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EDITORS

Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

VOLUME 3

Research Reports (Ja – Se)

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RESEARCH REPORTS

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A HELPING HAND IN OUTDOOR MATHEMATICS – THE ROLE OF GESTURES IN MATHEMATICS TRAILS

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Mathematics trails provide learners with opportunities to leave the classroom to discover and engage with mathematics outdoors, using real-world objects and structures. Students are immersed in mathematical problems that require them to think about and make connections between the mathematics learned in the classroom and how this knowledge can be applied in a novel context. In this qualitative study, participants engaged in a mathematics trail, during which we observed a significant use of gestures in the interactions between group members and the physical objects that formed the basis of the mathematical problems. Seeing a connection to the idea of embodiment, we pay particular attention to the types and functions of those gestures that emerged in the outdoor mathematics trail context.

INTRODUCTION

Compared to the confines of the classroom where students often remain seated while working at their desks, outdoor mathematics, a form of mathematics education that occurs in an outdoor environment, provides students with both movement and a less familiar context for doing mathematics. Mathematics trails are an approach to this that guide students along a predetermined route of mathematics tasks that use real objects along the trail (Gurjanow & Ludwig, 2018). By solving mathematics tasks outdoors, using real-world, physical objects, students engage in first-hand, out-of-class experiences of mathematics, which play a "central role […] in the learning process" (Kolb et al., 2000, p. 1) according to the Experiential Learning Theory (ELT). Given that gestures have been shown to be important to foster student learning (Sinclair & de Freitas, 2014), we wanted to study students' gesturing in this new, less constrained mathematics trails environment.

THEORETICAL BACKGROUND

Outdoor mathematics in the context of modelling and experiential learning

To solve a mathematics trail task, it is necessary to consider the object's outdoor context and to transfer it into the mathematical world. These processes are described in the modelling cycle outlined by Blum & Leiss (2007), where learners must first engage in "understanding" and "structuring/simplifying" the outdoor context before "mathematising" in order to transfer from the real-life object to the mathematical model. After students use the necessary procedures to solve the mathematics, the results must then be retransferred back to reality. This is referred to as "interpreting"

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the results before being subsequently “validated” and “presented” at the real object. To highlight the two domains, “Reality” and “Mathematics”, Figure 1 shows a modified version of the modelling cycle which emphasises these transfer processes specifically in the context of an outdoor mathematics task.

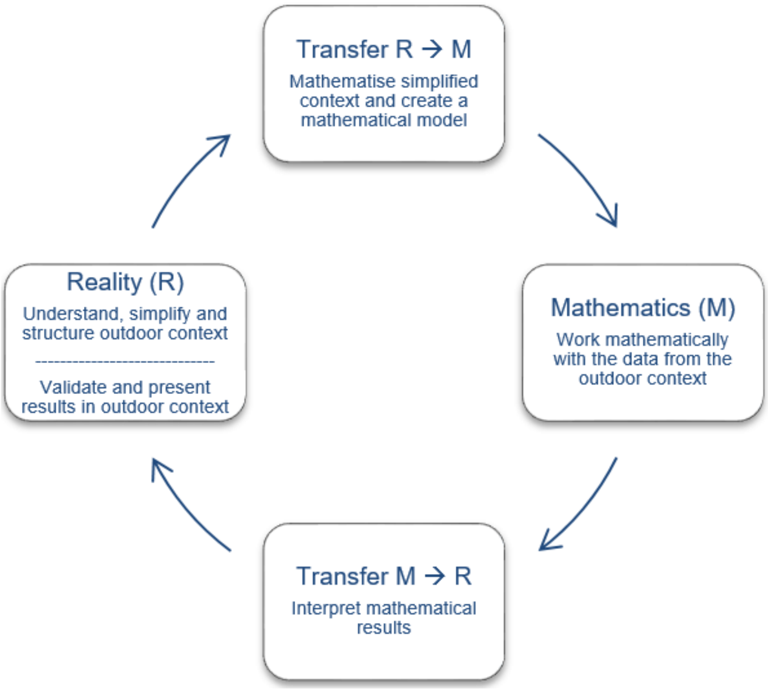

Figure 1: Modified modelling cycle for the outdoor context (Blum & Leiss, 2007).

In relation to ELT, activities that are linked to “Reality” seem especially relevant, since these first-hand experiences must necessarily take place at a physical object.

The role of gestures in outdoor mathematics education

From an embodied perspective, gestures, i.e. “hand movements that co-occur with speech” (Goldin-Meadow, 2003, p. 4), play an important role in learning and teaching mathematics concepts. “Children can express thoughts in gesture that they don’t even know they have. And those thoughts tend to be on the cutting edge of their knowledge” (p. 116). Hereby, gestures can occur in different forms, which McNeill (1992) categorised into four different types (see Table 1).

Gesture	Explanation
Iconic	Movements and shapes of body, objects, people in space
Metaphoric	Gestures that represent abstract ideas rather than concrete objects
Deictic	Indicate people, objects and locations in the real world
Beat	Gestures that beat musical time

Table 1: Types of gestures according to McNeill (1992).

Gestures, independent of their type, often occur unconsciously. However, in relation to mathematical concepts, they are used for different purposes, which Kita et al. (2017)

summarised as the four functions of gestures:

- Activating Spatial Information, i.e. focus on new/different information
- Manipulating Spatial Information, i.e. rearrange, translate, rotate, invert
- Structuring Spatial Information, i.e. organise information for the act of speaking
- Exploring Spatial Information, i.e. explore more complex situations and distinguish relevant and non-relevant information

Together with these functions, "gesturing may make it easier to link a speaker's words to the world. […] linking words and phrases to real-world objects, is required for comprehension" (Goldin-Meadow, 2003, p. 163). Finally, the positive impact of content-related movements and gestures in relation to learning and thinking about mathematical concepts are well known (e.g., Sinclair & de Freitas, 2014). In particular, these authors look at manipulating gestures in the context of a touchscreen environment. In addition to in-the-air movements, the authors also consider contact and touching gestures to be actual gestures. We follow this understanding of gestures.

From the connection between ELT and embodiment, we assume that gestures will be observed while learners engage in the solution process for mathematics trail tasks. Prior observations in the outdoor context confirm this hypothesis and show that deictic (particularly pointing) and iconic gestures in particular, seem to be relevant during the modelling steps of simplifying and structuring (Jablonski, 2021). Therefore, we assume that most gestures in the outdoor context are linked to "Reality" (see Figure 1) and to the work occurring at that task's object. The physical presence of the object at the task's location may influence the occurrence of gestures, which will be investigated in more detail.

Though in a different material-based mathematical context, the observations of Menz (2015) and Hare & Sinclair (2015) allow the hypothesis that the presence of real-world outdoor objects might extend the pointing, especially for deictic gestures, to an actual touching gesture. Therefore, we formulate the following research question for the outdoor mathematics context: *[RQ1] Does the presence of physical objects extend the use of deictic gestures from pointing to touching gestures during outdoor mathematics?*

The outdoor context makes it possible to interact with a task's object that is physically present. Therefore, students encounter a material-based situation, though the material object itself is not variable in its position and situation. This may lead to a more intense use of manipulating gestures in activities that are linked to "Reality" and prepare the actual "Transfer R → M". Based on this hypothesis, a second research question is taken into consideration: *[RQ2] What role do manipulating gestures play during activities linked to the real object of a mathematics trail task?*

METHODOLOGY

A qualitative study was conducted at Simon Fraser University (SFU), Canada during September and October 2021 to investigate the role of gestures in mathematics trails.

The sample is comprised of eleven participants (doctoral or master students and a professor) from the field of mathematics education at SFU, who volunteered to take part in the project. Each group of two to three participants solved eight tasks along a mathematics trail located on campus. The mathematics trail was created in the MathCityMap system, which guides students along the outdoor tasks, includes optional hints and provides immediate feedback on the quality of the entered solution (see Gurjanow & Ludwig, 2018). The groups required 60–90 minutes to solve all of the tasks and were accompanied by one of the researchers, who filmed the solving process and recorded observations relevant to the research questions.

The eight math trail tasks cover various topics of secondary school mathematics and required participants to gather the information necessary to solve the problem while at the outdoor location. Table 2 presents two tasks concerning outdoor objects, the task formulation and mathematical activities that could be useful for solving each task.

Object/Situation	Task Formulation	Mathematical Activities
	What is the volume of this pyramid? Give the result in m³.	Choose mathematical model (*regular pyramid*) and data (*height and base length*), take necessary measurements, use a formula
	Determine the height you will cover when you run up the hill. Give the result in meters.	Choose mathematical model (*total height as sum of height of each step*) and take necessary measurements (*individual steps*), addition

Table 2: Two mathematics trail tasks from the study.

The recordings of the solution processes used by each group resulted in more than four hours of video. For this paper, we chose the "Pyramid" and "Hill" tasks outlined in Table 2. These tasks were chosen for a comparative analysis due to the different foci on mathematics necessary to solve each task: the "Pyramid" task focuses on a characteristic of a physical object (i.e. its volume) and the "Hill" task is based on a real-life situation (i.e. running up the hill). In the *first step*, there were 37 sequences of gestures selected for analysis. Hereby, all noticeable hand movements are considered sequences and a single hand movement represents one sequence.

In the *second step*, the selected sequences of gestures, in conjunction with their accompanying speech between group members, were analysed in terms of the mathematical content (description of the task solution process) and outdoor context (categorisation of the sequence according to Figure 1). The latter identifies whether a sequence contains an activity that is either linked to "Reality", to "Mathematics" or to the "Transfer from Reality to Mathematics" or vice versa.

In the *third step*, the gestures within each sequence were coded using the categories from McNeill's (1992) *Type of Gesture* (iconic, metaphoric, deictic and beat) and Kita et al.'s (2017) *Function of Gesture* (activate, manipulate, structure and explore). Deictic gestures were further coded as pointing or touching. It seems legitimate to make reference to already existing categories of schemes, since gestures occurring outside should, hypothetically, be completely assignable. The categorisations were empirically confirmed in the context of the study with good values (Kappa between .72 and .8) of intercoder reliability. Finally, all analysed units are connected and taken into consideration for deeper analysis with regard to the research questions.

RESULTS

In the first part of the results presentation, we summarise the data for all identified sequences, while in the second part, we take individual sequences with either deictic (touching) and manipulating gestures into account.

General description of gestures

Analysis of the *outdoor context* shows that the majority of gestures occurred during activities that are linked to "Reality" (56.8%), followed by activities in the "Transfer from Reality to Mathematics" (37.8%), though they also played a minor role in "Mathematics" and "Transfer from Mathematics to Reality". These results are consistent with previous research findings in the outdoor context where gesturing occurred particularly while students were in the simplifying/structuring stage and during the mathematising stage of the modelling cycle (Jablonski, 2021). The different *types* and *functions* of the gestures that occurred during the 37 identified sequences are summarised in Figure 2.

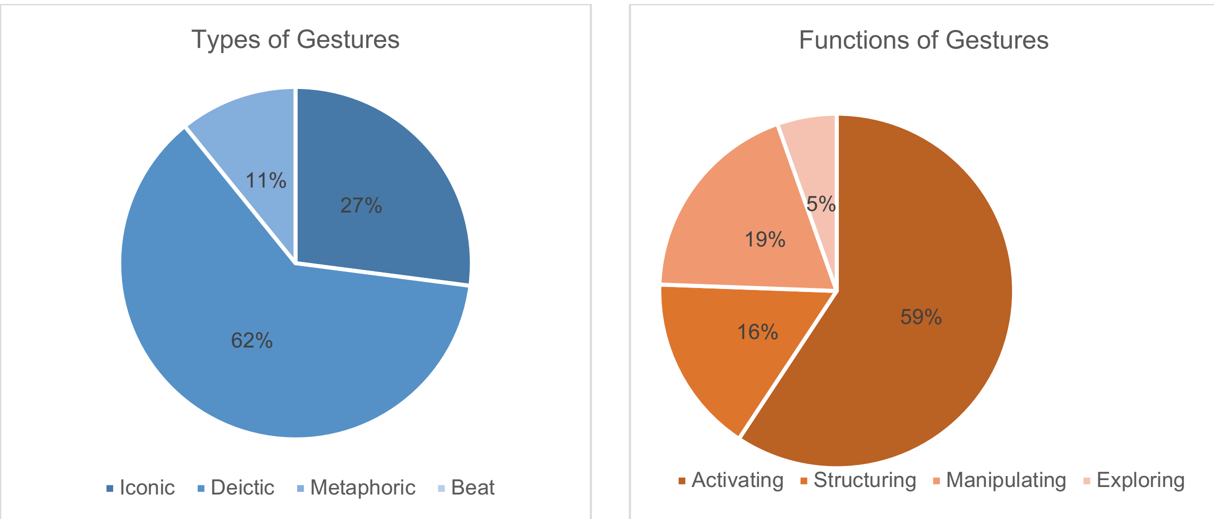

Figure 2: Type and function of the gestures.

The overall occurrence of *gesture types* also confirms Jablonski's (2021) previous findings of mainly iconic (27%) and deictic (62%) gestures used in the outdoor mathematics context, whereby deictic gestures, which contain both pointing (80%) and touching (20%) gestures, occur more than expected. The hypothesis that touching the real-world, physical object might be relevant in outdoor mathematics can be confirmed, however, it remains unclear in which situations and contexts that this gesturing occurs.

Before moving to a detailed analysis of individual sequences, we will consider the *functions of gestures*. The results show that most gestures are used for activating spatial information (59%), followed by gestures for manipulating (19%), gestures for structuring (16%) and gestures for exploring (5%) spatial information. As in the previous category, we can confirm the hypothetical relevance of gestures-for-manipulating through their occurrence in 19% of all sequences. Together with the deictic, touching gestures, we will focus on these gestures in more detail in the second part of the results presentation.

The role of deictic gestures

From the general overview, deictic gestures appear to play a major role while solving mathematics tasks with real-world, physical objects. Most deictic pointing gestures can be observed when participants are identifying relevant points, i.e. the pyramid's apex, and mainly occur in relation to activities related to reality, i.e. simplifying and structuring. Still, about 20% of the analysed deictic gestures involve physically touching the real object as described in this sequence from the "Pyramid" task. As the participants searched for a way to identify the height of the pyramid by using its characteristics, one participant's verbal proposal is accompanied by different deictic gestures (summarised in Table 3).

Act of Speech	Description of the Gesture	Analysis of the Gesture
We could measure this	Student moves left hand down the arm of the pyramid and *taps* it.	Deictic (both touching and pointing) indicating the arm of the pyramid as "this".
and knowing half of that	Student *points* with the right hand along the edge of the base to the bottom right corner.	Deictic (pointing) indicating the edge of the base as "that".
we could find that.	Student *points* upwards with the right hand toward the middle of the pyramid. With the index finger of the left hand the student makes a sweeping *point* at the bottom area of the pyramid towards the middle of the base.	Deictic (pointing) indicating the height of the pyramid as "that".

Table 3: Analysis of an act of speech with gestures.

Only the combination of the participant's act of speech, with the deictic gesture, makes his proposal – using the Theorem of Pythagoras to calculate the pyramid's height with only lengths that can be measured precisely – understandable. Instead of using the verbal mathematical expressions for identification, the participant touches and points while using the expressions, “this” and “that”. In particular, in the first part of the act of speech, we can observe the use of a touching deictic gesture when the participant refers to the arm of the pyramid. It seems like the participant wanted to highlight the actual physical appearance of this side length, which makes it easy to be measured precisely – in contrast to the height of the pyramid, which can only be imagined.

The role of manipulating gestures

Finally, we demonstrate the use of manipulating gestures in the following sequences related to the “Pyramid” task. The participants – we call them Kate and Paul – tried to figure out whether they could combine multiple pyramids into a rectangular prism using several in-the-air gestures to manipulate the object and situation.

Kate: If you put it in the middle [*motions up and down with an open palm, thumb up*] and flip this over [*using both hands parallel to each other, palms open, motions flipping the shape 90 degrees to the left*], does it not [*using right index finger, traces the left, top and right side of a rectangle shape*] follow a rectangular prism or am I way off?

Paul: It does not. Because that point [*points towards the apex with right index finger*] ends up down here [*points towards the bottom left corner of the pyramid’s base with right index finger*].

Kate: Down there [*points towards apex and then gestures down to the left base with her right index finger*] and then [*gestures back towards the apex and then moves her index finger across to the left*] it will cut right.

Paul: And the wide [*points both hands, index fingers towards the sky, hands and arms close together. Gestures outwards with both hands*] ends up there and you get this weird shape [*Moves index fingers down and inwards at an angle, then down and outwards at an angle, mimicking the hourglass shape made by two pyramids stacked on top of each other, apex to apex*] there.

We observe a series of manipulating gestures in this sequence. The rotation of the pyramid in particular, is explored through several manipulating gestures. Even though this idea is no longer relevant for their modelling process, we can observe similar considerations in relation to the geometric task of the pyramid’s volume. In contrast, the “Hill” task does not provoke manipulating gestures. Therefore, the role of manipulating gestures may depend on the context of the actual task and does not only rely on the existence of a real-world, physical object outside the classroom.

CONCLUSION AND OUTLOOK

The analysis of gestures that occurred in two different outdoor mathematics tasks provide insight into the embodiment nature of working mathematically with real-world objects. First, we can observe the overall appearance of gestures and conclude that embodiment and gestures are particularly relevant for the work in “Reality” and the “Transfer from Reality to Mathematics”. With deictic gestures being the majority, it can be seen that not only pointing, but also touching gestures appear (cf. RQ1).

Hypothetically, this happens in situations where the students want to bring the focus to the actual existence of the object and the use of its real, touchable characteristics. With the modelling process becoming more complex, i.e. imagining an altered or enriched situation, manipulating gestures can be observed (cf. RQ2). In our analysis, these occurred primarily during the geometric task. Additional work, however, is required to confirm this and should also be compared to similar task formulations inside the classroom without real-world objects.

ACKNOWLEDGEMENT

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ENGAGING WITH ONLINE ELABORATED FEEDBACK AS A MEDIATION TOOL IN THE MATHEMATICAL ARGUMENTATION PROCESS

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University of Haifa

Feedback may be more effective if students engage with its content. Almost no studies have examined the potential of online elaborated feedback to enhance students' mathematical argumentation. We designed a set of tasks requiring argumentation that must be supported by constructing examples in an interactive diagram using automatic verbal characteristics suggested by the technological environment. We explored whether and how a 7th-grade student's engagement with the mediation tools supports her mathematical argumentation of claims on the topic of comparing fractions. Our data were derived from a task-based interview and automatic analyses that the technological environment provides. The results show that there was an improvement in the student's argumentation in response to the feedback process.

INTRODUCTION

Feedback is a term most often used to describe information provided by agents in response to submitted work. It is considered as one of the most powerful ways of supporting learning processes (Hattie and Timperley, 2007). Shute (2008) pointed out that elaborated feedback, which addresses students' responses, particular errors, and examples, and provides guidance toward a correct answer, appears to enhance students' learning more than other types of feedback. Shute's review indicated that presenting too much information may result in superficial learning and cognitive overload. Feedback can support learners effectively if it is part of a process in which learners play a central role in sense making and use comments to improve their work or learning strategies (Carless and Boud, 2018). We distinguish between two meanings of feedback. Feedback as an object, which refers to the contents of feedback itself (the information), and the feedback process, which describes the student's interactions with the task and the feedback information. In this study, we used STEP (Seeing the Entire Picture) as a formative assessment environment. STEP tasks provide elaborated information presented as automatic verbal characteristics in the form of immediate feedback in response to the student's dragging action in the interactive diagram (ID) (Harel and Yerushalmy, 2021). To help students become aware and reflect on the provided feedback report, the same characteristics appear as claims in the tasks designed for the activity.

Mathematic argumentation is one of the meta-cognitive strategies, and it is defined as a process of drawing conclusions with the aim of demonstrating that a claim is true or false based on a set of relevant information. Argumentation has the potential to help

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students modify their understanding and refute misconceptions (Oh and Jonassen, 2007). Toulmin (1958) presented a model for analyzing arguments that consists of six statement types, each playing a different role in the argumentation process: the conclusion (C) is the statement that must be proven true or false; the data (D), which is relevant evidence for the claim; the warrant (W), which justifies the connection between the data and the conclusion (by appealing to a rule or a definition); the warrant is supported by the backing (B), which presents further evidence; the qualifier (Q), which qualifies the conclusion by expressing degrees of confidence; and the rebuttal (R), which potentially refutes the conclusion by stating the conditions under which it does not hold. Not all of these components must be explicitly verbalized in every argument. In our study, we used this model to identify the argumentation components in the student's transcribed interview while the student engaged in the feedback process.

Fractions are a central topic in the mathematical curriculum. Learning the concept of fractions poses a significant challenge, and it has been for a long time a focus of research of the mathematics education community. Students tend to develop misconceptions about fractions for the following reasons: (a) dealing with fractions as natural numbers, they compare fractions by looking at the values of the numerator and the denominator rather than considering the whole fraction, and assume that if the values of the numerator and the denominator are greater, the fraction is also greater than the other; (b) generalizing a given strategy to all fractions, for example, comparing fractions by comparing them with 1 is a strategy that students should use differently with fractions that are smaller or greater than one (Alacaci, 2014; Behr, Wachsmuth, Post, and Lesh, 1984). These misconceptions were the basis of argumentation claims that the student had to justify by constructing an example in the ID.

The novelty of this study lies in the type of designed feedback process stemming from interaction with online elaborated information and from the request for mathematical argumentation to be provided by the student. To this end, we designed a set of tasks requiring argumentation that must be supported by constructing examples in an interactive diagram using automatic verbal characteristics suggested by STEP. We conducted an empirical study to explore how students engage with such feedback, and we sought to identify the argumentation components reflected in this engagement. Our main research question was: How is the engagement of the student with the online elaborated feedback reflected in the student's examples and her mathematical argumentation?

METHODOLOGICAL CONSIDERATIONS

This study is part of a larger study in which we explore the use of online elaborated feedback by students and describes a small-scale experiment that enables qualitative analysis. One 7th-grade student worked on the tasks within the framework of a task-based interview. The student had learned fractions according to the national curriculum in a regular classroom. She was presented with a sequence of tasks delivered on STEP.

The interactive diagram (Figure 1) that the student used to solve each task is based on a representation similar to that described in the Arnon, Nesher, and Nirenburg's (2001) study. It displayed points in the coordinate system representing a fraction. The numerator was represented by the number that appears on the vertical axis, and the denominator by the number that appears on the horizontal axis. The student was asked to construct fractions by dragging the green and the red points only along the grey lines, where all the fractions reside for which the difference between the numerator and the denominator in absolute value is 1: fractions smaller than 1 on the lower line (labeled I) and fractions greater than 1 on the higher line (labeled II). Such fractions are given as an example of a fraction comparison strategy in the mathematics curriculum and books. The strategy requires the student to decide which fraction is greater depending on its distance from the 1 whole. This strategy is normally used in the case of fractions smaller than 1 and formulated as "the fraction that is closer to 1 is greater." Some students have a misconception that this formulation applies to all fractions, including to those greater than 1. Throughout the activity, the ID shows automatically which fraction is closer to 1 (Figure 1**).

The feedback that was reported automatically was generated in response to the dragging of the points in the ID. The characteristics were designed to reflect the ideas of the activity and the pedagogical mathematical goals of the task. Five characteristics, labeled 1-5, were included (Figure 1). They have the potential to be helpful in the student's attempts to explain and argue that the construction meets the requirements of the task: mathematical (characteristics 1 and 4); visual representation (characteristic 2); and the method for comparing two fractions (characteristics 3 and 5). If STEP identifies any characteristic in the example while the student performs a dragging action in the interactive diagram, it is highlighted in blue; otherwise, it is highlighted in grey. These characteristics were designed to reflect the curricular foci of the task and the basis for the student's engagement with the task, which may help students identify their misconceptions. The criteria for assessing changes between the examples in the course of the activity were based on a comparison of the automatically assessed characteristics, assuming that the change in the example space indicates changes in the student's concept of comparing fractions.

The flow of the activity. Each of the tasks 1-3 (Figure 1) contained the same four claims (labeled a-d), regarding which the student had to decide whether they are true, and support each true claim by constructing an example. In all tasks, claim d was true. The other claims reflected misconceptions stemming from the wrong generalization of the strategies for comparing fractions that the students had learned. Task 1, designed to reflect the initial conception, did not contain online elaborated feedback. Tasks 2 and 3 included five sets of mathematical characteristics each, which were given as online elaborated feedback. To help the student engage with the online feedback process, she was asked to characterize her example using this set of characteristics. Task 2 was formulated as follows: "Choose which are the true claims and support every true claim by an example. Below the interactive diagram, five characteristics can help you characterize the two fractions you have created. Check each characteristic that is present in your example before submitting it. Try to submit two fractions in a way that your submission should comply with as many characteristics as possible." If the student chose the true claim, the maximum number of characteristics was 3 (characteristics 1, 2, and 3). In task 3, the student was asked to submit examples that comply with as few characteristics as possible; the minimum number of characteristics was 2 (characteristics 1 and 3).

The representation line is a tool that student could choose whether to use (Figure 1*); it did not appear automatically. When the fractions were not equivalent, two distinct lines appeared, green and red; the line of the greater fraction was higher than that of the smaller fraction; the lines coincided when the fractions were equivalent. The use of the representation line could help the student connect it with the correctness of her answers.

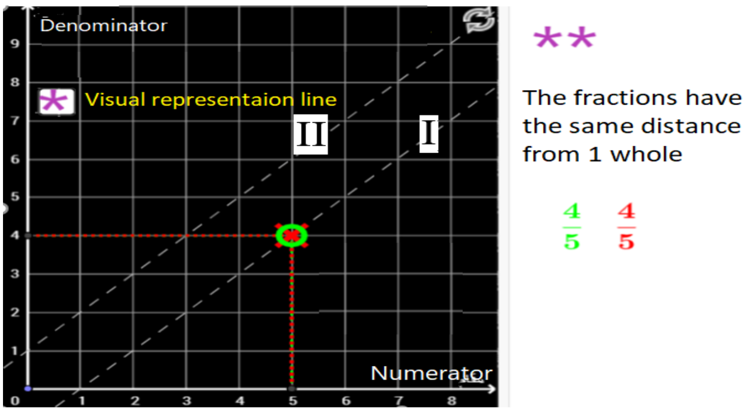
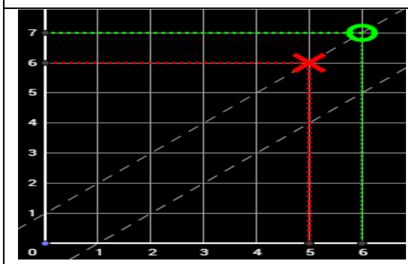
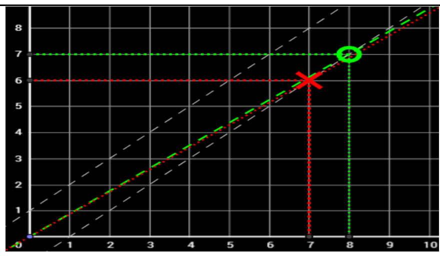
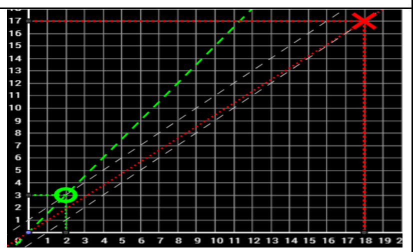
The interactive diagram	Claims
	a. The fraction that is closer to 1 whole is always the greater fraction. (If you agree with the claim, choose the green fraction as the greater fraction.) b. The fraction with the greater numerator and denominator is always the greater fraction. (If you agree with the claim, choose the green fraction as the greater fraction.) c. The fractions are always equal in this case. d. It is possible to find a fraction with a smaller numerator and denominator that is greater than the other fraction. (If you agree with the claim, choose the green fraction as the greater fraction.)
Online elaborated feedback 1. The green fraction is greater than the red one 2. The green line higher than the red one 3. One fraction is greater than 1, and the other fraction is smaller than 1 4. You chose two fractions that are smaller than one 5. The two fractions have equal numerators or denominators	

Figure 1. The interactive diagram, online elaborated feedback, and claims.

Data sources and analysis. The data were based on automatic information analysis by STEP, which we correlated with the segments of the student's transcribed interview that we analyzed qualitatively. Using Toulmin's model (1958), we sought indications of a connection between the student's engagement with the online feedback and the process of argumentation.

FINDINGS

After reading the instructions of task 1, the student chose Claim a (C), The fraction that is closer to 1 whole is always the greater fraction, and constructed the example 7/6>6/5 (D) (Figure 2-a). The student declared that she was confident in her choice (Q), she did not look for other true claims. The student did not realize that her choice of the claim was incorrect and that her submission did not meet the requirement of the claim she chose (the green fraction was not the greater one). The example that the student chose shows that the strategy she used stems from incorrectly generalizing a rule in Claim a. The student went on to task 2 and again chose Claim a. To meet the requirement of the task, she tried to support her claim by constructing an example that has as many characteristics as possible, based on the elaborated feedback. She chose two fractions that are smaller than 1, activated the line representation, and compared the characteristics in the elaborated feedback with the positions of the lines (Figure 2-b).

☑ Claim a	☑ Claim a	☑ Claim d
		
$\frac{7}{6}$ $\frac{6}{5}$ The green fraction is closer to 1 whole.	$\frac{7}{8}$ $\frac{6}{7}$ The green fraction is closer to 1 whole.	$\frac{3}{2}$ $\frac{17}{18}$ The red fraction is closer to 1 whole.
Figure 2-a. Submission of task 1	Figure 2-b. Submission of task 2	Figure 2-c. Submission of task 3

Below are some of the student's interactions with the feedback and her self-reflections:

1 S: (Dragging the two points and looking at the feedback characteristics) I see! When the green fraction is greater than the red one, the green line is higher than the red one. They (the characteristics 1 and 2 in the feedback) turn on and turn off together (means changing colors).

2 I: How do you know that the green fraction is greater?

3 S: It's written in characteristic 1, and I can see here (points to the statement that shows which fraction is closer to 1. The student chose 7/8>6/7).

4 I: Why did you choose these fractions?

5 S: Because this (points to characteristic 3) and this (points to characteristic 4) cannot turn on together (cannot be that both fractions are smaller than one, and at the same time one of them is greater than 1). I have to choose one of them. (The student ignored characteristic 5 in the feedback and submitted

her answer. The characteristics that were turned on in the feedback were 1, 2, 4.)

In line 1, the student realized that the connection between the representation lines and the value of the fraction provided her with a way to support Claim a (W). She chose an example that was suitable for the claim she thought was true and identified characteristics 1 and 2 in the feedback (D). In line 5, she realized that characteristic 3 and 4 were contradictory. She based her justification of the claim on evidence that supported it, but did not try to find examples that could refute her claim. The conclusion followed from the data. Although the student believed that it was not possible to refute the claim, there is way to do so by suggesting fractions greater than 1.

The student continued to task 3. She chose Claim a as true, read the characteristics trying to construct an example that would meet the task requirement of as few as possible characteristics, then dragged the points to 9/8>6/5. All the characteristics in the feedback were turned off. Below are some of the student's reactions to the elaborated feedback (Figure 2-c):

1 S: How it can be? The green fraction is closer to 1whole, but the characteristic 1 was turned off. (She activated the line representation.) The red line is also higher than the green one! (She wrote in the notebook.) If I do a common denominator for both fractions 9x5/8x5=45/40 and 6x8/5x8=48/40. The red is greater!

2 S: (She drags the green fraction to 8/7, then drags the red one over the grey line I). I understand now (changes her choice to Claim d).

3 I: Understand what?

4 S: I put the green fraction to be always greater than 1 (on the grey line II) and the red to be always smaller than 1 (on the grey line I). In this case the green will never be smaller than the red one. I thought that the green should always be closer to 1 but when I drag the red one, look, it's not the case here (the fraction that is closer to 1 changes color). Claim a is not true.

5 I: Why did you choose Claim d?

6 S: Claim c is not true for sure; the fractions are equal only when the two points are together. If Claim d is true, then Claim b for sure is not true because they are contradictory (chose 3/2>17/18; the characteristics that were turned on in the feedback were 1, 2, 3).

In line 1, the student constructed an example that supports her claim. She noticed that this example was refuted by the elaborated feedback characteristic (D). In line 1, the student calculated a common denominator for both fractions as an alternative strategy of comparing fractions (W) to check that the characteristic in the feedback was working. By this strategy, she discovered that her example was not correct (D), as the feedback showed, and refuted her claim, which made her decide that the Claim a was not true. The student tried another strategy to find the true claim: she used her previous knowledge about comparing fractions to 1 whole. This was the warrant (W), and the student tried to explain and investigate by dragging in the interactive diagram, simultaneously following the changes in the characteristics of the feedback (B) (lines

3-6). She chose Claim d as true (C). She could explain her argument by dragging the points and finding the relation between the examples (D), which were at the same time refuting the other claims (R). The conclusion regarding Claim d being true follows from the data that the feedback provided to the dragging action in the ID (Q). The student, however, did not mention specifically how a counterexample refuted the generalization in Claims a-c.

DISCUSSION AND CONCLUSIONS

The aim of this study was to examine the way the student engaged with online elaborated feedback on mathematical argumentation. One challenge was to help the student interact with the feedback information. To this end, we designed an activity based on a rationale for feedback and immediate linguistic mathematical characteristics offered by STEP, derived from argumentation requirements. The empirical results show that we could identify statements and student examples that were a response to the student's engagement with the feedback process. Through the feedback process the student changed her choice of true claims and examples. Mathematical argumentation was improved in response to engaging in the online feedback process. This improvement was apparent in the student's work and explanations based on the feedback process: she constructed examples as evidence or explanation for the requirement of the task and to the elaborated feedback that was part of it. The elaborated feedback helped her find the true claim; she refuted other claims by counterexamples, identified features that characterized her answers, compared characteristics, understood the relation between them, and connected them to other claims by way of a reasoning (for example, the connection between the comparison of the fractions and the representation line). The student used new strategies for comparing fractions to modify her choices (the representation line tool) and supported her argumentation by a warrant from her previous knowledge (the common denominator strategy). Engaging with the feedback process led the student to modify her concept and refute misconceptions. The feedback process served as an indicator that gave the student confidence with respect to the way she thought about and chose her claims.

The findings are consistent with the literature, which reported a strong potential for online feedback in the learning process (Harel and Yerushalmy, 2021). This is especially true for the improvement of mathematical argumentation (Gaona and Menares, 2021). To engage the student with the feedback process, we designed an activity in STEP that provided special elaborated feedback that was part of the task requirements. The student's engagement with the feedback activated her ability to raise questions, construct supporting or refuting examples, to find explanations connected to her existing knowledge, and to compare arguments. The engagement with the online feedback process led to improvement in the student's mathematical argumentation. These findings can serve as a basis for further research in the field of online elaborated feedback. The study was limited by one participant and should be reproduced with larger groups.

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USING ONLINE DISCUSSION FORUMS FOR THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

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We present a study of a model for professional development of mathematics teachers, based on their participation in a collaborative problem solving in online discussion forums, in two roles. At the first stage of the study, 47 high-school mathematics teachers participated in the forums as students. At the second stage, they mediated forums as mentors. The first stage of the study showed gradual development of group synergy among the teachers-as-students. The second stage showed that the experience of group synergy gained by the teachers at the first stage has supported the development of their mathematical fluency in teaching.

INTRODUCTION

There is a broad consensus in the mathematics education community that mathematical reasoning in problem solving, critical thinking, and the ability to work collaboratively are the key components of students' learning (OECD, 2019). This approach to students' learning implies that teachers should develop knowledge and skills of mathematical communication with students in real time, including the ability to listen, interpret and respond to the student's reasoning, and conduct effective mathematical discussion in the learning process. The proficiency in these skills is referred to as *mathematical fluency in teaching* (MFT) (Ball et al., 2008). In addition, MFT assumes the teacher ability to evaluate alternative solutions, understand students' unfinished ideas, and identify sources of their mistakes.

Studying the forms of teacher professional development (PD) that can contribute to the development of MFT is one of the priorities in the field of research on teaching mathematics (Hoover et al., 2016). Several studies have demonstrated the potential of PD models, in which teachers act as learners while tasting and developing the skills they would like to develop in students (e.g. Kramarski & Kohen, 2017). The current study makes one step further and examines a PD model based on teachers' participation in collaborative problem solving in online discussion forums while assuming two roles. At the first stage, the teachers participate in the forums as students. At this stage we target the growth of *group synergy* (Clark, et al., 2014; Stahl, 2021), which is referred to as continuous interaction among problem solvers who monitor and develop each other 's problem-solving ideas. At the second stage, the same participants assume the role of leaders of problem-solving forums (PSF henceforth). This study aims at testing

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the following hypothesis: the development of group synergy among teachers in the process of their participation in collaborative problem solving in PSF contributes to the development of their MFT.

THEORETICAL FRAMEWORK

In the last decades, many studies explored knowledge and skills that mathematics teachers need to develop (Chapman, 2015). Mathematical fluency in teaching (MFT) has been identified as one of the most important teaching skills (Ball, et al., 2008; Hoover, et al., 2016). It is broadly agreed that for the development of the MFT, it is necessary for the teachers to deepen their mathematical knowledge, in order to be in position to quickly navigate among approaches to understanding and solving mathematical problems that students may have. One of the methods of deepening mathematical knowledge is systematic engagement in solving challenging mathematical problems (Polya, 1945). Additionally, experiencing problem solving by teachers is necessary in order to strengthen their pedagogical skills for better understanding how students think (Chapman, 2015).

A number of PD models developed for deepening mathematical and pedagogical knowledge of teachers is described in the professional literature. For example, Koellner et al. (2007) described a PD model consisting of the following cycle: the teachers first solve mathematical problems, then analyse videotaped problem solving by school students who are given the same problems, and then discuss how they would use the problems in their classrooms. Koellner et al. (2007) showed that this model has undeniable potential for strengthening the link between the mathematical knowledge for teaching and teaching practice. However, the study did not attend to the exchange of mathematical ideas among the teachers in the problem-solving process, as well as to the enactment of the accumulated knowledge with students in real time.

A number of studies have demonstrated the potential of PD models, in which teachers act as learners, testing and developing skills that they would like to develop in learners (e.g., Kramarski & Kohen, 2017). The present study continues both of these directions: the development of mathematical knowledge of in-service teachers through problem solving and the testing of new teaching methods by teachers, on themselves as students. This article discusses the model of the PD of teachers in the process of their participation in the joint solution of mathematical problems in small groups in the role of students, with the subsequent transfer of the accumulated experience to teaching. According to many researchers, synchronous online forums are a conducive environment for successful group interaction due to more precise wording of arguments and a greater willingness of participants to express alternative views and critical ideas (e.g., Asterhan & Eisenmann, 2009; Stahl, 2021). For this reason, PSF were chosen as the environment in which two-stages discussions of mathematical problems took place in our study.

The question of the necessary conditions for productive collaborative work on tasks is broadly studied. In particular, Stahl (2021) studied interactions aimed at involving learners in "research participation" (Stahl, 2021, p. 493). In addition, the importance of interactions, in which learners attempt to understand each other thinking – so-called "other-monitoring" (Goos et all, 2002) has been pointed out. Over time, these types of interactions can lead to the emergence of group synergy. Interaction is considered a group synergy if it is a series of interrelated messages from different participants, in which they either continue and develop each other's ideas, or test the ideas expressed, based on theoretical knowledge and logical conclusions drawn from them. The result of such interaction is progress in understanding the problem and its solution, expressed in new ideas on the way to solving the problem or in the recognition of the fallacy of the proposed idea (Clark et al., 2014). Such cooperation presupposes the ability to delve into the mathematical ideas of colleagues in real time, quick reaction and the desire to reach mutual understanding about the ways of solving problems, that is, those qualities that determine MFT in communication with students and are the key to improving the mathematical education of teachers (Hoover, et al., 2016).

This study answers the following questions: (1) How does group synergy develop in interactions among teachers during their continued involvement in PSF as problem solvers? (2) How is teachers' own experience of group synergy reflected in the MFT of when the teachers interact with students as PSF mentors?

METHODOLOGY

Participants and research progress

The study was conducted as part of a PD program for mathematics teachers at the Faculty of Education in Science and Technology, Technion, Israel. The study involved 47 high school teachers with an experience of 5 to 20 years. At the first stage of the study, as part of the course "Foundations of Geometry. Plane Transformations", each teacher participated as a student in a group of 3-5 in six PSF meetings, mentored by the first author of this article. Each meeting was devoted to collaborative discussion and solving one challenging geometry problem. The second stage took place in the course "Methods of teaching mathematics", when each of the participants acted as a mentor (teacher) at two PSFs. The learners in these forums were students studying for B.Sc. in mathematics education. They also solved complex geometric problems. At this stage, the teachers were tasked with organizing and leading a discussion at the PSF. This article analyses the activities of one of the groups, consisting of 5 teachers. The group consisted of the same participants in all six PSF of the first stage. Then, the experience of one teacher from that group is tracked in his capacity of a PSF mentor. This group is quite representative of the other groups, as the data obtained for this group reflect similar learning processes.

PSF

The technological platform for the PSF in this study was the social network WhatsApp. A WhatsApp group was opened for each group of teachers in which meetings took place. The duration of each meeting was about one and a half hours. Each online meeting approximately consisted of 180 messages with an average frequency of 5 messages per minute. Most of the messages were text messages. Participants also posted photographs of drawings and, in some cases, resorted to short voice messages.

Data and data analysis

In the course of the study, 96 PSF protocols were obtained and analysed. Of these, 72 forum protocols in which teachers acted as students (12 groups with a permanent membership) and 24 forum protocols in which teachers acted as mentors. When analysing the protocols, the unit of interaction was a message (post) sent by one of the participants. In order to answer the first question of the study, the protocols of the forums in which teachers acted as students were analysed. To assess the dynamics of group synergy, we have defined the concept of "synergetic chain", which is understood as a block of interrelated posts of various participants concerning the discussion of one mathematical issue. An example of a synergistic chain is the following episode of the forum during the discussion of a geometric problem:

33	A.:	I think BE = EC
34	B.:	This is true since they are chords from equal inscribed angles
35	A.:	And also, triangle EHC is isosceles
36	C.:	Yes, because in it the height coincides with the median
37	B.:	Means BHCE kite. How have I not seen this before? This will help us a lot.

This episode refers to group synergy, as it contains several interrelated messages containing an element of monitoring (34, 36), the development of each other's ideas by the participants (37) and the progress of the group in understanding the task, since B. expresses his conclusions aloud, referring to the whole group (37). Each synergistic chain has its own length (the number of messages included in it). The length of the chain reflects the duration of the interaction between the participants. We use the average length of all synergy chains included in this forum as one of the characteristics of group synergy in it. In our study, this characteristic was named Syn1. So, in the given example, the length of the synergistic chain is 5. If four threads are found on the forum, containing respectively 5, 2, 7, 4 messages, then the Syn1 characteristic will receive the value Syn1 = 4.5, which shows the group's ability to long-term interaction. An additional characteristic Syn2 characterizes the share of group synergy among other interactions and is calculated as the ratio of the total number of messages included in a particular synergy chain and the number of messages in a given forum. So, if a forum containing 187 posts, contains 4 synergistic chains, 5, 2, 7 and 4 posts long, the Syn2 characteristic is calculated as follows: Syn2 = (5 + 2 + 7 + 4) /187=0.096 (9.6%).

To answer the second question of the study, a qualitative analysis of the content of the messages that the teacher published in the PSF, in which he was a mentor, was carried out. The situation in which the group was at the moment of the mentor's intervention was characterized. Examples of characteristics attended to are as follows: lack of activity in the discussion, the development of a wrong idea, or presence of a right idea that escapes the attention of the students. Then we inductively deduced from the above analysis which qualities of the MFT the teacher showed in his intervention. Finally, the forums in which the teacher-mentor acted as a student were characterized in order to identify situations that could be the prototypes of this intervention. Examples follow.

FINDINGS

Below are graphs showing the change in the indicators of group synergy in the selected group in the process of its participation in six PSFs as learners.

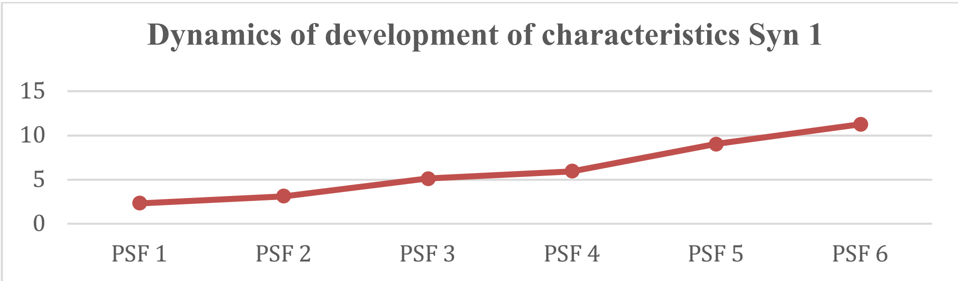


Figure 1. The development of group synergy (Syn1) in six forums

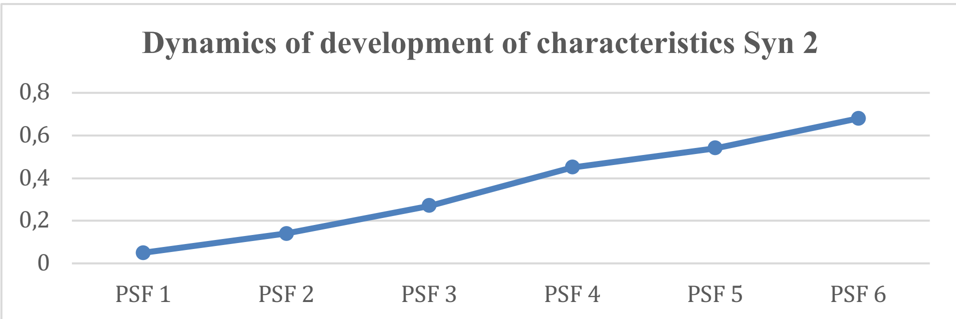


Figure 2. The development of group synergy (Syn2) in six forums

The graphs show an increase in the indicators of group synergy in this group, both in terms of the share of group synergy among the interactions of forum participants, and in terms of increasing the length of synergistic chains. In the last forum, group synergy becomes the main type of interaction, where 70% of messages are in synergy chains, that is, they are part of a brainstorming session. A similar pattern was observed in the other groups participating in the study.

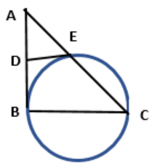
The analysis of the content of messages included in various synergistic chains led to the identification of different types of synergies. For example, the above episode demonstrates the complementarity of participants' mathematical ideas that propelled the group forward in solving the problem. As a result, together with other members of the group, everyone achieves more than he could achieve himself. Another type of group synergy refers to the case when one of the participants explains his idea, and other members of the group monitor it. Often, as a result of such a discussion, it turns out that the idea requires development or turns out to be incorrect. An example is the following snippet of the discussion:

62	A.:	I have proved the similarity of triangles in two corners.
63	B.:	What angles are equal?
64	A.:	There are two inscribed, resting on equal arcs
65	C.:	That's right, they are equal
66	A.:	More right angles. One inscribed at the diameter, and the second at the tangent point
67	B.:	Wait a minute, but, after all, we do not know whether the radius comes to the point of tangency.
68	C.:	It is definitely not a radius; it cannot go through the centre.
69	A.:	But then the angle is not right either. I think I was wrong.

Group synergy also arises when a group makes a collective effort to explain ideas it finds to a straggler or misunderstood comrade. Often during such an explanation, shorter paths are found or details are clarified. The final stage of the work is characterized by a group via reflective discussion of the problem.

The results of the analysis of messages, which supported the work of the PSF by teacher A. from the described group in the role of a mentor, illustrate the answer to the second question of the research. The following task was proposed for discussion in the forum:

A circle of radius R is given (see drawing). BC is the diameter of the circle, AB is the tangent to the circle at point B, D is the midpoint of the segment AB. The ACB angle is β. It is required to express the ratio of the areas of triangles ADE and ABC using R and β.



The following exchange of ideas took place between the students:

21	M.:	DE is the middle line of the triangle.
22	N.:	The figure shows that DE is equal to BD by the two-tangent theorem
23	K.:	Then β can be found. It is equal to 45 °.

All messages were received within one minute. The teacher was required to understand and evaluate the statements made in real time. That is, to show MFT skills. He should have noticed that N.'s statement (22) is true but requires proof. And the assertion M.

(21) is true only in the case $\beta = 45°$ and cannot be the basis for solving the problem in general. K.'s assertion (23) was based on trust in previous allegations, which could later lead the group in the wrong direction. After assessing the situation, the teacher had to make a decision about the usefulness and form of the intervention. He decided to intervene and sent a message: I don't fully understand why DE = DB? The success of the question from the point of view of organizing a mathematical discussion was proved by the subsequent reasoning of the students, during which they proved that DE is a tangent, but not necessarily a middle line. Between this episode and the episode described earlier, when the joint observation of A.'s statement (62) in the role of a student about the similarity of triangles led to an understanding of the fallacy of reasoning. It can be assumed that this experience was used by A. to stimulate discussion and monitoring of ideas while working as a mentor. Working in a group in the role of students, A. and his colleagues did not know whether the statement he proposed was true, and only a joint analysis led them to understand. In the role of a teacher, A. did not point out to the students that the ideas were correct or erroneous. Instead, he asked a specific question (similar to the way colleagues asked him why the angles he named were equal). Thus, with the help of a specific question, A. created a situation that entailed discussion and progress in understanding. One of the components of the MFT is the ability to conduct a mathematical discussion. In particular, it is necessary to involve students in the conversation, to push them to participate in the discussion. A. supported the discussion, using his own experience of participation in the PSF. For example, when there was a long pause at the beginning of the forum, A. stimulated the activity of the participants with the message: “Throw in ideas. The more ideas there are in the discussion, the more chances that some of them will lead to a solution”. A similar proposal was addressed to each other by members of group A. when they participated in the FOP as students. A.'s experience of participating in PSF as a student was also reflected in the fact that he supported and guided the discussion, using encouraging and guiding comments, which the instructor in his group encouraged the discussion. For example: “This is a great idea. You should discuss it "or" This is a good idea, but worth discussing if it is always correct. "

CONCLUDING REMARKS

Based on our findings, we concluded that PSFs are a conducive environment not only for collaborative learning, as shown in previous studies (Stahl, 2021), but also for the PD of teachers. Various forms of group synergy have been found to grow and develop with the continued participation of teachers in PSF as learners, demonstrating improvements in listening, critically analysing and developing others’ ideas in real time. Thus, teachers develop MFT skills, which are a necessary component of successful teaching of mathematics in the modern world (Ball et al., 2008; Chapman, 2015). The experience of mathematical communication acquired in the forums was used in the work of teachers as mentors of the forums, where MFT manifested itself in the ability to delve into students' ideas in real time, interpret them, and quickly choose the reaction that was most useful for learning. This study responds to a request for the need to study models of mathematics teacher PD that, on the one hand, will be relevant for teachers in terms of their work, and on the other hand, will correspond to the goals set for the mathematical education of teachers (Hoover, et al., 2016). The methodological contribution of this study is the quantitative method presented in this study for assessing group synergy in the joint solution of mathematical problems, which adds to methods of qualitative analysis developed in the past studies (Goos et al., 2002; Clark et al., 2014; Stahl, 2021).

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DEFINITIONAL AMBIGUITY: A CASE OF CONTINUOUS FUNCTION

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Definitions are an integral aspect of mathematics. In particular, they form the backbone of deductive reasoning and facilitate precision in mathematical communication. However, when an agreed-upon definition is not established, its ability to serve these purposes can be called into question. While ambiguity can be productive, the existence of multiple non-equivalent definitions for the same term can make the truth value of certain mathematical statements unclear. In this study, we asked mathematics educators to determine the truth of a definitionally ambiguous mathematical claim. Based on their responses, we identified several factors that influenced the teachers' choice of definitions. Finally, we consider the pedagogical implications of employing such a task in teacher preparation programs.

INTRODUCTION

In mathematics, definitions are paramount. As Edwards and Ward (2008) write, "the words of the formal definition embody the essence of and completely specify the concept being defined" (p. 223). Mathematics fixates on definitions for their importance in logical argumentation and proof. To make conclusive statements about mathematical objects, it is necessary that "we do not leave the meaning of a term to contextual interpretation; we *declare* our definition and expect there to be no variance in its interpretation in that particular work" (ibid., p. 224, emphasis in original).

Despite the widely acknowledged significance of definitions in mathematics, different definitions often exist for the same term. Ideally, these definitions are equivalent and any one of them may be chosen as "the" definition from which the others follow as theorems (Winicki-Landman & Leikin, 2000). Sometimes, however, the same term has different definitions that do not encompass the same class of objects. This introduces ambiguity into mathematical tasks. For example, the recent work of Mirin et al. (2021) discusses two different definitions of *function*, both acceptable in the mathematics community, that lead to opposite conclusions when one must decide whether a given function is invertible. In this paper, we present multiple, mathematically acceptable definitions of *continuous function* that can likewise lead to ambiguity. We then present the results of a study in which we asked teachers to decide on the truth value of a statement concerning this term, including the considerations they attended to when making their decisions.

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DEFINITIONS AND DEFINITIONAL AMBIGUITY IN MATHEMATICS

On the importance of definitions and their features.

Mathematicians and mathematics educators alike acknowledge the importance of definitions in teaching, learning, and exploring mathematics. One important feature of definitions is that they facilitate communication within a mathematical community; that is, they specify how a term is used in order to assure that interlocutors refer to the same concept when using that term (e.g., Borasi, 1992). Mathematical definitions are used to introduce new objects, to determine properties of what was defined and to assess the validity of statements related to the defined objects (Martín-Molina et. al, 2018). As such, mathematical definitions serve as a basis for mathematical proofs (e.g., Weber, 2002). Importantly, mathematical definitions are also used to classify—to distinguish between what is or is not a particular entity (e.g., Zaslavsky & Shir, 2005).

Within the disciplinary practice of mathematics, definitions are dynamic and adaptive and may undergo refinements in light of counterexamples and further developments (e.g., Martín-Molina et. al, 2018). However, in school, students are either presented with precisely worded existing definitions (e.g., Edwards & Ward, 2004) or work with mathematical notions in the absence of any provided definitions. To account for these two cases, drawing on the work of philosophers and lexicographers, Edwards and Ward (2004, 2008) distinguished between *extracted* definitions and *stipulated* definitions. Extracted definitions are deduced from the inspection of a body of evidence. Stipulated definitions are handed down to learners from a knowledgeable expert. This distinction is eloquently summarized by Edwards and Ward (2008) when they observe that "extracted definitions report usage while stipulated definitions create usage" (p. 224).

According to Leikin and Winicki-Landman (2000), *equivalent* definitions generate the same set of objects that satisfy the definition. However, when one set of objects satisfied by Definition-A is a proper subset of objects satisfied by Definition-B, then the two definitions are *consequent* definitions. Other times, when the sets of objects generated by two definitions have a nonempty intersection, but neither is a proper subset of the other, Leikin and Winicki-Landman (ibid.) refer to the definitions as *competing*.

Van Dormolen and Zaslavsky (2003) specify that *a criterion of equivalence* is necessary for equivalent definitions to be a fundamental part of a deductive system. That is,

> when one gives more than one formulation for the same concept, one must prove that they are equivalent. In practice this means that one has to choose one of the formulations as the definition and consider the other formulations as theorems that have to be proved. (p. 95).

However, we find no explicit direction for how, in practice, non-equivalent definitions of the same concept are to be handled. When consequent or competing definitions exist for the same mathematical term, the truth value of statements related to that term may become ambiguous. The focus of our study is on teachers' mathematical decision-making when faced with such ambiguity.

On ambiguity and definitional ambiguity

According to Byers (2007), "ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (p. 2). Byers considered ambiguity in mathematics as a source of creative development and argued against the popular perception that the logical structure of mathematics is definitive. Building on Byers' definition but interpreting it in the context of teaching and learning mathematics, Foster (2011) argued that *productive ambiguity* is an essential component of learners' engagement with mathematics. In particular, "ambiguity is necessary for ideas to move forward because it creates an instability in what is currently known that allows the formation of new knowledge" (p. 3). Foster also categorized different appearances of ambiguity. He distinguished between symbolic ambiguity, multiple-solution ambiguity, paradigmatic ambiguity, linguistic ambiguity and definitional ambiguity; the latter is of our interest in this study.

Definitional ambiguity, according to Foster (2011), arises "where there is more than one way of interpreting the meaning of a mathematical term." His example is the term "radius," which may refer to a geometric object or its length. In these cases, whether the reference is to a geometric object (as in a construction) or its size (as in the task, find the radius of a circle with a circumference of 5π cm) is clear in context. However, there are also situations in which definitional ambiguity is the result of different but non-equivalent definitions. We wondered how teachers resolve such situations. This led to the following research question: *What guides teachers' decision making in cases of definitional ambiguity*?

Definitional ambiguity: the case of "continuous function"

When searching for a definition of continuous function, either online or in calculus books, the most common results are definitions of continuity at a point or continuity on an interval. From these stipulated definitions, a possible extracted definition of a continuous function is "a function that is continuous everywhere." However, the meaning of "everywhere" can be interpreted differently and depends on which stipulated definitions this definition is extracted from.

Definition-1: A continuous function is a function that is continuous on all the points of the function domain.

Definition-2: A continuous function is a function that is continuous on all the real numbers.

We purposefully do not comment here on which definition we consider as correct. We do note that, using Definition-2, $f(x) = 1/x$ is not a continuous function as there is a discontinuity at $x = 0$. This interpretation corresponds to the naïve concept image of a continuous function that requires it to be drawn without lifting pen from paper. Using

Definition-1, $f(x)$ is a continuous function as it is continuous at all points of its domain, which excludes $x = 0$. Jayakody and Zazkis (2015) elaborated in detail on the inconsistent conclusions that can be reached by examining definitions of continuity in different sources. In particular, they noted inconsistency in referring to discontinuity at points where a function is not defined.

THEORETICAL UNDERPINNING: CONDITIONAL CONSTRUALS

Milewski et al. (2021) introduced the notion of *conditional construals* to describe teacher decision making in ambiguous situations that arose in mathematics classrooms. Conditional construals are described as “moments when teachers require additional context in order to judge whether a given teaching action is appropriate.” Milewski et al. (ibid.) used linguistic indicators, such as “it depends,” to identify instances of conditional construal. We note that, in these instances, the provided examples attended to teachers’ pedagogical decisions related to pedagogical scenarios. For example, in the exemplified responses, teachers conditioned their choices as depending on time constraints, the instructional sequence, or their familiarity with students.

We extend the notion of conditional construal to cases where a mathematical decision depends on implicit mathematical assumptions. To illustrate, consider the following statement: In division of 13 by 5, the quotient is 2. Do you agree? Your decision depends on your definition of a quotient, which in turn depends on the kind of division you consider. The statement is true when the division is of whole numbers, which implies a whole number quotient and remainder. The statement is not true if the division is of rational numbers, and the definition of quotient is taken to be the result of that division. The conditional construal is mathematical in nature. One may argue that this conditional construal also requires pedagogical context—however, we note that conversations about both whole number and rational division might occur in the same pedagogical context: a middle school classroom.

METHODS

Participants in this study were prospective teachers in the last term of their teacher certification program and practicing teachers enrolled in a professional development course ($n = 29$, referred to as T-1 to T-29). They were asked to respond, in writing, to the claim that $f(x) = 1/x$ is a continuous function. This response required the teachers to indicate their evaluation of whether the claim is true or false; to provide a justification, indicating any sources that informed their decisions; and to provide any hypothetical arguments that might be used by someone who disagreed with their evaluation. These responses served as a starting point to initiate a subsequent classroom discussion on definitions in mathematics.

Analysis of the written responses was conducted using the phases of reflexive thematic analysis. In particular, an inductive thematic analysis allowed for coding and theme development to be directed by the content of the data (Braun et al., 2019). In the first phase of analysis, each member of the research team familiarized themselves with the data. That is, they read and re-read the teachers' responses in order to become immersed in and intimately familiar with how they qualified both their justifications and any hypothetical disagreements. Then, each response was coded by multiple members of the research team to identify the conditional construals used as respondents conditioned their decisions. Initial codes were primarily semantic in that their creation was instigated by a teacher's explicit language choice—for example, the use of linguistic markers for conditionality such as "it depends." Later, these semantic codes were supplemented with latent codes that captured those instances in which conditional construals were implicit in the text (Braun et al., 2019). Members of the research team met regularly to discuss the generation and application of codes.

Next, the research team identified collections of codes—and, in some cases, especially prevalent single codes—that might constitute themes. These preliminary themes were examined in light of their ability to both answer the research question and meaningfully describe the dataset. Throughout this process, the research team members collaborated to refine ambiguous themes, merge redundant themes, and otherwise ensure that each theme contributed to the narrative of the data.

FINDINGS

A total of 12 out of 29 respondents identified the claim as a true statement, whereas 14 identified it as false. The final 3 respondents remarked that the claim could be interpreted as either true or false depending on additional assumptions made by the reader. Respondents' conditional construals were primarily centered on choosing a domain over which the continuity of the function should be considered. This decision was sometimes, but not always, tied to their choice of definition.

Choice of domain is dependent on the definition

Most often, participants chose a domain by choosing one definition of continuous function over another. To make this choice, many participants first chose a definition for continuity at a point, from which they extracted a definition of continuous function; this extracted definition tended to inherit its domain from the chosen stipulated definition. The definition would then prompt them to attend to either the entire real line or only those points where $f(x)$ is defined, in line with either Definition-2 or Definition-1 described above. Regardless of which definition they chose, respondents almost always acknowledged the alternative view as part of a hypothetical counterargument. For example, T-23 began her explanation of why the claim is false by "presuming that by continuous function we mean an everywhere continuous function." She later acknowledged that another reader might come to the opposite conclusion if they do not consider continuity at $x = 0$.

