need to conduct mathematical operations.

Figure 1: Sample item for measuring FDMS

Statistical analyses

To answer the research questions, we estimated linear mixed models, taking into account that students were nested in classrooms. We calculated two models with the word-problem solving test score as a dependent variable, one with all independent variables (general cognitive skills, language skills, basic arithmetic skills, and FDMS) and one without FDMS for comparison. We also calculated a model with FDMS as a dependent variable to disentangle, which variables predict FDMS.

RESULTS

RQ1: One question was, if FDMS could be measured with sufficient reliability. The reliability of the flexibility test was satisfying ($\alpha = .80$). The participants scored $M = 14.04$ points on average out of 20 total points, with a standard deviation of $SD = 4.12$, showing that the test instrument was relatively easy.

RQ2: All three predictors significantly predicted FDMS (general cognitive skills: $F(106.36, 1) = 4.24$, $p = .042$, $\eta_p^2 = .04$; language skills: $F(108.97, 1) = 7.53$, $p = .007$, $\eta_p^2 = .06$; arithmetic basic skills: $F(107.74, 1) = 27.56$, $p < .001$, $\eta_p^2 = .20$). About 5.6% of the variance that was not explained by the predictors was attributable to class membership.

RQ3: As expected, language skills, basic arithmetic skills, and FDMS were significant predictors of word-problem solving skills (see Table 1). However, general cognitive skills were not significantly predictive beyond the other predictors. When including FDMS into the model, marginal R-square values increased substantially (without FDMS: marginal $R^2 = .38$; with FDMS: marginal $R^2 = .45$), and indeed, FDMS contributed significantly to variance explanation (see Table 1) with a medium to large effect size. This indicates that FDMS may explain differences in word-problem solving skills beyond the other variables. About 3.1% of the variance that was not explained by the predictors was attributable to class membership. Effect sizes for language skills and, in particular, for basic arithmetic skills reduced substantially when including FDMS, indicating that FDMS might mediate their effects on word-problem solving skills.

	Model 1			Model 2		
	F-value	p-value	η_p^2	F-value	p-value	η_p^2
General cognitive skills	0.63	.430	.00	0.00	.990	.00
Language skills	11.19	.001**	.10	6.21	.014*	.05
Basic arithmetic skills	21.83	<.001***	.19	7.99	.006**	.08
FDMS				14.67	<.001***	.12
Marginal R^2	.38			.45		

Table 1: ANOVAs based on linear mixed models with and without FDMS (*: $p < .05$; **: $p < .01$; ***: $p < .001$)

DISCUSSION

The main goal of this contribution was to investigate, if FDMS can be measured reliably, how it relates to established predictors of word-problem solving skills, and whether it contributes to variance explanation in word-problem solving skills beyond these established predictors (Daroczy et al., 2015).

Although the instrument turned out to be quite easy for second graders, it captured inter-individual differences in FDMS reliably. Currently, the instrument only assesses receptive FDMS by comparing *given* descriptions of situations. It would be important to include productive FDMS as well, for example, by asking students to construct alternative (written) descriptions of a mathematical situation (Gabler & Ufer, 2021). This would come closer to what we assume is required during word-problem solving.

Inter-individual differences in FDMS were related to all three established predictors. Beyond general cognitive skills, language skills contributed to variance explanation, which reflects the close connection of FDMS to language skills (e.g., Prediger & Zindel, 2017). The largest contribution came, however, from basic arithmetic skills. This is particularly remarkable, since the flexibility test does not address any arithmetic calculations, and instead focuses on situational understanding. This relation might go back to the part of the arithmetic test that covered conceptual understanding of addition and subtraction. Thus, developing FDMS could be connected closely to developing conceptual understanding of arithmetic operations in classroom practice, for example, by not only using situations from everyday contexts or manipulatives to reflect on mathematical structures, but also to compare and contrast different perspectives and verbal descriptions of these situations.

Regarding inter-individual differences in word-problem solving skills, general cognitive skills did not contribute under control of language skills and basic arithmetic skills. This contradicts some prior findings (e.g., Jõgi & Kikas, 2016). Possibly, the effect of general cognitive skills is fully mediated by the other variables. Replicating results from prior research (e.g., Bjork & Bowyer-Crane, 2013; Vilenius-Tuohimaa et al., 2008), language skills as well as basic arithmetic skills predicted word-problem solving skills. As expected, FDMS contributed to variance explanation beyond the other predictors. These results indicate that being able to re-interpret situations flexibly may support students' word-problem solving processes (e.g., Kintsch & Greeno, 1985) by allowing them to consider alternative perspectives on the described situation, which might be easier to mathematize. This means that the new construct has explanatory power for inter-individual differences beyond existing constructs. Supporting students to develop FDMS might be a way to support their word-problem solving skills.

Although the results on the new construct are promising, further research will have to clarify, if and how it can be fostered, and if this has effects on students' word-problem solving skills. Moreover, future research will need to consider how the construct can be conceptualized beyond additive situations, for example in the light of multiplicative situations, possibly including proportional relations.

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EXEMPLIFYING AS DISCURSIVE ACTIVITY

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A mathematics teaching framework (MTF) tool is brought under a communicational discursive lens to capture the discursive activities that emerge when mathematics preservice teachers talk about examples, a key element of the MTF, in a lesson study (LS). Through the analysis of the LS participants' discussions as they planned the lesson to be taught— their reflective discourse— I show how three discursive activities emerge: mathematizing (talk about mathematics objects), MTFying (talk about MTF objects) and teaching (talk about teaching practices). I further evidence how these three discursive activities are intertwined. Key to these findings is the important role played by the knowledgeable other in the emergence of these discursive activities.

INTRODUCTION

Examples have always played a central role in mathematics learning and teaching (Bills et al., 2006). Adler and Venkat (2014) noted that the examples used in a lesson provide an analytical window into what is made available to learn through the design of teaching activities; hence, the value in studying their use in teaching. Essien (2021) has also argued for the significance of examples in preservice teacher learning programs, and for mathematics teacher education research to pay attention to how preservice teachers can be enculturated into the practice of exemplifying as they learn to teach. In this paper I draw from a wider study on teacher learning through participation in a lesson study (LS) where a designed mathematics teaching framework was a resource guiding their collaborative activity. I present the lens developed for conceptualizing and analyzing mathematics lesson study as discursive activity, and demonstrate how the teachers and the knowledgeable other (KO) talk about examples. This particular LS setting highlights the strength of the discursive approach in illuminating how the MTF was used discursively where the KO and the teachers are shown to contribute to the exemplifying discursive activity that is constituted in the LS reflective discourse.

EXAMPLES FROM MTF PERSPECTIVE

Adler and Ronda (2015) developed an analytic framework for describing and working on teaching, naming it MDI – Mathematics Discourse in Instruction. Emerging as this did in a research and teacher development project, the MDI was redescribed in a teacher-friendly form, named a Mathematics Teaching Framework (the MTF). MDI/MTF is underpinned by a view of teaching as always about something—a specific object of learning (OoL), and it is the teacher's role to bring about its learning by students. The MTF foregrounds three teaching practices—exemplification; explanatory communication and learner participation—that combine to enable

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mathematical learning (e.g. Adler, 2021). From the MTF perspective, examples are seen as a key resource for teachers to introduce and develop concepts. Following aspects of Variation Theory (Marton & Pang, 2006), in any lesson, examples, together with accompanying representations, and the tasks in which they are embedded can be selected, structured (and then mediated) to enable critical aspects of the OoL to come into focus and thereby provide learners with opportunities to learn specific mathematical concepts and capabilities. In a similar vein, Leinhardt (2001, p. 347) has argued "for learning to occur, several examples are needed, not just one; the examples need to encapsulate a range of critical features and examples need to be unpacked, with the features that make them an example clearly identified". The notion of critical aspects that I have used aligns with Leinhardt (2001) — and I focus in on mathematical routines (defined below) - together with the performer of those routines as the means to identify exemplifying in discursive form.

THEORETICAL FRAMEWORK

Although the MTF provides the field with what are the valued mathematics teaching practices to support mathematics teachers' work, little attention is paid to discursive acts of teaching (Mosvold, 2016). In the wider study from which this paper is drawn, the MTF was embedded in an overarching discursive framework as developed in Sfard's (2008) communicational theory. Sfard argued that communication is at the heart of her approach, based on the view that communication and thinking are aspects of a single entity termed discourse.

Discourses are types of communication common to particular communities. They are identifiable through four interrelated characteristic features: keywords, visual mediators, distinctive routines, and generally endorsed narratives. Most commognitive research to date has focused on mathematical discourses; however in this study I was interested in the discourse of teaching mathematics. From this perspective, teaching is defined as a communicative activity whose purpose is to bring students' discourse closer to a canonical discourse (Tabach & Nachlieli, 2016). The mathematics teaching discourse makes use of keywords, mediators, routines and narratives of mathematics, but also of teaching mathematics, much in the same way as MTF consists of mathematical and teaching practices. Thus, the MTF may be redefined as a discourse—MTF discourse. This particular discourse is characterised by names used to describe MTF objects such as examples, tasks, explanations, learner participation and OoL. The routine— repetitive patterns characteristic of this MTF discourse—included exemplifying, generating tasks, learner explaining, teacher explaining and teacher critical aspects of OoL (CA of OoL). This particular discourse used similar visual mediators as those found in mathematical discourse such as graphs and equations. Furthermore, this discourse is characterised by different narratives such as MTF narratives (talk about MTF objects and MTF routines), mathematical narratives (talk about mathematical objects and routines) and pedagogical narratives (talk about pedagogical routines).

Within communicational theory, the process of learning involves participating in discursive activities. The main discursive activity found in mathematics discourse is called mathematising—talking about mathematical objects (Sfard; 2008; Heyd-Metzuyanim, 2017). Drawing from this idea, talking about MTF objects can be viewed as a discursive activity which I have called "MTFying". Building on Mosvold (2016), participating in MTFying discursive activity refers to engaging in discursive acts of exemplifying, constructing learner tasks, explaining and inviting leaner participation with a goal of foregrounding what leaners need to know. To demonstrate this point, the empirical part of this paper analyses the discursive activities that emerge when the teachers use the MTF tool. I zoom in on the discursive act of exemplifying. I thus set out to answer the question: *How, do teachers talk about examples*?

METHOD

The study reported here involved preservice mathematics teachers participating in a LS that was designed for the study (Gcasamba, 2022). Data was collected through reflective discussions of four preservice teachers and the KO as they participated in LS sessions (lesson planning, enactments of the lesson and lesson reflection discussions). The object of learning teachers selected for the study related to the effects of y-intercept and the gradient of the linear function. A methodological approach based on Sfard's (2008) routine-use was used to analyze LS observation videos of the LS group—the four teachers and the KO. I report on collaborative engagements that took place during the first lesson planning session (LP), and exemplify with an extract from the LP transcript. The LP discourse is ideally a combination of teachers' and KO's discourse. However, the data from the LP observation showed a dominance of the KO in the LP discourse. This is not surprising since the KO was playing a role of introducing teachers to the MTF tool and its use during LP session, bringing knowledge of how the MTF tool may be used. As a way of concluding, I will comment on how the KO could be seen to be positioning herself as the leading interlocutor, and the implications of such a role in affording teachers opportunities to participate in the new discourse. The analysis began with a process of identifying instances which denoted participants' routine-use where I paid attention to what objects were explicitly named (e.g. mathematical objects—linear function and MTF objects—examples, tasks etc.); procedures (e.g. mathematical , MTF and teaching routines) and the intended performer of the procedures (e.g., teachers or learners). For example, in line 111, in the extract below, KO was talking about two categories of objects (as bolded) such as the MTF objects (example and tasks) and the mathematical objects of a linear function (gradient and intercept) and attended to mathematical routines (calculating). The focus on both the procedure and the performer enabled me to identify the MTF routines. For example in line 111, the phrase: ("we also want them to calculate"), the KO reflected about what the learners (as highlighted by the keyword "them" referring to the learners) were expected to do in relation to the mathematical routine of calculating the gradient. The intended focus in this utterance was on reflecting about the mathematical activity learners were expected to perform; hence this was identified as the MTF routine of constructing learner tasks. The exemplifying routines were identified when a particular mathematical routine was attached to the teacher as the performer and with intentions to illustrate a specific aspect of a mathematical object. For example in line 212, the KO, reflected on what teachers may do (performer) in relation to the mathematical routine of interpreting through a tabular representation in order to illustrate the effects of gradient (m) and y-intercept (c).

In the following extract I mark the objects in focus (**bolded**), the routine procedures (*in italics*), and mathematical representations (underlined). I highlight the performer (or intended performer) of mathematical procedures in grey. (M = teacher; T = teacher; KO = knowledgeable other).

109 KO: We know that some of the problems have something to do with *calculations*, so it means some of the **examples** must focus on *calculation*.

110 KO: So our **example space** is guided by errors, content analysis and also our previous knowledge.

111 KO: So what we are saying is that we need **examples** where they are *calculating*, and we also want them to *calculate* maybe **gradient or intercept**, these could be **tasks**.

112 KO: Other **examples** could be dealing with *translation*, i.e. *moving from graph to* table to equation.

113 KO: In our planning we need **examples** for *classification*.

208 KO: So let us start with our implementation. In our implementation we need **examples y equals two x** ($y = 2x$) I am thinking of kind of **explanations**

209 T: Let us change **the signs** (*writing* $y = -2x$)

210 KO: The e**xplanations** are going to be coming directly from the **examples.**

211 KO: We can start off the **explanations** by saying, "Now look at the value of a **gradient is 2**", so we would like *to illustrate* with more **example**s, with more explanations.

212 KO: For illustrations we can use tables. Will the **table** show any **effect on m and c**?

213 M: Not really, but the *drawing* a graph will show because **table and the equation** is more or less the same

214 KO: The table is going to help us to be able to *draw* a graph. It is part of our explanation.

215 KO: In a table we will be able *to see* when we are changing **the sign of your**

gradient.

216 KO: It is not that we will be wasting time with our *t*able; we can still use it to *draw* the graph. *So let's draw* it.

217 ALL: Yes

The analysis of this extract showed that the examples were talked about in two different ways. They were named explicitly; and they were linked to both the mathematical routines and the performer of those routines. I now discuss each of these.

Discursive act of explicit naming

As indicated in bold type, examples were explicitly named, and talk related to examples was linked to both mathematical and pedagogical routines. For example, in lines 109, 111 and 112, the explicit naming of the examples was accompanied by the mathematical actions related to "calculations", "translations" and "classifications". The explicit naming of examples was also used to bring to attention the pedagogical strategy to be used to construct the MTF narratives. For example, in line 110 the KO utterance suggested that the construction of examples was guided by the process of identifying CA (Lo, 2012). Further analysis of this extract showed that the explicit naming of examples was from the KO utterances, which suggests the KO's intention to foreground the MTF tool and its objects.

Discursive act of exemplifying

Further analysis of the extract above shows that the examples of linear functions were offered in different representations. For example, in the series of interactions from lines 208-217, the KO first highlighted an example of $y = 2x$ in a symbolic form. This was followed by T's proposal in line 209 where he suggested changing the sign of the coefficient of x which resulted in yet another example now of the linear function $y = -2x$. Further representational forms of the examples were proposed. For example, in line 212, the KO suggested an example in the form of a tabular representation, followed by M in line 213 who further suggested an example in the form of a graphical representation. This whole set of representations implied a set of different examples of linear functions aimed at helping the teachers with illustrations intended at providing substantiations about the effect of the gradient in the linear function. The different representations further implied different mathematical routines. For example, the graphical and tabular representation implied either constructing or interpreting routine, and the symbolic implied either calculating or interpreting routine, depending on the goal of the mathematical activity.

What needs to be noted is that the exemplifying talk across all participants' utterances evidenced that the examples were aimed for teachers' use. Each of these utterances highlighted the mathematical procedure together with the performer (the teacher, in this instance) for illustrative purposes. The combination of these two elements signified an exemplifying routine. Furthermore, this type of interwoven talk about mathematical procedure and performer as the teacher for illustrative purposes could be important for the identification of exemplifying routines. This finding has important methodological implications for developing an alternative lens for analysing MTF objects in discursive form without relying only on explicit naming of MTF objects. In other words, it provides an operational definition of MTF objects as discursive objects.

This analysis further showed that the exemplifying routines were connected to other MTF routines such as teacher explaining routines. It seems the teachers used exemplifying routines as teachers' resources to illustrate a specific idea about mathematical objects, as evidenced in lines 210-212, where the KO discussed how a chosen example $y = 2x$ could be used in the form of a tabular representation for illustrative purposes to enhance explanatory talk about the effects of the gradient and the y-intercept. This was further well evidenced in line 214. This finding agrees with Zaslavsky (2010) who claimed that examples carry a potential exploratory power, and that they carry important elements of explanations through illustrative representations and demonstrations of discursive activities. It is interesting to note that although the MTF components were separated analytically, they were intertwined in the discursive talk. The implication from this current study is that it highlights the blurred nature of boundaries between discursive acts of teaching related to MTF.

The analysis has also revealed that examples were associated with CA of OoL routines. For example, in line 109 where the KO remarked about the problems related to calculating that learner might experience. This suggested that the calculating routine was a critical aspect. Given the consideration of CA, KO further suggested a consideration of examples that were related to calculating (in line 111). The established association between CA of OoL and examples runs parallel with Leinhardt's (2001) report which showed that examples encapsulate a range of critical aspects. She further noted that such an association is a positive contributing factor to learning. It is worth noting that, after identifying the critical aspects, these were integrated into a set of examples. This finding seems to provide evidence of the connection between CA of OoL and examples.

The analysis of exemplifying routines implies that the goal of participating in this discursive activity was to construct new narratives related to examples. This was well evidenced in the analysis which showed the interwoven relationship between mathematical routines and the performer (teacher) for purposes of creating illustrations which resulted in a set of exemplifying routines.

In terms of roles played by all participants, the analysis showed that both teachers and the KO were engaging in the discursive activity of exemplifying as a collective, as seen in the interaction illustrated in the extract above. The repeated reference to keywords "we" and "our" confirmed this collective engagement. In this extract, teachers and the KO were seen to be contributing to the mathematizing actions aimed for illustrative purposes—discursive activity of construction of MTF narratives related to examples. Nevertheless, the strong authoritative voice of the KO cannot be ignored. Looking closely at these utterances, particularly from the KO, although the KO seemed to be creating an environment of collective engagement through reference to keywords “we” and “our”, her consistent reference to keywords such as “we would like”, “we can start” suggested acts of instructing what needed to be done in terms of routines. This suggests that the KO was positioning herself as the bearer for what teachers needed to do or focus on.

DISCUSSION

The analysis illustrated in the previous section highlighted the leading role played by the KO in providing point of focus and how things should be done. This was demonstrated by the use of various sources of MTF narratives. Firstly, the examples were signified by exemplifying routines which were characterised by various mathematical routines. Secondly, the analysis has further revealed that pedagogical routines such as identifying CA facilitated the construction of MTF narratives which resulted in the identification of exemplifying routines. To sum up, the established connection between examples, with different routines such as mathematical and pedagogical routines suggest that one cannot talk about the objects of the MTF without reference to doing mathematics (mathematical actions/routines) and/or teaching actions/pedagogical routines. I argue that doing so the KO would have afforded the preservice teachers an opportunity to view exemplifying as both the mathematising and teaching discursive activities. Furthermore, the analysis has revealed that, not only teachers were afforded opportunities to talk about examples, but also to link them to other MTF objects such as explanations and CA of OoL. This is an important aspect of teacher learning through exemplifying (Adler, 2021; Leinhardt, 2001).

A conclusion that can be drawn is that the communicational theory through focusing on analysis of discursive activities, provides a conceptual framework to capture various routines from different discourses (mathematical, MTF and pedagogical) that manifest themselves as the KO scaffolds the use of examples in the LS context. In this paper, I illustrated how the examples were talked about in terms of objects that were named, various routines that were used and further highlighted the participants’ roles (the KO in particular). These findings have implications about the nature of the MTF discourse with respect to ritual/explorative participation which was not the focus of this paper, hence, the limitation of this report. However, this aspect was taken into consideration in the wider study. In this paper I too have tried to model discursive practice, enriching the communicational framework with a new discursive activity such as MTFying.

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DEVELOPMENT OF ATTITUDES DURING THE TRANSITION TO UNIVERSITY MATHEMATICS – DIFFERENT FOR STUDENTS WHO DROP OUT?

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Students' attitudes are assumed to play a big role for successful learning processes and to differ substantially between students. To gain a better insight in which way attitudes at the start of a mathematics study program and their development influence study dropout, we asked 219 students to state their interest in university mathematics and their mathematical self-concept at the start of their studies and six weeks later. Applying a cluster analysis, we identified four development profiles which differ in both attitudes at the start of their studies and in the development of both attitudes. The dropout rate among students with different profiles ranged from 7 % to 44 %, highlighting that the development of attitudes in the first semester is of major importance for a successful start.

INTRODUCTION

High study dropout rates in mathematics study programs are a serious problem for individuals and for society. Noticeable is that many students drop out in the first year of study, in particular (Chen, 2013). Research assumes that beneath cognitive variables, such as prior knowledge (Rach & Heinze, 2017), motivational variables, such as attitudes, can explain why some students successfully complete their program whereas other students drop out (CHEPS, 2015; Di Martino & Gregorio, 2019).

In this contribution, we focus on the attitudes "interest" and "self-concept", which we define as follows: Interest in mathematics is a person-object-relationship (cf. Krapp, 2007) which includes a feeling-related component ("I enjoy mathematics") and a value-related component ("I value mathematics"); mathematical self-concept is the personal view of its own abilities in the domain mathematics ("I am fit in mathematics") (cf. Bong & Skaalvik, 2003). Research has shown that both attitudes at the beginning of a mathematics study program were related to study satisfaction (Bernholt et al., 2018; Kosiol et al., 2019) and to the attendance in final exams (Geisler & Rolka, 2018) which are (negative) indicators of study dropout. In addition, previous studies have documented that students' attitudes can develop during the study entry phase (e.g., Bressoud et al., 2013) and there are empirical studies which have assumed that the development of attitudes in the first semester influenced dropout (Di Martino & Gregorio, 2019) respectively have reported a relation between attitudes at the time point of dropout and actual dropout (Schiefele et al., 2007). Thus, beneath the level of attitudes at the start of one's studies also the development of attitudes in the first study

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year seems to play a big role for a successful transition in a mathematics study program.

This phenomenon could be explained by the ideas of Haak (2017). According to Haak, students are monitoring the fit between their own characteristics, such as attitudes or prior knowledge, and the characteristics of the learning environment of their study program. In case of a misfit, students can either adapt their own characteristics, for example by adjusting their attitudes, or they can leave the learning environment by dropping out or changing their study program. The latter is more likely, if they fail to adjust their own characteristics.

An initial fit between students' attitudes and the chosen mathematics study program is not self-evident, due to fundamental differences between mathematics at school and at university. Therefore, during the study entry phase students get to know a new kind of mathematics: whereas in school, new mathematical concepts are usually learned via experiences with real world objects or (counter)examples, in university formal concept definitions and rigorous proofs are used. Likewise, tasks at school often aim at applying mathematics to real world contexts and can be mostly solved via schematic calculations. In contrast, typical tasks at university involve deductive proving (Gueudet, 2008; Halverscheid & Pustelnik, 2013; Thomas & Klymschuk, 2012). With regard to these differences, researchers have argued that distinguishing interest in mathematics in school and in university mathematics helps to understand the role of interest for learning processes in the study entry phase (Liebendörfer & Hochmuth, 2013; Ufer et al., 2017). Indeed, in contrast to interest in school mathematics, interest in university mathematics strongly predicts study satisfaction (Kosiol et al., 2019). As academic self-concept is hierarchically organized (Bong & Skaalvik, 2003), it seems not necessary to split up mathematical self-concept in different facets.

To sum up, interest in university mathematics and mathematical self-concept at the beginning and its development during the study entry phase are probably important predictors for study dropout. However, it is questionable if a high level of these attitudes and a positive development are both important for being successful in a study program and if the development of interest *and* self-concept has to be positive. Instead, the positive development of one of the two variables could probably compensate the negative development of the other variable. Answering such questions calls for a person-oriented analysis, which is a well-known approach from analyses regarding learning strategies (Vanthournout et al., 2013).

RESEARCH QUESTIONS

The focus of this study is to describe students' attitudes at the beginning of the first semester, its development during the first six weeks, and the relation of students' attitudes to dropout. Precisely, we want to answer the following (exploratory) questions:

- In which way is it possible to identify different profiles of mathematical attitudes in the study entry phase?
- To what extend do students with different attitude profiles differ in their decision to drop out?

METHODS

The sample consists of 219 students in a pure mathematics bachelor program (n = 56) and a teacher education program (n = 163) at a large public German university. All students attended the same mathematics courses in the study entry phase and voluntarily participated on an informed consent in this study. Study dropout was measured at the beginning of the second year. 54 students, who were not enrolled in the program anymore, were called dropout students, the remaining 165 students were non-dropout students.

To measure interest in university mathematics and mathematical self-concept, the students rated statements on a five-point likert-scale from totally disagree (1) to totally agree (5) during the second week of the term (T1) and six weeks later (T2). The used items were taken from Kauper et al. (2009) and Ufer et al. (2017):

- Interest in university mathematics: scale of 5 items, "The kind of mathematics that is done at university is fun for me." (sample item), Cronbachs' α (T1) = .89, Cronbachs' α (T2) = .92.
- Mathematical self-concept: scale of 4 items, "I am very good in my study subject mathematics" (sample item), Cronbachs' α (T1) = .82, Cronbachs' α (T2) = .84.

The correlations between interest and self-concept and its development were weak to middle. Thus, it is adequate to apply a cluster-analysis to identify clusters which show similar attitudes or development patterns of attitudes. We included the following variables in the analysis: interest (T1), self-concept (T1), the development of interest (interest (T2) – interest (T1)), and the development of self-concept (self-concept (T2) – self-concept (T1)). All variables have been z-standardized. Applying the single-linkage procedure, we identified one outliner and deleted it from the data. The Ward-dendrogram indicated that a four cluster-solution is most appropriate to describe the data. A MANOVA revealed that the four clusters (profiles) differed significantly concerning their interest in university mathematics and their mathematical self-concept at the start of the program and concerning the development patterns of both attitudes ($F(12, 639) = 46.71$, $p < .001$, $\eta^2 = .47$).

RESULTS

Students' Profiles

We could identify four profiles which split the sample in rather similar big groups (see Figure 1).

- Profile 1 (n = 64, “average start but negative development”): Students belonging to this profile began their study program with an average interest and average self-concept which both decreased significantly in the first semester. In the middle of the semester, students with this profile reported the lowest interest and self-concept.
- Profile 2 (n = 53, “bad start but positive development”): Students with this profile began their study program with the lowest interest and self-concept but their attitudes developed positively during the first weeks.
- Profile 3 (n = 46, “average start and low development”): Students with this profile started with the second highest interest and self-concept and developed only little. Whereas their interest slightly decreased, their self-concept increased.
- Profile 4 (n = 55, “best start and low development”): Students belonging to this profile started with the highest interest and self-concept. Although their self-concept slightly decreased during the first weeks, it remained on the highest level of all profiles. Their interest even slightly increased.

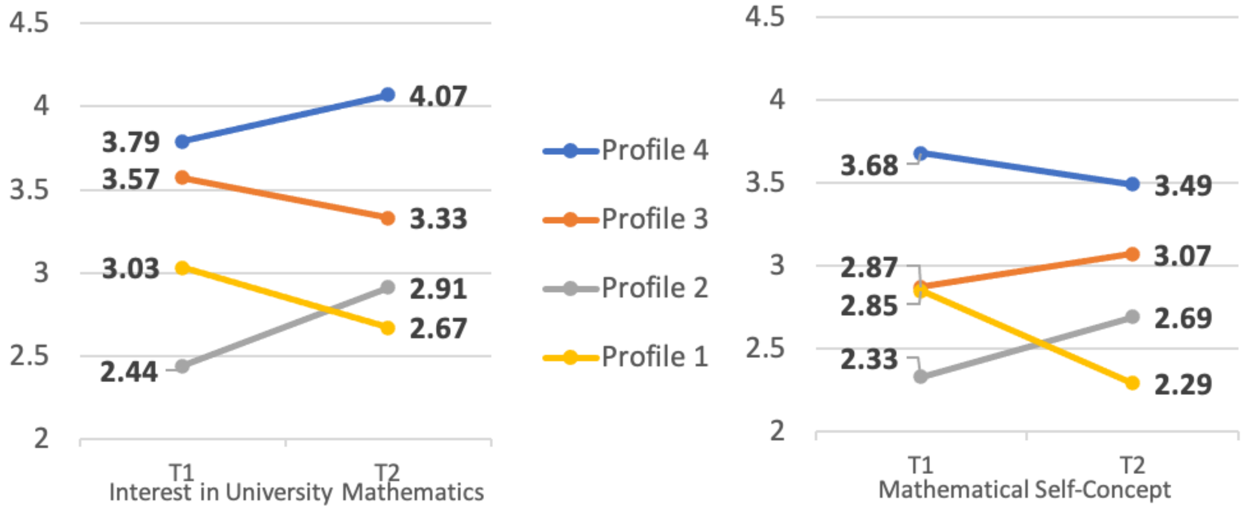

Figure 1: Interest and self-concept of the identified profiles. Statements rated on a five-point-likert scale from totally disagree (1) to totally agree (5) during the second week of the term (T1) and six weeks later (T2).

Female and male students were distributed nearly equally with regard to the identified profiles ($\chi^2(3) = 3.46$, $p > .10$). However, it is noteworthy that students from the pure mathematics bachelor and the teacher education program were not distributed equally with regard to the profiles ($\chi^2(3) = 9.93$, $p < .05$). Students of the teacher education programs were overrepresented in profiles 1 and 2, whereas students of the pure mathematics program were overrepresented in profiles 3 and 4.

Relation between students' profiles and study dropout

Significant differences in the dropout rate can be observed between the identified profiles (Table 1). Whereas profile 1 ("average start but negative development") has the highest dropout rate with 44%, profile 4 ("best start and low development") has the lowest rate with 7%. We expected such differences between profiles, which differed in their attitudes at the beginning of study. Noticeable is the big difference in the dropout rates between profile 1 and profile 2. These two profiles developed inversely in the first semester concerning their attitudes. Although profile 2 started their study with the lowest attitudes, it seems that the fit between students of this profile and the environment is better than the fit between students of profile 1 and the environment because students of profile 2 developed more interest in university mathematics and more mathematical self-concept. Profile 3 is an in-between profile because it shows average attitudes and developed only marginally in the first weeks of study.

	Profile 1 (n = 64)	Profile 2 (n = 53)	Profile 3 (n = 46)	Profile 4 (n = 55)	
Dropout rate in %	44%	17%	28%	7%	$\chi^2(3) = 8.65$, $p < .05$
N dropout students	28	9	13	4	

Table 1: Dropout rates of profiles.