Choice of domain is dependent on mathematical convention

When deciding on a domain, some participants attempted to align with what they perceived to be mathematical convention. For example, T-5 first presented a naïve conceptualization of continuity as a single unbroken line—but added that "we usually look at the domain (x-axis values) and or the range (y-axis values) of the function." Consequently, T-5 argued that the claim was true because $f(x)$ could be drawn as a single unbroken curve on each half of its domain. Of note is the fact that participants who appealed to a standard mathematical consensus sometimes disagreed about what exactly that consensus is. T-24 argued that the claim was false unless one disregards the discontinuity at zero, but that "by convention we do not restrict the domain in this manner, unless explicatively stated." T-25 made a similar assessment, adding that "since the domain in the claim is unspecified, it is assumed that we are talking about all real numbers." However, when considering hypothetical counterarguments to his conclusion that the claim was true, T-2 explained that only "purists would argue that all points $-\infty$ to ∞ should be shown to be continuous for a function to be continuous."

Choice of definition is dependent on personal preference

Some respondents selected from possible stipulated definitions based off of an underlying personal belief of what constitutes a continuous function. For example, T-11 examined multiple textbook definitions related to continuity. He admitted that he does not "like a definition of a continuous function that allows functions that are not continuous at all points," and ultimately rejected the Definition-1 as "overly-accepting." In contrast, T-9 chose Definition-1 because "I don't believe it makes sense to consider properties of functions when they are not defined." Finally, T-10 stated that "my understanding of a continuous function is that the function is continuous in its domain," but that someone might disagree because, "from their perspective, a continuous function must be continuous everywhere."

Choice of definition is dependent on visual intuition

Prevalent in responses to the claim were participants' underlying intuitions about what a continuous function should look like; such as when T-2 described a continuous function as "a function that does not have any abrupt changes in value across its domain." More often, participants described the naïve conceptualization of a continuous function as one that can be drawn without lifting one's pencil—although they did not often hold this conceptualization themselves, and instead acknowledged it as a hypothetical argument someone else might employ. For example, both T-13 and T-15 concluded that the claim was true but recognized that a counterargument might stem from the perspective that "it is obvious to the eyes of the reader that the function is not 'connected.'"

T-13 noted that the naïve conceptualization of continuity is "often an instructional language used by teachers and online to try and help students decide whether a function is continuous or not." Similarly, T-12 recognized that "the determination of continuity by drawing without lifting your pencil is an informal, practical way to determine the continuity of a function." Despite initially using this method herself, T-12 later used Definition-1 to argue that the claim is true. She found this to be "a more precise mathematical method which lends mathematical rigor to backing up the truth of the claim."

DISCUSSION AND IMPLICATIONS

Definitions are a pillar of mathematics, yet the notion of definitional ambiguity has not yet received significant attention in mathematics education research. Lack of an agreed-upon, formal definition can lead to cases of definitional ambiguity. In this study we focused on the existence of non-equivalent definitions for continuous function that could be extracted from related stipulated definitions for continuity at a point. The following observation made by T-28 summarizes, in part, the pedagogical implications from our study:

> As we were discussing a lot about how there is no agreed upon definition for many math claims and that different definitions can come up depending on where you are located for your learning. I never thought about this before. I always thought math was the one thing that was the same everywhere. But I am now seeing that math definitions change over time and location.

Participants reflected on their involvement with the task as an "eye-opening" experience, which, for some, changed their perceptions of mathematics. Several participants reported on their search for a "correct" definition, and their dissatisfaction with the ambiguity that they instead discovered.

As noted in previous studies (Foster, 2011; Marmur & Zazkis, 2021), productive ambiguity can be used to foster learners' knowledge and enrich classroom discussions. Involving teachers with cases of productive ambiguity, such as in the task described in this study, is a valuable pedagogical activity that can expand teachers' knowledge as well as enrich their appreciation of mathematics as a discipline. It can be used not only as a prelude for clarifying definitions and the importance of definitions in mathematical activity, but also lead up to a discussion on the nature of mathematics as a human endeavor and on ambiguity as a driving force in mathematical creativity.

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WHAT IS CONVINCING? – PRIMARY STUDENT TEACHERS UNDERSTANDING OF MATHEMATICAL ARGUMENTS

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A guideline for a semi-structured interview was developed that aims at understanding primary teachers' knowledge and conceptions towards the field of argumentation and reasoning mathematically. Apart from open questions, a repertory grid was implemented to evaluate students' reasons and arguments. This approach aims at connecting the teachers' evaluations of given students' arguments with their knowledge and conceptions of argumentation. The data analysis of 14 German student teachers for primary education focuses on the question: "What makes a convincing argument?" The student teachers' views of a convincing argument vary from mathematical correctness and rigour; understandable and clear statements to explanations and illustrations of mathematical phenomena within the argument.

THE CONTEXT OF THE STUDY IN THE FIELD OF ARGUMENTATION

Arguing, reasoning, conjecturing, and proving amongst others are essential activities in mathematics as a discipline and in mathematics in educational settings as they contribute to a strong foundation for understanding and learning mathematics. All of these activities describe ways of thinking mathematically and encountering mathematical problems. Fostering mathematical thinking in contrast to focus on mere calculation skills should therefore be one of the main aims in mathematics classrooms.

In the last decades, these cognitive processes and their role have been widely examined in educational settings, which underpins further their importance for teaching and learning mathematics. Each one of these habits show broad fields in the research of mathematical education. Besides mathematical proofs, argumentations or reasons (or the processes leading to these products) as objects of research, understanding and explaining the cognitions, affects, and behaviors of the persons involved are some of the main aims of research in these areas.

Regarding mathematical argumentation in school settings, the specifics of teachers' role in argumentation and proof is still object of discussion. Nevertheless, there is enough evidence showing that teachers are playing a significant part in argumentative processes taking place in the classroom. Teachers show responsibility for managing students' participation in an argument and in primary education especially, they initiate classroom argumentation. Educational settings that foster argumentative opportunities highly depend on the teachers. Consequently, examining teacher conceptions and their professional knowledge of argumentation is relevant to educational research.

Following an exploratory case study, a guideline for a semi-structured interview was developed that aims at understanding primary teacher conceptions towards the field of

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argumentation and reasoning mathematically. Apart from open questions in this direction, a repertory grid (Kelly, 1955) was implemented to gather criteria used by teachers to evaluate and discuss students' reasons and arguments. The structure of the interviews aims to highlight the connection between the conception, the knowledge and the classroom practice carried out by the teachers in argumentative settings as one facet of this complex phenomenon (Klöpping & Kuzle, 2019). The presented data analysis focuses on an essential component of argumentation. More precisely, it will be examined what the interviewees believe to be a convincing argument.

Therefore, the theoretical foundation of this study can be found in two different fields. On the one hand, teachers' mental structures are of importance especially regarding professional knowledge and conceptions of mathematical argumentation. On the other hand, paying attention to mathematical arguments, their structures, and their quality is inevitable for the methodological design and analysis of the data.

TEACHERS' PROFESSIONAL KNOWLEDGE AND CONCEPTIONS

When it comes to professional knowledge of mathematics teachers, there exist enough theoretical and empirical based models that adequately describe different facets of this construct. Under the assumption that the varying aspects of teachers' professional knowledge are interdependent, Kuntze (2012) elaborated a pragmatic notion of professional knowledge where beliefs and convictions are included. This understanding of professional knowledge suits the purpose of this study because socio-mathematical norms (Yackel & Cobb, 1996), and more specifically the acceptance of mathematical arguments, don't rely on either knowledge or beliefs alone. Rather than considering beliefs alone, an even broader structure like conceptions seems to be fruitful here. The term conception can be described as "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007, p. 259).

Related research has focused on teacher knowledge and beliefs in the domain of proof and proving, the teaching of conjecturing, and argumentation (e.g., Knuth, 2002). All of this research merely considers secondary and tertiary education showing a lack of recognizing the importance of teacher views for primary education.

Investigating the role of primary teachers on argumentative processes in the mathematics classroom, the combination and interplay of mental structures, knowledge and conceptions, is key to understanding this complex field.

ARGUMENTATION IN THE MATHEMATICS CLASSROOM

Considering a mathematical conjecture whose validity is yet uncertain, two main aspects lay in the interest of mathematicians. Firstly, verifying the conjecture and therefore examine if it holds true. Secondly, explaining why the validity holds to further make sense of the statement and the mathematics behind it. Both aspects seize on two fundamental functions of mathematical proofs: the verification of a statement and its explanation (Hanna, 2000). Hence, a mathematical proof considers primarily the validity of a statement. In the specific communication about this validity, both in dialogue with others and as an interior monologue, verification and explanation is expanded by the intention to convince somebody.

Deductive proofs certainly convince in most mathematical discourses. Nevertheless, “pre-formal” types of proof exist such as “action proofs” or pragmatic proofs (Balacheff, 1988) which present convincing arguments as well, especially at the level of primary education. The communication process about the validity of a mathematical conjecture shows a strong argumentative nature and conviction can be reached even without “rigorous” proofs. This is seen primarily in communication and argumentation processes observed in mathematics classroom. As these processes depend highly on the interaction of the persons involved, the acceptance of valid arguments is yet to be discussed in classrooms (Krummheuer, 1995). The acceptance of an argumentation, and this includes proving, and the persuasive power of the used arguments depend on the social context, on the persons interacting with one another. Such socio-mathematical norms are constituted in the classrooms but depend on the interaction of students and teacher, especially at the primary level (Yackel & Cobb, 1996). Representation, structure or correctness are therefore not the only factors of convincing proofs, arguments or reasons. The teacher’s view, their conception of argumentation and their knowledge, influences the socio-mathematical norm in the classroom and therefore the acceptance or refutation of an argument.

Assuming an influential role of teachers in argumentative processes and specifically in the acceptance of an argument, raises the question what qualities or characteristics, according to teachers, an argument should have to be convincing in the classroom. The research question focuses on the teachers’ perspectives and investigates what a convincing argument consists of.

METHODOLOGICAL CONSIDERATIONS AND DATA PROCESSING

The socio-mathematical norm (Yackel & Cobb, 1996), reflected in the acceptance of mathematical arguments, is formed by a personal understanding, experience, and conception of argumentation, reasoning, and proving in mathematics. In order to cope with this complex field a qualitative approach was chosen. Because of this, an interview guide was developed and later on, re-structured, and expanded methodologically based on the experience of a previous case study on mathematical argumentation. As knowledge, beliefs and attitudes as cognitive structures are intertwined and interdependent (Kuntze, 2012; Philipp, 2007), the methodological expansion led to a search for an adequate approach which then ended in the psychological theory of personal constructs by Kelly (1955). From his theory, Kelly (1995) derived a research instrument to describe “his” personal constructs: the so-called *Role Construct Repertory Test* (REP-test or repertory grid). In a repertory grid, persons, objects or situations, so called *elements*, are evaluated using *constructs*. Via a linking mechanism *elements* and *constructs* are connected. Integrating Kelly’s ideas to established approaches of qualitative research and adapting them for research on mathematics education doesn't mean to reject other methods, it should be rather seen as a synergy.

Structure of the interview and repertory grid

Hence, the interview guide consists of two main parts. Starting with open questions about argumentation in mathematics as a discipline (e.g., "What is the object of argumentation in mathematics?"), the subjects is narrowed to argumentation in the mathematics classroom (e.g., "When should students start to work argumentation tasks?"). Further on, the interviewees are invited to talk about their own experience with reasoning tasks and classroom argumentation (e.g., "Please talk about a situation you experienced in your lessons in which students had to justify their answers."). And lastly, the interviewees are asked to evaluate student explanations referring to the following mathematical statement on parity: For any positive integers a and b, if one of them is odd and the other summand is even then the sum a+b is an odd number. Healy and Hoyles (2000) followed a similar approach to explore students' views on given arguments but as a quantitative study it missed the opportunity to explain how the students' choices were made.

Adapting Kelly's theory (1955) for this particular study, it must be specified what the *elements* and what the *constructs* should be. Six student explanations and a self-written argument of the interviewed person comprise the *elements* of the grid. They are supplied and should show a broad variety. To reach this variety, Healy and Hoyles (2000) included in their questionnaire for instance three types of arguments: empirical arguments, narrative arguments, and algebraic arguments. Their theoretical framework is based on the taxonomy of proofs by Balacheff (1988) which can be applied to arguments and reasons as well. It distinguishes between naive empiricism, crucial experiment, generic example, thought experiment, and calculation on statements which fall into two categories: Pragmatic proofs and conceptual proofs (Balacheff, 1988). The *elements* selected for the repertory grid in this study are based on these categories. To incorporate a broad variety of arguments, at least one element covers each category. As an example, Figure 1 illustrates two provided *elements*.

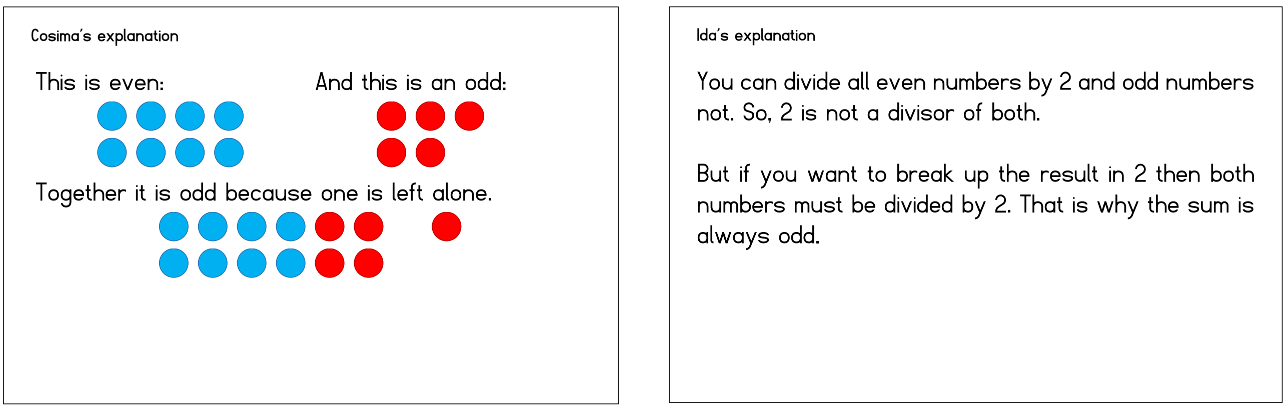

Figure 1: Student explanations as *elements* of the grid

The *constructs* can be understood as evaluation criteria consisting of two opposing poles, two *contrasts*. They are elicited using triads of the given arguments. With this triad, the participants are asked what two of these arguments have in common which differs from the third. These common characteristics are then employed as evaluation criteria. However, if possible *constructs* emerge already during the open question part of the interview they can be included as well. Finally, for each *construct* all explanations are evaluated on a five-point rating scale (see Figure 2).

Fiona

	elements								
constructs (5)	Anton's explanation	Cosima's explanation	Fynn's explanation	Gina's explanation	Hannah's explanation	Ida's explanation		explanation of the interviewee	*opposing pole (1)*
with tiles	*1*	*5*	*-*	*1*	*3*	*1*		*3*	*digits/numbers*
specific	*3*	*5*	*1*	*3*	*5*	*1*		*5*	*abstract*
a few examples	*3*	*3*	*1*	*5*	*3*	*1*		*5*	*no example*
universal validity	*1*	*3*	*5*	*2*	*1*	*5*		*4*	*a single situation*
logical math. chain	*2*	*3*	*1*	*2*	*2*	*5*		*4*	*no math. chain*
comprehensible	*5*	*5*	*3*	*5*	*5*	*3*		*2*	*more complex to comprehend*

Figure 2: Fiona's filled-out grid

As the evaluation criteria vary from one interviewee to another, the participants are asked to explain their decisions during the evaluation, which in addition simplifies the analysis as it is less open for interpretation.

Data sample

14 German student teachers for primary education, all of whom having at least some teaching experiences in primary mathematics, were interviewed. The recorded interviews lasted between 45 and 90 minutes, were transcribed and analyzed using computer supported qualitative content analysis. The repertory grids were digitalized as well and underwent a separate analysis. Combining both findings from the open section of the interview and from the grids is a key idea in the methodological approach.

WHAT IS CONVINCING? – RESULTS

Central to the research question is the data regarding the interview question: "What makes a convincing argument?" Before presenting all results, Fiona's and Greg's cases are given as an example of the data analysis and interpretation.

Interviewer: In general, what makes a convincing argument?

Fiona: First of all, others should be able to follow the thoughts, the persons to whom I'm presenting it. … So yes, I can take something from one topic and transfer it to a different topic. And it has to be logical, it has to make sense.

Fiona's remark on convincing arguments relates to the audience and depends therefore on the social context. What she means by "be able to follow the thoughts" is not further explained at this point. The other aspect in her understanding of convincing arguments refers to more abstract structures of an argument. A similar focus on the argument's structure can be found in Greg's answer:

Greg: Well, I believe, an argument convinces in the way that it is, as we call it, falsifiable. In the sense that no counter argument exists. So, an argumentation is appropriate, if it is logically structured, so that the argumentation is completely logical. And against this logical structure no counter argument can be found. Then it is for me convincing.

Later in the interview, when the interviewees are asked to evaluate the students' arguments, the *constructs* of the grid can refer back to those statements. Fiona picks up the notion of "following the thoughts" in arguments and explains her rating:

Fiona: Exactly, they all are totally easy to follow. For me, Hannah's, Gina's, Cosima's and Anton's reasons are all at 5.

Interviewer: Why?

Fiona: Because they are written with a lot of examples. In each given reason there is at least … one example. That makes it easy for me to follow.

If the argument consists of "a lot of examples" it is with Fiona's understanding "easy to follow" and fulfils one aspect of a convincing argument. During the evaluation with the grid she names this *construct* "comprehensible" whereas the contrasting pole is "more complex to comprehend" (see last row in Fiona's repertory grid, Figure 2). Fiona further clarifies that her own argument is still "hard to follow" because little explanation is given. Here it seems that a convincing argument for Fiona should include not only examples but also explanations.

Greg's understanding of a convincing argument is illustrated quite well with the data from his grid (Figure 3). The *construct* "convincing" is rated almost identical to the criterion "completely logical (generalization)" which is not surprising as his previously answer confirms exactly this. Interestingly, "justified terms" has a very similar rating, indicating that for him the proper use of mathematical terms is part of a convincing argument with a logical structure. On the other hand, the representation of an argument, whether it is "visual" or "symbolic", doesn't in Greg's understanding contribute to the conviction.

If Fiona would present a convincing argument in a classroom situation, one can from the data in the grid assume that this argument most likely considers different representations, is abstract, gives an explanation, consists of deductions, exemplifies its reason, and is "easy to follow" for the students. Distinctively, Greg would focus on the logical structure of an argument being general in nature and would justify the mathematical terms in use.

Greg

	elements								
constructs (5)	Anton's explanation	Cosima's explanation	Fynn's explanation	Gina's explanation	Hannah's explanation	Ida's explanation		explanation of the interviewee	opposing pole (1)
visual	1	5	2	1	4	2		5	symbolic/ numbers
convincing	1	5	2	4	3	5		5	not convincing
completely logical (generalization)	1	5	2	4	3/2	5		5	case/example
justified terms	2	5	1	4	4	5		5	no knowledge behind words

Figure 3: Greg's filled-out grid

Looking at the other participants, they can be grouped to similar aspects of convincing arguments. Fiona and three other student teachers highlight the idea of comprehensibility, where an argument is convincing if the train of thought can be followed. A group of four participants show a close understanding to the former, but demand an explicit explanation and justification of the given argument. Greg and another student teacher focus on the logical structure and on a sound reasoning. This might be the closest perspective to a general understanding of proofs. Mathematical correctness, the use of mathematical "facts" and "rules" is dominant in a group of three interviewees. Finally, there is one person stating that a convincing argument needs to give an insight in and illustrate the mathematical phenomena at hand.

FINAL THOUHGTS AND CONCLUSION

The general remarks on a convincing argument include comprehensibility, logical structures, mathematically correctness and even illustrations of the mathematics within the argument. Nevertheless, there are different approaches in each of the participants' answer. The analysis shows how the idea of a convincing argument is reflected in evaluating student explanations. More generally, it can be assumed that what is convincing differs, even if it is subtle, from teacher to teacher depending on their professional knowledge and conceptions of argumentation.

Furthermore, the analysis is an example of how a repertory grid might enrich existing research approaches. In a semi-structured interview, teachers' professional knowledge and conceptions can be explored but the repertory grid technique offers additional and helpful data to deepen the understanding of this complex field.

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DOWN THE ROAD: TEACHER'S PERCEPTIONS AND UPTAKE OF PD AFTER SEVERAL YEARS

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This study captured middle and high school teachers' perceptions of what they learned from professional development 3-4 years after participating in one of three NSF funded year-long professional development (PD) projects. We surveyed teachers (n=66) from three different PD projects on the types of content, pedagogy, and resources that they remembered learning and continue to use when teaching mathematics. Results indicate that teachers remember and use many aspects from PD experiences 3-4 years down the road especially those they find relevant to their current teaching position. Most residual learnings from PD also appear to be highly aligned with the goals and intentions of the PD developers and researchers and these learnings have evolved through colleague collaboration and other PD opportunities.

INTRODUCTION

One central challenge for the field of teacher professional development (PD) is how to design interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning (Jacobs, Koellner, Seago, Garnier & Wang, 2020). A number of researchers have worked to address this challenge and there is now a strong research base delineating critical design features of effective PD (e.g., Borko, Jacobs & Koellner, 2010). The consensus in the current PD discourse about features of effective PD include a focus on mathematics content, student learning of content, active learning opportunities for teachers, coherence, duration, and collective participation (Sztajn, Borko, & Smith, 2017). Although some PD programs that adhere to design recommendations by the literature have produced encouraging results (e.g. Franke, Carpenter, Levi & Fennema, 2001), others have proven less successful (e.g. Jacob, Hill & Corey, 2017).

It is not clear why there have been mixed results from rigorous empirical studies of PD incorporating these design recommendation that contradict conventional wisdom among the field. There are many reasons that potentially could account for these varying results such as: the content of the specific programs evaluated may have been inefficacious, fidelity to the materials or pedagogical practices may have deviated from the identified goals and practices, difficulties may have resulted from scaling the program to multiple sites with different facilitators, or issues may have arisen with the research design and methodology. An alternate perspective is most, if not all, of the impact studies that have been funded recently have been large-scale quantitative studies. Many have shown incremental change in teacher knowledge and practice one year following the intervention (Murata et al., 2012). This need for clarity may rest in

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an often-ignored issue related to the time allowed for funded projects to study the impact of PD on teachers and students. Many large randomized controlled designed studies look at pre post data across one year and at most, use a post-post measure one year out. We hypothesize this is not enough time to measure PD impact. We argue that for teachers incorporate new ideas and then to plan, implement, reflect and modify instruction may require more time to be reflected in practice and in research results than the typical one-year that is often related to funding cycles.

The Taking a Deep Dive (TaDD) research study examines the residual impacts of three different professional development models on teacher learning, specifically 3-4 years after the actual PD experiences. The project is conducting a rigorous cross case analysis across participants from the different projects across the US. This paper is focused on a survey that was given to participants in May 2019 which was 3-4 years after their PD experience. Although this study focuses on self-report survey data, findings contribute to the PD landscape of PD design and survey design. Findings identify indicators that seem to provide evidence of why some teachers might learn and implement more from a given PD compared to another (others). Our analysis also elucidates how a carefully designed survey focused on the constructs of content, resources, and pedagogy tell an important story related to the similarities and differences of the PD and some potential limitations.

THEORETICAL FRAMEWORK

PD models fall on a continuum from adaptive to specified (Borko, Koellner, Jacobs & Seago, 2011). On one end of the continuum are *adaptive* models, in which the learning goals and resources are derived from the local context and shared artefacts are generally from the classrooms of the participating teachers. In these models, the artefact is selected and sequenced by the facilitator and/or the participating teachers, and the related activities are based on general guidelines that take into account the perceived needs and interests of the group. On the other, specified models of PD typically incorporate published materials that specify in advance teacher learning goals and provide resources and guides to implement the PD. In video-based specified PD, the video clips are typically pre-selected and come from other teachers' classrooms.

The nature of what teachers take up and use across the continuum has the potential to shed light on factors that are associated with the teacher learning related to content and pedagogy. This study examines three professional developments that fall on different parts of the continuum. The goal is not to determine which types of PD are "best" because each has its affordances and challenges, but rather to better understand the *variance* of teacher uptake and use (in their classroom contexts) within and across these PD experiences. Understanding and unpacking variance among and between types of PD offers the potential to identify the factors that impact uptake and use from PD. This paper examines how teachers' self-reported uptake differs across PDs located at different points on the adaptive-specified continuum. Specifically, one is highly adaptive, one is highly specified, and one lands in the middle. We believe conducting a cross-case comparison aids in helping us understand the factors associated with uptake related to content, pedagogy, and resources.

OVERVIEW OF TaDD PROJECT

This three-year impact study, *Taking a Deep Dive* (TaDD), collects qualitative data from three large U.S. National Science Foundation PD projects in order to use cross case analysis to further inform what teachers take up in their classrooms 3-4 years after the initial professional development experiences. We want to explore how certain PDs get applied in specific educational contexts in different geographical locations. This paper uses a comparative case analysis and focuses on the portion of the TaDD study that investigates self-reported learning related to pedagogy, content and resources taken up and used from the following three NSF PD projects one to two years after the project and funding ended. In the next section, we briefly describe the three different PD projects.

Learning and Teaching Geometry (LTG) LTG is an efficacy study of the learning and teaching geometry professional development materials: Examining impact and context-based adaptations, sought to improve teacher's own knowledge and instructional strategies in transformations-based geometry. This PD consists of 54 hours of highly specified video-based PD grounded in modules of dynamic transformations-based geometry which is aligned with the Common Core State Standards in mathematics (CCSSM). Through video analysis, teachers work together to solve problems and further their knowledge in mathematics teaching in geometry. The PD allows teachers to better support students in their attempt to gain a deeper understanding of transformations-based geometry through activities like rate of change on a graph, scaling activities, and similarity tools to name a few. LTG is a specified PD as the content and pedagogical goals of the PD are clearly articulated for each workshop.

Collaborative research TRUmath and Lesson Study (LS) is a project that supports fundamental and sustainable improvement in high school mathematics teaching. LS is aimed to engage in design research to develop and implement a replicable model of teaching for a coherent, department-wide approach. In the PD, teachers collaboratively created focused and coherent lesson plans from their curriculum aimed at providing students the opportunity to gain a deep understanding of mathematics and the ability to make connections. The PD took a unique twist on lesson study by using the TruMath framework as a common observation tool that could guide teacher noticing and anchor discussions related to the lab lessons. The lab lessons are one teacher volunteers to teach a lesson and other participants in the LS observe quietly in the back of the classroom. The *TruMath* framework focused discussion and analysis of classroom interactions across five dimensions. Teacher teams identified a goal from one of the dimensions of the framework that they wanted to focus more deeply on. LS is an adaptive form of PD that utilized the TRU framework but allowed for teachers' ideas to guide the workshops.

Visual Access to Mathematics: Professional development for teachers of English Learners (VAM). The VAM PD, the focus PD of this paper, is a "60-hour blended, face to face and online course to build teachers' knowledge of and self-efficacy about LRT strategies to strengthen English Learners (Els) problem solving and discourse in middle grades" (De Piper et al., 2021 p. 491). The goals and intentions of VAM were to cultivate in teachers the fluent use of representations, anticipation of students' strategies, the ability to interpret and construct various mathematical solutions, and to reason with and across representations. Teachers learned how to strategically select and align VRs with their instructional goals, anticipate student thinking and misconceptions, and then implement lessons using these strategies in their classrooms. Once implemented they would share experiences and student work, and collaboratively and independently reflect on the teaching cycle in the VAM PD online workshops. VAM falls in the middle of the adaptive-specified framework as the face-to-face workshops had specified and intentional goals, and the online professional learning meetings were guided by the teachers using artefacts of practice to guide their discussions.

METHODOLOGY AND METHODS

Sixty-six participants took a 32-question survey (28 LTG, 25 VAM and 13 LS). Teachers also provided background information. All teachers held an undergraduate degree and 88% held a graduate degree, on average, but larger proportions of LTG (93%) and VAM (96%) teachers held graduate degrees compared to LS teachers (62%; $t=3.29$, $p<.01$). In addition, VAM teachers reported over 16 years of experience teaching, significantly more than LS and LTG teachers who reported approximately 10 and 12 years, respectively ($t=2.81$, $p<.05$ and $t=2.57$, $p<.05$, respectively). On average, 15% of teachers were currently teaching Geometry with no differences between groups.

The survey included both closed and open-ended questions that asked participants to reflect on their PD experience and characterize their past and/or current use of the PD content, pedagogy and materials as well as the support they received to implement new content and instructional practices. The survey included seven Likert scale questions. Participants responded to statements on a scale of 1-10, as well as eighteen follow up questions that allowed the participants to provide more details about their responses.

We coded the 18 questions on the survey from all 66 participants. We created a coding manual starting with apriori codes. The apriori codes were aspects of effective professional development from the literature (e.g. analysing student thinking, specific content, and representations used), supporting diverse learners. We then included emergent codes that appeared frequently and appeared relevant to the programs. We began with three researchers coding one survey from each project. We came together to discuss codes, add codes to the manual, and reconcile differences. We then continued this process with seven surveys from each project to achieve inter-rater agreement at 91%.

Once all surveys were coded, we calculated the amount of time a participant mentioned each code in their survey responses. For each of the four domains, we identified and averaged the specific codes included within that domain. For instance, we identified four codes that were related to content; these codes included GCSL (general content student learning), GCTL (general content teacher learning), SCSL(specific content student learning), and SCTL(specific content teacher learning). SCSL would refer to a comment on the survey that indicated specific content (e.g. dilations) and discussed a focus on student learning. Then we identified three codes related to pedagogy; these codes include MS (multiple solution strategies), SSDL(student strategies for diverse learners), and ST (student thinking). We identified six codes that were related to resources; these codes included GR (general resources), RSDL (resource to support diverse learners), RTL (resource for teacher learning), SR (student resource), TSML (technology support math learning), and V (mention of video to support noticing). Lastly, we identified four codes related to support; these codes include C (collaboration), FI (facilitator impact), CS (coach support), and PS (principal support). Finally, percentages of comments were created from the four domain averages and percentages of comments of the individual codes within domains were calculated for a deeper understanding of teacher responses.

ANALYSES

To analyse the data, we used descriptive statistics, paired samples t-tests, and analyses of variance and covariance with pairwise comparisons using the Bonferroni test to identify and understand the differences and similarities between uptake by project (LS, LTG, VAM). To control for pre-existing differences, graduate degree and years of experience teaching were included as covariates in the analyses of covariance. Measures of teacher undergraduate and graduate degrees and currently teaching geometry were included in preliminary analyses but found to be non-significant and dropped from subsequent analyses.

RESULTS

To identify what teachers remembered from their PD experiences 3 to 4 years ago and what they have continued to use related to that PD, we analysed the average percentages of comments made by teachers. Table 1 presents the percentages of comments within domains and across projects and the results of the analyses of covariance adjusted for teacher years of experience teaching.

Types of comments within projects. Within projects, paired samples comparisons within the LS group identified a significantly larger percent of comments focused on support compared to content ($t=6.70$, $p<.001$), pedagogy ($t=4.76$, $p<.001$), and resources ($t=4.62$, $p<.01$). While this group also commented more on resources than on content ($t=3.38$, $p<.01$), both LTG and VAM emphasized resources more than all other domains: content ($t=2.86$, $p<.01$ and $t=14.21$, $p<.001$, respectively), pedagogy ($t=10.70$, $p<.001$ and $t=17.89$, $p<.001$, respectively), and support ($t=4.14$, $p<.001$ and $t=12.82$, $p<.001$, respectively). LTG and VAM also focused more on content ($t=9.90$, p<.001 and t=3.80, p<.01, respectively) and support (t=8.29, p<.001 and t=9.48, p<.001, respectively) than on pedagogy.

To summarize, although the domain resources was somewhat emphasized in the LS project, content and pedagogy were emphasized far less. The LTG project, a specified PD, had the largest percentage of comments that were distributed among the categories. The largest percentage was related to resources and then percentages were fairly evenly distributed between content and support, but less so for pedagogy. The VAM teachers mostly emphasized resources followed by support and content and pedagogy.

Types of comments across projects. Comparing teacher comments across projects, results of the analyses of covariance identified distinct patterns of comments about PD experiences for each group (see Table 3). LS participants were significantly more likely to mention support and pedagogy compared to both the LTG (t=7.81, p<.001 and t=3.71, p<.01, respectively) and VAM participants (t=8.28, p<.001 and t=3.17, p<.01, respectively). Their comments included principal and coach support as well as colleague support. Support was the domain qualitatively discussed most throughout the survey.

LTG participants emphasized content significantly more than both LS (t=5.51, p<.001) and VAM participants (t=6.22, p<.001) and resources more than LS participants (t=4.35, p<.001). On the other hand, VAM participants mostly emphasized resources and did so significantly more than both LS (t=8.55, p<.001) and LTG participants (t=5.62, p<.001).

Domains of teacher comments	Lesson Study PD (LS, n=13)	LTG PD Efficacy Study (LTG, n=28)	Visual Access for ELLs in Math PD (VAM, n=25)	F	Pairwise comparisons
Content	10%	29%	10%	25.76***	LTG>LS*** LTG>VAM***
Pedagogy	13%	3%	4%	7.34**	LS>LTG** LS>VAM**
Resources	23%	43%	65%	37.56***	LTG>LS*** VAM>LS*** VAM>LTG***
Support	54%	25%	21%	38.89***	LS>LTG*** LS>VAM***

Total	100%	100%	100%

Note. Results from ANCOVA adjusted for years of experience teaching.
**p < .01; ***p < .001.

Table 1: Results of ANCOVA on percent of teacher comments across the four domain averages, by project (N=66)

Results indicate that the teachers' perceived uptake after 3-4 years was highly related to the goals and intentions of the PD projects. As the PD projects' goals and intentions were identified at different points on the adaptive – specified continuum, differences were highlighted based on comments related to content, pedagogy, resources, and support. In some ways this is not surprising that the different PD programs had different emphases, and these were revealed in the clusters of codes related to content, pedagogy, use of resources, and support yet it provides promising evidence that PD learning held residual value.

DISCUSSION

This study reveals that the teachers that participated in the three NSF funded PDs, 3-4 years before taking this survey, highlighted and wrote about the main goals and intentions of the PD that they attended. Although this may not be surprising that the teachers remember what the facilitator and PD developers intended, it shows promise that the PD's yielded high residue of teacher learning 3-4 years after the PD workshops especially when the content and the pedagogy of the PD were relevant, useable, and transferrable across the daily lessons of the teachers.

The LS teachers generally tended not to emphasize content, and when they did, they mostly discussed aspects of content that were generally related to teacher or student learning. In fact, they mentioned teacher learning more than VAM ($t=3.06$, $p<.01$) and student learning more than LTG ($t=2.50$, $p<.05$). When discussing pedagogy, most comments were related to working with diverse learners. If they were discussing a resource, they typically were discussing a specific resource, and did so more often than LTG ($t=3.71$, $p<.001$). Most likely, the specific resource they discussed was the TRU framework which was the centre piece of this project. LS teachers were significantly more likely to discuss specific resources. When talking about support, they mostly emphasized support from colleagues and more so than VAM ($t=2.71$, $p<.05$). Although only 21% of their comments were about coach support, this percentage was still significantly larger than for LTG ($t=3.05$, $p<.05$) and VAM ($t=3.09$, $p<.01$).

The LTG project, the most specified PD, had the most distribution between the four categories. Resources, both general and specific, were provided to participants including rich tasks, videotapes and applets to support the implementation of transformations-based geometry in middle and high school classrooms. LTG teachers commented specifically on the geometry content they learned and used in their classrooms which is not surprising since the PD was specified and the content new to many participants.

The VAM PD, is also a specified PD but the specificity did not only lie in the content but in the strategies, specifically using representations, to support emergent bilinguals. The VAM teachers commented on resources more than the other areas - content, pedagogy and support, and commented on resources more than teachers in the LS and the LTG PDs. The LTG and VAM projects did not solicit support from principals and coaches and these categories of support were not mentioned often by either group, but they did discuss the support they received from their colleagues and from the facilitators during the PD experience.

This study has a small sample size and results need to be taken with caution. The findings do provide some evidence that teachers remember and use aspects from a PD that they participated overtime and that there is residual knowledge that has endured. More research is needed to understand teacher learning over longer periods of time and perhaps to increase funding cycles for this to happen. Our next steps are to continue in this line of inquiry by conducting the cross case analyses from these projects. We will analyse how classroom practices related to the goals and intentions of the PD project are reflected in their teaching. We will conduct think aloud protocol interviews to understand teacher learning more fully and how this learning is evidenced in daily classroom practice through their voices.

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ANALYSING PROCESSES OF TRANSFER IN LEARNING BASIC FRACTION CONCEPTS: A DIDACTICAL APPROACH TO TRANSFER

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This paper reports selected findings from a study investigating processes of transfer in the development of basic fraction concepts. For this study transfer is conceptualised as a process based on the didactical model of Grundvorstellungen. This approach enables the analysis from both a normative and a descriptive perspective. Comparative interaction analyses of students working in dyads provide deep insight into the development of basic fraction concepts.

INTRODUCTION

Mathematical content is typically introduced by building on concrete experience with real-world activities that become progressively more abstract and symbolic (Scheiner, 2016). Thus, the learning of fractions usually starts with the idea of sharing or dividing real-world objects or their depictions into equal parts. This is supposed to form the basis for understanding the general concept of a fraction as a part of a whole. Learning then usually proceeds by sharing and dividing various other objects and their depictions in multiple ways. By varying the objects, representations, and distributions the learners are supposed to abstract the elementary production activity for any fraction, that is dividing a whole into *n* equal parts and duplicating a part *m* times to get a fraction $\frac{m}{n}$. This concept is then applied to various wholes, e.g. quantities (area, weight, length, money, etc.) and various iconic representations to extend the validity and applicability of this concept to a multitude of different situations (Behr et al, 1983; vom Hofe & Blum, 2016). Learning progressions like this are particularly based on processes of transfer in the sense that students are required throughout to transfer and connect their prior learning to new situations and applications and in this way extend and further develop their understanding of the concepts they are learning.

From this didactical perspective mathematics learning requires transfer of concepts, procedures, and structures in predominantly three types of situation (Kollhoff, 2021):

- Transfer between subject-related contexts and applications.
- Transfer between different representations and modes of representation.
- Transfer of concepts, procedures and structures to create, substantiate and justify mathematical connections.

The study reported in this paper investigates such processes of transfer. Research has provided rich evidence that the development of fraction understanding is difficult in general, requiring conceptual changes and a broader understanding of number and

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operation concepts (Gabriel et al., 2013). In particular, it can be highlighted that important obstacles in the learning process can be explained by a lack of transfer as well as negative and overzealous transfer that lead to misconceptions (e.g. "natural number bias, Ni & Zhou, 2005) as a consequence. However, very little is known about the underlying processes of concept development and to what extent these explanatory models for learning fractions apply to authentic classroom contexts.

A main aim of this study was to investigate processes of transfer in authentic student interactions in class. For this reason, a didactical approach to transfer was developed that builds on the model of Grundvorstellungen (vom Hofe & Blum, 2016). The applied conceptualisation of transfer will be introduced in the next section.

GRUNDVORSTELLUNGEN AND PROCESSES OF TRANSFER

Research on transfer has historically and epistemologically been predominantly conducted in controlled teaching experiments to compare and evaluate the efficiency of instructional methods. The mathematical content of these teaching experiments is often restricted to procedural skills and techniques that can be applied algorithmically. Only recently (cf. Hohensee & Lobato, 2021) research on transfer in mathematics-related contexts has focused more on semantically rich learning content, like the concept of proportionality (Lobato, 2012) or the empirical law of large numbers (Wagner, 2006) as well as investigating mathematics learning in school. Consequently, most theoretical perspectives on transfer are intended for application in research rather than teaching contexts und thus yield only limited guidance and directions for mathematics teaching and learning in school. In this respect, within the model of Grundvorstellungen (vom Hofe & Blum, 2016) transfer can be conceptualised in both didactical and empirical contexts.

Grundvorstellungen: Normative and Descriptive Perspectives

The concept of "Grundvorstellungen" (GVs) is deeply rooted in the German tradition of "Stoffdidaktik". GVs as didactical categories formulate prototypical mental models of mathematical concepts and procedures, which are supposed to: (i) Give meaning to a mathematical concept or procedure through connecting it to familiar knowledge and experience, or mentally represented activities; (ii) help learners develop adequate mental representations of a mathematical concept or procedure; and (iii) support the application of a concept or procedure in subject-related contexts, e.g. modelling or problem solving (vom Hofe & Blum, 2016). Thereby, GVs as normative categories characterise a mathematical concept or procedure and provide interpretations in various contexts (Kollhoff, 2021; Salle & Clüver, 2021).

Based on this normative perspective GVs provide a framework to characterise and describe didactically intended processes of transfer with respect to (Kollhoff, 2021):

- The conceptual core of a transfer, i.e. *what* concept, procedure or structure is supposed to be transferred?

- The required connections to be made between contexts, representations, modes of representation, or activities.
- The expected difficulties, mistakes, and errors based on empirical results and didactical experience.

Vom Hofe and Blum (2016) differentiate between two types of GVs: Primary and secondary GVs. Primary GVs are based on concrete activities with real-world objects. The corresponding concepts can thus be semi-isomorphically represented by real-world activities and hence have a representational character. In contrast, secondary GVs are based on mathematical operations with symbolic objects. The constituents of the corresponding mathematical structures are not real-world activities but imagined activities with (abstract) mathematical objects and means of representing these objects (e.g. number line, terms, function graphs). Therefore, secondary GVs have a symbolic character. This differentiation accounts for the didactical progression from concrete activities with real world objects to activities with abstract representations in the development of mathematical concepts.

The normative perspective of GVs is complemented by a descriptive perspective to reconstruct and analyse students' thinking and investigate their individual conceptualisations of mathematical concepts and procedures. As descriptive categories GVs build on Bauersfeld's (1988) theory of "domains of subjective experience" (DSE). This theory describes mathematics learning as a non-hierarchical, cumulative, and separated storing of an individual's experience in correspondence to their situated connections. DSE accumulate everything an individual has experienced and processed as subjectively important. For this reason, DSE are not static entities but are subject to a dynamic development through activation in various situations. Since DSE are characterised by very specific elements, e.g. meanings, language, affordances for activities, available routines, etc., learning can be described as the development of new DSE or mental models respectively. From this perspective, knowledge is cognitively and emotionally inseparable from the learning situation. In new situations DSE compete for activation and the dominant DSE is decisive for the interpretation of the situation. Although DSE are not organised hierarchically they can be connected to each other through the construction of a new coordinating DSE and form a network of "self-referential systems" (Bauersfeld, 1988). Transfer can thus be described as the process of connecting DSE and forming a new coordinative DSE. The dynamic nature of establishing these connections leads to the conceptualisation of transfer as a process (Kollhoff, 2021).

With respect to the normatively formulated Grundvorstellungen and the intended processes of transfer, the descriptive analysis and reconstruction of students' thinking regarding their explanatory models in use provide a ground for comparison to identify deviations. This enables detailed analyses of transfer processes from an empirical perspective on one hand, but also allows to consider specific measures of support for learning (in class).

The described approach to transfer is highly comparable to the Actor-Oriented Transfer (AOT) framework (Lobato, 2012) concerning the descriptive perspective. Like the AOT approach, it conceptualises transfer based on the learners' individual and subjective generalisations and interpretations of the learning content in contrast to expert models to reconstruct and analyse the learners' explanatory models in use. However, the Grundvorstellungen approach makes use of the expert models as normative guidelines that are applied in the construction of the learning materials and thus describe didactical intentions. This way, it extends the AOT framework by using expert models as a level of comparison, which therefore allows an evaluation of the normative models themselves.

INVESTIGATING PROCESSES OF TRANSFER IN FRACTION LEARNING

The study that is reported in this paper investigates processes of transfer in the progressive development of fraction understanding in an authentic classroom environment over the period of six weeks (Kollhoff, 2021). Within the limited frame of this paper, selected aspects of the study will be presented to illustrate the methodology and discuss selected results.

Research Questions

Among others, the main research questions were: (i) How do processes of transfer that are intended on a normative level project to the individual learning processes of the students? (ii) How are the students' processes of transfer related to their development of individual conceptualisations of fractions?

Methods

The study was conducted in an introduction to fractional numbers in grade 5 over the period of six weeks. Based on the normative perspective of Grundvorstellungen a curriculum with learning materials has been carefully constructed along a series of intended processes of transfer. Framed by a pre- and post-test, three learning sessions in which the students worked in dyads (28 students in 14 dyads) were filmed to record their interaction with each other. The data collection was spread over the six weeks to enable the reconstruction of the students' development in the progress of learning. In these sessions, the students worked on two initiating worked-out examples that were followed fade-out examples, in which the students had to reproduce the content on the worked-example. This was followed by a series of transfer problems that required transfer on various levels.

The qualitative data was analysed in interaction analyses (Cobb & Bauersfeld, 1995) based on transcripts of the students' interactions. The analyses primarily focused on the reconstruction and description of the individual students' processes of transfer and their explanatory models in use (Kollhoff, 2021). The results of these analyses were then taken into a comparative analysis on three levels: (i) The relation of the reconstructed and the intended processes of transfer, (ii) the results of other students/dyads, and (iii) processes of transfer in the context of different procedures and concepts over the period of the course.

RESULTS: TRANSFER OF THE PRODUCTION PROCEDURE OF FRACTIONS

The following samples illustrate the applied methodology as well as selected findings of the study.

In the beginning of the session the students worked on two worked-out examples that explained the production of the fractions $\frac{5}{8}$ and $\frac{3}{8}$ illustrated with the division of a circle and a rectangle. In the fade-out examples the students were expected to reproduce the production procedure for $\frac{6}{8}$ with a circle representation ("Explain how $\frac{6}{8}$ is produced and use the circle for illustration."). The intended processes of transfer were the application of the two production operators $:n$ and $\cdot\, m$ together with their illustrative meaning of dividing the whole into n parts and multiplying one part m times, which is represented by dividing the iconic representation into n equally sized parts and colouring m parts of the diagram. With respect to the development of Grundvorstellungen, this production process is prototypical for the production of any fraction and can be used as an illustrative model for the interpretation of fractions in various contexts.

As a transfer problem the students are later required to transfer this production procedure to represent fractions on a line segment. This representational transfer requires an adaption of the illustrative activity, because in circles and rectangles fractions are represented as areas while they are represented as lengths on a line segment. This means that the line segment needs to be divided into parts of equal length while circles or rectangles are usually divided into parts that are congruent or rather have the same area. Didactically, the representation of fractions on a line segment constitutes a first step to representing fractions on a number line.

Please note that the following transcripts are originally in German and have been translated into English.

Bennet & Julius

1	B:	Ah, look here, you have to divide this [a circle] by eight ... you have to write divided by eight, because then you have one eighth, and then you have to multiply by six, then you have six eighths.
2	J:	Minus six.
3	B:	Why minus six? You have to multiply by six.
4	J:	But when you divide … let's say you divide a cake into eight parts of equal size and you want to have six of them, you have to do minus six, because otherwise you wouldn't have them.
5	B:	But you don't know of what you have to do minus six.
6	J:	Hä? But you first divide it by eight and then you have to take six away – thus minus six.

7 B: Yes, but we have to write it down in these steps. First, we divide by eight and get one eighth, then multiply one eighth by six, and get six eighths.

In this episode Bennet and Julius discuss the production of $\frac{6}{8}$. While Bennet transfers the procedure analogically to the new set of numbers and interprets the production operators with their illustrative meaning as intended (1), his partner Julius does not. In contrast to his partner, Julius interprets the production process in the frame of the real-live activity of dividing a cake. He shares his partner's interpretation of the first operator to divide the whole (the circle) into eight parts, but then interprets the second production operator with the activity of "taking away" six parts or pieces, respectively (2). He thus wants "to do minus 6, because otherwise you wouldn't have them" (4). The interpretation of the second production operator as a minuend can be interpreted as a result of overzealously transferring the linguistic expression "divide by n and take m away", which is often used to illustrate the production procedure in the context of sharing a concrete object like a pizza, or in this case a cake. The students are thereby expected to differentiate between the mathematical and the real-world expression, which is likely to be misunderstood. This creates a conflict between two DSE, the real-world activity and the mathematical operation, that needs to be resolved by connecting the two DSE. Such a connection can probably only be established by reformulating and reframing the real-world activity without the idea of "taking parts away". Bennet refers to this conflict by questioning "of what you have to do minus six" (5). As Julius does not deviate from his perspective (6), Bennet tries to convince him by referring to the worked-out examples as he explains, that they have to write down the production procedure "in these steps" (7). By "steps" he refers to the production operators as he highlights their individual function on an abstract level, that eliminates the notion of "taking away".

After the first episode that is discussed above, Bennet and Julius worked on further fade-out examples in which they explained the production procedure for three other fractions and illustrated them in circles and rectangles. The conflict appears not to be resolved, yet, but Julius trusts his partner's explanations and seems to accept that he was wrong by interpreting the second operator as a minuend, although he shows no signs that he has understood the difference. In a later episode from the same learning session Bennet and Julius work on the representation of the fractions $\frac{1}{4}$ and $\frac{5}{6}$ on a line segment:

1 J: That's easy. You just need to divide this [the line segment] in four and then take one part. And then divide in six and take five of them.

2 B: Yes.

3 J: We first have to measure how long the line is. How long is it? […]

Julius refers to this task as "easy" (1). By "easy" he might indicate that he spontaneously has a solution plan in mind. He then uses a similar expression of "taking away" to describe the function of the second production operator as he did before (1). In contrast to the first task, the students are not required to describe the production process symbolically and are only asked to represent the fractions iconically on a given line segment. In this representational context Julius then correctly divides the line segment into four parts and colours one of them to represent $\frac{1}{4}$. He also proceeds as intended with the representation of $\frac{5}{6}$ and two other fractions. Since there is no calculation required in this task, the conflicted meanings of the second production operator remain hidden. Instead, the production operator is interpreted within the DSE of representing fractions iconically, in which it can be understood as colouring a specific number of parts. Julius' repeated use of the expression "taking away" can be interpreted as a means of not having resolved the cognitive conflict. Instead, the activity of representing fractions iconically offers a third interpretation of the operator as colouring a specific number of parts.

DISCUSSION

The selected samples in this paper illustrate two major findings from the study. Concerning the first research question, the study shows that the intended processes of transfer project to the learning processes of the students, but in a highly individual and subjective way. The single sample of the dyad Bennet and Julius shows a strong divergence of intended and not intended processes of transfer even at the very beginning of the learning progression. This finding becomes even more apparent in the comparative analyses on multiple levels (Kollhoff, 2021). Compared to his partner, Bennet does make all intended connections more or less spontaneously without any apparent difficulties. In contrast, the case of Julius is representative of a second major finding of the study concerning the second research question: Numerical procedures and the activities that are supposed to represent them appear to be treated, stored, and developed as separated DSE. The example of Julius illustrates the need to coordinate the separated DSE in the construction of a coordinative DSE that requires an active construction by the learner. Comparisons to other dyads in the study describe the consequences if no such coordinative DSE is being constructed (Kollhoff, 2021). In these cases, the students tend to develop robust misconceptions that interfere with their further learning. These students appear to not fully understand the procedures and concepts and as a consequence rely mainly on the numerical procedures that they try to apply algorithmically and often incorrectly in various situations. In later stages, the dominance of the numeric procedures and the lack of illustrative understanding of them lead to errors that can be compared to findings of studies that investigate the "natural number bias" (Ni & Zhou, 2005).

The conceptualisation of transfer as a process within the didactical model of Grundvorstellungen proved to be useful. The comparison of the normatively intended processes of transfer and the descriptive reconstruction of the students' actual processes of transfer revealed patterns of deviation that have to be taken into account in the design and planning of learning environments and progressions. In particular, it has been described that the connection of numerical procedures and their illustrative representations have to be actively supported since they will often not be constructed by the learners themselves. Furthermore, the findings point out specific difficulties of the representational models themselves (Kollhoff, 2021) that have to be accounted for in teaching fractions.

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MAKING REALISTIC ASSUMPTIONS IN MATHEMATICAL MODELLING

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Making realistic assumptions is an important part of solving open modelling problems and also a potential source of errors. But little is known about the difficulties that result from the openness of modelling problems and how they can be addressed in interventions. Here, we focus on two central solution steps that are necessary for making assumptions: noticing the openness and estimating the missing quantities. In a qualitative study with four ninth graders, we asked students to solve a modelling problem after informing them about the openness of the problem. We identified barriers that expand the two-step model (e.g., trouble integrating assumptions into the model). In addition, informing students about the openness of the problem improved their solution to the problem at hand but did not help them solve subsequent problems.

INTRODUCTION

Mathematics can help people solve problems from every day or professional life. These problems typically do not contain all of the information required to obtain a solution. To replace missing values and simplify the situation, it is often necessary to make assumptions so that a mathematical model can be set up and used to solve the problem. Hence, specific skills (e.g., estimation skills) are needed, and mathematics classrooms should foster these skills to prepare students to apply their mathematical knowledge in order to solve real-world problems. Galbraith and Stillman (2001) highlighted the importance of making assumptions as a genuine but underrated aspect of successful modelling and stressed the need for systematic research in this area. This need was recently recalled (Schukajlow et al., 2021) and is addressed in the present study. We analyzed (1) the difficulties students experience when making assumptions to solve open modelling problems and (2) how information about the openness of the problem helps them overcome these difficulties. Our findings contribute to a better understanding of the process of making assumptions and the kinds of information that might help students overcome their difficulties with regard to making assumptions.

THEORETICAL BACKROUND AND RESEARCH QUESTIONS

Making assumptions

Making an assumption means proposing that a statement is temporally true as a productive basis for subsequent activities (Djepaxhija et al., 2015). Assumptions are necessary to solve open problems because important aspects of the problem situation are not specified, and additional information is needed. Assumptions specify the missing information and help the problem solver find a solution under the restrictive conditions that come along with making assumptions. Two broad types of assumptions

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can be distinguished: Non-numerical and numerical assumptions. Non-numerical assumptions refer to assumptions about situational conditions, whereas numerical assumptions refer to assumptions about missing quantities. Both types require realistic considerations and extra-mathematical knowledge, but in order to make numerical assumptions, estimation skills may also be necessary (Chang et al., 2020). Estimations, which are rough calculations or judgments, can refer to different objects, including measurements (e.g., estimating length, height, or weight) and numerosity (e.g., estimating the quantity of objects) (Hogan & Brezinski, 2003). A number of studies indicate that estimation skills are difficult for students to acquire, and students often fail to estimate measurements with the appropriate accuracy (Jones et al., 2012).

Mathematical modelling competence and making assumptions

Mathematical modelling refers to the use of mathematics to solve real-world problems (Niss et al., 2007). The key aspect of modelling is that a real-world problem must be converted into a mathematical model that allows mathematical procedures to be applied to solve the problem. The mathematical result needs to be interpreted and validated with regard to the initial real-world situation. Thus, modelling can be considered a cyclic process that begins and ends in reality and passes through the mathematical domain. In mathematics classrooms, modelling problems are used to foster students' modelling competence. Figure 1 presents an example of a modelling problem.

Speaker
Maria bought the *Ultimate Ears BOOM* Speaker for 149.95 €. It has 360° sound with deep and precise bass. The speaker is 18.4 cm high.
Maria looks for a box with a cover for her speaker. On the web, she found a beautiful box. It is 14 cm wide, 10 cm high, and 14 cm deep.
Will the speaker fit in the box?

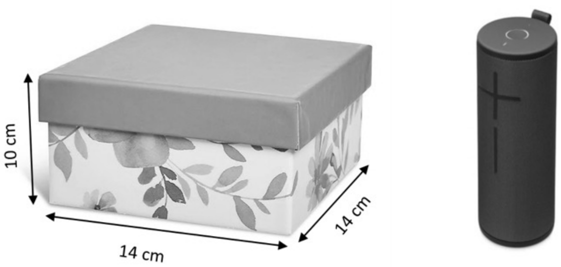


Figure 1: Modelling problem that requires assumptions to be made.

A characteristic feature of modelling problems is their openness as they often do not include all of the necessary information. To solve open modelling problems, two different solution steps are necessary (Krawitz et al., 2018): First, students need to notice the openness of the problem, and second, they have to estimate the missing quantities. For example, in the Speaker problem (Figure 1), students need to notice that the diameter of the speaker has to be taken into account and replace the missing quantity with an estimate (e.g., about 5 cm because, in the picture, the diameter is about one fourth of the height). Prior modelling research has shown that many students have trouble understanding, structuring, and simplifying the information given in modelling problems (Krawitz et al., 2021). Some of these challenges might result from the openness of modelling problems and the cognitive demands of making assumptions (Ärlebäck, 2009). An impressive body of research on word problems has demonstrated that students tend to neglect the realistic context of the problems, including the necessity of making assumptions, even if this leads to unrealistic responses (Verschaffel et al., 2000). In the Speaker problem, for example, an unrealistic response would be to ignore the fact that the diameter of the speaker has to be taken into account, calculate the diagonal of the box ($d = \sqrt{(14^2 + 14^2) + 10^2} = 22.18$), and conclude that the speaker fits because the speaker is shorter than the length of the diagonal. One potential reason for students' unrealistic responses is that they fail to notice the openness of the problem (Krawitz et al., 2018). In several interventions, researchers have tried to help students notice the openness, for example, by informing the students that the problems are tricky and cannot be solved in a straightforward way or by adding pictures to the problems (Dewolf et al., 2013), with little to no success. Students' restricted beliefs about word problems were found to be a reason for their difficulties (Djepaxhija et al., 2015). This finding indicates that the difficulties are persistent and hard to change. Initial indications for difficulties in noticing the openness of modelling problems came from a study conducted by Chang et al. (2020) where the failure to notice the openness was found to be a major barrier, whereas estimation skills seemed to play a minor role.

PRESENT STUDY AND RESEARCH QUESTIONS

The present study was conducted within the framework of the Open Modelling Problems in Self-Regulated Teaching (OModA) project, which is aimed at investigating cognitive, strategic, and affective conditions for the teaching and learning of open modelling problems. The research questions in the present study were:

RQ 1: What difficulties do students experience with respect to making assumptions when they solve open modelling problems?

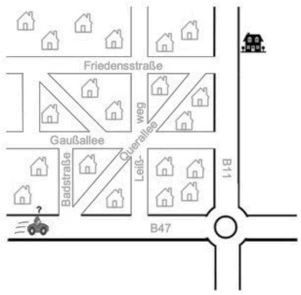
RQ 2: How does providing information about the openness of the problems help students overcome these difficulties?

METHOD

Participants and Data Collection

The sample involved four ninth graders (one female, all 16 years old) from two high-track schools (German Gymnasium). The students participated voluntarily in the study. Three of the participants were high achievers in mathematics (excellent grades), and one of them was an average achiever (average grades). In the following, the participants are referred to with pseudonyms. One of the participants (Andreas) stated that he had prior experience with open modelling problems, whereas the others did not. We used a qualitative approach to gather information on the underlying reasons for students' difficulties with open modelling problems and conducted individual sessions. The sessions consisted of three stages: problem solving, stimulated-recall interview, and semi-structured interview. In the problem-solving stage, participants were first given an open modelling problem (Shortcut Route Problem, Table 1) without information about the openness of the problem, a subsequent problem (Speaker Problem, Figure 1) with information about the openness ("To solve the problem, you must estimate the diameter of the speaker"), and finally another problem without such information (Tree Problem, Table 1).

Shortcut Route Problem: Mrs. Mai drives home on route B 47 and is running late. Fortunately, there is little traffic on the streets at night. She will soon come to the junction where the Street named Querallee branches off to the left. From there it would be another 1.5 km on B47 straight ahead, and from the roundabout another 2 km after turning left on B11 until she is home. Is the drive through the residential area worth it for Mrs. Mai so that she can get home earlier?



Tree Problem: Freshly planted trees are not yet rooted in the earth and need help attaching for the first few years. Support poles are often used to help. One end of the pole is hammered obliquely into the ground. A distance of 1.25 m from the tree is maintained so that the pole does not damage the roots of the fresh tree. The other end of the pole is tied to the tree with a rope at a height of 1.5 m. What is the length of the pole?

Table 1: Open modelling problems used in the study.

A quantitative pilot study with 143 students revealed that students rarely make realistic assumptions when solving these open modelling problems (percentage of solutions with realistic assumptions: 4.1% (Abbreviation problem), 3.2% (Speaker problem), 0.8% (Tree problem)).

Data Analysis

The video material was transcribed and sequenced. Sequences of the stimulated recall interviews were assigned to the related problem-solving sequences in order to collect more information about students' assumption-making processes. The sequences were categorized using qualitative content analysis (Mayring, 2014). In the coding process, noticing the openness and making assumptions were used as the main categories, and subcategories were inductively identified. Thereby, different types of assumptions (situational assumptions, numerical assumptions), purposes of assumption-making (simplify the situation, estimate missing quantities, interpret the result), and difficulties that could be attributed to the openness (noticing the openness, recognizing the possibility and necessity of making assumptions, integrating assumptions into the mathematical model) were distinguished. For example, the sequence "What is the diameter of the speaker? I would say, about as large as my water bottle. […] Okay, it is about 7 cm." Was paraphrased as "Estimated the length of the diameter of the speaker (7 cm)," and this was coded as a realistic numerical assumption.

FINDINGS

We analyzed students' difficulties that could be attributed to the openness of the problems. Table 2 gives an overview of the categories developed in the coding process.

Difficulties with:	Description
Noticing the openness	Not noticing the openness and consequently not making assumptions
Recognizing that assumptions might need to be made	Noticing the openness but not recognizing that making assumptions is a way to deal with it
Recognizing the need to make assumptions	Noticing the openness but thinking that it is not necessary to make assumptions
Integrating assumptions into the mathematical model	Not being able to set up an appropriate mathematical model that takes the missing quantities into account

Table 2. Overview of the difficulties that were attributed to the openness of the problem.

To answer the first research question, we analyzed students' solution processes for the first open modelling problem (Shortcut Route problem). Two of the participants (Tabea and Niklas) did not make any assumptions. Both calculated the distance without taking into account the different speed limits for the routes. Tabea did not notice the openness of the problem, whereas Niklas commented that he thought about the speed limits in his solution process but thought they were not important for the solution. Andreas directly recognized the need to make assumptions in the Shortcut Route problem. He made situational assumptions in order to simplify the real-world situation ("under the assumption that the street is perpendicular to the junction") and to specify his estimations ("because there are houses next to the road, the car has to look for pedestrians and cannot drive 100 km/h"). On this basis, he made realistic numerical assumptions about the speed limits (main road: 80 km/h; housing area: 30 km/h) and also defined situational requirements that did not need to be considered ("the speed while turning at the junction can be ignored"). Further, he used his assumptions to calculate the time that was needed to take the shortcut and to take the main road and completed the process by providing a realistic answer to the problem ("It is not worth it because of the speed limits"). In Christian's solution process, it was not clear at what point he noticed the openness of the problem. Christian did not make any assumptions and calculated the distances of both routes without considering the different speed limits. But his answer to the problem shows that he was aware of the fact that he neglected to consider this aspect in his solution ("The way through the housing area would be shorter but not necessarily faster"). His way of dealing with the openness of the problem was to acknowledge that his answer might not be valid. For Christian, noticing the openness did not lead him to make assumptions. Thus, simply noticing the openness is not enough for students to also recognize the need to make assumptions.

To address the second research question, we analyzed students' solution processes after they were given information about the openness of the problem (Speaker problem). We found that informing the students that a quantity was missing helped all participants in our study notice the openness. Two of four participants, Christian and Andreas, made assumptions about the missing quantity (here, the diameter of the speaker) and used their estimates to set up a mathematical model. One participant, Tabea, did not estimate the length of the diameter but took this quantity into account when interpreting her result ("It depends on the width of the speaker […] the maximum width would be 1.4 cm. I think this is too narrow."). Niklas also noticed that the diameter of the speaker was important but did not know how to use this information to solve the problem. Instead of estimating the diameter, he ended his solution process by simply guessing that the speaker would not fit into the box. His solution process exemplifies that integrating the missing quantities into a mathematical model can also be a barrier, in particular if the mathematical model becomes more complex when the additional information is included, as was the case for the Speaker problem.

To find out if the information also helps students notice openness while solving additional open modelling problems, the participants were given a third open modelling problem (Tree problem) without any information about the openness of the problem. None of the four participants noticed the openness of the problem. All of them neglected the fact that an assumption had to be made about the additional length of the support pole needed to fasten it to the ground in order to obtain a realistic solution (see Christian's solution in Figure 1). Consequently, the participants did not transfer their experience with the previous open modelling problem to the next one.

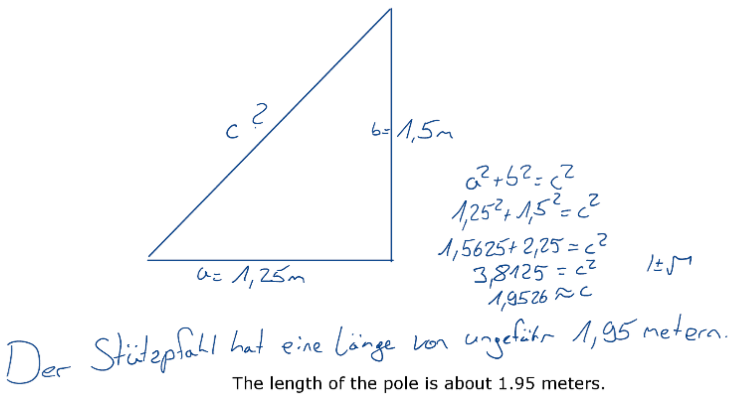


Figure 1. Christian's solution to the Tree problem.

Interestingly, Andreas and Bettina assigned the best value to their unrealistic solution:

Interviewer: Which of your solutions did you find the best?

Andreas: Best means that it is correct. Therefore, I would say the last one [Tree problem]. Because this is the one that really is correct. With the other, you have a greater inaccuracy because of the estimation.

In this excerpt, Andreas, who had previous experience with open modelling problems and was able to solve the Shortcut Route problem and the Speaker problem by making assumptions, states that he believes that his realistic solutions, which included assumptions, were less correct than his last unrealistic solution. He thinks the realistic solutions were less accurate due to estimation errors.

SUMMARY AND DISCUSSION

In line with previous research (Chang et al., 2020), noticing the openness of problems was revealed as a key difficulty. Further, noticing the openness did not automatically result in making assumptions. We identified three difficulties that prevented students from making assumptions after noticing the openness. First, making assumptions was not assumed to be necessary. Second, strategies or knowledge about how to deal with open problems were missing. Third, it was difficult to set up a mathematical model that took the missing quantities into account. Hence, our findings expand on the proposed two-step model for solving open modelling problems involving the steps of noticing the openness and estimating the missing quantities (Krawitz et al., 2018). These additional barriers should be taken into account in future studies investigating the role that making assumptions plays in mathematical modelling.

Contrary to studies that have revealed students' difficulties with estimation tasks (Jones et al., 2012), estimating the missing quantities did not hinder problem solving. Maybe the problems did not challenge our participants' estimation skills, or perhaps they failed at earlier stages in their solution processes so that we could not detect these difficulties.

Further, students' difficulties with noticing the openness could be overcome by providing information. However, the information helped only for the problem at hand, but it did not help students notice the openness of subsequent problems. Similar to research findings on word problems (Dewolf et al., 2013), students' difficulties with noticing the openness of a modelling problem seem to be persistent. Future studies should examine how the difficulties identified in the present study can be addressed in teaching methods. Students' restricted beliefs about word problems, in particular, the belief that every problem has a single numerical answer, were also found in our data and may have prevented students from making assumptions (Djepaxhija et al., 2015).

On a theoretical level, our study contributes to a better understanding of the process of solving open modelling problems and the challenges that are induced by the openness. Our findings provide a basis for developing teaching methods that address these difficulties in future research. A practical implication might be to provide more learning opportunities to deal with open problems in class so that students can acquire the knowledge and strategies that are necessary to deal with open modelling problems.

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SIMULATION-BASED LEARNING ENVIRONMENTS: DO THEY AFFECT LEARNERS' RELEVANT INTERESTS?

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The use of simulation-based learning environments to foster professional competences attracts more and more research. The role of participants' interest for learning is quite undisputable also in this context. Recent research findings emphasize that interest may trigger the activation of professional knowledge during participation in a simulation. Using data from N = 81 pre-service teachers who participated in four simulations over one semester, this contribution investigates how characteristics of the simulation (role-play vs. video) and participants' perception of the simulation affect the development of participants' interests. Results reveal that, beyond the perception of the simulation, development of participants' interests is weakly related to simulation characteristics.

INTRODUCTION

Diagnosing students' thinking is an important practice in teachers' professional life. In teacher education, learning environments based on role-play- or video-based simulations are applied to link conceptual knowledge with procedural components (Marczynski et al., in press). As learners' interests relevant in the simulation content can be seen as a "door-opener" for knowledge activation in such simulation-based learning environments (Kron et al., under review), the development of participants' interest becomes a crucial issue. The presentation may play a role here: While highly interactive simulation designs may increase interest, they run at danger of putting cognitive demand on participants, reducing the positive effects of interactivity.

Approximations of practice (AoPs) in pre-service teacher education

Simulation-based learning environments are special forms of approximations of practice (AoPs, Grossman et al., 2009). AoPs are intended to trigger knowledge activation in authentic, yet controlled situations. For example by using simulations, real-life situations are reconstructed to provide learning experiences, which are less cognitive demanding than real classroom situations, and reduce disruptive factors (Grossman et al., 2009). Especially in teacher education, AoPs are recommended to foster pre-service teachers' competences by allowing an application and extension of professional knowledge in authentic, yet not overwhelming situations (Codreanu et al., 2020). These competences entail cognitive as well as affective components, such as learners' interests (Heitzmann et al., 2019). As such, AoPs are discussed being effective tools for teacher training (Meletiou-Mavrotheris & Mavrou, 2013), for example to engage in the diagnosis of student thinking (Marczynski et al., in press).

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For the design of AoPs as learning environments, two criteria are highlighted to be relevant for learning: (1) the AoPs should be perceived as being authentic and (2) they should allow the learners to immerse themselves into the simulated situation (Goeze et al., 2014). However, the design of such authentic and immersive simulations is often at danger of increasing the extraneous cognitive load, which may impede their effectiveness (Sweller, 2010). Whereas teacher education commonly uses video-based simulations (e.g., Seidel et al., 2011), medical education has focused on role-play simulations, with trained actors as simulated patients (e.g., Stegmann et al., 2012). While role-plays may offer more opportunities for interaction and may thus be perceived as more authentic and immersive, video-based simulations with pre-structured options for action may lead to lower cognitive load during learning.

Development of learners' relevant interests in AoPs

Pre-service teachers' interest likely affect how they engage in such AoPs or other learning opportunities in university-based teacher education. Following Krapp (2002), interest is a relatively stable relation between a person and an object, reflecting the "tendency to occupy oneself with an object of interest" (intrinsic component, Krapp, 2002). Besides this "intrinsic component", interest also comprises a positive emotional relation to the object (Krapp, 2002), and ascribes a certain value to the object of interest (Schiefele et al., 1992). This person-object relation may change or develop whenever an individual encounters the object (Hidi & Renninger, 2006). An object of interest can be any entity from the individuals' "life-space" (Krapp, 2002), such as a professional practice, or a field of study. Relevant interests of pre-service mathematics teachers may, for example, address mathematics education content, or professional practices such as diagnosing student thinking.

Hidi and Renninger (2006) argue, that interest "as a motivational variable refers to the psychological state of engaging or the predisposition to reengage with particular classes of objects, events, or ideas over time". Research distinguishes between individual interest and situational interest (Hidi, 1990). Whereas situational interest is a temporary experience in a specific moment (Hidi, 1990), that results from "[…] an interaction of learners' and situational features" (Rach, 2021), individual interest refers to a relatively stable motivational trait. As such, situational interest has been found to enhance learning (Wade, 1992), whereas individual interest had positive effects on attention, recognition, and recall (Hidi & Renninger, 2006).

Thus, diagnostically relevant interests may play an important role when engaging pre-service teachers with AoPs on diagnosis, for example by playing the role of a "door-opener" for the activation of professional knowledge: Kron et al. (under review) report that the relation between pre-service teachers' professional knowledge and their performance in simulated one-on-one diagnostic interviews depended on their interest in mathematics education and diagnosis, This leads to the question how these interests may be developed in pre-service teacher education. Meaningful experiences in simulations may strengthen the person-object relation and lead to more intense interest in contents of the simulation and the simulated activities. However, research about how such learning environments contribute to the development of interest, is scarce.

Regarding this development, also Hidi and Renninger (2006) argue that experiences during a learning situation might trigger situational interest, if the learning environments are authentic and immersive, and "provide meaningful and personally involving activities". Beyond authenticity and immersion, cognitive load has been found to affect learners' situational interest negatively (Park et al., 2015). If sustained over time, situational interest may contribute to the development of individual interest (Hidi & Renninger, 2006). However, it is quite unclear how pre-service teachers' interest develops during repeated participation in simulation-based learning environments beyond short term effects of the simulation.

THE PRESENT STUDY

Despite the increasing research focus on interest development and the use of AoPs in teacher education, research linking these two fields is scare. We investigate the development of pre-service teachers' relevant situational interests during repeated participation in a simulation-based learning environment on diagnosis of student thinking. We compare role-play- and video-based presentation formats. Since role-play simulations may offer more opportunities for authentic and immersive experiences, but may also result in a higher cognitive load, we did not have specific hypotheses which presentation format would be more beneficial for interest development. We addressed the following questions:

RQ1: Does the presentation format of a simulation-based learning environment affect participants' relevant situational interests reported after the simulation?

RQ2: How do participants' initial individual interests and their perception of the simulation affect participants' situational interests after the simulation? We expected that higher initial interest, as well as perceiving the simulation as authentic and immersive, would go along with higher interest after the simulation, whereas higher extraneous cognitive load would decrease interest.

RQ3: Does the presentation format influence the development of situational interest over multiple simulations, after controlling for the perception of the simulation?

METHOD

To answer these questions, we used simulated diagnostic one-on-one interviews. Pre-service secondary school mathematics teachers at a large university in Germany were randomly allocated to one of the two parallel presentation formats (role-play: $N = 39$; video: $N = 42$). During summer term 2021, every participant participated in four simulations with a constant presentation format ($N = 324$ interviews, in total). The simulations were embedded in a web-based interview system. Initial individual interests were assessed before the first simulation. During each simulation, participants reported their perception of the AoP. We applied scales for situational interests directly at the end of each simulation session.

Simulation

Simulated diagnostic one-on-one interviews were developed (Marczynski et al., in press) as an AoP for mathematics teacher education. Pre-service teachers act in the role of a teacher, diagnosing the mathematical thinking of a 6th grader in the field of decimal fractions, by using a given set of diagnostic tasks. Four different student case profiles were constructed, with different profiles of mathematical understanding in the field of decimal fractions. Trained research assistants played the student role in the role-play format, while scripted videos of 6th graders were prepared for the video simulation. Whereas the participants of the role-play simulation interacted with the simulated student directly, participants of the video-version watched the provided videos. Each simulation contains four phases: (1) The participants got familiarized with the interview system, their role as the teacher, and reviewed the given set of diagnostic tasks (only first simulation). (2) The participants had 25 minutes time to interview the simulated student. They chose tasks from the given task-set, observed the student's response, and posed probing questions (in the video-simulation they selected from a range of possible probing questions). (3) After the interview, they prepared a diagnostic report about the interviewed student's mathematical thinking. (4) The simulation ended with a debriefing, providing informing about an expert's diagnosis of the student. Each participant conducted four simulations, one every two weeks.

Instruments

Interest: To assess participants' relevant interests, we adopted scales of Rotgans and Schmidt (2011), considering interest in mathematics education and interest in diagnosis to be relevant in the context of the simulation (three items per scale, five-point Likert scales from 0 = not true at all; 4 = very true for me; $\alpha_{math.ed} = .89$; $\alpha_{diagnosis} = .76$).

Perception of the simulation: Participants' perception of the simulation was assessed by established scales (e.g., Seidel et al., 2010) using three items for authenticity and four items for immersion on a five-point Likert scale (0 = not true at all; 4 = very true for me; $\alpha_{auth} = .88$; $\alpha_{immers} = .67$). Extraneous cognitive load was assessed by three items (five-point Likert scale; 0 = very easy; 4 = very difficult; $\alpha_{extr.load} = .75$).

Statistical analyses: All data were collected in log files by the web-based interview system. Due to the nested structure of the dataset (multiple simulations per participant), we used linear mixed models to estimate effects of the perception of the simulation, its presentation format, and repeated participation, on interest reports after each simulation. In a first step, only the effect of the presentation format was investigated. Then, participants' initial reported interest and the perception of the simulation were included. Finally, we added the number of the simulation (0 = first – 3 = last) as a metric covariate and its interaction with the presentation format. We used planned contrasts of estimated marginal means to investigate our research questions.

RESULTS

Average interest ratings after all four simulations were above the midpoint of the scale for mathematics education ($M = 2.44$, $SD = 0.81$) and diagnosis ($M = 2.78$, $SD = 0.66$).

Interest in mathematics education: *(RQ1)* Participants reported significantly higher interest in mathematics education after the video ($M = 2.61$, $SE = 0.11$) than after the role-play simulation ($M = 2.25$, $SE = 0.11$; $B = 0.36$, $p < .05$). *(RQ2)* These interest ratings were positively influenced by perceived authenticity ($B = 0.18$, $p < .001$) and immersion ($B = 0.15$, $p < .01$), and negatively by extraneous cognitive load ($B = -0.21$, $p < .001$). Initial interest in mathematics education did not predict the interest reported after the simulations significantly ($B = 0.04$, $p = .67$). Controlling for effects of the perception of the simulation, the difference in interest between the presentation formats, averaged over four simulations, was not significant anymore ($B = 0.20$, $p = .12$). *(RQ3)* Controlling for those effects of perception, the difference between the presentation formats was significant in the first ($M_{rp} = 2.32$, $SE_{rp} = 0.11$, $M_{vi} = 2.61$, $SE_{vi} = 0.10$; $B = 0.29$, $p < .05$), but not for the last simulation ($M_{rp} = 2.35$, $SE_{rp} = 0.11$, $M_{vi} = 2.46$, $SE_{vi} = 0.10$; $B = 0.11$, $p = .44$) due to declining interest in video simulation.

Interest in diagnosis: *(RQ1)* Participants did not report significantly different interest in diagnosis after the video simulation ($M = 2.83$, $SE = 0.09$) than after the role-play simulation ($M = 2.73$, $SE = 0.09$; $B = 0.10$, $p = .42$). *(RQ2)* These interest ratings were positively influenced by the perceived authenticity ($B = 0.13$, $p < .01$) and immersion ($B = 0.11$, $p < .05$), and negatively by extraneous cognitive load ($B = -0.13$, $p < .01$). Initial interest in diagnosis positively predicted the interest reported after the simulations ($B = 0.42$, $p < .001$). *(RQ3)* Controlling for the perception of the simulation, we observed a significant decline of interest ratings over the four simulations ($B = -0.04$, $p < .05$), which corresponds to a difference of $B = 0.13$ on the interest scale (0-4) over all four simulations. This decline did not differ significantly between the two presentation formats ($B = -0.03$, $p = .36$).

DISCUSSION

The aim of this contribution was to provide insights, how the presentation format of an AoP and the participants' perception of that presentation format affect their situational interest and its development, considering two different objects of interest. We intended to disentangle effects of situational experiences and developments of interest over time.

Pre-service teachers, who perceived the simulation as authentic and immersing, reported a higher level of interest directly after participation in the simulation (RQ2). These relations between authenticity and immersion and interest are in line with assumptions based on work by Hidi and Renninger (2006) on interest development. The negative relation of extraneous cognitive load and interest development confirmed results of Park et al. (2015). This highlights, that AoPs need to be designed in an authentic and immersing way, also considering potential sources of extraneous cognitive load. In fact, these requirements may run contrary to each other, as described on our assumptions about the two presentation formats (see also Codreanu et al., 2020).

While prior interest in diagnosis was substantially related to post-simulation interest in diagnosis, this was not the case for interest in mathematics education. Authentic encounters with the object of interest are assumed to contribute to interest development (Hidi & Renninger, 2006). Beyond lectures and exercise sessions, this was one of the first opportunities for the participants to apply their mathematics education knowledge in an authentic (though simulated) situation. These results may indicate, that participants re-evaluated their interest in mathematics education more strongly based on the situational perception of the AoP than their interest in diagnosis.

Without consideration of other factors, the video simulation triggered more positive ratings of interest in mathematics education than the role-play simulation (RQ1). According to our assumptions, this indicates that potential advantages of the video simulation in terms of lower extraneous cognitive load may have exceeded advantages of the role-play simulation in terms of higher authenticity and immersion (Hidi & Renninger, 2006; Park et al., 2015). Indeed, these situational perceptions explained almost all differences between the presentation formats. For interest in diagnosis, no differences in post-simulation interest by presentation format occurred. One interpretation of this finding could be that the presentation format was neutral regarding the emergence of situational interest in diagnosis, but not so for situational interest in mathematics education. The more structured interaction format of the video simulation (e.g., selecting from provided probing questions, instead of asking questions freely) might have helped participants to apply their knowledge from mathematics education and to experience it as helpful and valuable. In line with the idea of AoPs (Grossman et al., 2009) this result points to the importance of finding an appropriate level of complexity when designing AoPs.

Considering interest development under control of situational factors (RQ3), only one significant difference between the presentation formats occurred. The initially positive effect of the video-based simulation on interest in mathematics education vanished until the last simulation. This short-term effect may be due to the novelty of the video-based simulation format, which is rarely used at the university under study. Firstly, this indicates that the presentation format mostly affected situational interest, but that these effects did not transfer to long-term development. Apart from this decline for the video format, interest in mathematics education was stable over four simulations. In light of other studies usually finding declining interest in repeated measures designs (e.g., Rotgans & Schmidt, 2011), we take this stability of interest in mathematics education in our study as an encouraging sign. As in other studies on interest, we find a general decline of interest in diagnosis over the four simulations. Explicating the value of diagnosing student thinking was briefly addressed in the simulation activities, but more directed interventions, such as explicitly experiencing the value of diagnosis to design individual support, and reflecting on this value (Hulleman et al., 2010) might be necessary to develop pre-service teachers' interests in diagnostic activities.

The role of interest as a “door-opener“ for deep learning in general (Hidi & Renninger, 2006) as well as for knowledge activation in AoPs (Kron et al., under review) is undisputed. We contribute to understanding the emergence of situational interest during AoPs on the diagnosis students’ mathematical thinking. Systematic changes in situational interest over a longer time, under control of situational factors, can point towards possible developments of individual interest. Our findings indicate that current learning experiences shape participants’ interests, but that it is possible to identify developments over the course of a semester beyond these situational factors. It is crucial to disentangle pure novelty effects of new simulation formats from long-term developments of situational, and potentially also individual, interests. Further research should investigate effects of AoPs, but also explicit interventions regarding their potential to sustain and develop pre-service teachers’ relevant interests.

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ARGUMENTATION BASED ON STATISTICAL DATA AT THE VERY BEGINNING OF PRIMARY SCHOOL – EVIDENCE FROM TWO EMPIRICAL STUDIES

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Although students' argumentation is subject of high interest in mathematics educational research, specific studies on argumentation based on statistical data are still scarce, especially with a focus on young students at the beginning of primary school. Therefore, relatively little is known so far to what extent children starting school may already be able to engage in argumentation based on statistical data. Addressing this research need, evidence is reported from two empirical studies, which were conducted with N = 11 and N = 29 students during their first weeks in school. The results show that data-based argumentation is possible for many students from the beginning of primary school on, and provide insight into the broad spectrum of students' data-based arguments.

INTRODUCTION

From the beginning of primary school on, fostering students' argumentation is considered as an important aim of the mathematics classroom, which is reflected in several empirical studies (Sommerhoff et al., 2015), in a variety of literature promoting suggestions on how to foster students' argumentation in the mathematics classroom (Stylianides et al., 2016), as well as in curricula of many countries (e.g. NCTM, 2000). Even if the importance of argumentation is also highlighted frequently in the context of statistics education (e.g. Ben-Zvi & Sharett-Amir, 2005), it appears that data-based argumentation received relatively little attention so far, in particular in the discourse on primary mathematics education. In prior studies (e.g. Krummenauer & Kuntze, 2018, 2019), we have found that many older primary students were able to evaluate interpretations of data and to develop corresponding data-based arguments in different task contexts, including even relatively complex tasks which require considering statistical variation when developing data-based arguments. This raises the question, to what extent data-based argumentation is possible for younger students; in particular, what prerequisites related to data-based argumentation students have when starting school.

Addressing this research need, this paper is focused on the extent to which primary students are able to develop data-based arguments in different task contexts at the beginning of their first year in school. The empirical evidence reflected on in this paper has been gathered in two studies, applying an innovative study design. The results presented in this paper substantiate that data-based argumentation is possible for many

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students, in appropriate task contexts, already at the beginning of primary school, and give insight into the spectrum of complexity in students' data-based arguments.

In the following, the theoretical background of the research reported in this paper is presented, and the research interest is specified. Subsequently, the methodological background and empirical evidence from two studies are reported. The results and implications of both studies are discussed in the concluding section.

THEORETICAL BACKGROUND

When students encounter statistical data in real-life contexts, these data often are accompanied by different and sometimes conflicting interpretations. For dealing with statistical data and related interpretations it is, therefore, crucial that students are able to evaluate whether or not interpretations of data indeed can be substantiated by the respective data, and that students are able to justify their position based on data. We refer to this by the term *data-based argumentation*, which is considered as a specific case of argumentation in which statistical data are used to convince others that certain statements are true or false (Krummenauer & Kuntze, 2019). As presented in detail in a research report at PME 42 (Krummenauer & Kuntze, 2018), key requirements of data-based argumentation can be described from a theoretical perspective building up on psychological theories on children's *scientific reasoning* (e.g. Kuhn, 2011; Sodian et al., 1991; Zimmerman, 2007). In this perspective, interpretations of data have the status of hypotheses (in a broader sense, *theories*), while the statistical data these interpretations refer to represent the available *evidence*. When students develop data-based arguments, they are required to *coordinate* interpretations of data with the status of a *theory* and the statistical data with the status of *evidence* with each other, e.g. when evaluating whether interpretations are consistent with corresponding data or when basing own interpretations on data. In the literature, several strategies for coordinating theory and evidence (e.g. Zimmerman, 2007) are described, which are highly relevant for data-based argumentation: a fundamental strategy for coordinating theory and evidence is, for instance, to distinguish elements representing theory, such as claims or own beliefs, strictly from elements representing evidence (e.g. Kuhn, 2011); another scientific reasoning strategy, which is particularly helpful for data-based argumentation, is to search intentionally for counter-evidence (e.g. Sodian et al., 1991), instead of primarily searching for supporting evidence.

During the past decades, a large body of research on the development of scientific reasoning has emerged (Zimmerman, 2007). Several studies have shown that already children in kindergarten and primary school can be able to master tasks on coordinating theory and evidence (e.g. Koerber et al. 2005). However, at the same time, there is frequent evidence of insufficient strategies hindering the coordination of theory and evidence. For instance, Koerber and colleagues reported in the mentioned study that kindergarten children showed a tendency to be influenced by own assumptions when coordinating theory and evidence. Further, there are studies implying that young students tend to have difficulties to consider statistical variation when coordinating theory and evidence (Masnick & Morris, 2008). In conclusion, the available studies on scientific reasoning imply that students at the beginning of primary school may already have some cognitive preconditions for data-based argumentation; at the same time, it needs to be expected that difficulties regarding the coordination of theory and evidence may cause difficulties in data-based argumentation.

In empirical studies with older primary students specifically targeting on students' data-based argumentation (e.g. Krummenauer & Kuntze, 2018, 2019), many participants were able to evaluate interpretations of data and to develop arguments based on the data for substantiating their evaluation; in the case of the study reported in Krummenauer and Kuntze (2018), this required students even to take into account statistical variation of the data. These studies also revealed that some students gave answers indicating specific difficulties, which appear to be interrelated with difficulties in students' scientific reasoning; for instance, some students used only aspects of data for argumentation which were in line with their assumptions but did not consider disconfirming data (Krummenauer & Kuntze, 2019).

RESEARCH INTEREST

Building up on the research with older primary students, the studies reported on in this paper were conducted in order to investigate the extent to which data-based argumentation is possible already for primary students starting school. In particular, the research presented in this paper is targeted on the following research question:

To what extent is it possible for students at the beginning of the first grade to evaluate data-related statements and to develop data-based arguments in order to justify their evaluation?

STUDY I

Design of the Study

As there had been hardly any specific research on young primary students' data-based argumentation so far, a first exploratory interview study has been conducted (Krummenauer et al., 2020). In preparation for this study, an interview design needed to be developed, which addresses the specific needs of young students. As it cannot be expected that children produce data-based arguments spontaneously, an elicitation method was developed, implemented in a one-to-one interview design. For that, a set of tasks had been adapted specifically to the needs of students at the beginning of primary school. In the interviews, the tasks were presented to the students one after another, following a highly standardised interview guideline. Each task consists of a data set (two examples are given in Figure 3) visualised by means of pictograms, in combination with corresponding statements expressing interpretations of the data (e.g. "Most students like chocolate ice cream" in case of the data set in part b) of Figure 3). In the interviews, the task context and the data as well as a statement to be evaluated were presented to the students. Subsequently, the students were asked to evaluate the statements and to justify their evaluation, so that the students were required to develop data-based arguments. In this first study, *N* = 11 students (6 girls, 5 boys) were interviewed during their first weeks in school. There had been no prior intervention and the interviewer carefully avoided giving any examples or hints. The transcribed interview data were subjected to a dichotomous top-down coding in order to find out whether the students developed consistent data-based arguments in respondence to the tasks. To be rated as "consistent data-based argument", answers had to contain a correct evaluation of the statement (e.g. "no, that's not true") and a reference to aspects of the data which allow to substantiate the given evaluation; sample answers fulfilling these criteria are presented below in detail. Answers not meeting these requirements were subjected to a further bottom-up analysis (overall inter-rater reliability: κ = .96) investigating types of students' difficulties, which is reported in Krummenauer et al. (2020); in the present report, we deepen the analysis regarding the top-down analysis in order to gain deeper insight into the qualitative spectrum of students' successful data-based arguments identified in the top-down analysis.

Results

Figure 1 gives an overview of the number of consistent data-based arguments for each student. All participants were able to develop at least one consistent data-based argument, and most of the students developed consistent data-based arguments in more than half of the tasks. In one case (S6), a student provided consistent data-based arguments for almost all 11 tasks. To give insight into the coding and into the spectrum of students' successful answers, two sample answers differing in their complexity are discussed in the following, beginning with an example with relatively low complexity, but still fulfilling all above-mentioned criteria of data-based arguments.

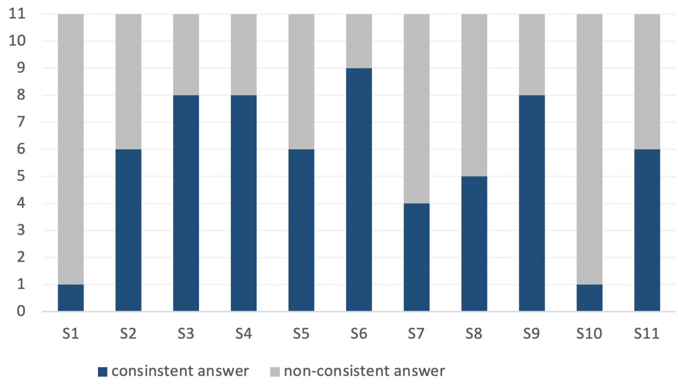

Figure 1: number of answers containing data-based arguments per student (cf. Krummenauer & Kuntze, 2020)

The following transcript (translated from German) is related to a task, which is about a fictive competition in which the drivers of four cars meet once a week for a race. The diagram in part a) of Figure 3 displays the number of trophies won by each driver. The transcript starts after the interviewer had introduced the data set and its context. In (1), the interviewer presents (by means of and in the name of a hand puppet) the statement which shall be evaluated based on the data.

(1) hand puppet: If I would take part in the race, then I would take the red car, it looks the fastest.

(2) student: But it isn't. The green car is the fastest, because it has the most trophies.

In (2), the student rejects the hand puppet's statement ("But it isn't"), i.e. the student gives a negative evaluation of the statement. The student then substantiates this evaluation by correcting the statement ("The green car is the fastest") and connecting it with the term "because" to the number of trophies, i.e. aspects of the data which support the student's evaluation of the hand puppet's statement.

The next sample answer refers to a – in terms of coordinating theory and evidence – more complex task, which is about a school excursion with two participating classes ("hedgehog class" and "mouse class"). During the excursion, each student was allowed to order one scoop of ice cream; the two data sets (part b) of Figure 3 represent the number of scoops of ice cream ordered in each class.

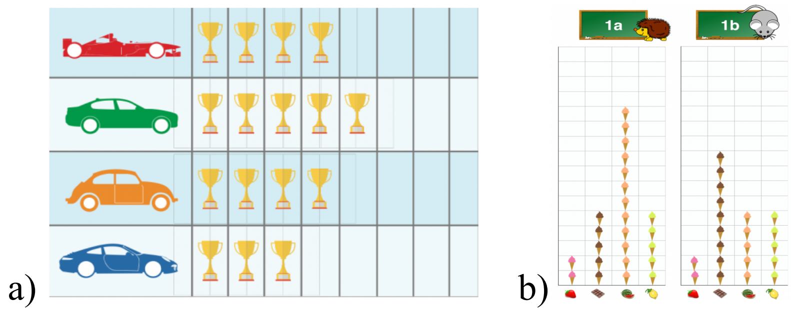

Figure 3: task examples (Krummenauer et al., 2020, p. 5; 7)

The transcript starts after the task context was introduced.

(1) hand puppet: In the hedgehog class are more children than in the mouse class.

(2) interviewer: Is this true?

(3) student: (agrees).

(4) interviewer: How do you know that?

(5) student: (points to the data in the diagram) look, here are two. Then here are two. Look, both are five, that is both five / So, this is two times five, this is two times five, this has one times two and this [the bar of chocolate scoops] has this height, but this here [the bar of melon scoops] is a bit higher.

After the statement to be evaluated was presented in (1), the student indicates in (3) a positive evaluation of the statement. After the interviewer asked for a justification, the student substantiates the evaluation in (5) based on the data: the student identifies and matches bars with the same height in both data sets (bars with the height 2 and bars with the height 5) and shows that the remaining bar in the diagram of the hedgehog class is higher than the remaining bar in the diagram of the mouse class, so that more students need to be in the hedgehog class than in the mouse class. In comparison to the first sample answer, this argument has a much higher complexity in terms of coordinating theory and evidence, as the student needs to relate the data for all sorts of ice cream to each other. This results in an argumentation with multiple steps, while developing an argument in the first example only requires to relate fewer elements of the data with the statement being evaluated.

STUDY II

Based on the first study, which had shown a relatively broad spectrum in students' data-based argumentation – both in regard to the number of data-based arguments per student as well as in regard to the complexity of students' arguments – a second study was conducted recently in order to investigate in more detail and with a larger sample size the qualitative spectrum of students' data-based argumentation at the beginning of primary school.

Design of the Study

For this, the methodology of the first study was further developed. In order to make the full spectrum of students' data-based argumentation visible, the set of tasks was systematically further developed in order to be able to provide a spectrum of tasks to students, differing in their complexity under the perspective of coordinating theory and evidence. The tasks were implemented in a similar interview design as in the first study and were administered to $N = 29$ primary students at the beginning of their first year in school, again without any prior intervention. In the following, we reflect in detail on the quantitative results related to three tasks, which provide further insight into the spectrum of students' data-based argumentation at the beginning of primary school. The inter-rater reliability of the top-down coding conducted for this analysis is $\kappa = .88$.

Results

At first, we would like to put the focus on the task in part a) of Figure 3, which had the highest rate of successful answers in the study; 82.8% of the students were able to develop a consistent data-based argument in respondence to this task. The task is about the number of marbles of three children displayed in the diagram. The statement to be evaluated in this task by the students is “Jana has got three marbles”. Compared with the tasks presented above, the complexity in terms of coordinating theory and evidence is reduced, as the data which is needed for evaluating the claim can directly be taken from the diagram; no further steps, such as comparing different data sets, as required in the case of the task on ice scream scoops shown above, are necessary.

The data set on ice cream scoops had also been used in the second study, combined with a modified statement (“in the mouse class, more children like chocolate ice cream than in the hedgehog class”). In contrast to the marble task, this task requires to compare data from two data sets in order to gain the relevant evidence for evaluating the statement. Empirically, the increased complexity is reflected in a lower success rate of 48.3%.

Beyond such tasks, we implemented further, more complex tasks in which coordinating theory and evidence does not only require to take into account and to compare several data points, but also to consider that the given data may vary to some extent. A sample task is shown in part b) of Figure 3. The task includes two diagrams displaying how many deers have been observed during the past five days in a forest (right diagram) and in a city park (left diagram). In the task context, the students had to evaluate the statement (claimed by a character of the context story) "If I really want to see a deer, I should go to the park". As the data imply that the number of deers can change from day to day, and as the statement is about the future, the task requires students to take into account that the data may vary, which needs to be addressed when developing a corresponding data-based argument. In our study, several students compared the number of deers and argued, that it would be better to go to the forest as the number of deers in this diagram is higher; however, no student in the sample considered that the data may vary, which appears to be a challenging requirement for the participating students.

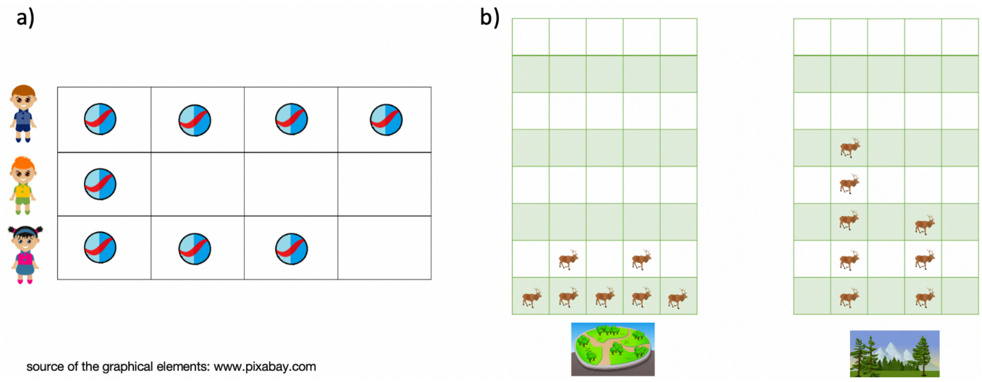

Figure 3: sample tasks

DISCUSSION AND CONCLUSIONS

Both studies have shown that the participating school starters were in many cases able to evaluate given interpretations of data and to develop consistent arguments based on the data in order to substantiate their evaluation. Although the samples of the studies are clearly not representative, the qualitative and quantitative analyses revealed a broad spectrum of students' data-based argumentation, both regarding the frequency as well as the complexity of their arguments. Against the background that both studies had been conducted without any prior intervention, it appears that young primary students have a high potential related to data-based argumentation, which should be addressed and fostered in the mathematics classroom during primary school (and beyond). As implied by research on children's scientific reasoning (Masnick & Morris, 2008), the students showed difficulties in tasks which require considering statistical variation when developing data-based arguments. Fostering students in this regard, e.g. by providing learning opportunities which allow for experiences in dealing with statistical variation, may therefore be a promising approach for fostering students' data-based argumentation, which is planned to be evaluated in an intervention study.

Acknowledgements

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TEACHING AND LEARNING ADDITION AND SUBTRACTION BRIDGING THROUGH TEN USING A STRUCTURAL APPROACH

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An eight-month-long intervention based on the idea of using a structural approach to addition and subtraction, and particularly bridging through ten, was implemented in Swedish Grade 1. A goal was that at the end of Grade 1, students would solve tasks like 15–7= using part-whole relations of numbers. In this paper, we report on learning outcomes from task-based interviews with intervention and control groups before, immediately after and one year after the intervention, to investigate long-term effects and whether students used a structural approach when solving tasks in Grade 2. Results show that students in the intervention group increased their learning outcomes the most and to a larger extent solved tasks in higher number ranges using a structural approach.

INTRODUCTION

A structural approach in arithmetic has been advocated by several scholars as means to facilitate students in developing powerful and sustainable ways of solving arithmetic problems (e.g., Cheng, 2012; Ellemor-Collins & Wright, 2009). It is, however, not only the way arithmetic is taught, but how the student experiences arithmetic tasks as structure based in part-whole relations, that is highlighted. For example, Ahlberg (1997) concludes from empirical research that "[w]hen children handle numbers by structuring they do not count on the number sequence in order to keep track of the numbers, but rather structure the numbers in the problem in parts and the whole in order to arrive at an answer" (p. 70). This way of seeing arithmetic learning and understanding challenges the view dominated by cognitive science (Baroody, 2016; Fuson, 1992) that young students learn addition and subtraction through acquisition of basic counting strategies, e.g., counting from the first addend, emphasizing counting as a primary arithmetic strategy. To bring clarity to the long-term effects of these differing approaches to arithmetic learning, we implemented an intervention program based on the idea of using a structural approach to addition and subtraction and particularly emphasizing part-whole relations and the ten-base unit in four Grade 1 classes in Sweden during one school year. The research question we answer in this paper is: What are the effects of a structural teaching approach on students' learning of addition and subtraction?

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 83-90). PME.

LEARNING ADDITION AND SUBTRACTION BRIDGING THROUGH TEN

Experiencing numbers as part-whole relations is considered to be critical for development of arithmetic skills (Cheng, 2012; Resnick, 1983), since being aware of part-whole relations may allow students to make use of powerful strategies, such as decomposition (c=a+b), commutativity (a+b=b+a), and the complement principle (a+b=c then c–a=b), when solving addition and subtraction tasks (Zhou & Peverly, 2005). Piaget (1952) states that "[a]dditive and multiplicative operations are already implied in numbers as such, since a number is an additive union of units, and one-one correspondence between two sets entails multiplication. The real problem, if we wish to reach the roots of these operations, is to discover how the child becomes aware, when he discovers that they exist within numerical compositions" (p. 161). Empirical research has however shown that this discovery of numbers' part-whole relations and how to operate with them, especially when bridging through ten, is not easily done by young students. A substantial number of students frequently and successfully use counting strategies instead of retrieval-based strategies for simple addition (Hopkins, Russo, & Siegler, 2020). Furthermore, there are hardly any reports of students using for instance the "subtraction by addition" strategy (e.g., Heinze, Marschick, & Lipowsky, 2009; Selter, 2001), which is considered to be a powerful and sustainable way of completing arithmetic tasks, building on conceptual understanding of numbers' part-whole relations. The scarce use of retrieval-based and structure-based strategies among students has been explained in terms of a lack of understanding of the underlying complement principle between addition and subtraction, i.e., if students do not understand that one part–part–whole combination refers both to the components of a subtraction problem a–b=c and to its complementary addition problem c+b=a, it hinders their discovery and use of the subtraction by addition strategy and other structure-based ways of reasoning (Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). In a study on three-digit addition and subtraction, Selter (2001) concluded that many students appear to be "blind" to the relations between given numbers in a task, and execute a stable pattern of methods and strategies, regardless of the task. Selter further suggests that students' sense for number relations does not develop independently of instruction. Consequently, students should be encouraged to consider the nature of the problem type before trying to solve the problem. Young students' ways of experiencing or "seeing" a task have also been shown in a recent study (Kullberg & Björklund, 2020) to be related to their developing arithmetic skills. Those who experienced numbers represented *both* as one set (e.g., a finger pattern of five fingers on one hand) and a composed set (e.g., composed of two and three fingers on different hands) were more likely to develop known number facts from a long-term perspective. Thus, there does seem to be more to solving arithmetic tasks in powerful ways than making use of certain strategies – it seems to include a way of experiencing the task and numbers in the task as relational.

THE INTERVENTION

The intervention was built on findings from previous studies with 5-6-year-olds (Kullberg, Björklund, Brkovic, & Runesson Kempe, 2020) and on principles from phenomenography and its extension, variation theory (Marton, 2015). Results from these studies demonstrate that there are certain aspects that must be discerned to be able to experience and handle elementary arithmetic: modes of number representations, ordinality, cardinality, and part-whole relations (the latter has four subcategories: differentiating parts and whole, decomposing numbers, commutativity, and inverse relationship between addition and subtraction). The discernment of these *critical aspects* presupposes an experience of variation in the focused aspect against a background of invariance. A goal was that students would be able to structure numbers and solve tasks like 15–7= using part-whole relations and ten as a benchmark, at the end of Grade 1. Finger patterns, as a way to represent numbers, were used by teachers and students from the start, and played an important role in the intervention to show numbers and part-whole relations. The teachers were told to avoid single unit counting in their teaching. Throughout the intervention, the teachers elicited parts and wholes of number relations. Aspects assumed to be critical for student learning, identified from Interview 1 and previous research, that were elicited in activities were: 1) Seeing numbers (seeing finger patterns or an amount of objects without counting), 2) Understanding the ordinal and cardinal aspect of numbers, 3) Experiencing that numbers can be partitioned, 4) Understanding that numbers can be represented in different ways (e.g., by different finger patterns), 5) Experiencing place value, 6) Experiencing operations as part-whole relations, 7) Experiencing commutativity in addition, but seeing that it is not true for subtraction, 8) Experiencing the complement principle (a+b=c, c–a=b), 9) Seeing 10 as a benchmark in an operation, 10) Seeing parts in parts, 11) Experiencing counting "up to ten" *or* "down to ten" when solving a subtraction task bridging through ten (e.g., 13–5= could be solved as 5+**5+3**=13, or 13–3–2=**8)**. Ten activities were enacted several times in each class during the eight-month-long intervention and were video recorded, so it was possible to analyze whether the aspects were elicited in the activities. Two of the activities are described briefly to exemplify features of the intervention. The activity "Partition numbers", into two and three parts in many different ways, was a key activity, since this was seen as foundational for being able to solve addition and subtraction tasks bridging ten.

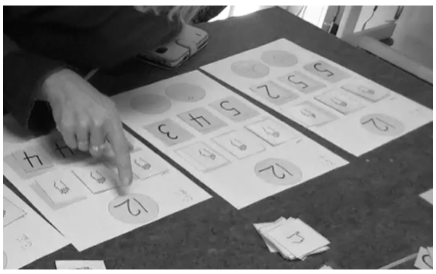

Figure 1: The same number (12), partitioned into two and three parts, was made possible to experience simultaneously by means of numerals and finger patterns.

Hence, in order to subtract 12–7=, students need to be able to partition one part (7) into two smaller parts (2 and 5) in order to bridge 10. Figure 1 shows how students (in pairs) work with partitioning 12 into two and three parts (with numerals and with pictures of finger patterns), in different ways on the same assignment. This makes it possible for the students to experience how the same number (invariant) can be partitioned differently (parts varied). Another activity "Subtraction bridging through ten using the 15-snake" involved discussions about tasks bridging ten, 13–8= and 13–5= (as well as a task not bridging ten, 13–2=), and the part-whole relations illustrated on the board using ten as a benchmark. Based on discussions of how the students solved the tasks, primarily two different ways ("up to ten" *and* "down to ten") of bridging ten were made possible to experience (13–8= as 8+**2+3**=13 and 13–3–5=**5),** where the subtrahend (8) and the difference (5) were shown as composed/decomposed units at different places (varied) on the 15-snake on the board, although the task remained invariant.

Figure 2: The subtrahend in 13–8=, was made possible to perceive as 13–3–5=.

METHOD

Four experienced teachers from three different schools and their students participated in the Intervention group. The teachers met three researchers every other week during a period of eight months to plan, analyze and revise lessons in the intervention. The teachers enacted the collaboratively planned lessons in their classes and video recorded them. Three experienced teachers from two other schools and their students were part of the Control group. One of the researchers met with the teachers from the Control group (six times) and video recordings from their teaching were collected and discussed at meetings in their schools. The participating teachers and the legal guardians of the students had signed a written consent for participation. In this paper, results of analysis of 363 video-recorded interviews, from three points in time (before, immediately after, and one year after the intervention), conducted individually with each student are reported (Intervention group N=86, Control group N=35). Each interview lasted for 20-30 minutes. The interview tasks were a mix of orally presented story problems (8+5=, 15–7=, 6+_=13, 24–_=15, e.g., A baker baked 24 buns, and left the buns on a tray. When he came back there were only 15 buns left. How many buns were missing?) and tasks with numerals (11=5+_, 6+_=13, 16+_=23, 14–_=6). Follow-up questions were posed to the students on all tasks, e.g., "How do you know it is x [the answer]?" and "Please show me what you did when you solved the task?". The interview tasks were coded in two ways: for correct and incorrect answers and according to the strategy used (*structure* or *single unit counting*). For example, we coded it as single unit counting when a student counted backwards: "15–7, fourteen, thirteen, twelve…". It was coded as structure when a student used larger parts (than ones) of number to arrive at an answer, saying e.g., "I have 15 and take away 7, then I have 3, because I thought about the 10 there, then I have 3 left and 5 from the other [5 in 15], and then I take 5+3".

RESULTS

The Intervention and Control groups showed similar results on eight tasks on addition and subtraction bridging through ten before the intervention started (Interview 1). In order to test the effectiveness of the intervention, we conducted mixed ANOVA analysis, with Interview occasion (Interview 1, Interview 2, Interview 3) as within- and Group (Intervention-Control) as between-group factor. The interaction (Figure 3) between Group and Interview occasion was significant ($F(2,239) = 4.579$, $p=.011$) showing that the profile of change in results was different for control and intervention groups, i.e. that the Intervention group results over time increased more than those of the Control group. This suggests that the intervention had a positive effect on the results of the Intervention group.

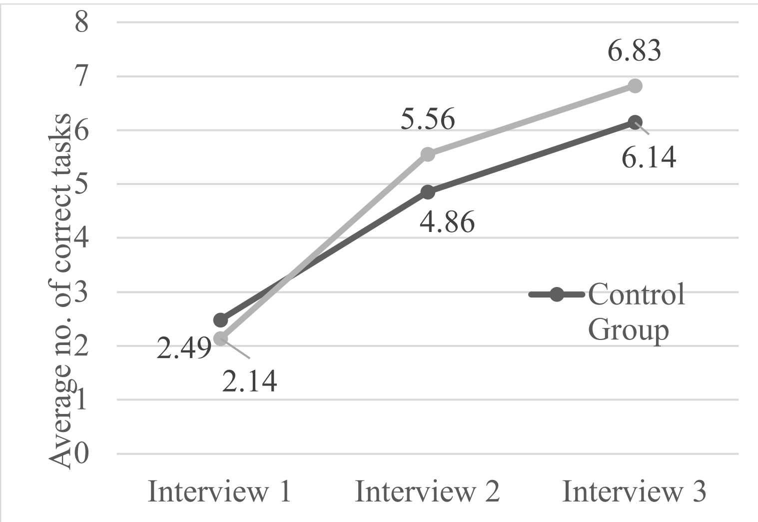

Figure 3: The average number of correctly solved tasks for Intervention and Control groups across three interview occasions.

Figure 4 shows the percentage of students with correct answers from Interview 3 on ten items for the Intervention and Control groups. We see a small difference in percentage of correct answers between the groups on 8+5= and 15–7=, two straightforward tasks in a lower number range that were used in all three interviews. However, when the number range increases, more pronounced differences between the Intervention and Control group are visible. The largest differences are found on the subtraction tasks, 32–25= and 83–7=, solved by 51% and 73% of the students in the Intervention group, compared to 31% and 49% of the Control group. We also find large differences on the addition tasks 15+17= and 28+44=, and items with a large subtrahend, 204–193= and 132–78=.

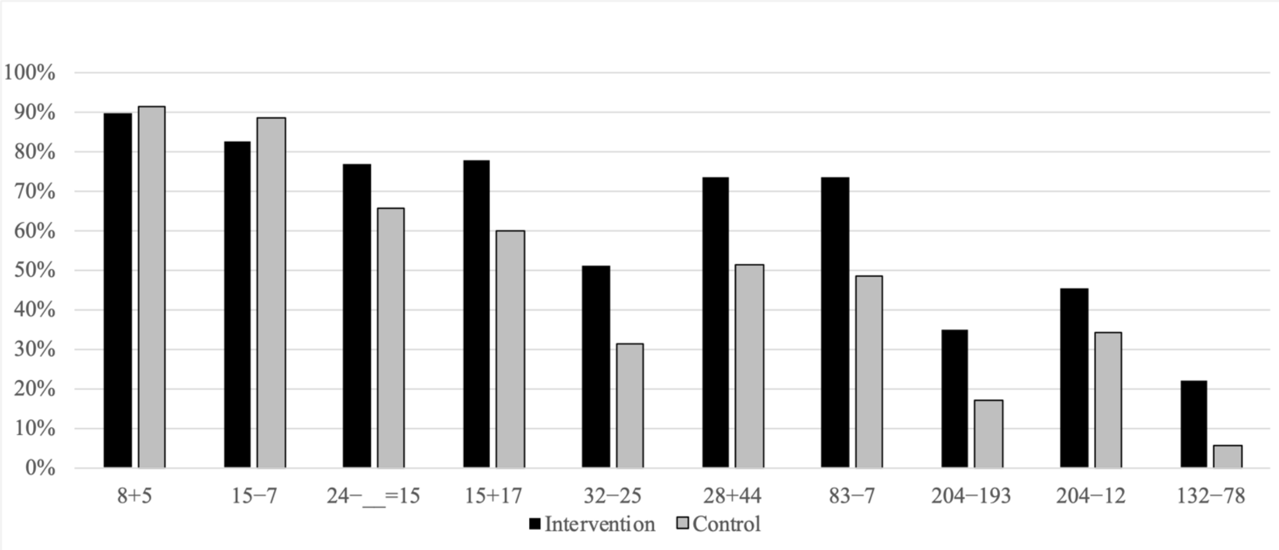

Figure 4: Percentage correct answers on orally presented story problem tasks (first five items) and tasks with numerals (last five items) in Interview 3.

Figure 5 shows how students solved 83–7=. It was coded as structure if a student was able to partition 7 into two parts to solve the task (83–3=80, 80–4=76). We found that more than 60% of the students in the Intervention group used structure to solve the task and ended up with the correct answer, compared to about 30% in the Control group.

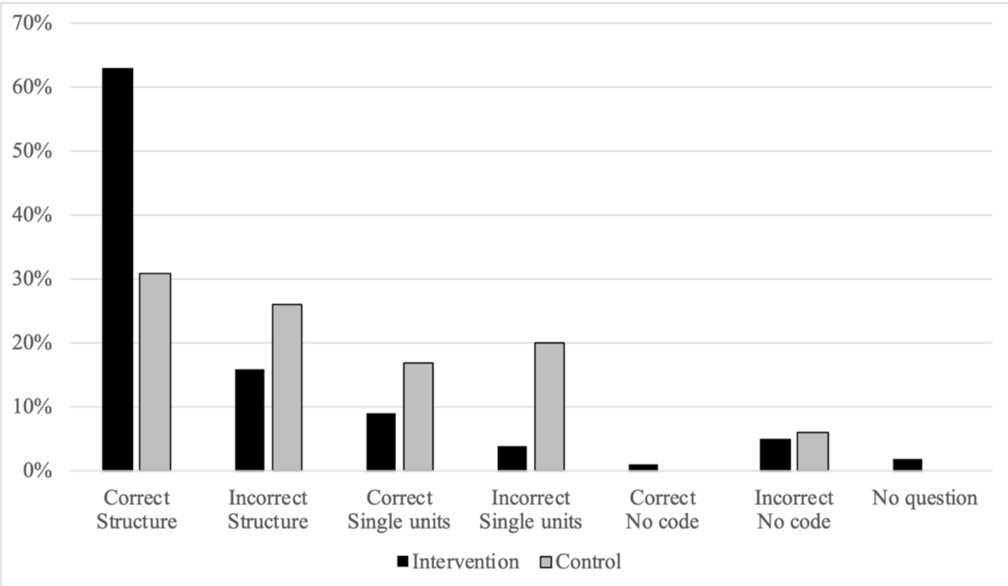

Figure 5: How students in Intervention and Control groups solved 83–7, using structure or single unit counting in Interview 3. Not possible to code=No code.

There were also students who tried to structure the task but did not end up with a correct answer as their first answer, or not at all (Figure 5, Incorrect structure). For example, Mia (Intervention group) first answered “66”, but when explaining how she solved the task she changed her mind. “I had 83 and then I took away 3, and then I had only 80 left, then I took away 4, and then I saw that it was, no, 76”. Hence, she was able to partition 7 but did not get the ten right from the start. Students from both groups counted in single units backwards to solve the task, 9% from the Intervention group and 17% from the Control group, and succeeded in solving the task. However, 20% of the students in the Control group used single unit counting and failed to solve the task. This is most likely due to difficulties counting seven steps backwards and at the same time keeping track of the counting sequence.

DISCUSSION

Our research question concerned what effects a structural teaching approach can have on students’ learning of addition and subtraction. We suggest that the improvement in results on student learning outcomes for the Intervention group in Grade 1, and on more difficult tasks in Grade 2, is most likely an effect of the intervention. Students in the Intervention group were taught to structure numbers, and used this knowledge to solve tasks, in higher number ranges also. When encountering a higher number range, students in the intervention group seemed to be able to generalize what they had learned about (e.g., number relations, decomposition of numbers and using ten as a benchmark) in a lower number range. Although more students in the Control group (89%) were able to solve 15–7= compared to the Intervention group (83%), more students in the Intervention group (72%) were able to solve 83–7= compared to the Control group (49%). We also find a greater span of strategies used in the Control group than in the Intervention group, where a majority of the students used structure. We find it striking that almost 40% (20% incorrect) of the students in the Control group used single unit counting for solving a task like 83–7= in Grade 2. Students using single unit counting most likely do not experience numbers in the same way as students who are able to use structure to arrive at the answer. The results of our intervention suggest that learning to experience numbers as structural relations from the start seems to be helpful. Our findings support previous studies suggesting that a structural approach is beneficial for student learning (Ellemor-Collins & Wright, 2009; Venkat, Askew, Watson, & Mason, 2019). In addition, our findings indicate that counting as an arithmetic strategy may hinder students’ ability to solve tasks in a higher number range (cf. Cheng, 2012; Hopkins et al., 2020). Further research is needed to investigate how students’ ways of solving arithmetic tasks affect future learning.

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"HELPING LEARNERS" – PRE-SERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF LEARNING SUPPORT THROUGH THE LENS OF THEIR SITUATED NOTICING – A VIGNETTE-BASED STUDY

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Learning support is a key aspect of the teaching profession. In particular, promoting mathematics-related learning is a goal when mathematics teachers respond to students' questions or comments during their work on tasks. "Helping" learners in this sense should be (A) adaptive to the learner's needs and (B) carry the potential to elicit further learning - both core aspects of learning support informed by a teacher's noticing in the interaction with the learner. Pre-service teachers' noticing in this area can be assumed to be still under development and there is hence a need of empirical studies investigating the learning support they suggest to provide. Consequently, this paper presents results from a vignette-based study with n=116 pre-service teachers, providing insight into their difficulties and also perspectives for improvement.

INTRODUCTION

Mathematics teachers should be able to help learners in building mathematical knowledge and in using such knowledge for solving tasks. "Help" in this sense can be described as individual *learning support* (e.g. Krammer, 2009; cf. Schnebel, 2013). A reaction to the learner should at least (A) take into account the specific individual needs of the learner (i.e., *adaptiveness aspect* of learning support) and (B) carry the potential of facilitating and/or eliciting further individual learning (i.e., *progress aspect* of learning support). Providing individual learning support hence requires mathematics teachers to analyse the learner's mathematical thinking in order to identify potential individual difficulties, in order to find stimuli for further learning and understanding in an adaptive way and to communicate them to the learner. Such *analysing* (Dreher & Kuntze, 2015) can be understood in the framework of teacher noticing (Amador et al., 2021; Choy, 2014; Fernández, & Choy, 2020) as a *knowledge-based reasoning* process (Sherin et al., 2011; Berliner, 1991; Dreher & Kuntze, 2015). Accordingly, the teacher has to notice possible difficulties in the student's understanding, such as incomplete conceptual knowledge, for instance, and to identify a reaction which can support the individual learner to build up or strengthen the mathematical knowledge needed. For this complex and multi-step process, professional knowledge (Shulman, 1986; Kuntze,

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 91-98). PME.

2012, cf. Kuntze, Dreher, & Friesen, 2015) is needed, including *content knowledge* (CK) and *pedagogical content knowledge* (PCK).

All in all, being able to provide effective individual learning support in classroom situations can be seen as a key aspect of mathematics teacher expertise. The requirements of adaptiveness and (content-specific) progress as introduced above show that the particular classroom situation plays a key role – also for research which aims at finding out about how competent teachers are in providing individual learning support. Vignette-based research can help to investigate such situation-specific noticing and to respond to a need of empirical studies in this area. In particular, evidence about *pre-service* teachers' analysis and their ability of providing adequate learning support is highly relevant, in order to find out about professional development needs and to describe pre-service teachers' growth empirically.

Consequently, this paper focuses on whether and how pre-service teachers can provide learning support in a learning situation in the context of divisibility, which is a content area from the pre-service teachers' training in a university course. Through the lens of the pre-service teachers' noticing, i.e. analysis and their suggested learning support, the results can also give insight into how they conceive of "help" to learners.

THEORETICAL BACKGROUND

There is a large consensus that mathematics teachers' reactions to learners' questions or comments should support them in their further learning (e.g. Krammer, 2009; Schnebel, 2013), such reactions should hence respond adaptively to learners' needs and provide them with stimuli for their further construction of mathematical knowledge and understanding. Research about teachers' noticing and analysis (e.g. Sherin, Jacobs, & Philipp, 2011; Amador et al., 2021; Choy, 2014; Fernández, & Choy, 2020; Dreher & Kuntze, 2015; cf. Kersting et al., 2012) has focused continuously on aspects of mathematics teacher expertise related to these requirements: in such research, the teachers' situation-adaptive knowledge-based reasoning and decision-making related to possible situated reactions is typically in the focus. Methodologically, related empirical studies mostly use representations of practice (Buchbinder & Kuntze, 2018), i.e. vignettes (Skilling & Stylianides, 2020; Kuntze et al., in press), for eliciting the teachers' noticing. Beyond a situated scope, there are studies which describe ways of inferring from teachers' situated noticing to more general aspects of their expertise (e.g. Kersting et al., 2012; Friesen & Kuntze, 2016).

For successful noticing, teachers need to draw on their professional knowledge (Shulman, 1986); their instruction-related views, which are also considered as components of their professional knowledge (Kuntze, 2012), can interfere in this process. For providing adaptive learning support, both CK and PCK is needed in order to mathematically analyse requirements of a task, a learner's thinking, and possibilities to provide learning support (Vondrová & Žalská, 2013). In the noticing process, teachers can draw on professional knowledge components from different levels of situatedness (Dreher & Kuntze, 2015; Kuntze, 2012). Figure 1 gives a model-like overview of noticing related to providing learning support in the sense of the framework introduced above. In an analysis cycle as described in Kuntze and Friesen (2018), the task requirements, the learner's thinking (Fernández et al., 2018), and potential difficulties or needs of the learner have to be analysed against the background of the teacher's professional knowledge and situation-related observations. Based on this analysis cycle and again drawing on professional knowledge, possible reaction(s) have to be identified and a reaction which corresponds to an optimal adaptive learning support (Hardy et al., 2019) has to be chosen.