DISCUSSION

At the transition to university mathematics programs, research has indicated that mathematical interest and self-concept predict study success respectively dropout (Di Martino & Gregorio, 2019). However, the interplay between these attitudes as well as its development and students' decision to dropout has not yet been clear. By applying a person-oriented analysis approach, we identified four profiles, which differ in both attitudes at the beginning of study and its development in the first semester. Whereas profile 4 ("best start and low development") showed the lowest dropout rates, profile 1 ("average start but negative development") showed the highest rates. It seems that both aspects – the level of attitudes at the beginning and the development during the first semester – played a big role for students' decision (not) to drop out. Noticeable is that neither interest nor self-concept stand out to predict students' dropout and there are no clear indications that the positive development of one attitude variable can strongly compensate the negative development of the other variable.

As this study is an exploratory one, we had no clear hypotheses that we could test with our study. Our results should be confirmed in follow-up studies. In addition, the study took place at one university and only students, who had participated in the lecture of the second week as well as in the lecture in the middle of semester, are included in the analysis. Thus, the results are restricted to students who did not drop out in the first semester weeks and who regularly attended the lectures.

Besides these limitations, the results of our study support the assumption of Haak (2017) that students monitor the fit between their attitudes and the environment and then decide to adapt their attitudes, for example to develop interest in university mathematics (see profile 2), or to leave the environment, by dropping out, as nearly half of the students with profile 1 did. Thus, a cluster-analysis enables a more differential perspective on the study entry phase: We identified a group of students (profile 2) with growing interest in university mathematics and mathematical self-concept in the first week of study (see Kosiol et al., 2019). Even if students with this profile started their studies with the lowest interest and self-concept, the dropout rate in this profile is rather low. Growing interest and self-concept can be understood as a first adaption to the learning environment in the sense of Haak (2017). Students with profile 1 ("average start but negative development") started with average interest and self-concept but underwent an unfavourable development. Likewise, the dropout rate was highest amongst students with this profile, since the pattern of development can be interpreted in the sense that students of this profile did not adapt their attitudes to the learning environment.

In practice, it seems to be not sensible to sort out students according to their attitudes at the beginning of the first semester. Instead, students need a chance to get used to the university mathematics. As many students get to know mathematics as a formal and deductive discipline for the first time at university (Halverscheid & Pustelnik, 2013), it can take some time to develop joy and value and a positive image of its own abilities concerning this form of mathematics. Lecturers can support this development by highlighting the advantages of this form of mathematics and by building bridges to school mathematics (Weber et al., 2020) which could be helpful for students in teacher education programs, in particular.

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USING ONLINE PLATFORMS TO IMPROVE MATHEMATICAL DISCUSSION

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The difficulties encountered during the COVID19 emergency have led to reflection on teaching methods and the introduction of new digital tools. In this paper we highlight how the use of a digital platform can support mathematical discussion, playing a fundamental role in the construction of meaning of new mathematical objects during laboratory activities. We qualitatively analyse the results of a teaching experiment involving a discussion based on comparison of different solutions to the same problem, which is recognised as a powerful pedagogical activity but also a challenge for both teachers and learners.

INTRODUCTION

The recent emergency due to the COVID-19 virus forced students and teachers in Italy, as in many other countries, into the world of distance learning (DL). The spring of 2020 was a very critical moment for the Italian educational system: the sudden transition to DL led to a rapid digitalisation of teaching and learning processes. To support teachers in this transition, the EdTech centre Future Education Modena developed the M@t.abel 2020 project, which was followed by an online community of more than 1,000 teachers. The aim of the project was to re-think mathematics laboratory activities from the perspective of digital technologies, allowing their implementation also in DL situations. In these activities, students must deal with problem situations that enable them to achieve the construction of meaning of mathematical objects, and the mathematical discussion between the teacher and the students is a fundamental element of this approach. In this paper, we present a research study aimed at exploring how mathematical classroom discussion of different solutions to the same problem can be supported using online platforms, and we report data relative to an experiment based on M@t.abel 2020 materials.

THEORETICAL FRAMEWORK

Mathematical discussion is recognized as fundamental in mathematics teaching at all school levels (Crockcroft, 1982). It can be defined as a "purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions" (Pirie & Schwarzenberger, 1988, p.461). Thus, it has a specific purpose related to mathematics contents and processes: students participate actively in the discussion, proposing reasoning to achieve the solution, and this leads to other students' reactions (via new verbal intervention but also with different interaction modes, e.g., gestures, changes of attitudes). One specific type of mathematical discussion is that based on the comparison of alternative solutions to the same problem, and, as highlighted by

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 291-298). PME.

Richland and colleagues (2017), it is an effective pedagogical practice. The mental exercise of comparing different representations and solutions and identifying relationships between them, is particularly important for developing relational thinking in mathematics and thus for reaching a deep mathematical understanding of concepts, which is necessarily a relational understanding (Skemp, 1976). Richland and colleagues (2017) identify the reasons why a discussion based on comparison of different solution can be challenging for students and teachers, proposing possible strategies to overcome these difficulties. They base their analysis on two main concepts: Working Memory (WM), defined as "the cognitive resource that enables humans to hold information in mind and to manipulate that information without losing it", and Executive Functions (EF) resources, which are an important requirement for relational reasoning and control "what information should go into WM […], inhibit attention to irrelevant information and update their WM with new information" (Richland et al, 2017, p.43). An excessive effort requested in terms of WM and EF leads students to identify fewer relationships between the solutions compared, to more distracting errors and to greater difficulties in comparing different representations. Furthermore, teachers must consider firstly the large variability in their students' EF, and then their different needs in terms of time and support during the discussion. They identify possible strategies to overcome these obstacles*: Sequencing instruction and selecting analogy, Providing explicit cues to compare, Making compared representations visible simultaneously, Using spatial organisation to highlight key relations*, *Using linking gestures to move between spatial representation*.

The implementation of digital technologies, which is a necessity during the COVID emergency, can provide a useful instrument to support teachers in this contest but is itself a challenge for many teachers (Drijvers et al., 2013). The idea of *instrumental orchestration* introduced by Trouche (2004) is "the teacher's intentional and systematic organization and use of the various artefacts available in a - in this case computerized - learning environment in a given mathematical task situation, in order to guide students' instrumental genesis" (Drijvers et al., 2013, p.1350). The instrumental orchestration consists of three elements: a *didactic configuration* (configuration of the teaching settings and artifacts involved), an *exploitation mode* (the way the task is introduced to and approached by the students, also considering the use of artifacts and timing organisation) and the *didactical performance* (decisions taken during the teaching). Different types of orchestration could be identified; in this paper, we elaborate the *Sherpa-at-work orchestration* to show how a student uses technology to socialise his/her work while the teacher acts as a mediator between the Sherpa-student and the class.

RESEARCH QUESTION

In this research study, we connect two different well-known challenges in teaching mathematics: the first is the integration of digital tools into mathematics education, which is a "non-trivial issue" because it affects all aspects of education (Drijvers et al., 2013), while the second is the promotion of significant mathematical discussion based on comparison of different solutions to the same problem in order to reach a deeper conceptual understanding of mathematics (Richland et al., 2017). Our hypothesis is that the use of an online platform can support the mediator role of the teacher in the orchestration of mathematical discussion. Our research questions are:

- How can the use of a digital platform support mathematical discussion?
- Which of the specific difficulties related to comparison of multiple solution during a discussion could be overcome thanks to use of a digital platform?

METHODOLOGY AND EXPERIMENT PLAN

The task-related stimulus for the discussion

The problem situation proposed to the students comes from the original M@t.abel activity "The Picture": in observing the picture of a boy (Luca), students are asked to estimate his height considering the possible references supplied by the picture and other data driven by external sources (e.g., personal experience). We decided to focus our research on this task because previous experimentation highlighted that it allows students to find multiple and different solutions and the discussion based on comparison of students' hypotheses can be rich and fertile. This task was used for fostering a two-step discussion: the discussion starts on a digital platform (Padlet), before being resumed and developed in the classroom.

Data collection

The research involved 10 grade 7 classes from different schools with different backgrounds and from 7 regions in Italy. The problem was proposed by each teacher, using a Padlet prepared to present the task and record students' solutions. The task was reported in the Padlet in a written form ("This is a picture of Luca at age 5. How tall was he?") which accompanied the picture but also through a video of Luca who is now 35 and directly asks to help him reconstruct how tall he was. Students had to reflect on the problem as homework, individually, and then post their hypothesis on the Padlet using the modality of their choice (written posts, pictures, audios, small videos, or links). The Padlet was set up so as to enable the teacher to moderate the posts and collect all students' answers before making them visible. Once all the students had posted their solution, the teacher made all the posts visible before setting the second task (again as homework), which consisted in reading all their classmates' solutions and commenting on them (in written form or in the form of 'likes'). The posts in the Padlet were set as anonymous but students had to sign them with a nickname that was shared with the teacher and classmates only during the final discussion. Then, in the following lesson, the teacher displayed the Padlet on the interactive whiteboard along with all students' solutions and their comments on other solutions, and the classroom mathematical discussion was based on the previously-collected information in the Padlet. Due to the different pandemic situation of each region, in 6 classes the classroom mathematical discussion took place during a lesson in person, while in the other 4 the discussion was held online.

For each class, the following data were collected and analysed: Padlet completed by the class; video or audio recording of the classroom discussion; transcript of the mathematical discussion (open data available: Giberti, 2022). All data were analysed via qualitative methods, schematising interactions in a diagram and labelling teachers' and students' interventions in terms of obstacles related to the development of the mathematical discussion and strategies to overcome these obstacles (Richland et al., 2017). When an obstacle/strategy appeared strictly related to the use of Padlet, this was marked. If we consider the three elements of the instrumental orchestration, in our research the *didactical configuration* and the *exploitation mode* were defined in advance and shared with teachers, who had the possibility only of adjusting them to their classes in term of timeline organisation and online/distance mode of the final discussion. We then compared the findings that emerged from the *didactical performance* of the teachers and how this performance is supported by the fact that the mathematical discussion had already been launched on the online platform.

RESULTS AND DISCUSSION

The request to compare alternative solutions could be a huge effort for students in terms of WM because if solutions are explained by other students verbally, the listeners must pay explicit attention to grasp that information, think about it and retrieve it for future considerations (Richland et al., 2017). In our experiment, all the solutions were collected in the Padlet, and during the discussion all the teachers displayed the Padlet using the whiteboard (if the lessons were in person) or sharing their screens (if they were in DL mode). Then, when a student (the *Sherpa-student*, Drijvers et al., 2013) explained verbally his/her own solution, this solution was displayed for all the other students, thus helping them to grasp and retain key information. The earlier collection of students' strategies on the Padlet also limits situations in which a student may modify his/her solution while explaining it to the class, thereby limiting situations that require huge effort in terms of cognitive task related to WM. For instance, in one class the discussion dealt with the concept of height and whether the height of a person is the same if he's standing upright or leaning; this led one student to intervene and modify his own strategy. In this case, the presence of the Padlet helped both the student and the teacher to refer explicitly to the post:

Student: I was thinking about what I had written in the post, and I have now considered that my hypothesis was wrong regarding the guardrail because I did not consider the inclination instead

Teacher: ok, well, we can always add it

Student: yes, the distance between the point and the guardrail if it is straight is not quite right

Teacher: thanks to the posts of your classmates, you can add an element to your post!

When students are faced with a strong request in terms of WM and EF load, they struggle more in integrating the relationship between different representation, and the support of the Padlet helps them clarify some elements discussed. A paradigmatic example is the intervention of this student who, at the end of the discussion compares three strategies:

Student: in my opinion, A.'s reasoning is right; E.'s is right too, but only if she had had the measurements and the photos had been taken at that moment… in my opinion it would have been correct reasoning and also maybe a bit different, a bit like T. did, when he positioned the kid a number of times. He did it with the guardrail while she would have done it with the kid.

Moreover, following the work of Richland and colleagues, we also identified other reasons for using Padlet to help students during a mathematical discussion.

Sequencing instructions, selecting analogies, providing explicit cues to compare

In the development of comparative thinking, teachers must consider the huge variability in students' EF and the different support requirements necessary to identify key relationships. The collection of students' solutions in the Padlet and the homework task of reading and commenting classmates' strategies helped in this process because the introduction of these previous steps mediated by the teacher gave all the students the time necessary to read, reflect and start comparing the solutions. For instance, at the beginning of the discussion, one of the teachers proposed: "*So, since you have already seen them, already read them, let's try to see if the posts you have written have anything in common, ok?*". Then all the students identified the use of the guardrail as a reference point to find the height of the children as a strategy adopted by many students. In a previous work, based on a subset of the same data, we observed that this activity was particularly inclusive and all the students, including the one with special needs, actively participated in the Padlet, which led to a more inclusive mathematical discussion (Giberti et al., under review).

The earlier collection of solutions in the Padlet also gave the teacher more time to make decisions about the instructional sequence, reflect on all students' solutions and representations and then identify those to compare during the mathematical discussion. The following diagram (Fig. 1) summarises the intertwining of interactions in a Padlet discussion.

This would be helpful for teachers also in relation to another strategy suggested by Richland and colleagues (2017): teachers might organise in advance the problem solutions into hierarchies of smaller sub-goals in order to highlight the structure of the solutions and similarities between them. Indeed, the task of noticing the relevance of a comparison and making relevant inferences is often difficult for students; also, in many cases, careful planning by the teacher is not sufficient, but the possibility of identifying sub-goals helps teachers to provide explicit clues to compare, helping learners to recognise the relevant structure of the problem (e.g., some teachers grouped solutions based on external data taken from the students' experience and online research activity).

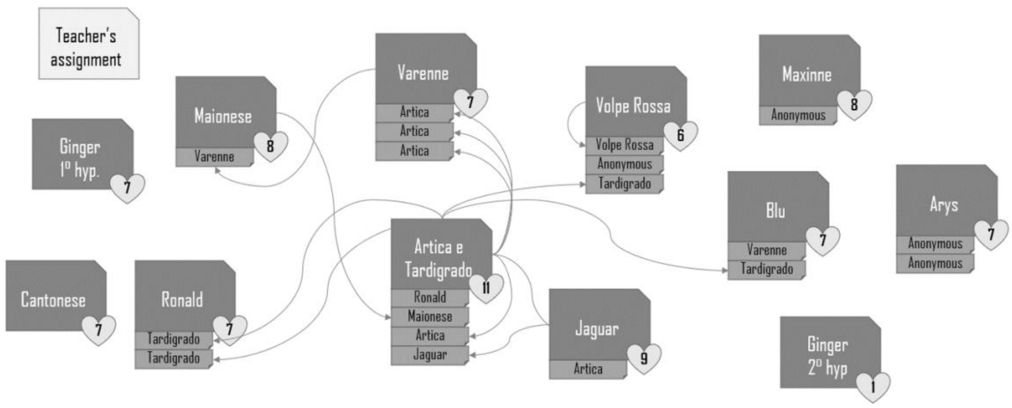

Figure 1: Synthesis of interactions between students in the development of the discussion on Padlet (the students' nicknames are reported).

Making compared representations visible simultaneously

During the final discussion, all the teachers shared the Padlet in order to have all the discussion solutions visible simultaneously: this helped in reducing cognitive load and promoted relational reasoning as detailed by Richland and colleagues (2017).

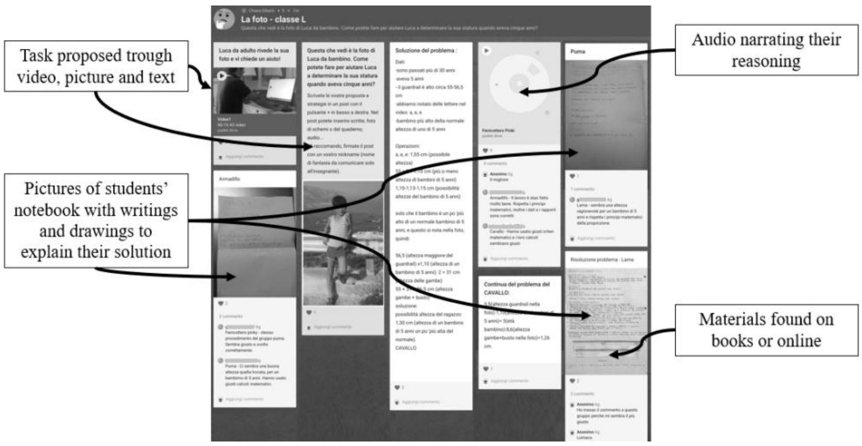

Figure 2: An example of how Padlet appears

The screen-sharing of all the responses led students to identify both general considerations ("*in my opinion one of the most common mistakes [...] was not to average out the guardrail, which has different heights*") and comparison between two ("*then, in my opinion, the reasoning is right, I did it too in that way, albeit I was 2 cm shorter, but I still did that strategy*"). It is interesting to note that, thanks to the simultaneous visualisation of different solutions (Fig. 2), a teacher could show that the same problem may have different solutions, yet all correct (commenting on the use of different proportions and ratios).

Using spatial organisation to highlight key relations

Again to reduce the demand for attention demands and facilitate relational thinking, Richland and colleagues suggest that teachers use spatial organisation to compare the different solutions and representations. In this way, students are assisted in making the correct correspondences during the mathematical discussion. Padlet revealed itself to be a perfect instrument to do so: posts can be moved and put side-by-side, furthermore teachers can also zoom in on a specific part of the Padlet and focus students' attention on specific comparisons.

Using linking gestures to move between spatial representations

Finally, Richland and colleagues (2017) also promoted the use of gestures to explain the importance of a comparison to students. For instance, in one class, gestures accompanied the comparison between desk height (in the classroom) and guardrail height (in the Padlet), with one student observing "*at 5 years old he cannot be twice the height of the bench!*". The role of gestures was analysed regarding one of the classes in which the discussion took place in presence; the results are explained in detail in a previous work (Giberti et al., under review).

CONCLUSIONS AND FURTHER PERSPECTIVES

We observe that mathematical discussion based on comparison of alternative solutions could be supported by the introduction of a digital tool that collects students' hypotheses and comments (Padlet). This helps overcome the fact that a traditional classroom discussion is necessarily linear and unidimensional, whilst a digitally supported class allows multidimensional development with "simultaneous voices and counterpoints" that can be orchestrated by the teachers. The Padlet helps in overcoming difficulties due to excessive effort related to WM and EF (which often arises during this kind of discussion). Furthermore, most of the strategies suggested by Richland and colleagues (2017) can be promoted through the use of Padlet. It guarantees all students the necessary time to think and make an initial comparison of other solutions, while also giving the teacher time to select points/comments to compare and prepare the didactical performance. Padlet is useful during the classroom group discussion because it presents all solutions simultaneously; it could also help teachers to organise (also spatially) solutions in groups and help students in reaching a relational understanding. Another interesting aspect which emerged (thanks to the fact that the discussion is based on the answers and comments collected anonymously), is that sometimes the discussion focuses spontaneously on specific posts and subsequently the author of the post is called on to explain her/his solution; the role of *Sherpa-student* in this case is assigned by the classmates.

Finally, the use of online platforms such as Padlet could be an important tool for collaboration in the mathematical discussion between different classes, also in different geographical regions. We are now considering the possibility of a multilevel hybrid learning discussion in which the solutions proposed by a class could be discussed by another class and vice versa. Our hypothesis is that this kind of collaboration beyond the borders of the class could help overcome obstacles caused by the didactic contract and influences of the milieu (Brousseau, 1988).

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LINKING LEARNING AND INSTRUCTIONAL THEORIES IN MATHEMATICS EDUCATION

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There is a dilemma between knowledge enquiry and transmission in general instructional theories, which are usually supported by constructivist, objectivist or a combination of constructivist and objectivist learning theories. Considering the epistemological, ontological, and semiotic assumptions about mathematical knowledge of the Onto-semiotic Approach, in this paper we describe a theoretical model of mathematical instruction that articulates the constructivist and objectivist approaches to learning in order to optimise the didactic suitability of the teaching and learning processes.

INTRODUCTION

The problem-situations designed to contextualise and provide meaning to curricular contents, the ways of interaction in the classroom and the resources used are, among other, factors that determine students' learning opportunities. The complexity of instructional design explains the existence of different instructional theories supported by differing psychological and pedagogical assumptions (Reigeluth et al., 2016). The family of instructional theories known as "Inquiry-Based Education" (IBE), "Inquiry-Based Learning" (IBL), "Problem-Based Learning" (PBL), advocate for inquiry-based learning, with little guidance from the teacher (Artigue & Blomhøj, 2013).

Contrary positions, such as those of Mayer (2004) and Kirschner et al., 2006) point to a wide range of research that concludes the greater effectiveness of instructional models in which the teacher and the transmission of knowledge are given a greater role. The problem about teaching models focused on the student or on the teacher can be related to the inquiry and transmissive instructional models, as well as to the debate between constructivism and objectivism (Jonassen, 1991), respectively. Consequently, a dilemma arises as to which paradigm is more effective in promoting the learning of mathematics: objectivism or constructivism? The objectivist model is based on the assumption that there is a real world and that the purpose of education is to map the entities of that world in the learner's mind. The constructivist paradigm relies on the premise that reality is constructed during interaction with the environment and classmates and that knowledge is both individual and collective.

> Radicals of each camp argue that is impossible to mix the two paradigms. [...] However, dominant paradigms, in both the physical and social sciences, [...] rarely

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> replace each other by falsification. Instead they tend to co-exist and are used whenever they are appropriate" (Vrasidas, 2000, p.12).

In this paper we first analyse the tension between student-centred didactic models, which are based on constructivist theories of learning, and teacher-centred models, which are supported by objectivist theories; in other words, we pose the dilemma between student enquiry and teacher transmission of knowledge. We then discuss a mixed didactic model, based on the epistemological, ontological, and semiotic assumptions from the Onto-Semiotic Approach (OSA) to mathematical knowledge (Godino & Batanero, 1994; Godino et al., 2007), through which the moments of transmission and enquiry are articulated to optimise the didactic suitability of the mathematics teaching and learning processes.

LEARNING AND INSTRUCTIONAL THEORIES

Theories of learning are often distributed along the objectivism - constructivism continuum (Jonassen, 1991), two extremes that are hardly ever proposed in isolation as the psychological underpinning of educational methods. Whether constructivist or otherwise, learning theories, in themselves, do not entail a theory of teaching; their implications for guiding educational processes are not prescriptive, but merely indicative. Ernest (2010) sees them as philosophies or generic orientations on how people manage to understand and appropriate knowledge, but without the characteristic of theories, which is the requirement of experimental contrast for their possible falsifiability. The implications for mathematics teaching of the four different versions of constructivism described by Ernest (2010) may legitimately be addressed by teachers who base their pedagogical practice on any of the learning theories "As a grand theory, or perhaps a paradigmatic theory, constructivism is too general to reach to the classroom directly" (Confrey & Kazak, 2006, p. 320).

The debate between direct teaching, linked to objectivist positions about mathematical and scientific knowledge, which supports the teacher's central role in guiding learning, and the minimally guided teaching, usually related to the constructivist model of teaching, is not clearly settled in the research literature. For Zhang (2016) the tissue between these two instructional positions is not whether one or the other is in favour of providing more or less guidance or support to students, but between explicitly discussing solutions with learners or letting learners discover them. "For the advocates of direct instruction, explicitly presenting solutions and demonstrating the process to achieve solutions are essential guidance" (Zhang, 2016, p. 908). In constructivist positions, although a certain amount of transmission of information from the teacher to the learner is accepted, it is still essential to hide some of the content.

For supporters of direct instruction who assume the cognitive load theory with an emphasis on worked examples, it is essential to provide the solutions. Authors such as Mayer (2004) and Kirschner, et al. (2006) claim that empirical research over the last half century provides overwhelming and clear evidence that minimal guidance during instruction is significantly less effective and efficient than guidance specifically designed to support the cognitive processing necessary for learning. In a similar vein, Radford states: “Indeed, it does not seem reasonable to expect that the child (working alone or in collaborative groups) would be capable of reconstructing by him/herself the complex theories featured in the curriculum” (Radford, 2012, p.103).

ONTOSEMIOTIC APPROACH TO MATHEMATICAL KNOWLEDGE

The problem of what mathematics should be taught and how to teach it is being addressed from various theoretical perspectives. The OSA framework problematises the very nature of mathematical knowledge, as does the Theory of Didactic Situations (Brousseau, 1997) and the Anthropological Theory of the Didactics (Chevallard, 1992). In the OSA, a double dimension of mathematics is considered, as an activity of people involved in the resolution of some kind of problems, and as a system of historically and culturally shared objects. The OSA ontological postulates are in line with those formulated in Wittgenstein's philosophy of mathematics:

> Concepts/definitions and propositions are regarded as “grammatical” rules of a certain kind. From this point of view, mathematical statements are rules (of a grammatical kind) governing the use of certain types of signs, since that is precisely how they are used, as rules. They do not describe properties of mathematical objects with any kind of existence that is independent of the people who wish to know about them or the language through which they are known, even if this may appear to be the case (Font et al., 2013, p. 110).

The central notion of *semiotic function*, as a relationship between mathematical objects and systems of practices, together with the proposed typology of mathematical objects and processes allowing to articulate the operational-pragmatist and referential positions of meaning and to reveal the onto-semiotic complexity of mathematical knowledge (Font et al., 2013; Godino et al., 2021).

The theory of mathematical knowledge embodied in the OSA on anthropological (Wittgenstein, 1953), pragmatist (Peirce, 1931-58) and semiotic (Hjelmslev, 1969/1943) foundations entails crucial implications for educational-instructional processes, by providing articulation elements between the theories of learning and mathematical instruction. The theory of didactic suitability (Godino et al., 2016), as a module of OSA, recognises the complexity of educational processes by taking into account not only the cognitive - affective (learning) and the instructional (interactions and resources) facets, but also the epistemic (content) and ecological (context) dimensions, as well as the interactions between these facets.

Didactical suitability is defined as the degree to which an instructional process (or part of it) meets specific characteristics that qualify it as optimal or adequate to achieve the alignment between the personal meanings achieved by learners (learning) and the intended institutional meanings (teaching), while considering the available circumstances and resources (environment). This involves the coherent and systemic articulation of six criteria related to the facets involved in an instructional process (Godino et al., 2007, p. 133):

- *Epistemic suitability*, expressing the degree of representativeness of the institutional meanings implemented, with regard to a reference meaning.
- *Ecological suitability*, referring to the degree to which the instruction is in line with the school and society educational project and the environmental conditions in which it is developed.
- *Cognitive suitability*, describing the degree to which the meanings implemented correspond to the learners' zone of potential development, as well as the closeness of the personal meanings achieved to the intended meanings.
- *Affective suitability*, indicating the learners' degree of involvement (interest, motivation, etc.) in the instructional process.
- *Interactional suitability*, indicating the degree to which the types of didactic configurations implemented, and their articulation help to identify and settle the potential semiotic conflicts that arise during the teaching process.
- *Mediational suitability* is the degree of availability and adequacy of the material and temporal resources necessary for the development of the teaching-learning process.

Achieving educational-instructional processes with high didactic suitability involves a complex human activity that requires coherent articulation of teaching and learning activities with curriculum development and teacher education activities, all of which are carried out within an institutional and ecological background that supports and conditions them.

SUITABLE MATHEMATICAL INSTRUCTION

Within the OSA framework, a didactic model has been developed seeking to locally optimise mathematics teaching and learning processes by considering the triple dialectic between the teacher`s and students' work and the mathematical content (Godino et al., 2019; Godino & Burgos, 2020). In line with the principles of cultural/discursive psychology (Lerman, 2001), it is assumed that autonomy and creativity in problem solving by students is a progressively achievable goal, rather than a precondition for learning.

The OSA's epistemological and onto-semiotic assumptions provide a rationale for designing and implementing a mixed instructional model, including inquiry-based cooperative, dialogic and transmission-based phases (Figure 1). Students have to learn numerous rules (concepts, propositions, procedures) as well as the conditions under which they can be applied. The learner proceeds from known rules and produces new ones, which have to be shared and be compatible with those already established in the mathematical culture. Such rules (knowledge) have to be stored in the subject's long-term memory and put to work at the appropriate time in the short-term memory.

In the moments or phases of the student's first encounter with a specific meaning of a mathematical object, a dialogic-collaborative configuration can optimise learning. In this type of interaction, the teacher and the students work together to solve problems that bring the intended knowledge into play in a critical way. The first encounter with new knowledge should be supported by explicit explanations and input from the teacher. These transmissive (somewhat magisterial) didactic configurations can be meaningful if they refer to the problem situation they are studying collaboratively. The teaching-learning process could thus achieve higher epistemic and ecological suitability (Godino et al., 2007). Under an inquiry-based teaching model, with minimal teacher guidance, students are exposed to the risk of not finding the solutions in the first encounter phase, with the consequent rejection and frustration. “Even if the students find the solutions on their own, they do not know the most effective procedures as they have to wander around in the problem searching process, not to mention the cognitive loads they are imposed” (Zhang, 2016, p. 909).

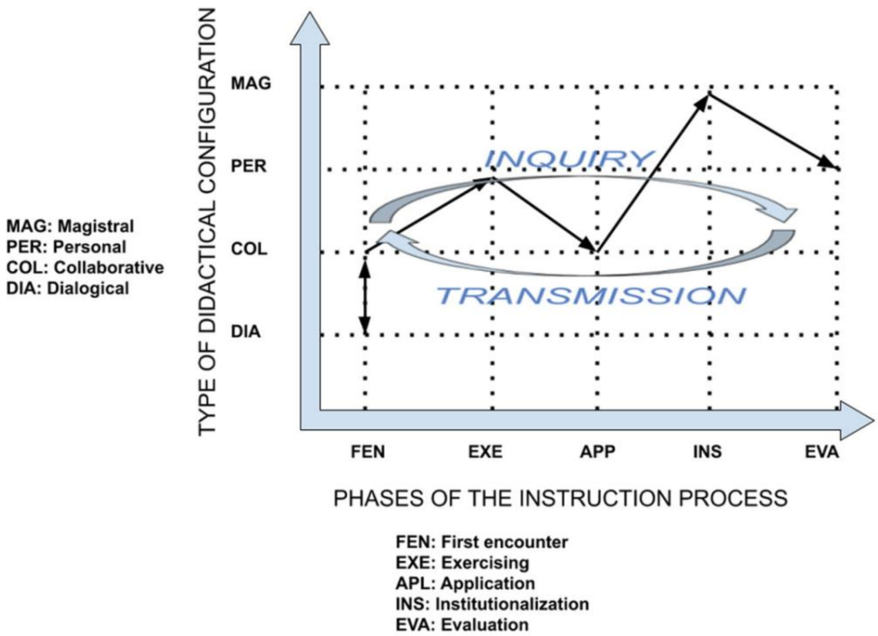

Figure 1. Transmission-inquiry dialectics according to the instructional process phases (Godino & Burgos, 2020, p. 16)

When the application rules and circumstances that characterise the object of learning are understood, it is possible to move towards higher levels of cognitive and affective suitability by offering more in-depth study of the content (exercise and application situations), through didactic configurations that progressively and in a controlled manner grant more autonomy to the learner.