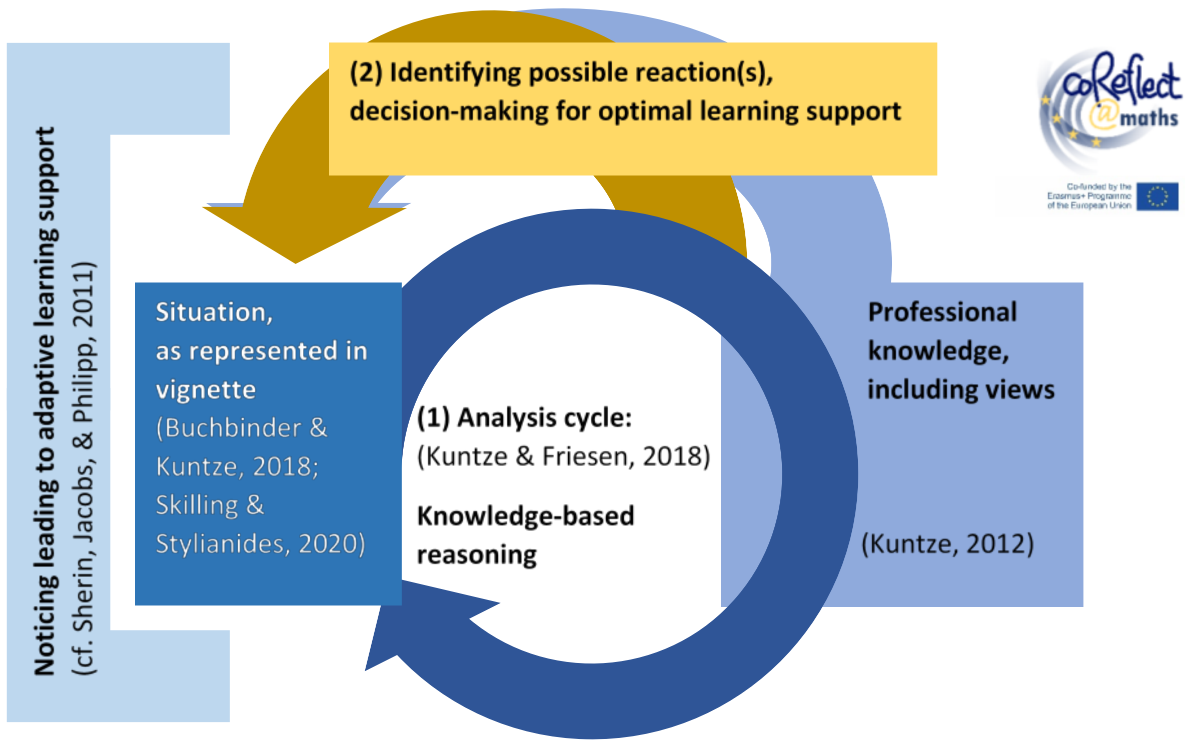

Figure 1: Model-like overview of noticing related to providing learning support.

Teachers' views related to "helping learners" i.e. to forms of learning support are assumed to influence this process and its results. When a learner struggles with finding a correct solution to a problem, learning support may consist of *directly providing information* such as the task solution, parts of it or a standard solution algorithm or rule, so that the learner can learn from this solution, rule or algorithm. However, learning support may also take the form of *feedback*, for example telling the learner that her/his reasoning is not correct or providing a counter-example, with or without indicating a further pathway for a correct solution. Moreover, rather *procedural* learning support can focus on stimuli to the learner for discovering a correct solution on her/his own, such as encouraging the learner to try out specific strategies or to challenge and check her/his thoughts on her/his own by using example values. A teacher's preference for such different forms of learning support may indicate this teacher's views about learning support. For pre-service teachers in particular, such views may influence in which direction they develop their professional knowledge and instruction-related experience further (cf. e.g. Kuntze, 2012). In conclusion, mathematics teachers' views can be reflected in their noticing and analysis of vignettes.

RESEARCH INTEREST AND RESEARCH QUESTIONS

In particular for pre-service mathematics teachers – who are in the process of their professional development – relatively little is known from vignette-based empirical studies about how they provide learning support and to which extent they encounter obstacles when having to "help" learners, such as lacking CK. Such vignette-based research can not only indicate potential pre-service teachers' professional development needs, but also inform vignette use in pre-service teacher education and related evaluation research. This corresponds also to the aims of the Erasmus+ project coReflect@maths ("Digital Support for Teachers' Collaborative Reflection on Mathematics Classroom Situations", www.coreflect.eu).

For this reason, this study aims at analysing pre-service teachers' answers to a vignette in the content area of divisibility with respect of the following research questions:

(1) To what extent are pre-service teachers able to provide learning support in a vignette-based setting showing a fictitious situation in the content area of divisibility?

(2) What role does their content knowledge (CK) play in this context?

(3) In which form do they suggest to provide learning support and is it possible to infer to their conceptions of "helping learners" from the findings?

DESIGN AND METHODS

In order to answer the research questions introduced above, a vignette-based questionnaire was designed by the team of co-authors of this paper, using representations of practice (Buchbinder & Kuntze, 2018). For the vignettes, the style of concept cartoons (Samková, 2020) was chosen, in order to be able to present different learners' thoughts and to implement a variety of learning support requirements. The instrument focused on problems from the content area of divisibility, in line with the learning content of the target group. In this way, it could be assured that beyond their prior CK, all pre-service teachers had been given a set of opportunities for CK-related learning in the topic area of divisibility beforehand. One of the vignettes in concept cartoon style is shown in Figure 2. The research instrument with this vignette was administered to n=116 pre-service teachers preparing to teach at primary schools (18% male, all in their first year of studying mathematics) enrolled at a University of Education in southern Germany.

This study is part of a larger set of empirical studies carried out in the framework of the Erasmus+ project coReflect@maths. In the case of the results reported here, more analyses will be carried out in the future on the base of more data, also from groups of Spanish and Czech pre-service teachers who had worked with vignettes from the questionnaire as well, in the framework of an international research approach in coReflect@maths.

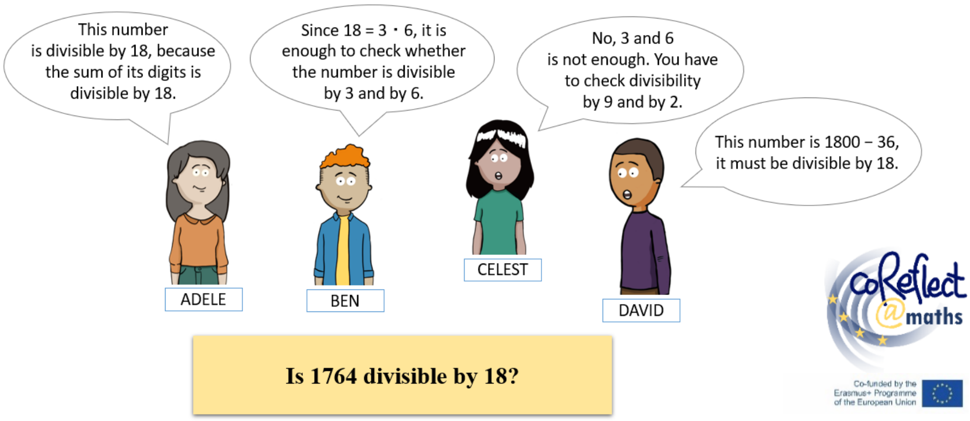

Figure 2: Vignette in the style of a concept cartoon (translated).

The participating pre-service teachers were first asked to analyse the thinking of the persons in the cartoon (Fig. 2). Then, they were asked to think of a reaction: The key vignette question for the analysis corresponding to the research aims of this paper was "How could you help the student teachers (1) to correct their answers or (2) to improve their argumentation?". In this way, the questions required analysing the vignette learners' thinking and providing the vignette learners with adaptive individual learning support.

The vignette in Figure 2 contains two answers with a mistake (Adele, Ben) and two answers that can be interpreted as incomplete (Celest, David) in the sense that the corresponding argumentations can be improved. As the above-mentioned question requires that "help" should be provided to all persons represented in the vignette, the learning support (A) should fit to the needs of the respective person (*adaptiveness aspect* of learning support) and the (B) "help" should lead further on the content level (*progress aspect* of learning support). Consequently, a top-down coding (cf. Mayring, 2015) was applied according to these two aspects: For each vignette person,

- code (A) describes whether there is an adaptive content-specific connection of the answer with the given vignette person's comment (dysfunctional attempts of adaptive connections with an observable aim of connecting to the cartoon character's thinking were coded as such, e.g. in case of mathematically inadequate connections or (partial) misinterpretations of the cartoon characters' thinking),
- code (B) describes whether the content of the answer could somehow advance the vignette person's learning or understanding.

Additionally, the form of suggested learning support was coded in a bottom-up approach (cf. Mayring, 2015), in which a set of different categories emerged, which will be reported together with the respective frequencies in the results section.

RESULTS

Figures 1 and 2 display results of the coding introduced above and the relative frequencies of the respective categories. Research question (1) focuses on the extent to which the 116 pre-service teachers were able to provide learning support to the four vignette persons. The results indicate that a considerable number of pre-service teachers struggled with CK difficulties, which inhibited both the adaptiveness (code A) and progress aspect (code B) of the individual learning support they suggested. For research question (2), there is more than half of the answers with evidence of CK difficulties, except for answers to David with a lower frequency of CK difficulty codes.

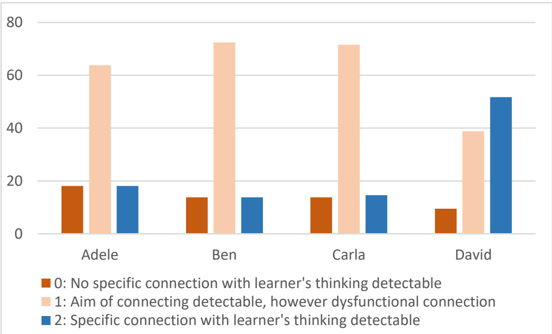

Figure 3: Relative frequencies for Code (A) (in per cent).

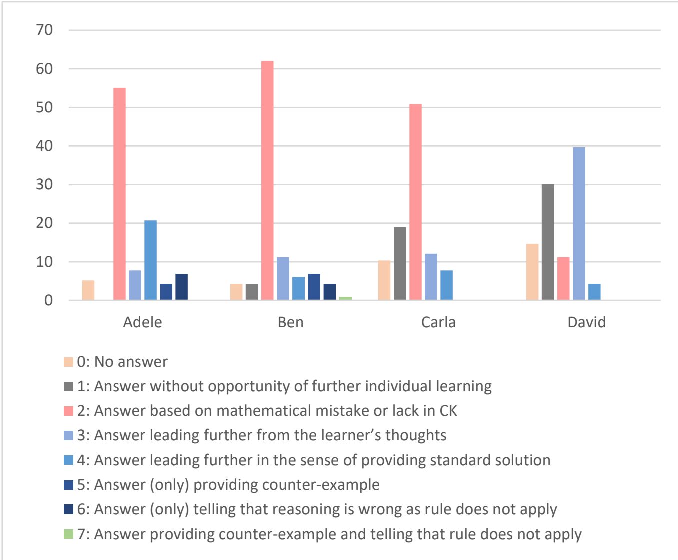

Figure 4: Relative frequencies for Code (B) (in per cent).

Research question (3) concentrates on forms of suggested learning support. In particular the results shown in Figure 5 indicate that the pre-service teachers mainly chose forms of presenting or providing information, even if incorrect.

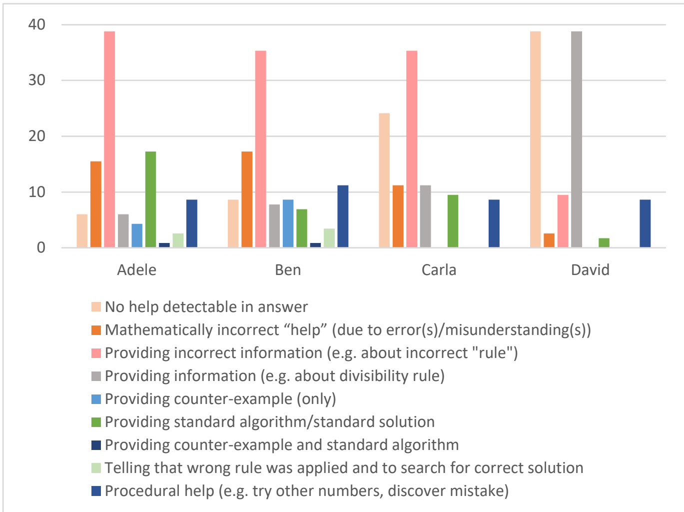

Figure 5: Relative code frequencies for the form of learning support provided (in per cent).

Only in around 10% of the cases, procedural help, emphasising a comparably more active role of the learner, was suggested. The large majority of answers falls into categories that reflect a conception of "help" that consists in providing information about rules, standard solutions, or feedback in the form of counter-examples.

DISCUSSION AND CONCLUSIONS

Even if the evidence should be interpreted with care, given that the sample is not representative for German pre-service teachers, the research questions could be answered and provide insight into the participants' noticing and professional knowledge, especially as far as CK-related needs for professional development are concerned. More than half of the answers to the cartoon characters Adele, Ben and Carla were marked by CK deficits, so that the ability of providing learning support appears in need of improvement for many pre-service teachers.

As far as forms of learning support are concerned, the data shows a predominance of telling the learners about rules (including attempts with evidence of mathematical, i.e. CK deficits) or standard solutions and algorithms. This might be a consequence of the pre-service teachers still being in a learning process related to divisibility contents, possibly leading them to rather focus on evaluating the vignette persons' thinking and on newly learned rules and standard procedures. The evidence however also might reflect the pre-service teachers' conceptions of "helping learners" through the lens of their noticing: For many of them, "help" might rather consist in directly providing information or hints related to procedures than in stimulating the learner's thinking and activities in the direction of learner-centred experience and reasoning. This differs from conceptions of learning support in literature (e.g. Schnebel, 2013; cf. Krammer, 2009). In this sense, the results also point to needs in the development of pre-service teachers' instruction-related views (cf. Kuntze, 2012). Future further analysis also of additional data from Spanish and Czech pre-service teachers promises further insight here, also on a cross-cultural level.

As far as methodological approaches are concerned, the study highlights the potential of vignettes to elicit mathematics teachers' noticing: On a situation and content-specific level, pre-service teachers' analysis of learners' thinking and decision-making related to learning support can be made accessible to research and evaluation by teacher educators by asking the pre-service teachers to comment on vignettes. In line with the potential of vignette-based formats for pre- and in-service mathematics teacher professional development, the project coReflect@maths will further focus on corresponding research and development needs.

Acknowledgements

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UNDERSTANDING THE ‚AUXILIARY TASK' CONCEPTUALLY – DISCRETE VERSUS CONTINUOUS CARDINAL OBJECTS

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For conceptually understanding the ‚Auxiliary Task', learners have to understand the compensation process. Yet, since the strategy is highly complex compared to other mental calculation strategies, an important question is how the conceptual understanding of the strategy can be fostered and for this purpose, ordinal as well as cardinal learning environments were developed and evaluated in a design-based-study (which is part of the mixed-methods MaG-Project). Prior analyzes showed that especially the cardinal learning environment leads to more thorough conceptual discourses. In this paper, qualitative insights into the use of specific forms of cardinal representation – discrete versus continuous – and its interpretations by four 11-year-old German primary school learners' will be given.

STARTING POINTS AND THEORETICAL BACKGROUNDS

The ‚Auxiliary Task' and its relevance as a mental calculation strategy

In the last decades, there has been a shift in the perception of the importance of mental calculation strategies: Mental calculation strategies are not seen as mere pre-steps for the full algorithms anymore but have an important role in the emergence of flexible calculation processes (Heinze, Marschick & Lipowsky 2009). At the same time, a problem of over-emphasizing specific mental calculation strategies is visible: Students tend to use the HundredsTensUnits(HTU)-strategy, where they calculate in an order being structured by the hundreds, tens and units of the first and the second number, or the Stepwise-strategy, where they calculate by dividing the second number into hundreds, tens and units (see Selter 2001). Different and more complex mental calculation strategies like the ‚Auxiliary Task' are mostly not activated by learners. The ‚Auxiliary Task' differs from the HTU- and Stepwise-strategy insofar as that learners have to utilize compensation rules and have to see specific numerical properties before using the strategy, leading to a so-called ‚analytical noticing' (Threlfall 2002): When calculating 332 – 118 for example, learners using the ‚Auxiliary Task' have to recognize the proximity of the 118 to 120, thus modifying the second number by rounding it up to 120 through adding + 2 and compensating the modification at the interim result by adding back what was taken away too much (+ 2) (see figure 1).

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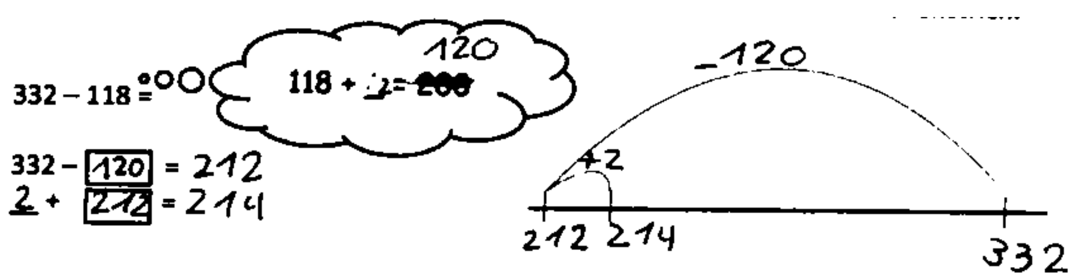


Figure 1: The ‚Auxiliary Task' as sequenced task and non-numbered line (Kuzu 2021).

Thus, the ‚Auxilary Task' is complex in terms of processing and mental noticing since the learners have to see the option to modify and compensate, but it also mediates a crucially different view on numbers: They have to be perceived as flexible, modificable objects, where one is allowed to change them, *if* every modification is compensated (equally) by adding the modification amount to the (interim) result – a different view on numbers which is fostering (pre-)algebraic aspects like the perception of indeterminant, flexible parameters (see Kuzu & Nührenbörger 2021).

Cardinal versus ordinal ways of teaching the ‚Auxiliary Task'

From a didactical standpoint, the teaching of the ‚Auxiliary Task' thus seems to be of high relevance in the transition from primary to secondary school, but there is a need for explorative research especially concerning the design-related question on *how* to teach the ‚Auxiliary Task' conceptually (see ibid.): In most learning environments, the conceptual understanding is fostered by using ordinal representations (f.e. through the use of non-numbered lines, see figure 1) and only very few learning environments do utilize cardinal means of representation, although conceptual aspects – like the compensation process – can be represented through cardinal manipulatives in a more meaning-related way, for example when taking-away and putting-back an equal amount of objects (see Britt & Irwin 2011). This leads to a specific, design-related research gap: The development and evaluation of a learning environment utilizing a cardinal representation of the ‚Auxiliary Task'. Prior analyzes conducted in Kuzu & Nührenbörger (2021) indicate specific hurdles on the conceptual as well as linguistic level: Interpreting and explaining a cardinally represented compensational process is highly complex due to the sequence of steps, which have to be visualized in a coherent, intuitive and relational way, but it is worthwhile to do so since interestingly, the cardinal representation led to more and thorough conceptual discourses. In comparison, the ordinal representation led to a faster transition to procedural discourses (with non-viable notions not being discussed as much as with the cardinal representation) (see ibid.). What is yet unclear is the effect of using *different* cardinal representations of the ‚Auxiliary Task' – discrete versus continuous – since both forms of representing a cardinal amount are of relevance in primary school (see Greer 1992). This is the research question to be focussed in this paper: *How do learners interpret discrete versus continous ways of cardinally representing conceptual facets of the ‚Auxiliary Task' in the context of the designed learning environment?*

METHODS OF THE LEARNING-PROCESS STUDY

Research context and data corpus of the study. The research question was pursued in a design-based-study (see Prediger, Gravemeijer & Confrey 2015) that was part of the larger mixed-methods project MaG. The aim of the study was to develop a learning environment fostering the conceptual understanding of the ‚Auxiliary Task' for all four arithmetics and to generate local theories about the effects of the design principles and design elements by analyzing students' learning processes (see ibid.). In groups of 2-3 learners and with an iterative research design, a learning environment consisting of two 60 minutes sessions was developed and conducted. The data corpus consisted of n = 18 learners from age 11-14 and at the end of the second iteration, a total of 520 minutes of video material was cumulated (the learners being analyzed in this paper were 11 years old). The use of a continuous cardinal representation was a design element of the learning-environment from the first iteration, whereas the discrete representation was an adaption made for the second iteration.

Methods for qualitative data analysis. The transcripts were analyzed with respect to students' epistemological processes when interpreting and explaining the ‚Auxiliary Task' with cardinal manipulatives and representation. For this purpose, two analytical steps were followed: In the first step, a turn-by-turn interpretative analysis was conducted, in which the researcher analyzed students' utterances and interactions being based on the research questions of the study. These analyzes were discussed in teams of researchers and the main aim was to get carfully reflected insights into students' processes of interpretation (Schütte, Friesen & Jung 2019). In a second, complementary step, an epistemological analysis was conducted for deepening the analysis in specific transcript sections with so-called epistemological triangles (Steinbring 2006). In this paper, mainly the turn-by-turn analyzes will be focussed and sign-related utterances will be interpreted verbally without depicting the full epistemological triangles.

The design of the learning environment. The learning environment was designed based on two design principles: 1. Fostering of a conceptual understanding through a content-and-language-integrated approach preceding procedural calculation and the 2. Fostering of generalization processes through demanding verbal explanations. For the research question of this paper, especially the first design principle is of relevance since the use of different cardinal representations for fostering the conceptual understanding is focussed (see figure 2).

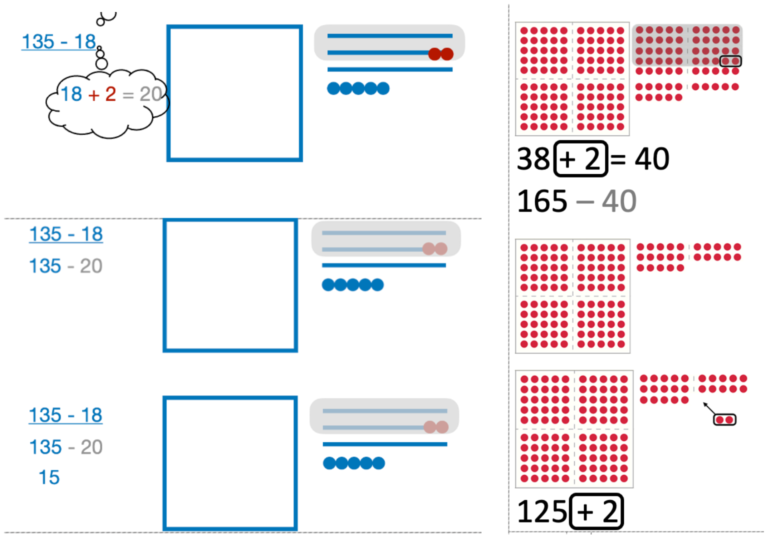


Figure 2: The continuous-cardinal representation from iteration 1 (left side) versus the discrete-cardinal representation from iteration 2 (right side).

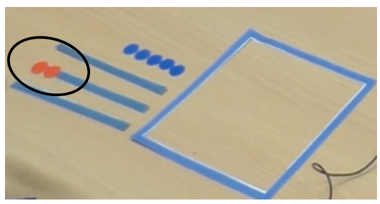
On the left side of figure 2, a mainly continuous cardinal representation with continous tens and hundreds and discrete ones being based on Oehl (1962) is visible. The representation focussed the rounding-up of the second number and three colors were used: Blue for the task, red for the added number and grey for taking away the rounded-up number. On the right side, a discrete cardinal form of representation, where tens, hundreds and ones are discretely depicted, is visible and was used with a similar color coding (red for the task, a black rectangle for the added number and grey for taking away the rounded-up number). The cardinal objects in both variants are visible in an iconized form on the worksheet, but they were also available in form of manipulatives and for discussing the meaning of the ‚Auxiliary Task', an enactive approach with manipulatives was the first step to re-understand the fictive student Max' use of the ‚Auxiliary Task'.

EMPIRICAL INSIGHTS INTO STUDENTS' INTERPRETATIONS

Sequence 1: Insights into the interpretation of the continous cardinal material

The students S1 and S2 try to explain the ‚Auxiliary Task' for the task 135 – 18 with continuous cardinal objects. The task sheet with the iconic representation of Max' procedure (see figure 2) is laid visibily on the table. They have available the continious cardinal material consisting of a hundreds-square, tens-lines and ones-dots.

2	I	Okay, sorry- let's put it on the table again as it was earlier so that it looks as in Max' picture *[the Interviewer had nudged the objects by mistake]*.
3	S1	Yes *[puts two ones-dots on the right side of the second tens-stripe]*
4	I	And then, and the we have learned not to relocate the material *[smiling]*. Yes, like that, thank you.
5	S1:	Two he takes on it. But then he takes away



these two *[pointing at the two red dots]*. And the twenty, he takes away also.

6 I: Aha.

[Turn 7-13: Organizational discourse. Continuation in Turn 14.]

14 S2 So, he also takes away these two? *[pointing at the last two blue dots]*

15 S1 No. He has, one, these are five. Then he has seven *[pointing at the five blue dots first and then on the two red dots]*. And then, when he as eighteen, plus two makes twenty. Then he takes away the twenty. And then it is ten minus two I believe.

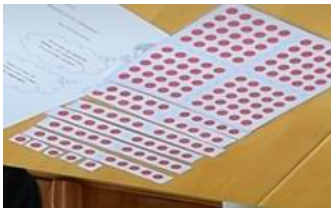
From Turn 3 to 19, the learners S1 and S2 try to interpret the continuous cardinal material. In Turn 3, they put it on the table as it was depicted on the task sheet (see figure 2). After that, in Turn 5, a first individual interpretation of the cardinal material becoms visible: S1 interprets the two blue-tens lines with the two red dots on it, as visible on the picture in Turn 5 (the grey box being not visible very good), as twenty, where *"two he takes on it… then takes away these two. And the twenty, he takes away also"*. It seems that she infers a double-subtraction process, a non-related tie between the two red ones-dots and the two blue tens-lines since she uses a paratactic structure in her sentence and verbalizes the sequence with the language means "then" and "and", indicating non-related processes instead of interpreting the two red dots (the rounding amount) as an integral part of the twenty (18 + 2 = 20), the rounded-up number. S2 seems to be irritated and asks in Turn 14, if he (possibly the fictive student Max from the task) *also* takes away the red ones-dots. S1 at first neglects the presumption from S2 in Turn 15 and hints at the total number of dots on the table (five blue dots and two red dots), but after that she gives a similar answer to her explanation from Turn 5: That it is eighteen plus two, which makes twenty to take away, *and then* ten minus two. Especially the last part of her utterance, with the emphasis of a last following step ("and then"), where she describes the taking-away of *another* two (10 – 2) beside of the two being taken away within the twenty, shows again a double-subtractional notion.

From an epistemological perspective, the mathematical signs being interpreted here are mainly the continuous cardinal representation on the task sheet, the objects on the table and the numerical representation. Especially the objects on the table are discussed and interpreted in a mathematically non-viable way, but S1s interpretation seems to be rooted in the ambiguity being related to the material-design: The placement of two ones-dots on a continuous tens-line seems not self-explanatory and leads to a perturbation (irritation): Instead of interpreting the two red ones-dots as part of the twenty, thus as being part of the whole of the second number, a distinct interpretation of the ones-dots and tens-lines becomes fostered.

Sequence 2: Insights into the interpretation of the discrete cardinal material

The students S3 and S4 try to explain the way the ‚Auxiliary Task' is used for calculating the task 165 – 38 with the given discrete cardinal material. While doing so, they try to explain the ‚Auxiliary Task' (which was demonstrated by the fictive student

Max, see figure 2). They have ones-dots, stripes of tens-dots and squares of hundreds-dots (consisting of ten stripes of tens-dots). The iconic representation of Max' procedure (see figure 2) is visibly beside the amount on the table.



21	I	Which number did you now put on the table?
22	S3	Ehm these *[points at the amount on the table]*
23	S4	165 *[puts the laste ones-dots to the discrete amount]*
24	S3	Yes.
25	I	Okay, very good. So, what does Max do now? How does he proceed?
26	S3	38 plus 2 *[takes stripe of tens into his hand]*
27	S4	That is 40. Well- I'll just make it like this, plus two *[puts two ones-dots on the right side of the hundreds-dots]*- although, more like under it *[indicates to put the ones-dots beside the other ones-dots under the amount]*
28	S3	Let's just do the result.
29	S4	Well okay. Then it is 40.
30	S3	And then 40 minus *[has four stripes of tens-dots in his hand and holds them next to the stripes of tens-dots on the table]* five *[looks at the five ones-dots]* now this comes away *[takes away four of the stripes of tens-dots and pushes the ones up right under the hundreds-dots]* Plus… these two *[puts two ones to the ones under the 165]* that makes hundred- *[5 seconds]* 117.

In sequence 2, a similar interactional process to sequence 1 is visible: The learners S3 and S4 discuss the meaning of the ‚Auxiliary Task' with cardinal means, but S3 and S4 discuss a discrete representation here (see figure 2). Being asked what they put down on the table in Turn 21 by the Interviewer, S3 deictically points at the cardinal representation of 165, which S4 verbalizes and finalizes in Turn 22 by putting down the last ones-dots. In Turn 26 then, after being asked what the fictive student Max may have thought in Turn 25 by the Interviewer, S3 and S4 begin to verbalize their interpretation: S3 verbalizes the numerical task (38 plus 2) but at the same time takes stripes of tens-dots into his hands. S4 then finishes S3s task in Turn 27 by saying *"that is 40"*, but more importantly, S4 here also adds two ones-dots on the right side of the material on the table, an action matching the rounding amount of *"plus two"* in the task (38 + 2). The numerical utterance thus is accompanied by the analoguous enactive action of putting down the matching number of discrete cardinal material. From Turn 28 to 30 then, especially in Turn 30, S3 shows again enactively, what he seems to have meant in Turn 26: By holding four stripes of tens-dots, which he took into his hand already in Turn 26, beside of four stripes of tens-dots already on the table, he indicates first, how many stripes have to be taken away (minus 40), and then takes away these. After that, he pushes the ones-dots up so that there is no gap, whereafter he finalizes his calculation.

From an epistemological viewpoint, the mathematical signs being interpreted here seem less ambiguous when compared to the signs from sequence 1: The ones-dots and stripes of tens-dots as well as squares of hundreds-dots are interpreted by S3 and S4 in a more coherent and relational way, visible by the utterances being accompanied by parallel and matching enactive action (see Turn 27 and 30). It seems that S3s and S4s interpretation does not differ from the task as S1s interpretation in sequence 1: The two ones being put down on the table in Turn 26 and 27 are not taken away twice but once. The extra ones-dots thus seem to be interpreted by the learners as part of a viable compensative thinking being analoguous to the cardinal and numerical representation on the task sheet. Generally, the learners seem not to be perturbed through the cardinal material and representation if compared to the interaction in sequence 1.

DISCUSSION OF RESULTS AND LIMITATIONS

With regard of the research question of this paper, the analysis of both sequences shows an important difference in the interpretation of the ‚Auxiliary Task' with discrete versus continuous cardinal material: The continuous cardinal objects seem to be ambiguous in terms of their meaning since putting two dots onto the continuous tens-line leads to a non-relational, non-integral interpretion of the rounding amount (+2) and the rounded-up number (18 + 2 = 20), resulting in a non-viable interpretation of the ‚Auxiliary Task' as a double-subtraction. In contrast, with the discrete continuous material, the rounding amount and the rounded-up number seem to be interpreted in a more integral way by verbalizing the compensation process more directly in an unequivocal way and by accompanying it with analoguous enactive actions (see sequence 2). This hypothesis can be verified by broadening the analysis to all n = 18 learners: The learners from iteration 2, where the discrete cardinal material was used, seem to understand the compensation process more viably than the learners from iteration 1, where a lot of sign-related irritations could be reconstructed in the qualitative analyzes. An important local theory for designing a learning environment, which is utilizing a cardinal approach to explaining the ‚Auxiliary Task', thus is that the use of discrete material may lead to a more relational, more viable interpretation of the compensation process due to lesser sign-related ambiguities. For the continuous material, at least a redesign of the double layered objects with hidden versus visible elements seems to be necessary, for example by using shortened tens-lines, where the ones-dots are not *on* the line, hiding a part beneath it, but *beside* it, but then another ambiguity would occurs: Tens-lines with a shorter and a "normal" length. This leads to the conclusion that the use of discrete objects seems to be more adequate for explaining the ‚Auxiliary Task' with cardinal means.

What the analyzes do not show is if a discrete representation is better than a continuous representation generally: The insights are local, meaning they are closely connected to the designed learning-environment about the ‚Auxiliary Task'.

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STUDENTS WORKING ON MODELS; AN ON-GOING EXPERIMENTATION IN MATHEMATICS AND CHEMISTRY

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This paper focuses on modelling at upper secondary level. The objective is to give students an understanding of mathematical concepts and methods in close relationship to a domain of reality, as well as to give them insight into the contribution of models of different kinds. This has led to the development of a framework for modeling activities based on the Mathematical Working Spaces theory. The questions at stake concern the operationality of this framework. To what teaching situations can it lead? How do students work in these situations? We examine these questions through an on-going experimentation on models of acid-base transformations.

APPROACHES TO MODELING IN MATHEMATICS EDUCATION AND SCIENCE DIDACTICS

Beginning in the 2000s, interest in involving real-world contexts has grown in mathematics education. The ICMI 14 study (Bloom 2002) kicked off a lively stream of research and focused on mathematics as an important activity in society. This stream seems to us marked by two lines of force, problem solving and the modeling cycle as a theorization of the modeling activity. Many authors indeed characterize modeling activities as solving authentic problems, but the activities they propose focus more on the solution to a contextualized problem than on models. For example, in Blum and Ferri (2009), the task involves a lighthouse of a given height and students have to find a value for the visibility distance that is valid specifically for that height. The authors identify steps in students' problem solving consistent with the modeling cycle, but these steps lead to the solution rather than to a model. Overall, the modeling cycle remains close to a classical resolution scheme where the problem and the solution are expressed at the extra-mathematical level and solving is done at the mathematical level. The cycle specifies steps and transitions and this allows, among other things, the interpretation of students' trajectories in their complexity. Nevertheless, the "real" and the "mathematical" remain two levels insufficiently intertwined to account for how working on a model articulates mathematics and real-world objects and phenomena (Czocher 2018).

In experimental science didactics, the main concern is the relationship between an "empirical referent" (Sanchez 2008) made up of objects and phenomena as they are perceived and spontaneously mobilized by the students, and a "scientific referent" consisting of theoretical elements. By appropriating a model, the students can relate these two referents and thus progress both in their perception of everyday objects and

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phenomena and in their understanding of scientific concepts. Nevertheless, the mathematical aspects of the model are generally not questioned as such, and science didactics privileges models where these aspects are minor for fear of complexity.

Thus, dominant approaches in mathematics education emphasize problem solving rather than working on models, and science didactics favors appropriation of a model while leaving aside mathematical aspects. Drawing on science didactics, we aim to engage students in explicit work on models, but we also want to include mathematics in this work. The experimentation we are carrying out starts from a laboratory technique taught in the chapter about acid-base reactions of the chemistry course in the non-vocational upper secondary stream in France. Students often describe this technique as “a cooking recipe”, since mathematical methods are used, but not explained with reference to acid-base reaction models. The purpose of the experimentation is then to look for ways to make students study models both in their chemical and mathematical aspects and get a better understanding of these aspects.

APPROACH AND FRAMEWORK

Lagrange et al. (forthcoming) distinguish between modelling and mathematization of a domain. While mathematization is global in scope, modeling aims to account for certain aspects of the domain in order to understand it, even partially, and to act on it. A corollary is that there is not a single model: several models are as many ways to approach a reality. Modeling thus has a subjective and social dimension: all models can be useful, but each one must be discussed and confronted with others. In each model, the contribution of mathematics results from a specific mathematical work, in collaboration with experts in the field, in order to make the model more intelligible and facilitate its use. Thus, there is not a real model on one side and a mathematical model on the other side, but a plurality of models, each with a specific implication of mathematics into the same domain of reality. We consider students’ activity in modelling as a work of appropriation of two or more models, and as a work of uncovering relationships between models, in order both to get better insight into the domain and to progress in the mathematical concepts used; for instance Lagrange (2018) proposed to consider four models of a suspension bridge for a high school teaching project, one based on a study of tensions, a second one on arithmetical relationships, a third one being a computer simulation, and finally a fourth model based on notions in real analysis (functions, integration, etc.).

Modelling implies collaboration between experts with different viewpoints (Lagrange et al. forthcoming) and that is why the above approach has led to organizing students’ work in a "jigsaw classroom". The work starts from a question. For the present study, the question will be about how a model of the reaction justifies the laboratory technique. There are four phases: (1) Presentation of the question and work on prerequisite concepts (2) Expert groups: each group works on a model from a specific viewpoint (3) Jigsaw groups: each group gathers experts from each expert group and progress in understanding the models (4) Whole class discussion and conclusion by the teacher.

Regarding the notion of work in educational settings, we refer to the theory of Mathematical Working Spaces (MWS). According to Kuzniak et al. (2016), a MWS is an abstract space that is organized to support mathematical work in an educational setting[1]. The theory of MWS distinguishes three levels:

- A reference working space (WS) is a space in which somebody educated in a specific domain is expected to do the work in this domain.
- A suitable WS helps manage the work for beginners in a teaching project.
- A personal WS is particular to individuals.

As Menares-Espinoza and Vivier (forthcoming) explain, beginners approach a new domain with their prior knowledge and cognitive processes. Teaching must design tasks to help students' personal WSs evolve towards the reference level and this requires designing suitable WSs. Here a reference WS is what allows for scientific thinking about the laboratory technique. Models on which this thinking can be based are described below. This paper focuses on suitable WSs, both a priori with reference to models of the reaction and a posteriori from student observation, leaving for further research a study of personal WSs.

The research question follows. RQ: What are the suitable WSs that provide a conceptual basis for a mathematical-chemical approach of acid-base reaction models? How do they predict students' behavior and cognitive processes?

MODELS, WORKING SPACES AND TASKS

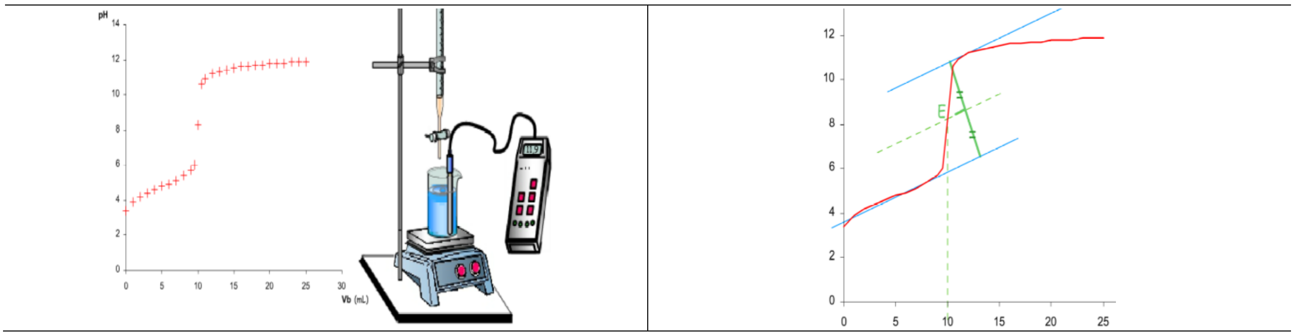

Table 1: The laboratory technique. Empirical procedure and tangent method.

The laboratory technique we start with is titration, i.e. the use of a solution of known concentration (titrant) to determine the unknown concentration of another solution (analyte). The titrant is added from a graduated buret to a known quantity of analyte. In case of an acid-base reaction, the analyte is an acidic solution, characterized by a preponderance of oxonium ions and the titrant is a basic solution characterized by a preponderance of hydroxides ions. During the titration, the oxoniums and the hydroxides react and then the pH (minus the decimal logarithm of the concentration

1 Because of limited space, we do not insist on the three dimensions that structure a MWS: semiotic, instrumental and discursive. However they are important in the domain of modelling, ensuring that mathematics are not simply considered as a "language".

in oxonium) of the mixture increases and a table of values (volume added, pH) is obtained (Table 1 on the left). The experimental curve is a sigmoid whose inflection point (called neutralization point) corresponds to a volume added for which the mixture is neutral, i.e. has equal concentration in oxoniums and hydroxides corresponding to pH 7. The position of the neutralization point allows to know the quantity of hydroxides added and consequently the concentration in oxoniums of the analyte. A geometrical technique (method of tangents) is used to determine this position. It is based on the quasi-symmetry of the experimental curve with respect to the inflection point (Table 1 on the right). As said before, no theoretical justification in chemistry and mathematics is given to students. The underlying model of the reaction is the evolution of the pH based on empirical observation, increasing and almost symmetrical with regard to the neutralization point. This is Model 1.

Titrant: 8 ml acid, oxinium concentration 0.0125 mole/liter.
Analyte: base, hydroxide concentration 0.01 mole/liter.

Oxinium concentration for x ml base< 10:

$$Ca(x) = \frac{10^{-4} - x \cdot 10^{-5}}{(8+x) \cdot 10^{-3}} = \frac{10 - x}{100 \cdot (x+8)}$$

pH for x ml base < 10:

$f(x)$= - log(10-x)+log(8+x)+2

Hydroxide concentration for x ml base> 10:

$$Cb(x) = \frac{x \cdot 10^{-5} - 10^{-4}}{(8+x) \cdot 10^{-3}} = \frac{x - 10}{100 \cdot (x+8)}$$

pH for x ml base> 10:

$f(x)$= log(x-10) – log(8+x) + 12

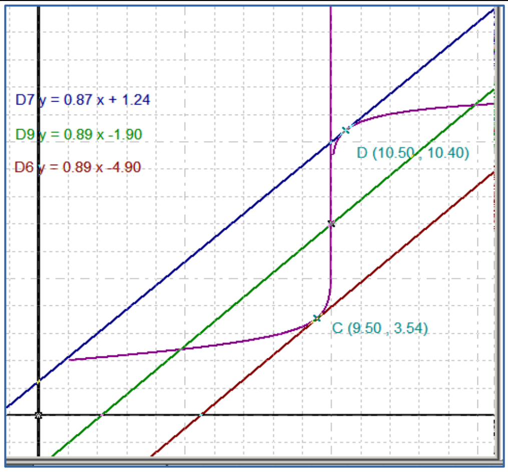

Table 2: Mathematical function based on assumption: mixture contains only one type of ions.

Model 2 is another model in which the titration curve is obtained by the calculation of a mathematical function (Table 2). It is based on a simplifying assumption: hydroxide and oxonium ions neutralize so that the mixture contains only one of the two types of ions. This assumption allows to obtain a function whose curve is coherent with the empirical curve, except in the vicinity of the neutralization point where the function tends towards infinity. This is a consequence of a phenomenon called partial dissociation of water, that contradicts the assumption: acidic solutions contain hydroxides and basic solutions contain oxoniums, but this is negligible except in the vicinity of the neutralization point. Because tangents are drawn at a distance from the neutralization point, the model is a basis for a mathematical work to justify the method. Mathematics also allows to question the simplifying assumption, and thus students working on the models should progress both in chemistry and mathematics.

In accordance with the RQ, the experimentation aimed to build and evaluate suitable

WSs thanks to which students could recognize Model 1 and 2 as the foundations of the titration technique, and compare the two models. These WSs are designed for the two group phases (expert groups and jigsaw groups). There are three expert groups. Students in group Ea should become experts in Model 1, students in Eb should become experts of the mathematical component of Model 2 and students Ec should develop an expertise in quantifying the evolution of concentrations throughout the titration. WSEa, WSEb and WSEc are suitable WS, each representing the respective expertise targeted in each group and WSJ is the suitable WS pour the Jigsaw groups. A presentation of the WSs and associated tasks follows, summarized in Table 3.

WSEa: Acid, base, neutralization (visual), pH (reading), measure, proportion, curve (experimental), tangents (visual).	**Task Ea:** Appropriate a simulation software. Simulate for given data. Operate the tangent method for varied positions and compare accuracy.
WSEb: Functions (symbolic), curve, tangents (software), decimal logarithms.	**Task Eb:** Study the function *f* (Table 1); growth, limits. Trace the curve and tangents at abscissas 10-x and 10+x for varied values of x ; observe the mid-lines.
WSEc: Ions, concentration, neutralization, pH. Volumes, ratios, formula.	**Task Ec:** Calculate hydroxides and oxynium concentration for varied added volume. Calculate pH for these values and draw curve.
WSJ: Idem Ec + symbolic calculations	**Task J:** Compare curves (Tasks Ea, Eb,Ec). Show how *f* (Task Eb) models the pH as a function of the added volume. Justify the tangent method.

Table 3: Suitable working spaces and tasks in the group works.

WSEa is suitable to work on Model 1 and thus includes elements of chemistry (ions, pH, concentration, etc.) but also of mathematics (measurements, curve, tangents, ratios, etc.) We choose to have Ea students work on computer software simulating titrations. The purpose of the software is to help systematize and reflect on the method: students must enter the data of a given titration; they obtain a curve simulating the empirical curve, they can choose points to perform the tangent method and observe the accuracy of the method (Task Ea).

WSEb is the space for a complete mathematical study of the Model 2 function. The signs, the theoretical frame of reference as well as the use of a software for functions belong to high school calculus, with the exception of decimal logarithms and the piece-wise function which are unfamiliar. Task Eb is a study of properties of the function. It is classical in the form, but the function is unusual.

WSEc is the space for students to numerically compute concentrations along the titration, using the assumption of complete neutralization. The elements of chemistry are the same as in Model 1, but they must be systematically quantified; pH formulas must be used. From a mathematical point of view it is an arithmetic space. Task Ec requires students to have a good mastering of chemistry notions, as well as ratios and unit conversion. The calculation of the concentration must take into account the quantity of ions, but also the increase of the volume along the titration.

WSJ correspond to the jigsaw groups in Phase 3. This is the space for students to understand Model 2, make the connection to the assumption that the mixture contains only one type of ions and make sense of the tangent method at a symbolic level. Task J should lead students to perform a computation similar to Task EC, but at a symbolic level as in Table 2, and to justify that two parallel tangents, one on each branch, are nearly symmetrical with regard to the neutralization point.

IMPLEMENTATION AND ANALYSIS

In the context of the pandemic in 2020, the experimentation was carried out in the form of a "lock-down online jigsaw classroom", i.e. the six participants were physically separated, communicating on a platform that allowed either all students to be gathered together (Phases 1 and 4) or split into groups (three groups of two in Phases 2, and two groups of three in Phase 3). The students were of average level, some more proficient in mathematics, other more in experimental sciences. They had previously performed titrations on real solutions. The platform allowed the recording of exchanges and productions. This data was completed by a e-mail survey. In the analysis of the data we leave aside the aspects related to the online work, underlining only that even online, the jigsaw classroom kept its potential for collaborative work.

This is the analysis of the group work phases (about one hour each). Table 4 presents extracts of the reports made by the three experts groups (Ea, Eb, Ec, Phase 2) and the two jigsaw groups (J1, J2, Phase 3). Ea appropriated the simulation software after having difficulty understanding menus and data needed. They were able to use the tangent method for several values and compare the accuracy. Eb’s study of the function remained partial as the log decimal function was unfamiliar. As shown in their report they were also not comfortable with a piece-wise function. They noted an undefinitness, but did not mention infinite limits that would have shown inconsistency with the experimental curve. Pseudo-symmetry and use of parallel tangents for the position of the center were discussed. Ec was comfortable with the chemistry concepts and the various calculations, but had difficulty taking into account the variation in the volume of the solution. They made the connection with the titration curve. Overall, Ea and Ec’s work can be seen as fitting in the suitable WSEa and WSEc after initial difficulties. This is not the case for Eb. The students were not comfortable with the function of Model 2, which is different from routine functions they had been trained with. They were able to draw curves and tangents thanks to the software. It was consistent with WSEc only for the use of software.

Group Ea	We used a software to enter the data, and we got curves for concentration and pH . We saw clearly the turning point that makes it go from acidic to basic. We used the tangent method to get the pH.

Group Eb	We had two functions, one for x<10 and the other for x>10, but it was the same function. The two functions are not defined in 10. We drew the curves with GeoGebra. The pH curve, was symmetrical with regard to the neutralization point. We placed the points A and B according to the given abscissas then we traced the tangents and the mid line on the computer to find the neutralization point.
Group Ec	We calculated the quantity of oxoniums for each added volume. Then we applied the formula m/V after converting the volume into liters to get the concentrations, and the formula -log(Ca) for the pH values. We did similar calculations for the hydroxides after neutralization. We saw that values increased and that it looked like the empirical curve.
Groups J	**J1** We observed that when we apply the formulas of group Eb, we obtain the data found by group C. We can therefore deduce that the curve representing pH are related to the function of group Eb. We calculated the derivatives to get the slopes of the tangents. When the difference between these is close to 0, we get more accurate results. **J2** We had to develop a formula that actually calculates everything at once. We made calculations like Ec did but with a variable *x* and we got the function of group Eb for the acidic part.

Table 4: Extracts of reports of students' group work.

Both J groups observed that the curves obtained by the three experts groups were similar. Group Eb's remark (the two functions are not defined in 10) did not lead students to observe a discrepancy between models near neutralization. Group J1 concluded that the similarity of curves is sufficient evidence that *f* is a model of pH evolution. They started a study of the slope of the tangents with regard to the accuracy of the method, using the derivative of *f* with difficulty. Group J2 looked for a an analytic proof that *f* is a model and succeeded only for the acid part. The behaviors in both groups show students' partial appropriation of WSJ, the suitable working space that should provide mastery of Model 2. One shortcoming is that students' symbolic calculation skills taught in math class were poorly enacted. Another shortcoming is that they were not able to recognize a discrepancy between the models. In Phase 4, after the J groups reported on their work, the teacher emphasized the use symbolic computation and the discrepancy between models near neutralization, which he explained by the dissociation of water. Answers to post-questions by email showed that these points were partially understood by students.

CONCLUSION AND PERSPECTIVES FOR RESEARCH

The suitable WSs prepared for this experimentation allowed the students some appropriation of the models both in their chemical and mathematical dimensions. A critical look highlights achievements and gaps. WSEa did not help students distinguish Model 1 from Model 2. WSEb was too demanding in symbolic calculation. WSEc seemed appropriate, with students Ec completing the task and contributing to the work in Phase 3. WSJ was affected by Ea and Eb's shortcomings.

Another implementation was then carried out in 2021 with a variation of the tasks. In Task Ea the simulation was a computer program that the students could read and interpret. In all tasks it was asked to get values of the pH (Tasks Ea and Ec) or of the function (Task Eb) very close to the neutralization point. The discrepancy between values obtained by Ea on one side and by Eb and Ec on the other side brought a discussion in groups J. The students did not reach a consensus. Some students emphasized the validity of the model underlying the computer program and its conformity with empirical observations, and others maintained that Model 2 was more reliable, stressing the inaccuracy of empirical observations compared to mathematics. These results may be somewhat surprising and unsatisfaying but they confirm the interest of this situation and provide insights for further experimentation.

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DIMENSIONS OF PRE-SERVICE TEACHERS' DIAGNOSTIC JUDGEMENTS OF STUDENT SOLUTIONS

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Judgments are part of teachers' daily practice and crucial for students' educational careers. Previous evidence indicated that judgments are informed by various criteria. But how pre-service teachers (PSTs) judge student solutions and how these judgments are structured are still open questions. In two studies we shed light on the construct. First, we investigated PSTs' judgements of an exemplary student solution regarding the applied categories (n_1=110). Based on the results, we then constructed items and investigated the structure of the construct by applying EFA and CFA (n_{2a}=168, n_{2b}= 209). The results revealed the following judgment dimensions: understanding, solution quality, presentation of procedure, and motivation. In addition to evidence on the structure of the construct, we gained an instrument to measure PSTs' judgments.

INTRODUCTION

When planning lessons and making daily decisions regarding instruction, teachers rely on their diagnostic judgment of students' knowledge and potential. Diagnostic judgment informs not only the assessment of students' performances, but also their grades and transition recommendations and is therefore crucial for students' academic development (Zhu et al., 2018) and their educational careers (Fischbach et al., 2013). Thus, teachers' diagnostic judgment plays an important role and must be given special attention during teacher education (Ready & Wright, 2011). Judging student solutions against the background of learning goals, such as gaining conceptual and procedural knowledge, is crucial in all school subjects. Especially in mathematics, teachers often struggle with judging the variety of student solutions as tasks allow for multiple solution pathways (Durking et al., 2017). During teacher education, emphasis is thus put on pre-service teachers' (PSTs) judgments with respect to identifying the potential in students' solutions. Up to now, some evidence on how PSTs notice students' mathematical thinking as a pre-requisite of their judgments (Crespo, 2000; Talanquer et al., 2015; Baldinger, 2020) exists. Also, Loibl et al. (2020) contributed a framework focussing on the cognitive processes underlying diagnostic judgments. So far, no studies examined what teacher diagnostic judgments of student solutions are actually based on and how they are structured. Particularly, we are interested in exploring whether a content-related perspective is taken or, rather, a generic viewpoint.

In our first study, we utilized an open response instrument to assess the variety of criteria PSTs used to judge an exemplary student solution and reconstructed judgment criteria by content analysis. In our second study, we developed items based on the

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 115-122). PME.

aforementioned results that were assumed to represent the criteria. We then equipped the student solution with these rating scales and assessed two different groups of PSTs to examine the dimensional structure of PSTs' diagnostic judgments.

THEORETICAL FRAMEWORK

As indicated by social cognitive (dual process) models (Grawonski & Creighton, 2013, Loibl et al., 2020), judgments can arise from automatic and spontaneous or from controlled and reflected strategies of processing information. In many countries, educational standards postulate competencies that students should acquire and thus can serve as a normative framework against which teachers judge student solutions. For the learning of mathematics, gaining conceptual and procedural knowledge is important (Goldin, 2018). Students need to acquire procedural knowledge, thus knowledge about how procedures, algorithms, or methods are to be applied, as well as conceptual knowledge, in the sense of a content-related understanding of essential concepts and procedures and their interrelationships (Rittle-Johnson & Schneider, 2015). Thus, mathematics teachers are requested to assess students' products with regard to the extent to which procedures were applied appropriately and correctly to the tasks and whether conceptual knowledge has been acquired.

Previous research on PSTs' judgments of students' products revealed that PSTs use three strategies when judging students' products: mathematical reasoning, pedagogical (content) reasoning, and reasoning through self-comparison (Baldinger, 2020). Furthermore, judgments are often restricted to describing students' work instead of sense making of students' ideas (Talanquer et al., 2015), merely evaluating instead of interpreting, and not building inferences on students thinking (Crespo, 2000). Also, studies showed that students' errors resulting from a lack of conceptual understanding were interpreted by PSTs as lacking procedural understanding (Son, 2013). As a consequence, PSTs tended to directly respond to students' utterances or to correct their mistakes instead of asking questions to reveal their mathematical thinking (Cai et al., 2021). However, findings from intervention studies imply that learning opportunities can strengthen PSTs' judgments towards a more detailed investigation of students' thinking (Monson et al., 2018).

In sum, previous studies revealed that PSTs seem to focus on content-related aspects, but base their judgement on rather surface characteristics as describing students' solution instead of drawing on deep structure characteristics such as student understanding. That is, the evidence provides insights into the variety of judgment criteria and suggests a multidimensional structure of the construct. Against that background our study was guided by the following aims and research question.

AIMS AND RESEARCH QUESTION

To investigate the dimensions of PSTs' diagnostic judgments of student solutions we combined a qualitative and a quantitative approach. We first approached possible dimensions inductively (study 1). Based on these findings, we then constructed scales and checked the dimensionality of the construct (study 2). Particularly, we pursued the following research questions: RQ1: What judgment criteria can be detected from PSTs' diagnostic judgments of an exemplary student solution? RQ2: What dimensions structure PSTs' diagnostic judgments of an exemplary student solution?

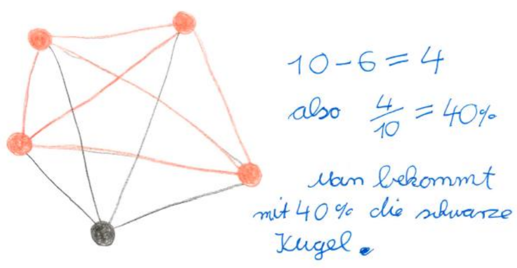
METHODOLOGY

To reveal the variety of judgment criteria, in study 1 we used an exemplary student solution of a probability task (see Figure 1) that allows for a diagnostic judgment with different focuses and by using different categories.

Task

There are 5 balls in a box. Four balls are red, one ball is black. Sarah pulls two balls out of the box with her eyes closed. What is the likelihood that Sarah will pull out the black ball?

Student solution



Translation of the solution:
10-6=4
thus 4/10=40%

You get the black ball with 40%.

Figure 1. Task and student solution.

The task was submitted to a sample of n_1 = 110 PSTs of a primary teacher education program, who attended a lecture on probability and stochastics.

The PSTs were asked to judge the result as well as the solution process and to justify their judgments. The open response data were analyzed by means of a step-wise inductive approach. First, the data-set was split into three subsets. One researcher analyzed one subset to identify a first set of categories that the PSTs used to rationalize their judgments. The research group then intensively discussed the categories. Second, code labels, definitions and examples were applied and revised through three rounds of coding and recoding, to identify the coding scheme that fits the data best (Kuckartz & Rädiker, 2019). Four categories of criteria were finally derived to code the whole dataset by two researchers. One sentence or more sentences with consistent meaning served as idea units and were coded to one category, if possible, or several categories, wherever necessary.

As a next step, items were constructed by extracting the most typical statements of each category. In study 2, we then combined the constructed items with a six-point response scale (from completely agree to completely disagree) to rate the student solution and submitted it to n_{2a} = 168 PSTs of two universities, enrolled in a master's program and who previously attended a lecture of probabilities and stochastics. To analyze the dimensional structure of the items, we first conducted an exploratory factor analysis (EFA) and, second, cross-checked the structure by applying confirmatory factor analyses (CFA). As factor loadings implied to add items, we constructed additional items based on the findings gained in study 1, and repeated the data analysis. The revised itemset was submitted to another sample of n_{2b} = 209 PSTs at the end of their bachelor teacher education program to validate the gained dimensionality. Again, CFA was carried out to analyze the structure of the construct. Quality of model fit was investigated by interpreting common fit indices (Hu & Bentler, 1999). Thereby, McDonalds ω was estimated as indicator for reliability (Hayes & Coutts, 2020).

RESULTS

The analysis of the open response data (RQ of study 1) revealed four categories PSTs used when judging the student solution. They pursued a focus on understanding, procedure, presentation, or motivation. PSTs with a focus on understanding usually emphasized that the student was able or not able to grasp the problem correctly (e.g. “student’s solution shows that he or she understood the problem well”). PSTs who showed a focus on procedure pointed to details of how the student proceeded in either a correct or incorrect way or in a complete or incomplete way (e.g., “calculates correctly, converts to fractions, and gives correct percentages”). A focus on presentation was coded for judgments based on how the solution process was presented, arguing that the student created a picture of the problem, or wrote down a solution path and an answer or did not (e.g., “solution is not clearly arranged”). When PSTs recognized merely the student’s effort to solve the problem, we coded it as focus on motivation (e.g., considering that he or she has a solution, and strained him- or herself). The four categories were thus coded regardless whether the PSTs pursued a deficit- or strength-based perspective.

In study 2, the EFA of the items constructed based on the most typical statements of each category indicated a three-dimensional model, that was proved by CFA against a four-dimensional structure (*presentation* and *procedure* modeled as two different factors in the second model, AIC = 5411,88, BIC = 5546,21, X^2 = 111.88, df = 47, p = .00; CFI = .89; RMSEA = .09 [.07 ; .11]; SRMR = .08). The results revealed a three-dimensional model as more appropriate than a four-dimensional model (AIC = 3812,44, BIC = 3909,28, X^2 = 37.25, df = 23, p = .03; CFI = .96; RMSEA = .06 [.02 ; .09]; SRMR = .06). Hence, *presentation* and *procedure* were building one factor which we labeled *presentation of procedure*, showing a high reliability (McDonalds ω = .80), in addition to the factors *understanding* (McDonalds ω = .78) and *motivation* (McDonalds ω = .79).

Each factor was presented by items with substantial loadings higher than .57. However, a closer look at item quality and loadings led to a revision of the dimension *understanding*. Two of the items of the dimension rather addressed the quality of the student’s solution, e.g., “the solution is a smart one”, with high loading on the factor.

Consequently, we added items based on the data gained by study 1 to test whether an additional dimension needs to be modelled.

Again, we conducted a CFA with an additional data set that revealed a four-dimensional model (see table 1) to fit the data best (see table 1). *Understanding* and *quality of solution* were building two different factors in addition to the factors *presentation of procedure* and *motivation* (X^2 = 86.10, *df* = 48, *p* < .01; CFI = .97; RMSEA = .06 [.04 ; .08]; SRMR = .05). The model fit was considered good. That is, the four dimensions each showed high reliability (McDonalds ω between .85 and .97). Also, they were represented by three items with a loading higher than .55.

The factors correlate moderately with each other (from r = .21, p < .01 to r = 59, p < .01), except for *understanding and quality of solution* with a high correlation (r = .82, p < .01). Nevertheless, the CFA confirmed a four-dimensional model as more appropriate than a three-dimensional model (ΔCFI = .14).

Dimensions	Items
Understanding $\omega = .97$	The student's solution shows that he or she understood the problem well. The student's solution indicates that he or she delved the problem. The student grasped the problem.
Quality of solution $\omega = .94$	The student carefully considered the solution. The student solution is smart. The student skilfully solved the problem.
Presentation of procedure $\omega = .85$	The student should have structured the solution better. The student should have chosen a different notation. The student solution does not show how he or she proceeded.
Motivation $\omega = .90$	The student tried hard to understand the task. The student strained to solve the task. The student gave thought to find a solution.

Table 1: Dimensions of PSTs' diagnostic judgment.

DISCUSSION AND CONCLUSION

In two studies, we shed light on the "black box" of PSTs' diagnostic judgments when confronted with an exemplary student solution – which will later be an important part of their daily practice. First, we could show that they pursued a focus on understanding, procedure, presentation or motivation. The results are in line with previous research, indicating the relevance of content-related aspects (Baldinger, 2020). However, some PSTs restricted their judgements to rather generic aspects when elaborating on how the solution was presented (Talanquer et al., 2015) or merely acknowledging motivational aspects such as the effort the student made. Thus, the diagnostic judgment criteria PSTs applied are of different quality with respect to fostering students' learning.

In the second study, we used the results of the qualitative study to further explore the dimensions of the construct. The results confirmed a multi-dimensional factor structure. Beyond the results of previous studies that identified content-related dimensions, our studies revealed that students' motivation as a generic dimension needs also to be considered as representing PSTs' judgments. Our results further revealed that the focus on procedure and on presenting the solution formed one dimension, in line with previous evidence on PSTs' judgments, showing that PSTs who focus on procedure take a rather descriptive than an interpretative view, not building inferences on students' thinking (Crespo, 2000). Furthermore, we discovered that the factor quality of solution needs to be considered in addition to the factor understanding. The factors presentation of procedure and motivation indicate a more surface view on student solutions as it was implicated by prior research (Talanquer et al., 2015). In contrast, the two content-related factors of understanding and quality of solution indicate a rather deep structure view, meeting the requirement to build inferences on students thinking (Rittle-Johnson & Schneider, 2015).

Our study on the one hand contributes to the field of teacher education by understanding the diagnostic judgment criteria PSTs use and how the construct is structured. On the other hand, we gained a standardized instrument to measure diagnostic judgment criteria PSTs apply when they judge an exemplary student solution. So far, we could conduct an additional study to test whether the identified diagnostic judgment dimensions fit the judgment of a student solving an arithmetic solution, proving the independency of the dimensions from the concrete task used. As a next step, starting from the study of Monson et al. (2020) who could show that PST learning opportunities can contribute to a stronger focus on students' thinking, we will apply the instrument to examine whether and what learning opportunities can affect a shift from focusing on surface to deep structure and content-related characteristics.

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AN OUTDOOR ACTIVITY TO LEARN OPERATIONS WITH INTEGERS

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This article describes an example of research-informed teaching to help students to understand the operations with integers. The approach used is that of Inquiry and Embodied cognition in outdoor context, with the help of sagittal axis. The activity involved 15, eight grades (aged 13/14), students from a middle school in Trieste, Italy. The results were tested by proposing to the same students' different types of exercises and problems. 73% have obtained positive results with 72% of which very good. Finally, we investigated through Mentimeter the students' appreciation of the outdoor activity. 100% of students found the activity fun and helpful.

INTRODUCTION AND THEORETICAL BACKGROUND

Learning outside the classroom essentially can be defined as use of resources out of the classroom to achieve the goals and objectives of learning (Knapp, 2010; Smith & Walkington, 2020). Recently there has been an increased interest in the development of outdoor and adventure education programmes (Fägerstam & Samuelsson, 2012). The constant focus on textbooks and formal mathematical practice might invoke a view among students that mathematics is abstract, distanced and only useful in a in classroom context. Existing research on outdoor learning in mathematics indicates positive affective outcomes and possible academic benefits from learning mathematics in an out-of-school context (Daher & Baya'a, 2012; Moffett, 2011). Moreover, outdoor environments, are real-life contexts enabling children to internalise, transfer and apply mathematical ideas and provides direct experience, the students need to be active in the learning process (Moffett, 2011). It lends itself to the Inquiry-based mathematics education, a student-centred form of teaching whose guiding principle is that the students are supposed to work in ways like how professional mathematicians work (Artigue & Blomhøj, 2013; Dorier & Maass, 2014): they must observe phenomena, ask questions, look for mathematical and scientific ways of answer these questions, interpret, and evaluate their solutions, and communicate and discuss their solutions effectively. Cooperative learning gives the opportunity to discuss and reason with others and justify one's mathematical thoughts on how to solve different mathematical problems. Cooperative outdoor learning in mathematics gives the possibility to observe that a task at hand can be solved in more than one way and that more than one "right" solution to the problem may exist. The sensorimotor experiences arising from the environment also play a paramount role in learning (Wilson, 2002).

Embodied cognition is described as a bodily sense of knowing, expressed through physical movement and sensory exploration with environments (Merleau-Ponty, 2002;

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Varela et al., 1991). There is complexity in the processes that may be involved in the development of embodied cognition as “knowledge depends on being in a world that is inseparable from our bodies, our language, our social history” (Varela et al., 1991, p. 173). According to Glenberg (2010) perception and how memory works is affected by how people move their bodies. The role of gestures as semiotic tools, contributing to deeper understanding of mathematical concepts (Arzarello et al., 2009).

Fluidity with integer operations marks a transition from arithmetic to abstract algebra. They do not correspond to any of the pre-existing cognitive structures and destabilize the perceptions - established since elementary school - of students on numbers and operations. It is difficult for them to perceive that -27 is less than -1 or that addition can cause a reduction, while subtraction can cause an increase. Moreover, negative numbers are conceptually difficult because students spend much less time learning them. Not being able to attribute natural objects or quantities to them, they try to recall the rules that do not guarantee the validity of their results (Vlassis, 2002; Bofferding, 2014; Badarudin & Khalid, 2008). The key to a successful method is not to let them memorize a bunch of rules before they understand. Instead, students' understanding can be enhanced by using images or manipulating tools, to enable them to translate concepts into images. Additionally, giving students the opportunity to explore multiple representations of a particular mathematical concept can strengthen their conceptual understanding.

Numbers are closely related to space both in action and in thought. A now classic finding is the “spatial numerical association of response codes (SNARC) effect”: among literate individuals from cultures who read from left to right, smaller numbers induce dispositions to act in the left space and larger numbers in the right space. Negative integers also induce spatial arrangements, although the task requires influence whether they are “left” of zero, in line with their relative numerical magnitude, or mixed with positive integers based on their absolute value. Spatial arrangements can also play a role in more complex tasks: mental arithmetic, for example, induces systematic arrangements to respond spatially, with addition-bias responses to the right and subtraction bias responses to the left (Knops at al. 2009; Marghetis & Youngstrom, 2014). Anelli at al. (2014) found a significant “congruency effect” where subjects performed more correct addition operations when moving horizontally rightward (the inferred orientation for addition in cultures that read left to right). Citing earlier work on bodily movement and mathematical processes, these researchers offer more “evidence about the influence of active body movements on the calculation processes of additions and subtractions,” evidence which reveals, “…the direction of body motion can influence not only number magnitude in a number generation task, but also the more complex process of calculations that leads to a numerical magnitude” (2014, p. 4).

Typically, negative numbers are interpreted as a continuation of a horizontal number line, or number sequence, where numbers to the left of zero are negative and numbers to the right of zero are positive. Sometimes, they can mimic vertical number lines for example, a temperature gauge. Additionally, the meaning of the minus sign, the symbol most fundamental to integers, is ambiguous. Common meanings include the meaning of an operation (take away), a value (negative), and "the opposite of," and learners often apply multiple meanings during manipulation (Lamb et al., 2012).

Little attention has been given to the arrangements along the sagittal axis, which runs from behind the body forward. Things ahead can be seen, heard, touched; the things behind it are much more difficult to access. Furthermore, the sagittal axis is associated with another abstract domain: Time. Recent studies have shown that negative numbers are spontaneously associated with the space behind the body and positive numbers with the space in front. These spatial arrangements were evident only when the task involved both the positive and negative numbers. Whole reasoning, therefore, is not entirely abstract, but induces systematic dispositions to action (Marghetis & Youngstrom, 2014).

The purpose of this article is twofold: it is intended to show how an outdoor activity should be presented with a view to the Embodiment, the Inquiry and with the use of sagittal axis; to test, as first exploratory study, whether an hour and a half of outdoor activity was enough for the students to understand concepts and if the activity was appreciated by them. The study involved 15 students, eight grades (aged 13/14) 8 boys and 7 girls, from a middle school in Trieste, Italy.

THE METHODOLOGY

In the first part we describe those steps that led students to the discovery of properties regarding the addition and subtraction with integers. The approach used is that of Inquiry and Embodied cognition in an outdoor context. The activity takes place in the "Classroom under the sky" https://www.youtube.com/watch?v=lGJbz_d7OUs&t=80s (for another example see Lepellere & Gasparo 2021). The environment is already welcoming in itself: a small pond right on the edge of a laurel grove, an open lawn that converges to the maple tree in the centre of the space, under which a blackboard and seats for students are placed. The students can also make use of portable shelves, to support books and notebooks. The activity middle school in Trieste, Italy. The results were tested by proposing, to the same 15 students, just after an hour and half of outdoor activity 72 exercises on operations and five different problems. Finally, we investigated through Mentimeter the students' appreciation of the outdoor activity.

THE ACTIVITY

The straight line of numbers is represented by the stairs, the increasing direction to the right is not as intuitive as climbing the steps (positive numbers) or descending them (negative numbers). After having identified the zero point on the landing, we start by drawing positive and negative numbers on the wall next to the stairs. To further help visualization and memory, it is possible to paint negative numbers in red and positive numbers in blue, placing the relative sign in front of them. The students take their chosen position on the various stairs, after which we introduce some rules.

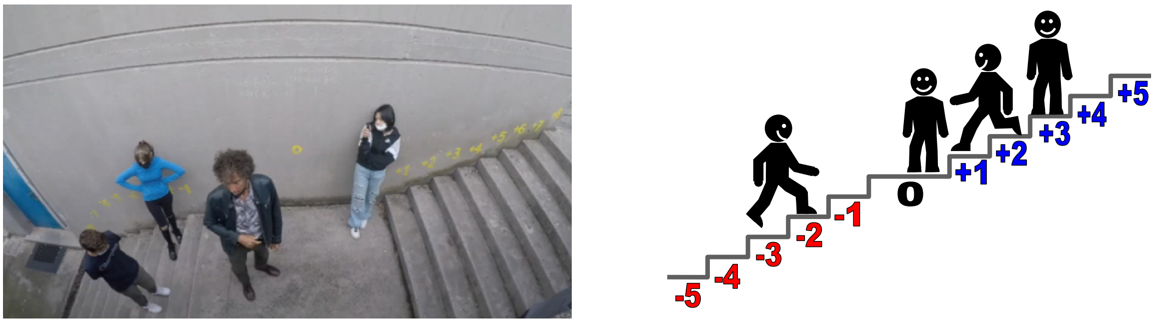

Figure 1: Stairs real and imaginary.

We invite students to discover the first oddity: how is it possible that by subtracting two numbers we get a larger number? I point out that subtracting means making a difference, so we invite students to calculate the difference between the +2 step and -4 step, that is, let's see how many steps we must do to go down from +2 to -4. We discover that there are 6! So, +2 - (-4) = + 6. Now we establish another rule that we have to do with the starting position that a student must take. Upon departure, a student stands in a neutral position, towards the teacher or fellow who gives the commands. Then he or she behaves like this: to add something they must turn upwards (positive numbers) and to subtract they must turn downwards. In Figure 1 on the right, we see the first student who passes from -3 to -2: he is turned upwards in the addition operation: -3 + (+1) = -2. The second stays in neutral position on 0, the third drops from +2 to +1, i.e., it is turned down in the subtraction position +2 - (+1) = +1. The fourth is neutral on +3 and wait instructions. It is time for the second rule: in front of a number there are two signs, one for the operation and one that indicates whether the number is positive or negative. If the number following the first is positive, we move forward and if the number is negative, we move backwards. We invite the students to move like fleas, amplifying the gesture with a jump. During the first calculations we often see some pupils simulating these jumps with their fingers in the notebook.

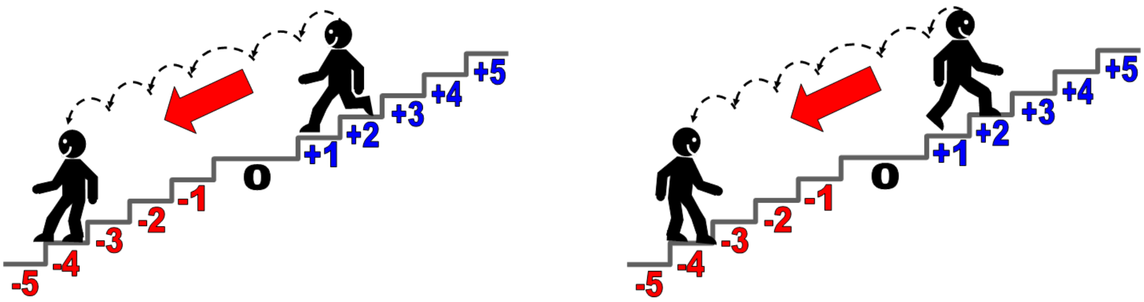

Figure 2: From step +2, we can get to step -4 with two methods.

Together we discover that starting from step +2, we can get to step -4 with two methods: turn left (-) and go forward (+) doing +2 - (+6) = -4 (Figure 2 on the left) or turn right (+) and go back (-) doing +2 + (-6) = -4 (Figure 2 on the right). At this point a series of games began: a team assigns an operation to the opponents who must solve problems such as -2 - (- 3). Then we ask how, starting from -1, we can get to +4. They start with simpler procedure; they turn left and go forward by 5. Then the students also discover the other system: they turn right, down, and go up to shrimp back by -5. The variations to the game are many: it is left to the students to find some. The whole outdoor activity lasted an hour and a half.

THE TEST

The next day and without any prior notice, students were offered a challenging test consisting of 72 operations to perform and 5 problems to set and solve. The operations were of type: $-20+15=\boxdot$; $12+(-16)=\boxdot$; $-4+\boxdot=10$; $-7+\boxdot=-2$; $\boxdot+(-3)=7$; $\boxdot+13=-5$ and so on in such a way as to cover all possibilities. Moreover, we give them the following 5 problems involving a vertical schema as temperature and sea level, horizontal schema as movement and timeline and finally a neutral schema as loans. Problem 1. Temperature: Lara looks at the thermometer: the temperature is -2. In the afternoon, however, the temperature rises by 11 degrees. What temperature do we have now? Write down the operation you did. Problem 2. Movement: Cristian walks 20 meters ahead and then returns 14 meters back. Where is he in relation to the starting point? Write down the operation you did. Problem 3. Sea level: Sofia is in a submarine with her friend Matteo. They are 150 meters below sea level! If the submarine rises 100 meters, what level is it now? Write down the operation you did. Problem 4. Timeline: Luca was a prominent Roman emperor, before reincarnating as a student of the Caprin. He was born in 510 BC. In the twentieth reincarnation he became a swallow, which died in 220 AD. How many years has he lived in these 20 reincarnations? Write down the operation you did. Problem 5. Loans: Isabel and Giada go to buy a sweatshirt from Scarface. Giada has 15 euros with her, but the sweatshirt costs 23. How much money does Isabel have to lend her for the purchase? In other words, how much money does Giada owe? Do you have to put + or - in front of the number? Write down the operation you did.

In Table 1. we show the results obtained. The 15 students numbered from 1 to 15 are placed in the column. The scores obtained from the operations are represented in the column “Operation Scores”. One point has been assigned to each operation. The scores on the problems are reported from the third to the eighth column: 1 if it was correct and 0.5 if it was formulated correctly but the operation was wrongly made. The “Difficulty” column also shows the difficulty perceived by the students of the test (0 easy to 5 very difficult) and in the subsequent ones the preference between the schemes used (0 dislike 3 like very much), horizontal, vertical, scaled or the use of the rules. The last 2 columns contain the total score and the grade (in tenths) achieved by each individual student. The last row of the table contains the arithmetic means of the various results.

	Operation	Problems					Difficulty (1-5)	Preference Schemas or Rules (1-3)					
	Scores	1	2	3	4	5		Orizzontal	Vertical	Stairs	Rules	TOTAL	SCORE
Total Score	**72**	**1**	**1**	**1**	**1**	**1**						**77**	
1	**33**	1	1	1	1	0	3	2	1	2	3	37	**5-**
2	**46**	1	1	1	1	1		2	1	3		51	**6/7**
3	**12**	1	0,5	0	0	0						13,5	**3**
4	**35**	1	0	0	0,5	0,5	3	3	2	3	1	37	**5-**
5	**71**	1	1	0,5	0	0	3	3	0	2	3	73,5	**9/10**
6	**60**	1	1	1	0,5	1	3	2	2	3	3	64,5	**8/9**
7	**61**	1	1	1	1	1		3	3	3	0	66	**8/9**
8	**66**	1	1	1	1	1	2,5	0	2	1	2	71	**9+**
9	**57**	1	1	1	0	0	2,5	1	1	3	0	63	**8**
10	**55**	1	1	1	0,5	0	2	1	0	3	0	58,5	**7/8**
11	**66**	1	1	1	1	1	1	3	3	3	3	71	**9+**
12	**72**	1	1	1	1	1	1	1	2	3		77	**10**
13	**72**	1	1	1	1	1	1	3	3	3	3	77	**10**
14	**66**	1	1	0,5	1	1	2	1	0	0	3	70,5	**9**
15	**41**	1	1	0,5	0	0,5		0	0	3	0	44	**6-**
Avarage	**54**	**1**	**0,9**	**0,8**	**0,7**	**0,6**	**2,2**	**1,8**	**1,5**	**2,5**	**1,9**	**54,1**	**7**

Table 1: Results.

Immediately it is noted that only 3 out of 15 students obtained an insufficient total grade of which only one very negative (6/7 stands for 6.75 and so on). The first temperature problem, which is also what students must do most in real life, was solved correctly by all students, following the movement and sea level problem. The timeline exercise had an additional difficulty related to intrinsic knowledge of the subject and therefore less skill was expected. More unexpected is the result on the problem about loan and would need further study. The test was perceived as not very difficult even if 4 students did not answer to the question about it. As for the use of the schemes, the scale scheme (10 students gave preference grade 3), was appreciated more than the scheme in the horizontal (5 students gave preference grade 3) and vertical (3 students they gave preference grade 3). Finally, the use of the rules received some appreciation (6 students gave preference grade 3). Cases of non-response were not taken into consideration in the calculation of the mean.

Mentimeter was used to test student appreciation of the activity. First, they were asked to write the first 5 words that came to mind when thinking about the outdoor activity. Figure 3a shows the results. The words funny combined with fun, nice, beautiful are the most used. But understanding-related words such as interesting, simple, focus, ease, easy, and intelligence were also highly rated. We find the words numbers, scales, errors, comparison too. When asked to indicate on a scale from 0 to 5 (0 not at all and 5 very much) how much they liked the activity 6 students gave score 5, 5 students score 4 and 1 score 3, scores 0, 1 and 2 are not been voted on (Figure 3b.).

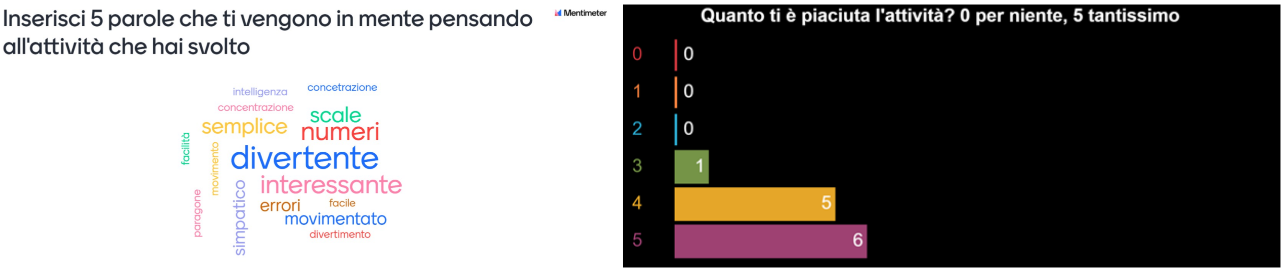

Figure 2: Mentimeter results.

It was also asked to indicate from 0 (not at all) to 5 (a lot) how much the activity on the stairs was of help compared to the study in class on the blackboard and the notebook. Here, too, 11 out of 13 young people who replied said that the activity was very helpful. Finally, it was asked whether by carrying out this activity the student was able to discover some rules on his own. 6 students voted yes and 7 no.

CONCLUSIONS

The lack of cognitive prerequisites based on personal experience, sensory deprivation, the lack of direct experiences are elements of risk that we detect in today's young people and that lead them to have difficulties even when they need to analyse, deduce, abstract. The term "educate" derives from the Latin ex ducere, "to lead out", in the sense of trying to get the best out of each student but it can also be interpreted as "to lead out" from the classroom. Here, movement can represent a stimulus to learning if practiced in serenity and even more in an open environment (Moss, 2009). Covid-19 launches a challenge to schools today in a strong crisis and that of outdoor schools is a real way that connects students with reality, nature, dexterity, art and a new responsibility towards creation, others, themselves. The reduction of opportunities for socialization has led to various psychological disorders in adolescents: panic attacks and anxiety. It should therefore come as no surprise that fun and socializing activities are of interest and approval. The survey carried out anonymously at the end of the outdoor lesson shows that 100% of students found the activity fun. Several used terms such as "beautiful, joy, happiness". After only an hour and a half of outdoor lessons most of the students obtained a very high score in a demanding test consisting of 72 operations of different types and 5 problems. This is a first investigative intervention that will lay the foundations for future experimental work.

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PROSPECTIVE TEACHERS' CONFIDENCE AND MATHEMATICS CONCEPTUAL KNOWLEDGE FOR TEACHING IN SOUTH KOREA – FRACTION DIVISION

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We focused on both prospective teachers' (PTs) confidence about their knowledge preparation and the extent of their knowledge of the specific topic of fraction division. The results revealed how these PTs' confidence may or may not be supported by their knowledge preparation for teaching fraction division, a concept they would be expected to teach as part of the country's curriculum standards. The results also illustrated the importance of specifying knowledge components in mathematics in order to help build or support PTs' confidence for classroom instruction.

Accumulated research findings in past decades have led to the understanding that teachers' knowing mathematics for teaching is essential to effective classroom instruction (e.g., Li & Howe, 2021; Li & Kaiser, 2011). Corresponding efforts have also been reflected in teacher preparation programs that call for more emphasis on prospective teachers' learning of mathematics for teaching (CBMS, 2012; Li, Ma & Pang, 2008; Li, Pang, Zhang & Song, 2020). Such efforts can presumably increase the quality of teacher preparation and prospective teachers' confidence and ultimate success in their future teaching careers. However, previous studies (Li & Kulm, 2008; Li & Smith, 2007) revealed a wide gap between sampled prospective middle school teachers' high confidence and their limited mathematics knowledge needed for teaching fraction division in the U.S.. Much remains to be learned about the extent of knowledge in mathematics and pedagogy that prospective teachers have and what else they may need to know for building or supporting their confidence. As part of a large research study of elementary school teachers' mathematical preparation, this paper focused on a group of PTs' confidence and knowledge of mathematics and pedagogy on the topic of fraction division in South Korea.

The topic of fraction division is difficult in school mathematics not only for students (Li, 2008), but also for prospective teachers (Li & Kulm, 2008; Simon, 1993). Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as "invert and multiply." However, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, or real world contexts (Greer, 1992; Li, 2008). The selection of the topic of fraction division, as a special case, can provide a rich context for exploring possible depth and limitations in prospective teachers' knowledge in mathematics and pedagogy. Specifically, this study focused on the following two research questions:

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 131-138). PME.

(1) What is the confidence of prospective elementary school teachers regarding their knowledge preparation for teaching?
(2) What is the extent of prospective elementary school teachers' knowledge in mathematics and pedagogy for teaching fraction division?

CONCEPTUAL FRAMEWORK

To be able to help students learn mathematics with understanding, teachers need to have mathematics conceptual knowledge for teaching (MCKT; Li et al., 2020). By MCKT we mean *topic-based conceptual knowledge packages that are needed for understanding, explaining, as well as teaching specific mathematics content topics with connections*. It can be specified as containing the following three topic-based knowledge components that can and should be acquired by mathematics teachers:

(a) Having knowledge and skills directly associated with a specific content topic;
(b) Being able to connect and justify the main points of a content topic, and to place it in wider contexts;
(c) Knowing and being able to use various representations for teaching the content topic, and being able to teach the relations between them.

Clearly, specific MCKT varies from one content topic to another. The task of specifying MCKT is needed but enormous for different content topics. Nevertheless, teachers' acquisition of MCKT would enable them to develop a profound understanding of mathematics content topics they teach as termed by Ma (1999). Given the dramatic variations across mathematical content topics, we focus on the MCKT that teachers would need to have for teaching fraction division.

The conceptual complexity of the topic of fraction division is evidenced in a number of studies that documented relevant difficulties prospective and practicing teachers have experienced (e.g., Borko et al., 1992; Simon, 1993; Tirosh, 2000). Although both prospective and practicing teachers can perform the computation for fraction division, it is difficult for teachers, at least in the United States, to explain the computation of fraction division conceptually with appropriate representations or connections with other mathematical knowledge (Ma, 1999; Simon, 1993). Teachers' knowledge of fraction division is often limited to the invert-and-multiply procedure, which restricts teachers' ability to provide a conceptual explanation of the procedure in classrooms (e.g., Borko et al., 1992). Because the meaning of division alone is not easy for prospective teachers (e.g., Simon, 1993), fraction division is even more difficult (Li & Kulm, 2008; Ma, 1999). The findings from previous studies help provide specifics of these three components of MCKT as follows:

(a) Having knowledge and skills about fraction division, including conceptual and procedural knowledge (e.g., Borko et al., 1992), and solving problems involving fraction division (e.g., Greer, 1992).

(b) Mathematical connections and justifications of main points related to fraction division, including fraction concept; addition, subtraction, and multiplication of fractions (e.g., Ma, 1999; Tirosh, 2000).

(c) Representational variations and connections for teaching fraction division such as explaining the computational procedure for "division of fraction" with different representations (e.g., Li & Huang, 2008; Li & Kulm, 2008).

The specifications of these three components of knowledge provided a framework for the current study and served as a guideline for selecting items to examine the extent of PTs' knowledge and specific difficulties with fraction division.

METHODOLOGY

Subjects

The participants were prospective elementary school teachers sampled from four national universities that offer 4-year B.A. or B.Sc. teacher preparation programs in South Korea. They had already taken the required mathematics courses and were completing the mathematics methods course at the time of their participation in this study. A total of 221 responses were collected and used for analyses and reporting, with 135 (61%) of responses from juniors, 86 (39%) responses from seniors.

Instruments and data collection

A survey was developed for this study, containing two main parts with three items for Part 1 and seven items for Part 2. Part 1 contains items on elementary teachers' knowledge of mathematics curriculum and their confidence in their readiness for teaching. Part 2 has seven main items that assess elementary teachers' three knowledge components of MCKT on the topic of fraction division. Most items were taken from previous studies (Li, Ma, & Pang, 2008; Li & Smith, 2007), with some items adapted from school mathematics textbooks and others' studies (e.g., Tirosh, 2000). Given the limited page space, only three items (note: each item containing two questions) from Part 2 and PTs' responses to these items are included for analyses to provide a glimpse of sampled PTs' confidence and MCKT.

The survey was administrated at regular class time by instructors in four institutions. Participants were notified that the survey was for research purposes only and should be completed anonymously.

Data analysis

Both quantitative and qualitative methods were used in analysing and reporting the participants' responses. Specifically, responses to the items in Part 1 were directly recorded and summarized to calculate the frequencies and percentages of participants' choices for each category. To analyse participants' solutions to the items in Part 2, specific rubrics were first developed for coding each item, and subsequently, the participants' responses were coded and analysed to examine their solutions/answers.

RESULTS AND DISCUSSION

In general, the results showed interesting relationships between PTs' confidence and their mathematical preparation for teaching fraction division, which illustrates the importance of specifying knowledge components in mathematical preparation in order to help build or support PTs' confidence for classroom instruction.

For PT's confidence, the results from the survey indicated that (1) participating PTs in South Korea did not know well about their national curriculum standards in general; (2) the majority of these PTs were confident in the knowledge preparation they received for future teaching careers; but (3) they knew very well about the topic placement of "multiplication and division of fractions" in mathematics curriculum. The results suggested that these PTs tend not to feel over confident.

For specific knowledge components of MCKT, these PTs' performance revealed that their mathematical preparation was sound in the content topic itself, especially in the procedural and pedagogical aspects, and relatively weak conceptually in connecting the content topic with other topics mathematically. The seemingly mixed results in their responses actually suggest that these PTs' confidence was built upon or supported by what they know that can and should be specified in concrete terms or knowledge components. The following sections are organized to present more detailed findings corresponding to the two research questions.

Prospective teachers' confidence in elementary school mathematics

The following items are from Part 1 of the survey to illustrate PTs' confidence of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the National Mathematics Standards? On a scale of four choices (High; Proficient; Limited; Low), 55% and 9% of the participants chose "Limited" and "Low", respectively. Relatively small percentages of the prospective teachers felt to have high (8%) or proficient (29%) understanding of their national mathematics standards.

For item 2-(2): Choose the response that best describes whether elementary school students have been taught the topic – *Multiplication and division of fractions.* On a scale of five choices (Mostly taught before grade 5; Mostly taught during grades 5-6; Not yet taught or just introduced during grades 5-6; Not included in the National Mathematics Standards; Not sure), 93% participants indicated that the topic is "mostly taught during grades 5-6" (a correct choice), and most of the remaining (5%) chose the first response ("Mostly taught before grade 5", a partially correct choice if only fraction multiplication is considered). The results, in contrast to the participants' response to item 1, suggested that these PTs know very well about the content topic placement in mathematics curriculum, although the majority did not feel confident in knowing about their national mathematics standards.

For item 3-(2): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of "Number –

Representing and explaining computations with fractions using words, numbers, or models?" On a scale of three (Very ready; Ready; Not ready), 67% of the participants thought they were "ready", while 7% chose "very ready," and 25% "not ready." The results indicated that the majority of these PTs were confident in their preparation for teaching fraction computations, including fraction division. There was also a large percentage of PTs who are not confident. The diversity in responses suggested the need of learning more about their confidence and possible connections with their knowledge preparation.

Taking together, PTs' responses to the Part 1 suggested that these PTs in South Korea tend not to feel over confident, although they actually knew very well about some specifics. In fact, the results are consistent with what has been reported about in-service mathematics teachers in East Asian countries (Mullis, Martin, Gonzalez, & Chrostowski, 2004) and PTs in China (Li, Zhang & Song, 2019). The consistency in the general response pattern between PTs in the current study and elementary teachers in other studies suggested that culture likely plays an important role in expressing confidence by teachers in East Asia including South Korea.

The extent of prospective elementary school teachers' preparation in MCKT for teaching fraction division

These PTs' responses to Part 2 allowed a closer look at the participants' three knowledge components of MCKT, especially on the topic of fraction division. Results indicated that these PTs do very well on items related to fraction division computation and problem solving (MCKT knowledge component 1). For example, for the problem "Say whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without solving. Explain your reasoning.", 96% of these PTs answered the problem correctly (i.e., the first numerical expression is greater than the second one). The most common explanation is that 2/3 is smaller than 3/4. Some showed why 2/3 is smaller than 3/4 by comparing these fractions with 1 (i.e., 1– 1/3 vs 1– 1/4), converting them to equivalent fractions with the same denominator (i.e., 8/12 and 9/12), or drawing a picture to represent 2/3 and ¾ for comparison, etc. About 26% mentioned, "If the divisor is the smaller, the result of the division (or quotient) is bigger." About 5% who got the correct answer changed the division of the given numerical expressions into multiplication and mentioned that 3/2 is greater than 4/3, implying "the greater the multiplier, the larger the product."

Moreover, these PTs also had great performance in solving multi-step word problems that involve fraction division. For example, 93% participants solved the following problem correctly.

> Johnny's Pizza Express sells several different flavour large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was 3/4 of the number of pepperoni pizzas sold, and 2/3 of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?

Specifically, 70% used a multi-step computation method to get the answer (e.g., $24\times\frac{3}{4}=18$, $\square\times\frac{2}{3}=18$, $\square=18\times\frac{3}{2}$, $\square=27$), about 6% used a combined computation method (e.g., 24 x ¾ ÷ 2/3 =27), 5% adopted an algebraic approach to set up and solve an equation for solution, 6% provided a correct answer with no explanation, 5% provided simple explanation without variables (e.g., 3/4 of 24 is 18, 2/3 of Deluxe is 18, so (the answer is) 27), and about 1% (2 respondents) provided something else. The remaining PTs either did incorrectly (15, 7%) or provided no answer at all (1 PT).

For the knowledge component 2 of MCKT, PTs were asked to explain "the meaning of fraction division, and how fraction division relates to other content topics" that aims to assess their knowledge of fraction division and ability of connecting and justifying possible association between fraction division and other content topics. The results suggested that 41% provided correct explanations to the first sub-question. the most common explanation was the measurement interpretation of fraction division (39%), followed by partitive interpretation (29%). In addition, more than 10% of sampled PTs were able to provide other meanings of fraction division such as the inverse of multiplication (10%) or determination of a unit rate (15%). Note that 28% of the PTs were able to explain the meanings of fraction division in two ways or more. Among the incorrect answers (46%), the most common explanation (32%) was to describe the meaning of fraction division as division with fraction (i.e., division with the divisor and/or the dividend as fractions). About 13% provided no answer or simply stated "I don't know". For the second sub-question, about 69% were able to relate fraction division to other content topics. The most common content topic related to was fraction multiplication mainly because the multiplicative inverse of the divisor is used in fraction division. Note that both measurement interpretation and partitive interpretations used in answering the first sub-question are related to the meaning of *division* and more than 29% of the PTs were able to provide these interpretations. In contrast, only 15% of the PTs related fraction division to whole number division and 13% related it to the division of decimal numbers. About 8% PTs failed to provide a correct explanation, and 22% provided no answer or simply stated "I don't know".

There were several items used to assess PTs' knowledge component 3 of MCKT. As an example, PTs were asked to explain how to explain/teach given computations of fraction division. In particular, the problem of "How would you explain to your students why $\frac{2}{3}\div 2=\frac{1}{3}$?; Why $\frac{2}{3}\div\frac{1}{6}=4$?" (adapted from Tirosh, 2000) was included in the survey. For the first fraction division (i.e., explaining why 2/3 ÷ 2 = 1/3?), 99% provided valid explanations for dividing a fraction by a whole number (i.e., 2/3 ÷ 2 = 1/3). The dominant explanation (>54%) used drawing to show that if you equally divide 2/3 into 2 pieces, you get 1/3. Other respondents (11%) used the meaning of division or fraction without drawing. Some respondents (5%) used the common denominator and others (5%) used an algorithmic approach (i.e., dividing a number equals to multiplying its reciprocal). About 11% of these PTs provided valid explanations in two or more ways. In doing so, drawing was often used as a basic approach. For the second fraction division (i.e., explaining why 2/3 ÷ 1/6 =4?), 97% provided valid explanations and the dominant explanation was based on drawing to show the meaning of measurement division. Even though the drawings were different from explaining the first fraction division, the main idea was to display how many 1/6s are included in 2/3 (or 4/6). Additionally, 19% of these PTs explained the meaning of the measurement division in words or numerical expressions without drawing. 15% of these respondents used an algorithmic approach of using the inverse number. About 10% of these PTs were able to provide two or more kinds of explanations. Again, drawing was the most prevalent approach included.

The results from these PTs' responses on MCKT items revealed their strengths in many aspects of MCKT, as specified in the framework. However, PTs' strengths across these aspects varied to a certain degree. It appeared that these PTs have solid performance on items related to fraction division computation itself, especially in the procedural aspect and pedagogical explanation, but relatively weak conceptually in connecting the content topic with other topics mathematically.

CONCLUSION

The findings from this study helped shed a light on the relationships between these PTs' confidence and their mathematical preparation for teaching fraction division. Specifically, these PTs didn't feel over-confident about their understanding of national mathematics standards, but they knew very well about the curriculum placement of selected content topics. They also had better confidence in terms of their readiness to teach elementary school mathematics. Such confidence was likely supported by their solid knowledge and skill directly associated with fraction division, a knowledge component that is also typically required for school students. At the same time, their relatively weak performance on items that are conceptually demanding in mathematics likely failed to support their confidence in readiness for teaching. Such knowledge differentiations, as specified in the MCKT framework, help provide an important and feasible lens for us to know the strength and weakness of teachers' knowledge. For the case of South Korea in this paper, the results suggested that PTs likely gain much more on mathematics and mathematical pedagogy, and certain limits on connections of mathematical ideas through their program studies. In turn, such results helped illustrate what teacher preparation programs need to do more in mathematical preparation in order to help build or support PTs' confidence for classroom instruction.

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THE POTENTIAL OF TASKS FOR MATHEMATICAL LEARNING AND ITS USE IN INSTRUCTION –PERSPECTIVES OF EXPERTS FROM GERMANY AND TAIWAN

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The potential of tasks to stimulate students' mathematical thinking and the adequate use of this potential in instruction are prominent indicators for instructional quality. Since the assessment of a task's potential depends on the aims of instruction, it may be argued that corresponding perspectives vary between cultural contexts. However, so far, this has not been systematically investigated in cross-cultural comparisons. In this study, we investigate whether Western (German; N=17) and East Asian (Taiwanese; N=19) professors of mathematics education have different perspectives on the potential of word problems for students' learning and the use of this potential in instruction by means of vignettes from a cross-cultural research project. We illustrate how differences reflect cultural aspects of mathematics instruction.

THEORETICAL BACKGROUND

The potential of tasks for students' mathematical learning and the use of this potential in teaching (the *potential of tasks and its use*) have been shown to be crucial factors for students' learning. Across cultures, there is a consensus that competent teachers are able to identify tasks with high learning potential, and, in addition, implement them in a way that uses this potential (e.g., Stein & Lane, 1996). However, it is well known that Western and East Asian perspectives on mathematics teaching and learning are different in many aspects (Leung, 2001). Hence, it is questionable whether research focusing on the evaluation of a task's potential and its use can be cross-culturally valid (Clarke, 2013) and it is thus important in our inter-cultural research community to seek corresponding evidence. Consequently, this research report investigates how professors of mathematics education (experts) from Taiwan and Germany (representing an East Asian and a Western perspective) evaluate the potential of tasks and its use in instructional situations. We focus on a very common kind of task that is used in mathematics instruction across grades and cultures: word problems with links to real-life situations.

Word problems, their learning potential and use in Germany and Taiwan

Generally, mathematical tasks are considered to have a high potential for students' learning, if they are focused on the instructional content, aligned with the teaching aims, and suited to stimulate students to work mathematically. Word problems, in particular, often have features that are considered to promote learning, such as their potential to provoke multiple solutions or require explanations (Stein & Lane, 1996).

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However, word problems are used with many different functions (e.g., Verschaffel et al., 2020). For example, word problems can be used to practice mathematical procedures, to discover new mathematical concepts, or to engage in mathematical modeling. Hence, it is an interesting question whether a certain word problem's potential for supporting students' learning may be evaluated differently. As it is known that the use of word problems varies between cultural contexts (e.g., Chang et al., 2020), this question is especially relevant for cross-cultural comparative research.

Mathematics teaching in Germany and Taiwan has typically different priorities such as meaningful learning vs. high procedural fluency (Leung, 2001), which may impact the perspective on word problems and their potential for learning. The German curriculum is literacy-oriented and clearly stands in a Western tradition. Engaging in mathematical modeling processes is hence an important practice (Chang et al., 2020). There is a focus on using real-life situations to encourage students to draw on their world-knowledge to understand them and validate solutions against the situation (Verschaffel et al., 2020). In Taiwan, word problems are used with a strong focus on the application of foundational knowledge and procedures (Chang et al., 2020; Pratt et al., 1999). Consequently, Taiwanese students were consistently found to outperform Western students in comparative studies where word problems were used for assessment, benefiting from a sound knowledge base and flexible use of procedures, that may result from high perseverance in studying (Leung, 2001).

Based on these differences, it can be assumed that there may be different perspectives in Germany and Taiwan on what constitutes a high potential of word problems for students' mathematical learning, and, consequently, how this potential should be used in mathematics instruction. Particularly, there are indications that word problems with real-life contexts are used with different aims in Germany and Taiwan: While in Taiwan such problems are primarily seen as opportunities to apply mathematical concepts and procedures to deepen mathematical understanding, in Germany they are seen as opportunities to learn mathematical modeling as a specific practice.

Eliciting culture-specific norms using of vignettes

To elicit and contrast perspectives on teaching quality across cultures, we follow approaches that use classroom vignettes to assess professional noticing (Dreher et al., 2021). Professional noticing with respect to teaching is described as a process of attending to aspects of classroom situations that are relevant for instructional quality (selective attention) and interpreting them by drawing on corresponding professional knowledge and other resources (knowledge-based reasoning) (Sherin, 2007). Typically, instruments to assess noticing use text- or video-based vignettes as representations of practice. A common "operational trick" in these approaches is to design or select vignettes in which something happens that does not meet the expectations of "good" teaching, i.e., they include a *breach of a norm* regarding some aspect of instructional quality (Dreher et al., 2021). The vignettes are shown together with a prompt to evaluate the depicted classroom situation and to give reasons for the evaluation. A person's reaction to the critical incident serves then as the indicator for the noticing; the reasoning can be used to infer what knowledge and beliefs guided the noticing process.

Up to now, such vignettes have mainly been used to assess noticing. One could, however, also use them to investigate whether the noticing of experts from different cultures reflects differing norms regarding aspects of instructional quality. To do so, one would need vignettes that potentially show breaks of culture-specific norms. However, in comparative studies, such culturally sensitive instruments are usually avoided as much as possible in order not to jeopardize the comparability of the results. This does not solve the problem that seeking the highest possible comparability may be detrimental to the validity of the instruments precisely when conceptions of instructional quality differ across cultures (Clarke, 2013). To the best of our knowledge, this has not been systematically investigated for the instructional quality regarding task potential and its use, as the corresponding instruments were lacking.

RESEARCH QUESTIONS

Against this background, we ask: Do mathematics education experts from Taiwan and Germany have different perspectives on the potential of word problems and its use as represented in vignettes authored in Germany or Taiwan?

CONTEXT AND METHODS OF THE STUDY

The reported study is part of the binational research project "Teacher noticing in Taiwan and Germany" (TaiGer Noticing) aiming at investigating the role of culture-specific norms regarding aspects of instructional quality. To this end, we developed a set of text vignettes reflecting potentially culture-specific norms regarding aspects of instructional quality (Dreher et al., 2021). Due to the prominent role of tasks in mathematics teaching, one of these aspects is the potential of tasks and its use. To validate whether the developed vignettes reflect indeed norms regarding this aspect in the respective countries, all vignettes were evaluated by experts in Germany and Taiwan. This report uses the responses regarding two of the vignettes (task2, task4). Vignette task2 was developed in Germany and vignette task4 in Taiwan. Both included a breach of a norm from the perspective of the authoring national team members. Due to the sophisticated method of a concurrent vignette development process in the research project (Dreher et al., 2021), we could ensure that the resulting vignettes represent classroom situations that may occur in secondary mathematics education of both countries (ecological validity).

Instruments

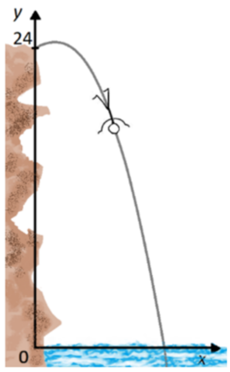
The two vignettes have a similar structure: First, a task that is considered to have a high potential for mathematical learning from the perspective of the authoring national team is presented. Second, a classroom situation is described (approx. 230 words of a fictitious transcript).

In detail, vignette tasks2 builds on a task "cliff-jumping" (topic of quadratic functions, Figure 1, left). It requires students to understand a real-life situation (presented as graphically supported text), make an educated guess about the solution based on the real-life context, and determine the solution with the help of a given mathematical model. The German authors saw the potential of the task for learning especially in its clear focus on the known difficulties of students to understand and interpret the connection between the real-life situation and mathematical models. They would expect teachers to use the educated guesses or the given visual representation to validate mathematical solutions and support students' modeling processes.

Vignette task4 builds on a task "student camp" (topic of systems of linear equations, Figure 1, right). It requires students to understand a real-life situation (presented as text) and set up a system of linear equations to determine the solution. The Taiwanese authors saw a specific potential for students' learning of this tasks, as it is suited to discuss pros and cons of different possibilities to assign variables: Assigning x and y to be the numbers of groups of students leads, for example, to a simpler calculation than assigning x and y the numbers of students in congruence to the unknowns in the word problem. The Taiwan team members would hence expect the teacher to discuss how different ways of variable assignment lead to systems of equations with different characteristics so that students acquire abilities to use different strategies flexibly for effective solutions.

Cliff divers jump from high rocks into the sea. In order to determine whether it is safe to jump, it is important to know where they will hit the water (entry point).
Imagine that the rock is approximately vertical and 24 m high.

1. How far is the entry point from the cliff? Please make an educated guess.
2. An optimal jump trajectory can be described by the function $f(x) = -(x-1)^2 + 25$.
   How far is the entry point from the cliff in this case?



In a student camp with a total of 60 secondary and primary students, students are grouped with others of the same school type. If every group consists either of 9 secondary students or of 4 primary students, then there is a total of 10 groups.
How many secondary school students and how many primary students are in the student camp?

Figure 1: Task "cliff-jumping" (vignette task2, Germany); Task "student camp" (vignette task4, Taiwan).

The classroom situations represented by the vignettes task2 and task4 were designed to depict non-optimal use of the potentials of the tasks from the perspective of the authoring team (breach of a norm). In the vignette task2, the teacher works in an interactive manner with the students but makes no advantage of the task's potential to focus on mathematical modeling processes. In the vignette task4, the teacher presents two different ways of assigning variables (x, y groups of students; x, y numbers of students) and labels the first one as resulting in a simpler calculation, but does not use the potential of the task to discuss the pros and cons of the different ways of assigning variables. During the development of the vignettes, the team members from Germany as well as the members from Taiwan already experienced that seeing the specific tasks' learning potentials and, subsequently, their non-optimal use can be difficult for the members of the other culture.

Sample and procedures

Participants were recruited from professors of mathematics education who were active in mathematics education research and in preparing future secondary mathematics teachers. As we aimed for a sample of 15 experts in each country and assumed a participation rate of at least 50%, in Germany, a random sample of 30 professors out of the full list of persons meeting these criteria was contacted. In Taiwan, these criteria yielded a list of only 32 professors and thus all of them were contacted. In total, a sample of $n_1 = 19$ Taiwanese professors (6 female, 13 male) from 10 universities and a sample of $n_2 = 17$ German professors (7 female, 10 male) from 13 universities worked on the vignettes (completion rates were TW 59%, GER 56%). To capture the experts' perspectives on the tasks' potentials and their use, the experts were given the following open-ended prompt: "Please evaluate the teacher's use of the task in this situation and give reasons for your answer."

Both vignettes were administered to experts in both countries online in their native language (German resp. Chinese). Responses were translated into English as the common language within the research team and analyzed with respect to two main aspects: 1) Did the experts evaluate the teachers' use of task as inadequate? And if so: 2) What were their reasons? We coded whether the experts saw a breach of the same norm as the authors. In addition, we expected that experts may see further reasons why the task implementation can be criticized, so we extracted further reasons inductively from the answers. More than one reason could be assigned to an answer.

RESULTS

In this research report, we summarize the coding as follows (Table 1): First, we give the number of expert responses showing no negative evaluation of the classroom situation depicted in the vignette (no breach). We count the number of responses where experts saw the intended breach of a norm. In the remaining responses, the experts only gave other reasons for their negative evaluation. To answer our research question, we focus here on the perspectives of the majority of experts in each culture on the given vignettes. With this approach, we highlight what can be considered a norm within each culture (perspective shared by a majority).

	N	Task2			Task4		
		No breach	Intended breach of a norm	Only other reasons	No breach	Intended breach of a norm	Only other reasons
GER experts	17	4	**9**	4	2	4	11
TW experts	19	2	3	14	1	**11**	7

Table 1: Summary of Coding.

We present these findings for each vignette, highlight the differences, and illustrate them with sample answers, as far as this is possible within the space limits of this report. Regarding vignette task2, the majority of the German experts saw the breach of a norm as intended and evaluated the vignette negatively as the teacher did not make optimal use of the opportunity to focus on modeling processes (see GER1_8). In the Taiwanese sample, only 3 out of the 19 experts saw the intended breach of a norm. Some German experts, as well as Taiwanese experts, criticized the dealing with the algebraic demands or the appropriateness of the task implementation in respect to practical concerns, for example, whether the classroom discussion should better be complemented by written notes. Unlike any German expert, six experts from Taiwan were concerned about the structure of the teaching sequence from a content perspective, for example, whether it is appropriate to mix up questions of quadratic functions and quadratic equations or whether the teacher managed to focus on flexible use of different solution strategies (see TW27).

GER1_8: T focuses obviously on solving the quadratic equation, while the modeling aspects contained in the task are hardly or not at all addressed. The following questions are therefore not clarified: - Mark in the illustration what is to be calculated. - How did you come up with your educated guesses? Can the illustration be used to justify which educated guess is particularly realistic? - Why is the approach of S1 correct? - What is described by the solution -4? What is the difference between the real-life situation and the descriptive function?

TW27: [...] The key message that the problem was to solve a quadratic equation with one variable and that there is not only one solution strategy was not delivered.

Regarding the vignette task4, the majority of the Taiwanese experts saw the breach of a norm as intended and criticized that the teacher did not make optimal use of the opportunity to discuss the pros and cons of variable assignment (see TW28). In the German sample, only 4 out of the 17 experts saw the intended breach of a norm. As other reasons for a negative evaluation, Taiwanese experts, as well as German experts, mentioned that the teacher does not build enough on students' thinking or that s/he works out relevant steps instead of the students. Unlike the Taiwanese experts, 8 German experts saw a lack of focus on the equivalence of the two systems of equations that resulted from different variable assignments (see GER2_13). As above, we found hence a kind of reasoning within the German responses that we did not see in the Taiwanese responses.

TW28: 1. The last line of teacher T's statements ran too fast. It was obvious that some students expressed their preference for the second method, the teacher insisted that everyone uniformly learned the first method, and the lesson immediately progressed to solving the problem without spending time on discussing how to choose "groups" to set the unknowns. 2. Some students preferred the second method, maybe because they could only set the unknowns based on what the problem asked. Although the first method was easy to solve, the students did not know how to choose which variables in the problem were appropriate to set the unknowns. The teacher must spend time discussing with the students how to set the unknowns rather than skipping and proceeding to solve the system of equations.

GER2_13 The teacher discusses the two models exclusively under the aspect of computational simplicity. The central phenomenon of equivalent modeling of a situation and the interesting insight that both models are algebraically identical is not addressed. In addition, the problem arises that the two systems do not emerge through one of the usual ways of transposing an equation, but through substitution. This is obscured by the identical naming [note: the teacher uses x, y in both systems of equations with different meanings] and is not discussed further.

DISCUSSION

This study shows that despite the international consensus regarding the relevance of tasks' potential for mathematical learning and its use, the specific understanding may differ between cultures. First, our symmetric approach of designing vignettes within the national research teams in Germany and Taiwan differs significantly from typical approaches in cross-cultural research, as it is aimed at culturally sensitive vignettes. The presented study on two such vignettes with a sample of experts from each country explored whether the vignettes reflected indeed different culture-specific norms (and not only the particular view of the authors). By means of two vignettes focusing on word problems, we showed that perspectives of German and Taiwanese experts are different, but a) within each culture in line with the expectations of the research team members. Moreover, b) the differences in reasoning between the German and Taiwanese experts are in line with described cultural differences: In the case of task2, the concerns exclusively found in Taiwan resonate with the focus of East Asian mathematics education on the mathematical content and the product-oriented perspective on establishing flexible solving strategies. In the case of task 4, the unique German reasoning referred to a perceived potential of the task for the aim of a meaningful understanding of relations between different mathematical models of a situation rather than its potential to apply specific strategies of variable assignment.

The study also has some limitations. First, a study based on two vignettes regarding word problems in secondary algebra is, of course, not generalizable, but may rather serve as a proof of existence for cultural differences that call for further research. However, the overarching research project TaiGer Noticing could also uncover culture-specific norms of responding to students' thinking between Taiwan and Germany. Second, the brevity of this report allows only a first analysis based on the distinction between answers that reflect the intended breaches of norms and other reasons. An in-depth analysis of professional knowledge and other resources that shape the experts' evaluation is still missing and could substantiate our findings.

Despite these limitations, the study shows that the understanding of the potential of a task for mathematical learning and its adequate use may be inflicted by cultural differences. To illustrate possible consequences for research: If we would have used our data for assessing the noticing of the experts (note: the data was not collected for this purpose), the German experts would have largely missed the noticing target of task4, what was easy for their colleagues from Taiwan to notice, and, at the same time, the Taiwanese experts would have been outperformed by the German experts on task2. It should be discussed how these findings can inform future comparative studies, for example, of instructional quality or teacher noticing, where researchers always face the challenge of balancing the validity of instruments within cultures and their comparability across cultures.

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A DEVELOPING DISCOURSE ON TRANSITIONS BETWEEN DIFFERENT REALIZATIONS OF THE SAME FUNCTION

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The paper analyses the developing discourse of high school students on transitions between different representations of linear and quadratic functions. Using a commognitive perspective, we conducted an exploratory study on how routines and sense-making could support students in recognizing different visual mediators as possible realizations of the same function. Results support the hypothesis that fostering students' production of narratives on multiple visual mediators could provide an important entry point to mathematical discourse.

INTRODUCTION

Research in mathematics education has highlighted that considering different representations as being "the same" mathematical object is central in learning mathematics, but it is also one of the most challenging learning achievements for students, especially in the case of functions (Sfard, 2008; Nachlieli & Tabach, 2012). This difficulty seems connected with the strong procedural emphasis through which function representations are usually introduced at school (e.g., Thompson & Carlson, 2017). The paper intends to contribute to this line of research by adopting the commognitive lens (Sfard, 2008). Recently, the discursive approach has been widely adopted in mathematics education and, especially, in studies focusing on both physical and digital representations of functions (e.g., Antonini et al., 2020). In particular, Baccaglini-Frank (2021) has highlighted how the students' concern of making sense to scholar procedures triggers a wider participation into the mathematical discourse. Building on this finding, we present an exploratory case study involving three dyads of high school students that were interviewed while they were trying to match different representations of the same function. The fine-grained analyses of students' discourse we have conducted show that the dyad who succeeded in making all the transitions between the proposed representations is the one that was more engaged in an attempt to making sense of the procedures.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

According to the theory of *commognition* (Sfard, 2008), learning mathematics is the process of changing one's mathematical discourse. Lavie and colleagues (2019) describe this process in terms of routinization of students' actions. In particular, students in a given *task situation* model their present actions, which constitute the implemented *procedure*, on what they learnt and did in the past, and it results in

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patterns of actions called *routines*. The past situations that are identified as acceptable and which allow them to act in the new task situation are called *precedents*.

Taking this perspective, a mathematical object is the *signifier* together with its *realizations* (Sfard, 2008). For example, a Cartesian graph and an algebraic expression can be two different realizations of the same function. Experts are able to make transitions between them, that is, to construct a mathematical discourse about the Cartesian graph and to translate it into a mathematical discourse about the algebraic expression. Differently, for students entering mathematical discourse on functions these are not initially realizations of a same signifier, but they play the role of *visual mediators*. Visual mediators are objects of symbolic, iconic, gestural nature to which we refer in mathematical discourse to mediate the communication about discursive objects (Nachlieli & Tabach, 2012). The evolution of students' discourse towards experts' discourse is the main objective underlying teaching and learning processes. This evolution can be grasped through the identification of students' attempt of *sense-making*, revealed by their production of "consistent, comprehensive and cohesive" narratives (Baccaglini-Frank, 2021, p. 295).

The following two research questions led our investigation: a) Do the students succeed in addressing certain given visual mediators as possible realizations of the same signifier? b) If so, how do they make transitions between the different realizations of the functions in focus?

METHODOLOGY

The case study involves a convenience sample of three dyads of students who took part in an interview as volunteers. The students attended the third year (age 16-17) at a vocational high school for "economics and commerce". A specific teaching sequence was implemented in their class by the regular math teacher, who introduced functions through the use of interactive dynamic mediators (Antonini et al., 2020). At the end of this teaching sequence, a researcher, who had never met the students before, conducted a *task-based interview* (Goldin, 2000) with the three dyads. The data collected consists of video recordings, showing what the students write, their gestures and nonverbal expressions. For this study, we focus on a task in which three lists of different realizations of functions are presented: list A is made up of Cartesian graphs, list B of algebraic expressions, and list C of input-output machines. The dyads are asked to match as many items as possible from the three lists.

The regular math teacher was also interviewed for gaining further information about the implementation of the teaching sequence on functions and it emerges that students repeatedly interacted with different realizations of parabola and line. In light of that, for this paper we select from the task the realizations of a parabola (#1 in Fig.1) and a line (#2 in Fig.1), since we expected that all dyads could succeed in making the three expected associations by moving among the given realizations. In particular, the transition between realizations belonging to list A and list B could be made recognizing A1 as a realization of a parabola with vertical axis of symmetry, whose algebraic expression contains *x* squared, while A2 as a realization of a line, whose algebraic expression is linear. Therefore, students could manipulate the algebraic expressions with calculation or work with translations in the Cartesian plane. Another possible way consists of choosing a point that belongs to the graph in list A and verifying that its coordinates satisfy the algebraic expression in list B. To make the transition from realizations belonging to list C and list B, a variable 1's value could be selected from a specific entry of a table (list C) and then assigned to the *x* in an algebraic expression (list B). The value obtained for *y* from the calculation should be equal to the variable 2's value from the same row of the table; this should be done for each row. Finally, to make the transition between realizations belonging to list C and list A, students could verify that each point, whose coordinates are pairs of values of the form (*variable 1, variable 2*), taken from the same row of a table (list C) belongs to the Cartesian graph (list A).

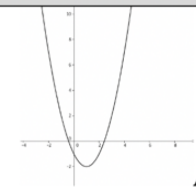
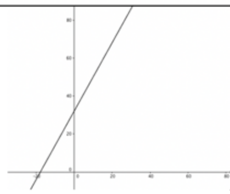
#	List A	List B	List C
1	 A1	$y = (x-1)^2 - 2$ B3	variable 1 | variable 2 −1 | 2 0 | −1 1 | −2 2 | 1 3 | 2 4 | 7 C3
2	 A3	$y = \frac{9}{5}x + 32$ B6	variable 1 | variable 2 −10 | 14 0 | 32 25 | 77 30 | 86 40 | 104 45 | 113 C5

Figure 1: Expected triplets of realizations of parabola and line.

How the analyses were conducted

Data were analysed passing through two rounds of analysis of the students' discourse. The first round is aimed at identifying the associations between different realizations made by the three dyads, whose pseudonyms are Sara-Nico, Ida-Lisa, Tina-Lena. We developed an analytical tool that is a flowchart showing which associations are actually addressed in the dyad's discourse. For example, starting from an empty flowchart composed by the correct triplets of labels for each function (i.e., A_n, B_n, C_n in Fig.1), when a dyad's discourse focuses both on a certain Cartesian graph (list A) and on a certain algebraic expression (list B) we coloured the corresponding labels and added an arrow between them (Fig. 2). This phase addresses the first research question, by looking for emergent recurrent patterns or evident differences among the dyads.

The second round of analysis is aimed at identifying the features of each dyad's discourse that are useful for answering the second research question. We adapted the analytic scheme developed by Baccaglini-Frank (2021), by specializing the focus according to the specific task the dyads were solving. We concentrated on the following aspects: *objects in focus* (What is the conversation about? Which realizations of the same signifier are used?); *routine*, as a pair of *task* (What is the association in focus?) and *procedure* (Are the students able to identify a precedent? Is this to perform a procedure?); *sense-making* (Are there signs that the participants were concerned with the consistency, comprehensiveness, and cohesiveness of their narratives?). Since space is constrained, we have only reported on the questions guiding the analysis (in brackets); all the details are available in (Baccaglini-Frank, 2021, p. 299).

CASE ANALYSIS

This section reports on the analysis carried on through the use of flowchart, and the analysis of students' discourse that describes some transitions prompted by the task.

Instances of associations of visual mediators in students' discourse

For constructing the flowchart, we considered all the excerpts of the interviews in which the students associate different visual mediators from the given lists.

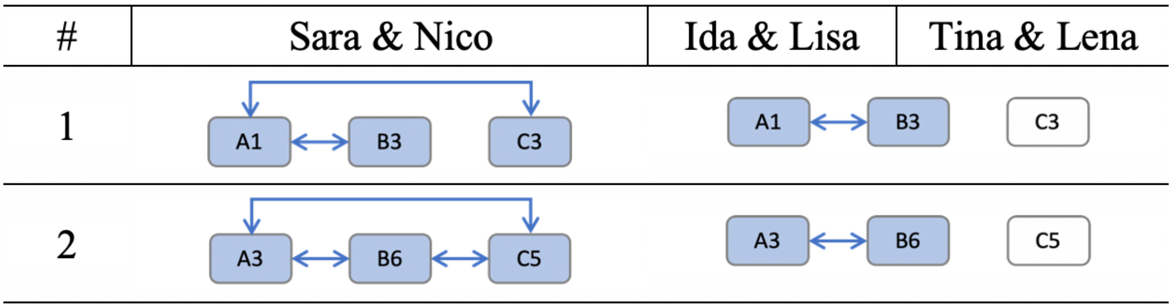


Figure 2: The ABC flowchart of the three dyads with respect to the two functions.

Looking at the ABC flowchart (Fig. 2), involving a parabola (#1) and a line (#2), we can notice both instances of complete associations of the triplets and instances of partial associations. In particular, the transition between the realization of functions as graphs (list A) and as algebraic expressions (list B) is made by all three dyads, while the association with the realization of functions as input-output machines (list C) is carried out only by one dyad. Moreover, the arrows show that in the case of parabola the graph seems to be the realization that takes on a special role for Sara and Nico, since all the associations pass through it in their discourse. Whereas in the case of line, this dyad associates all visual mediators of the triplet, suggesting they perceive them as different realizations of the same signifier. We further investigate this point in the next section.