CONCLUSSIONS

Instruction is usually understood as the combined activity of teaching and learning, i.e. Instruction = Teaching + Learning. In this paper, we add to this equation the content to be taught/learned, as we consider that how a content should be taught depends substantially on the nature of what is intended to be taught/learned. Moreover, it depends on the context and circumstances of the teaching process and the subjects involved, so that the optimisation of such activity has local components (actions of teacher and students in the classroom) and global components (didactic transposition processes of the specific content). A theory of learning does not suffice to understand and make decisions about instructional practices, rather a theory of content/knowledge is also required. Learning theories do not prescribe what instruction should look like, but that role should be played by content theories and theories of intended skills, or rather derive from the three-way dialectic between content, learning and teaching.

Hudson et al. (2006) also justify the implementation of mixed instructional models that adapt and mix explicit instruction (teacher-centred) with problem-based instruction (learner-centred) because of the need to make curricular adaptations to the diversity of students' abilities. As we can see, several authors propose blended instructional models that combine enquiry and knowledge transmission. In this paper we also defend the greater suitability of these mixed models, but we add some details on how the content type influences the greater or lesser presence of enquiry or transmission in the mixed model. The reasons given are basically linked to the onto-semiotic complexity of the intended mathematical activity, which complement the cognitive reasons highlighted by other authors (Kirschner et al., 2006).

The challenge of education is to optimise the teaching and learning process by balancing the learning of all students at their own pace and, at the same time, proposing educationally valuable content. This is a challenging task for the teacher and other educational agents who have to select and develop the right instructional resources for each circumstance. In our opinion, posing the dilemma between student-centred and teacher-centred education, between constructivist and objectivist learning theories, is naïve since the optimisation of the educational process requires the coherent and appropriate articulation of the principles of these educational theories and models.

Achieving mathematics education with high didactical suitability requires the coherent and balanced articulation of various facets and components (Godino et al., 2016), not only at the local level, i.e., within the classroom where the interaction between students, teacher and content takes place, but also at the global level. Two essential components of the ecological facet that crucially condition and support educational processes are the curriculum, where decisions are made about the content and educational resources that are made available to teachers, and the teacher education system, which is responsible for the preparation and ongoing support of teachers.

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CONCEPTUAL AND PROCEDURAL MATHEMATICAL KNOWLEDGE OF BEGINNING MATHEMATICS MAJORS AND PRESERVICE TEACHERS

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In light of known challenges in the transition from school to university in mathematics, we investigate differences in the (mathematical) prerequisites of mathematics majors and preservice mathematics teachers. Results show that although there are no significant differences in high school grade point average, mathematical prerequisites of mathematics majors are significantly better than those of preservice mathematics teachers. Differences are higher in conceptual than in procedural knowledge with medium effect sizes between mathematics majors and preservice higher secondary teachers and (very) large effect sizes between mathematics majors and preservice lower secondary or primary school teachers. These results are discussed regarding transition challenges and the fit of prerequisites and chosen study program.

MATHEMATICS PRESERVICE TEACHERS AND MAJORS COMPARED

In Germany teacher training at university is organized in separate degree programmes, but not always in separate lectures. While preservice higher secondary teachers attend advanced mathematics courses together with mathematics majors, primary and lower secondary teachers usually attend specific mathematics courses with more basic mathematical content (Gildehaus et al., 2021). The credits to be taken in mathematics are accordingly less in lower-secondary and primary programs.

In joint courses with mathematics majors, preservice teachers perform slightly lower on exams (Göller et al., 2022) and report less satisfaction with their studies than mathematics majors (Kosiol et al., 2019). Mathematics (preservice) teachers in general, tend to question the relevance of advanced mathematical content for their teaching profession (Gildehaus & Liebendörfer, 2021; Zazkis & Leikin, 2010).

However, it is unclear whether these differences arise from acculturation at the university or are rooted in different prerequisites already at the beginning of the studies: For example, differences in dissatisfaction can be partly explained by different interest profiles at study entrance (Kosiol et al., 2019). Preservice teachers are in mean more interested in school mathematics (especially in using calculation techniques) and less interested in university mathematics (e.g., proof and formal representations) than mathematics majors (Ufer et al., 2017).

In addition to such affective variables, cognitive variables are relevant factors for academic success and related study satisfaction. The high school grade point average (HSGPA) has empirically proven to be one of the best indicators for predicting study

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success across different study programs (e.g., Richardson et al., 2012; Schneider & Preckel, 2017; Westrick et al., 2021). Mathematical knowledge assessed in entrance tests is found to be an even better predictor of later academic performance in mathematics courses (Eichler & Gradwohl, 2021; Greefrath et al., 2017; Halverscheid & Pustelnik, 2013; Rach & Ufer, 2020).

In terms of such cognitive prerequisites, the differences between mathematics preservice teachers and majors are less evident: Blömeke (2009) found no differences in high school grade point average (HSGPA) between mathematics preservice teachers and mathematics majors, however, mathematics majors performed better in a mathematics test at study entrance than mathematics preservice teachers (Pustelnik & Halverscheid, 2016). To elaborate on these findings, we report on a study following the idea that differences in students' mathematical interests (Ufer et al., 2017) might be mirrored in different types of mathematical knowledge.

CONCEPTUAL AND PROCEDURAL MATHEMATICAL KNOWLEDGE

In mathematics tests for beginning university mathematics students, mathematical knowledge is usually conceptualized and surveyed as a unidimensional construct. To investigate whether different types of mathematical knowledge, corresponding to the different mathematical interests of students (Ufer et al., 2017), can be empirically distinguished, we build on the subdivision of mathematical knowledge into *conceptual knowledge*, which is thought as a network of relationships connecting different pieces of information, and *procedural knowledge*, which comprises knowledge about algorithms or a series of steps for completing mathematics tasks (Hiebert, 1986). Although conceptual mathematical knowledge seems to be theoretically (Gray & Tall, 1994; Gueudet & Thomas, 2020) as well as empirically (Hailikari et al., 2007; Rach & Ufer, 2020) more important for later academic success in mathematics at university, many of the mathematics tests used at study entrance rather measure procedural knowledge, such as basic arithmetic skills (Heinze et al., 2019). We do not know of any study that explicitly examines differences between mathematics preservice teachers and majors in terms of conceptual and procedural mathematical knowledge. We thus explore the following research questions:

RQ 1: Can conceptual and procedural knowledge of mathematics students at study entrance be empirically distinguished?

RQ 2: How do mathematics students of teacher and non-teacher study programs differ in their (mathematical) prerequisites at the beginning of their studies?

METHODS

To answer these questions, we refer to data from a medium-sized German University with 310 participants in a pre-university mathematics course in September 2021, about one month before the start of their studies. Participants can be subdivided into three groups (with regard of their different study programs):

- *Group 1* (majors) consists of 15 mathematics and 70 computer science majors.
- *Group 2* (higher secondary teachers) consists of 55 preservice higher secondary mathematics teachers enrolled in a study program with joint mathematics courses with mathematics majors (Group 1).
- *Group 3* (primary & lower-secondary teachers) consists of 170 preservice primary and lower-secondary school mathematics teachers enrolled in a study program without joint mathematics courses with mathematics majors (Group 1).

The participants self-reported their high school grade point average as well as their last math grade from school (1 = best, 6 = poorest) and worked for 60 minutes on an online mathematics test with 21 tasks (12 (complex-)multiple-choice items, 9 with open numerical input) of which 11 were classified as conceptual items and 10 as procedural items. Conceptual items comprised tasks that required connecting different pieces of information such as changing between different representations (e.g., relating terms and graphs, modelling) or using given information for argumentation (see Figure 1).

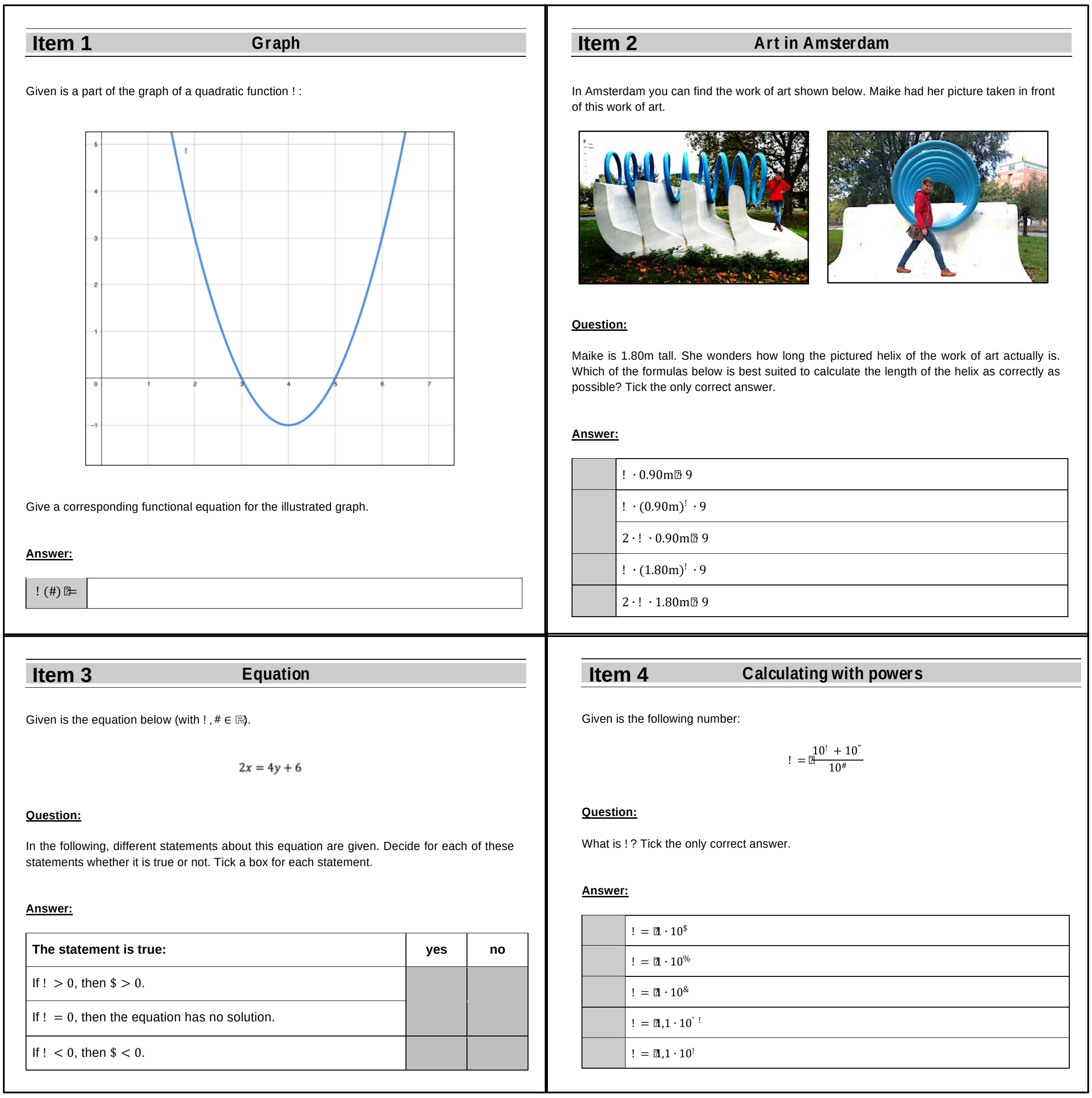

Figure 1: Examples of test items. Items 1,2,3 conceptual items, Item 4 procedural item. Items translated from German by the authors (cf. Besser et al., 2020)

Procedural items comprised tasks that required algorithms or a series of (calculation) steps to be completed. Examples of procedural items are e.g. calculating the derivative of $f(x) = (3 - x^2)^6$ or simplifying the expression $\frac{x^{k-n}}{y^{2n}} : \frac{x^{2k-n}}{y^{n-1}}$ for $x, y \neq 0$ and collecting the variables (Hochmuth et al., 2019) as well as Item 4 of Figure 1.

Participants' answers were coded dichotomously (0 = not correct, 1 = correct; missing answers were coded as not correct) and analyzed using the R-package "mirt" regarding a unidimensional 2-parameter logistic IRT model as well as a two-factor 2PL IRT model distinguishing conceptual and procedural items. Person scores were extracted and further analyzed using analyses of variance in SPSS.

RESULTS

Addressing RQ 1, both considered models show acceptable to good model fit statistics (cf. Table 1). Noteworthy, the two-factor model that distinguishes conceptual and procedural knowledge fits the data significantly better.

	RMSEA	TLI	CFI	AIC	BIC	$\chi^2(5)$	p
Unidimensional	0.044	0.941	0.947	7667.14	7824.07		
Two Factors	0.029	0.976	0.979	7621.56	7797.18	55.576	<.001

Table 1: Fit statistics of the unidimensional and the two-factor (conceptual-procedural) IRT model

The latent factor correlation of conceptual and procedural knowledge is r = .70, indicating that they measure different (yet correlated) constructs. Table 2 shows the bivariate correlations below the diagonal and reliability measures on the diagonal. Also, the bivariate correlation of conceptual and procedural knowledge indicates different (yet correlated) constructs with r = .51 for the IRT person scores.

	1	2	3	4	5
1. HSGPA	-				
2. Math Grade	.65*	-			
3. Unidimensional Model (IRT Unidim)	-.23*	-.28*	.83		
4. Conceptual Knowledge (IRT Concept)	-.21*	-.30*	.87*	.73	
5. Procedural Knowledge (IRT Proced)	-.18*	-.20*	.86*	.51*	.74

Table 2: Bivariate correlations of school grades and IRT person scores below the diagonal and empirical reliability measures on the diagonal. *p < .01

Regarding RQ 2 we first give some descriptive statistics in Table 3. In the mean, students solved approximately half of the tasks. Means of Group 1 (mathematics majors) are the highest for all measured variables, followed by Group 2 (preservice higher secondary teachers), while means of Group 3 are the lowest.

	Full Sample				Group 1		Group 2		Group 3	
	Min	Max	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
HSGPA	1.00	3.60	2.28	0.53	2.21	0.62	2.22	0.56	2.34	0.47
Math Grade	1.00	5.00	2.32	0.91	2.04	1.00	2.18	0.84	2.51	0.85
Sum Test	1.00	21.00	11.46	4.62	14.82	4.32	12.75	4.07	9.37	3.73
Sum Concept	0.00	11.00	6.25	2.83	8.28	2.52	7.05	2.50	4.98	2.36
Sum Proced	0.00	10.00	5.21	2.42	6.54	2.27	5.69	2.20	4.39	2.23
IRT Unidim	-2.23	2.21	0.00	0.91	0.68	0.90	0.25	0.81	-0.42	0.70
IRT Concept	-1.89	1.64	0.00	0.85	0.62	0.76	0.25	0.74	-0.39	0.70
IRT Proced	-1.97	1.95	-0.01	0.86	0.46	0.84	0.16	0.81	-0.30	0.77

Table 3: Descriptive statistics. For HSGPA (high school grade point average) and Math Grade (1 = best and 6 = poorest). Sum Test = Sum of correctly solved items, Sum Concept = Sum of correctly solved conceptual items, Sum Proced = Sum of correctly solved procedural items. Group 1 = majors, Group 2 = higher secondary teachers, Group 3 = primary & lower-secondary teachers

The results of the ANOVA (Table 4) show that the means of the three groups do not differ significantly regarding high school grade point average (HSGPA). Mean differences in the last mathematics grade are only between Group 1 (math majors) and Group 3 (preservice primary & lower-secondary school teachers) significant with medium effect size. Mean differences in the math test are higher, especially for conceptual knowledge and the total test, with medium effect sizes between Group 1 and Group 2 (preservice higher secondary teachers) and (very) large between all other groups. Differences in procedural knowledge are somewhat smaller, with Group 3 again performing significantly lower than the other two, with medium to large effect sizes.

	F	p	η^2	d_{G1-G2}	d_{G1-G3}	d_{G2-G3}
HSGPA	2.11	.123	.014	-0.01	-0.24	-0.25
Math Grade	8.76	<.001	.054	-0.16	-0.53*	-0.39
IRT Unidim	59.84	<.001	.280	0.49*	1.42*	0.92*
IRT Concept	59.04	<.001	.278	0.50*	1.45*	0.89*
IRT Proced	27.04	<.001	.150	0.35	0.95*	0.59*

Table 4: Results of the ANOVA. d_{Gi-Gj}: Effect size (Cohen's d) of the mean difference between Group i and Group j. *Differences are significant (post hoc tests with Bonferroni correction, $p < 0.05$)

DISCUSSION

The results show that conceptual and procedural knowledge of mathematics students at study entrance can be empirically distinguished and measured (RQ 1). Furthermore, they show that although differences between mathematics preservice teachers and majors are not significant for HSGPA and rather small regarding the last mathematics grade from school, they differ significantly with regard to the math test scores, with medium to very large effect sizes (RQ 2). Overall, these results suggest that students chose (have chosen) a study program that fits their mathematical abilities, as reflected in the mathematics test but not (barely) in their school grades: Preservice higher secondary mathematics teachers (Group 2) who attend joint mathematics lectures with mathematics majors (Group 1) are almost at the same level with their performance while preservice primary and lower secondary school teachers (Group 3) who attend mathematics lectures on a less advanced level start their study on a significantly lower mathematical knowledge base. Nevertheless, the mathematical prerequisites of the preservice higher secondary mathematics teachers (Group 2) are lower than those of the mathematics majors (Group 1) which might contribute to explanations of preservice teacher' dissatisfaction with university mathematics contents (Gildehaus & Liebendörfer, 2021) as well as their slightly lower performance in mathematics exams compared to mathematics majors (Göller et al., 2022).

Noteworthy, the differences in mathematics performance are higher in conceptual than in procedural mathematical knowledge. On the one hand, this is in line with preservice teachers interests who are in mean more interested in school mathematics (especially in using calculation techniques) and less interested in university mathematics (e.g., proof and formal representations) than mathematics majors (Ufer et al., 2017). On the other hand, this suggests that university pre- and bridging courses should focus (even) more on building conceptual knowledge in order to compensate for inequalities and to prevent frustrations or other difficulties accompanying the transition from school to university in mathematics (Göller & Gildehaus, 2021).

When interpreting the results, the following limitations, to name but a few, should be taken into account: 21 Items of a one-hour test cannot capture overarching constructs such as mathematical, conceptual, or procedural knowledge in their entirety, which means that the results are of course influenced by the operationalization of the mathematics test used for this study. Since participation in the pre-course, in which the test was taken, was voluntary and test-taking was anonymous, and thus performance on the test had no consequences for the participants (apart from feedback for themselves), selection effects are possible, which are likely to be influenced in particular by the interest of the participants. In addition, the relatively small sample should be considered, which consists of students from only one university.

Accordingly, further research is desirable to better understand the (mathematical) prerequisites of university students and, based on this, to advance the teaching and learning of mathematics at the university.

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TOWARDS A NEWER NOTION: NOTICING LANGUAGES FOR MATHEMATICS CONTENT TEACHING

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In this report, we revise and connect our approaches to mathematics teacher noticing and to the classroom language of the teacher for content teaching in the attempt: 1) to articulate mathematics teacher education knowledge from research on noticing and on language around a newer notion of noticing languages for content teaching; and 2) to apply the articulated knowledge to the design of content-specific materials oriented towards enhancing the development of noticing processes with primary school student teachers in mathematics teacher training programmes. We propose processes of identification, interpretation and decision on languages for content teaching aimed at reducing school learning challenges, and developmental work at the levels of specialised word names and content-related explanatory and exemplifying sentences.

PUTTING TWO APPROACHES TOGETHER

Professional teacher noticing has gained notable traction in research (see, e.g., the *ZDM* issue by Dindyal et al., 2021), and its development is considered important in teacher training programmes (Jacobs & Spangler, 2017). Mathematics teacher noticing refers to what mathematics teachers attend to in classroom situations and how they interpret their observations in order to make instructional decisions. While the focus on noticing mathematical thinking of students is distinctive, we strategically shift the focus onto noticing classroom languages for content teaching. We propose this shift in the context of progress and expansion of mathematics education research on language and teacher preparation (e.g., Shure et al., 2021). This said, relatively few studies have examined language responsiveness in mathematics teacher education (MTE) through networked collaboration of researchers who think about mathematics learning and teaching from different theoretical traditions.

In this report, we challenge and connect our respective approaches to mathematics teacher noticing and to the classroom language of the teacher for content teaching in the attempt: 1) to articulate MTE knowledge from research on noticing and on language around a newer notion of noticing languages for content teaching; and 2) to apply the articulated knowledge to the design of content-specific materials oriented towards enhancing the development of noticing processes with primary school student teachers in mathematics teacher training programmes. Following this introduction, in the first section we summarize some crucial parts of the sources of knowledge we build on. In the second section, we present a number of decisions adopted during our collaboration regarding the practical understanding of language-and-learner responsiveness for mathematics content teaching. We finish with the discussion of what comes next.

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NOTICING LANGUAGES FOR MATHEMATICS CONTENT TEACHING

The development of noticing in teacher training programmes

Sherin (2007) characterises teacher professional noticing as two groups of processes regarding: a) the selective attention to mathematics teaching and learning situations, and b) the knowledge-based reasoning allowing to make sense of what is attended to. In line with the decision processes introduced in Jacobs et al. (2010), we see teacher professional noticing as learning outcomes and related processes of: 1) identifying relevant aspects in mathematics teaching and learning situations; 2) interpreting these aspects according to knowledge of mathematics and mathematical pedagogies; and 3) taking teaching action decisions informed by the adopted interpretations.

The survey work in Fernández and Choy (2020) outlines research conducted on the development of noticing in mathematics teacher training programmes. This latter study also reviews the production of design strategies and materials that have been shown to support student teachers on what and how to notice. Different names, which reflect distinct theoretical lenses, are used to imply the material resources or documents aimed at facilitating noticing processes in the work with student teachers. A widespread strategy, however, is to provide tasks that consist of theoretical materials completed with illustrations of practice (Ivars et al., 2019). The theoretical materials provide linked knowledge of mathematics and mathematical pedagogies informed by mathematics education research. In this way, they offer theoretical lenses that are generally represented in the form of classroom situations (e.g., transcripts of teacher-students interaction) in which student teachers are asked to identify, interpret and decide on selected aspects of mathematics teaching and learning.

Language use for mathematics content teaching

Research on approaches to language as a resource in MTE has documented various relevant aspects to focus on in the classroom languages of teachers for content teaching (Planas, 2019). Within the sociocultural framing in Halliday (1985), Planas (2021) presents content-specific developmental work with secondary school teachers at the word and sentence levels of language. Drawing on Halliday (1978, p. 195), where a register is "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings", and strengthening the emphases on words and sentences, and on school learning, we consider three interconnected tools or resources in language for content teaching:

- *Naming*, or giving word names from content registers oriented towards reducing content learning challenges.
- *Lexicalisation*, or giving sentences with encoded explanations of content-related meanings oriented towards reducing content learning challenges.

- *Exemplification*, or giving sentences with encoded variations of content-related elements oriented towards reducing content learning challenges.

If we think of the common challenge of viewing fractions as numbers, at the level of words, the use in teacher talk of the name *terms* to refer to the numerator and the denominator, and of the name *number* to refer to fraction may be considered learner-responsive. These words can then be put into sentences with the potential function of explaining meanings in order to overcome the learning challenge, such as: *Fractions are numbers expressed in the form of a relationship between two terms.* From the perspective of variation theory (Marton et al., 2004), teacher talk can also produce sentences with the function of exemplifying variations, such as: *The size of one quarter is two if the whole size is eight, but it is three if the whole size is twelve*. This sentence would contribute to supporting the difficult understanding of the fraction size and the elements or facts that make it vary. Names, explanations and variations are, therefore, practical dimensions of naming, lexicalisation and exemplification. Their use in teaching can be approached as intersecting conditions of content languages oriented towards reducing specific learning challenges.

Particularising a language-informed notion of noticing

At the interplay of the two approaches presented above, we particularise a language-informed notion of mathematics teacher noticing: *noticing languages for content teaching*. Since the first author has researched the challenges faced by primary school students when learning fractions (e.g., González-Forte et al., 2020), we illustrate the theoretical and practical work around this newer notion specifically linked to this content. Considering the three processes in our approach to professional noticing, here illustrated for variations only, noticing languages for content teaching refers to:

Identifying mathematically relevant names, explanations and variations in languages for content teaching. Given, e.g., the situation of a teacher who is talking about the division of the unit into equal-size parts while drawing different rectangle models on the board, we want student teachers to develop the ability to *identify* the importance, in the language-responsive teaching of fractions, of using sentences to exemplify *variations of the shape of equal-size parts*, alongside other resources like drawings.

Interpreting names, explanations and variations in languages for content teaching with regard to their potential for reducing school learning challenges. Given the above-mentioned situation, we want student teachers to develop the ability to *interpret* the importance, in the language-and-learner responsive teaching of fractions, of *variations of the shape of equal-size parts* in order to help learners to challenge the frequent thinking of the equal-size parts of the continuous whole as always equal-shape.

Deciding language-and-learner responsive names, explanations and variations in languages for content teaching. Given the same learning challenge and a similar situation, now with a teacher who is talking about dividing the unit into equal-size parts and who draws one rectangle divided into equal-shape parts, we want student teachers to develop the ability to *decide* on the importance, in the language-and-learner responsive teaching of fractions, of alternative representations of rectangles based on *variations of the shape of equal-size parts* in order to challenge learners' thinking.

DESIGN OF MATERIALS TO ENHANCE PROCESSES OF NOTICING LANGUAGES FOR CONTENT TEACHING

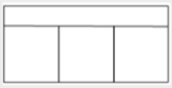
The ultimate objective in our collaboration is to enhance, in teacher training programmes, processes of noticing languages for content teaching aimed at resourcing school content learning. The consideration of appropriate materials and how to design them is thus key. On the one hand, we consider the design of preparatory theoretical documents that would guide student teachers in identifying relevant aspects of languages for content teaching, and in interpreting them in relation to knowledge of (school) mathematics and of learning challenges. On the other, we consider the design of representations of practice in the form of transcripts of either real or fictional languages of teachers in content teaching, together with prompting questions. The latter serve to identify, interpret, and take knowledge-based decisions as to which names, explanations and variations could improve the teaching languages in the transcripts.

A preparatory theoretical document

This document is designed to explain and illustrate the potential of language and some of their verbal tools for the teaching of fractions in the primary school. Operational definitions of naming (names or vocabulary within the school register of fractions), lexicalization (explanations of mathematical meanings regarding fractions) and exemplification (variations of elements related to fractions) are given with short instances of fictional languages for teaching fractions. Some of these instances intentionally miss opportunities of addressing learning challenges documented in the specialized literature. Table 1 reproduces two extracts that have been translated from the original document, one for naming (the names chosen are *equal-size parts*, *numbers* and *fractions*, and *nonequal equivalent fractions*), the other for exemplification (the chosen variations refer to *the size of the parts, and the pairs of fractions to be compared*). In each teaching situation, one instance is language-and-learner responsive (e.g., B2 or G2 are a model of more precise languages of fractions, which in turn respond to specific, well-documented and somehow predictable learning challenges), and the other instance (e.g., B1 or G1) is less responsive regarding the missed opportunities to situate the language within the content register more clearly and/or to address learning challenges. Moreover, each pair (e.g., B1 and B2) come with reflective questions on whether both instances would equally support school learners when facing the enunciated learning challenge.

Three teaching situations that illustrate the introduction of vocabulary		
A1. *If we eat one part of a pizza divided into four, we leave three parts uneaten.*	B1. *Which is larger, 2/3 or 4/5?*	C1. *One half is two quarters.*
A2. *If we eat one part of a pizza divided into four* ***equal-size parts****, we leave three parts uneaten.*	B2. *We have two* ***numbers****, 2/3 and 4/5. Which of the* ***fractions*** *is larger?*	C2. *One half and two quarters are* ***nonequal equivalent fractions****.*

LEARNING CHALLENGE
Understanding that a fraction is a number, and they are not two numbers separated by a slash.

Three teaching situations that illustrate the variation of examples		
G1. *This is a rectangle divided into four equal-size parts.* 	H1. *In 2/5 the parts into which the whole is divided are larger than in 5/7.*	I1. *Which is larger, 2/3 or 4/9?*
G2. *This is a rectangle divided into four-equal size parts.* 	H2. *In* ***1/5*** *the parts into which the whole is divided are larger than in* ***1/7****.*	I2. *Which is larger,* ***2/3*** *or* ***7/8****?*

LEARNING CHALLENGE
Understanding that the parts dividing a continuous whole are of equal area, that is, equal in size and variable in shape.

Table 1: Extracts of a translated version of the theoretical document

Short extracts of two fictional dialogues of one teacher with one primary school student each are included at the end of the preparatory document. These dialogues show the teaching of the equal-size condition of the parts in the part-whole relationship. The first teacher names the unit, the equal-size parts and the equitable sharing, amongst other specialised forms of vocabulary within the register, and gives explanations and mathematically relevant variations to help the student to overcome concrete learning challenges. The second dialogue serves to present a contrast. The teacher here misses several opportunities to use names, explanations and variations that support the learning of fractions. Although these dialogues are representations of practice, they are shown in the theoretical document to illustrate how names, explanations and variations represent intersecting tools in the classroom language of the teacher, rather than discrete elements working in isolation. Moreover, these dialogues situate words, words into sentences and sentences within the broader level of discourse.

A document for professional practice

This paired material contains three fraction comparison classroom situations which are represented through transcripts of interactions (dialogues) between one primary school teacher (Carlos, Patricia and Raquel) and one student each (David, Roberto and Lucía). The student teachers have the basic information at their disposal in the theoretical document and this allows them to engage in the three intended noticing processes. In that preparatory document, instances regarding fraction comparison are illustrated with respect to language-and-learner responsive names, explanations and variations, anticipating content knowledge and knowledge of school learning challenges.

The teachers' languages are designed to show different emphases on the use of names, explanations and variations. Carlos uses relevant names and explanations but does not offer variations that could challenge David's reasoning, which is biased by natural number thinking. Patricia uses relevant explanations and variations but does not offer names that could challenge Roberto's reasoning based on the difference between numerator and denominator. Raquel uses relevant names and variations but does not offer explanations that could challenge Lucía's reasoning based on choosing the fraction with the smaller denominator as the larger fraction. The student teachers have to read each dialogue and answer five questions focused on our noticing processes: Q1) *Identify*. What mathematically relevant names, explanations, and variations are used by the teacher? Q2. *Interpret.* What learning challenges may this talk help to reduce by means of these… names? (Q2.1) explanations? (Q2.2.) variations? (Q2.3.) *Decide*. Drawing on your answers, what other names, explanations or variations would support the learning of fractions? Choose a teacher intervention and propose a change.

Below we reproduce an English version of the dialogue between Carlos and David. Instead of two thirds, e.g., we write 2/3 because the teacher and the learner are supposed to say the names and to write them symbolically on the board. For clarity in this report, we mark the content-relevant names (except for names of fraction representatives such as two thirds) in bold and underline the explanations in teacher talk. In our design of this dialogue, the variations intentionally fail to support David when interrogating the validity of his reasoning. The natural number thinking bias here is the situated reference for the identification and interpretation of names, explanations and variations, whether explicit or absent, that would increase language-and-learner responsiveness in the teacher talk. Carlos misses the opportunity to introduce variations of the numerators and denominators of the fractions to be compared that would allow to question the understanding of these terms as natural numbers.

Carlos: I propose a challenge. I give you pairs of **fractions** and you compare the **fraction size**. Let's take 2/3 and 7/9. Which **fraction** is larger?

David: 7/9!

Carlos: All right, David. 7/9 is larger than 2/3. How did you come to it so quickly?

David: It's very clear. I didn't calculate anything.

Carlos: What did you know? Can you explain?

David: Yes. I saw the numbers. I mean, I know it because of the numbers.

Carlos: What **numbers** are you referring to?

David: Well, I am referring to numbers two and three, and numbers seven and nine. I always look at the two numbers... if they are bigger. Since seven is larger than two, and nine is larger than three, I know that 7/9 is larger.

Carlos: <u>But you have to remember that a **fraction** is a **number**, not two numbers separated by a slash. When you say numbers, you are actually referring to **terms**, the **numerator** and the **denominator**</u>. So, if we have **numbers** 1/4 and 5/9, which **fraction** is larger?

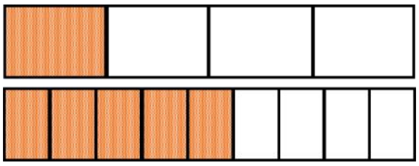
David: Now 5/9 is larger.

Carlos: How do you know?

David: Same reason. Numbers five and nine are larger, so that is the largest fraction.

Carlos: Could you make a graphical representation of both **fractions**? <u>Remember that we must represent **fractions** using the same **whole** for them to be **comparable**. Otherwise, they are **not comparable**</u>.

David: Yes. Here they are.



Carlos: So, was your comparison right? Is 5/9 larger than 1/4?

David: Yes, it is clear that 5/9 is larger. The larger the numbers, the larger the fraction.

Carlos: <u>Remember that a **fraction** is a **number** that expresses a relationship between two **terms**, the **numerator** and the **denominator**. These terms are **not comparable** like natural numbers. Which **fraction** is largest depends on the **quantity of equal size parts** dividing the **whole**, but also on the **quantity of parts taken**</u>.

LOOKING FORWARD THE NEXT COLLABORATIVE STEPS

We reported here the results of our collaborative study which seeks to theorise and prepare materials for developmental work on processes of noticing languages for content teaching in MTE settings. We anticipate that the introduction of the types of materials presented, covering a range of mathematical contents, will provide student teachers with basic professional knowledge and allow them to understand how the use of language can play an essential role in their teaching of mathematics. We expect mathematics teacher educators other than ourselves will use the material outputs of our study in their teaching. It is therefore important to initiate the implementation and evaluation of the materials in our contexts so as to explore the learning opportunities and challenges generated in MTE practice. Empirical insights stemming from the implementation of the materials in university classes will help to improve the materials, and to continue refining the theoretical tools and design processes.

We may have seemed to assume that mathematics teachers from other contexts will simply reuse the tasks we provided. On the one hand, the documents will need a careful meaning-responsive specialised translation if they are to be redesigned in a language other than the original. On the other, while student teachers may have developed basic knowledge of the mathematical contents, they may not be accustomed to producing explanations and reflecting on variations. While the preparatory document introduces explanations and variations, and hence involves some indirect teaching of them, it is not primarily designed to promote student teachers' learning or practising of the discursive practices embedded. Developing the ability to respond to critical uses and omissions of mathematically relevant explanations and variations in teachers' content languages may therefore become problematic. At some point in the collaboration, the studying and working with language tools at the levels of words and sentences will require specific attention and training in order to sequence explanations and variations at the level of classroom discourse and mathematical discourse practices. Further directions of work can still be planned within the granular levels of words and sentences. In the dialogue with Carlos and David, we foresee the potential of studying tools in language for giving sentences with encoded interrogations of content meaning.

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INVESTIGATING THE LEARNING PROCESS OF STUDENTS USING DIALOGIC INSTRUCTIONAL VIDEOS

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The need for high-quality remote learning experiences has been illustrated by the COVID-19 pandemic. As such, there is a need to explore instructional videos that go beyond expert exposition as the main pedagogical approach. An emerging body of research has begun to investigate instructional videos that feature dialogue. However, this body of research has focused primarily on whether such videos are effective. In contrast, the purpose of our study is to investigate the dialogic learning processes involved as students viewing dialogic videos develop mathematical meaning. We employed a Bakhtinian perspective to analyse the learning of a pair of Grade 9 students who engaged with dialogic instructional videos. The results focus on ventriloquation as a learning process.

BACKGROUND AND PURPOSE

The explosive growth in the number of online mathematics videos and the dramatic need for such videos during the COVID-19 pandemic has allowed educators to reimagine how students can learn mathematics. However, the effort to increase access to high-quality learning experiences through online videos has been limited by their uniformity in expository presentation, emphasis on procedural skills, limited attention to mathematical argumentation, and missed opportunities to address common student difficulties (Bowers, Passentino, & Connors, 2012).

In response, our research team created online math videos featuring the dialogue of secondary school students. Alrø and Skovsmose (2004) define dialogue as a conversation that involves the quality of inquiry, referring to an interaction that aims to generate new meanings or ways of comprehending. Our videos are unscripted to capture authentic student confusion and resolution of dilemmas. Each video shows a pair of students (called the talent) next to their mathematical inscriptions (Figure 1), which allows other students viewing the videos (called vicarious learners, or VLs) to see both the talent and their work. “Vicarious” refers to indirect participation in the dialogue of others (Chi, Roy, & Hausmann, 2008). A teacher guides the talent and can be heard but is not seen, so that the focus remains on the talent’s reasoning. The videos also feature annotations of the talent’s work and occasional voice-overs that highlight key mathematical ideas the talent have voiced.

Dialogic videos have been used in a small body of interdisciplinary research, much of which has focused on quantitative studies of the effectiveness of learning vicariously (e.g., Muldner, Lam, & Chi, 2014). Much less work has sought to understand how this learning occurs. Observing the voicing of common misconceptions seems to play an

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important role (Muller, Sharma, & Reimann, 2008), as does the inclusion of an authentic learner who displays confusion and asks questions (Chi, Kang, & Yaghmourian, 2017). The purpose of our study is to contribute to this work by investigating the dialogic learning processes involved as VLs develop mathematical meaning.

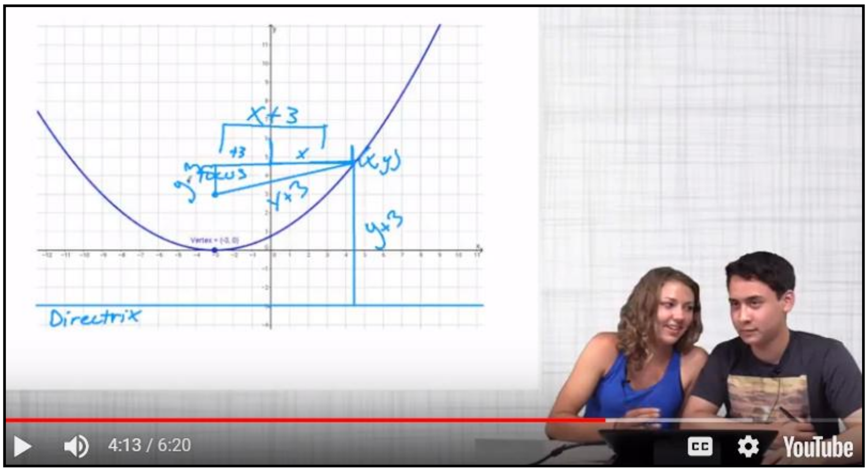

Figure 1: Screenshot for an online dialogic mathematics video

THEORETICAL FRAMEWORK

To investigate the learning processes as the two VLs engaged with the dialogue of the videos, we turned to Bakhtin's (1981) theory of dialogism. According to Bakhtin, the origin of our personal ways of reasoning is others' expressed thoughts, what he calls *voices*. Specifically, voices are words or actions and their associated meanings (Kolikant & Pollack, 2015; Silserth, 2012). As learners express their own voices, Bakhtin (1981) would suggest the words they use are only partly their own. "The word in language is half someone else's. It becomes, 'one's own' only when the speaker populates it with his own intention, his own accent, when he appropriates the word, adapting it to his own semantic and expressive intention" (p. 293). As this quote suggests, Bakhtin's claim is not simply that learners mimic the words and phrases of others, though at times this can occur. Rather, learners appropriate another's voice, including the meanings associated with that voice, for their own purposes. In this way, this voice influences their thinking.

The learning process in which a learner appropriates a voice is called *ventriloquation*. We are particularly interested in instances of ventriloquation when learners go beyond a simply repeating the words or actions of others, but when the voices of others begin to influence the learner's thinking. In these instances, the learner needs to integrate the new voice with their collection of previously internalized voices—those that already influence their thinking. This collection of voices forms the learner's *personal narrative*.

The process of integrating a new voice with the voices in a learner's personal narrative is not always straightforward. Learners may resist a new voice. For example, Taylor

(2003) illustrated resistance during a math methods class for preservice elementary teachers, in which she had been trying to shift the focus of the class from memorizing procedures to proving and justifying. She recounted how a student, Lee, who did not yet see the purpose of or need for proving flouted the teacher's request for a proof and instead responded with a sarcastic remark. The student's demeanour, facial expressions and tone indicated resistance to the integration of "proving" with the voices forming her personal narrative around the nature of mathematics.

METHODS

We investigated the ventriloquation process as two grade 9 students (14-15 years old) watched dialogic instructional videos we created. The two VLs recruited for this study, Desiree and Belinda, had participated in a previous study, which focused on the range of orientations that a larger group of students (26 students) had towards these video lessons. In the previous study, Desiree and Belinda had made good mathematical progress with the first lesson, suggesting they would be good candidates to the continue on with the lessons. They were recruited from an ethnically diverse school in the United States. In their regular Algebra 1 class, they were earning grades in the B to D range. Both were fluent in English and Spanish.

The videos the VLs watched were part of a 10-lesson instructional unit on parabolas. The overarching goal of the video unit was to support the derivation of the vertex form of a general parabola as $y = \frac{(x-h)^2}{4p} + k$. The unit began with the talent being given the geometric definition of a parabola, which is the set of points that are equal distance from a fixed point (called the focus) and a fixed line (called the directrix). Over the course of the 10 lessons, the talent used this definition to first, find the equation for the family of parabolas with vertex at the origin, namely $y = x^2/(4p)$, where p is the distance from the vertex to the focus, and then leverage this equation to derive the general equation of a parabola, with vertex at (h, k). A major focus of the unit was on quantitative reasoning, where a quantity is one's conception of a measurable attribute of an object (Thompson, 2011). In particular, the quantitative meanings of the variables and parameters in the derived equations as distances were emphasized in these lessons.

The VLs participated in 9 research sessions, which occurred after school in a classroom at the VLs' school. Each session lasted 75-90 minutes. In these research sessions the VLs were asked to engage in mathematical tasks that mirrored those given to the talent in the instructional videos. When the talent's task was complex, the same task was given to the VLs. In other instances, similar tasks were given (e.g. some numerical values were changed) to ensure a high level of problem solving for the VLs.

Two researchers participated in these sessions. One operated two camcorders, one focused on the VLs' written work and one focused on the VLs as they interacted with each other. The other researcher interacted with the VLs. She sat on the other side of the room from the VLs while they worked on the math tasks and watched videos, but would come over when they were finished or stuck so the VLs could explain their thinking. The researcher's purpose was to understand the VLs' reasoning as they engaged with the dialogic instructional videos, rather than to provide new information. As such, she left many areas of confusion unresolved.

Analysis began with the creation of descriptive accounts of the 9 research sessions (Miles & Huberman, 1994). From these descriptive accounts, we identified 9 candidate topics where the VLs showed progress in their understanding. We selected the VLs' meaning for the parameter p in the equation $y = x^2/(4p)$, because it is complex for learners and is also important mathematically. We focused our analysis on Research Session 6, because it was during this session that the VLs were able to articulate the meaning for p as the distance between the origin and the focus. Prior to this, the VLs had used p as they derived the equation $y = x^2/(4p)$, but not with this quantitative meaning. Instead, they managed other meanings for p, such as a particular number used in the calculations when finding the equation for a specific parabola or the y value of the focus.

In Research Session 6, the VLs were given two p values, $p = 1.5$ and $p = 2.5$, and were asked to graph the parabola, write its equation, and label p, the focus and the directrix. This tasked mirrored a task that was given to the talent in the instructional videos, though the talent were given $p = ¼$ and $p = ½$. In the previous research session, the VLs had derived the general equation for a parabola with vertex at the origin, $y = x^2/(4p)$.

We analysed the VLs' learning process using a Bakhtinian lens. In particular, we identified the voices that the VLs used as they worked on the task. We also coded for aspects of ventriloquation from the literature, such as repetition, resistance, and integration.

RESULTS

In this section, we will present evidence to support the claim that the VLs came to understand the p-value of a parabola as a distance by ventriloquating a voice of "p as a distance" from a dialogic instructional video. However, this process was not simple. It only happened after they had watched the video twice. The first time they watched the video they were unable to engage with the voice "p as a distance," as they said the video was too confusing. We will provide evidence that they were eventually able to make use of the voice after watching the video a second time, but only after initially resisting the voice and relying on their personal narrative, repeating the voice, and finally integrating it with their personal narrative.

Description of voices in the dialogic instructional video

In the video that the VLs watched, the talent grappled with a task similar to the one given to the VLs, in which they were to place the focus of a parabola with $p = 1/4$. Initially, the talent were confused about how to do this, but they eventually decided that the focus should be placed at (0, 1/4), since this is 1/4 units away from the vertex at (0,0) and $p = 1/4$. The idea that p is the distance from the vertex to the focus was then reemphasized in a voice-over and with an annotation over the talent's work. In the video, the talent expressed two voices that the VLs eventually repeated. The first is "once you know the p value, you know the focus's location" and the second is "p as a distance." The talent expressed both voices as they determined the location of the focus.

Resisting the voice "p as a distance" and relying on personal narrative

After the VLs finished watching the video the second time, the researcher asked them what they noticed about p, the focus, and the directrix. Desiree responded, "It's getting harder, intense." Then, instead of engaging with p, the VLs graphed the parabola with p = 1.5 using point substitution. They did so by finding the equation for the parabola by plugging 1.5 into the general equation $y = x^2/(4p)$ for p, which yielded $y = x^2/6$. They then used this equation to generate a table of x and y values that satisfied the equation and then plotted the points. This systematic process to graph the parabola was a significant detour from locating p, taking about 5 mins and 30 seconds to complete.

We see this departure from engaging with p as an instance of resistance in that the VLs set aside the voice expressed in the video of "p as a distance," in favour of a voice that appeared to already be a part of their personal narrative, that of "graphing a parabola by point substitution." Notably, graphing the parabola did not help them locate p. However, it was a familiar voice, unlike the voice "p as a distance."

Repetition of two voices related to p

Once the parabola was graphed, the VLs read the task prompt again, which asked them to mark in p, the focus, and directrix. In response to the task, Desiree suggested that the focus would be at (0, 1.5), justifying the location by saying "because it's 1.5," presumably in reference to the p value. She then labelled the point (0, 1.5) as the focus and drew in a line at y = -1.5, which she labelled the directrix. However, the VLs had not yet marked in p, as the task requested. At this point they explained their reasoning to the researcher, recounting their point substitution method. The researcher noted that they were able to find the equation and graphed the parabola, but asked where p would be in their graph.

Researcher: Where is p?

Belinda: This [sweeping gesture from the origin to directrix] and this [sweeping gesture from origin to focus].

Researcher: Can you label those [the p values]?

Desiree: You do that [hands Belinda the pen].

Belinda: This is p [draws and labels line segment from origin to line y=-1.5] and this is p [draws a labels line segment from origin to focus at (0,1.5)].

The researcher then asked what they had learned and Belinda explained why knowing p is useful, claiming "when you're given p, you know the focus and directrix".

In this episode, the VLs began to engage with p as they repeated two voices from the video. The first is "once you know the p value, you know the focus's location." This voice was expressed by Desiree when she justified the placement of their focus at (0, 1.5) and by Belinda when she said "when you're given p, you know the focus and directrix." They then further engaged with p as they repeated the action associated with the voice of "p as a distance" when they drew in segments from the origin to the focus and from the origin to the directrix to indicate p, which are similar to the annotations in the video. However, while the voice of p as a distance was beginning to emerge, the VLs were still struggling to articulate verbally that p is a distance.

Integrating the voice "p as a distance" with the VLs' personal narratives

The VLs then moved on to another task, where p = 1/2. In contrast to their previous method, they seemed to use p immediately to find the focus and directrix. In response to the task, Belinda promptly drew in the directrix at y = -0.5 and labelled the segment between the directrix and vertex as p. Similarly, she then drew in the focus at (0,0.5) and labelled the segment between the focus and vertex as p. Meanwhile, Desiree started to create the graph using the point substitution method. Belinda then described their work to the researcher.

> Belinda: Since we know that p is the distance between the focus [and vertex] and directrix [and vertex] we put .5, because ½=.5.

In this episode the VLs clearly articulated the voice of p as a distance. Belinda said "p is the distance..." and marked in and labelled the extents between the focus and the vertex and between the directrix and vertex as p. While the VLs made similar inscriptions while working on the previous task, they made the inscriptions more quickly in this episode. Furthermore, not only are the VLs able to describe p as a distance, but this meaning for p seemed to influence their thinking. Belinda explained that knowing p allows one to efficiently locate the focus and directrix. Instead of first needing to graph the parabola with a point substitution method, they quickly mark in the focus and directrix, using the fact that p = .5. At same time, however, they did not abandon the point substitution method, as Desiree continued to use it create a graph. Rather, these voices were integrated together as the VLs used both to engage in the task. As such, we claim this voice was integrated with their personal narrative, not only because it became influential in their thinking, but also because it was coordinated with another voice that was already part of their personal narratives.

DISCUSSION

The results of this study illustrate the complexities of the ventriloquation process. Rather than simply adopting a voice as it was presented in the videos, the VLs first resisted the new voice. However, unlike in Taylor's (2003) example, the VLs in this study did not seem to be actively antagonistic towards the voice expressed in the video. Rather, they seemed confused by the voice and were unable to make use of it until they invoked their personal narrative. By evoking voices in their personal narratives, they made space that allowed them to integrate the voice of "p as a distance" with those. We see personal narratives as a tapestry of voices, meaning learners need to understand how the voices in the personal narratives relate to one another. By using the point substitution method to create a graph alongside the voice "p as a distance," the VLs had the opportunity to begin to explore the connections between the equation, the graph, the focus, the directrix, and p.

IMPLICATION

As video developers create dialogic instructional videos, they should consider how to support VLs in ventriloquation. VLs need opportunities to integrate new voices into their existing personal narratives. The results from this study suggest that this may mean they need opportunities to evoke voices in their personal narratives and explore the connections between these voices and those presented in the videos. One support for the VLs in this study to explore these connections seems to have been the opportunity to explain their thinking to a partner. We suggest that VLs be given opportunities to discuss with other students, whether in-person or virtually, how they are reasoning about tasks related to those explored in the video and the connections between how they are reasoning and the ways of reasoning presented in the videos.

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DIFFERENCES IN TEACHER TELLING ACCORDING TO STUDENTS' AGE

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Teacher telling can support but also hinder learning. In inquiry activities, telling that removes productive struggle may be problematic. In this study, different aged students experimented in a digital learning environment to build rules for a balance beam. We examined how the amount of teacher telling vary according to students' age and the sophistication level of the rule. We collected video data from 21 pre-service teachers when each of them guided eight, 10, and 12 year old students. We found that the amount of teacher telling generally was not related to students' age. However, when considering teacher guidance for more sophisticated rules, teacher telling was related to students' age. Thus, the focus of the guidance is an essential factor affecting telling and teachers may have pressure for guiding students towards a high-level product.

INTRODUCTION

There seems to be consensus that students need some support or *guidance* in inquiry activities (Lazonder & Harmsen, 2016). One essential dimension in guidance is the degree of students' *autonomy* (Vorholzer & von Aufschnaiter, 2019). On one hand, detailed instructions in performing something may remove student autonomy. On the other hand, sometimes non-specific guidance such as open or general questions do not offer enough help for students. Olsson and Granberg (2019) presented evidence that students are more able to perform an inquiry activity when having detailed instructions than those who received more open task. However, they also found that the learning results were more durable for those who were able to perform the open task.

One type of detailed guidance is *teacher telling*, in which the teacher provides full information or explanation to some issue leaving no autonomy for students in examining or working on this issue. Teaching mathematics solely through telling is against many recommendations (e.g., NCTM, 2000) and, particularly in inquiry activities, it hinders the underlying idea of students investigating mathematics. However, also completely avoiding telling is problematic. Smith (1996) suggests that avoiding telling may affect negatively on teachers sense of efficacy. Furthermore, Chazan and Ball (1999) point out several instances in which teachers may need to tell. Indeed, Baxter and Williams (2010) found that two teachers' teaching aligned with reform mathematics in many ways and the teachers at times strategically engaged in telling. Furthermore, Ding and Li (2014) suggest the need to flexibly use both direct guidance and facilitating guidance.

Thus, the literature suggests that productivity of telling depends on the context including the students, the mathematical issue, the purpose of telling and the point in

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time. *Productive struggle* is a concept that may help to consider telling. According to Hiebert and Grouws (2007), in productive struggle, students expend effort to make sense of mathematics. Thus, we can consider whether telling increases or decreases productive struggle. In an inquiry activity, telling may increase productive struggle, for example, if the teacher tells the meaning of a mathematical concept and students ponder how to apply the concept. On the contrary, telling may decrease or even remove productive struggle, if the teacher tells the steps how to achieve a particular result.

Finally, Lobato et al. (2005) suggest considering the function of telling instead of the form of telling. The teacher may introduce information in the form of questions or in the form of declarative statements. For example, a series of questions can introduce an idea to students.

An underexamined issue in teacher guidance seem to be the relation between guidance and students' age. In their review of studies of inquiry-based learning in mathematics and science, Lazonder and Harmsen (2016) cautiously note that younger students may benefit from more specific guidance. They call for more studies in teacher guidance of different aged students, particularly when a same task is used with students of different age. Songer et al. (2013) found that while some kind of guidance was used similarly with younger and older students, more specific guidance (e.g., turning an open-ended question into a few multiple-choice options) was used with younger students. As there exists suggestions that more specific guidance might be suitable for younger students, it may be that teacher telling is used differently depending on students' age. In this study, we focus on this issue taking into account Lazonder and Harmsen's (2016) recommendation of using the same task with students of different age.

In this study, the same inquiry activity was used in grades 2 (8 year old), 4 (10 year old), and 6 (12 year old). In the activity, the students experimented in a digital learning environment to build rules that describes an equilibrium state for a balance beam. As several rules of different sophistication level are possible, the activity is suitable for the different grades and teachers may need to guide students differently. In this paper, we focus on teacher telling that decreases productive struggle by removing student autonomy in considering a particular rule as we hypothesize that this may happen more often with younger students and with more sophisticated rules. The following research questions guided the analysis:

- How does the amount of teacher telling vary according to students' age?
- How does the amount of teacher telling vary according to sophistication level of the rule with different aged students?

METHODS

Context

We developed a digital learning environment involving dynamic representations. In this environment, students work to construct a rule or several rules that can be used to find an equilibrium state for a balance beam. Using dynamic representations made with

GeoGebra, students can experiment with a balance beam where two birds with varying weights can be placed on different sides of the fulcrum at different distances from the fulcrum (Fig. 1). The environment contained a laboratory where students could explore rules in an open setting and tasks in which they were supposed to use their rules. Usually, students first build less sophisticated rules such as 'same weights and same distances' before more sophisticated rules such as 'the product of the weight and the distance on both sides are equal'.

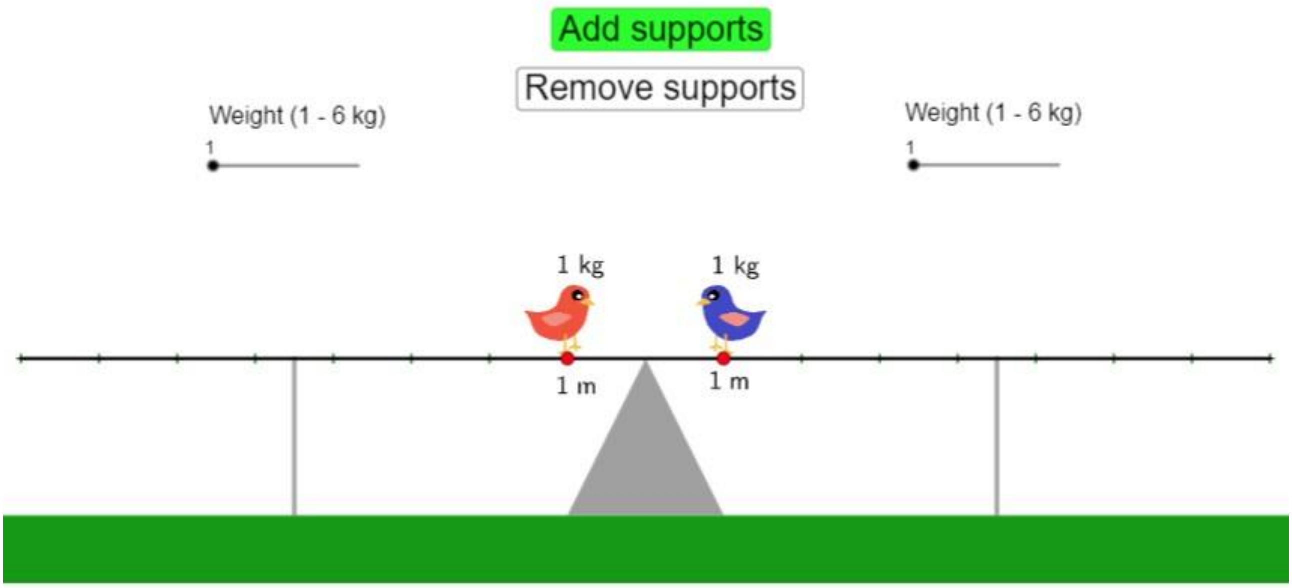

Fig. 1: The dynamic representation for exploring the rule for balance

The students worked in groups of three students for 40 minutes. Each group had one pre-service primary school teacher guiding their work. Each pre-service teacher guided one second grade, one fourth grade and one sixth grade group at different times. The pre-service teachers participated in a course in which they were prepared to guide students. For example, they used the same environment as students and discussed various kinds of rules for balance that are possible to build. Discussion also included pedagogical ideas, such as building on students' thinking even though the students would not be heading towards the most sophisticated rule.

Data collection

Altogether 21 pre-service teachers (hereafter shortly teachers) participated the study. Thus, data was collected from 21 second grade, 21 fourth grade, and 21 sixth grade groups that had three students in each group.

The screen of each student group's laptop was recorded using screen capture software. The software also captured audio from the laptop microphone and video from the laptop webcam in sync with screen capture. In addition, a small action video camera recorded the group from the side to enable the recognition of gestures and the person who is talking. All these data sources from each group were synchronized in one video file.

Data analysis

Data was transcribed, and the data analysis used transcript and video in parallel. First, we identified episodes in which students either tried to balance the beam or reflected on the result of trying to balance the beam. For each episode we coded whether the rule was expressed (partially or completely) or not. We further coded the sophistication level of the expressed rule as presented in Table 1.

	Description	Examples
Proportional rule	The rule consists of a correct relation between the proportion of masses and proportion of distances expressed in any form with or without symbols or variables.	Weight times distance equals on both sides. The weight is halved, and the distance is doubled. 9 kg / 3 kg = 3 and 6 m / 2 m = 3
Other rule	The rule consists of correct qualitative properties or the rule consists of non-generalizable relations between the variables.	The heavier bird is closer, and the lighter bird is further away from the fulcrum. If the weight is doubled, the distance between the birds is increased by one.

Table 1: Codes for the sophistication level of the rules for balance

Then, we selected episodes in which a rule was expressed and the teacher guided the students in building or using the rule. Thus, we omitted other kinds of guidance that could relate to, for example, use of the environment. If an episode contained teacher guidance related to a rule, we coded whether the guidance included telling that removes student autonomy related to building or using the rule (Table 2).

The reliability was tested with two coders. In all the dimensions (episodes, expression of rule, rule type, autonomy level), Cohen's kappa coefficient was above 0.80, which indicate that the reliability is very good.

	Description	Examples
Telling	The teacher guidance leaves no choices for the students in building or using the rule. The teacher lays out the essential components of the rule.	Instead of adding, you can divide these two and these two. Series of questions: What could you do to these numbers? Could you multiply them? Then, what about these numbers?

Guiding without telling	The teacher guides building or using of the rule but leaves some choices for the students. The students produce at least some essential component of the rule.	Revoicing a rule that the students expressed. Why did it stay in balance? Does the same rule work here?

Table 2: Codes for teacher telling that removes student autonomy

RESULTS

Table 3 gives the frequencies and percentages of the episodes in which the teacher guided with telling or without telling and the guidance was related to any kind of rule. Table 3 also includes episodes in which the rule was not eventually expressed. A chi-square test of independence showed no statistically significant relation between students' grade and the amount of teacher telling, $X^2(2, N = 752) = 2.449$, p = 0.294. In all the grades, less than 1/5 of the episodes contained telling. In addition, the number of episodes in which teachers guided the students was about the same across the grades, which indicates equal amount of guidance in all the grades.

	Telling		Guiding without telling	
	f	%	f	%
2nd grade	35	16	181	84
4th grade	30	13	202	87
6th grade	35	12	269	89

Table 3: Episodes of teacher telling or guiding without telling related to any rule

To examine if the sophistication level of the rule affected telling, we examined separately episodes in which proportional rules were expressed and episodes in which other rules were expressed. Table 4 gives the frequencies and percentages of the episodes in which the teacher guided with telling or without telling students related to proportional rules. A chi-square test of independence showed statistically significant relation between students' grade and teacher telling, $X^2(2, N = 140) = 8.138$, p = 0.017. In case of proportional rules, telling existed more often in second grade. Half of the episodes in second grade contained telling. In addition, the total amount of episodes differed across the grades, which indicate that younger students less often considered proportional rules.

	Telling		Guiding without telling	
	f	%	f	%
2nd grade	7	50	7	50
4th grade	5	17	25	83
6th grade	17	18	79	82

Table 4: Episodes of teacher telling or guiding without telling related to proportional rules

Table 5 gives the frequencies and percentages of the episodes in which the teacher guided with telling or without telling students related to other rules. A chi-square test of independence showed no statistically significant relation between students' grade and teacher telling, $X^2(2, N = 387) = 2.797$, p = 0.247.