Focusing on some transitions: a description of the dyads' discourse

By analysing students' mathematical discourse in more detail, we can describe how the transitions between the different realizations of the functions in focus occur. The starting approach to the given task is similar for all the dyads, who look at list A and choose the parabola or the line, that are described as familiar graphs ("So, let's start from this one that is simple [they look at A3]"). So, the *task* they are solving consists in looking for a suitable algebraic expression for a certain Cartesian graph. Moreover, the students' discourse shows how the visual mediators from list A (*objects in focus*) are not just unrealized symbols for them. For example, focusing on the first function (#1 in Fig. 1), the signifier parabola is explicitly mentioned at the very beginning of each interview ("If meanwhile we look for this function [they point at A1]" "Ok, parabola function") and it is then addressed many other times throughout the interview.

All the dyads succeed in identifying a precedent which prompts different *procedures*. For example, in the case of parabola, the Cartesian graph plays the role of precedent identifying classroom experiences involving quadratic expressions containing a term in the form ax^2 ("Like…squared is a parabola"). The same happens in the case of linear function ("It is a line [...] so, without x squared"; "It is a line because x is not squared"). *One of the prompted procedures* involves translations of elementary functions within the Cartesian plane. The most telling example is provided by Sara and Nico's discourse about the parabola: "We know that *x* minus one squared was…if it was *x* squared, it went through the origin and it is a parabola like this [gesture in Fig. 3a]"; "it should be moved to the right". We observe their use of past tense verbs, suggesting that they are guided by precedent identifiers that are external features of the algebraic expression and of the graph they see, which in return prescribe specific actions in the given task situation. *A different procedure* can be observed in Ida and Lisa's discourse. After B3 has been identified as a possible match for A1, they mention a "solving procedure" suggesting their intention of making some calculation on the algebraic expression ("Then I can try to solve the…the function, the equation, that one, and then see if there are some points that could fit"). Then, the students manipulate B3, reaching the form "$y=x^2-2x-1$" (Fig.3b), and go back and forth among A1 and B3 to check the association ("I try doing…I mean, I replace *x* with a number that, eventually, we can see on the parabola, but a bit…[they look at A3]"), thus repeatedly changing the *objects in focus*.

The *procedure* of manipulating the algebraic expression is applied also in the case of linear function. For example, for making the transition between A3 and B6, Tina and Lena focus on the visual mediator B6 and manipulate the expression to reach the form $y–32=95x$:

Lena: "But, wait… It goes that way, it becomes minus thirty-two [see the gesture on Fig. 3c], plus thirty-two, anyway it should intersect, right? Because it passes through... I mean... it touches the y-axis at thirty-two [points at (0, 32) on the graph]."

This form seems to be more familiar to the students and it plays the role of precedent identifier that tells them to enact the *procedure* of finding the intersection point of the line with the *y*-axis. The procedure seems to combine the use of translation of elementary functions, suggested by the gesture of moving 32 leftwards, and the associations of pairs of values with points on the Cartesian graph. However, Tina and Lena check only one value, that is (0, 32). Their effort goes in the direction of remembering the sought-after procedure for checking the association ("The slope…the slope of the line with respect to *y*, mmm *x*…Which was…I don't remember"), rather than producing consistent, comprehensive, or cohesive narratives.

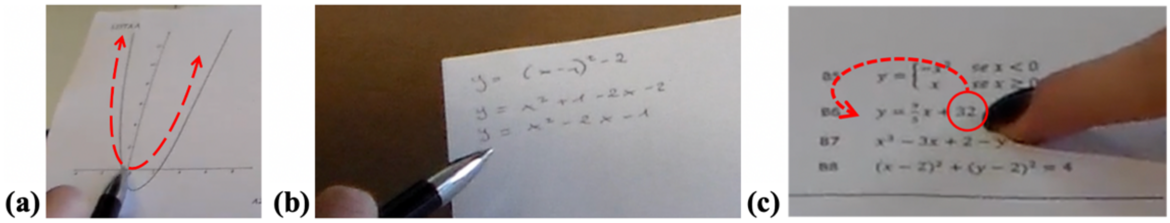


Figure 3: Visual mediators in students' discourse: Sara (a), Ida (b), and Lena (c).

Till now we reported on similarities in the three dyads' discourse, showing how the students exploit routines involving a specific realization in order to address another realization. However, for the transitions from and towards realizations within the list C things are slightly different. The flowchart (Fig. 2) shows that only one dyad's discourse involves the items of list C. We now analyse Sara and Nico's discourse about the line to deeper investigate how the transition between the realizations is operated. Here is the corresponding excerpt which starts right after the students have identified the pair A3-B6, thanks to considerations on the algebraic expression that are quite similar to those made by all the other dyads ("It is linear, so a line should fit well").

1	Nico	But, if we wanted to be meticulous
2	Sara	We check
3	Interv.	Eh, if you wanted to check better, how could you better check?
4	Nico	It should be this [he points at x in B6]
5	Sara	We put an x-value and we check [...] We give zero to the x [Fig. 4a] and it turns out thirty-two [Fig. 4b]

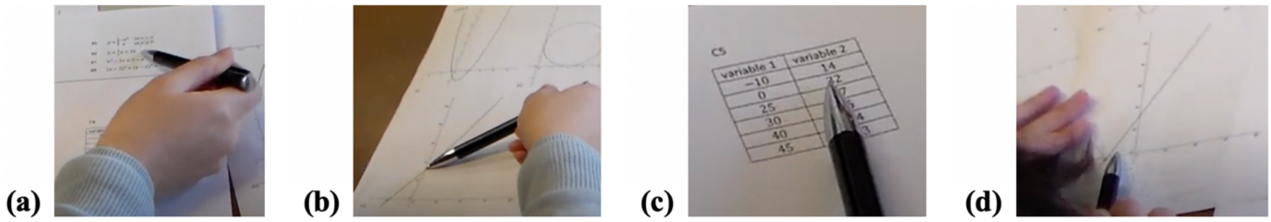


Figure 4: Gestures performed by Sara and Nico during the interview.

Differently from the other dyads, Sara and Nico seem to be concerned with *making sense* of the accuracy of the match A3-B6. Indeed, Nico suggests "to be meticulous" [line 1] and Sara supports this idea, proposing a *new task*, also endorsed by the interviewer [line 3], which consists in checking some pairs of values. The procedure of assigning input values to the x-variable and finding the corresponding outputs [line 5] seems to be a precedent identifier of the realization of functions as input-output machines, which were presented during math lessons. For the sake of brevity we do not report the entire excerpt, but the dyad continues assigning many other values to the x-variable. Thanks to this *procedure* they endorse the proposed association A3-B6. Overall, Sara and Nico's discourse shows that they recognize the outcome of the input-output procedure as related to their new task (*sense-making*). The following excerpt demonstrates how they complete the triplet of realizations.

6	Nico	Here, for example, we gave twenty and we know that the intersection would be at sixty-eight, I mean the quantity
7	Sara	Shall we look for twenty and sixty-eight? [...] Zero thirty-two [Fig. 4c], sorry!
8	Nico	Zero thirty-two, that's true! [enthusiastic tone]
9	Sara	Twenty-five sixty-seven, we're already there [points at A3]
10	Nico	That's true, well done!
11	Sara	Minus ten and fourteen, here we are [gesture on Fig. 4d]
12	Nico	Yes, yes
13	Sara	It should be this [the pen is on B6]
14	Nico	Let's go [writes down A3 next to C5]. That's all right!

The procedure carried out by Sara and Nico to verify the chosen relation A3-B6 allows them to make the new transition towards the realization in list C [line 7]. Indeed, they decide to solve a *new task*, that consists in comparing the pairs of values previously calculated [line 6] with the ones given in the tables of list C [line 7]. In this way, they indirectly associate B6 and C5 by constructively using the outcome of their procedure. The pair (0, 32) is recognized as a useful outcome to be reinvested in the new task [lines 7-8], demonstrating the dyad is developing consistent narratives on different realizations of lines (*sense-making*). Finally, we also find instances of the association A3-C5 in the students' discourse because they check the other values in table C5 with the corresponding points in the graph A3. At this stage, they complete the triplet [line 14] and fully answer to the task given at the beginning of the interview.

CONCLUDING REMARKS

Although the limited number of students that were interviewed and the narrow focus on two specific functions, we can draw some promising conclusions that could represent the starting point for further research.

Analyses reveal that all the dyads succeed in associating a graph with an algebraic expression of the same function, through the use of several routines that are activated by a specific realization for recognizing another one. In doing this, a common signifier ("parabola" and "line", respectively) is explicitly mentioned as the main object in focus that allows the dyads to identify a precedent which prompts different procedures for making the transition. Most of the expected transitions are actually made by the students, except for the visual mediators of list C that are less addressed in the dyads' discourse, suggesting that tables are not recognized as realizations of functions by all the dyads. This is quite surprising, because the teacher claimed that the discourse on functions was established in the class starting from input-output machines as possible realizations. In this scenario, the analysis of Sara and Nico's discourse provides an interpretation of this emerging finding. Nico's concern of making sense to the procedures that were established in the class brought the dyads to produce an intermediate realization that bridges the gap between the Cartesian graph and the unrealized numbers of the given table. Despite the routine of constructing input-output machines seems to be shared by all students, during the interview it is only exploited by Sara and Nico in their search for sense-making. In other words, in their attempt to make sense of the procedures they come to recognize the visual mediator of list C as a realization of the same function realized by the already associated graph and algebraic expression.

Our findings are in line with the strand of research that considers routines as windows onto students' learning (Lavie et al., 2019) and, especially, support the hypothesis that the necessity of making sense to scholar procedures triggers students' engagement into a mathematical discourse closer to the experts' one (Baccaglini-Frank, 2021). Moreover, the task designed for the interview revealed to be valuable for fostering students' production of meaningful narratives on different realizations of the same signifier, as demonstrated by the dyads' rich discourse. Although the learning path of these students may still be long, discourse about the transitions between different realizations might constitute a step towards experts' mathematical discourse and, in general, a form of participation in this discourse.

Acknowledgments

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STUDENT INTIMIDATION IN THE MATHEMATICS CLASSROOM

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This study adopts the Contextual System Model as it utilises the "Draw your mathematics classroom" test as a methodology to explore student intimidation in the mathematics classroom. As part of a larger investigation, the study included thirty participants, aged 5-7, who attended three different schools located in rural Cairo. Mathematics students were asked to create drawings in response to the aforementioned prompt. The drawings were qualitatively analysed according to the criteria of the toolkit in conjunction with a follow up semi structured student group interview. Findings indicated student intimidation in classrooms, a process that seemed to be perceived by students as constructive disciplinary practice. The study suggests tackling deep rooted questions of agency and perceived teacher authority.

INTRODUCTION

In their work with children, Shumba (2013) presented different forms of emotional abuse that take place in classrooms as teachers mal-interpret their authority as educators. Younghusband (2010) also elaborated on different forms of abuse that happen by teachers in classrooms worldwide. Younghusband (2010) discussed physical, verbal, administrative and system abuse as ways to intimidate students, thereby rendering them as easier to govern in the classroom. In the interest of exploring the context of rural Cairo, this study focuses on early learners' experiences in the mathematics classroom. Details of the study are presented in the next sections.

LITERATURE REVIEW

This section outlines the literature stance about student intimidation in the classroom at early learning stages as resulting from a distorted teacher-student power dynamic. The impact of student intimidation on the mathematics classroom is then presented. The Contextual System Model is also presented as a theoretical framework for this study.

Student intimidation in the classroom

Sansanwal (2019) conceptualises three types of teacher-student relationships that are particular to the early years, namely: warm teacher-child relationships, conflicting teacher-child relationships and dependent teacher-child relationships. The former builds on concepts of the attachment theory (Bowlby, 1982) and presents itself as a relationship where the child feels safe to trust the teacher and where retrospectively this trust is not abused. The latter two represent relationships, where either one or both parties express mutual anger (conflicting teacher child relationship) or where one party takes advantage of its superiority, thereby creating an imbalanced sense of child

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 155-162). PME.

attachment (dependent teacher child relationship). The latter two types of relationships, when experienced at early learning stages result in a distorted image of teacher authority, leading to a potential classroom experience of student intimidation.

The impact of student intimidation on the mathematics classroom

According to scholarship (Stylianides & Stylianides, 2014), the mathematics classroom needs to be a space where students are engaging in problem solving activities. This process of problem solving entails a series of steps starting from identifying the problem all the way through finding different routes that are underpinned by mathematical concepts to analyse and offer alternative solutions in response to the framed problem statement (Baars, Leopold & Paas, 2018). In order for this process to happen, the teacher needs to be able to step back to a facilitator role, thereby enabling the student to assume authority and ownership over the problem at hand. A recent study conducted in the Egyptian context revealed how a distorted image of teacher authority might result in a tightly controlled mathematics classroom experience, where students barely navigate their way through the procedures provided by the teacher and hence are crippled to act as problem solvers (Makramalla, 2021).

THEORETICAL FRAMEWORK: THE CONTEXTUAL SYSTEM MODEL

In his attempts to unpack how children form patterns of relatedness and interactions, Pianta (1999) coined the contextual system model. The model brings together two contextual units; namely the child-family unit and the child-school unit. For the scope of this paper, I focus on the latter. Pianta (1999) presents the school as a complex integrated system that a child navigates already at early stages. This complex system includes multiple strands such as peer relations, teacher relations, school coordinator relations, school policy and infrastructure and so on. In an attempt, to navigate their identity within the complex multi-stranded school system, the child at early stages of their learning, looks up to the teacher as the compass for navigation (Figure 1).

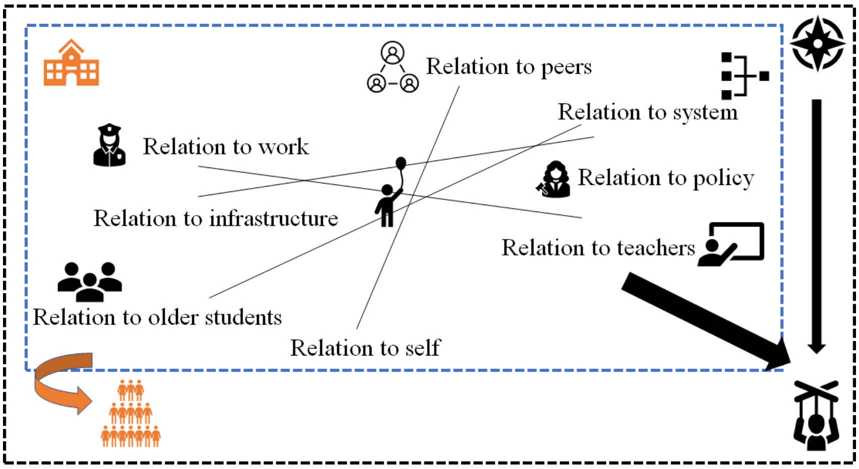

Figure 1: Multi-stranded Navigation System.

Figure 1 shows how the child struggles to navigate between the different strands of the complex schooling system, which is a sub-system of the wider societal system.

According to Pianta (1999), at the early learning stages, the child cannot be held responsible for balancing this multiplicity of relations and therefore, the quality of the interactions that the student has with the school as a system needs to be fostered by the teacher. In other words, it becomes the teacher's responsibility to aid the student as s/he navigates their role, identity, and agency in relating to the different players of the school system. At the early stages, the teacher frames the student's sense of relation to key players in the schooling unit in ways that align with the teacher's own beliefs and practices about their own relation to students. As the teacher directs the student to navigate their way in this complex system, this navigation according to Dewey (2013) translates into how the learner navigates the wider complex system of society.

RESEARCH DESIGN

In this section, I utilise the drawing toolkit as a methodology for exploring subconscious student perceptions of the teacher-student relationship. The double stranded data collection protocol, which utilises semi-structured group interviews as a build-up on the student drawing activity is also presented along with the context and the selected sample for the study. Finally, the analytical framework for correlating the student drawing analysis to form relevant subsequent interview questions is presented.

Analytical Framework: Conceptualising student drawing filters as a toolkit

In their study, Thomas, Pederson and Finson (2017) have identified the act of drawing as a suitable tool for uncovering how students subconsciously related to their teachers as authority figures. According to the authors (Thomas et.al., 2017), the act of drawing lowers the guards in students' minds and accordingly more accurate data can be extracted in terms of how they relate to their teachers. Figure 2 presents a student drawing that has been depicted by a primary stage student in Sweden.

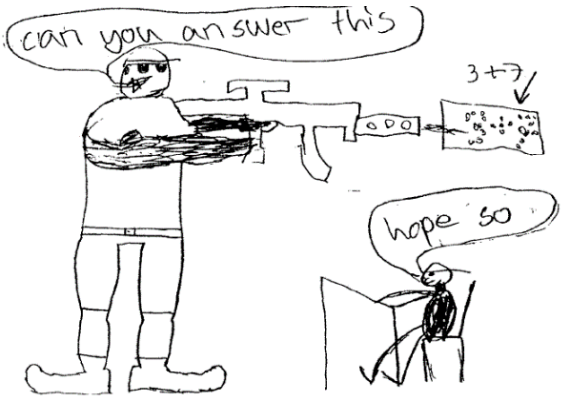

Figure : Student Drawings in previous studies (Picker & Berry, 2000).

As evident from Figure 2, the student drew themself as very small compared to teacher size. Also, the student has depicted the teacher as holding a threatening tool. These two features have been conceptualised along with other features as part of a drawing filter analysis toolkit, which in turn acts as an analytical framework to check-mark student drawings for the purpose of uncovering indicators of student intimidation in the classroom (Picker & Berry, 2000; Thomas et al., 2017). Based on the drawing analysis, a sub-sequent semi-structured student group interview took place to triangulate the findings of the drawings. For the case of the drawing depicted in Figure 2, for example, the subsequent interview revealed that there was no physical threatening tool in the classroom. The student has expressed their sense of intimidation by mentally depicting the classroom in this way. The analysis of the subsequent semi-structured group interview utilises the same filtration tool to complement the analysis of the drawings.

Framing the Research Question

The current study utilises children's drawings as tools to uncover how students relate to their mathematics teacher as an authority figure in the classroom. The study aims to answer the following question: How do student drawings inform us about student intimidation in the mathematics classroom for early learners in rural Cairo?

METHODOLOGY

This section outlines the context of the study, followed by the data collection protocol and the analytical framework, which has been adjusted to fit the scope of this study.

Context of the study

As part of a larger study (Makramalla, 2021) that investigated student perceptions of schooling, a group of thirty students, aged five to seven, of mixed genders were asked to respond to the prompt: Draw your mathematics classroom. The students were not offered additional elaborations of the prompt and were not assisted during the act of drawing. This took place at a summer school that brought together students from three different schools within the same district of rural Cairo. The idea was to capture data from different sources within the same context for triangulation purposes (Yin, 2011). Prior to the study, consent was attained of the summer school leadership and of the students' legal guardians. Additionally, a semi-structured group interview took place. Students were at ease throughout the entire data collection process.

Data Collection Protocol

The data was collected over four stages. Firstly, students were provided drawing tools along with a drawing prompt. Secondly, the drawings were filtered in accordance with the reduced analytical framework, presented below. Thirdly, based on this preliminary filtration, a semi-structured group interview re-emphasised the rationale behind the some drawing features that were depicted by the students and which corresponded to the filtration features. Finally, students were rewarded for their participation.

Analytical Framework

A previous study (Makramalla, 2016) has conceptualised a sequential double filtration analytical framework in conjunction to the prompt: Draw your mathematics classroom. The framework was designed with the target of exploring indicators for student intimidation in the mathematics classroom. This study utilises this framework as an underpinning analytical tool for assessing the student drawings and interviews. For the scope of this exploration, two particular filtration features will be emphasised, namely drawing dimensions and symbols indicating physical abuse (such as a cane or the like).

FINDINGS: STUDENT DRAWINGS

As already mentioned, for the scope of this paper, I present the analysis of the drawings and subsequent student interviews as mapped against two main features:

- Dimensions; i.e. depicted student size reference to depicted teacher size
- Symbols of physical abuse in the classroom

Figure 3 presents a sample student drawing that depicts the teacher holding a cane and the students depicted in distorted dimensions reference to teacher size.

Figure 3: Sample of student drawings.

As evident from Figure 3, the students depicted herself as much smaller in size compared to the teacher and she also depicted the teacher as holding a cane. Traces of distorted dimensions, similar to the one in this example, were evident in 63% of the student drawings. A cane was depicted in 82% of student drawings.

When asked, during the follow up group interview, whether the teacher was maybe utilising the cane as a pointer, students clearly responded that the cane was used “to beat up the student that gave wrong answers”. Students were asked whether they thought that being beaten would help them to find the right answer, to which student responded that it was “the teacher’s role to discipline those who gave wrong answers”.

Students finally indicated that the teacher was the main player in the classroom, that s/he “knew better” and that therefore they “needed to follow the instructions as given”. Students seemed to have a sense of inferiority and threat as they think of their mathematics classroom. This mental depiction would naturally result in a deprivation from developing problem solving skills as will be further discussed in the next section.

CONCLUSION

As already noted, the contextual system model acts as a theoretical framework for discussing the findings of this study. According to this framework, at stages of child early development, the teacher sets the pace for how the child ought to relate to their teacher as an authority figure. The findings of this study show how the student relates to teachers as oppressive and intimidating authority figures. The findings also show that this abuse of teacher authority is normalised to the extent that students consider it to be best practice. In line with the contextual system model lens, it seems to be that teachers set this tone in terms of classroom power dynamics early on, which in turn could result in a lifelong distorted perception of authority, which replicates itself at each level of the authority dissemination ladder (Herrera & Torres, 2006).

Violence in the classroom as a form of student discipline.

Given the worldwide attention to violence in the classroom (Pinheiro, 2006), it is not surprising to find indicators of physical violence, expressed through student drawings. Despite the advancement of pedagogical practice, different forms of violence (Younghusband, 2010) are still manifested in different cultures as part of the daily classroom routine. The more alarming reality is that students identify with violence as being a form of "discipline", claiming that the teacher would be doing it with the best interest of the student in mind. This distorted image of discipline seems to be penetrating students' minds, making them unaware of the harm being enforced on them and making them relate positively to their abusers. Scholars (Solomon & Sekayi, 2010) have studied reproductive cycles of teachers acting in ways that they themselves experienced as learners. Teachers are often unaware of the harm they are causing as in their minds; they relate to their own teachers as role models. In other words, there is a danger that teachers might not even be aware that they are harming the students and might even themselves believe that this practice was in the best interest of the students.

Violence as normalised practice in the mathematics classroom.

Building on findings from a previous study (Makramalla, 2021) conducted in a different Egyptian context, it seems that the student relation to the mathematics teacher as an autocratic authority figure has often been normalised in classroom practice to the extent that students would have no negative connotation to this authoritarian relationship but would instead consider it to be normal.

In their study of abuser-abusee relationship, scholars often refer to this as the Stockholm Syndrome (Fabrique, Romano, Vecchi & Van Hasselt, 2007). In alignment with the contextual system model, where the teacher sets the pace for how the student would relate to authority figures, the Stockhom Syndrome refers to the status, where the victim develops emotions of trust and affection towards the abuser. Building on this understanding, it becomes clearer, why students relate positively to the teacher holding a cane. This presents a potential explanation of why the presence of violence instruments in the classrooms is normalised as part of the classroom infrastructure.

Drawing as a vehicle to unveil mathematics classroom power dynamics

In response to the research question, this study aligns itself with previous studies in mathematics education scholarship (Makramalla, 2021; Picker & Berry, 2000) in confirming the power of the arts to act as a filtration tool for identifying student perceptions. The wider focus group discussion that resulted from the drawing activity would have been very difficult to trigger, if the students were not prompted by a drawing or a similar form of self expression first. Both the drawings and the subsequent conversation in the group confirm a distorted power balance that is instilled by the teacher as a trusted authority figure in early years of learning. Based on the contextual system model, this study also shows how this distorted mental image of mathematics teacher authority becomes normalised.

Particularly focusing on mathematics instruction, this distorted image of classroom power dynamics prevents the process of creativity and problem solving as students perceive themselves as inferior, uncapable to solve problems without the guidance of a dominant authority figure. This mentality blocks the sense of autonomy at an early stage, thereby creating learners that can very well recall procedures but that would find it very hard to analyse problems or create solutions.

IMPLICATIONS

This study aligns itself with previous studies in scholarship as it presents the dominant case of normalised violence in the classroom. Despite the advancements in pedagogy, worldwide mathematics students are still intimidated by their teachers. Implications of this study call for awareness raising, policy formations and training of teachers, in order to re-envision the creative dimension of the mathematics classroom that is impossible to foster in an atmosphere where intimidation is normalised.

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DIMENSIONS OF VARIATION IN TEACHERS' APPLIED MATHEMATICS PROBLEM POSING

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This study suggests eight different dimensions through which products of teachers' applied mathematics problem posing (AMPP) can be modified to achieve different pedagogical goals: authenticity, correctness, compactness, mathematical diversity, multiple data representations, answer format, generalization, and students' agency and decision making. The dimensions were identified from a qualitative multiple-case study using variation theory as a theoretical framework. We compared items and versions of secondary teachers' AMPP products during professional development (PD). The resulting model informs teacher educators and researchers in planning and implementing AMPP in teacher PD, can serve as a basis for an assessment model of AMPP product, and enhance teachers' learning in task design environments.

INTRODUCTION AND THEORETICAL BACKGROUND

Modern teachers are expected to choose and design their instructional resources (Jones & Pepin, 2016; Remillard & Heck, 2014). Pepin et al. (2015) claimed that adapting, assimilating, and designing mathematical tasks improve teachers' pedagogical knowledge for teaching mathematics. However, a recent literature review on teachers-as-designers found that while studies on the potential of teacher-designed resources for student learning in mathematics education are common, studies that focus on teacher learning are almost absent (Pepin, 2018). We address this lacuna by situating our study in the context of teacher professional development, where the design of reality-related and applied mathematics problems for achieving various pedagogical goals is the primary vehicle for teacher learning.

Teachers' task design for students in mathematics education can take the form of problem-posing (PP, Cai & Hwanºg, 2020; Koichu, 2020). Per Koichu, PP of teachers includes reformulation and generation of new tasks to advance students' problem-solving performance (Koichu & Kontorovich, 2013; Koichu, 2020). Our conceptualization of PP as an activity for teachers is close to Koichu. In our study, PP of in-service teachers is an authentic mathematical activity that arises from teachers' pedagogical need to develop students' mathematical competencies through applying mathematics to realistic situations (c.f. Gravemeijer, 1999) and from a corresponding aspiration to develop teachers' design capacity (Brown, 2009). We focus on teachers' applied mathematics problem posing (AMPP) products and perceive variations in designed tasks as indicators of teachers' learning (Brown, 2009; Pepin, 2018).

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Theoretical foundations for dimensions of applied mathematics problem posing

Maaß (2010) developed a classification scheme for modeling tasks (i.e., tasks applying mathematics to realistic, open problems) based on existing theory (e.g. Blum, 2002). This scheme reflected different features of modeling tasks, offering guidance in the task design and selection processes for specific aims and predefined objectives and target groups. The model incorporated nine classifications. Among them, three are generic (applicable for different types of tasks): openness; cognitive demand (e.g. mathematical reasoning); mathematical content (topic, level). Six classifications are pertinent to modeling tasks: the focus of modeling activity (all modeling cycle or part of it); data (excessive or lacking data); nature of the relationship to reality (authenticity); situation (e.g. personal, scientific); type of model (descriptive or normative); type of representation (e.g. textual or pictorial). The classification developed by Maaß (2010) was used to characterize modeling tasks developed by educational specialists for a particular target group of students. Specifically, these tasks were not intended to be modified. We used Maaß's (2010) classification as an initial framework for characterizing dimensions through which teachers may modify their AMPP products to achieve different pedagogical goals. To identify more dimensions of variations in the context of AMPP, we applied variation theory (Lo & Marton, 2012).

Variation theory and research question

Variation theory relies on the premise that learning is always directed at something (phenomenon, skills, or certain aspects of reality) and conceptualized as a qualitative shift in the way of perceiving this "something" (Marton & Booth, 2013). To see or experience an object of learning in a certain way requires the learner to be aware of its specific aspects and discern these aspects simultaneously. Lo and Marton (2012) emphasize that awareness is stimulated by experiencing difference (variation) between two values as a contrast. When we become aware of a value by contrasting it with another value (e.g., large vs small), the value is separated from the object of learning, and a dimension of variation is realized (e.g., size). Then, the object is perceived with its value (feature) and its dimension of variation, and the learner can focus on the value alone, naming it and even changing it (Lo & Marton, 2012). Our study is focused on teachers' AMPP products as the objects of learning. We use variation theory as a methodological tool to discern the dimensions of variations within these products to discuss further their relationships with pursuing different pedagogical goals.

Therefore, our research question is: Across which dimensions do secondary teachers' AMPP products vary?

METHODOLOGY

This paper derived its data from the first year of a three-year PD program in which secondary school teachers designed applied mathematics tasks. The PD was conducted as a community of practice (Hodges & Cady, 2013) in which teachers, teacher educators, and researchers collaborate to achieve specific goals (Cooper & Koichu, 2021). Addressing Israeli students relatively low achievement in OECD's Programme for International Student Assessment (PISA; OECD, 2019), the project's overarching goal was to improve students' mathematical competencies by applying mathematics to realistic situations. We identified that one barrier for students' success is teachers' unpreparedness to use AM tasks even if they are provided with them. We hypothesized that through designing their own AM materials, teachers would (a) develop their capacity to use mathematics knowledge and skills in real-life challenges; (b) gain ownership over the materials they develop and an inner motivation to implement them in their classrooms (Brown, 2009; Koichu, 2020).

There were three cycles of AMPP development in the PD, each constituted of PP initial design, receiving feedback from the PD community, redesigning, classroom testing, and final revisions. In the first year, eight experienced teachers participated, each from a different school in Israel, with teaching experience ranging from 5 to 25 years. The teachers reported they never engaged in composing tasks independently and used only textbook problems or tasks found online in their classrooms. Five more participants in the community were three experienced teachers serving as community leaders (one of them the first author of this paper), a mathematics education researcher (the second author), and an assistant researcher. The leading team did not have any formal experience in AMPP and therefore perceived themselves as part of the learning community – as facilitators and not as instructors.

Data collection and analysis

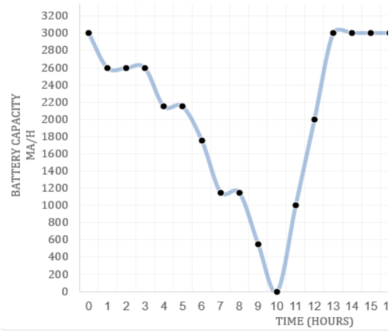
The data comprises 22 AMPP products composed by teachers, each including multiple drafts generated throughout PD community sessions (see examples in Figures 1, 2). In addition, we collected teachers self-reported considerations on their problem-posing attempts. We used variation theory to compare different task items within individual tasks and across different tasks made by different teachers. Since our goal was to identify the dimensions of variation (Lo & Marton, 2012), whenever we identify a case of dissimilarity between two items, we categorized it as a potential dimension of variation upon mutual agreement between the authors. If this dissimilarity repeated itself throughout the data, we established it as a prominent dimension of variation. Teachers' reports during community meetings regarding their considerations and pedagogical goals they sought to achieve through a particular design decision were used as complementary data. This process is exemplified by the two tasks (Figures 1, 2) that manifest diversity across all the identified dimensions.

Shira is preparing for a 24-hour annual school trip. She is taking her cellphone, which has several applications, each consuming different battery life in milliampere-hour (mA·h), as shown in the table. The capacity of her phone battery is 3000 mA·h.

1. What percentage of the battery will Shira consume if she hears music for one hour?
2. Shira played a game for five minutes, used Instagram for 20 minutes, and listened to music for two hours. What percentage of battery did she have left?

3. Shira wants to use all the applications listed in the table during the trip. Suggest a reasonable usage that will leave Shira with 10% battery life at the end of her trip.
4. At 8:00 p.m., Shira's mother called her. How long can they chat so that Shira's phone will have 8% of the battery charge after the call? *Assuming there were 30% of the battery left at that time.*
5. *The graph below presents Shira's battery charge in mA·h during the trip. Write down the applications in the order she used them.*
6. *The pie-chart below presents the distribution of Shira's application use in percentage. What is the probability that Shira will be playing a game when her mother calls?*

Application	mA·h
WhatsApp	300
Instagram	400
Music player	450
Phone calls	500
Gaming	600



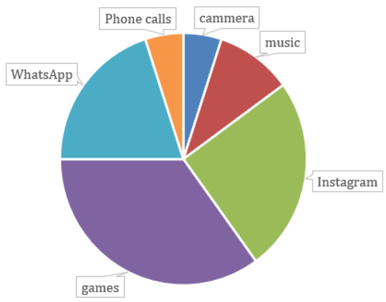
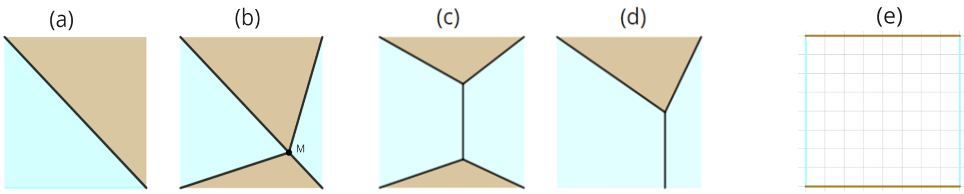


Figure 1: Rona's task (in italics: text added to a revised version after receiving community feedback).

A jewelry store owner wants to redesign an 8-meter-wide square ceiling in her store. She received several alternative designs from the architect, each combining glass panels (blue) and plaster (brown) with aluminium beams (black) between them.



A1. The store owner would like to have as much area under the glass roof to provide natural light as possible to save energy. She would also like to reduce the overall length of the beams to save their cost. Which of these designs (a-d) would you recommend her to choose?

A2. The store owner asked the architect to relocate the middle beam in designs (c) and (d) to minimize the cost of the beams. Is that possible? Justify your answer.

B1. Calculate the total area of glass panels, plasterboards, and beams length in the project (a).

B2. In the architect's second sketch of the store's ceiling, four aluminium beams from the corners are connected at a single point M, which appears three meters from the left side and one meter from the upper side. Sketch the design using the grid (e) and calculate the areas of glass and plaster panels as well as the total length of the beams.

B3. In design (e), are there any other places that point M can be repositioned so that the total area of the glass and plasterboard panels will be equal?

B4. The store owner chooses which model is cheaper, (c) or (d). Which of the following will affect her decision?

(i) The cost of a glass board per square meter.

A3. If the side of the square ceiling was x meters long and the length of the middle beam was y meters, what was the ratio between the glass area and the plaster area?	(ii) The cost of a plasterboard per square meter. (iii) The cost of an aluminium beam per meter.

Figure 2: Suha's task. A – an initial task, B – task after reflection and redesign .

RESULTS – THE DIMENSIONS OF TASK VARIATION

We present the task variation dimensions found in products of teachers' AMPP and illustrate them with the analysis of problems composed by the teachers Rona and Suha.

Rona's task was developed while preparing for her annual school trip. She realized how worried her students were about their cell phones having enough battery power. Her task is thus based on a relevant topic for her students. The mathematics it entails (capacity word problems) arises naturally, and Israeli textbooks do not contain such an up-to-date context. These considerations were widespread in teachers' tasks. Thus, the first dimension is *authenticity.* This dimension is divided threefold and refers to the extent to which the context of the problem is (a) relevant, appealing, and motivating for students; (b) nonartificial, and (c) original and not overused in textbooks.

Many teachers' tasks, especially early drafts, contained mathematical mistakes, missing data, or excessive prerequisite students' mathematical knowledge. For example, item 4 in Rona's task is unsolvable without the text added in the revised version, and Suha's A2 necessitates knowledge in calculus that her target students do not yet possess. These items illustrate the dimension of *correctness* – the requirement of the task to be solvable without out-of-scope mathematical knowledge, where given information befits the real-life context.

Comparing Rona's items 3 and 4, one can see that they both demand almost similar reasoning and calculations. This phenomenon was commonly observed in our data: different items invited repeated enactment of the same mathematical procedures. Thus, we identified the *compactness* dimension – the extent to which each item stimulates non-repeating mathematical procedure(s) and processes. Note that, unlike correctness, *compactness* is not as crucial and, in some cases, repetition is even worthwhile.

We found that teachers in our study tended to incorporate several and usually distinct mathematical topics in a single task. For instance, Rona's task combines percentages, capacity word problems (items 1-3), the notion of slope (item 5), and probability (item 6). Suha's task mainly concerns geometric concepts and includes an optimization problem (A2) and the concept of ratio (A3). We, therefore, identified the *mathematical diversity* dimension, which captures the extent to which different and distinct mathematical sub-domains are combined in the same task.

Some teachers' tasks included different kinds of data representation, summoning opportunities for transformations between them. In Rona's task, the added items (4) and (5) diversify the repertoire of data representations to include a graph and a pie chart. Suha's item B2 exemplifies a request to transform data from one form to another (text to diagram). We call this dimension *multiple data representations*.

The teachers in our study have included items of different formats in their tasks: single vs multiple correct answer(s) (as Rona's items 1 vs 3), open-ended items (Suha's A1) vs multiple-choice items, (Suha's B4). The teachers intended their students to solve by applying exact calculation (B1) aside from evaluating quantities (B4). We identified it as the *answer format* dimension.

One can see that Rona's Items 1 and 2 are calculations of specific cases, and item 3 generalizes them. Suha's A3 is about producing a generalization formula, and B3 and B4 strive at generalizing which features of the ceiling design impact its cost. These items helped us identify the *generalization* dimension that indicates to what extent students are requested to formulate a general statement or concept, obtained by inference from specific cases.

Both teachers present a need for calculations and mathematical reasoning by positioning students as consultants in decision-making. Suha is recruiting student mathematical efforts to help a jewelry retailer choose an economical ceiling design for her store (item A1). Rona asks students to help Shira plan a reasonable usage of her cellphone battery (item 1-3). However, Suha's item B1 requests calculations supporting no apparent decision. These differences led us to identify a dimension of *students' agency and decision-making*.

DISCUSSION

The research question of our study was: Across which dimensions do secondary teachers' AMPP products vary? We identified eight dimensions through which products of teachers' applied mathematics problem posing (AMPP) can be modified to achieve different pedagogical goals: authenticity, correctness, compactness, mathematical diversity, multiple data representations, answer format, generalization, and students' agency and decision making. We exemplified each dimension based on two tasks composed by teachers, each with initial and final versions. Six of the identified dimensions are close to characteristics of modeling problems (see the classification of Maaß, 2010). However, since our model stemmed from products of teachers' and not educational specialists like Maaß's (2010) classification, it brings force dimensions such as *correctness, compactness*, and *students' agency and decision making.* As for dimensions corresponding to some of Maaß's classifications, they stress different meanings and aims. For instance, Maaß's *mathematical content* class corresponds to our *mathematical diversity* dimension. However, while Maaß's model only states the mathematical content involved, we stress combining different mathematical topics in the same task.

Our dimensions of variation enrich the theory of teachers as designers by suggesting specific considerations teachers can make to compose or adjust mathematical problems to fulfil various pedagogical goals. Of the dimensions, correctness is the only one that is a sine qua non – we cannot afford a final AMPP product that is incorrect. Other dimensions are less compulsory - variation is allowed and even advantageous. For instance, not all tasks or items must have multiple data representations. However, those who have it may be pedagogically richer, containing more opportunities for students to benefit from their engagement in the task. Rona's task does not have items that vary along the generalization dimension (as in Suha's A3 and B4); this observation can inform her design, suggesting a direction for adding a generalization item.

Adding variation along one dimension may well be at the expense of another dimension. For instance, the added items in Rona's task (items 5 and 6) vary the data representation and the mathematical diversity dimensions. However, it also demands more working time (hazarding the correctness dimension) and reduces students' agency in decision making (the answers requested support no decision). So, when a teacher-designer varies a task along one dimension, she should also observe the other dimensions maintained in balance, seeing any change as a potential tradeoff.

Our eightfold dimension model could also be used to assess teachers' design products and capabilities (Brown, 2009). These dimensions can constitute a standard for designers to serve for evaluation. For instance, if one strives to present mathematics as a unified, powerful toolkit for problem solving, tasks blending various mathematical topics around the same real-life context with multiple data representations may serve her pedagogical goal. By setting the *mathematical diversity* and *multiple data representation* dimensions as essential aspects of task design, one could evaluate teachers' products to the extent these dimensions are enacted.

Although the perplexity entailed, we deliberately decided to include an initial version for each teacher's task. The changes from the initial tasks to the final versions were made by Rona and Suha themselves after receiving feedback from the PD community. Behind each change lies a realization about a particular shortcoming in the initial design to achieve some pedagogical goals. For instance, during a few PD sessions, Suha gradually realized that A1 was too complex for her students to solve because it was overloaded with geometric concepts unknown to them (e.g. area preservation, adding auxiliary constructions). She understood she could break down A1 into a gradual sequence of items (B1-4). Her capacity to modify the task along the *correctness* dimension instigated mathematical and didactical insights for her to learn. We hope that our eight-dimension model would help teachers learn more profoundly from processes of iteratively designing tasks. It may help them recognize dimensions across which their items can be composed and modified.

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TEACHING STATISTICS WITH PROJECTS AT THE UNIVERSITY: A CASE STUDY

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We present the implementation and analysis of a study and research path (SRP) in Statistics for business administration students. An SRP is an instructional proposal developed within the Anthropological Theory of the Didactic based on the inquiry of open questions. Our analysis focuses on the in-process evolution of the SRP, as well as the qualitative a posteriori analysis of its implementation. The results help describe interesting instructional devices for their design and management and identify some critical challenges that explain the difficulties of their dissemination in university education. We emphasize the need for educational research to focus on the conditions and constraints that enable and hinder the existence of project-based learning activities in current educational systems.

INTRODUCTION

Statistics is a branch of applied mathematics that is undergoing a significant evolution in the last 30 years. During the 1990s, it has started to be defined as data science and the emphasis began to be put on the importance of the empirical data and the context they come from. Wild and Pfannkuch (1999) point out the need for data, their visualisation and the reasoning within the statistical context. Such a shift in the professional area of statistics, followed by substantial technological developments enabling the collection, treatment and analysis of a big amount of data, posed a challenge to the education system as well. The teaching of statistics tends to or at least should tend to follow the recent changes in scholarly knowledge. However, the initiatives and reforms of education take time and do not follow the developments in the professional areas instantly.

As one way of adapting the teaching of an area to its professional evolution, Knoll (2014) detects the emergence of the Project Method as an instructional approach for the training of architects in the 16th century in Italy. During the centuries, teaching methods with similar ideas grew and were reshaped, for today to still be adapted and used in education at all levels and around the world. Nowadays, the most common expression for the successor of the Project Method is *project-based learning* (PjBL) (Harmer, 2014). In statistics, Batanero et al. (2013) suggest implementing such an instructional approach to connect the mathematical concepts and the statistics environment, therefore developing a "statistical sense" of the students.

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Related to the PjBL movement, we are here considering a proposal coming from the Anthropological Theory of the Didactic (ATD) based on the continued inquiry of problematic questions, named *study and research path* (SRP) (Chevallard, 2015). We can consider SRPs as a broad instructional format that encompasses PjBL and provides a methodological framework for its design and analysis. In the case of statistics, literature on the PjBL commonly focuses on the students' learning and perceptions towards statistics before and after the project implementation, while the SRP approach puts a strong emphasis as well on the questioning of the statistical activities that are taught, the planning of the learning process and especially the in-process observations and analyses (Markulin et al., 2021a). In this paper, we will illustrate this through an implementation of an SRP in statistics for business administration at the university.

THEORETICAL FRAMEWORK

Throughout the past 15 years, the ATD yielded a line of research to study the conditions needed for a change from the prevailing pedagogical *paradigm of visiting works* towards the one of *questioning the world*. In the former paradigm, the syllabi are usually a list of themes, topics or disciplines to learn, without necessarily knowing their raison d'être. The latter paradigm considers knowledge as a tool to question the world and elaborate answers to the questions raised. To analyse the conditions for transitioning to the new paradigm, the ATD proposes to design, implement, analyse and develop a new instructional proposal, the SRP. This proposal considers open questions as the central activity of the teaching and learning processes. An initial question generates an arborescence of derived questions to be answered by the students under the direction of a teacher or a team of teachers. In the pursuit of elaborating the answers to the questions, different *research* activities will appear (searching for information, collecting data, comparing the information collected, producing partial answers, etc.), as well as *study* activities to understand, acquire and put into practice the new knowledge and analysis tools (Chevallard, 2015).

Until now, several applications of different SRPs have been implemented in university education for students in engineering, chemistry, medical sciences, economics and business administration (Bosch, 2018; Lucas, 2015; Markulin et al., 2021b, Parra & Otero, 2017; among others). Those implementations vary in duration and moment in the course when they take part in. They share however some crucial aspects that, taken together, specify them among other PjBL proposals. First, the fact that students work in teams and that teams collaborate to address the same problematic question: the SRP's *generating question*. Second, the use of particular instructional strategies and tools, such as maps of questions and answers, the elaboration of intermediate reports, the search for new information and data and their corresponding study, and the presentation or defence of the final answer (Barquero et al., 2021). Finally, the use of a specific methodology – the *didactic engineering* (see below) – that provides a general framework to design the SRP in relation to the global structure of the course.

We will describe the implementation of such an SRP and its analysis through the teachers' observations and the students' answers to a questionnaire and semi-structured interviews. Our focus will be on the integration of the SRP in the global Statistics course organisation and the interaction between both.

THE SRP IMPLEMENTATION

In the academic year 2020/21, we implemented an SRP in a 6 ECTS Statistics course for second-year business administration students. It was the third year to implement an SRP to that particular course by two researchers in mathematics education who had the teaching responsibility. The Statistics course has a duration of one semester and is organised in two parts that are intertwined: the traditional part combining lectures with case studies, and the SRP-project part that mainly occurs during the last three weeks.

The course syllabus consists of describing datasets with descriptive statistics and graphs, relationships between variables, models of distributions, inference, and hypothesis testing. All statistical analyses are performed with R Commander, a basic graphical user interface for the statistical program R. The first part of the course is a mixture of theory and genuine practice. It is organised in bi-weekly terms centred in case studies to yield some descriptive statistics of the data given, introduce models of distribution, inference analysis and hypothesis testing. Each case is based on a different dataset being analysed using different statistical tools. These tools are progressively introduced according to the analysis needs.

The 2020/21 implementation of the SRP started by posing an initial question coming from an association that proposed an exploration of the city residents' consumer behaviour and their intention to participate in the set-up of a cooperative supermarket. The project was proposed at the beginning of the semester and was retaken in the middle of it for an intermediate report on the city's different districts using official statistics. This first step was to help organise the survey's implementation and to check the quality of the sample afterwards. This study was elaborated using Excel, the software that students were quite familiar with. The activity turned out to be quite challenging, especially because it coincided with a switch to a completely online modality of the classes due to the COVID19 situation.

Later, the partial exam took place and the bi-weekly cases continued with different topics. During this period, students could collect the answers to a survey elaborated specifically to answer the association's demands. When approaching the end of the classes and data were collected, three weeks (6-7 sessions) were left only for the project work. In this last project period, students were asked to submit two more intermediate reports, one on the analysis of the sample (the survey dataset) and the other one on the preliminary results of the analysis of the consumer behaviour of the respondents. All three intermediate reports (one about the official city statistics and the two just mentioned) obtained detailed feedback from the teachers for students to continue their work. During the online classes, interactions and discussions with the teachers mostly occurred on the students' demand and were rarely forced by the teachers. Even though there were two available teachers to address students' questions, not all the project work could have been finished only during the official session time slots. With the online modality of the course, the students got used to the online work rhythm, and that also facilitated a more flexible and approachable way for the student team members to meet and continue the work "out of the class".

Students were required to present their answers to the association during the last session of the course. The exposition of the presentations was attended by the whole student group and a three-member jury formed by one of the statistics teachers and two teachers from different school departments (marketing, accounting, ethics, quantitative methods) that were not familiar with the project topic. A more thorough description of the 2020/21 implementation can be found in Markulin et al. (in press).

RESEARCH QUESTIONS AND METHODOLOGY

Our research focuses on the role of SRPs as instructional facilitators of the shift between the pedagogical paradigm of visiting works towards the one of questioning the world. This contribution's specific research questions are:

> RQ1: What elements of the course organisation facilitated the implementation of the SRP? Which ones hindered it? What other constraints appeared?
>
> RQ2: How can the identified constraints be related to the *paradigm of visiting works* prevailing in university education and what consequences can be drawn towards the general dissemination of PjBL proposals?

To develop some answers to the posed research questions, we considered the implemented SRP as a case study and rely on qualitative research as part of the *didactic engineering* (DE) methodology (Barquero & Bosch, 2015). For the *a posteriori analysis*, the DE last phase, the source of our data are the naturalistic observations of the statistics teachers (who are also researchers in mathematics education), as well as semi-structured interviews to a small sample of students, and a questionnaire passed to the students after the course. The interviews were done with 5 students from different groups and having different Statistics final grades, while the questionnaire was anonymously answered by a sample of 71 students up to 113. The interviews and the survey addressed all the parts of the SRP and its relation to the course organisation: interest of the generating question; data collection process; clarity of the project aim; survey composition; availability of statistical tools; classes organisation (online and synchronous); students' teamwork; final presentation of results; calendar and duration of the project; relationship with the case studies; etc. Our hypotheses were:

H1. *Generating question and project aim*. The fact that the association representatives formulated the generating question and students had to present the final results in front of a jury brought realism to the project. However, the initial question did not seem to be considered real enough by the students during the analyses.

H2. *Project survey and data collection*. Social networks provided students with facilities to collect data, despite their reduced mobility due to the COVID19 situation.

However, exploiting the data to get interesting answers to the questions raised required more time and students' investment.

H3. *Integration of the SRP in the Statistics course*. The classes previous to the project provided tools and knowledge for the project work. They also prepared students to use inquiry strategies and resources, like data collecting and cleaning, raising questions, summarising results, etc. that are part of the "statistical sense" (Batanero et al., 2013).

H4. *SRP organisation and management*. Even if the SRP took place during the last three weeks, some learning resources, like teamwork and report writing, were introduced in the previous case study sessions. The schedule of the project (at the beginning, middle and end of the course) gave visibility to the SRP all along the course, even if the global time devoted to the data analysis seemed too short.

RESULTS AND DISCUSSION

In the following paragraphs, we present the interviews and questionnaire results associated with the different hypotheses (students from the interviews are denoted as S1, S2, S3, S4 and S5).

H1. The students confirmed our impression that the project's initial question was clearly posed. However, unlike the teachers' observation of the students' detachment from the core issue, they considered themselves well immersed in the matter. S2: "The question is well-posed and is exploited because the whole project was based on that question…later we started to understand, we saw examples, the advantages from there to continue, we started to integrate and understood what the association was pursuing"; S3: "I was not considering it [the initial question] during the project but in the end, our final presentation, it was then when we focused everything on what the association needed". Regarding the teachers doubt whether they guided the students' analyses too much, and unintentionally making an open problem too academic, the questionnaire students answered resulted with a mean of 2.5 on a scale from 1 to 5 as an answer to the question: "The teachers guided us too much: (1 being not at all, 5 being too much)". About the final presentations, they were mostly proud moments for the students where they could present and defend their results in front of their colleagues as well as the evaluating jury. S1: "In our case, we did not have to elaborate much the presentation. It was a subject that we already had so well established and so integrated that in the end, it came out on its own."; S4: "I liked it. I think the presentation could have been a little longer, maybe there was a lot of information and it had to be reduced. But overall, it was good." Moreover, according to the questionnaire, the students seemed to appreciate the assessment by a jury. As an answer to the question "I think it is important that there are external evaluators for the final presentation (1 being totally disagree, 5 being totally agree)" the mean of the answers was 3,8 and the median a 4.

H2. About the data collection, the process went indeed without major issues, probably thanks to the fact that the respondents could have been of all age and social groups (unlike some previous projects that demanded specific respondents' characteristics)

S1: "It is true about the COVID19 issues, but nowadays, with the social networks and the level of expansion, I think we reached the same people as we would have without the COVID19 issues". Moreover, some students suggested simplifying the survey to improve the project results. S5: "A simpler and easier to answer survey would have gotten the most reliable results for sure". However, the questionnaire gave us a mean of 2.8 and a median of 3 on a scale from 1 to 5, a quite symmetric distribution, as an answer to the question "The data collection was easier than I expected: (1 being totally disagree, 5 being totally agree)" which is satisfying since it shows that data collection is a part of statistical work that is not trivial but can be simplified if the survey is well composed and the target respondents are not a very reduced group of people. In what concerns the lack of students' engagement with the survey's richness, S4 mentioned the responsibility they felt for the project, but also that it was not shared by all the teams: "If all our colleagues from the group really thought they had a responsibility, it could have made a difference in the form of how the project was to be developed. At least it was what I expected."

H3. The combination of theory, case studies and a project in a Statistics course has shown as engaging but at the same time providing a steady base for the application of learnt skills as well as acquiring some new ones along the way of analysis of the real-world data. S1: "It was nice to learn a bit about the tools like R that are widely used in companies and at universities, as well as the theory. Everything has its application"; S3: "Maybe I learnt some things that I ended up not using for the project, but as more as we learnt the more we could apply, and it helped us explain more things eventually"; S5: "I think that I learned to do it while doing the reports because it was a progressive thing with the cases. For the project later, we had the basics to start with".

H4. The students appreciated the teamwork. S1: "In the end, no matter how much you want to, you will have to work in a team at some point or another. This is why it is very important that you adapt to different ways of doing things"; S3: "I think that last year, in the first year of the degree, I did not have the same ability to communicate in a group as I have now"; S4: "I think it has to be done as a group, first of all, because in statistics many topics are touched upon. There are many questions and some of us is going to be better in certain problems while the other team members will be more skilled with another type of analysis." In what concerns the intermediate reports, even if they had a tight schedule, the students confirmed that they helped them focus their analyses. S1: "Handling the deliveries...we were there for a month, almost every week...we were on top of everything. And the feedback from the teachers was more or less fast. Sometimes we expected more, but it was not a must, either."; S3: "In the end, we had a hard time grasping what you wanted to get from us. Then, in the end, the last reports, we saw much more what you wanted and it was easier for us." Finally, the calendar of the project was mainly well organised and did not disrupt the flow of the theory classes and the case studies but was reminding the students that the big project is to be kept alive in their consciousness and they could think about the project ideas while learning the theory. S5: "I think it was good that at the end it was more squished in time because you were working on it quite regularly and you did not lose the hang of things, you knew what was going on. Maybe the only thing I would say is there was a void between the project presentation by the company and the first report, so I remember some people saying: Oh, what was this association like? What is happening with that?"

All in all, the general impression the students stated, which was satisfying for the teachers as well, is that they appreciated the connection of a university course to a real company and found it useful for their future professional and personal experiences. S1: "I think it is a way that helped us to do a little bit of market research, which in the short term will be something we will have to do when we are working."

CONCLUSIONS

Concerning RQ1 (facilitators and constraints in the implementation of the SRP), if we only focus on the SRP, the presented experimentation did not add substantial elements to the existing investigations. We corroborate the students' difficulties in raising questions and taking them "seriously" (Hypothesis 2), as well as searching for validation tools outside the teachers' approval or feedback. We also confirm the need to base SRPs on the study of real generating questions to ensure that it is the question that leads the inquiry and not the knowledge tools needed to elaborate an answer (Barquero et al., 2021). However, if we consider the SRP together with the course implementation, the experimentation reinforces the interaction between both teaching strategies. The students' ease in searching and cleaning data, preparing summaries, working in teams, elaborating written reports and defending their results in oral presentations can be explained by the inclusion of these activities repeatedly since the very beginning of the course. They cannot be learned in a three-week activity.

About RQ2 (constraints related to the paradigm of visiting works and consequences for PjBL proposals), the main lesson we can draw affects the unit of analysis that is considered by research on project-based teaching. Our experience illustrates how the structuring of the course cannot be considered – from a design nor a research perspective – as separated from the project. An SRP is not a longer activity the teacher includes at the end of the course, as an "application" of some previously visited works of knowledge. On the contrary, it is part of a course globally designed to provide students with practical competencies in dealing with data and, therefore, culminates with the study of a real managerial question requiring its approach through data collection and analysis. We consider that research on PjBL would gain in delimiting broad units of analysis that include the courses where the projects are implemented, instead of detaching them from the PjBL strategy. Probably, many of the constraints hindering the dissemination of PjBL instructional proposals do not come from the proposals themselves but from the global teaching activity that integrates them.

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THE RELATION BETWEEN RIEMANN SUMS AND DOUBLE INTEGRALS: RESULTS OF A SECOND RESEARCH CYCLE

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Following a first research cycle of student understanding of the relation between Riemann sums and double integrals, we proposed a model (genetic decomposition) of how students may construct their understanding of these notions. Didactical materials were designed and implemented in the classroom to help students do the proposed constructions. This is a report of a second research cycle in which the effectiveness of these materials was studied. The same interview instrument was used in both cycles. Interviews' responses from eleven students who did not use the materials and eleven who did were analyzed. Results show that students using the proposed materials and a collaborative didactical strategy, constructed a deeper understanding of the intended relation than those attending a lecture-based course.

INTRODUCTION

Multivariable calculus is of great importance in mathematics and its applications. In particular, research on integral multivariable calculus is sparse (Martínez-Planell & Trigueros, 2021). Student understanding of the relation between Riemann sums and double integral was studied in Martínez-Planell and Trigueros (2020). This study produced a model (genetic decomposition) of how students may construct knowledge about basic aspects of this relation. The model led to the design of didactic activities to help students do the proposed mental constructions. This is a report of students' construction found after students had used these activities in class.

LITERATURE REVIEW

McGee and Martínez-Planell (2014) proposed a specific semiotic chain to help students understand double and triple integrals and their relations to Riemann sums. To describe the semiotic chain, we use an example of an integral in rectangular coordinates: Essentially, given a symbolic, tabular, or verbal (as a density) representation of a function defined on a rectangle (e.g., $f(x,y) = x^2 + y$ on $[0,2] \times [1,2]$), and given a partition into a small number of sub-rectangles with a chosen point in each sub-rectangle (e.g., $\Delta x = 1$, $\Delta y = \frac{1}{2}$, and the point closest to the origin) students were first asked to draw in space the collection of prisms resulting from the corresponding Riemann sum (e.g., Figure 1); to give a numeric representation of the Riemann sum as an expanded sum (e.g., $f(0,1)(1)\left(\frac{1}{2}\right) + f(1,1)(1)\left(\frac{1}{2}\right) + f(0,1.5)(1)\left(\frac{1}{2}\right) + f(1,1.5)(1)\left(\frac{1}{2}\right)$); to write a symbolic representation in expanded

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 179-186). PME.

form (e.g., $f(x_1, y_1)\Delta x\Delta y + f(x_2, y_1)\Delta x\Delta y + f(x_1, y_2)\Delta x\Delta y + f(x_2, y_2)\Delta x\Delta y$); as well as a symbolic representation using sigma notation (e.g., $\sum_{j=1}^{2}\sum_{i=1}^{2}({x_i}^2 + y_j)\Delta x\Delta y$); they were also asked to generalize the Riemann sum approximation to a limit of Riemann sums (e.g., $\lim_{m.n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n}({x_i}^2 + y_j)\Delta x\Delta y$); and finally to express and compute the limit as an iterated integral and the double integral (e.g., $\int_1^2\int_0^2(x^2 + y)dxdy$). The same semiotic chain was used when introducing double integrals in rectangular and polar coordinates, and triple integrals in rectangular, cylindrical, and spherical coordinates. McGee and Martínez-Planell (2014) found that students who used this semiotic chain throughout a semester seemed to show a deeper understanding of Riemann sums and their relations to double integrals than students in a lecture-based course who did not use those special activities.

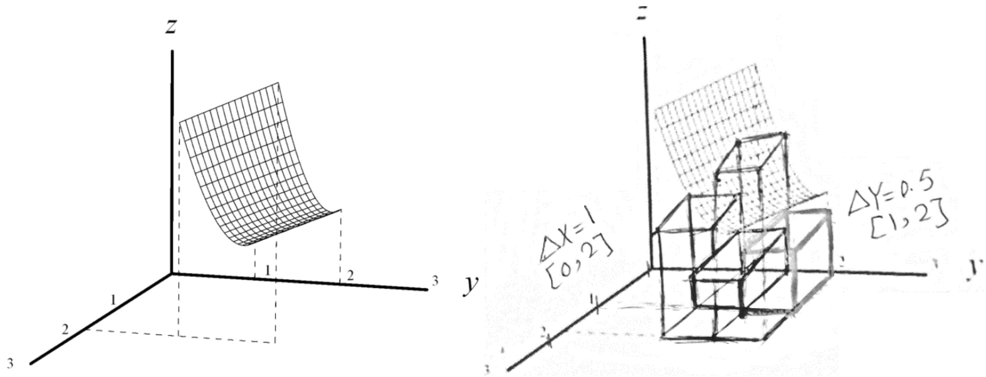

Figure 1: Sample geometric representation of a Riemann sum.

Although Sealy (2014) studied integrals of one-variable functions, her work influenced our treatment of the integral calculus as it may be extended to the case of multivariable functions. She proposed a framework for characterizing student understanding of Riemann sums and definite integrals, consisting of five "layers". Interpreting these layers in the context of double integrals resulted in an "orienting pre-layer," in which students would attend to the individual meanings of $f(x_i, y_j)$, Δx, Δy, and $\Delta x\Delta y$; a "product layer" where students are to make sense of the product $f(x_i, y_i)\Delta x\Delta y$ where $f(x, y)$ could be given as a density; a "sum layer" to help students to make sense of $\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i\,, y_i)\Delta x\Delta y$, and both a "limit layer" and a "function layer" which we are not considering in this study.