	Telling		Guiding without telling	
	f	%	f	%
2nd grade	27	21	104	79
4th grade	23	16	123	84
6th grade	14	13	96	87

Table 5: Episodes of teacher telling or guiding without telling related to other rules

Based on tables 4 and 5, teacher telling with 2nd grade students was more common in proportional rules than in other rules.

For example, in the following excerpt a teacher tells second grade students the proportional rule in one case when the students have balanced the beam with 12 kg in 1 m distance on the left side and 6 kg in 2 m distance on the right side.

Teacher: Good, yes. Why did it stay in balance? Let's write this down. Juliana, would you write this?

Juliana: Yes.

Teacher: So, why do you think that it stayed in balance?

Alex: Well, because the other was six, and then because 6 + 6 is 12 (inaudible) it was like half.

Teacher: Really good observation.

Alex: And 2 meters.

Teacher: So 6 + 6 is 12 and 1 + 1 is 2. [Points the screen.] Thus, this is two times the weight of this one and this is two times the distance of this one. Isn't it? Really good. You solved it.

The student noticed the proportion of weights being 1/2 but did not yet connect this to distances. Directly after this, the teacher introduced the proportional rule that included both variables and thus, removed the opportunity for productive struggle related to this rule. After this, the group continued balancing the beam without mentioning the proportional rule.

DISCUSSION

In this study, we examined how the amount of teacher telling varies when the same teachers use the same inquiry activity in different grades. We found that the amount of teacher telling generally was not related to students' age. However, when considering teacher guidance for more sophisticated proportional rules, teacher telling was related to students' age. Thus, it is important to consider the focus of telling when examining telling. As Lazonder and Harmsen (2016) pondered, researchers may accommodate the inquiry tasks to the capabilities of the age group, which hinders the possibilities of noticing age-related differences in guidance. In this study, the task was the same across the grades, which allowed noticing that age-related differences existed when the teachers focused on more advanced issues.

The finding that telling is used more often with second grade students when focusing on the proportional rule, may be an indication of the teachers' pressure to reach the high-level rule. The teachers were introduced to various kinds of rules and were instructed to build on students' thinking even though the students would not be heading towards the most sophisticated rule. Nevertheless, the teachers were aware of the proportional rule and may have felted the need to guide students towards that. When the students have major difficulties, it is challenging to help students but still leave space for productive struggle. If only avoiding telling, the teacher may do nothing to assist the students (Chazan & Ball, 1999) or just asks general questions that do not help students (Hähkiöniemi & Francisco, 2019). However, in case of open problems, that have multiple correct solutions, there is also an option of focusing on less advanced solutions. Similarly, Hähkiöniemi et al. (2013) reported that in an open problem, a teacher directed students to consider an easier subproblem to support student reasoning. In the context of the activity used in this study, teachers could focus on less sophisticated rules if the proportional rule is too challenging for the students. This would still allow the students to engage in the inquiry activity in meaningful way and have productive struggle in building lower-level rules.

Finally, we emphasize that telling can also support inquiry, for example, by reminding students of previous knowledge or by introducing standard notation for students' ideas. We only question the productivity of telling that removes productive struggle from students inquiry. Afterall, the inquiry process is more important than the outcome of the inquiry.

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A DESIGNATED PROFESSIONAL DEVELOPMENT PROGRAM FOR PROMOTING MATHEMATICAL MODELLING COMPETENCY AMONG LEADING TEACHERS

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Mathematical modelling is an important component of STEM education in the 21st century. This study examines how a designated professional development program impacts teachers' perceptions of mathematical modelling instruction and their mathematical modelling competency. The perceptions were assessed by a pre-post questionnaire and their modelling competency was measured by their solutions to modelling tasks, over three different timepoints. The results show a positive change in teachers' perceptions of modelling instruction and a positive trend in their competency to apply certain stages of the mathematical modelling cycle. In the study, methodological and practical contributions are discussed with respect to promoting and assessing mathematical modelling competence among mathematical teachers.

INTRODUCTION

Mathematics is considered as the foundation of all the STEM fields, yet studies indicate that there is a gap between the relevance of mathematics as taught in classes, compared to the applicability of mathematics in real-life, particularly in STEM-related fields (Blum, 2015; Kaiser, 2017; Kohen & Orenstein, 2021; Verschaffel et al., 2020). The use of mathematical modelling (MM) provides a method for demonstrating students the applicability of mathematics, as it reflects a transition from a real situation to a mathematical model. MM is a cyclic process that begins and ends with real-world situations unrelated to mathematics, in which a translation is made from the real-world context into mathematical terms toward a mathematical solution to the real-world situation (Blum & Leiß, 2007; Kaiser, 2017; Perrenet et al., 2012). Yet, students face a variety of challenges when it comes to MM, as they are faced with questions that arise from the reality, which they must apply mathematical knowledge to (Ferri, 2017). MM instruction, particularly when it involves a STEM-related context is a significant challenge for mathematics teachers as well. As teachers, they are required to deal with the difficulties of their students who are unfamiliar with modelling as part of formal math lessons (Verschaffel et al., 2020), as well as deal with the same difficulties themselves in applying modelling skills in solving modelling problems with different contexts (Kramarski & Kohen, 2017). Also, since this sort of instruction is not often addressed in formal math classes, it is important that teachers have positive perceptions towards MM instruction (Kohen, Orenstein, & Nitzan, 2019). It is therefore imperative that teachers be trained to have these skills, both as learners and as teachers, through a

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supportive professional development (PD) environment (Darling-Hammond et al., 2017).

THEORETICAL FRAMEWORK – MATHEMATICAL MODELLING

Blum and Leiß (2007) presented an MM cycle which consists of seven stages that reflect the transition from reality and mathematics (see Figure 1). This model describes the actions that a solver must perform in order to solve a real-world problem using mathematical methods. The modelling cycle suggested by Blum and Kaiser's has four main steps. The first two steps involve idealizing of a real-life situation and making it into a realistic model. Mathematization that is the third step is the transition from reality to the mathematical world upon choosing a mathematical model to solve the real-world model, and involves investigating of the mathematical model through the use of mathematical algorithms, routines, and procedures. The last three steps involve ensuring that the results of the model are comparable with reality by interpreting them. The MM cycle closely resembles PISA's mathematical literacy cycle. PISA defines mathematical literacy as the ability to think mathematically in order to solve problems in a variety of real-world contexts (OECD, 2018) (see Figure 2). In terms of the PISA conceptual framework, mathematical literacy includes three main stages that fit to the MM cycle: Formulate, Employ, and Interpret. These matches to PISA’s cycle are visualized in figure 1 as follows: formulate related stages are marked with orange frame, employ related stages are shown in blue frame, and whereas interpret related stages are shown in green frame.

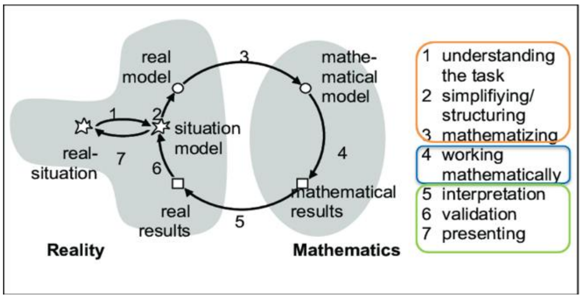


Figure 1. The modelling cycle (Blum & Leiβ, 2007)

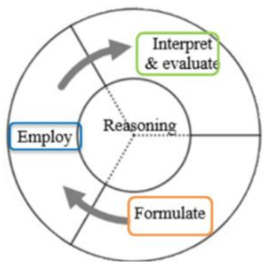


Figure 2. PISA’s Mathematical literacy model (OECD,2018)

The modelling tasks applied in this study reflect the narrowed modelling cycle. However, as these tasks are suited for formal school mathematics (Kohen & Orenstein, 2021), the idealizing stage that takes place within the reality is explicitly provided in the task, thus reflecting a more constrained type of a modelling problem. With that, these modelling tasks have much similarity with the PISA framework, which led us to use the PISA conceptual framework to represent the MM stages.

This research is based on MM tasks with a real-world context, that is retrieved from technology and engineering authentic applications, and an example of that is the 'Iron Dome' task. 'Iron Dome' is an advanced defence radar system that can detect the trajectory of rockets and can calculate the expected impact zone. As soon as the rocket enters the free fall stage, the Iron Dome system uses the information it receives to trace its trajectory, by using mathematical calculations that are based on a quadratic equation. In accordance with the MM processes, formulating is realized in this task with the use of a main question that reflects the transition from reality to the mathematical world: 'How can the rocket trajectory be predicted?' Then, the employing stage involves applying mathematical procedures and graphic representations, to derive the mathematical solution, that is based on the identification of three points in the rocket trajectory and calculating a quadradic equation. In the interpretation stage, based on the mathematical solution and the predicted landing area, students can estimate whether or not the Iron Dome will intercept the rocket.

The purpose of this study is to explore the impact of a designated PD program on the advancement of teachers' MM competencies, as well as teachers' perceptions toward modelling-based instruction. The research question is: What are the changes (if any) in teachers': a) perceptions towards MM instruction, and b) MM competencies?

METHODOLOGY

The context of the study - a PD program for modelling-based instruction.

The current study was conducted as part of a designated 60-hour PD program for leading mathematics teachers, which is held for the goal of training teachers to apply modelling-based instruction. During the PD program meetings, the coaches introduced the modelling framework, and its correspondence to the PISA's framework, introduced modelling tasks with real-world technology or engineering context, and discussed pedagogical content to support the adaptation of modelling-based instruction.

Participants were about 40 math leading teachers who took part in the PD program. Some of the teachers are math coordinators, instructors, or hold key positions in the Ministry of Education in Israel. The teachers have varied teaching experience, with most of them having more than seven years of experience in the education system.

Research Tools and analysis.

The study uses two main tools. The first tool was a pre-post self-reported questionnaire for measuring teachers' perceptions toward MM instruction, on a six-level Likert scale (1, not true at all, and up to 6 - almost always true). The questionnaire aimed to assess teachers' perceptions toward the application of the various modelling processes, i.e., formulate, employ, and interpret in their instruction during math lessons. The questionnaire was distributed to participants at the beginning and the end of the PD program. The second tool was a solution to a modelling problem, which aimed to assess the teachers' MM competency. The teachers were asked to explicitly write all the phases of their solution. Below is an example of a MM problem, that was retrieved from the 'Iron Dome' task (see Figure 3).

The teachers were asked to solve three modelling tasks throughout the PD program in three different time points. The Iron Dome task described above was applied at the beginning of the program as a starting point. Four months later, the teachers solved a problem retrieved from the 'Autonomous Car' modelling task, which involves using ultrasonic sensor technology, that is based on sound speed, and is related to a motion problem where the distance equals time multiplied by speed. Towards the end of the program, the teachers were asked to solve a problem retrieved from the GPS modelling task which deals with how satellite signals are received and analysed in order to determine a GPS receiver's location, which solution is based on a motion problem followed by a Pythagorean theorem.

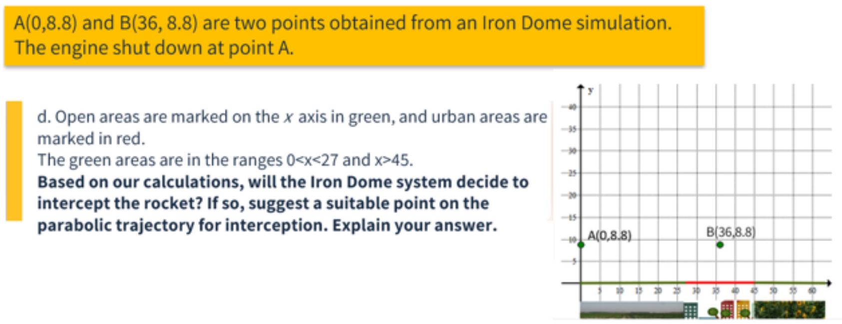


Figure 3. An example for a MM problem, retrieved from the 'Iron dome' task

DATA ANALYSIS

Quantitative data retrieved from the questionnaires was analysed using dependent T-test to determine changes over time. Further, for assessing the teachers' MM competency, we analysed their solutions to the modelling tasks, and graded them in a 3-level process as described below. We then conducted One-Way ANOVA with repeated measures to evaluate the change over time in teachers' modelling competency, as measured referring to the three investigated modelling tasks. We demonstrate the analysis process, based on the modelling problem that was retrieved from the iron dome task, and is presented in figure 4.

Phase 1 – Assessing the level of the various modelling components of the task.

This phase was conducted prior to the tasks being responded to by the participants and was designed to objectively evaluate the modelling competencies the tasks require. Based on a valid rubric (Kohen & Gerrah-Badran, in press) for assessing an authentic MM task, each task was evaluated. This rubric allows to determine the MM competency that are summoned in modelling tasks, referring separately to each of the modelling processes, which is given a grade on a scale of 1 (low level of modelling competency) to 3 (high level of modelling competency).

For the 'Iron Dome' task, the coding was as following. Formulating was assigned to level 2 (medium) since the problem requires working efficiently with two representations (graphic and algebraic), while taking assumptions, such as “falling in an open area” means no interception, so there must be a cut point with axis X. However, there is no requirement to create a new representation, but to work with a familiar one, so the level is medium and not high. Employing was assigned to level 3 (high) since the problem requires planning strategies for a solution while reasoning the mathematical solution such as choosing the cut point on axis X and then finding the quadratic equation. Finally, interpreting was assigned to level 2 (medium) since the problem requires reasoning to provide justification and adaptation to a representation of a situation in the real world, such as the selection of the appropriate point for interception. As there is no requirement to explain the process of drawing conclusions, the level of this stage is medium.

Phase 2 – Using indicators for recognizing modelling components in teachers' solutions.

This phase is based on evaluating the teachers' responses to the various tasks. An indicator (0/1) was given for each component of the modelling process that was recognized in teachers' solutions, based on an indicator that was developed for each task specifically. The indicator development included a validation process performed by math-education experts. It should be noted that the formulating component of the modelling process is a component without which the employing cannot be reached. Therefore, even if the formulating process is not expressed in the written answer, but the employing process was carried out correctly, it was assumed that the teacher went through the formulating process while thinking about the solution. The following example demonstrates the solution of Michael (pseudo) to the iron dome task, and the indicators that were given for this solution, referring to each of the modelling components: "After finding the equation of the function (by placing it in the vertex representation 18,25) it is possible to select a point whose X-rate is for example 15 that will need interception as it enters a built-up area (24.55, 15)".

In this solution, Michael goes directly to the mathematical procedures and explain what procedures should be performed to solve the mathematical aspect of the question. Thus, an indicator of 1 was given to the employing component. In this case, it can be assumed that the formulating process was conducted within his mind, thus this component was also marked with indicator '1'. Then, there is a reference in his solution of returning to the real-world context of the task, but it seems as if Michael did not fully understand the question (the determination weather the Iron Dome system will or won't intercept the rocket, based on its expected fall location). Thus, an indicator of '0' was given to the interpreting component as he reached a mathematical solution but failed to draw conclusions out of it.

Phase 3- Determining a grade for the teacher's MM competency.

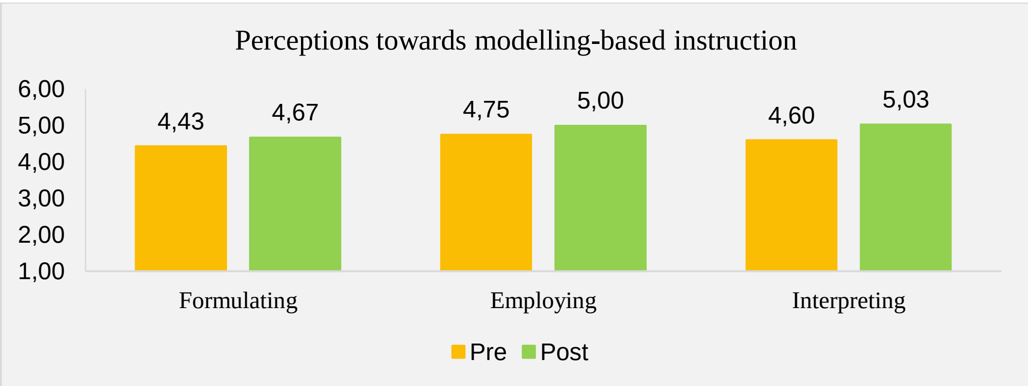
In this phase, a merge of the two previous phases was conducted to determine the modelling competency of the teachers that was reflected in each of the investigated tasks. For each component of the modelling process, we multiplied the objective grade that was given to each modelling component on phase1 by the indicator that was given to teachers' solutions on phase 2. Then we summed up the results and divided it by the sum of grades from phase 1 for the purpose of normalizing the score, so the grades ranged between 0 to 1. Table 1 presents the assessment of Michael's MM competency, as was determined based on the problem retrieved from the Iron Dome task.

Modelling stage	Phase 1	Phase 2	Phase 3	Grade
Formulating	2	1	$2 \cdot 1 = 2$	$\frac{2+3+0}{7} = 0.71$
Employing	3	1	$3 \cdot 1 = 3$	
Interpreting	2	0	$3 \cdot 0 = 0$	

Table 1. The assessmnet of Michael's MM competency

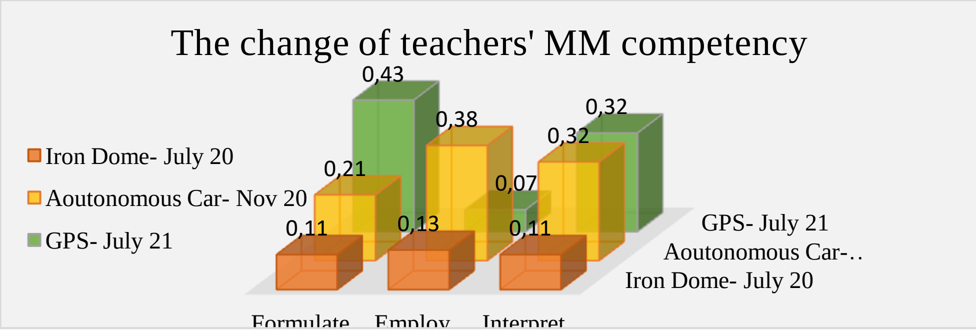
FINDINGS

Findings revealed no significant difference in teachers' perceptions toward modelling-based instruction, with respect to all modelling components, $-1.91<t<-1.05$, $p>.05$. Yet, post-hoc analysis according to Cohen's d effect size indicated that the teachers demonstrated more positive perceptions toward modelling instruction that is based on the formulate ($d = 0.36$) and employ ($d = 0.42$) modelling processes, and particularly with respect to modelling instruction that is based on the application of the interpret process ($d = 0.67$). As Graph 1 below demonstrates, there is a positive trend in teachers' perceptions before and after participating in the program.



Graph 1. Teacheres' Perceptions towards modelling-based instruction, before and after participating in a PD program

Graph 2 below presents the change over time in the teachers' modelling competency. Findings revealed a significant multivariate effect for the three latent variables as a group in relation to three times of measures, indicating higher modelling competency toward solving the third modelling task, $F(3,40) = 10.83$; $p<.0001$, $\eta^2=.619$. Simple main effect tests with Bonferroni adjustment indicated that teachers' competency of formulate and interpret during solving the third task was significantly higher that the competency they demonstrated during solving the first task, and the second task (with respect to merely the formulate competency). For the employ competency, they demonstrated an improvement of this competency during the second task, which decreased during solving the third task.



Graph 2. The change in teachers' MM competency

CONCLUSION AND CONTRIBUTIONS OF THE STUDY

The study findings indicate an improvement in both teachers' modelling competency and their perceptions toward instruction-based modelling. Based on previous studies conducted on the field with the aim of improving the MM capabilities of students, Niss (2001) concluded that applications and modelling capabilities can be learned. Teachers play an essential role in promoting modelling competency among their students (Doerr & English, 2003). However, for learning to occur, teachers must devote time and effort to implementing modelling tasks. Thus, their positive perceptions towards modelling-based instruction, as well as their own modelling competencies are significant in promoting MM among their students.

The most significant improvement in teachers' modelling competency was detected in the formulating and interpreting stages of the process. These two stages represent the main difference between a standard mathematical word problem and a MM one, as they reflect the transition from the real world to the mathematical one (Ferri, 2017; Kaiser, 2017; Perrenet et al., 2012). This finding reinforces the importance of supporting teachers' modelling competency as leaners, through PD programs. However, it remains to be seen whether the employing stage is directly impacted by the PD program or if it is primarily influenced by teachers' previous knowledge that is required to solve the MM task during the employing stage.

An effective PD program allows teachers to progress professionally and changes the way they apply new or improved methods of instruction (Darling-Hammond et al., 2017). The practical contribution of this study is reflected in the presented designated PD program that was found to be effective in enhancing the participating teachers' MM competencies and their perceptions toward modelling-based instruction. In terms of the study's methodological contribution, we produced a tool for measuring teachers' modelling competency, which can be also valuable as a practical tool for teachers and other researchers.

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FOCUSING ON NUMERICAL ORDER IN PRESCHOOL PREDICTS MATHEMATICAL ACHIEVEMENT SIX YEARS LATER

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The development of numerical ordering of number symbols, unlike numerical ordering of other stimuli, such as sets of everyday items, has recently gained growing research interest. Here, we report a nine-year follow-up study with 36 three-year-old children. We investigated how children's focusing on numerical order develops alongside number sequence production and cardinality recognition skills. Results showed large individual and developmental differences in children's focusing on numerical order from the ages of 3 to 6 years. Preschool focusing on numerical order and spontaneous focusing on numerosity predicted curriculum-based math achievement at 12 years of age.

INTRODUCTION

Children learn many important mathematical skills at preschool age. At around 3 years of age, they already practice reciting number word sequence and start to recognize cardinal values for small sets of items by subitizing (Fuson, 1988). Next, at roughly 3.5 years of age, children begin to learn to recognize the numbers of items by counting (Wynn, 1990). Counting skills develop gradually, and there are many subskills involved (Fuson, 1988; Sarnecka et al., 2015). It has been well established that these early mathematical skills predict later academic achievement (Duncan et al., 2007), and a great number of studies have described how early mathematical skills develop (Clements & Sarama, 2007).

Children's tendencies to spontaneously focus their attention on mathematical aspects have drawn increasing research interest (Verschaffel et al., 2020), following research on spontaneous focusing on numerosity (SFON) (Hannula & Lehtinen, 2005). Numerous studies show developmentally significant and domain-specific individual differences in children's tendency to focus on exact numerosity in situations that are not explicitly mathematical, even if they have the mathematical skills to do so (for a review, see McMullen et al., 2019). Children with a higher SFON tendency have been shown to have an advantage in the development of early and later mathematical skills in elementary school (for a review, see Verschaffel et al., 2020).

Studies of numerical order development have found that children who are better at deciding whether three numbers are in numerical order or not have better arithmetic skills later in life (Malone et al., 2021; Attout & Majerus, 2018; Lyons et al., 2014). Children's ability to process numerical order is a unique predictor of later arithmetic skills, and the predictive force seems to increase from Grade 1 to Grade 6, exceeding

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cardinal skills as a predictor of arithmetic skills (Lyons et al., 2014). However, studies regarding numerical order have mainly focused on the order of number symbols; only a few studies have used non-symbolic stimuli (Spaepen et al., 2018), and even rarer are studies using play-based methods with sets of toys.

Numerical order is not just some artificial order related to number symbols. It is a consequence of the corresponding cardinal values, which have the order of each cardinal value being exactly one more than the previous one (Spaepen et al., 2018). Recent studies have shown that even if children acquire counting and cardinal skills, they still lack the generalized knowledge of how counting up the number sequence is related to cardinal value increasing by one (Spaepen et al., 2018; Cheung et al., 2017), which is the mechanism of how all natural numbers are constructed. Understanding how cardinal and ordinal aspects of numbers are integrated during early development is a question that has yet to be answered.

THE PRESENT STUDY

A developmentally important aspect of numeracy has been neglected in previous studies: the recognition of the numerical order of nonsymbolic items. Here, we report longitudinal data on a novel task that was developed for measuring children's focusing on numerical order in the context of a three-year longitudinal study on early numeracy (Hannula & Lehtinen, 2005) with a follow-up study that took place nine years after the first testing. The data sought to emphasize that the use of numerical order in action first requires focusing on the exact numerosity of items in sub-sets, followed by recognizing the number of items in the sub-sets, only after which can the focusing on and recognition of numerical order of sub-sets take place. Focusing on numerical order may thus require well-integrated cardinal and ordinal aspects of numbers (Anderson & Cordes, 2013), and it may thus appear only after children have learned to fluently recognize the cardinality of a set by counting. The research questions are as follows:

1) How does focusing on numerical order develop from the age of 3 to 6 years?
2) How do preschool mathematical skills, such as spontaneous focusing on numerosity, focusing on numerical order, number sequence production, and subitizing-based enumeration, predict math achievement at 12 years of age?

METHOD

Participants

Thirty-six Finnish children (18 girls and 18 boys) with no developmental delays from Finnish-speaking families in daycare participated in this study. The mean age of the children was 3.0 years (*SD* = 1.5 months) at the start of the first data collection.

Procedure and tasks

Preschool data collection took place at the ages of 3, 4, 5, and 6 years. Children were tested for their focusing on numerical order, number sequence production, and cardinality recognition skills at every time point. In addition, the children were tested for their subitizing-based enumeration skills at the age of 5 and their SFON tendency at the age of 6 years. A follow-up was conducted at the age of 12 years, where the children were tested for their curriculum-based math achievement.

Children's *focusing on numerical order* was assessed with a novel, previously unreported caterpillar ordering task. In this video-recorded task, the child was shown similar boxes. Each box contained a caterpillar with a unique number of legs and a picture of a matching number of socks (see Figure 1). At the ages of 3 and 4 the child was shown five boxes (1–5 legs) and seven boxes (1–7 legs) at the ages of 5 and 6. First, the experimenter helped the child notice that each caterpillar had its own box of socks with as many socks as the caterpillar needed (Figure 1a.). The boxes were placed on the table and the caterpillars next to their own boxes. Then, the experimenter took the caterpillars away from their boxes and said, "Let's organize these boxes of socks like this. Every box has its own place." With the socks being visible, the experimenter organized the boxes in a vertical row in an increasing numerical order (Figure 1b.), and the child was asked to remember where each box was. The child was left to notice themselves that the boxes were in numerical order. Then, the experimenter closed the boxes, handed the caterpillars in front of the child, and asked the child to show each caterpillar where its own box of socks was (Figure 1c.). The highest number of caterpillars in the correct order was recorded and regarded as the score in the task.

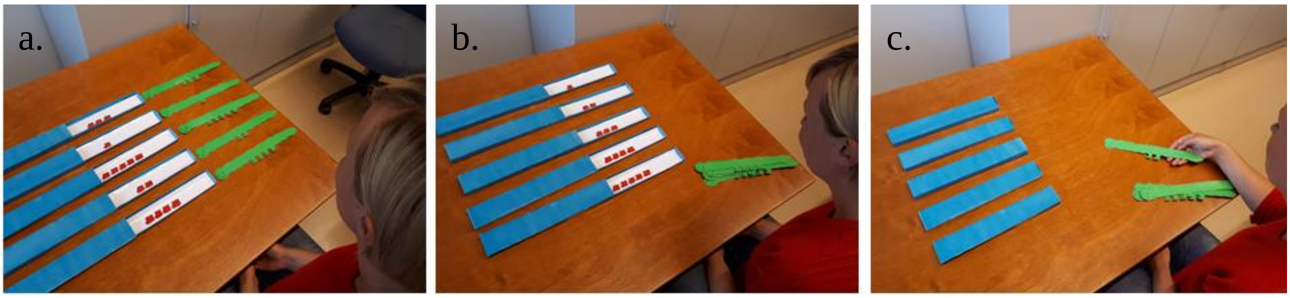

Figure 1: Caterpillar ordering task used to measure children's focusing on numerical order.

Children's *SFON tendency* was measured at the age of 6 years with a sum score of Imitation, Model, and Finding tasks (for details, see Hannula & Lehtinen, 2001; 2005) in which it was assessed how frequently the child noticed and used exact number of items spontaneously, i.e. without any guidance or explicit task instructions. *Cardinality recognition skills* were assessed at the ages of 3 and 4 with the caterpillar task ("Bring the caterpillar as many socks as it needs", max 10 [see Hannula & Lehtinen, 2001; 2005]) and at the ages of 5 and 6 years with an object counting task ("Count aloud how many 'turtles' there are on the table", max 23 [see Hannula & Lehtinen, 2005]). In the *number sequence production* task, children were asked to count as far as they could, or until fifty, where they were stopped (for details, see Hannula & Lehtinen, 2005). Children's *subitizing-based enumeration skills* were measured with a computer-based test, in which the child was asked to identify which of the four groups with different numbers of dwarves had stolen the groups of objects that had earlier been flashed on the laptop for 120 ms (see Hannula et al., 2007). Curriculum-based *math achievement* was measured using the Finnish standardized RMAT test (Räsänen, 2004), which includes 56 items of multi-digit arithmetic and simple algebra.

RESULTS

The results in Table 1 show large individual differences and how children's focusing on numerical order developed from 3 to 6 years, where most participants did not really understand the task at 3 years old to almost all of them mastering it at 6 years old in the caterpillar ordering task.

	Highest number of correct order produced							
Age	0	1	2	3	4	5	6*	7*
3 years	87	10	3	0	0	0		
4 years	44	18	13	3	0	23		
5 years*	10	18	8	5	5	0	0	54
6 years*	0	2	8	10	0	0	0	80

Note. * The total number of boxes used was five at the ages of 3 and 4, and seven at the ages of 5 and 6.

Table 1: The highest number up to which the boxes were numerically correctly ordered in percentages of the sample (N = 36).

Figure 2 demonstrates the differing ranges for children in number sequence production, cardinality recognition (or object counting), and focusing on numerical order. Importantly, this is only indicative of children's number ranges in real life due to the task maximums differing in various measures. However, a closer look at the individual children's developmental data indicates that the developmental order of the skills followed the same pattern. First, children learned to recite a list of number words, then to recognize small numbers of items and develop object counting skills for the first three or four numbers; only then did they start to notice the numerical order of items. The differences in the numerical ranges of the children's skills mirrored the developmental order of the skills as well.

Next, we explored the associations between preschool mathematical skills (number sequence production, subitizing-based enumeration, and focusing on numerical order at the age of 5, and SFON tendency at the age of 6) and math achievement at 12 years old. We found significant correlations within preschool mathematical abilities, and also between all preschool mathematical abilities except for subitizing-based enumeration and general mathematical abilities at 12 years (Table 2).

We further investigated whether preschool mathematical skills predict mathematical skills at the age of 12 by means of multiple linear regression analysis. We included number sequence production, subitizing-based enumeration, focusing on numerical order at 5 years, and SFON tendency at 6 years as predictors in the model. The results presented in Table 3 indicate that there was a collective significant effect between preschool mathematical skills and general math achievement by the age of 12. A closer look at the predictors indicates that only focusing on numerical order at 5 years and SFON tendency at 6 years were significant predictors of general mathematical abilities, unlike number sequence production at 5 years or subitizing-based enumeration at 5 years old.