In another of the few publications dealing with the multivariable integral calculus, Jones and Dorko (2015) studied how students generalize from their one-variable function integral conceptions to construct their multivariable integral ideas. They observed that ideas generalized from the "area under a curve" conception seem to be favored by students, that none of the ten interviewed students showed an understanding based on the Riemann integral, and that the understandings shown by students suggested they struggled with setting up and using integrals in contextual situations. These authors suggest that it may be necessary for instructors to help students pay careful attention to the conceptual aspects of the Riemann sum and the Riemann integral during multiple integral instruction. Our proposed genetic decomposition and didactical activities follow this suggestion by implementing Martínez-Planell and McGee's (2014) semiotic chain while keeping Sealy's observations. The present study was designed to test the effectiveness of this approach complemented by the use of APOS Theory (Arnon et al., 2014).

THEORETICAL FRAMEWORK

The study used APOS theory (Arnon et al., 2014), the same theoretical framework used in the study of Martínez-Planell and Trigueros (2020). In APOS Theory, an Action is a mathematical transformation that is perceived as external. It could be the rigid application of an explicitly available or memorized procedure. As the individual repeats and reflects on an Action, it might be interiorized into a Process. A Process is perceived as internal and allows the individual to omit steps and anticipate results. Different Processes may be coordinated or reversed. When the individual is able to think of a Process as an entity in itself and is able to apply or imagine applying Actions on the Process, then one says that the Process has been encapsulated into an Object. A Schema is a coherent collection of Actions, Processes, Objects and other previously constructed Schema. We will not make explicit use of Schemas in this study.

An important methodological component in APOS is a genetic decomposition (GD). This is a model of how an individual may construct a specific mathematical notion. It is expressed in terms of the structures (Action, Process, Object, Schema) and mechanisms (interiorization, encapsulation, coordination, …) of the theory. The following is the portion of the genetic decomposition for integrals of two-variable functions that was tested in the first research cycle (Martínez-Planell & Trigueros, 2020) and that was again used for this second research cycle. Due to space limitation, we only present two of the four parts of the GD in detail. The whole GD is available in Martínez-Planell and Trigueros (2020). We must stress that the genetic decomposition we use in this study is only part of a more comprehensive genetic decomposition of multivariable integrals. Note that it responds to the ideas of McGee et al. (2014) and Sealey (2014).

1. Recognition of rectangle and function (details omitted)

2. Forming one term of a Riemann sum

Given a function of two variables, a rectangle in its domain, and a point in the rectangle, the student does the Action of evaluating the function at the given point and multiplying it by the length, Δx, and width Δy, of the rectangle to form a product of the form $f(a,b)\Delta x\Delta y$. Students do the Action of interpreting the factor $\Delta x\Delta y$ as the area of the rectangle. These Actions are interiorized into a Process which can be coordinated with conversion Processes between different representations of function, rectangle, and given point, into a Process whereby the product can be interpreted as the volume of a rectangular prism in space. This Process makes it possible for students to recognize the product's units when functions $f(x,y)$ are given as rates (densities in the tasks for this study). This part of the GD describes constructions needed to give meaning to the separate factors $f(a, b)$, Δx, Δy, $\Delta x\Delta y$ and construct the meaning of $f(a, b)\Delta x\Delta y$.

3. Forming a partition and corresponding Riemann sum

Given two small specific positive integers (not in symbolic form, but actual numbers), n and m, the Action of subdividing given intervals $[a, b]$ and $[c, d]$ into subintervals of equal length $\Delta x = \frac{b-a}{n}$ and width $\Delta y = \frac{d-c}{m}$ both numerically and geometrically is done, in order to obtain a subdivision of the rectangle $[a, b] \times [c, d]$. These Actions are interiorized into a Process of subdivision of rectangles so that the student can imagine how, for any given positive integers, n and m, the respective subdivisions of $[a, b]$ and $[c, d]$ give rise to a subdivision of the rectangle $[a, b] \times [c, d]$ without having to explicitly do so for any other specific values of n and m. Given a continuous function f defined on the rectangle, the Action of choosing a prescribed point (x_i, y_j) on each sub-rectangle of the given partition and producing the products $f(x_i, y_j)\Delta x\Delta y$, the Actions of forming the corresponding sum of the products, and the Actions of interpreting this sum geometrically (as a collection of rectangular prisms in space), numerically (as a sum of numbers), symbolically as an extended sum, symbolically using sigma notation, and verbally (interpreting the products in terms of its units) may be interiorized into a Process that enables imagining forming such sums of products in different representations for the collection of sub-rectangles of any given partitioned rectangle.

4. Recognition of underestimate, overestimate, and exact value (details omitted)

METHODOLOGY

We designed didactical activities based on the GD. They were used in a section of an introductory multivariable calculus course as part of the ACE didactical strategy (**A**ctivities worked in small groups of students, **C**lass discussion, and **E**xercises for the home). This section will be called the “APOS” section. The experience took place in a university in Iran where the course was taught by one of the researchers. Eleven students were chosen for semi-structured interviews so that three were considered “above average”, five “average”, and three “below average” as chosen by the professor, based on their course grades. Another eleven students were similarly chosen from another section of the same course. This “regular” section was taught through lectures. Both sections used the same textbook, but in the regular one, students did not use the specially designed activities, and it was taught by a different experienced professor. The regular section students were chosen so that their course grades were as comparable as possible to those of the students in the APOS section. Each interview lasted approximately one hour. The interviews were audio and video recorded, transcribed, translated to English, analysed individually, and discussed as a group. Differences in opinion were negotiated. Student responses were also graded for their mathematical correctness from 0 to 2 points: 0 being mostly incorrect, 1 partially correct, and 2 correct. This was used as an aid to look for patterns. However, the individual and group analysis concentrated on trying to ascertain the mental structures (Actions, Processes, Objects) evidenced by students in regards to the GD.

The interview instrument had nine questions; they all considered a function defined over a rectangle and a partition consisting of only the rectangle itself (Figure 1, left): (1) Represent the domain of f in the figure; (2) Let $g(x,y) = x^2 + y$ be a function with domain restricted to the set with $0 \leq x \leq 2$, $1 \leq y \leq 2$. Represent the domain of the function in a given 3D coordinate system; (3) Given Figure 1 and if $\Delta x = 2, \Delta y = 1$, interpret graphically $f(0,1)\Delta x\Delta y$; (4) Compare $f(0,1)\Delta x\Delta y$ to the double integral over the rectangle -an underestimate-; (5) Compare $f(2,2)\Delta x\Delta y$ to the double integral over the rectangle -an overestimate-; (6) Does there exist a point (a,b) for the function in Figure 1 such that $f(a,b)\Delta x\Delta y$ is equal to $\iint_D f(x,y)dA$?; (7) Let $\Delta x = 1, \Delta y = \frac{1}{2}$, and $D = \{(x,y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$. Consider the Riemann sum $f(0,1)\Delta x\Delta y + f(0,1.5)\Delta x\Delta y + f(1,1)\Delta x\Delta y + f(1,1.5)\Delta x\Delta$ of the integral $\iint_D f(x,y)dA$. What does that Riemann sum represent geometrically and how does its value compare with $\iint_D f(x,y)dA$?; (8) Interpret one term of a Riemann sum and a double integral in a given contextual situation; (9) Explain the relation between Riemann sums and double integrals.

RESULTS

The following table summarizes the results of the graded interview questionnaire of students in the APOS and regular sections. It suggests that there was a fundamental difference in the way students in those sections were able to do the constructions proposed in the GD, and thus in their understanding of the relation between Riemann sums and double integrals. Note that the percentage obtained by APOS section students is much higher in all the questions of the instrument. Also, although we don't have space to include details, six students from the APOS section showed all of the constructions in the GD so they seemed to have constructed a Process conception of the relation between Riemann sums and double integrals. Two other students only missed constructions in the contextual situation of problem 8. Only one student from the regular section constructed or seemed close to constructing a Process conception.

Prob./ Sect.	1 geo dom	2 symb dom	3 undr	4 undr & $\iint$	5 over & $\iint$	6 exact	7 four-term	8 thin plate	9 $\iint$&R	Total
APOS	95%	95%	80%	82%	80%	68%	77%	45%	82%	80%
Reg.	18%	36%	30%	18%	27%	14%	14%	7%	11%	19%

Table 1: Question by question percentages of APOS and regular section students.

We only have space to show a few examples comparing the typical response of students in APOS and regular sections. For example, consider the drawings in Figure 2 interpreting graphically $f(0,1)\Delta x\Delta y$ as produced by students A6 of the APOS section (left) and student R4 of the regular section (right).

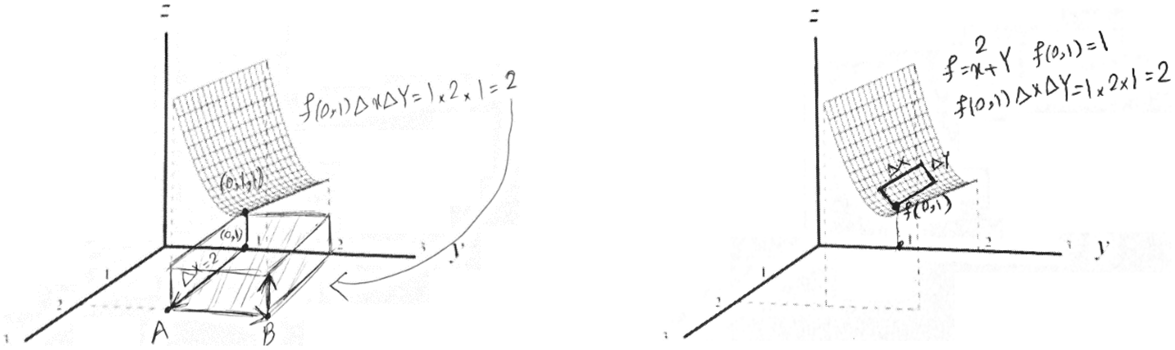

Figure 2: Drawings of one term of a Riemann sum by A6 (left) and R4 (right).

The response of R4 is similar to those of students from the first research cycle (Martínez-Planell & Trigueros, 2020), who had not used the GD-based activities.

I: What does it represent graphically?

R4: I think it shows the area of a small piece of the surface $f(x, y)$ at $f(0,1)$ on the surface, umm it's the area of a square, sorry not a square because Δx and Δy aren't equal to each other, so the area of a small rectangle at $f(0,1)$ on the surface.

I: If we consider a unit, for example centimeters, which will be the unit of $f(0,1)\Delta x\Delta y$?

R4: Umm, its unit will be centimeters cubic, umm so $f(0,1)\Delta x\Delta y$ doesn't show the area umm it shows the volume.

I: Can you show this volume?

R4: I don't know which volume it shows exactly umm but it shows a volume.

The drawing by student A6 of the APOS section depicting the four-term Riemann sum in question 7 appears in Figure 1 (right), and that of regular section students R3 in Figure 3 (left) and R4 in Figure 3 (right). These last two students considered the rectangular prisms shared the same base. It is noteworthy that R4 produced his drawing just after the interviewer explained to him the graphical interpretation of $f(0,1)\Delta x\Delta$.

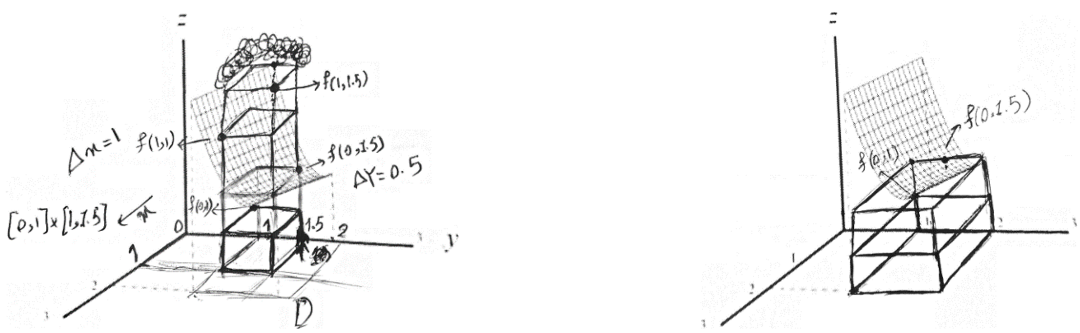

Figure 3: Drawings of a four-term Riemann sum by R3 (left) and R4 (right).

I: Before you solve this question, let me explain you about $f(0,1)\Delta x\Delta y$ in question 4 [the interviewer went on to produce a drawing and explain in detail] … Now use this idea to solve question 7.

R4: To use this idea in question 7, we have four terms, umm the first one is a box in the region D, umm its height is $f(0,1)$. The second term is again a box but higher than the previous box because its height is $f(0,1.5)$ which is more than $f(0,1)$ [Figure 3, right].

I: In this question we have $\Delta x = 1$ and $\Delta y = \frac{1}{2}$, you should use these values in your answer. As I just explained you, we had $\Delta x = 2$ and $\Delta y = 1$ in the question 4 but here they are different.

R4: I don't know how to use the given delta x and delta y, umm what are their roles here? In the two boxes that I have already drawn the bases are rectangles with sides 2 and 1, but how can I change them to 1 and $\frac{1}{2}$? umm sorry I don't know.

The above discussion suggests that an instructor's explanation and drawing may not suffice to help students' construct the "pre-orienting" and "product" layers of Sealey (2014). Students would need to reflect on activities where they give sense to the quantities $f(a,b)$, Δx, Δy, $\Delta x\Delta y$, and $f(a,b)\Delta x\Delta y$ that appear in Riemann sums. Perhaps more telling than the previous examples is the typical response for the question which asked students to explain the relation between Riemann sums and double integrals. We include the responses of student A3 from the APOS section and regular section students R3, R4.

A3: If we consider our context as the volume under the surface, then each of the terms of the Riemann sum is the volume of a prism and the Riemann sum itself is the sum of the volumes, umm if we have many sub-rectangles inside the domain D, then we have also many prisms and so their sum will approach to the exact volume under the surface, therefore if the number of rectangular prisms goes to infinity, umm like limit, the answer of the sum of the volumes will be the volume under the surface, I mean this is the relation between the Riemann sum and the double integral

R3: I remember the limit of sum is equal to integral but I can't justify it, umm I don't know how to see this relation geometrically.

R4: … about integration I can't find any relation, umm but I just remember in the single variable calculus course we learned that the limit of sum was equal to integral and it was like the sum of many areas of rectangles was equal to the area under the surface, umm but here the context of the function of two variables and we are working with volumes and I don't know how to generalize the ideas of area, I mean integral, to volume, I mean double integral.

DISCUSSION AND SUMMARY

The research cycle in Martínez-Planell and Trigueros (2020) together with the present study show how cycles of research in APOS can be used to progressively improve student understanding of important mathematical ideas, in this case, the relation between Riemann sums and double integrals. A genetic decomposition serves as an initial hypothesis that is tested with student interviews, refined with the resulting data, used as a basis to design didactic activities, and then, after using the activities in a new didactical intervention, can serve as a renewed initial hypothesis. The present study shows evidence of APOS section students' improved understanding. In the first research cycle students had not used didactic activities based on the genetic decomposition, just like students in this cycle's regular section. In this research cycle, comparison with the regular section shows the advantages of collaborative work and the use of APOS based activities. There is much that is left to be done. In this cycle we found that the activity sets have to be improved to better account for contextual situations. Future research cycles need to test for resulting improvement in student applications of integration in contextual situations and for other portions of the GD not included in this article, like the portion dealing with the double integral as a limit, double integrals in non-rectangular domains and in polar coordinates, and triple integrals in rectangular, cylindrical, and spherical coordinates.

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LESSON STUDY IN INITIAL TEACHER EDUCATION: DRAWING CONCLUSIONS FROM TWO PORTUGUESE EXPERIENCES

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This research aims to understand which situations created opportunities for prospective Mathematics teachers to develop their knowledge. Following a design-based research with two cycles, data was collected through participant observation, document collection, and individual interviews. The results show that establishing a focus on prospective teachers' learning created opportunities to develop their knowledge through discussing empirical articles, teaching the planned lessons, and reflecting on students' learning.

INTRODUCTION

Initial Teacher Education (ITE) must provide opportunities for prospective teachers (PTs) to develop knowledge about how to foster their students' learning. However, a big problem identified by PTs is "putting both theory and content into practice" (Chen & Zhang, 2019, p. 568). Lesson study, which is a collaborative and reflexive professional development process focused on students' learning, has shown benefits in ITE. This process encourages PTs to prepare lessons in detail and discuss their ideas, and allows them to enact what they planned, and to reflect on their practice. However, a better understanding is needed about how to structure lesson study, maintaining its efficacy and integrity, to create opportunities for the PTs to develop their knowledge being able to locate theory into practice (Ponte, 2017). Bjuland and Mosvold (2015) point out that is particularly important to pose "a research question targeting the student teachers' own learning" (p. 89). Thus, this research is based on establishing a focus for the lesson study regarding the PTs' learning. It pays particular attention to the opportunities created for PTs to develop their knowledge, in two Portuguese universities, answering the research question: Which situations created opportunities for the development of PTs' knowledge?

THEORETICAL FRAMEWORK

The PTs face several challenges in locating into the teaching practice what they learn in the ITE courses, raising a theory-practice gap (e.g., Bjuland & Mosvold, 2015; Chen & Zhang, 2019; Ni Shuilleabhain & Bjuland, 2019). For instance, a big challenge is related to planning and teaching lessons where "emphasis was placed on the use of cognitively demanding tasks … the encouragement of productive interactions … and the importance of listening respectfully to students' reasoning" (Stein et al., 2008, p. 316). Its core is the exploratory activities in which students are involved, based on demanding tasks as starting points for whole-class discussions. These *exploratory lessons* are usually structured in three phases: first, the teacher proposes a task; then, the students work autonomously in small groups; and, finally, the teacher selects some

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students' strategies to discuss with the class, fostering their justifications, and does a final summary of the main ideas. However, selecting tasks that allow students to work autonomously, and fostering productive interactions in the classroom is not simple for PTs. They usually lack proper knowledge about what can happen during the lesson and "are regularly surprised by what students say and do, and therefore often do not know how to respond to students in the midst of a discussion" (Stein et al., 2008, p. 321).

To bridge this gap, it is important the PTs have school-based field experiences so they can learn "how to teach, and … experience the difficulty of teaching" (Chen & Zhang, 2019, p. 551). Usually, ITE includes field experiences where the PTs put into practice their ideas and enact the lessons, under the mentorship of an experienced teacher. However, since they are developing knowledge about students and teaching practice, it is a challenge to plan and orchestrate lessons to foster students' learning (Bjuland & Mosvold, 2015). Thus, ITE has a responsibility of creating opportunities for PTs to develop their knowledge, based on field experiences and reflection on those experiences (Chen & Zhang, 2019; Ni Shuilleabhain & Bjuland, 2019).

Lesson study is a professional development process that aims to improve teaching and learning through planning, teaching, and reflecting on students' learning, through a collaborative work of a group of teachers. They start by identifying an issue in students' learning (e.g., a common difficulty) and study the related curriculum and research results, defining a learning goal. Then, they plan a *research lesson*, selecting, solving, and adapting tasks to be suitable to the lesson, anticipating students' strategies and difficulties, and preparing their interventions (e.g., questions to pose). After, a teacher teaches the lesson, and the others observe gathering notes to reflect on students' learning. By a deep exploration of different aspects related to students' learning, this reflexive environment allows the PTs to discuss their ideas about teaching practice, promoting the development of their knowledge through field experiences (Ni Shuilleabhain & Bjuland, 2019; Ponte, 2017).

Still, integrating lesson study in ITE demands adaptations, considering the settings of each university. The process tends to be simplified, which may compromise its unique characteristics and benefits. But, without adaptations, it is not possible to carry it out because of the specifies of each program and the aims of preparing PTs (Ponte, 2017).

There are several experiences of lesson study in ITE, with different contexts and designs, using diverse theoretical frameworks (see Ponte, 2017). These lesson studies usually have aims concerning professional or didactical aspects, as the development of PTs' knowledge or their reflective practice. However, "such aims are not indicated in an explicit way in most studies" (Ponte, 2017, pp. 173-174). In addition, the lesson studies usually focus on the planning phase, on the reflection phase, or both, which establishes the main activities during the process (Ponte, 2017). But these foci seem to emerge by the required adaptations of integrating lesson study in each specific ITE program or seems to be a choice of teachers educators. For instance, in Chen and Zhang's (2019) study, "lesson planning … is a central focus in learning how to teach"

(p. 550). The authors structured the lesson study in two courses and asked the PTs to plan a lesson based on the knowledge they developed by being taught about "the process of lesson planning, the frame of a lesson plan, and the specific guiding of each aspects" (p. 556), to teach that lesson for their colleagues, through microteaching.

Although the focus of a lesson study is students' learning, it must happen under the development of teachers' knowledge (Lewis et al., 2019). So, it is important not only to define a learning goal regarding students, but also to establish a broad long-term goal for the teaching and learning process, focused on the PTs' learning. For Bjuland and Mosvold (2015), "identifying a research focus for the students teachers' own learning" (p. 88) is an important element of lesson study in ITE. This is an idea also shared by Lewis et al. (2019) for in-service teachers: "[the research theme] helps reconnect educators with the goals that are really vital to them" (p. 21). Therefore, by investigating a certain aspect of their own learning, PTs can guide the lesson planning to the research focus and think of it as "an empirical investigation of their own teaching and learning" (Ni Shuilleabhain & Bjuland, 2019, p. 3). Thus, it seems to be important to define two main dimensions of goals for lesson study in ITE. The first dimension is related to the *definition of a learning goal for the research lesson*, regarding students' learning. It can be related to a usual difficulty they have on a specific topic, or it can be focused on fostering their skills as reasoning processes. The other dimension concerns a broad long-term goal which implies an *establishment of a lesson study focus regarding PTs' learning*, to foster the students' learning based on the development of PTs' knowledge.

METHODOLOGY

This research follows a qualitative approach as a design-based research (Cobb et al, 2016) with two design cycles, in two Portuguese universities. It aims to provide insights on how teacher educators can create opportunities to promote the development of PTs' knowledge, establishing a focus for lesson studies regarding PTs' learning. So, the interventions were structured on lesson study experiences with a particular focus on creating opportunities for the PTs to develop their knowledge. At the end of Cycle I, a retrospective analysis was done considering the data collected, similar empirical studies, and confronting with theoretical perspectives.

In each university, the participants were secondary school Mathematics PTs supervised by a teacher educator who accepted to carry out the lesson study. They observed several lessons taught by experienced teachers, at the field practice. The facilitator role was shared by teacher educators and the first author (also as researcher). In Cycle I, Mónica and Olívia, planned a first lesson for Olívia to teach, and then both planned a second lesson that both taught to different classes. Sílvia, Lila and Maria, in Cycle II, planned and taught three lessons each, in different classes.

Data collection includes participant observation by the researcher (with researcher's journal and audio recordings), document collection (lesson plans and written reflections), and individual interviews at the beginning and at end of the lesson studies.

All the necessary permissions were requested, and the names are all pseudonyms.

The data analysis is organized as thematic episodes from the two lesson study experiences. First, it is presented the focus of the lesson study regarding the PTs' learning, describing how it was established. Then, it is discussed the lesson planning work, having as background the established lesson study focus regarding PTs' learning. Finally, considering the PTs' reflections, it is pointed out what can be improved for the next lesson study experiences and what knowledge the PTs developed.

TWO LESSON STUDY EXPERIENCES IN TEACHER EDUCATION

Cycle I

Establishing a lesson study focus regarding PTs' learning. To prepare the intervention, the researcher and the teacher educator met to define the aspects to explore and to organize the sessions attending to the field practice agenda. Considering that exploratory lessons usually raise challenges for the PTs, the teacher educator suggested exploring this teaching approach. Therefore, the facilitators proposed the analysis of the curriculum, and the analysis of an article regarding a teacher's learning about exploratory lessons. During the discussion of the article (Session 2), Mónica and Olívia showed a superficial understanding of exploratory lessons:

> Olívia: The students' previous knowledge is important. It's forwarding to a reflection.
>
> Mónica: Instead of the teacher being the main guide of the lesson, he guides the students to… he is no longer the main figure in the lesson, the students are.

Additionally, they drew the first lesson plan describing what should happen in the lesson and did not plan moments for students to work autonomously on the proposed task or to share and discuss their mathematical ideas:

> Four students will be selected randomly. It will be explained that each one will have to build a cube. … As soon as they finish it, the selected students will have to go back to their places. The task will the solved by the whole class. (Mónica and Olívia's first lesson plan)

Thus, the lesson study focused on exploratory lessons' structure and purposes to promote the development of PTs' knowledge about it, based on discussions on empirical articles and careful lesson planning work for the research lesson.

Planning a lesson based on the lesson study focus. The facilitators wanted to promote the development of PTs' knowledge about planning exploratory lessons. Since the task proposed is the starting point in an exploratory lesson, they suggested the PTs select different tasks to critically analyze their strengths and weaknesses. They aimed the PTs to be able to design questions that allow several solving strategies to foster students to share and to justify their ideas during the whole-class discussion. By analysing the tasks, the PTs showed their concern about students' motivation:

> Does this motivate the students? … there are things that we see right away they don't care about. And there are others they are interested in. (Mónica, Session 4)

So, they investigated their students' interests and sociocultural contexts, to design an interesting task for them, developing their knowledge about the students.

The facilitators also suggested discussing another article about aspects that should be considered when planning an exploratory lesson. After the discussion, the PTs reformulated their lesson plan structure, organizing it in the three moments of an exploratory lesson and using the scheme proposed which involved tasks and learning activities, expected duration, students' activities and possible difficulties, teachers' answers and aspects to pay attention, and goals and assessment. Also, the PTs paid attention to some aspects disregarded on the first lesson plan, as students' difficulties, as well as the preparation of teachers' interventions to foster the students' justifications. So, the discussion of the article seemed to influence the PTs' lesson plans and, consequently, the development of their knowledge about planning exploratory lessons.

Reflecting on the lesson study experience. At the end of the lesson study, Mónica stressed out in the final interview that "we should have better prepared the communication part … For example, in a specific [students'] question, what will be the keyword to use to help them? … We had only written 'the teacher must guide to…'". The teacher educator also mentioned that the whole-class discussion needed to be better prepared and added that "[the reflection phase] should take longer … it should be more systematic". Thus, the retrospective analysis arose two main issues to be considered in the next lesson study: promoting a careful preparation of whole-class discussions and creating more opportunities for the PT to reflect on students' learning.

Notwithstanding, Mónica highlighted she learned "to handle everything, … whether it is an exploratory lesson or not, I think [I've learned] the teaching approach for each lesson.". Also, Olívia said she developed their knowledge about "The planning part … I used to plan as running text and it was very confusing... and there were many pages! And so, in a table, it became more succinct". So, for these PTs, knowledge about teaching approaches and lesson planning was developed during the lesson study.

Cycle II

Establishing a lesson study focus regarding PTs' learning. To prepare the intervention, the researcher conducted interviews with the PTs and shared the principal ideas with the teacher educator, as well as some issues that arose from the previous experience. Thus, the lesson study was structured considering the PTs' themes for the Final Report, namely to foster students' reasoning processes and classroom communication and the two main issues pointed out in the previous lesson study.

To promote a careful preparation of whole-class discussion, the facilitators asked the PTs to prepare their interventions considering as starting point a detailed anticipation of students' strategies and difficulties. Then, they suggested design tasks that allow students work autonomously in small groups, using different solving strategies and several representations, to foster their reasoning processes and explanations during the whole-class discussion. To create opportunities to reflect on students' learning, the facilitators proposed pre-lesson and post-lesson reflection guides, focusing on PTs' themes for the Final Report. They also proposed an additional written reflection, based on lesson video recordings, for the PTs reflect on students' learning and look for strategies for improving their practice. They also encouraged them to plan three lessons each, to be taught, so they could improve their practice based on their reflections.

Thus, the lesson study focused on foster students' reasoning processes and classroom communication, to promote the development of PTs' knowledge about it, based on reflections developed before and after the lessons to improve their teaching practice.

Planning a lesson based on the lesson study focus. Encouraging the PTs to plan three lessons each created lesson study microcycles: plan the lesson based on a pre-lesson reflection guide, teach the lesson, discuss the lesson based on the post-lesson reflection guide, write a reflection considering the students' learning during the lesson to look for strategies for improving their practice, and repeat the process for the next lesson.

The PTs began to select tasks more suited to the learning goals and to adapt them to foster students' reasoning processes and communication, namely during the whole class discussions. They also began to prepare their interventions to support students in the whole-class discussion, challenging them to confront their mathematical ideas with their colleagues, without validating their reasonings. For example, in her first lesson plan, Lila proposed a task with a quadratic function for students to determine the maximum value. If they have difficulties, Lila would "ask them to search on their notebook how to calculate the parabola's vertex", giving them the procedure. In her third lesson plan, in a similar situation, she wrote "I will suggest them to sketch the situation", promoting different representations to foster students' justifications.

The written reflections also seemed to have contributed significantly to the development of PTs' knowledge, as we can read on Maria's third written reflection:

> These moments [of discussion] are often scarce … there may be a tendency of the teacher to explain … we must try to make the students express orally what they thought.

When they reflected on specific lesson moments and wrote it down, confronting with what was planned, the PTs tried to realize what happened and its influence on students' learning, drawing conclusions to improve their practice. So, these microcycles seemed to influence the development of the PTs' knowledge, namely about lesson planning.

Reflecting on the lesson study experience. In the final interviews, the PTs pointed out the biggest problem of being involved in a lesson study experience is managing the time with the field practice. Maria also added that "even when I read some articles about lesson study, I didn't realize it. Knowing the process is very different than carrying it out". For her, it would be important having other lesson study experiences or attend conferences about it. Notwithstanding, they valued the reflections they made, before and after the lessons, as Maria wrote in her first written reflection:

> This reflective process that I must do, to justify myself to other people and to explain why I took those options, makes me go to the lesson with much more appropriate activities.

Focusing on students' reasoning processes and classroom communication, made the PTs rethink aspects as the tasks to foster the students' justifications as well as how to challenge the students to foster classroom communication. In particular, Lila highlighted in her written reflection that "during the whole-class discussion, the anticipation of the different students' solving strategies … allowed me to be better prepared and to support students more easily". Maria and Sílvia also valued the anticipation of students' work, pointed out the benefits of having the opportunity of redoing it, as Sílvia said in the final interview:

> I began not realizing what difficulties students might have. For the second lesson plan, [anticipating students' difficulties] became easier because I had already taught a lesson.

For these PTs, knowledge about students and lesson planning was developed during the lesson study microcycles, based on the systematic improving on anticipation work and considering their pre and post-reflections on students' learning.

CONCLUSION

Through this research, some conclusions emerged about carrying out lesson study in ITE, namely situations that created opportunities for PTs to develop their knowledge.

Discuss empirical articles. In Cycle I, the diagnosis of the PTs' lack of knowledge about exploratory lessons emerged by the discussions on an article, which became the focus regarding PTs' learning. Those discussions led these PTs to carefully select and adapt tasks to the lesson and to rethink how to plan this kind of lesson.

Teaching the lessons planned. Teaching the lessons led the PTs to put into practice their planning and to observe students' work. In particular, the microcycles (Cycle II) created opportunities for the PTs to develop knowledge about students' difficulties.

Reflecting on students' learning. These experiences allowed the PTs to reflect on the lessons taught based on students' learning rather than a simple description of what happened. In Cycle II, they were able to identify what should be improved in the next lessons plans, namely the preparation of whole-class discussions. In Cycle II, considering their themes to write down their reflections on students' work gave them further data on students' learning. Additionally, in this cycle, the regular moments of reflection encouraged the PTs to redefine strategies to improve their practice.

The results show that planning, teaching, and reflecting on a lesson are important opportunities for the PTs to develop their knowledge. Nonetheless, the cooperating teachers were not able to attend the sessions, which was a limitation of this research. Results also show that a lesson study cycle, for itself, does not give an immediate effect on the PTs' knowledge. Thus, the development of PTs' knowledge may benefit from more than one lesson study cycle, enhanced by the discussions and its reflexive nature. In Cycle II, the PTs benefited from the reflection guides, as it incited them to justify their choices for the lesson considering students' learning, and led them to view their formative process as an empirical investigation of their own teaching and learning.

The adaptations on lesson studies were made having in mind the ultimate goal: foster the students' learning by developing PTs' knowledge. Establishing a lesson study focus regarding PTs' learning was important to structure the process considering these PTs' lack of knowledge and their wills to foster students' learning. So, it can help the teacher educators to focus the lesson study activities, by being established at the beginning of the lesson study, considering some identified PTs' lack of knowledge or an issue they want to develop. This lesson study focus can be related to a specific Mathematical content topic, or it can be drawn by aspects related to the teaching practice (e.g., selecting appropriate tasks, planning lessons, orchestrating students' ideas, fostering students' reasoning processes, or even improving teaching through technologies).

This research provides insights on how lesson study may be carried out. The lesson studies were prepared aiming to locate the theory of the university modules in the practice of planning, teaching, and reflecting on students' learning. Despite the significant differences, the specificities of each experience provide insights on how teacher educators can create opportunities to promote the development of PTs' knowledge, establishing a focus regarding their learning.

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RELATIONSHIPS BETWEEN PERCEIVED TEACHER AND PEER SUPPORT ON MOTIVATION AND ACHIEVEMENT IN HIGH SCHOOL MATHEMATICS

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One hundred fourteen US students were surveyed to test a model of the relationships among motivational variables resulting from students' first experiences as they transitioned from middle school to high school, and math achievement. Key malleable factors impacting motivation and achievement included perceived supportiveness of respondents' teacher and peers. Longitudinal Path Analysis revealed that teacher support can impact students' beliefs about the supportiveness of their peers, but that these social factors are mediated through students developing personal interest in mathematics to ultimately impact achievement.

INTRODUCTION

Mathematics engagement has been characterized as the attention, interest, investment, and effort students expend in the process of learning mathematical content (Marks, 2000). Psychologically, engagement is associated with a sense of belonging in the social functioning of the classroom, as well as the behavioral, emotional, and cognitive characteristics of one's mathematical thought and actions. Research shows that different aspects of engagement interact with each other.When all aspects of engagement are at optimal levels, students tend to expend more effort (behavioral engagement), enjoy their experiences more (affective engagement), employ more efficient and effective study and problem-solving strategies (cognitive engagement), and both help and receive help from their peers (social engagement) (Middleton, Jansen & Goldin, 2017). Because of this complexity, however, it is still somewhat of a mystery how each of these aspects of engagement contributes to achievement, separately and in conjunction. Not all students who are engaged highly in each or all of these facets achieve at optimal levels, and some students who may lack in one form of engagement may utilize other forms to compensate and achieve (Skilling et al., 2016).

The purpose of this study is to examine aspects of engagement longitudinally, modeling the longer-term effects of these variables on each other—i.e., on "growth" of mathematics engagement in a course, and their mutual influence on achievement. As part of this model, we are also especially concerned with the perceived supportiveness of the teacher, and the perceived supportiveness of peers. These factors are hypothesized to contribute to classroom climate in such a manner that students' engagement may be impacted positively.

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Teacher Support and Student Engagement in High School Mathematics

Recent views about engagement from the motivation literature, as well as from emotion research and teacher education hold that engagement is largely a function of the context within which students learn (Strati, et al., 2017). In mathematics classrooms, the teacher can influence norms of interacting that enhance students' sense of belonging, as well as their cognitive and affective engagement with the mathematics. This is accomplished through the instructional support a teacher provides such as the selection and orchestration of mathematics tasks, scaffolding discussions, and providing assistance and feedback. Strati, et al., term this type of support to be *instrumental* in that it is directly associated with the mathematics content and its experiencing.

When students perceive that their teacher is supportive in this manner, they also tend to report greater efficacy and effort, lower anxiety, and greater intrinsic motivation in math. These motivational effects in turn, appear to directly effect achievement and ongoing commitment to schooling (Klem & Connell, 2004).

In short, when teachers are perceived as helpful, providing feedback, *and* caring with fair, respectful treatment of their students, students seem to respond positively, engage deeper cognitively, emotionally, and behaviorally, and achieve better as a consequence.

Peer Support and Student Engagement in High School Mathematics

Like teacher support, peer social support has been shown to impact students' beliefs about and patterns of engagement in mathematics. In a highly cited report, Mata, et al,, (2012) studied the perceived peer support as reported by 1,719 Portuguese students, from fifth-to-twelfth grade and their interest and enjoyment of the subject, and with their perceived competence in mathematics--a construct nearly identical to mathematical self-efficacy. Across those grade bands, they found that peer social support, measured by items such as "In math class students want me to do my best in math work," was positively associated with their perceived competence, interest and enjoyment in mathematics, and notably, the perceived support of the teacher. These results along with others (Froiland & Davison, 2016) show that peers influences their friends' interest in mathematics and through that, their mathematics achievement (see also Ahmed, et al., 2010).

Variables Making Up Student Engagement

There is considerable evidence that peer support and teacher support together create a learning environment that facilitates the development of self-regulation strategies, positive mathematical self-efficacy, and personal interest in mathematics (Cleary, et al., 2017; Hidi and Renninger, 2006). fThis robust self-efficacy and personal interest in mathematics, in part, influences achievement positively.

The remainder of this manuscript will describe a longitudinal study examining the relationships among social engagement domains--teacher and student support—and student engagement factors in the cognitive and affective domains including mathematics self-efficacy, interest/enjoyment in mathematics, and mathematics self-regulation.

METHOD

Participants

One hundred fourteen students assented and received parental consent to participate in the study during the 2018-2019 academic year. All students were drawn from schools in a large urban school district in the Southwest US. 47 percent of the students identified as male, 53% identified as female. 82% of the students identified as Hispanic/Latinx, 18% identified as Caucasian/White, 5% identified as Black/African/African American, 4% identified as American Indian/Alaska Native, 3% identified as Asian or Asian-American. sAll students were enrolled in a mathematics course designated as "first-year high school mathematics," focused on traditional Algebra 1 content.

Instrument

The Long-Term Engagement Survey consists of items that assess many aspects of student engagement. A full description of the psychometric properties of this instrument can be found in (Zhang, et al., 2019).

Four scales were utilized as indicators of mathematics motivation: (1) math *personal interest* (comprised of thirteen 7-point Likert scale items, $\alpha = 0.91$). (2) mathematics *self-regulation*, (comprised of thirteen 7-point Likert scale items, $\alpha = 0.84$; and (3) mathematics *self-efficacy*, i.e., the extent to which students feel capable of doing math (comprised of eighteen 7-point Likert scale items $\alpha = 0.87$).

A teacher support scale consisted of 12 Likert items assessing instrumental and emotional support, and care. Example items included "My math teacher tries to understand how I see things before suggesting a new strategy" (instrumental support), and "My math teacher recognizes us for trying hard" (emotional support).

The student support scale consisted of 7 Likert items assessing belonging and classmates' interest and caring. Example items included "My classmates in my math class care about how well I learn." Reliability of the teacher and peer support scales were high (Teacher Support $\alpha = 0.95$, Student Support $\alpha = 0.84$).

The Achievement Measure consisted of the state-mandated, multiple-choice, high school mathematics proficiency examination, administered at or near the end of the Spring Semester, 2019. The measure covered content from the traditional High School Curriculum through second year Algebra.

Procedure

The Long-Term Engagement Survey was administered twice in the students' first-year high school mathematics course: Once near the beginning of the course to assess students' incoming sense of engagement, peer and teacher support, and once near the end of course, but prior to state-level achievement testing.

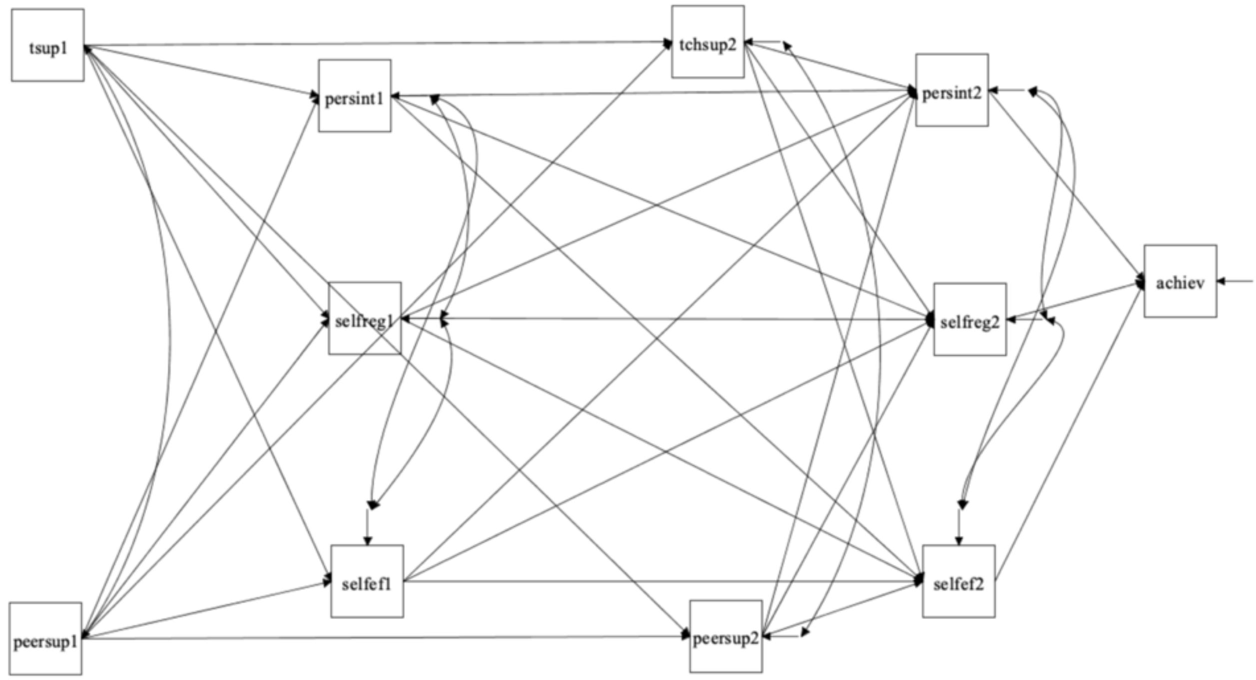

Figure 1: Hypothesized relationships among teacher and student support, mathematics engagement, and achievement over time.

RESULTS

Our model of the relationships and flow of effect of engagement factors maps the hypothesized influence of students' perceptions of their earlier experiences in mathematics with variables labeled with subscript 1, on their later experiences, labeled with subscript 2 (see Figure 1). The flow of time in Figure 1 is from left to right, with prior beliefs impacting subsequent beliefs. Achievement is hypothesized to be dependent primarily on engagement as it is manifest at the end of the year, just prior to the state-level assessment being administered (see Davis, 1985). Teacher and peer support are hypothesized to be reciprocal effects in both time periods (e.g., Klem & Connell, 2004), and engagement variables are hypothesized to influence each other and are therefore modeled as covariates.

Longitudinal Path analysis was performed with the proposed model defining the regression paths. With the relatively small ratio of sample size to parameters being estimated, this facilitates model convergence at the price of lost sensitivity. With the excellent reliability and factor structure of our instrument, we assess this to be an acceptable tradeoff. All models were estimated in MPlus Version 8 (Muthen & Muthen, 2017).

Table 1 shows the standardized regression coefficients for the hypothesized path model. Figure 2 illustrates the significant paths for the model, with coefficient estimates and their respective standard errors.

	Dependent Variables										
	Fall 2018					Spring 2019					
Independent Variables	Teach Supp 1	Peer Supp 1	Pers Int 1	Self Reg 1	Self-Eff 1	Teach Supp 2	Peer Supp 2	Pers Int 2	Self Reg 2	Self Eff 2	Ach
Teach Supp 1		0.96*	0.23*	0.11	0.06						
Peer Supp 1			0.21	0.29*	0.07						
Pers Interest 1								0.58*	-0.05	0.04	
Self Reg 1			0.38*					-0.06	0.41*	-0.03	
Self-Eff 1			0.20*	0.22*				-0.16	-0.12	0.28*	
Teach Supp 2	0.63*	-0.16					0.50*	0.02	0.08	0.09	
Peer Supp 2	0.23*	0.32*						0.47*	0.34*	0.21*	
Pers Interest 2											6.56*
Self Reg 2								0.15*			-3.70
Self Eff 2								0.10*	0.10*		-8.94

*Significant $p<0.05$.

Table 1: Standardized Regression Coefficients for Hypothesized Paths.

With regards to the impact of students incoming feelings of math engagement on their feelings at the end of the year, we can see significant direct effects of Personal Interest, Math Self-Regulation, and Math Self-efficacy on their respective counterparts at the end of the year. Within each time point, these variables are strongly correlated, but across time, they appear to primarily impact within-variable change.

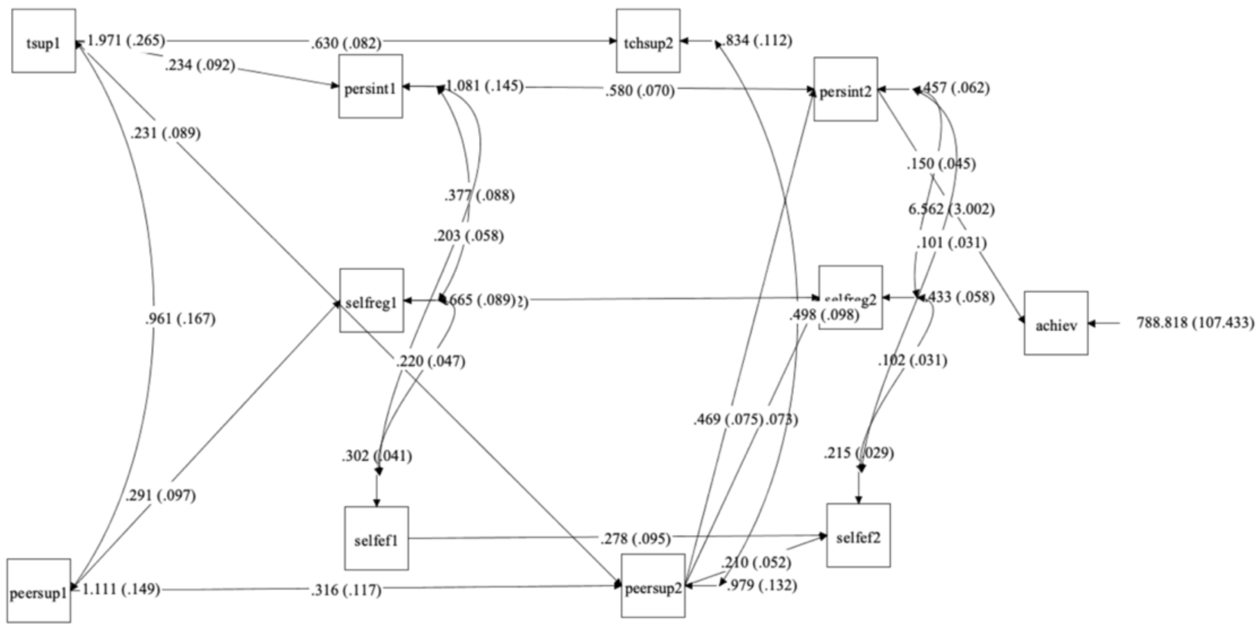

Figure 2: Final Path Model showing significant paths.

The impact of perceived Teacher Support and Peer Support shows strong evidence of mediation effects. The regression coefficients between Teacher Support and Peer Support at each time are very high. At the beginning of the Fall semester, teacher support showed a moderate impact on Personal Interest and Math Self-efficacy, with non-significant relationships for Self-Regulation. For its part, Peer Support in the Fall Semester appeared to impact Self-Regulation primarily. With the strong relationship between Peer and Teacher Support, it is unclear exactly how direct these paths may be.

Likewise in the Spring of 2019, we find a strong relationship between feelings of Peer and Teacher Support. But in Spring, 2019, it is apparent that Teacher Support is mediated through Peer Support. Peer Support shows strong relationships with Personal Interest and Self-Regulation, with a moderate relationship with Self-Efficacy.

Finally, with respect to mathematics Achievement, Personal Interest in Mathematics appears to be the strongest impact, of the measured variables. This is consistent with prior research showing that Personal Interest in mathematics is among the most influential determinants of math Achievement.

The model tested showed excellent fit (see Hu & Bentler, 1999). The Chi-square to degrees of freedom ratio was 1.49. CFI was estimated at 0.98, and TLI was estimated at 0.95. RMSEA was a bit high for this analysis at 0.066. However, this measure becomes inflated at lower degrees of freedom. When the standardized coefficients are assessed, the SRMSR is within acceptable limits at 0.049.

DISCUSSION

Taken together, results indicate that teacher and peer support are mutually impactful in the high school classroom, interacting with each other to create a classroom climate that can be facilitative or obstructive to the development of productive mathematics engagement. The impact of these variables are mediated in a number of ways as students negotiate the first year of high school, but Peer Support especially appears to become more important over time as a potential determinant of mathematics engagement.

Achievement as an outcome in freshman mathematics is impacted in a highly complex manner by these interacting facets of the classroom climate. Evidence from this study supports earlier reports that as students transition into the comprehensive high school, their attention to peers, their status, and the social aspects of schooling become more important than the perceived influence of the teacher (Reindl et al., 2015). Our results suggest that teacher support can impact students' beliefs about the supportiveness of their peers, but that these social factors are mediated through students developing personal interest in mathematics to ultimately impact achievement.

At this time in students' lives, it appears that math Self-efficacy appears to mediate teacher support, influencing subsequent mathematics achievement as well as interest. The current study adds to our understanding of how these incoming beliefs play out as the new norms of high school mathematics are introduced and reinforced in students' first year. Yu & Singh (2018) suggest that positive interactions among teacher and students influence students building stronger beliefs about their cognitive capability (i.e., self-efficacy), and enhancing their personal interest in the subject matter. This increase in efficacy in turn further reinforces interest, which directly supports their achievement in mathematics.

Some caveats must be stated about the interpretation of these results. First, we modeled the variables in this study as measured variables, not as latent variables. This was for practical reasons due to low sample size relative to the number of parameters we estimated in the model. Inevitably due to this lack of power, some of the hypothesized paths may not have been detected, constituting Type II errors. Second, the sample itself is unique, reflecting urban classrooms in large public high schools in the Southwest US and may not reflect the motivation or classroom culture evidenced in other regions.

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NUMBER STRUCTURE IN LEARNER WORKBOOKS

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This paper reports on the extent to which different representational modes in current learner workbooks conceptually signal the structuring of number (especially base-ten thinking) which research shows to be vital for learners to shift from counting to calculating strategies. Tasks contained in two learner workbooks currently used in Grade 1 classrooms across South Africa, i.e. DBE and Bala Wande, were contrasted in light of the conceptual signalling contained in the representations used. Analysis of these workbooks showed that the Bala Wande workbooks had more explicit conceptual signalling for working with number structure, which helps to address the wide-spread use of counting strategies that underpin poor learner attainment on the ground.

INTRODUCTION

Children usually start solving simple additive problems by counting in ones and also develop more sophisticated calculation strategies that are not based on counting, like near-doubles (e.g. 6 + 7 = double 6 + 1) or bridging through ten (e.g. 6 + 7 = 6 + 4 + 3 or 13 – 7 = 13 – 3 – 4). Developing calculation strategies builds on learners' facility with *structuring number*, which can be described as the skilful organisation of numbers using number relationships, number patterns and various combinations and partitions of numbers (Wright et al., 2006; 2009). Developing learners' facility with structuring number can be supported through the use of structured representations, that is, representations that can be 'read' as embodying a certain mathematical structure, like base-ten or doubles (Venkat, Askew, Watson & Mason, 2019). The importance of structuring number for enacting calculation strategies, and the access to structuring provided by structured representations, underlies this investigation into the nature of representations used in learner workbooks.

The enquiry reported on here is set in a context where an over-emphasis on counting in ones and an over-reliance on the use of concrete/unstructured materials hampers progression from counting to calculating strategies (Hoadley, 2012). Empirical research shows widespread use of unstructured representations of number (e.g. counters) in South African Foundation Phase classrooms (Grades 1 to 3, i.e. 6-9 year old) by teachers and learners (Ensor et al., 2009), the result of which is the confining of learners to counting-based strategies, which are inefficient and error-prone when working beyond the 1-10 number range.

NUMBER STRUCTURE AND BASE-TEN THINKING

A key form of number structure that children learn in the early grades is the base-ten structure arising from our use of the base-ten decimal number system and positional notation (Cobb & Wheatley, 1988). Activities that build learners' awareness of base-

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 203-210). PME.

ten include learning the bonds of ten (e.g. 6 and 4, 8 and 2), adding and subtracting 10 to/from any number (35 + 10, 62 – 10), and flexibly splitting numbers into tens and ones (i.e. seeing 64 as 60 + 4 and as 30 + 30 + 4).

Base-ten thinking is a term related to structuring that stems from the body of work developed by proponents of RME. According to Wright and colleagues (2012), Freudenthal believed that "to make any progress in mathematics children must be inducted into base-ten thinking, developing a skilful habit of organizing numbers and calculations into 1s, 10s and 100s." (p.16). Freudenthal further argued that children need to skilfully structure numbers and calculations using base-ten thinking in order for them to get a handle on working with numbers larger than twenty.

A similar view about knowing the base-ten structure of number for working in higher number ranges is held by Anghileri (2006) who argues that learners who can add 10 to any number as a known fact is ready for 2-digit addition and subtraction. Learners who do not know that 32 add 10 is 42 as a known fact will need more practice with structured materials to establish these patterns of incrementing and decrementing by ten before attempting 2-digit additive problems (Anghileri, 2006).

It is widely accepted that children need to be facile in the use of the composite unit in base-ten representations of number and this facility is considered to be well within the reach of children in the first years of school, if not earlier (Perry & Docket, 2002). As noted above, such fluency is not attained by many learners in South Africa and this paper contributes to our understanding of why this might be so.

CONCEPTUAL SIGNALLING

In their evaluation of a workbook as a curriculum tool, Hoadley and Galant (2019) use *conceptual signalling* to refer to the extent to which the concepts/content/skills underpinning tasks or activities in the workbook are made explicit. Explicit conceptual signalling is communicated through explanatory notes, headings, sub-headings, text boxes or teacher notes. In the absence of explicit signalling, the likelihood of learners (and possibly teachers) becoming aware of the underlying mathematics is greatly reduced.

Drawing on the work of Hoadley and Galant (2019), we extend the notion of *conceptual signalling* to argue that various representations of a particular problem or concept may clearly signal some structure, with other representations not making the structure as 'transparent'. Structured representations can be effective at signalling number structure like doubles, base-five or base-ten. For example, the pairwise ten-frame in Figure 1 can signal the concepts of doubling and base-ten. Doubling, because eight can be seen as double 4, and base-ten, because eight can be seen as two away from ten – which can be linked to the number sentences '4 + 4 = __' and '8 + __ = 10', respectively.

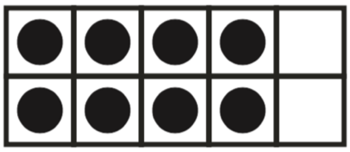


Figure 1.

By extending Hoadley and Galant's sources of explicit conceptual signalling, we argue that representations in learner workbooks that exhibit mathematical structure (e.g. Part-part-whole (P-P-W) diagrams, images of 10-frames or bead strings) can also send explicit conceptual signals.

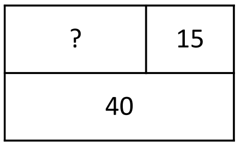
REPRESENTATIONS

Representations play an important role in teaching and learning mathematics (Heinze et al., 2009) and come in many forms such as drawings and inscriptions, pictures, symbols and concrete objects. Researchers agree that learning how to structure number can be supported by the use of structured representations, initially presented as concrete materials and gradually 'faded' until these representations merely need to be imagined by learners to support their mental calculation (Anghileri, 2006; Wright et al., 2012). Examples of structured representations usually found in Foundation Phase classrooms include: number tracks, number charts, number lines, part-part-whole (P-P-W) diagrams, arrays, triad diagrams, 5-frames and 10-frames.

Figures 2 and 3 below are two different representations of the same missing addend problem.

Figure 2. $\square + 15 = 40$

Figure 3.



In Figure 3, the part-part-whole representation may enable a learner to see the structure in the problem as not being limited to finding a missing addend and become aware of the possible use of subtraction to find the solution. In the symbolic representation of the problem (Fig. 2) the possibility of using subtraction to find the missing addend is much less noticeable to young learners (who frequently add the given numbers). The P-P-W diagram (Fig. 3) thus conveys a stronger 'conceptual signal' about the relationship between addition and subtraction. Thus conceptual signals arising from structured representations can serve as an affordance to meaning making.

The caveat must be kept in mind that children do not simply 'see' a number concept or relationship embodied in a representation, but rather become aware of these concepts and relationships as they use these representations in their number work (Ellemor-Collins & Wright, 2009). One can say that learners become aware of number structure through the process of structuring number. For example, a learner who shows 32 on an abacus by counting in ones is not aware of the 10-ness embodied in the design of the abacus. But, when encouraged to show 32 by counting "ten, twenty, thirty, thirty-two" while moving over 3 lots of 10 beads and 2 beads on the abacus, the learner becomes more aware of the 10s structure of the abacus and can learn over time how to use this 10s structure for efficient enumeration. The structured representation and structuring actions on the abacus lead to understanding structure.

The importance of using certain modes of representation to encourage the structuring of number prompted using the notion of *conceptual signalling* to investigate to what extent representations used in workbook tasks might facilitate learners' use of base-ten thinking.

RESEARCH AIM AND QUESTIONS

In a context where learners' main access to mathematics structure is through workbooks, the aim of this study was to examine to what extent do workbooks conceptually signal base-ten thinking through the representations used in tasks. To achieve this aim we examined current Grade 1 mathematics workbooks used in South Africa, namely, the Department of Basic Education (DBE) Workbook and the Bala Wande (BW) Workbook (or Learner Activity Book). The selection of these workbooks was guided by the fact both workbooks are currently used in government schools: the DBE workbook nationally, the BW workbook in 3 of SA's 9 provinces. The specific research questions guiding this investigation were:

- How do two current workbooks used in Grade 1 conceptually signal base-ten thinking through the representations used?
- What are the implications of different types of conceptual signalling?

METHODOLOGY

Following Mason and Johnston-Wilder's (2006) distinction between task and activity, a 'task' in this report refers to what is presented in the pedagogical text as the focus of attention (these can be broken down into smaller parts) while the 'activity' describes what happens in the enactment of the task. The DBE and BW Grade 1 mathematics workbooks for 2021, covering all four school terms, were analysed and contrasted for this report; these workbooks were obtained in print and digital format.

The BW workbooks for Grade 1 are clearly divided into weeks and days. The 5th day of every week is for assessment and/or consolidation. All the activities planned for one lesson are seen as different parts of one task, therefore 1 day = 1 task. Most of the tasks in the BW workbooks consist of worksheets stretching over 2 pages. The BW workbook series for Grade 1 consists of 185 tasks in total, divided across Terms 1 to 4 as 45, 50, 50 and 40 tasks, respectively. The bilingual Sepedi-English version of the BW workbooks were used for consistency when referencing page numbers.

The DBE workbooks for Grade 1 are presented as two volumes: Book 1 for the first two terms and Book 2 for the last two terms. Book 1 and 2 each contain 64 discrete tasks: 32 tasks per term spread over eight weeks, i.e. four tasks per week (DBE, 2011 as cited in Fleisch, et al., 2011). Most tasks in the DBE workbooks consist of worksheets covering 2 pages (a few extend to 4 pages).

To start the process of analysis, each workbook series was carefully read to determine their overall structure. Using the task as the unit of analysis, a proforma was developed to capture the following data in tabular form: the total number of tasks per term, and the constitution of tasks: topic, intended activities and nature of representations used (i.e. structured or unstructured). This data was captured for DBE Workbook 1 (for Terms 1 and 2), DBE Workbook 2 (for Terms 3 and 4) and the Sepedi-English version of the four BW workbooks. Tasks in the BW workbooks that were used for assessment purposes were omitted because there are no comparable assessment tasks in DBE workbooks.

After this preliminary data capture, we re-looked at each task that used a structured representation and recorded additional information about the task: the number range used, the purpose of the structured representation/s (for illustration or for learners to act on/use to calculate an answer) and the number structure or number relationships signalled by the structured representation/s. We also made a note of tasks that used more than one structured representation for the same activity.

We were also interested in the frequency with which various structured representations were used in the workbooks. To this end we counted the number of times different structured representations were used across each term in both workbook series. If the same representation is used more than once in a task – e.g. BW Term 1 Week 5, Day 2, P-P-W diagrams are used in the whole class activity (p52), and the same representation is used again in the independent activity (p53) – this is counted as one instance of P-P-W used.

FINDINGS

The number of tasks in each workbook are shown per term, side-by-side in Table 1. Also recorded in Table 1 are the number of tasks that make use of an image of a structured representation and the number of tasks that use more than one structured representation for the same activity.

	Bala Wande				Total	DBE				Total
Term	1	2	3	4		1	2	3	4	
Number of tasks	45	50	50	40	185	32	32	32	32	128
Tasks with 1 struc. rep.	28	18	34	16	96	3	11	16	10	40
% of structured reps.	62	36	68	40	52%	9	34	50	31	31%
Tasks >1 structured rep.	13	0	5	0	18	0	4	2	3	9
% using multiple reps.	29	0	10	0	10%	0	13	6	9	7%

Table 1: Info on tasks in DBE and BW Workbooks.

From Table 1 it is evident that 96 of the 185 tasks in BW workbooks (+/- 52%) use an image of a structured representation while 40 of the 128 tasks in the DBE workbooks (+/- 31%) do so. About 10% of tasks in BW workbooks use more than one structured representation for one activity while about 7% tasks in DBE workbooks do so.

Table 2 shows the various structured representations present across both series of mathematics workbooks and the number of times these were used in tasks. Some tasks used more than one representation, thus the number of tasks that use an image of a structured representation (Table 1) does not match the number of instances a certain representation was used (Table 2). All nine structured representations used in the analysis are present in the BW series whilst five are present in the DBE workbooks. The 10-frame is the most frequently used structured representation in the BW series (50 instances) but not used at all in DBE workbooks. Other structured representations present in BW workbooks but absent from DBE workbooks are the array, P-P-W, triad diagram and 5-frame.