Figure 2: Children’s accurate number ranges in number sequence production, cardinality recognition/object counting, and focusing on numerical order.

Variable	*M*	*SD*	(2)	(3)	(4)	(5)
(1) Number sequence production at 5 years	23.42	6.53	.47**	.53***	.46**	.39*
(2) Subitizing-based enumeration at 5 years	2.64	1.38	--	.39*	.40**	.09
(3) Focusing on numerical order at 5 years	4.61	2.84		--	.51**	.54***
(4) SFON tendency at 6 years	2.58	1.76			--	.54***
(5) Math achievement at 12 years	36.03	6.53				--

Note. * $p < .05$, ** $p < .01$, *** $p < .001$.

Table 2: Descriptive statistics and correlations between preschool mathematical skills and general math achievement at the age of 12 years (N = 36).

Variable	*β* (*SE*)	95% *CI*
Number sequence production at 5 years	.14 (.08)	[-.12 – .28]
Subitizing based enumeration at 5 years	-.27 (.75)	[-2.81 – .23]
Focusing on numerical order at 5 years	.36* (.39)	[.04 – 1.64]
SFON tendency at 6 years	.40* (.61)	[.23 – 2.71]
Total R^2 = .44***		

Note. * $p < .05$, ** $p < .01$, *** $p < .001$ ($N = 36$)

Table 3: Regression analyses predicting math achievement at 12 years.

CONCLUSION AND DISCUSSION

The current study investigated the three-year development of children's focusing on numerical order and its predictive effect on general mathematical abilities. We found evidence that children have individual differences in focusing on numerical order as well as an increase in focusing on numerical order from 3 to 6 years of age. Our finding seems to be in line with an earlier study that showed that cardinal recognition skills develop before the ability to order sets numerically (Spaepen et al., 2018). In fact, Cheung et al. (2017) have suggested that only after acquiring cardinal skills can children place sets of objects in correspondence with the number sequence, which is also reflected in our data.

Next, we found that focusing on numerical order and SFON tendency were significant predictors of children's math achievement, even after controlling for number sequence production and subitizing-based enumeration skills at preschool age. This indicates that focusing on numerical order might be an important aspect of early mathematical skills. Interestingly, number sequence production, which many studies have reported to be a strong early predictor of mathematical skills (Koponen et al. 2016), did not significantly predict later mathematical skills when the two numerical focusing tendencies were included in the model.

These results may indicate a similar reciprocal development between early mathematical skills and focusing on numerical order, as was reported in earlier studies of SFON (Hannula & Lehtinen, 2005). Initial number skills enable the spontaneous use of these skills in various situations, in this case noticing and making use of numerical order, which subsequently leads to enhanced mathematical skills. In this study, the caterpillar ordering task had hints toward the numerical nature of the task, so the task did not yet measure children's *spontaneous* focusing on numerical order. In addition, our sample was small, and the numerical order focusing task had only one test item. Thus, our results need to be treated as suggestive. Future studies with larger samples and more tasks should investigate whether focusing on numerical order could be another member of the "spontaneous mathematical focusing tendencies" (McMullen et al., 2019), called Spontaneous Focusing On Numerical Order (SFONO).

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THE PROCESS OF MODELLING-RELATED PROBLEM POSING – A CASE STUDY WITH PRESERVICE TEACHERS

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In real life, problems emerge from situations and often need to be posed before they can be solved. Despite the ongoing emphasis on the processes involved in solving modelling problems, little is known about the process of problem posing. To help fill this gap, the current study examined (1) what activities are involved in modelling-related problem posing and (2) the sequence in which they occur. For this purpose, we invited seven preservice teachers to pose a problem based on given real-world situations and analyzed their problem-posing activities. We identified the five most frequent activities that occurred in the sequence: understanding–exploring–generating–problem solving–evaluating. These results contribute to the uncovering of important activities and contribute to theories of modelling and problem posing.

INTRODUCTION

In mathematics classrooms, the ability to solve problems in the real world (i.e., mathematical modelling) is a key competency that needs to be learned to be able to function as a responsible citizen in society (Niss & Blum, 2020, p. 2). However, in the real world, problems often need to be identified and posed first before they can be solved. Therefore, posing problems in given real-world situations (i.e., modelling-related problem posing) is an important competency. In the past, a great deal of research has been conducted on mathematical modelling (Schukajlow et al., 2021). However, only a few studies have analyzed modelling-related problem posing. Posing one's own problems is a demanding process that has to be learned (Cai & Hwang, 2002). To improve the teaching and learning of problem posing, knowledge about the activities involved in the process is needed (Cai et al., 2015). Research on the activities involved in posing problems based on given real-world situations has largely been missing so far. To help fill this gap, we aimed to analyze the activities involved in modelling-related problem posing from a cognitive perspective.

THEORETICAL BACKGROUND

Problem Posing

Research in mathematics education has been focusing more on problem posing in recent years as it can be gainfully used for teaching and learning mathematics (Cai et al., 2015). Problem posing can be defined as the generation of new problems and the reformulation of given problems that can take place before, during, or after problem solving (Silver, 1994). Stoyanova (1997) differentiated between structured problem posing, which is based on an initial problem, and unstructured problem posing, which is less restricted and is based on a holistic description of a situation. The connection to

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reality of the given stimuli is another important characteristic of problem posing. Based on the classification of problems with and without a connection to reality (Blum & Niss, 1991), the stimuli given for problem posing can be intramathematical descriptions or real-world situations. The focus of the present study is on problem posing as the generation of new problems based on given real-world situations before solving them. In the following, we will refer to this type of problem posing as modelling-related problem posing. An exemplary real-world situation that can be used as a stimulus for modelling-related problem posing is presented in Figure 1.

Cable Car

For more than 90 years, the *Nebelhorn* cable car has taken numerous guests up into the heights. Now it can go into well-earned retirement. Beginning in the summer of 2021, a new cable car will transport enthusiastic outdoor fans to the Nebelhorn mountain. The aim of the project is to avoid long waiting times, provide seated transportation with an optimal view from every seat, and increase carrying capacity.

Technical data for the old cable car:

Model:	Large-cabin aerial tramway
Weight empty cabin:	1600 kg
Weight full cabin:	3900 kg
Height valley station:	1933 m
Height top station:	2214.2 m
Horizontal difference:	905.77 m
Speed:	8 m/s
Carrying capacity:	500 people/h
Power Unit:	120 PS

Figure 1: The real-world Cable Car situation

Problem-Posing Activities

Based on the given real-world situation, a variety of real-world problems can be posed (Galbraith et al., 2010; Hartmann et al., 2021). An exemplary problem that can be posed using the given real-world situation in Figure 1 is: What is the best way to reconstruct the cable car? To pose such a problem, creative thinking is necessary (Bonotto & Santo, 2015). Wallas (1926) used a four-phase model consisting of the phases *preparation (exploration)*, *incubation*, *illumination*, and *verification* to describe creative mathematical thinking process.

Some studies analyzed the activities that occur when a problem is posed (Baumanns & Rott, 2021; Christou et al., 2005; Pelczer & Gamboa, 2009). First, the situation has to be explored with respect to possible problems that can be posed in the given situation. This activity is called *editing, selecting* by Christou et al. (2005), *transformation* by Pelczer and Gamboa (2009), or *analysis, variation* by Baumanns and Rott (2021). Second, problems can be generated by formulating them. This activity is called *translating* (Christou et al., 2005), *formulation* (Pelczer & Gamboa, 2009), or *generation* (Baumanns & Rott, 2021). Third, the posed problems can be evaluated with respect to individual criteria (e.g., solvability or appropriateness) (Baumanns & Rott, 2021; Pelczer & Gamboa, 2009). Previous studies indicated that the sequence of posed problems was typically guided by the employed problem-solving strategies (Cai & Hwang, 2002). Therefore, thinking about a possible solution might already be part of problem posing. Moreover, some students might develop a possible solution plan while problem posing (Baumanns & Rott, 2021). Overall, it can be assumed that the problem-posing process consists of *exploring*, *generating*, and *evaluating* activities and might already involve *problem solving*. However, the studies revealed that the activities involved are by no means linear and that the process is instead characterized by jumping back and forth between the individual activities (Baumanns & Rott, 2021; Pelczer & Gamboa, 2009). Prior studies on problem posing used unstructured problem posing with intramathematical stimuli (i.e., graphs, tables, equations, and dressed-up stories) (e.g., Christou et al., 2005) or structured problem posing with intramathematical and dressed-up word problems (e.g., Baumanns & Rott, 2021; Pelczer & Gamboa, 2009). Regarding modelling-related problem posing, only a little is known about the activities that take place when posing problems based on given real-world situations. In problem posing based on real-world situations, students should understand and explore the situations, generate possible problems, and evaluate the problems regarding their solvability (Bonotto & Santo, 2015). However, these theoretical considerations have yet to be empirically evaluated.

RESEARCH QUESTIONS

The goal of the present study was to examine the modelling-related problem-posing process by investigating the activities involved in posing problems based on given real-world situations. For this purpose, we asked the following research questions:

1) What activities are involved when preservice teachers pose problems based on given descriptions of real-world situations, and how can these activities be described?
2) In which sequences do the problem-posing activities occur?

METHOD

Sample

Seven preservice mathematics teachers between the ages of 20 and 26 (M = 22.86, SD = 1.95) from a large COUNTRY/REGION university participated in our study (4 women). To select the sample, we used heterogeneity sampling regarding different mathematics performance levels, experience in problem posing and modelling, and participation in different university programs. Two of the participants studied in a middle-track secondary school teacher program and five of them in a higher track secondary school teacher program. All participants were experienced in solving modelling problems and six of them in posing problems. The study was approved by the ethics committee of the faculty.

Procedure and Instruments

To identify the cognitive processes and to gain deep insights into the processes of problem posing, we used a qualitative study that included thinking aloud and the stimulated recall method. The preservice teachers were instructed to first pose a problem based on the given real-world situations, and after posing each problem, to solve it. For both posing and solving, they were instructed to think aloud during all activities. All responses were videotaped. The videos included their voice, gestures, writing, and facial expressions. To initiate problem posing, we used three real-world situations as they are described in modelling problems and enriched them by adding further authentic information to allow a variety of problems to be posed. An example of a real-world situation is presented in **¡Error! No se encuentra el origen de la referencia.**.

Data Analysis

To analyze the recorded videos, we first transcribed the material from the problem-posing process and the subsequent stimulated recall with regard to content-bearing semantic elements and then analyzed them using Mayring's (2015) content analysis. The coding scheme is based on the theoretically assumed problem-posing activities (exploring, generating, evaluating, problem solving) described in the literature and was extended inductively on the basis of the given material by using subsumption.

Transcripts were coded by the first author. To test for interrater reliability (measured as Cohen's kappa), over 50% were coded by a well-trained second rater. Cohen's kappa was at least moderate ranging from $\kappa = .81$ to $\kappa = .95$ (Cohen, 1960). To gain an overall picture of the activities involved in modelling-related problem posing, we analyzed the data with respect to the realization of the individual activities, and then for the second research question (sequence of activities), we focused on the number and frequency of changes in activity.

RESEARCH FINDINGS

With regard to our first research question, which was aimed at describing the activities that take place when learners engage in modelling-related problem posing, the analysis revealed the involvement of the five activities understanding, exploring, generating, problem solving, and evaluating. **¡Error! No se encuentra el origen de la referencia.** gives an overview of the observed activities.

Category	Description
Understanding	Comprehending and understanding the given real-world situation and information based on the description of the situation.
Exploring	Discovering and gathering relevant information to develop possible problems and organizing the information.
Generating	Posing, formulating, and defining possible problems.
Evaluating	Evaluating possible problems on the basis of individual criteria (solvable, meaningful, complete, appropriate formulation and difficulty, suitable for a particular target group).
Problem Solving	Finding a solution plan to the self-generated question.

Figure 2: Activities involved in modelling-related problem posing

In the following, we focus on the realization of the individual activities:

Understanding involved building an understanding of the situation. Thereby, students read the given situation, summarized information, asked comprehension questions, and related the given information to their personal experiences. Exemplarily, Lisa questioned her understanding of the horizontal distance in the following excerpt.

> Um theoretically, I'm wondering right now if the horizontal distance really means that it's sort of between the valley station and the top station.

Exploring involved exploring the given situation for possible problems that could be posed. This included identifying relevant, irrelevant, and missing information, organizing the identified relevant information, and expanding the context with further information. In the following excerpt, Lisa identified relevant information (height of the mountain and valley station, horizontal difference) and linked them by making a drawing of the situation (see Figure 3).

> So, I'm sort of making a drawing for this because I know that I have here, let's say the (draws in a first point), the um top ah the valley station and the valley station here (draws in a second point).
>
> And I know that the height here at the valley station (labels one point) is 1933 m, and the top station (labels the other point) is 2214.2 m.

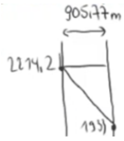

Figure 3: Lisa's drawing

Generating was aimed at posing and writing down a problem. Thereby, possible problems were posed. From these, one question was then selected, formulated, and written down. In the following excerpt, Theo generated an idea for a possible problem based on the information he considered to be relevant.

> The goal of the project is to avoid long waiting times, seated transportation with an optimal view. Ok there you can perhaps consider how many people can realistically fit into such a cabin, so that each person sits at the window and has an optimal view and then consider whether you are exceeding the weight of a full cabin or not.

Evaluating included an assessment of the posed problems and referred to the assessment of appropriateness, solvability, and formulation. For example, Lea evaluated the appropriateness of her question as the following:

> So, you could somehow ask something about the weight in any case. But then the information is not relevant for whether we need a new one.

Problem Solving included solution plans for the self-generated problems. Thereby, mathematical operations or possible solution steps were identified. In the following excerpt, Max described a rather less detailed plan for solving the problem.

> You have to work through different steps bit by bit in order to solve it because I don't think you can come up with the solution directly in a calculation.

To find out more about the sequences in which the activities occurred (RQ 2), we analyzed the changes in activities (Table 1). All activities except understanding by evaluating followed each other at least once.

Followed by	Under-	Exploring	Generating	Evaluating	Problem

	standing				Solving
Understanding	-	*36 [73%]	5 [10%]	0 [0%]	2 [4%]
Exploring	18 [20%]	-	53 [58%]	7 [8%]	6 [7%]
Generating	1 [1%]	22 [19%]	-	59 [52%]	6 [5%]
Evaluating	2 [3%]	16 [20%]	38 [48%]	-	12 [15%]
Problem Solving	1 [3%]	6 [20%]	3 [10%]	12 [40%]	-

Table 1: Overview of the number of activity changes (Note: *Understanding was followed by exploring in 36 sequences, 73% of all understanding sequences.)

Regarding frequencies, understanding was predominantly followed by exploring and rather rarely by generating and problem solving. Exploring occurred frequently before generating but less frequently before exploring, evaluating, and problem solving. Generating was predominantly followed by evaluating, less frequently by exploring, and rather rarely by problem solving and understanding. Evaluating primarily occurred before generating, less frequently before exploring and problem solving, and rather rarely before understanding. Problem solving was followed most frequently by evaluating, less frequently by exploring, and rather rarely by generating and understanding. If we consider only the activities that follow one another most frequently, the idealized process model of a problem-posing route emerges (Figure 4). It presents a hypothesized process model for describing the idealized process of modelling-related problem posing. Importantly, the sequences of the activities while posing a specific problem by an individual (called individual problem-posing routes) are not linear and vary significantly (i.e., switching between different activities in the process model).

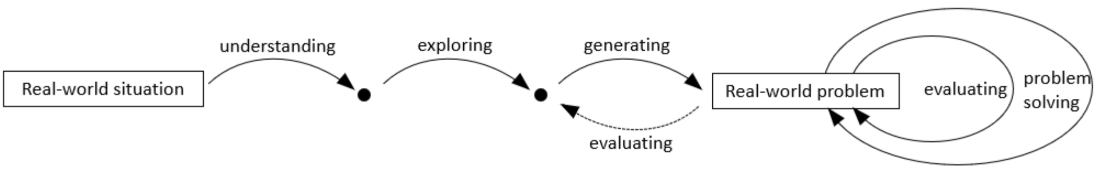

Figure 4: Hypothesized process model for modelling-related problem posing

DISCUSSION

Modelling-related problem posing included the activities understanding, exploring, generating, evaluating, and problem solving. These findings are partly in line with the activities found in prior studies on intramathematical problems and word problems (Baumanns & Rott, 2021; Christou et al., 2005; Pelczer & Gamboa, 2009). The activities of exploring, generating, and evaluating were observed in this and other studies. This finding indicates the commonalities between modelling-related problem posing and other problem-posing processes. In addition, the analyzed processes involved an activity in which possible solution steps are planned, similar to a study on structured problem posing based on a given word problem (Baumanns & Rott, 2021). This finding supports Cai and Hwang's (2002) assumption that problem posers are already thinking about a possible solution when posing a problem. However, modelling-related problem-posing activities differ in some ways from the activities found in prior studies. First, we were not able to identify the activities *transformation* and *variation* as described in studies on structured problem posing (Baumanns & Rott, 2021; Pelczer & Gamboa, 2009). A possible explanation is that stimuli had a different structure. As modelling-related problem posing is not based on a given initial problem, it is not necessary to transform the given problem. Second, we identified the activity understanding as being a part of problem posing. Understanding is an essential activity in the well-established models of the solution process of modelling problems, and it is important for problem posing as well (Niss & Blum, 2020, p. 17). However, prior studies on structured problem posing did not identify the activity understanding. A possible explanation could be that structured problem posing begins with the solution of the initial problem, and students already understand the initial problem before problem posing. In our study, two activities—exploring and evaluating—which were described in Wallas' (1926) model of creative mathematical thinking, were observed. Consequently, problem posing was revealed to be a creative process (Bonotto & Santo, 2015). However, we were not able to observe the activities incubation and illumination, probably because these processes are described as occurring subconsciously (Wallas, 1926), and hence, we were not able to capture them with our research method. Future research with methods such as eye-tracking or narrative interviews are needed to find out whether problem posing involves incubation and illumination.

Due to the design we chose, our study has some limitations. We used a qualitative research approach with a small sample to identify the process of modelling-related problem posing. The aim was to uncover problem-posing activities and develop an idealized hypothetical model of modelling-related problem posing. These findings must be verified in future studies. Additionally, limitations result from using specific real-world situations. Overall, our study contributes to research on problem posing from a cognitive modelling perspective. Our findings can be used to improve the teaching and learning of modelling-related problem posing by taking into account problem-posing activities and their ideal sequence.

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THE ALGORITHMS TAKE IT ALL? STRATEGY USE BY GERMAN THIRD GRADERS BEFORE AND AFTER THE INTRODUCTION OF WRITTEN ALGORITHMS

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Solving addition and subtraction problems efficiently is an important goal of elementary school mathematics education. However, after the introduction of written algorithms, many students exclusively use these procedures to solve arithmetic problems, even if they are inefficient and error-prone. We explore the assumption that the dominance of written algorithms is due to the fact that students already previously had only used a very limited repertoire of strategies, which was then replaced by the written algorithms. We used data from a study of 222 German third graders. Sixty students received a brief training on computational strategies at the start of the school year and showed a broader strategy repertoire than their peers before the introduction of written algorithms. After learning the algorithms, the trained students still used a broader strategy repertoire (including short-cut strategies). We assume that students can succeed in flexibly using a broad strategy repertoire even after the introduction of the algorithms if they are supported in doing so from the beginning.

INTRODUCTION AND THEORETICAL BACKGROUND

One central goal of arithmetic education in the elementary school is the acquisition of computation skills. Meanwhile, arithmetic curricula in many countries also address number-based computational strategies (e.g., stepwise strategy, split strategy, compensation strategy, indirect addition), although the digit-based written algorithms continue to play an important role (Mullis et al., 2016). Skills in flexible use of strategies should help students to solve arithmetic problems efficiently with an appropriate strategy instead of using the same strategy for all problems. At the same time, learning different strategies is considered to promote conceptual understanding of numbers (e.g., Baroody, 2003; Verschaffel et al., 2007) and internalized computation strategies can be helpful to solve specific types of multi-digit arithmetic problems by purely mental calculation without paper-and-pencil computations.

In this report, we focus on strategies for multi-digit addition and subtraction problems. As mentioned before, these strategies can be categorized as digit-based (standard) written algorithms and number-based strategies. The latter can be further divided into universal number-based strategies, which are suitable for all addition and subtraction problems (stepwise: 462 + 299 via 462 + 200 = 662, 662 + 90 = 752, 752 + 9 = 761;

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split: 462 + 299 via 400 + 200= 600, 60 + 90 = 150, 2 + 9 = 11, 600 + 150 + 11 = 761), and short-cut strategies, which are very efficient for specific problem types (compensation strategy: 462 + 299 via 462 + 300 = 762, 762 - 1 = 761; simplifying strategy: 462 + 299 = 461 + 300 = 761; indirect addition: 702 - 697 via 697 + 5 = 702). Sometimes, students mix different strategies and if they have some routine they might also use short versions of the universal number-based strategies stepwise and split by combining sub-steps (e.g., 462 + 299 via 462 + 200 = 662, 662 + 99 = 761).

Although different computation strategies have already been implemented in curricula and textbooks in several countries for about 20 years, elementary school students show a low variation in applying different strategies and especially in applying short-cut strategies (e.g., Csíkos, 2016; Heinze et al., 2009; Hickendorff, 2020; Torbeyns & Verschaffel, 2016; Torbeyns et al., 2017). This indicates that acquiring skills in the flexible use of strategies is challenging for students. However, empirical research also suggests that these skills can be promoted through instruction (Hickendorff, 2020; Heinze et al., 2018; Nehmet et al., 2019; Sievert et al., 2019; Torbeyns et al., 2017).

Students' strategy use after the introduction of the written algorithms

Studies examining the development of students' strategy use in regular elementary school mathematics classes revealed that the use of number-based strategies decreased substantially after the written algorithms were introduced (e.g., Hickendorff, 2020; Nehmet et al., 2019; Selter, 2001; Torbeyns & Verschaffel, 2016; Torbeyns et al., 2017). Many students used the written algorithms almost exclusively to solve addition and subtraction problems, and there was little variation in the use of the strategies across the problems. Different possible explanations for this observation can be derived from empirical studies in the research literature. This research report takes a closer look at two of them which might apply to different groups of students.

A first possible explanation is that most students have used only a few strategies already before the introduction of the written algorithms. Empirical results suggest that there is a high proportion of students who initially use only one or two universal number-based strategies, like the stepwise and/or split strategy (e.g., Csíkos, 2016; Heinze et al., 2009; Torbeyns et al., 2017). Thus, there is also little flexible use of strategies before students learn the written algorithms. After the introduction of the written algorithms, the exclusively used universal number-based strategies are then replaced by the universal digit-based written algorithms. As a result, these students always use those universally applicable strategies that they learned last.

A second possible explanation is that students' skills in using strategies flexibly is not stable. Some students may have learned various number-based strategies (including short-cut strategies) in mathematics class before the introduction of the written algorithms. Then the written algorithms were explicitly introduced by the teacher and practiced intensively by the students for a longer period of time. Afterwards, on the one hand, students' knowledge and skills about the number-based strategies may have decreased again and, on the other hand, the algorithms may have gained a great importance in the students' perception. The findings of Nehmet et al. (2019) can be interpreted in this direction. They taught one group of students in the usual way, that is, the number-based strategies first and then the written algorithms. A second group of students learned all strategies interleaved. After the intervention the written algorithms were used significantly less and the short-cut strategies significantly more often in the second group than in the first group. Thus, if students spend long periods of time working exclusively on written algorithms, they may lose skills in other strategies.

PRESENT STUDY AND RESEARCH QUESTIONS

To examine the previously mentioned explanations, we use existing data from the intervention study of Heinze et al. (2018). This study monitored students of several school classes over the course of grade 3. A subsample of students was trained on number-based strategies and their flexible use at the start of the school year. In the second half of the school year, the written algorithms were introduced by the teachers in the regular mathematics class. Thus, the dataset covers two subsamples of third-graders. One subsample of students which participated only in the regular mathematics classroom and one subsample from the same classes which were briefly trained at the start of the school year. The latter showed better knowledge and skills of short-cut strategies and their flexible use than their peers before the introduction of the written algorithms.

Using data from this study, we explored the following research questions:

RQ1: What strategies do third-graders from a regular German mathematics classroom use before and after the introduction of the written algorithms?

RQ2: What strategies do third-graders use before and after the introduction of the written algorithms if they possess advanced knowledge and skills of short-cut strategies and their flexible use?

RQ3: What impact does more frequent use of short-cut strategies by students before and after the introduction of written algorithms have on the performance in addition and subtraction (in the sense of correct solutions)?

The third research question provides information on whether the two groups of students show a comparable arithmetic performance before the introduction of written algorithms. Further, we obtain information about whether the different use of strategies affects the solution rates.

METHODS

To investigate the research questions, we use data from Heinze et al. (2018) for a secondary analysis. In Heinze et al. (2018), 17 Grade 3 classes from Germany were considered. We selected those students who participated in all three tests we needed for our analysis. The sample comprised 222 third-graders (9-10 years old) from 15 classes, 162 of whom participated only in regular mathematics instruction, while 60 students received an additional training for the flexible use of computational strategies. The design on the study is presented in Figure 1.

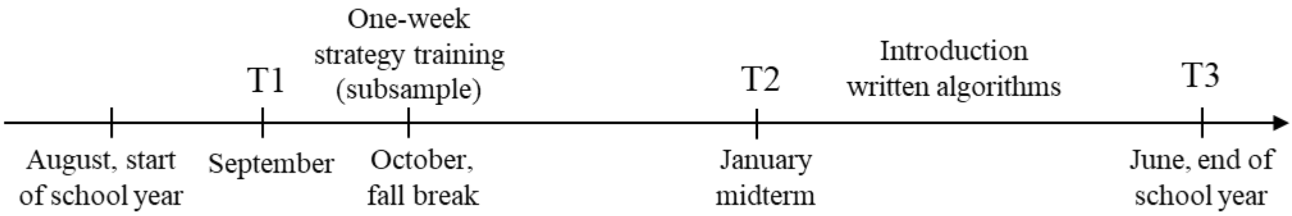

Figure 1: Design of the study, data collection at T1, T2 and T3 by test

According to the German Grade 3 curriculum, the number domain is extended up to 1000 and students learn addition and subtraction strategies for three-digit numbers. In the second half of grade 3, the standard written algorithms are introduced. The one-week training during the fall break was advertised in several schools. Students from each of the 15 classes participated voluntarily. They were taught five strategies (stepwise, split, compensation, simplifying, indirect addition). In the original study in Heinze et al. (2018), two instructional approaches were compared. Because their effectiveness did not differ, they are not distinguished in the current analysis here.

Data for strategy use was collected by trained university assistants in all 15 classes with a first test at the start of the school year (T1), a test 3 months after the training, but before the introduction of the written algorithms (T2), and a test at the end of the school year after students had learned the written algorithms (T3). Each test consisted of 8 multi-digit addition and subtraction tasks suggesting especially the short-cut strategies as efficient solutions. A core of 4 items was part of all tests (403-396, 1000-991, 398+441, 502+399). The item solutions were analyzed two times: firstly as correct or incorrect, and secondly by categorizing the applied strategies for the given task. For the latter, a bottom-up procedure to develop a category system with 21 strategy categories was applied (e.g., the ideal-typical strategies, as well as observed short versions and mixtures of these strategies). The assignment of a strategy to a category was judged independently by two persons with an acceptable inter-rater reliability ($\kappa > .70$). In case of different coding a consensual agreement was achieved after a discussion. In this report, we present a coarser category system in which the 21 categories have been combined into 5 categories (Table 1). We used χ^2-homogeneity tests to analyze the data for research questions 1 and 2, and a t-test as well as ANCOVAs for research question 3.

RESULTS

Table 1 presents the strategies the students used in the three tests. A comparison of columns No. 1 and 2 in Table 1 indicate that there was no significant difference in strategy use between the students of the training group and their peers at the start of the school year. The significant effects of the one-week training in October becomes apparent at T2 in January (columns No. 3 and 4): the trained group used much more short-cut strategies and less universal strategies than their peers.

Column No.	1	2	3	4	5	6
	T1 - start of school year		T2 - before introduction written algorithms (midterm)		T3 - after introduction written algorithms (end of school year)	
	Regular class	Regular class & training	Regular class	Regular class & training	Regular class	Regular class & training
Written algorithm	42 (3.4%) [a]	23 (5.0%)	168 (13.4%)	31 (6.5%)	583 (45.5%)	179 (37.8%)
Number-based universal strategies	692 (56.0%)	251 (54.7%)	558 (44.5%)	169 (35.5%)	280 (21.8%)	76 (16.1%)
Short version of number-based universal strategies	248 (20.1%)	79 (17.2%)	252 (20.1%)	57 (12.0%)	193 (15.1%)	29 (6.1%)
Number-based short-cut strategies	118 (9.6%)	57 (12.4%)	221 (17.6%)	211 (44.3%)	213 (16.6%)	185 (39.1%)
Not assignable	135 (10.9%)	49 (10.7%)	56 (4.5%)	8 (1.7%)	13 (1.0%)	4 (0.8%)
Total[b]	1235 (100%)	459 (100%)	1255 (100%)	476 (100%)	1282 (100%)	473 (100%)
χ^2	$\chi^2(4, N = 1694) = 6.47$		$\chi^2(4, N = 1731) = 139.41$		$\chi^2(4, N = 1755) = 109.28$	
p	.166		< .001		< .001	
Cramér's V[c]	.06		.28		.25	

[a] Percentages are column percentages, [b] Different total numbers due to a few missing solutions; theoretical maximum number of solutions was 1296 for regular class and 480 for regular class & training, [c] Interpretation of Cramér's V: weak association: < .20, moderate association: .20-.50 and strong association: > .50

Table 1: Number of applied strategy types for students in regular class and in regular class with additional training at start, midterm and end of school year

To analyze research question 1, we compared columns No. 3 and 5 which show the strategy use of the 162 untrained students before and after the introduction of the written algorithms. As expected, the use of the written algorithms drastically increased whereas the use of the number-based universal strategies decreased. The small amount of short-cut strategies remains stable (17.6% to 16.6%). For research question 2, we compared columns No. 4 and 6 and found a similar development for the trained students: strong increase of written algorithms, decrease of number-based universal strategies, and the amount of short-cut strategies remains more or less stable (44.3% to 39.1%). However, the difference to the untrained students is that the trained students used much more short-cut strategies before the introduction of written algorithms (44.3% to 17.6%) and the use of these strategies remains stable at T3 (39.1%).

For research question 3, we considered the test scores of the students (1 point for each correct solution). Table 2 presents the results for the different tests as well as the reliabilities. The t-test revealed no significant difference at T1, the start of the school year ($t(220) = 1.9$, $p = .066$, $d = 0.27$), despite the trained students ($M = 5.10$, $SD = 2.41$) showing higher scores than the untrained students ($M = 4.51$, $SD = 1.99$).

Accuracy strategy use (max 8 points)	T1 - start of school year *M* (*SD*)	T2 - before introduction written algorithms (midterm) *M* (*SD*)	T3 - after introduction written algorithms (end of school year) *M* (*SD*)
Regular class	4.51 (1.99)	4.98 (2.17)	5.44 (1.91)
Regular class & training	5.10 (2.41)	5.53 (2.18)	6.02 (1.81)
Total	4.67 (2.12)	5.13 (2.18)	5.60 (1.90)
Cronbach's α	.70	.74	.66

Table 2: Accuracy of applied strategies (mean values and standard deviations for correct solutions) for trained students and their peers at T1-T3

We ran two analyses of covariance with T1 as covariate and T2 as well as T3 as dependent variable. Neither at T2 ($F(1, 219) = 0.66$, $p = .417$, part. $\eta^2 = .003$), nor at T3 ($F(1, 219) = 1.84$, $p = .176$, part. $\eta^2 = .008$) significant effects occurred.

DISCUSSION

The results in Table 1 (columns No. 1 and 3) are consistent with previous findings that students without a specific support use only few strategies and, in particular, hardly use any short-cut strategies (e.g., Csíkos, 2016; Heinze et al., 2009; Hickendorff, 2020; Torbeyns & Verschaffel, 2016; Torbeyns et al., 2017). Table 1 (column No. 5) replicates findings that the written algorithms are dominant after their introduction (e.g., Hickendorff, 2020; Nehmet et al., 2019; Selter, 2001; Torbeyns & Verschaffel, 2016; Torbeyns et al., 2017). Regarding the previously mentioned two possible explanations for the dominance of the written algorithms, our findings support the first explanation. In the untrained group, mostly universal number-based strategies were used, which were then replaced by the written algorithms (Table 1, columns No. 3 and 5). The second possible explanation for the dominance of the written algorithms cannot be supported. The training group had used a high proportion of short-cut strategies before the introduction of the written algorithms (Table 1, column No. 4). This proportion remained essentially stable after the introduction of the written algorithms (Table 1, columns No. 4 and 6). Thus, it can be assumed that if students show skills to use short-cut strategies, this kind of strategy use will be maintained and short-cut strategies will not be replaced by written algorithms. Finally, we could show that the use of a variety of strategies (including short-cut strategies) is not at the expense of the correctness of the solutions.

Limitations

There are several limitations of the study. The analysis is based on tests consisting of only eight items, which in turn all suggested short-cut strategies. A longer test would be desirable, including items where short-cut strategies did not provide an efficient solution. Second, the items were the unit for analysis in Table 1; an analysis with the students as the unit will still be conducted. Third, there is no information about the mathematics instruction in the 15 classes. Given the weak results for the untrained students, we assume that there was not a strong emphasis on short-cut strategies. Fourth, the trained students participated voluntarily in the training during fall break. It might be the case that these students are more interested in mathematics. However, the data we presented above does not indicate that these students are only high-achieving students. Finally, there may be other possible explanations for why the written algorithms become dominant. For example, socio-mathematical norms perceived by the students could also play a role.

Educational practice and further research

Despite the limitations, suggestions for teaching practice can be derived from our study. For example, we found that promoting the use of different strategies (including short-cut strategies) before the introduction of the written algorithms leads to the retention and further use of these strategies after the learning of the written algorithms. We assume that the flexible use of different strategies can be further increased if it is addressed again after the introduction of the written algorithms. An appropriate range of tasks in textbooks could have impact on teacher action (Sievert et al., 2019). Such an approach and also approaches of interleaved learning of strategies (Nehmet et al., 2019) should be investigated in further studies.

Acknowledgement

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CHALLENGING DISCOURSES OF LOW ATTAINMENT: USING THEY POEMS TO REVEAL POSITIONING STORIES AND SHIFTING IDENTITIES

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Mathematical identity work is defined as the way a person self-positions in relation to the domain of mathematics as well as how others position them. The stories that another chooses to share about a student labelled as low prior attaining (LPA) are influential voices that create an ascribed mathematical identity. However, identity work is malleable and fluid and hence the positioning stories of others can shift. This study focuses on the positioning stories told by one teacher about a student who was labelled as LPA in mathematics in order to consider a possible counternarrative to the dominant negative discourses. By introducing the poetic structure of a 'They poem' into the Listening Guide process of analysis, I was able to identify voices that indicate a positive positioning within the stories and shiftings in the teacher's narrative data.

INTRODUCTION

In the United Kingdom (UK), it is common practice for schools and colleges to group students into separate classes based on attainment levels. For students who are labelled as low prior attaining (LPA), this grouping can lead to an ascribed mathematical identity, with learners who *obtain* a low score becoming categorised as students who *are* low attaining, the acquired label becomes the inherent state, which can lead to a homogeneous perception of their classroom identity work. This research report arises from a broader study that aimed to find a counternarrative to the dominant discourses around students labelled as LPA, through foregrounding the voices of those influential in a student's identity work, the student themselves and their teacher. Focusing on the teacher's positioning of a student labelled as LPA, I introduce the use of a novel structure called a 'They poem' into the Listening Guide method (Gilligan et al., 2006) to highlight the voices that told positioning stories including historic shifts and shifting in the moment.

MATHEMATICAL IDENTITY WORK

The concept of identity can be viewed from two standpoints, as an acquisition, something one has, or as an action, something one does (Darragh, 2016). However, in the domain specific discussions around mathematical identity, the field is maturing towards a dominant view of identity, or identity work, as an action within the socio-cultural perspective (Graven & Heyd-Metzuyanim, 2019). This perspective considers mathematical identity work as multi-voiced, socially situated, and domain specific as well as ambiguous, fluid, and unstable (Gee, 2000; Hand & Gresalfi, 2015; Verhoeven et al., 2019). Self-positioning within Mathematical identity work is found

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in the way a person talks, acts, and thinks about themselves, the stories being told, in relation to mathematics (Bishop, 2012). Sfard and Prusak (2005) go further in stating rather than stories as *describing* identity work, the stories themselves *are* identity work, the work of identity is happening in the story telling process. However, alongside the self-positioning stories about and as identity work, the stories that an influential other (for example a teacher) tells about a student co-authors the joint accomplishment of identity work and therefore should be seen as significant (Gutiérrez, 2013; Hand & Gresalfi, 2015). For students who are labelled as LPA, their enacted mathematical identity work, the way they talk and act in the classroom, is often ascribed, that is they are positioned as kinds of people (Gee, 2000) through dominant discourses of passivity and disengagement. Some authors highlight that the positioning other can use a deficit lens when viewing students labelled as LPA which can lead to a restricted pedagogy (see for example Alderton & Gifford, 2018; Marks, 2014). Other authors consider the overlooked potential of students labelled as LPA and the impact on the positioning by the teacher, often in relation to an innovation in teaching practice (see for example Coles & Brown, 2021; Watson, 2002). For students who have to continue to study mathematics in their post-16 education in England, due to not gaining a good pass in their General Certificate of Secondary Education (GCSE), there is some contemporary research that highlights shifts in identity work, although this relates to the self-positioning voice of the student rather than that of the teacher who positions them (see for example Bellamy, 2017; Boli, 2020; Hough et al., 2017). There is less research into the shifts and shifting in the teacher's positioning stories about a student in a 'business as usual' situation, that is where there seems to have been no change in practice or intervention on the part of the teacher. The study from which these findings are drawn intends to deepen conversations in this infrequently studied area of identity work research.

METHODOLOGY

The data described in this paper was part of a broader project into identity work in the context of low prior attainment. The participants were a student who attended a post-16 college, for whom I use the pseudonym Claire, and her classroom teacher. Before attending the college, Claire had been allocated a grade 3 by her previous school in her GCSE, the summative examination taken at age 16 in England. (In the summer of 2020, due to Covid-19 restrictions, students had been allocated a grade by their schools, rather than physically sitting the examinations). The grading system goes from grade 1 to grade 9, with grade 9 being the highest, and if a student has a grade 1, 2 or 3 they are considered to have not gained a good pass in their GCSE and must continue to study mathematics at college in order to improve their grade. In November 2020, Claire was able to sit the examination whilst at college but achieved a grade 3. The teacher had been teaching the class in which Claire was a student since September of 2020, at times either face-to-face or online, depending on the changing guidance from the UK government. The teacher and I met online on three occasions over the time period of December 2020 to July 2021. In these peer-to-peer conversations, called a teacher-researcher partnership, we discussed the teacher's observations of Claire, both from the classroom and wider college life, reflected on her work in the November external examination and internal examinations (set by the college in March 2021), as well as my analysis of the data I had collected from Claire's email interviews.

Research questions

There were two research questions in the study, that are addressed in this report by focusing on the teacher participant only:

RQ1: What stories are shared about/as enactments of identity in the context of low prior attainment in mathematics?

RQ2: What patterns of identity are evident when attention is given to the (self)positioning of students, through the work of a teacher-researcher partnership?

The Listening Guide method

The teacher's narrative data was analysed using an extension of the Listening Guide (Gilligan et al., 2006). This voice-relational analysis method considers listening to (rather than reading) data as an entry point into the inner world of another, foreground the voice of the narrator over that of the researcher. The method is a four-stage process that focuses the listener onto a person's ways of speaking, highlighting the different voices that coexist in a person's narrative. Having listened for the overall plot in the first stage, within the second stage the researcher creates an 'I poem', a found poem from the narrative, by identifying and extracting the use the first-person pronoun, that is how the narrator talks about themselves. During the pilot work for the main study, I extended the Listening Guide method by introducing a 'They poem' into the method, allowing me to focus on the way another talks about, and hence positions, the protagonist, in this case how the teacher talked about Claire (Helme, 2021a). The construction of the They poem started with inspecting the narrative and underlining wordstrings that used the relevant pronoun for the protagonist (as well as proper nouns that were replaced with the pronoun), along with the verbs and any other seemingly important words, as can be seen in this extract:

Teacher: i am halfway through [the explanation] and <u>she's interrupted the chat</u> and said 'can i get on the work now'{pause}<u>that's the first time</u> <u>she's kinda pushed it</u> where <u>it is almost like she's like 'i can do this now</u> i want to move on' but instead of having say more manners than anything else {pause} because <u>normally she would wait until i finished speaking</u>

The underlined wordstrings were extracted and placed in an ordered list, to create an interim structure called a long phrase form:

<u>she's interrupted</u> the chat

that's the first time <u>she's kinda pushed it</u>

it is almost like <u>she's like 'i can do</u> this now'

<u>normally she would wait</u> until i had finished

The long phrase form acted as a reference point, enabling me to stay as close as possible to the original meaning without the noise of the whole transcript. For the final iteration, using the long phrase form, I focused specifically on the pronoun, accompanying verbs, and other seemingly important words (for a more detailed account of this process and guidance used to create the form of the final structure below, see Helme, 2021b). Hence, I created the final aligned They poem below:

	she	’s interrupted
	she	’s kinda pushed it
	she	’s like ‘i can do…’
normally	she	would wait

The extract here is small portion of the full poetic structure, shown as an example of the construction process, however using the complete They poem, I then began to identify the different coexisting voices that were evident in the poem. This process involved reading and rereading the They poem, identifying which verbs seemed to work together to create tones of voice by attending to cadences and rhythms, shifts in meaning, and associated streams of consciousness that were present.

In the final two stages of the Listening Guide, I used the voices identified in the They poem to return to the full narrative, for the first time in relation to the research questions. I began to identify possible markers for each voice, listening and relistening to hear the different layers of positioning voices and stories, as well as what was unspoken. Finally, all the separate listenings were brought back together to compose the final analysis.

ANALYSING THE DATA

Focusing first on RQ1, when listening for the overall plot, I noted that the teacher used a positive tone when talking about Claire, seemingly emphasizing success over failure, with a repeated refrain that she was good at algebra and geometry. This positivity was echoed in the They poem, where it was evident that verbs were predominantly used in their positive form, rather than negate with a “not”. Moving to the They poem, I identified the presence of two different voices. Firstly, the teacher spoke about what Claire did, a doer of mathematics, using phrases that implied levels of proficiency, such as “knows”, “strong”, “really good”, “struggle”, and “misunderstood”. However, he also talked about the inner thinker:

i don’t know whether	she	said
or whether	she	reflected
	she	’s really really reflective
that’s how	she	was thinking

in this extract, the use of the verbs “reflected” and “thinking” indicate they thought of Claire as a contemplative student. There are other examples used in the poem such as “beliefs”, “confidence”, “pleased”, “enjoy”, “happy” as well as “concerned”, “confused”, “crumbling”, and “falling apart”. It seemed that they saw more than observable actions, Claire the doer but also Claire the thinker, a holistic view of a learner of mathematics.

The second voice I identified arose from the use of verbs that indicated effort:

how much effort	she	's put into
	she	's still working
	she	's got about
which	she	's cracked on with

the use of the word "effort" itself and other related phrases such as "still working", "cracking on with", and "having a stab" indicate that the teacher saw Claire as a hard worker. Returning to the full narrative, it was evident that the teacher wished to draw attention to Claire's work ethic:

> Teacher: we looked at common areas for development *which she cracked on with yesterday and completed that* and then I gave her intervention work on the stuff I thought she could do better but hadn't in the paper and again so there you go look *over an hour's worth of work there and completed them both*

In this extract the teacher was describing Claire's willingness to engage and work on improving in a timely fashion. However, they also used the phrase "there you go look" which suggested that they particularly wanted to highlight to the listener her attitude to work. There was a tone of voice that demonstrated pride through the repeated use of variations on the phrases "there you go" and "look at this". As the teacher reflected on Claire's work, pleasure in her successes was evident:

> Teacher: she knows that if 40% was 56 she then knew that 10% was 14 so she scaled it up and fair play to her you know *that's lovely, what she's done there is nice*

In the full narrative, the holistic voice and the proud voices intertwined as the teacher described Claire in terms of her capability, talking what she was able to do but also sharing their own emotional responses to these successes.

As I began to trace the voices in the full narrative, I identified a third voice that had not be evident in the first listenings. When the teacher was telling stories of Claire's less successful attempts at mathematical work, they would often offer a mitigation:

> Teacher: she's struggling with *the drawing tools and stuff* again I don't worry about that because when they do graphs on paper they can draw them

the reason given for Claire not being successful on a question was attributed to difficulties using the online software tools rather than her own a lack of understanding. The teacher also commented on poorly worded questions, inconsistencies in the marking examination papers, and curriculum content that had not been taught yet. Furthermore, the teacher used language that suggested that they saw unsuccessful attempts as a minor issue, using the phrase "simple mistake" to suggest that the miscalculation did not present a global difficulty but minor error that could be easily corrected. There are other examples where the teacher used the phrases "not quite", "a little bit", "not a huge error", "some gaps" and "partial strength" as well as "everyone in the class", "a recognised thing" and "caught lots of people out". The voice mitigated within the stories of Claire's less successful attempts at mathematics work by framing in terms of small issues that were downplayed and could be easily overcome, common issues, or barriers that were outside of Claire's control.

Shifting stories

Moving on to RQ2, what patterns of identity emerge, I reviewed the narrative for shifts in the way the teacher positioned Claire. I identified two types of stories that recounted shifts in historic thinking. The first type of story related to the positioning of Claire's behaviour in the classroom:

> Teacher: she would disrupt the take up time by shouting out an answer so and it was trying to get her to recognise you know discreetly about the behaviour for learning and everyone gets a voice etc and *then slowly she started to get used to that and then I knew once she started to get used to* that I knew if she doesn't understand something I could then probe her to find out what you know her level of understanding

the teacher had observed changes in Claire's behaviour which in turn impacted their own conduct in the classroom. They went on to tell stories about shifts in her self-efficacy saying that "she's evolved" in a way that they believed would impact her wider college and personal life.

The second type of story related to the teacher's own assumptions about Claire:

> Teacher: you know when you see somebody who is quite weak with number *that's why I thought you know perhaps she has been over graded but no she backed it up* with her algebra stuff so I was quite happy with that

the teacher reflected on their own previous assumptions about a possible misallocation of her GCSE grade by her school. They used versions of the phrases "over graded" and "I did not believe she was a grade 3" alongside "I misjudged her" and "I got her wrong" repeatedly during our discussions to highlight that their initial assumptions had shifted. The teacher's positioning stories about Claire had changed over time, and it seemed that they felt these were important stories to tell.

However, alongside the teacher own stories of shifts, there was evidence of shiftings happening during the teacher-researcher discussions, as can be seen in the example below:

> Teacher: if its non-monetary she's got no problems at all but well there's a common theme I think and *I'm seeing it more now as we go through the mock* that you know when we're doing the you know with money we're doing the ratio with money she hadn't got {pause} and I think that money is the crux of it here yeah *that's me discovering it this afternoon*

as we reviewed Claire's work together, the teacher experienced shiftings in knowledge and assumptions about Claire. In other examples they talked about seeing her errors in new ways, that is an issue with processing information rather than of misreading, and realising that she may have previously followed a more advanced curriculum compared to what she was being offered at college. The teacher also acknowledged that they would have interpreted Claire's found image of two pathways, that she had chosen to represent her current experiences of mathematics, as indecision rather than as two opportunities, both of which Claire saw as positive choices.

CONCLUSION

The aim of this paper is to introduce a poetic structure called a They poem into the Listening Guide as a means to identify the positioning voices and stories of one teacher about a student labelled as LPA. Using the novel structure, I was able to identify a counternarrative of positive positioning by the teacher, as well as evidence of shifts and shiftings in the teacher's stories which suggested that they were willing to challenge their own assumptions. The findings highlight that as teachers and researchers we should carefully reflect on the lens through which we view the identity work of students labelled as LPA and consider a counternarrative of hard work, constructive attitude, and positive affect. Although literature discusses the impact of the significant other in identity work, the findings suggest the positioning stories of the other are themselves influenced by the self-positioning of identity workers within the complex melee that is identity work. Students are not passive vessels that should be labelled, sorted, and filled with new knowledge, but active agents in their mathematical identity work with a capacity to both be transformed, and transform others, and as such as teachers and researchers we should also be open to change by recognising and challenging our own assumption driven positioning stories.

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DEVELOPING A MODEL OF MATHEMATICAL WELLBEING THROUGH A THEMATIC ANALYSIS OF THE LITERATURE

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Globally, many students experience low mathematical wellbeing, defined here as the fulfilment of one's core values, accompanied by positive feelings and functioning in the mathematics classroom. To increase positive feelings about and engagement in mathematics, there is a need to better understand students' values and align practices to supporting these values. We report on a scoping review of 40 mathematics education publications. Student values in mathematics education could be categorised into seven wellbeing dimensions, namely accomplishment, cognitions, engagement, meaning, perseverance, positive emotions, and relationships. The resulting seven-dimensional mathematical wellbeing model points to target areas to build student mathematical wellbeing.

INTRODUCTION

Mathematics promotes human flourishing through greater educational and career opportunities, and more informed decisions regarding health, wellbeing, and socioenvironmental issues (Su, 2020). Unfortunately, Australian students' achievement relative to other countries is declining, with a lower proportion of students selecting advanced mathematics courses in upper secondary school (Kirkham, Chapman, & Wildy, 2020; Thomson et al., 2019). These declines have occurred despite the introduction of various policies, curricula, teacher training, and classroom practices over the past several decades to support mathematics performance (Su, 2020). But less attention has been paid to students' subjective experiences in the classroom. For many students, mathematics education is far from a positive experience. Studies indicate that students value social learning, caring relationships, and engaging and meaningful pedagogies (e.g., Hill, Kern, Seah, & van Driel, 2021), but these values are not being fulfilled within mathematics education for many students, resulting in disengagement, anxiety, and boredom being commonly reported by students (Attard, 2013). That is, many students are experiencing low wellbeing in mathematics.

Wellbeing in mathematics education – or 'mathematical wellbeing' (MWB) – is defined here as the fulfilment of core values (Tiberius, 2018) within the learning process, accompanied by positive feelings (e.g., enjoyment) and functioning (e.g., engagement, accomplishment) in mathematics. That is, MWB is not only feeling and functioning well (Huppert & So, 2013), but is a positive state of functioning that results from students' experiences in the classroom aligning with their personal values. For example, a mathematics student who values enjoyment, personalised learning support,

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 379-386). PME.

and solving challenging mathematical problems will likely feel good and engage more with the subject when they enjoy their learning, experience one-to-one teacher support and are given challenging tasks. In contrast, that student might feel unwell and disengage from learning when the learning is perceived as boring and they lack personal teacher support.

For many students, mathematics is a challenging school subject. Students with high MWB are more likely to see the challenge as doable and engaging, whereas students with low MWB are more likely to be overwhelmed by the challenge, further contributing to low MWB. That is, the challenge of the subject is less of an issue than incorporating pedagogies that help students value their learning and thrive through that challenge. We suggest that to improve students' experiences at school, we must attend to their MWB, beginning with understanding and attending to what students value in mathematics education.

To support understanding of these values, we undertook a scoping review focusing on literature documenting student values in mathematics education, exploring conditions associated with positive learning experiences and aligning these with wellbeing dimension proposed in the literature. We defined values in mathematics education as the aspects students consider to be important in the process of teaching and learning mathematics (Hill et al., 2021). Across the 40 publications included in our review (see Hill, 2022), we discovered students' mathematics values aligned with seven wellbeing dimensions. These dimensions were also observed to transcend different student ethnicities and grade levels.

BACKGROUND AND THEORETICAL FRAMEWORK

The concept of wellbeing has many uses and conceptualisations across different disciplines (Chia et al., 2020). Here we focus on students' subjective experiences of feeling and functioning across different dimensions (e.g., cognitive, emotional, and social). Various models of subjective wellbeing have been proposed. For example, Seligman (2011) proposed five wellbeing dimensions: positive emotions, engagement, relationships, meaning, and accomplishment (PERMA). Kern and colleagues (2016) proposed the EPOCH model of adolescent wellbeing, which includes engagement, perseverance, optimism, connectedness, and happiness dimensions.

The value fulfilment theory (VFT) of wellbeing (Tiberius, 2018) asserts that individuals' experiences of wellbeing depend on their values, which can differ across personal, cultural, and contextual conditions (Alexandrova, 2017). For instance, what a student values in mathematics likely differs to what they value in physical education or arts, and thus wellbeing looks different across these subjects. Values are hierarchal. At the highest level, 'ultimate values' are core values that are valued for their own sake. At the next level, 'instrumental values' are the things that are valued to achieve more ultimate values (Tiberius, 2018).

To our knowledge, only two publications have explicitly investigated wellbeing specific to mathematics education. Clarkson and colleagues (2010) proposed a three-dimension MWB model (i.e., cognitive, affective, and emotions), arguing that high MWB was achieved through development in all three dimensions. Part (2012) explored adult learners' MWB in terms of capabilities (valued doing or beings) and functioning (valued outcomes). According to Part, high MWB encompasses students feeling both capable and believing they hold the skills to function well. While these two models are a helpful starting point, both models ignore the important social aspects of mathematics learning and lack corresponding measures. Both were derived from mostly Western ethnic backgrounds. They are also theoretically based rather than incorporating students' perspectives. Yet considering MWB is subjective, students' perspectives are important and necessary. Attending to the criticisms of current MWB models helped inform our search strategy.

METHODS

A scoping review of the mathematics values literature was undertaken guided by Arksey and O'Malley's (2005) scoping review framework. Our guiding research questions were: (RQ1) What types of values are espoused by primary and secondary students in mathematics education that positively impact on their mathematics leaning experiences? (RQ2) To what extent do students' values in mathematics education align with wellbeing dimensions proposed in philosophy, positive psychology, and mathematics education research? And (RQ3), what might be an updated model of MWB that addresses some of the limitations of existing models?

Five databases were searched: Academic Search Complete, Education Research Complete, Education Resources Information Centre (ERIC), ProQuest, and PsycINFO. Our inclusion criteria were that the article was published between 2011 and 2021 (corresponding to the period in which the majority of values research in mathematics education was published); that it focused specifically on mathematics education; that primary or secondary student cohorts were involved; and that students specifically reported their values.

In total, 2,252 publications were exported into Covidence, a review management software. Titles and abstracts were screened as per the inclusion criteria leaving 135 publications. Full texts were then read leaving 40 values publications to be analysed. These 40 publications were then imported into NVivo12 and thematically analysed using a combined inductive/deductive strategy (Braun & Clarke, 2006). We began with a bottom up (inductive) approach to generate data-driven themes (RQ1) with subsequent theoretically driven top down (deductive) analysis to categorise these themes according to the wellbeing literature (RQ2). For RQ1, initial codes were inductively generated. For example, qualitative methodologies were coded from student quotes. Quantitative (survey) methodologies were coded from students' highest ranked values. For RQ2, using a deductive strategy, we aligned the emergent value themes (from RQ1) with seven welbeing dimensions from the literature categorising the values (identified for RQ1) into one of the seven dimensions, rather than including values across multiple dimensions. Finally, we present an updated model based on the

Deductive WB Themes (RQ2)	Description	Dimension Source	Example Inductive Value Themes (RQ1)
Accomplishment	Valuing achievement, reaching goals, confidence or mastery completing mathematical tasks and tests	PERMA	Accuracy, high marks, goals, confidence
Cognitions	Valuing knowledge, skills, and/or understanding required to do mathematics at school	MWB	Efficiency, recall, prior knowledge, understanding
Engagement	Valuing concentration, absorption, deep intertest, or focus when learning/doing mathematics	PERMA, EPOCH	Attention, interesting work, novel learning, autonomy
Meaning	Valuing direction in mathematics; feeling mathematics is valuable, useful, worthwhile or has a purpose	PERMA	Maths agency, real world links, utility, task value
Perseverance	Valuing drive, grit, or working hard towards completing a mathematical task or goal	EPOCH	Challenging maths, perseverance, practice & hard work
Positive Emotions	Valuing positive emotions when learning/doing mathematics e.g., enjoyment, happiness, or pride	PERMA, EPOCH, MWB	Minimal anxiety, fun, safe climate, pride
Relationships	Valuing supportive relationships; feeling valued, respected and cared for; connected with others; or supporting peers in mathematics	PERMA, EPOCH	Belonging, group work, family support, teacher explanations, teacher warmth & care, peer support

Table 1: Deductive themes, descriptors and accompanying indictive value themes. PERMA: Seligman, 2011; EPOCH: Kern et al., 2016; MWB: Clarkson et al., 2010.

dimensions that aligned between the values and wellbeing literature (RQ3), identifying the percentage that each dimension was mentioned, overall and separated across demographic characteristics (age and jurisdiction).

RESULTS

We found 90 unique emergent value themes which could be deductively categorised according to seven wellbeing dimensions. Table 1 presents the final MWB model, with the deductively identified themes (column 1), descriptions identified from the literature (column 2), and sources for the deductive model (column 3), along with example value themes identified within the 40 publications included in the scoping review (column 4; see Hill (2022) for full set of coded themes). Across all publications, the most frequent value themes (RQ1) were mathematical *understandings* (12% of total value theme count); *practice, hard work and effort* (12%); *meaningful and relevant learning* (12%); *sharing ideas and peer explanations* (10%); and *teacher explanations* (9%).

Demographic characteristics	Acc	Cog	Eng	Mean	Pers	PosE	Rel	Value Theme #
Overall (*n* = 40 publications)	13%	18%	11%	15%	14%	10%	19%	189
Primary students (7)	11%	17%	11%	17%	17%	8%	19%	36
Secondary students (21)	12%	16%	14%	14%	12%	11%	19%	97
Primary & Secondary (12)	16%	21%	5%	14%	16%	7%	20%	56
Europe (8)	8%	16%	13%	18%	16%	13%	16%	38
Australia/NZ (13)	15%	17%	13%	15%	10%	8%	22%	60
Asia (8)	15%	24%	3%	12%	18%	6%	21%	33
Africa (5)	14%	23%	5%	18%	23%	0%	18%	22
North America (3)	11%	11%	17%	17%	11%	17%	17%	18
South America (2)	17%	17%	17%	8%	8%	17%	17%	12
Mixed countries (1)	17%	17%	17%	0%	17%	17%	17%	6

Table 2: Student demographics, % of value theme mentions by each row/demographic, and total theme count across each row. *Note.* Acc = Accomplishment, Cog = Cognition, Eng = Engagement, Mean = Meaning, Pers = Perseverance, PosE = Positive emotions, Rel = Relationships

Table 2 summarises the percentage of themes identified in the 40 studies across the seven themes overall and by age and jurisdiction. We found relationships was reported most frequently (19% of total count), followed by cognitions (18%) and meaning (15%). Positive emotions were mentioned least frequently (10%). Some differences by age and jurisdiction did occur. For instance, meaning and perseverance were mentioned more by younger than older students. Across ethnicities, notable differences included Europeans valuing accomplishments less often than other ethnicities, Asian and African students reported greater cognitive values than other ethnicities, Africans valued perseverance most, South Americans did not value meaning to a great extent, and positive emotions were rarely mentioned by African students.

DISCUSSION

Here we undertook a scoping review of the mathematics values literature, thematically coding for emergent value themes. Based on VFT, if student wellbeing is about the fulfilment of values, we interpret these mathematics values as conditions or experiences that support student wellbeing in mathematics education. We discovered 90 unique mathematics values themes which aligned with seven wellbeing dimensions proposed in the literature (Clarkson et al., 2010; Kern et al. 2016; Seligman, 2011).

Values relating to relationships in mathematics were mentioned most frequently, which included references to teachers, peers, and families, as well as general belonginess and support. This aligns with research showing students mostly refer to teacher and peer relationships when describing factors supporting their wellbeing in mathematics (Hill et al., 2021). Also, relationships and feelings of connectedness are central to students' conceptions of their own wellbeing (Powell et al., 2018). Cognitions were mentioned second most frequently; this included values relating to mathematical skills and understandings. Students associated cognitions with both positive and negative emotions, suggesting some overlap across the dimensions. For instance, misunderstandings often contributed to anxieties and disliking of mathematics (e.g., Larkin & Jorgensen, 2016). Successful problem solving, and accuracy contributed to pride and enjoyment (e.g., Martínez-Sierra & González, 2014). The progressive yet linear nature of most mathematics teaching and learning can contribute to fear or anxieties about being left behind in a fast-paced curriculum (Gesist, 2010). The cognitive dimension is absent from generalised wellbeing models and was sourced from Clarkson et al. (2010) MWB model. This suggests a generalised approach to student wellbeing might overlook crucial subject specific variations, speaking to the need for greater subject specificity for wellbeing models.

Because wellbeing is value dependent (Tiberius, 2018), how wellbeing is experienced likely differs across student demographics. This was somewhat confirmed in our data. These differences likely reflect students' cultural values. For example, the valuing of perseverance by African students may reflect the high social inequities in Africa and working hard may help transcend these adversities. Yet all seven dimensions, with one exception (i.e., Africans' valuing of positive emotions), were cited by students across cultures and grades. What this implies is that these seven dimensions are still likely important for culturally diverse student cohorts.

A limitation of our review is that we categorised values into one of the seven dimensions. Yet, values often reflect multiple wellbeing dimensions. For instance, valuing hand-on, and practical mathematics learning might align with cognitions (e.g., practical tasks facilitate better understanding), engagement (e.g., they are interesting) or positive emotions (e.g., they are enjoyable). To determine the best category, we would pose the question, *what is the true purpose this value serves for this student?* This was often not possible for survey responses; however, for literal student quotes it generally involved exploring the wider context of the students' experiences. Evidence suggests wellbeing dimensions are interconnected and complementary (Kern, 2021). For example, feeling accomplished or having meaningful experiences are also generally enjoyable. Similarly, a single value might serve multiple purposes, the same value differently enhancing wellbeing across different life domains. Future reviews might consider what emerges when values are allocated across multiple categories.

CONCLUSION AND IMPLICATIONS

Guided by VFT, our review revealed seven dimensions associated with MWB. This model provides a practical solution to explore and potentially build student MWB. Teachers often struggle describing and implementing wellbeing strategies in individual subjects (Waters, 2021). This MWB model might provide teachers with tangible and measurable dimensions which they can apply in their mathematics teaching. For example, they might consider how to foster positive emotions during mathematics or consider ways to enhance teacher-student relationships. Future studies will look to quantify MWB through surveys guided by this model.

For many students, mathematics learning is far from a positive experience, and often, it is the negative aspects of mathematics that students (and teachers) focus on. This study offers a more positive approach to mathematics learning by focusing on what experiences might enable students to thrive in the study of the subject, rather than the source of their failings.

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A PRELIMINARY STUDY EXPLORING THE MATHEMATICAL WELLBEING OF GRADE 3 TO 8 STUDENTS IN NEW ZEALAND

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In recent years, there has been an increasing focus on student wellbeing in schooling. Despite evidence of disengagement and anxiety related to mathematics, how wellbeing is experienced in individual subjects is vastly under researched. This research paper presents the findings of a study which explores the 'mathematical wellbeing' of 1281 grade 3 to 8 New Zealand students participating in their first of a multiyear mathematics teacher learning and development intervention programme. Findings indicate a decline in reported wellbeing as student grade level increases as well as examining both the strengths and weaknesses of students' mathematical wellbeing. The study highlights the importance of exploring subject-specific wellbeing and provides an eight-dimensional model to measure wellbeing specific to mathematics.

INTRODUCTION

For many students, mathematics education is potentially a negative experience with high incidences of disengagement, boredom, and mathematical anxiety (e.g., OECD, 2013). Further challenges are faced by diverse groups of students from marginalized communities given the perpetuation of negative stereotypes in relation to beliefs about mathematics ability and who has the potential to achieve highly. For instance, in New Zealand, Māori and Pacific Islander (Pāsifika) students often experience lower teacher academic expectations, and themselves underachieve in mathematics compared to students of other ethnicities (Rubie-Davis & Peterson, 2016). These negative experiences point to a poor sense of student wellbeing in many mathematics classrooms – or 'mathematical wellbeing' (MWB).

Despite student wellbeing emerging as a priority for educational institutions around the world (Water, 2021), particularly following the COVID-19 pandemic, subject-specific wellbeing is vastly under-researched. Given wellbeing is context-specific, persons, cultures, or organisations, may espouse different values (Tiberius, 2018). Applied to school education, what a student values in mathematics potentially differs to what they value in history or visual arts. Thus, varied conditions across school subjects could be necessary for maximising student wellbeing and alternatively generalisation of wellbeing across all subjects masks important individual subject variations. This calls for a greater investigation into subject specificity for student wellbeing.

It appears there has been limited research related to student wellbeing in individual subjects including mathematics education. In this paper, we investigate the MWB of students participating in their first of a multi-year teacher learning and development intervention programme called *Developing Mathematical Inquiry Communities*

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 387-394). PME.

(DMIC). An aim of DMIC is to, ultimately, both improve students' achievement and MWB, in part through enhanced support with mathematics learning and by teachers aligning pedagogy with and fulfilling students' values.

DMIC PROFESSIONAL LEARNING AND DEVELOPMENT INITIATIVE

DMIC is a whole-school formative professional learning and development (PLD) initiative focused on supporting teachers to develop ambitious mathematics pedagogy (Kazemi Franke & Lampert, 2009) and culturally responsive/sustaining teaching aligning with cultural values (Gay, 2010). This work is implemented in schools across New Zealand with a particular focus on schools serving marginalized Māori and Pāsifika communities. Key aspects of DMIC PLD include the use of mathematical tasks which draw on the funds of knowledge of students, their families, and communities; instructional practices that align with students' cultural values and support respectful social interactions; and the development of key mathematical practices like questioning, explaining, and justifying (Hill, Hunter & Hunter, 2019). The pedagogical process requires shifts in both teachers and students' roles. Specifically, students are required to participate and engage in ways of learning that privilege different forms of knowledge and interaction. Aligning the teachers' pedagogical values with students' values underpins DMIC with the aim of promoting positive learning outcomes like achievement, and engagement. We conjecture that in schools where the pedagogical practices advocated in the DMIC PLD are implemented with high fidelity over time, students' MWB will potentially be enhanced.

THEORETICAL FRAMEWORK

Conceptualisations of subjective wellbeing often centre around two distinct, yet related, constructs – hedonism and eudemonism. Hedonism equates wellbeing with maximum pleasure (or happiness) and minimal distress (Waterman, 1993). Eudemonia, coined by Aristotle, is a life lived in accordance with our *daimon* or true self, full of meaning, virtue, and personal growth (Waterman, 1993). Dual wellbeing models (combining hedonia and eudemonia) are currently the most widely accepted (e.g., Huppert & So, 2013; Kern et al., 2016) and often simplified to "feeling good and functioning well" (Huppert & So, 2013, p. 839) across multiple life domains. Example dual models include Seligman's (2011) PERMA: positive emotions, engagement, relationship, meaning and accomplishment. Also, Kern et al., (2016) adolescent wellbeing model: engagement, perseverance, optimism, connectedness, and happiness (EPOCH). Despite wellbeing models proposing different conditions of wellbeing, each model makes assumptions concerning what is important for wellbeing and thus is dependent upon values. The value fulfilment theory of wellbeing (Tiberius, 2018, p. 34) asserts that wellbeing is "the extent that we pursue, and fulfill or realize, our appropriate values...when we succeed in terms of what matters to us emotionally, reflectively, and over the long term". For instance, if we value enjoyment, relationships, and accomplishments then our life goes well when we enjoy what we are doing, have close relationships, and experience 'success'.

Applied to mathematics education, we define MWB as the fulfilment of students' values (Tiberius, 2018) within the learning process accompanied by positive feelings (e.g., fun) and functioning's (e.g., engagement) (Huppert & So, 2013) in mathematics. Previously, values have been shown to support and align with student wellbeing in mathematics education (Hill, Kern, Seah & van Driel, 2021).

Wellbeing is enhanced when someone's values are congruent with the values prevailing in their environment, or between persons. In contrast, illbeing occurs when one's values conflict with his/her environment (Sirgy; 2021). For instance, subjective wellbeing is enhanced when tertiary students' align their values with their course values or with peers; when employees values align with their organisation or co-workers (Sirgy; 2021). Similarity for mathematics education, teachers aligning their personal or pedagogical values to align with their students' values may enhance student MWB.

A thematic analysis of the mathematics education values literature revealed seven conditions (or dimensions) supporting student MWB (Hill, Seah, Kern, & van Driel, 2022) described in Table 1: accomplishment, cognitions, engagement, meaning, perseverance, positive emotions, and relationships (Table 1). These dimensions were tested and confirmed with Australian (Hill, Kern, Seah & van Driel, 2021) and Chinese (Seah & Hill, 2021) students. Because wellbeing is value dependent, to be relevant for culturally diverse and minority students like New Zealand Māori and Pāsifika we propose an eighth dimension – cultural wellbeing – for the current study.

METHODS

Whilst our larger study explores changes in students' MWB across multiple years in DMIC, here we report on students' MWB during their first year of DMIC, in 2020, providing a baseline of MWB at the beginning of the PLD. Our research question asks: What is the MWB of students during their first year of the DMIC PLD?

Participants included 1281 students (50% female/male) in grade 3 (n = 127), grade 4 (214), grade 5 (266), grade 6 (256), grade 7 (99), grade 8 (85), or grade unknown (234), and attending one of 11 schools ranging from low sociodemographic (deciles 1 – 3, n = 874), medium sociodemographic (deciles 4 – 6, n = 109) or unknown school demographic (n = 298) and located throughout New Zealand.

We designed an online survey consisting of 20 Likert style questions covering the eight wellbeing dimensions (ranging from 0 for low to 4 for high MWB). Question wording was mostly based on general wellbeing surveys (e.g., Kern et al., 2016) with amendments specifying mathematics education. For example, *I finish whatever I begin* (p. 591, Kern et al., 2016) was changed to *In maths, I finish what I begin.*

Survey responses were imported into SPSS 27 for statistical analysis. To determine which dimensions were rated the highest and lowest across all students, one-way repeated measure ANOVA (using Greenhouse-Geisser corrections if needed) with Bonferroni post hoc tests were used. Univariate ANOVA with post hoc Tukey tests determined if individual MWB dimensions were rated significantly higher or lower by certain groups of students (genders, grades, and school decile/demographics).

Dimension	Description	Example survey questions
Accomplishment	A sense of achievement, reaching goals, or mastery completing mathematical tasks/tests	I feel like I am making progress towards my goals in maths
Cognitions	Having the skills, and understanding to undertake mathematics	I have the maths skills to complete my maths work
Cultural	Acknowledging and respecting one's cultural identity in mathematics	In our maths lesson it feels good to be [culture]
Engagement	Concentration, absorption, deep interest, or focus when learning/doing mathematics	When I am doing maths I get completely absorbed in what I'm doing
Meaning	A sense of direction, feeling mathematics is valuable, worthwhile or has a purpose	I feel my maths learning has a purpose and is meaningful to me
Perseverance	Drive, grit, or working hard towards completing a mathematical task/goal	In maths I finish what I begin
Positive Emotions	Positive emotions in mathematics e.g., fun, gratitude, enjoyment	When I am doing maths I have a lot of fun
Relationships	Supportive relationships, feeling valued/cared for, connected with others, or supporting peers in mathematics	I have help and support from my maths [teacher/peers] when I need it

Table 1: Eight dimensions of mathematical wellbeing

RESULTS

Figure 1 summarises the mean scores across the MWB dimensions also the grand mean or total MWB score. Overall student mean scores significantly differed across the eight MWB dimensions (F(6.3, 7931.46) = 24.76, p < 0.001). The relationship dimension had the highest mean score, followed by accomplishment, meaning and perseverance with mean scores displayed within the boxes in Figure 1. These four dimensions did not significantly differ from one another, however, they were each significantly higher than the four lowest rated dimensions, positive emotions, engagement, cognitions, and culture. The engagement dimension had the lowest mean score and was rated significantly lower than relationships, meaning, accomplishment and perseverance dimensions.

For the MWB difference between student genders, only relationships differed significantly (F(1, 1173) = 13.41, p < 0.001) with girls (M = 3.15, SD = 0.04) rating the relationship dimension higher than boys (M = 2.95, SD = 0.04). Across school demographics (or decile), significant differences were found for positive emotions only ($F(1, 979) = 6.13$, $p = 0.01$) with students from lower school deciles reporting more positive emotions than medium decile schools (low $M = 2.9$, $SD = 0.04$; med $M = 2.64$, $SD = 0.1$). Notably, the greatest differences in mean scores were found across student grade levels with all eight dimensions and total MWB $F(5, 1036) > 4.49$, $p < 0.001$ showing a statistically significant downward trajectory, particularly from grades 3 to 5 then grades 6 to 7 shown in Figure 2. Positive emotions showed the greatest mean score decrease ($M = -0.97$ from grades 3 to 8) compared to all other dimensions. Between grades 5 and 6 the mean ratings for most dimensions improved slightly, however, none of these were statistically significant. Then from grades 7 and 8 the mean ratings for culture, engagement and relationships dimensions also improved however again none reached statistical significance.

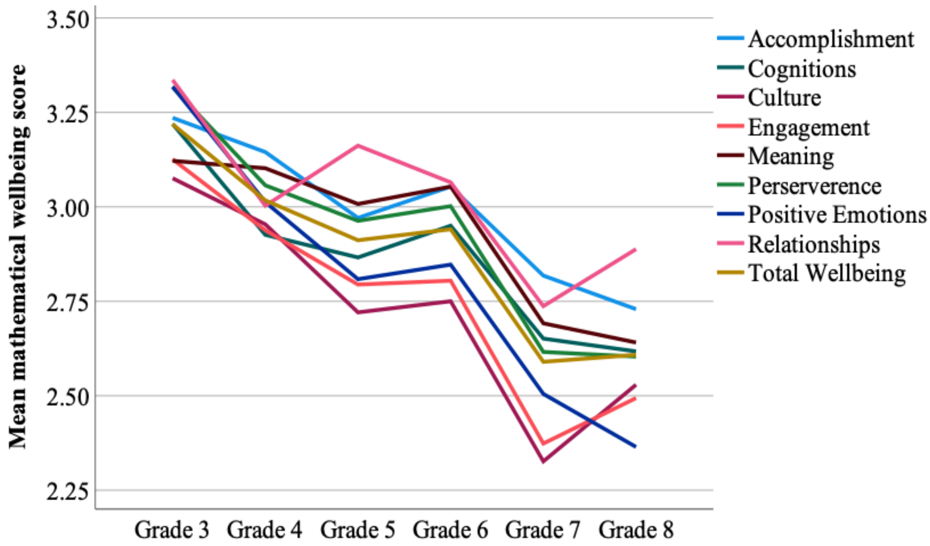

Figure 2: Mean MWB scores across grades

DISCUSSION

Despite student wellbeing increasingly becoming a priority across schools and in many countries, there have been few studies that have specifically explored wellbeing within specific subjects, including mathematics. This study attends to this gap by surveying 1281 New Zealand students from grades 3 to 8 to explore their MWB during their first of a multiyear DMIC programme. Using an eight-dimensional model of MWB we found, on average, that students reported a 'normal flourishing' range for all dimensions, with each reported mean value between 2.8 to 3.06 as aligned with the PERMA survey cut off values (Kern, 2017).

Students flourished most in terms of their relationships, sense of accomplishment, meaningful learning experiences, and mathematical perseverance positively enhancing students' overall MWB. Culturally diverse students often cite relationships as most important for their MWB (Hill, Kern, van Driel & Seah, 2021). Accomplishments are a core aim of education and often highly valued by students in mathematics (e.g., Hill, Kern, Seah & van Driel, 2021). Meaningful, relevant, useful, or real-world tasks are often engaging and interesting for students (Attard, 2013). Additionally, many students equate mathematical success with perseverance and effort (Hill, Hunter & Hunter, 2019).

Students rated engagement, cultural wellbeing, and positive emotions significantly lower than the highest four dimensions. Globally, over a third of fifteen years old's feel helpless or anxious in mathematics (OECD, 2013). Student boredom in mathematics is widespread (Larkin & Jorgensen, 2016) and linked to surface learning strategies, poor achievement (Ahmed et al., 2013), and unpleasant emotions (Larkin & Jorgensen, 2016). Acknowledging students' cultural values particularly for minority students supports students' relationships, their cultural identities, and engagement with mathematics (Hill, Hunter & Hunter, 2019). These four lowest rated dimensions are potential MWB weaknesses and point to target areas to improve students overall MWB.

Relationships were rated significantly more positively by females than males reflecting patterns from other general wellbeing surveys (Bulter & Kern, 2016). Emotions towards mathematics were rated more positively across lower decile schools contrasting with earlier studies asserting higher student demographics generally experience more positive attitudes and less anxiety in mathematics (Grootenboer & Marshman, 2016).

Notably, there was a significant decline in MWB across all grades and wellbeing dimensions with the sharpest decline between grades 6 to 7. This coincides with the typically stressful primary to secondary school transition. Much research attests to the sliding mathematics affective response particularly over middle school from years 5 to 8 (e.g., Grootenboer & Marshman, 2016). Increasing pressures to achieve, overreliance on textbooks, out of field teachers in secondary classes, and less opportunities for peer collaboration are potential sources of this affective and wellbeing slide in mathematics education.

FUTURE DIRECTIONS AND IMPLICATIONS

Here we present preliminary findings exploring student MWB in New Zealand providing a baseline to measure subsequent changes in MWB coinciding with longitudinal experience of the DMIC programme. Future investigations will explore ethnic variations, MWB at the individual level, also broaden the sample to include all grades and school deciles. The eight-dimensional model proposed here provides discrete measurable entities to measure MWB, pointing to both strengths and challenges to students' MWB whilst highlighting target areas to improve students' experiences in mathematics. Teachers are increasingly tasked with supporting student wellbeing, particularly during the COVID-19 pandemic, yet many struggle to contextualise wellbeing principles to specific subjects (Waters, 2021). Using this eight-dimensional model as a framework teachers might feel better prepared to recognise and communicate about student wellbeing specific to mathematics.

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THE CONNECTION BETWEEN MATHEMATICS AND OTHER FIELDS: MATHEMATICIANS' AND TEACHERS' VIEWS

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This study investigated: (1) what secondary school teachers, who participated in an academic program that included applied mathematics, learned about the connections between mathematics and other fields, and how this knowledge contributed to their teaching, and (2) what mathematicians, who taught in that program, wanted to teach teachers about those connections. Data source included interviews with five research mathematicians and 14 teachers. Analysis revealed that the mathematicians wished to teach teachers about the contribution of mathematics to other fields as well as the reciprocal contribution of other fields to mathematics. Yet, the teachers enriched their knowledge only about the former, and used their new knowledge to raise students' interest and motivation to learn mathematics, but not for doing mathematical work.

INTRODUCTION

Applied mathematics, which links between mathematics and other fields (e.g., physics, computer science, engineering, economics and biology) is an integral and essential part of the discipline of mathematics. Its central role in the discipline is reflected in the growing number of areas of applied mathematics research conducted by mathematicians at prominent research universities around the world (e.g., Department of Mathematics at ETH Zurich, 2022; Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022; School of Mathematical Sciences of Fudan University, 2016; University of California, Berkeley, n.d.). These include, for example, spectral and dynamical problems of quantum mechanics, population genetics, image processing and medical imaging, mathematical finance and quantitative risk management.

An important characteristic of applied mathematics is that the interactions between mathematics and other fields are often bi-directional. One direction is from mathematics to other fields, denoting the contribution of mathematics to solving problems in various fields; the other direction is from other fields to mathematics, denoting the contribution of other fields to the development of mathematics, as explained in the following: "This interaction is often bi-directional: mathematical concepts and techniques are used to model and solve concrete problems in other fields. Reciprocally, scientific progress raises new mathematical problems, and motivates the development of new mathematical concepts and tools" (Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022).

In contrast to the central role that applied mathematics has in the discipline of mathematics, the professional education and development of mathematics teachers

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rarely include opportunities to learn about this key aspect of the discipline (Cai et al., 2014; Greefrath & Vorhölter, 2016; Novotná, 2019; Schmidt et al., 2008). For example, Schmidt et al. (2008), who examined the structure of secondary mathematics teacher preparation programs in six countries from three continents (Bulgaria, Taiwan, Germany, South Korea, Mexico, and the US), found that applied mathematics courses were not included in the academic mathematics content courses that prospective teachers were required to study. Novotná (2019) reported similar results regarding the Czech Republic, and Greefrath & Vorhölter (2016) revealed that mathematical modelling is not a compulsory content in teacher education programmes at universities in German-speaking countries.

With the current broad consensus regarding the importance of promoting applications and mathematical modelling in schools (e.g., Galbraith et al., 2007; Kaiser, 2020), the vast attention given to mathematical literacy and the relevance of mathematics to real life (e.g., COMAP & SIAM, 2019; PISA, 2018), and the growing interest in STEM education (e.g., Li et al., 2020; Maass et al., 2019), the need to attend to this deficiency in the professional education and development of mathematics teachers is further enhanced.

Our study addresses this issue by examining what secondary mathematics teachers may learn about the connections between mathematics and other fields in an academic program that comprises a focus on applied mathematics. Furthermore, as learning is shaped by teaching, we also examine what mathematicians that teach in such a program wish to teach teachers about these connections. The research questions are:

1. What mathematicians, who teach in an academic program for secondary teachers that includes applied mathematics, want to teach teachers about the connections between mathematics and other fields?
2. What teachers, who participate in an academic program that includes applied mathematics, learn about the connections between mathematics and other fields and how this knowledge contributes to their teaching?

METHODS

Setting and Participants

The study was situated in a unique master's program for practicing secondary school mathematics teachers, in which academic-level mathematics and research in mathematics education are the main components. The mathematics component comprised eight courses, designed and taught by research mathematicians. Four of these courses dealt with topics in the school curriculum at an advanced level: algebra, analysis, geometry, and probability and statistics. Three courses dealt with topics in applied mathematics, presenting modern use and application of mathematics in computer science, natural sciences, social sciences and everyday technologies. One course appraised the history and philosophy of mathematics. In addition, a final project was carried out under the guidance of a mathematician.

Five of the seven mathematicians who taught in the program participated in the study (M1, M2 … M5). They taught all the mathematics courses in the program but two: algebra and the use of mathematics in computer science. All were prominent research mathematicians. The research interests of three of them involved applied mathematics. The teachers participating in the study were 14 program graduates (T1, T2 … T14). All held a bachelor's degree in mathematics or in a mathematics-related field before starting the program. Their teaching experience ranged from 5 to 23 years.

Data Source and Analysis

The main data source included individual semi-structured in-depth interviews with the mathematicians and the teachers. These interviews were conducted as part of a comprehensive research program that examine the relevance and contribution of academic mathematics studies to secondary school mathematics teachers' knowledge about the discipline of mathematics, and how that knowledge contribute to their teaching (Hoffmann & Even, 2021, 2018). The mathematicians were asked about their teaching goals in the program, first in general and then specifically regarding what mathematics is. Correspondingly, the teachers were asked whether they learned something new about what mathematics is from their mathematical studies in the program, and if they did, whether that knowledge contributed to their teaching. Additionally, the teachers were presented with eight phrases that appear in the literature in relation to characteristics of the discipline of mathematics (e.g., mathematical definitions, thinking in mathematics, formal presentation in mathematics, the connection between mathematics and other disciplines), and were asked to choose three for which their mathematical studies in the program enriched their knowledge, and to describe what they learned.

For the purpose of this study, the interviews were analyzed qualitatively in an iterative and comparative process, aiming to identify what, if at all, the mathematicians wished to teach teachers about the connections between mathematics and other fields, what the teachers learned, and how this new knowledge contributed to their teaching.

FINDINGS

Mathematicians

All five participating mathematicians expressed in their interviews a wish to enrich teachers' knowledge about the connections between mathematics and other disciplines. No differences were found in this regard between the two mathematicians who taught applied mathematics courses and the three that taught the other mathematics courses. For example (interviewer denoted by I):

5	I:	I'd be happy if you could elaborate on your goals. You say that you sat down and thought about what this program should be. Could you elaborate?
6	M1	…we, at least I and some of my colleagues, felt that there is a need to show the connection between mathematics and other disciplines and everyday world.

The mathematicians referred to the connection between mathematics and other disciplines as bi-directional, reciprocal contributions. One direction is from mathematics to other fields, when mathematics is used to solve problems in other fields. The other direction is from other fields to mathematics, when work on solving problems in other fields raises new mathematical problems which then promotes the development of mathematical concepts and methods and thus advance mathematics.

When referring to the contribution of mathematics to other fields, the mathematicians stressed that they wished to expand teachers' knowledge about the practical worth of mathematics. At times they associated it with what they viewed as deficiencies in the contents of the high-school curriculum. For example,

> One of the things that... bothers me very much in high school mathematics... is that the mathematics that is taught in high school is not related to life at all. It usually viewed by students as an annoying exercise that is meant to upset them, and it doesn't look like something that has any value... That is, they teach students calculus, for example, for no reason... That is, why is there a derivative? and why is there an integral?... Where on earth did it come from? …And why did they develop it?... Quite a few of the mathematics teachers... don't know that a derivative is related to speed, that is, acceleration... there is a kind of thinking here that mathematics is a philosophical field that has nothing to do with science, and nothing to do with technology, and nothing to do with anything... I think that it is simply unacceptable that they would talk about derivatives and wouldn't know why Newton developed it. (M4)

The mathematicians emphasized that they would like to show teachers that mathematics is not just a theoretical science with no connection to reality. Instead, it can be used to develop the world. For example,

> Mathematics is a tool to describe the world around us... Moreover, you can use it too. What does it mean? After describing the world, it can be used to make a better world. (M3)

They explained that the usefulness of mathematics is conveyed through its contribution to solving problems in various disciplines and diverse areas of life, such as physics, medicine, economics, biology, engineering, geography, computer science, communication, navigation, etc. Often, they drew on examples from their own teaching in the program. For example, M5 reported that he presents in his course contemporary uses of mathematics, such as, GPS, search engines, encryption, robotic movements; and M4 described the emphasis she puts in her course on the power of mathematical models in solving problems from different disciplines:

> I start the discussion with an applied problem, and in fact, I also define for them this field of applied mathematics, which is building models. How to build a mathematical model for a problem... So, I think it gives them this beautiful connection that I keep emphasizing in the course, that mathematical language is a language that can be used to describe lots and lots of different problems in the same language. Once you have the mathematical tools, then you can answer questions from different disciplines. (M4)

When referring to the contribution of other fields to mathematics, the mathematicians associated it with the idea that the development of the discipline of mathematics lies in mathematical work related to questions. They stressed that questions may originate in mathematics as well as in other disciplines. Referring to mathematical work on questions originated in other disciplines, the mathematicians emphasized that work on such questions is central not only to the development of other disciplines, but also to the development of the discipline of mathematics itself:

> A great many of the developments in mathematics had real motivation... questions from life, not from mathematics, and the mathematical way helped solve them, and *then they developed mathematics* [emphasis added]. (M4)

Exemplifying the contribution of work on such questions to the development of mathematics, the mathematicians mentioned, for instance: calculating area and volume that contributed to the development of calculus; navigation that accelerated the development of geometry and trigonometry; and computer science, electricity, and electronics that promoted the development of graph theory. For example,

> Graph theory is an example of a mathematical field that has developed on its own, from… mathematical questions... but also from practical questions of applied mathematics. Because graph theory is the internet, graph theory plays a role more or less in all fields of science today. So, graph theory has very much developed in recent years. (M4)

The mathematicians further stressed the dialectic relationship between the development of the discipline of mathematics and that of other disciplines:

> This cycle that there is a problem that starts with something real and then goes through many degrees of abstraction and becomes something completely theoretical, and at the end, it goes back to something real, happens a lot in mathematics. (M1)

Teachers

Twelve of the 14 participating teachers reported in their interviews that academic mathematics studies contributed to their knowledge about the connection between mathematics and other disciplines; 10 specifically chose the phrase “the connection between mathematics and other disciplines” when asked to select topics for which their mathematical studies in the program enriched their knowledge. For example, T12 picked up the card with this phrase and repeated things she mentioned before:

> The connection between mathematics and other disciplines. In this regard this program gave me a lot. Physics, philosophy, in computers... as I said, nature, science... It enabled me to see that everything in nature can be organized in a mathematical way… (T12)

All 12 teachers connected their new knowledge regarding the connection between mathematics and other disciplines to two of the applied mathematics courses: mathematical applications in natural sciences, and mathematical applications in social sciences and everyday technologies. Three other mathematics courses – probability and statistics, application of mathematics in computer science, and the history and philosophy of mathematics – as well as the final project were also mentioned by some teachers.

In contrast to the mathematicians, none of the teachers referred to the connection between mathematics and other fields as bi-directional. Instead, they described a uni-directional contribution only: from mathematics to other fields. For example,

> The mathematical applications course was an eye-opening course… Image compression, how it is expressed mathematically... I have never seen how there is at all a connection between mathematics and these things... Google search, how it works… and a medical problem that you can find how to model it mathematically. And how mathematics can help not only to develop thinking and the mind and to enjoy mathematics but… to discover its use. And I tell about it to my students as a motivation. (T7)

Similarly, T2 described how the course in probability and statistics enriched her knowledge regarding daily life usage of mathematics.

> Let me give you an example, say, [lecturer's] probability course. I remember his first two lessons. He sent us home to look in the paper for all sorts of things of probability... say, what the poverty index is in the country, how they are calculated, and to check their correctness… I felt that it was ... something that is very connected to us. Like, where in our lives we find this connection to mathematics…
>
> Last year I gave it to the 11th grade, middle-level track. I asked them to look for all kinds of statistics and how they were obtained. And they brought it to class. After that, I talked with them about the role of statistics in our lives… At the middle-level track, there is always this question: "What does it give me?" and "Where does it accompany me in my life?" (T2)

Nine teachers of the 12 teachers who reported on new knowledge regarding the contribution of mathematics to other disciplines, reported also that this new knowledge was relevant to their teaching work. All explained that it helped them increase students' interest and raise their motivation to learn mathematics, as was illustrated above in T7's and T2's quotations.

CONCLUSION

Applied mathematics, a key aspect of the discipline of mathematics, entails bi-directional interactions between mathematics and other fields (Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 2022). One direction is from mathematics to other fields, when mathematics is used to answer questions in other fields. The other direction is from other fields to mathematics when work on problems in other fields promotes the development of new mathematical concepts, tools, questions and theories.

The literature suggests that the professional education and development of mathematics teachers rarely include opportunities to learn about applied mathematics (Cai et al., 2014; Greefrath & Vorhölter, 2016; Novotná, 2019; Schmidt et al., 2008). Situated in a professional development program for practicing secondary school mathematics teachers that comprises a focus on applied mathematics, our study provides important information regarding the potential contribution of such programs to teacher knowledge and practice related to the connections between mathematics and other fields.

Our findings suggest that the participating mathematicians aimed to enrich teachers' knowledge about the connection between mathematics and other disciplines, and the participating teachers considerably advanced their knowledge on one aspect of this connection, namely, the contribution of mathematics to other fields. Correspondingly, the teachers acquired a rich repertoire of contemporary examples of authentic use of mathematics to solve important problems in various areas of life, which they then used in their teaching to raise students' interest and motivation to learn mathematics. Yet, such use in teaching mainly involved informing students about fascinating uses of mathematics in the real world without actually doing any mathematical work. This result might be connected to factors, such as, not having teaching materials on which the teachers could draw in order to incorporate their newly acquired knowledge with the school mathematics curriculum, lack of support from their work environment for doing so, and more.

Additionally, in contrast to the participating mathematicians' wish for teachers to learn about the bi-directional connection between mathematics and other fields, the teachers did not mention in their interviews the contribution of other fields to mathematics. This result appears to be in line with the way applied mathematics is often dealt with in mathematics education, emphasizing the use of mathematics for solving problems in real-world contexts (e.g., Cai et al., 2014; Kaiser, 2020). As the other direction has been central to the development of mathematics, and continues to be so today, a question then arises whether this aspect needs to be more explicitly incorporated into the professional education and preparation of mathematics teachers.

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