	BW				Total	DBE				Total
Terms	1	2	3	4		1	2	3	4	
Array	6	-	-	-	**6**	-	-	-	-	**-**
Hand or foot	2	-	-	-	**2**	-	4	7	1	**12**
Number chart	-	-	-	2	**2**	-	2	2	9	**13**
Number line	10	-	10	4	**24**	3	6	10	5	**24**
Number track	2	4	5	-	**11**	-	2	7	4	**13**
P-P-W diagram	8	13	6	10	**37**	-	-	-	-	**-**
Triad diagram	9	2	1	-	**12**	-	-	-	-	**-**
5-frame	5	-	-	-	**5**	-	-	-	-	**-**
10-frame	18	3	25	4	**50**	-	-	-	-	**-**
Total	60	22	47	20	**149**	3	14	26	19	**62**

Table 2: Structured representations used in workbooks.

Taken together, Tables 1 and 2 show that the BW series of Grade 1 mathematics workbooks contain a wider range (9 to 5, respectively) and a higher frequency (52% to 31%, respectively) of structured representational use across all tasks compared to the DBE workbooks.

DISCUSSION AND IMPLICATIONS

The use of structured representations of number can signal important number concepts or relationships that support learners' use of calculation strategies that are not based on counting. For many low attainers, who rely on inefficient and error-prone counting strategies, the use of structured representations can be the bridge to structuring number which in turn provides access to more sophisticated calculation strategies and working in higher number ranges. Using structured representations in an every-day resource like a workbook is one way of ensuring that learners have multiple opportunities to notice and use number structure.

In our investigation into how two current learner workbooks conceptually signal number structure, especially base-ten thinking, we found that the Bala Wande workbooks explicitly signal number structure by using structured representations more frequently than the DBE workbooks. This implies that learners who used the Bala Wande workbooks had greater access to representations that foreground number structure, and therefore had a greater chance of structuring number and using sophisticated calculation strategies, than learners who used the DBE workbooks.

Research shows that learners who are exposed to multiple representations of a concept, and who learn to seamlessly shift between representations, develop deeper conceptual understandings (Heinze, et al., 2009). Bala Wande workbooks provide stronger conceptual signals of representational flexibility than the DBE workbooks because they use a wider range of representations and use multiple representations for one task to a larger extent than their counterpart. The implication is that learners who used the former had a greater chance of shifting between multiple representations and developing deeper conceptual understandings than those who used the latter.

CONCLUSION

"Tasks lie at the centre of learning and teaching mathematics" (Askew, 2016, p1) thus their importance cannot be downplayed. In this report, tasks in Grade 1 learner workbooks were considered in light of the affordances provided through the conceptual signalling of number structure in the representations used. By focusing solely on the design of tasks, this report cannot claim that the presence or absence of a textual feature implies adequacy or inadequacy in the teaching/learning associated with such tasks. This report highlights the affordances and opportunities that are provided through workbook tasks that make use of a specific textual feature (i.e. structured representations), irrespective of implementation. This sends a strong message to workbook designers about being aware of the conceptual signalling in representations selected for tasks and ensuring that these signals align with their intentions for tasks. This is also a wake-up call to consumers of workbooks (teachers, parents, etc.) - closer attention must be paid to conceptual signals in representations used in tasks as this affects children's opportunities to learn.

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INTERLINEAR MORPHEMIC GLOSS (IMG) OF FRACTION NAMING CONVENTIONS IN ISIXHOSA

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Many learners of mathematics struggle with fractions, frequently considering fractions to be two different unrelated numbers. In this paper four mathematics texts, translated From English into isiXhosa, are analysed in order to describe isiXhosa fraction naming conventions. Interlinear morphemic glossing (IMG) provides addition information, which goes beyond the idiomatic translation, providing relevant information for both the English and isiXhosa texts. The analysis shows that there are two primary conventions. One convention is expressed with the denominator first as in 'fifths.of-which-there-are-4', the other is more similar to the English '4 over five' but makes explicit a relationship between 4 and 5. The affordances and constraints of both isiXhosa fraction naming conventions are described in relation to the English naming conventions.

INTRODUCTION

A research agenda for 'Language in Mathematics Education' (LiME), has recently been outlined in Sfard (2021) as comprising of six interrelated themes: (1) linguistic mechanisms that generate mathematical objects; (2) The role of language in the historical emergence of mathematical discourses; (3) linguistic relativity of mathematics, (4) linguistic changes in the process of learning mathematics (5) linguistic gaps in the classroom and (6) dialogic engagement as a protection from falling into linguistic gaps. This research agenda asserts that mathematics is not universal, but is always expressed in a language. How mathematics is expressed differs remarkably across language groups. Previously language was generally considered to be the background noise, or transparent, in mathematics classrooms. This was partly because mathematics was expressed and studied in Indo-European language groups which have much in common with each other, and in largely linguistically homogenous classrooms. However, as the world has globalized, mathematics education research has become alive to language diversity. Mathematics education research now attends to the need to teach mathematics to English Language Learners (children who learn mathematics in English but draw on another language at home and for thinking), as well as to the richness evident when attending to the diversity of mathematics expressions which enriches both the mathematics and the mathematics pedagogy.

Fractions are internationally recognised as being difficult for learners (Charalambous & Pitta-Pantazi, 2007). One of the primary reasons why learners struggle with fractions is because of their tendency to see fractions as two different whole numbers with no particular relationship (Charalambous & Pitta-Pantazi, 2007). In order to reduce the

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possibility of this misconception developing, the South African Foundation Phase Curriculum and Assessment Policy Statement (CAPS) states that

When writing about many fraction parts. e.g. 3 halves, 3 quarters, write this as the figure and the word. The expression 3 over 2 or 3 over 4 is meaningless and it is best to leave this symbolism to the Intermediate Phase. (Department of Basic Education, 2011, p. 255).

This naming convention differentiates the fractional part from the number of the fractional parts. While this naming convention and its affordances are available in English, it doesn't stand to reason that the convention and its affordances are available in other languages, such as the other official South African languages.

Even though there is a small body of research pertaining teacher preparation for teaching mathematics in isiXhosa (Roberts & Alex, 2020) and the naming of whole numbers in some of South Africa's other official languages (e.g. Feza, 2016, Poo, 2017, Mostert, 2019), there is no research about the naming of fractions. This paper sets out to address this research gap by considering the naming of fractions in one African language, namely isiXhosa. In particular this paper will answer the following question: How are fractions named in isiXhosa and what are the affordances and constraints of the different naming conventions?

By contrasting two languages with contrasting structures, we aim to reflect on two of Sfard's (2020) LiME themes: (1) "linguistic mechanism used to generate the mathematical object" of 'a faction', and (5) make explicit the related "linguistic gaps" which are likely to arise in isiXhosa, and English dominant classrooms. In so doing, we hope to better equip teachers of mathematics in English, to better support the both the English speakers and English Language Learners in their classrooms. Building on the examples given by Edmonds-Wathen (2019), this study provides an example of how Interlinear Morphemic glossing (IMG) can be used when comparing translations of canonical texts with the original language and the value of such a comparison.

THEORETICAL FRAMING

We briefly discuss the concept of a fraction as it is construed mathematically before discussing naming fractions naming in particular languages.

The mathematical concept of fractions is made up of a number of subconstructs, where Kieren (1980) proposed five subconstructs – fraction as part-whole relation, as ratio, as quotient, as measure and as operation – all of which were necessary to understand the concept of fraction. The two subconstructs relevant to this paper are 'part-whole' and 'measure'. The part-whole subconstruct is the central construct and is closely linked to partitioning. The measure subconstruct views a fraction as unit that can be repeated, and is therefore closely related to fraction addition (and subtraction) (Charalambous & Pitta-Pantazi, 2007).

Different languages have different naming conventions for fractions. One overarching difference between naming conventions is the word order – whether the numerator or the denominator is read first (Bartolini Bussi, Baccaglini-Frank, & Ramploud, 2014).

European languages read fractions such as $\frac{4}{5}$ from top to bottom, saying the numerator first and then the dominator [N-D]. As such these languages mention the number of pieces to be taken first, and then number of pieces the whole should be partitioned into. As an example in English, the numerator is read first, and the dominator follows [N-D]: '4 fifths'. This is an example of discursive "compactification" (Sfard, 2008) where the mathematical process "start with a whole, partition the whole into 5 equal parts; then take 4 of those parts" is reified to become an object "4 fifths".

In contrast, many Asian languages read the denominator first and the numerator afterwards [D-N]. It is possible to express a fraction name in English where the denominator is read first: 'fifths.of-which-there-are-4' (or 'fifths.that-are-4'). The latter is not in common use, but serves to demonstrate to the English speaker, how a fraction could be expressed with the denominator following the numerator. Bussi, Baccaglini-Frank and Ramploud (2014) suggest that the denominator first articulation might help learners conceptual understanding, as they know how to partition the whole, before choosing the parts.

METHODOLOGY

In this paper the isiXhosa translations of four mathematics education texts for grades 1-3 are analysed: the curriculum and assessment policy standards (CAPS); and three sets of the learner workbooks (produced by the Department of Basic Education, the National Education Collaboration Trust (NECT); and the Nelson Mandela Institute (NMI). All four texts were written in English and translated into isiXhosa.

To answer the question, examples of named fractions were identified in each text. Similarities and differences between the apparent naming conventions were noted in order to categorise the conventions. Then a simplified version of IMG - a form of translation used in linguistics in order to present the structure of the source language and not only the meaning - was applied. IMG provides a means of presenting the structure of a phrase from the source language in a target language. If only an idiomatic translation is provided, the structure of the source language is often lost (Edmonds-Wathen, 2019).

We draw on Edmonds-Wathen (2019) to apply a four level IMG. In this study, the top level gives the source language, isiXhosa, in sentence form. The second level gives the isiXhosa morphemes (the smallest unit of a language that has its own meaning), the third level gives the morphemic gloss in English (the target language), and the final level gives a free translation in English. The following provides an example of IMG for the number word 37:

Level 1: amashumi amathathu anesixhenxe

Level 2:	*ama-shumi*	*ama-thathu*	*a-ne-sixhenxe*
Level 3:	tens	that.are-three	that.are-with-seven
Level 4:	'thirty seven'		

Note that the italics denotes isiXhosa words or morphemes. We then attended to the noun classes and adjectival stems used for numbers. This provided further detail on whether, and when the name of fraction was being construed, as either a process (a whole partitioned into five equal parts, then take 4 parts) or as an object in its own right (4 fifths).

To identify affordances and constraints, both isiXhosa naming conventions were analysed in terms of the five different conceptions of fractions as originally formulated by (Kieren, 1980).

FINDINGS AND DISCUSSION

For clarity we focus particularly on the fraction name for '4 fifths'. This is intended to enable the English reader, who is unfamiliar with isiXhosa, to engage with the isiXhosa language structure. To do so we draw on Mostert (2019) to inform the reader of the relevant number names for whole numbers in isiXhosa.

In isiXhosa the number 'four' is expressed as a numeral noun: *isine*, as an adjectival stem: *-ne*, or as said in the forward number word sequence (the counting song) as *(zi-)ne*. The number 'five' is expresses as a numeral noun: *isihlanu*, as an adjectival stem *-hlanu*, and in the forward number word sequence as *(zi)ntlanu.*

An analysis of textbooks and workbooks reveals that there are a number of ways of referring to fractions in isiXhosa. There are however two primary naming conventions which correspond to the two English naming conventions and are discussed in terms of their affordances and constraints.

Naming convention A: Number of fractional pieces naming (D-N)

The first isiXhosa fraction naming convention articulates the denominator first. $\frac{4}{5}$ is named 'izintlanu ezine' which can be directly translated as '*fifths*-they.are-four' and which means 'four *fifths*'. The interlinear morphemic gloss for this is as follows:

Level 1:	izintlanu ezine	
Level 2:	izi-ntlanu	ez-ine
Level 3:	(things)-they.are-five	(things)-they.are-four
Level 4:	'four fifths'	

This convention using the basic isiXhosa noun in its plural form, drawing on noun class 10, -izinto (things). The prefix izi- denotes the plural form, in contrast to the prefix isi- for the singular form. The word 'ezine' replaces 'isine' as a result of the pronunciation from the vowel sound (at the end of -hlanu).

In this convention, as in English, the fraction piece (or unit) is an object that can be operated on. In this sense the naming convention can support the measure subconstruct of fractions. But notice the invariant structure evident at " Level 3: (things)-they.are-five (things)-they.are-four". Working from the isiXhosa towards the English, with the knowledge that this phrase refers to fraction names, lead to a Level 4 translation of 'four fifths'. However in the absence of the context knowledge that what is being translated is a fraction, this could lead to a Level 4 translation of 'four fives'. Whether one is referring to 4 fives, or 4 fifths has to be deduced from the broader context. 4 fifths is clearly mathematically distinct from 4 fives, yet is not discernible linguistically (in the absence of broader context). We think this is a linguistic gap for isiXhosa speakers which ought to be explicitly discussed in isiXhosa mathematics classrooms.

Naming convention B: Number 'in relation' to number (N-D)

The second isiXhosa fraction naming convention articulates the numerator first. $\frac{4}{5}$ is named 'isine *kwisihlanu*' which can be directly translated as 'four *in/on/at/by* five' and which means 'four *out of* five'. This is a 'compactification' (Sfard 2021) of the mathematical process 'take four out of the five equal parts' which is 'objectified' (Sfard 2021) to be a noun depicting its number symbol: 'four-over-five'. The IMG for this is as follows:

Level 1:	isine kwisihlanu	
Level 2:	isi-ine	ku-isihlanu
Level 3:	a.four	on-a.five
Level 4:	'four over five'	

This convention uses the isiXhosa numeral noun (a number). The prefix ku- becomes kwi- when before an i, and denotes a relationship (numerator <u>on</u> denominator).

It is difficult to learn from a single example, and to note the underlying structure of each convention. We therefore tabulated unitary fractions as named using each naming convention. We constrained the number of fractional pieces to one, and varied the fractional parts sequentially.

Number of fractional pieces		***X out of Y***	
[D-N]	[N-D]	[N-D]	[N-D]
isiXhosa	English	isiXhosa	English
ihafu enye	one **half**	isinye *kwisi***bini**	one *out of* **two**
isi**thathu** esinye	one **third**	isinye *kwisi***thathu**	one *out of* **three**
isi**ne** esinye	one **fourth**	isinye *kwisi***ne**	one *out of* **four**
isi**hlanu** esinye	one **fifth**	isinye *kwisi***hlanu**	one *out of* **five**
isi**thandathu** esinye	one **sixth**	isinye *kwisi***thandathu**	one *out of* **six**

Table 1: Unitary fractions named according to two naming conventions, in isiXhosa and in English.

Creating this tabulation revealed the lack of regularity in the naming of fractions in English for the ‘number of fractional pieces’ convention. In English the fractional pieces are named using ordinal numbers (third, fourth, fifth etc)’ however this is not applied consistently in the case of 2 and 4. Using the ordinal number naming convention, ‘1 half’ should be ‘1 second’. Similarly, ‘1 quarter’ should be ‘1 fourth’. Given the use of ‘second’ both as an ordinal number name of 2, and as a unit of time, this is likely “linguistic gap” (Sfard 2021) for English Language Learners. Attention should be drawn to the English words – half, second, fourth, and quarter to mitigate against this gap.

Next we tabulated the two conventions for different numbers of fifths. We constrained the fractional parts to fifths and varied the number of fractional pieces sequentially.

Number of fractional pieces		**X *out of* Y**	
[D-N]	[N-D]	[N-D]	[N-D]
isiXhosa	English	isiXhosa	English
isi**hlanu** esinye	one **fifth**	isinye *kwi*si**hlanu**	one *out of* **five**
izi**hlanu** ezimbini	two **fifths**	isibini *kwi*si**hlanu**	two *out of* **five**
izi**hlanu** ezintathu	three **fifths**	isithathu *kwi*si**hlanu**	three *out of* **five**
izi**hlanu** ezine	four **fifths**	isine *kwi*si**hlanu**	four *out of* **five**
izi**hlanu** enzintlanu	five **fifths**	isihlanu *kwi*si**hlanu**	five *out of* **five**

Table 2: Different numbers of fifths, named according to two naming conventions, in isiXhosa and in English.

In Table 2, the English fraction names are regular for both conventions. The isiXhosa fractions names are consistent within each convention, but when comparing the two conventions with each other there are notable differences. For the ‘number of fractional pieces’ convention the nominal noun class is used (*izi-*), for a generic set of “things” (noun class 1), however for the ‘ number in relation to number’ or ‘x out of y’ convention the numeral noun is used for both the number of parts, and for the fractional part. These whole number names are related by the prefix *kwi-* which denotes the relationship (‘on’ or ‘over’) between them.

Affordances and constraints of the two naming conventions

The first affordance for the ‘number of fractional pieces’ convention in both isiXhosa and in English, is that the fraction piece (or unit) is an object that can be operated on. In this sense the naming convention can support the measure subconstruct of fractions referred to by Kieran (1980). The second affordance relates to word order. Because adjectives follow rather than precede nouns in isiXhosa, the word order in this convention is denominator-numerator (D-N). This is similar to many Asian fraction naming conventions (e.g. Japanese, Korean), which first specify how to partition the whole before specifying how many pieces to choose. However, this may in fact being a constraint as within the South African context where learners need to transition to English, and the isiXhosa D-N word order is the reverse in English, thus potentially making it more difficult for isiXhosa learners to transition to English. A second constraint relates to the ambiguity of the names of the fractional parts. In English, fractional parts are named using ordinal numbers (third, fourth, fifth etc) which is an affordance, although this is constrained in English by the irregularity relating to the naming of halves and quarters. In isiXhosa the nominal form of the number (*isibini, isithathu, isine* etc) is used instead of the more common 'adjectival form' (*zimbini, zintathu, zine* etc) (Mostert, 2019). However this naming is ambiguous as '*izihlanu* ezine' can be translated both as 'four *fifths*' and as 'four *fives*', which is the third constraint.

The 'number in relation to number' convention in both isiXhosa and in English, relates to the part-whole subconstruct, particularly the variation which specifies that parts are being referred to, 'four pieces out of five pieces'. The process by which a fraction is created is explicit. This is the fundamental subconstruct that all others rest on, this is one affordance of this naming convention. Another affordance is the fact that the word order (D-N) is the same as both English naming conventions, thus potentially making the transition to English easier. However, the word order is also one of the constraints of this convention as it follows the Western convention of first specifying the number of parts before specifying the size of those parts. Even though the isiXhosa naming convention is similar to the English '4 over five', the fact the prefix *kwi-* is used establishes a relationship between the two numbers, thus possibly reducing the likelihood that fractions are seen as two separate numbers. However, this naming convention does not support the measure subconstruct and therefore adding fractions is not as easy.

CONCLUSION

This paper has provided an initial description of two isiXhosa fraction naming conventions, appearing in four foundation phase texts. The two conventions differ both superficially – the order in which the numerator and the denominator are referred to is reversed – and more fundamentally. The first convention points to the part-whole and the measure subconstructs identified by Kieran (1980), while the second convention points to the part-whole subconstruct. The two naming conventions have affordances and constraints. If South African learners who are taught in African languages are to fully benefit from home language instruction, it is important that teachers are equipped to leverage these affordances and mitigate these constraints. It is also important that teaching and learning texts are consistent in the way in which they name fractions.

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TEACHER LEARNING ABOUT EXEMPLIFICATION IN GEOMETRY THROUGH LESSON STUDY

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We explore aspects of Malawian teachers' learning in their first encounter with Lesson Study (LS) professional development. Experienced secondary mathematics teachers from two schools participated in a theory-guided LS focused on geometry. Using data collected during the first LS cycle, we examined dimensions of variation of geometry examples made available, and changes in example sets. Findings show teachers' take-up of two dimensions of variation in the initial lesson plan, with the third dimension coming into focus during lesson reflection. We argue that theory-guided LS can support teachers to strengthen their knowledge and use of example sets in geometry.

INTRODUCTION

In Malawi, geometry learning is considered as very challenging (Ministry of Education Science and Technology [MoEST], 2020). One of the aspects of teaching geometry is using diagrams that exemplify geometric objects and properties. Exemplification has been advocated as an important teaching practice (e.g. Adler & Alshwaikh, 2019; Watson & Mason, 2006), suggesting possibilities for supporting the teaching of geometry. The broad question we pursue in the LS project is: how can we organise professional development (PD) to support teachers' learning of exemplification in geometry? PD using Lesson Study (LS) is relatively new in Malawi and to date mainly conducted with primary mathematics teachers and teacher educators (Fauskanger, Jakobsen & Kazima, 2019). In LS, teachers undertake collaborative research to reflect on and improve their teaching (Lewis et al., 2006). As a PD practice, LS has been adapted and implemented in many countries and in geometry (Fujii, 2014; Huang & Leung, 2017). We build on these studies and respond to the call for theory-guided LS by adapting and using a Mathematics Teaching Framework (MTF) that structured LS in algebra in low-income South African secondary schools (Adler & Alshwaikh, 2019) to introduce LS in secondary level geometry in Malawi. MTF includes a focus on exemplification and draws directly from variation theory (e.g. Marton & Tsui, 2004) to enhance generalising about an object of learning through focusing on what changes (variance) amidst what remains the same (invariance) across an example set (Watson & Mason, 2006). Building on Adler & Alshwaikh (op cit), we will argue that LS is a productive context for learning exemplification as a mathematics teaching practice, here in the context of geometry. We focus on variation in diagrams, geometric examples in our terms, as they are vital for enhancing learners' geometric reasoning (Al-Murani, Kilhamn, Morgan & Watson, 2019; Huang & Leung, 2017). The specific questions addressed in this paper are: 1) what dimensions of variation in examples do

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the teachers make available, and 2) how do these dimensions unfold in successive lesson plans over a LS cycle? We begin by describing the MTF framework and the LS model used.

MATHEMATICS TEACHING FRAMEWORK (MTF)

MTF is a lesson planning, observation and reflection tool developed from the Mathematical Discourse in Instruction (MDI), an analytical framework for describing and evaluating the quality of mathematics made available in teaching (Adler & Alshwaikh, 2019; Adler & Ronda, 2015). MDI draws from key tenets of socio-cultural theory, and thus a view of mathematics as an interconnected and hierarchical network of scientific concepts, and teaching/learning as goal directed and mediated (Vygotsky, 1987). The starting point of MDI/MTF is that teaching/learning, is always about 'something' which in Marton and Tsui's (2004) language is called the 'object of learning', and the work of the teacher entails bringing that 'something' into focus – its mediation (Adler & Ronda, 2015). The object of learning is what learners are expected to be able to know and do at the end of the lesson, which in our case is establishing and applying the exterior angle of a triangle theorem. In MTF, the object of learning is mediated by the teaching practices of exemplification, explanatory communication, and learner participation. Exemplification includes and distinguishes examples, tasks and representations as semiotic mediational means. In geometry, however, a diagram can be viewed as both an example and a representation. Thus, MTF required adaptation to clarify how patterns of variation in example sets can be described in geometry. We draw on the constructs of dimensions of variation and the range of change (Watson & Mason, 2006). The features of diagrams that vary constitute dimensions of variation, and the extent to which they are varied is the range (Al Murani et al., 2019; Watson & Mason, 2006).

In this paper, we describe three possible dimensions of variation in geometric diagrams each of which was made available in the lesson plans we analysed: angle measures, complexity, and orientation of diagrams. We describe the range of change in orientation as between standard or non-standard. For example, standard orientation means that a triangle is drawn in its prototype position, i.e., a triangle ABC drawn with vertex A on top and vertices B and C on the bottom and horizontal side BC extended to E to form exterior angle ACE. Non-standard orientated triangles are those drawn in atypical orientations e.g. with two vertices on top and one on the bottom with one of the top sides extended to form an exterior angle. The range of change for diagram complexity is between basic (if it does not require decomposition to do calculations or proof) e.g. one triangle with one or more exterior angles. A diagram is complex if it requires decomposition, for example a diagram comprising overlapping triangles with an exterior angle of one triangle being an interior angle of another triangle.

METHODOLOGY

Malawian secondary mathematics teachers' challenges in teaching geometry are both in content and pedagogy (MoEST, 2020). Teachers were thus introduced to both geometry content using MTF and to LS mode of PD in a two-day PD with secondary mathematics teachers from two schools. After the workshop, teachers from each school met to decide on problems that they each wanted to focus on in their LS. The Malawian LS proceeded as illustrated in Figure 1 below. There were two initial planning sessions (LP1A and LP1B), followed by teaching 1, reflection 1 and lesson planning 2, then teaching 2, followed by reflection 2 and lesson planning 3. One knowledgeable other (first author here – KO1) participated in all stages of the LS cycle and video recorded the sessions. During LP1A session, there was minimal input from KO1 as teachers discussed their choices of examples. We aimed to identify the aspects of MTF from the PD session that teachers had initially taken up and included in their plans on their own. All authors (as knowledgeable others KO1, 2 and 3) commented on the plans, and KO1 discussed the comments with the teachers LP1B planning session.

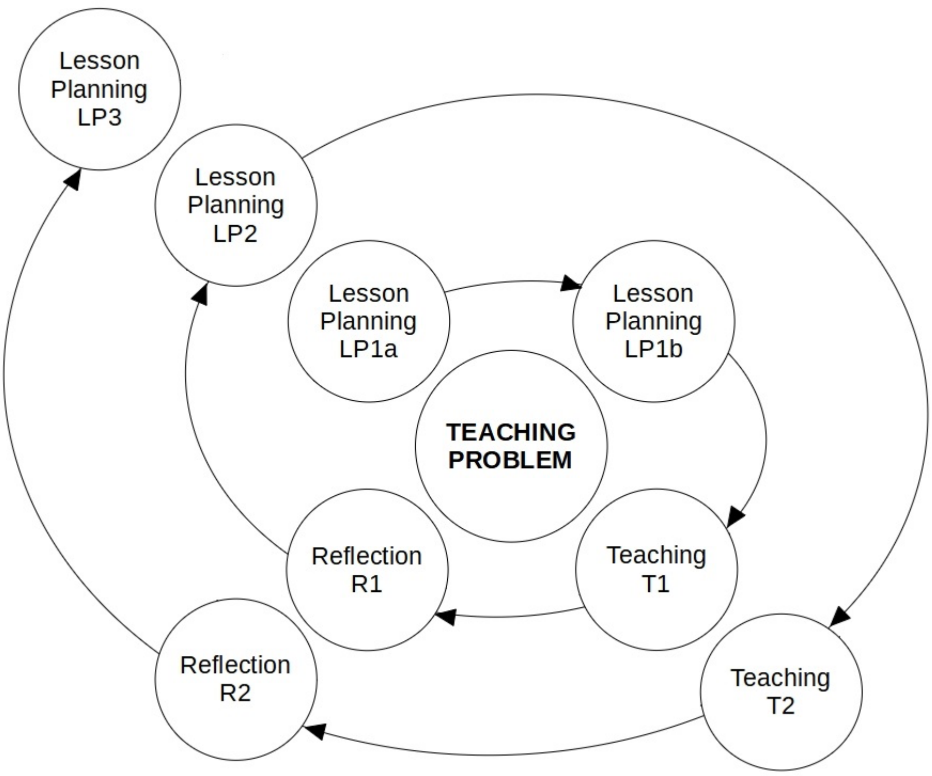


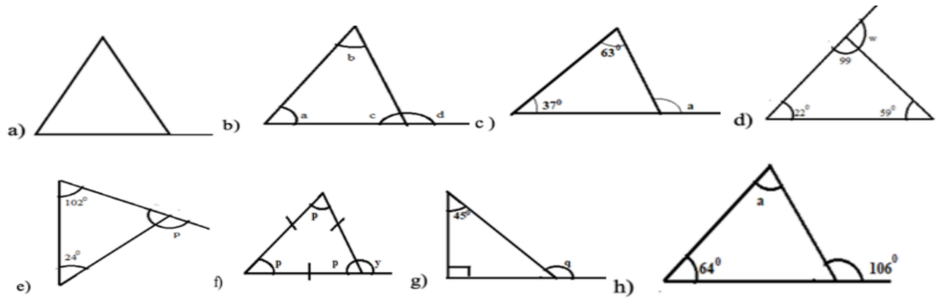
Figure 1: Lesson Study Cycle.

In this paper we focus on one school and conduct content analysis on a) transcripts of lesson planning sessions 1A, 1B and 2; b) written lesson plans; and c) transcripts of reflection 1 session. We began by analysing the whole set of examples across each lesson plan to examine the dimensions of variation made available and the range of change in each dimension. We then compared examples in all plans to identify what the teachers maintained, added or removed. We simultaneously examined the transcripts for the planning sessions and lesson reflection sessions for an in-depth understanding of teachers' rationales for changes made in the example sets.

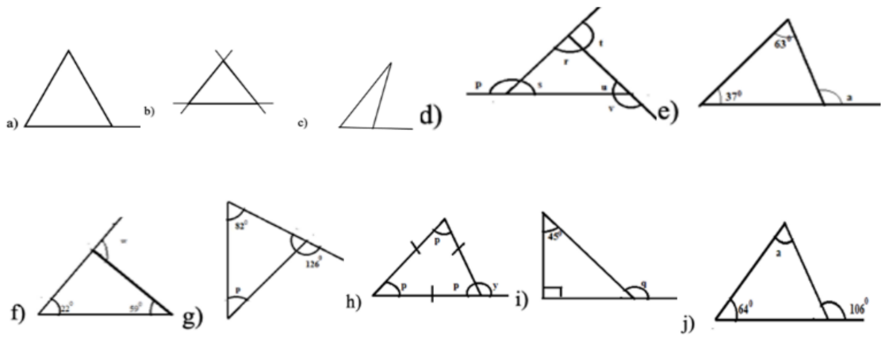
RESULTS

The teachers decided that the problem they wanted to take for LS is the challenges that learners face in understanding the relationship between the exterior angle and the two opposite interior angles of a triangle. They specifically mentioned that students were not able "to form equations with interior and exterior angles of a triangle". After the PD workshop, they were encouraged to plan a lesson, using the MTF as a guide, that could bring this relationship into focus with learners. We describe the examples planned by teachers in LP1A, LP1B and LP2 sessions using Table 1. We have not shown example space for LP3 because it is like that of LP2.

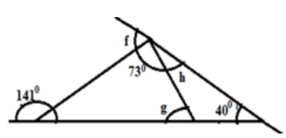
LP1A



LP1B



LP2 Examples a, b, c, d, e, g, h and i are the same as in LP1B but another example was added as shown



Assessment examples for LP1B and LP2

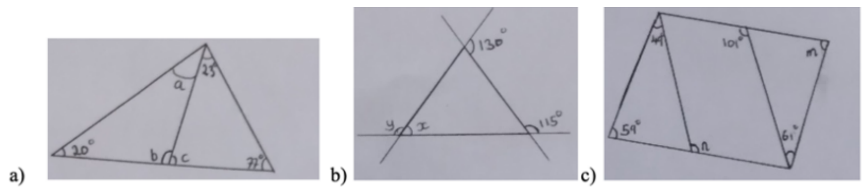


Table 1: Examples for LP1A, LP1B and LP2.

In each lesson plan, examples were to be used for different tasks. For example, in LP1A examples a and b were to be used for empirical activity of measuring angles to derive the theorem, and all the other examples were used for applying the theorem to calculate measures of angles. While more can be discussed about the tasks, in this paper, we have backgrounded them to focus on the examples. Analysis of the example space in LP1A (shown in Table 1) show variation in two dimensions: the orientation of the diagrams in terms of the position of the exterior angle, and variation of the angle measure to be calculated. We view this as learning initiated by the PD workshop as there was no contribution from the KO in relation to examples at this stage. As Table 1 shows, the third dimension of varying the complexity of the diagram was not in focus in LP1A as the example space contained only simple diagrams of a triangle with one exterior angle with varied orientation and angle measure to be calculated. In LP1B, KO1 suggested that the teachers consider using several examples with varied orientation and measures of interior and exterior angles to derive the theorem, and to consider including complex examples in the example space to enable building to generality through different dimensions of variation. As shown in Table 1, the teachers took up these suggestions, adding examples b and c for doing empirical activity, and examples a, b and c in the assessment section in LP1B to LP2. So, from LP1A to LP1B, the learning in terms of variation in examples was initiated by KO1 through the suggestions on varying the diagram orientation and complexity.

In contrast, from LP1B to LP2, changes made were initiated by teachers' observations from their teaching of lesson 1 and assessing the learners at the end of the lesson. In LP2, teachers removed some examples from LP1B. In the discussion during LP2, teachers paid attention to rushing of the teaching of the application examples. They agreed to drop examples f and j from LP1B because they were like examples e and g respectively, and added a new complex example shown in Table 1. The similarity was in terms of angle measures to be calculated and the difference was the orientation of the diagrams. The inclusion of complex example in LP1B resulted from their discussion during reflection 1 (see transcript below) on student responses to a complex diagram in the assessment task given at the end of the lesson.

T represent teacher, and KO represent knowledgeable other.

87. T1: The triangles that we gave them (in the assessment tasks) are different in complexity. Different from the examples we did during the lesson.

88. T4: Mmm, exactly.

89. KO: How different were they from the examples you did in the lesson?

90. T1: Like triangle number one, in the example (a), we didn't have that kind of diagram, there are three triangles in fact in the first diagram if we were to count them. There was no triangle that had a line inside (in the lesson examples).

91… KO: Within another triangle?

94… T1: So that was somehow a challenge because during the lesson, the exterior angle wasn't inside the triangle. But here we see angle b is interior to one triangle and is also exterior on the other triangle.

603. T4: So, in terms of what we need to do differently, we discussed that on the examples, there's need for an example where the exterior angle is also inside the … bigger triangle.

From line number 87 to 94, T1 refers to examples a and c under the assessment section in Table 1 to explain the gap between examples used for teaching and those used for assessing learners. He explained that while the examples used during the lesson were simple, they contained only one triangle and one exterior angle, the examples used for assessment were complex because there was an overlap of at least two triangles, making some of the exterior angles to be embedded in a bigger triangle. In line 603, T4 explains that they would address this gap by including a complex example where an exterior angle of a triangle is embedded in another triangle. They agreed to include complex example in LP2 to ensure that the learners engage with complex examples during the lesson, thus reducing the gap between the lesson examples and assessment examples. Thus, through the teaching of their lesson, the teachers learned what worked well through the example space and what did not and made changes to improve on what did not work well. We infer from their changing choices on example spaces that those dimensions of variation appeared to make sense to the teachers, and they used these to reflect on their own example spaces.

Of further interest to us in terms of the teachers' use of the MTF for working on their teaching, in the focus group interviews (data also not presented here) at the end of the cycle, the teachers talked about using the MTF in their planning to compare the examples from the five different prescribed textbooks to identify the textbook that contained varied examples, and constructed some of their own to produce an example set that contained all the variations that they were looking for.

DISCUSSION AND CONCLUSION

Our intention for introducing the teachers to MTF was that their discussions during lesson study, that is, lesson planning, teaching, and lesson reflection would be informed by MTF, resulting in enhancing their learning to improve the quality of teaching geometry. As the findings have shown, the teachers were able to take up the exemplification aspect of MTF in a substantial way, mirroring findings in Adler and Alshwaikh (2019). From the first lesson planning session, the teachers showed that they were developing a deeper understanding of how dimensions of variation in the selected examples could be infused into their own classroom practices to benefit student learning. As the teachers worked with the examples, they also gained new insights into the mathematical content. For example, teachers realised that a triangle could have up to six exterior angles and not only three exterior angles as indicated in some of the textbooks that they use. While developing their own examples to enhance different patterns of variations, the teachers noticed that exterior angles that are formed from a common vertex of a triangle are vertically opposite and so equal. We therefore add that the moments of constructing, critiquing and revising example sets using variation provide teachers with mathematical content learning opportunities as well.

In the lesson reflection transcripts, we also noticed that the teachers used variation of examples as an analytical tool for reflecting on the quality of their teaching. One clear indicator of this was their noticing and then addressing the gaps in the examples space spread across teaching and assessment. Therefore, the findings show that MTF was used to structure discussions during planning and reflection in terms of choice and use of examples (Adler & Alshwaikh, 2019). We regarded the teachers' attention to analysing and discussing what varied and what remained the same in the example spaces as knowledgeable choice of examples by the teacher.

In conclusion, in this study, we explored aspects of what teachers learned in their first encounter with LS type of PD by examining the possible dimensions of variations of geometry examples made available, changes in the example sets, and how these changes come into focus. The findings reveal that the teachers quickly picked up two dimensions of variation during the PD workshop and implemented them in their initial plan. The third dimension of variation came into focus through the knowledgeable other and through lesson reflection on learners' assessment examples. In conclusion, the paper contributes to the confirmation of prior work on exemplification in algebra and builds on it by locating it in geometry through introduction of LS form of PD in Malawi.

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EXPLORING CHANGE IN SECONDARY MATHEMATICS TEACHERS' NOTICING OF ARGUMENTATION

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This study examined changes in secondary mathematics teachers' noticing of argumentation through experiencing a peer assessment cycle. Sixty-one teachers participated in the cycle comprised of (a) analyzing a written argumentative classroom situation (ACS) by using a report format, (b) collaboratively assessing peers' ACS reports using an ACS rubric format, (c) providing feedback to peers, (d) receiving feedback from peers, (e) individually refining the initial ACS reports, and (f) reflecting on their experience. Analysis of teachers' initial and refined ACS reports revealed changes in teachers' noticing of various dimensions associated with argumentation. The study provides evidence of the potential of the peer assessment process for teachers' learning to notice key aspects of argumentation.

INTRODUCTION

The importance of students' engagement in argumentation in the mathematics classroom has been well recognized. It has been shown that participation in argumentation that requires students to explore, confront, and evaluate alternative positions, voice support or objections, and justify different ideas and hypotheses, promotes meaningful understanding and deep thinking (Weber et al., 2008). Research demonstrates that mathematics teachers have difficulties integrating argumentation into classroom practice (Ayalon & Hershkowitz, 2018) and that argumentation in the mathematics classroom is not yet a common practice (Umland & Sriraman, 2020). It appears crucial to investigate how best to devise effective professional learning for enhancing argumentation in the mathematics classroom. We addressed this issue by building on teacher noticing research to explore a particular type of noticing, which we call noticing of argumentation. Noticing is one of the central skills that determine teachers' proficiency, involving three interrelated skills: attending, interpreting, and responding (Jacobs et al., 2010). We assume that teachers who are better able to notice argumentation possess the skills necessary to begin to promote argumentation in the mathematics classroom. We also drew on evidence from research on the potential of using peer assessment techniques for effective learning (Topping, 2010). This study explored the potential of using peer assessment strategies to develop secondary school mathematics teachers' (SMTs) noticing of argumentation.

THEORETICAL PERSPECTIVE

The commonly accepted definition of argumentation is that of van Eemeren and Grootendorst (2004), who maintained that argumentation is "a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a

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standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint" (p. 1). Following Jacobs et al. (2010), and based on the educational literature on argumentation, we conceptualize the noticing of argumentation as a set of three interrelated skills: attending, interpreting, and deciding how to respond. The study adopted a theoretical perspective according to which productive argumentation (Asterhan & Schwarz, 2016, p. 167) is characterized by co-constructing arguments, critically and respectfully listening to others' ideas, identifying the weaknesses and strengths in each idea, and searching for alternative ideas while working toward consensus building. *Attending* relates to identifying salient characteristics, structural and dialogic, of the argumentation in a classroom situation (McNeill & Pimentel, 2010). The structural aspect focuses on the proposed claim and the justification for the claim, and in our context, in accordance with the accepted types of justification in the classroom community (Yackel & Cobb, 1996). The dialogic aspect relates to the above-mentioned productive argumentation characteristics of co-constructing of arguments, critiquing arguments, mutual respect, and working toward consensus building (Mueller et al., 2012). *Interpreting* relates to making sense of the argumentation in the classroom situation while considering factors that may enable or inhibit the argumentation. We consider four main factors associated with teaching that create opportunities for students to participate in argumentation, as expressed in the literature: (a) task characteristics, for example, implementing open-ended tasks that invite multiple representations and strategies (Mueller et al., 2014); (b) teaching strategies, such as encouraging students' participation and thoughtful questions (Mueller et al., 2014); (c) students' cognitive and affective characteristics, such as prior knowledge, common ways of thinking, and argumentation skills, as well as self-confidence, interest, and enjoyment (Knuth & Sutherland, 2004); and (d) socio-cultural characteristics, such as recognizing the value of argumentation and expectations of critique, collaboration, mutual respect, and socio-mathematical norms related to the kinds of justifications accepted in the classroom (Mueller et al., 2014; Yackel & Cobb, 1996). Finally, *deciding how to respond* relates to what one would do assuming that one was the teacher in that situation, to promote argumentation. Figure 1 summarizes our conceptualization of argumentation in the mathematics classroom and of noticing of argumentation. We used this framework in building the research tool and in analyzing the data, aiming at exploring changes in (SMTs) noticing of argumentation through experiencing a peer assessment cycle.

RESEARCH QUESTIONS

RQ1: What change occurs in SMTs' noticing of argumentation, if any, through experiencing peer assessment strategies?

RQ2: What factors promoted or inhibited the change in SMTs' noticing of argumentation, from the teachers' point of view?

RESEARCH CONTEXT AND PARTICIPANTS

The study was conducted in Israel at the beginning of a course focused on argumentation in mathematics teaching, as part of a master's degree in mathematics education. It is part of a larger research exploring the development of secondary in-service and pre-service mathematics teachers' skills of noticing of argumentation during their participation in a course focusing on analysis of argumentation classroom situations (ACSs), which serve as both research and pedagogical tools. An ACS is a written representation of an instructional situation that took place in the mathematics classroom, which provides teachers with opportunities to attend to structural and dialogical aspects of argumentation. ACSs also allow teachers to offer interpretations for the argumentation sequence in the situation, and to address factors that seem to enable or inhibit the argumentation. A group of 61 SMTs participated in a peer-assessment cycle comprised of (a) individually analyzing an ACS using a report format, (b) collaboratively assessing peers' ACS reports using an ACS rubric, (c) providing feedback to peers, (d) receiving feedback from peers, (e) individually refining the initial ACS reports, and (f) reflecting on their experience. The teachers in this group were not formally exposed to argumentation before the study.

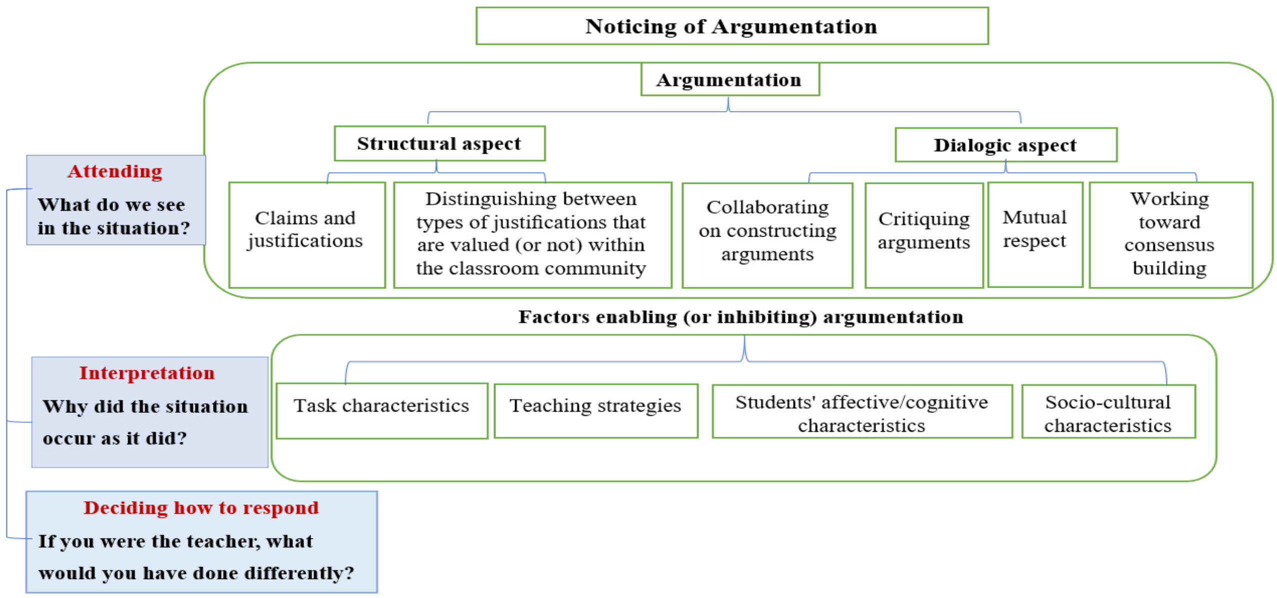

Figure 1: Theoretical perspective of argumentation in the mathematics classroom and the components of noticing of argumentation.

RESEARCH TOOLS

(a) an ACS focusing on the issue of "Abbreviated multiplication formulas" in a 9th grade class; (b) an ACS report format (adapted from Jacobs et al., 2010) that includes prompts related to the three skills of noticing of argumentation: *attending prompts* (a request to describe in detail those parts of the ACS that the SMTs deem important for argumentation, with reference to structural and dialogic aspects); *interpreting prompts* (a request to provide possible answers to the question "why did the ACS occur as it did?" by referring to possible factors that enabled or inhibited the sequence of argumentation); and *deciding how to respond* (a request to offer warranted alternatives to the teaching in the ACS, aiming to promote student participation in argumentation); (c) ACS rubric format, developed during a previous pilot course (Table 1). (The research tools and illustration of their use will be presented at the conference).

Noticing skill	Levels of noticing
Attending to structural aspects	1. Identified correctly some claims and justifications; identified types of justification partially or not at all; or incorrectly identified some of the types of justification. 2. Identified correctly all claims and justifications; identified correctly all types of justifications.
Attending to dialogic aspects*	1. Paid no attention to the dialogic aspect. 2. Paid attention to the dialogic aspect, lacking or general description of how the aspect is manifested in a given situation. 3. Paid attention to the dialogic aspect, detailed description of how the aspect is manifested in a given situation.
Interpreting**	1. Did not address the factor. 2. Addressed the factor, the response is mostly descriptive or evaluative, little or no use of evidence to support claims. 3. Addressed the factor, some evidence to support claims. 4. Addressed the factor, robust evidence to support claims.
Deciding how to respond	1. Offered no ideas for alternatives, or offered ideas for alternatives that were unconnected to the situation. 2. Offered ideas for alternatives that were relevant to the situation; provided some evidence to support claims. 3. Offered ideas for alternatives relevant to the situation; provided robust evidence to support claims.

* for each of the four dialogic aspects mentioned above

** for each of the five factors mentioned above

Table 1: Coding framework of SMTs' noticing of argumentation.

DATA COLLECTION

(a) SMTs' reports focusing on analysis of the "Abbreviated multiplication formulas" ACS. Each SMT submitted a report in Phase 1 of the peer-assessment cycle (initial ACS report) and a refined report in Phase 5, after giving and receiving feedback to and from peers (refined ACS report). The reports served as the main data source for characterizing the participants' skills of noticing of argumentation, and the change in skills following their participation in the peer-assessment cycle (RQ1); (b) written reflections, focusing on SMTs' experiences through the sequence of activities, their perceived strengths and difficulties, the similarities and differences between the initial and refined ACS reports, and what caused these. The reflections served as a source for identifying the factors affecting the change in SMTs' noticing of argumentation, from their perspective (RQ2); (c) individual, semi-structured interviews with 20 SMTs conducted to gain more insights related to the findings and the factors affecting the change in noticing of argumentation, from the SMTs' perspective.

DATA ANALYSIS

For RQ1: In Stage 1, we used the rubric to analyze the initial ASC reports, by applying the quality levels presented in the rubric format, focusing on what and how the components of the ACS report were noticed. In Stage 2, we applied the same process to analyze the refined ACS reports. Stage 3 focused on measuring the change in the participants' noticing of argumentation using the assessment obtained in the previous two stages. We used percentages to describe the distribution of responses in the initial and refined ACS reports. For statistical inference, we applied non-parametric methods because of the ordinal nature of the variables examined. We used McNemar's test to determine whether there was a change in SMTs' *attending* to structural aspect of argumentation because only two scores were used (1 and 2). We used the Wilcoxon signed-rank test to determine whether there was growth in the other components of SMTs' noticing of argumentation (RQ1). For RQ2: Stage 4 focused on exploring the factors affecting the changes in SMTs' noticing of argumentation, from the SMTs' perspective. We conducted interpretive and in-depth qualitative analysis of the written reflections and interview transcripts. Using inductive line-by-line coding, we sought descriptions of the factors that shaped the change in SMTs' noticing of argumentation.

FINDINGS

Change in SMTs' noticing of argumentation

To determine whether there was a change in SMTs' attending to structural aspects of argumentation, we used McNemar's test to compare the scores of the initial and refined ACS reports. The results indicated a statistically significant change ($p=0.001$): 18% of SMTs increased their score of *attending* to structural aspects of argumentation from level 1 to level 2. The Wilcoxon signed-ranks test, applied to the other components, indicated a statistically significant change in the three SMTs' skills of noticing of argumentation: attending to dialogic aspects, interpreting, and deciding how to respond, between the initial and the refined ACS reports (Table 2). We also found that about one third of the teachers attended to more dialogic aspects in the refined ACS report than in the initial one. Similarly, in their interpretation, many teachers addressed more factors in the refined report than in the initial one. We found variation between the levels of interpretation among the factors after the intervention: most teachers reached high levels (3&4) with respect to the teaching strategies, student cognitive characteristics, and task characteristics factors. By contrast, only about half the teachers reached high levels of interpretation when addressing the factors affective students' characteristics and socio-cultural characteristics. Regarding *deciding how to respond*, most teachers offered ideas for alternatives relevant to the situation; some provided robust evidence to support claims (Level 4), while others provided some evidence to support claims (Level 3). (Findings will be presented and illustrated at the conference).

Noticing skills	Different aspects of argumentation	Time	Mean	Z	Effect size (r)	Percentage of increase
Attending	Co-building of arguments	Initial	2.52	3.58^{***}	0.46	24.6%
		Refined	2.84			
	Critique arguments	Initial	2.39	3.64^{***}	0.47	26.2%
		Refined	2.75			
	Mutual respect	Initial	2.21	4.40^{***}	0.56	36.1%
		Refined	2.64			
	Working toward consensus building	Initial	2.11	4.51^{***}	0.58	41%
		Refined	2.72			
Interpreting	Task characteristics	Initial	2.07	4.53^{***}	0.58	42.6%

		Refined	2.84			
	Teaching strategies	Initial	3.30	3.94^{***}	0.50	29.5%
		Refined	3.67			
	Students' cognitive characteristics	Initial	2.70	4.45^{***}	0.57	39.3%
		Refined	3.28			
	Students' affective characteristics	Initial	1.57	4.77^{***}	0.61	47.5%
		Refined	2.54			
	Socio-cultural characteristics	Initial	1.51	5.02^{***}	0.64	52.5%
		Refined	2.46			
Deciding how to respond		Initial	2.20	5.2^{***}	0.67	44.3%
		Refined	2.64			

***p<0.001

Table 2: Percentage of increase, Wilcoxon signed-ranks test, and effect size for initial/refined ACS reports.

Thematic analysis of the transcripts of the SMTs' written reflections and interviews

The analysis process of the written reflections and interview transcripts resulted in a coding scheme with ten themes grouped into three main types. (1) Seven themes related to **factors associated with the peer assessment experience,** which according to the SMTs contributed to their noticing of argumentation: being exposed to a variety of peer reports, discussing their assessment with peers, and the assessments received contributed to (a) improvement in attending a wide variety of details and aspects of the situation; (b) developing flexibility in interpreting a given situation, different from one's initial interpretation; (c) developing skills of providing evidence of interpretations; (d) increasing awareness of the distinction between quality levels; (e) increasing the motivation to look for and analyze the expressions of the various aspects of argumentation in the given situation; (f) increasing knowledge of argumentation, for example, what counts as acceptable justification and teaching strategies for encouraging argumentation; finally, (g) as the group discussion of assessing peer reports was argumentative, it contributed to understanding the concept of argumentation. (2) Two themes related to **teacher factors,** which according to the SMTs, enabled, but also constrained their noticing of argumentation: (a) the SMTs' views on teaching and learning that promoted (or restricted) opportunities for addressing some aspects of argumentation. For example, a teacher reflected that her thinking that students' cognitive skills are vital in determining the argumentation process, whereas social and emotional factors are much less critical, restricted her interpretation process; and (b) the SMTs' self-confidence in analysis (for example, hesitation in discussing students' characteristics) promoted (or restricted) their interpretation of certain aspects. (3) One theme related to the **specific ACS characteristics**: the specific ACS enabled but also restricted certain opportunities for addressing some aspects of argumentation, for example, students' affective characteristics were not prominent in the given situation.

DISCUSSION

The results of the study provide evidence of growth in SMTs' noticing of argumentation following their participation in a peer assessment process. A significant change took place in the three skills of SMTs' noticing of argumentation: *attending to structural aspects and dialogic aspects* (co-constructing arguments, critiquing arguments, mutual respect, and working toward consensus building) *of argumentation*; *interpreting* the argumentation in the situation through various factors that may enable or inhibit the argumentation, including task characteristics, teaching strategies, cognitive and affective students' characteristics, and socio-cultural characteristics; and, *deciding how to respond*. These findings suggest the possibility of developing the teachers' skills of addressing at the same time multiple dimensions of argumentation in a given situation. This contrasts with a previous study showing that many teachers focused on one dimension of argumentation and had difficulty noticing multiple dimensions (Ayalon & Hershkowitz, 2018). We found that teachers have difficulty offering an interpretation for how the students' affective and the socio-cultural aspects may have shaped the argumentation in the situation. Such factors adhere to important notions of argumentation that promote learning (Asterhan & Schwarz, 2016) and therefore deserve attention. After the assessment process, most of the teachers provided alternatives relevant to the situation to encourage argumentation, but some teachers still had difficulty providing robust evidence to support their alternatives. The findings suggest that there is still a way to go in improving SMTs' skills of interpreting the argumentation by using various perspectives and offering possible responses. Considerations should be given to how to design research interventions that promote these skills.

Our research design does not allow making firm claims regarding the reasons for change in participants' noticing of argumentation, but analysis of the SMTs' reflections provides some indication of the factors that supported or constrained their noticing. According to the teachers' responses, three types of factors were involved. Prominent were factors relating to the SMTs' experience of the peer assessment process. Giving and receiving feedback using the rubric—considered critical in effective formative assessment (Swan & Burkhardt, 2012)—seemed to support their noticing. Through negotiation with peers about the rubrics and assessments they noticed various details related to argumentation in the classroom situation, which they had not considered before, and attended more reflexively to their practice in interpreting the situation. These findings resonate with those of research indicating that when learners analyze the work of others, they have access to a variety of examples that help them better see nuances in quality of the work (Topping, 2010). From the teachers' reflections we also learned that teacher factors, such as views on teaching and learning, and confidence in analyzing the situation, also shaped their noticing, in particular, their interpretation of the situation. A few SMTs pointed at the specific given ACS that helped them address several aspects of argumentation, and at the same time hindered the noticing of some other aspects, such as student's affective and socio-cultural characteristics.

This study's findings contribute to the literature on professional learning, specifically on developing teachers' noticing of argumentation, by providing evidence of the potential of the peer assessment strategy for teachers' learning and noticing of key aspects of argumentation practice. Exploration of the change that occurs in SMTs' noticing of argumentation, even for a short duration and with only one peer assessment cycle, enabled us to consider some of the likely advantages and challenges associated with using peer assessment as a learning tool in teacher preparation courses. One of the limitations of the study is that we do not know whether the change in noticing following the peer assessment process will remain. Further research is needed to explore the ways in which the effect of participation in professional learning of this type can be sustained, and whether it is realized in classrooms.

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LEVELS OF GENERALIZATION IN THE OBJECTIFICATION OF THE RECURSION STEP

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In this study, we focus on the students' objectification of the recursion step, intended as a process of generalization in the sense of Radford (2001). Building on Radford's model, we elaborate levels of generalization of the recursion step and use them to analyze the processes activated by secondary school students during collaborative activities with geometric recursive sequences. The analysis allows us to identify different levels reached by the students in grasping the recursion step and their transitions between these levels.

INTRODUCTION

This study is part of wider research on mathematical induction and its teaching/learning through significant and exploratory tasks (Antonini & Nannini, 2021; Telloni & Malara, 2021). It has been recognized that recursion and induction are strongly related and a deep understanding of recursion could support a meaningful learning of mathematical induction (Leron & Zazkis, 1986; Harel, 2001). Indeed, recursion, as a process which contains itself as a subprocess, has been intended as a more accessible and "executable version of induction", hence a good "stepping-stone" to teach and learn mathematical induction (Leron & Zazkis, 1986, p. 28). In tune with this idea, we focus on the students' objectification of the recursion step (RS), i.e. their understanding of the possibility to describe a sequence from a given basis in terms of a generic step which connects two consecutive elements of the sequence. In this study, we interpret the objectification of the RS as a form of generalization in the sense of Radford (2001): according to a process moving from particular to general, the RS in a geometric recursive sequence could be recognized as a placeholder for each specific step from a figure to the subsequent one.

The research goal we pursue is to investigate how the objectification of RS emerges in collaborative activities involving secondary school students.

THEORETICAL FRAMEWORK

Assuming a socio-cultural semiotic perspective, Radford (2001, 2003) distinguished different levels of generalization which novice students accomplish when they are involved in generalization of geometric patterns. These levels of generalization, called *factual*, *contextual*, and *symbolic*, are revealed from semiotic means of objectification, i.e. linguistic and non-linguistic signs conveying relations between particular and general, which students use in mathematical generalizing processes. The *factual level* consists in a generalization of iterated actions on concrete objects, possibly linked to

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numeric operations. The *contextual level* no longer operates on specific objects, but on *concrete non-material objects*, such as “the figure”. At this level students perceive the emergence of a general structure, as a mathematical object, genetically arising from the actions performed. The *symbolic level* arises when students produce a non context-bounded explanation and the mathematical meanings are elaborated in general terms. The levels of generalization introduced by Radford refer to the generalization of geometric patterns, aimed at describing the n^{th} figure of the sequence. In this study, we build on Radford’s model to elaborate levels of generalization concerning the objectification of the RS, focused on the link between figure n and figure n+1 in a sequence. According to our perspective, the objectification of the RS in a geometric recursive sequence can be interpreted as a generalization process: from viewing many different steps connecting a figure with the next one to grasping one generic step as a placeholder which represents all the steps. In tune with Radford (2010, p. 55), we expect that the generalization of the RS consists of: 1) “noticing commonality” in some given transition from a figure to the subsequent one; 2) “forming a general concept” of the transition from a figure to the subsequent one by generalizing the noticed commonality to all the transitions within the sequence.

In the following, we rewrite the levels of generalization of the RS and identify the corresponding semiotic means of objectification. At the factual level of generalization, the students, although focused on each individual step to pass from a figure to the next one, begin to identify common features of the different steps. These features emerge from the actions to be done on a specific figure to obtain the next one. This is highlighted by expressions such as “always” and “so on”, referring to something continuing in space and time, which can be iteratively described. Other semiotic means characterizing the factual level are ostensive signs and verbs indicating actions or perceptions. Moreover, the rhythm of the utterances and the ostensive movements can create a cadence revealing the factual level of generalization in a non-linguistic way. Within the contextual level, the different RSs are grasped in terms of a generic representative step, dynamically viewed as a transition process between any two consecutive figures. The semiotic means revealing the contextual generalization are locative and generic terms such as “the next/previous figure”, referring to non-material objects, although spatially situated. The language is hybrid, including abstract and situated elements. The grotesque pointing to concrete objects, typical of the factual level, becomes a refined pointing to non-specific objects, which testifies a new perception field. The symbolic level is revealed by a depersonalized description of the RS, independent from individual actions and perceptions, without reference to space and time. At this level the transition from a figure to the subsequent one is algebraically represented as a static connection between figures n and figure n+1.

We notice that the evolution from the contextual to the symbolic level of generalization of the RS is in line with the transition described by Harel (2001) from the *inference step view*, typical of “quasi-induction”, to the *inference form view*, typical of mathematical induction. The link between the above described levels of generalization also recalls the passage from the *local implications* "P(1)→P(2)", "P(2)→P(3)",..., to the *generic implication* "P(k)→P(k+1)" and finally to the *general implication* "P(k)→P(k+1) for all natural numbers k" discussed in Telloni & Malara (2021) to foster an aware and meaningful learning of the Principle of Mathematical Induction.

METHODOLOGY AND TASK DESIGN

This is a qualitative study, whose data were collected during an educational path carried out in distance learning. The path involved 24 voluntary students (grade 11^{th}-13^{th}, from 7 secondary schools in the centre of Italy) in a series of activities, preliminary to the introduction of mathematical induction. Participants, who agreed to take part in an experimental study and to be video-recorded, were divided in 6 groups of 3-5 students. Students of each group interacted through Meet platform under the supervision of a researcher; they were provided with a shared board and a shared document, where they were required to provide their solutions to some tasks. The researcher did not intervene during the activity, except to give the tasks and to answer specific questions by the students. Collected data consist of the recordings of the whole development of students' activities and not only the final products: the video-calls, the shared board and shared text document. The two researchers separately analyzed the video-recordings, with specific attention on speeches, inscriptions and gestures as semiotic means of objectification of the RS at specific levels of generalization. Then they discuss the outcomes of the individual analyses up to reach an agreement.

In this paper we focus on the first activity of the path, concerning geometric patterns. Specifically, the first five figures of a recursive sequence were given to the students (first task, Figure 1a), followed by two separated requests: (1) "Draw the figures 6 and 7 which continue the sequence", (2) "Describe the sequence of figures in a way that, a reader, following your words, could draw as many figures as she wants starting from fig.1". Later, the first five figures of another recursive sequence were presented to students (second task, Figure 1b), followed by the same requests (1) and (2).

A few comments on the task design are necessary. In the request (1) we asked students to draw the *two consecutive* figures which continue the sequence. We hypothesized that students could draw figure 6 and *then* figure 7 by using the previous drawing, perhaps with some 'copy-and-paste' strategies. In other terms, we thought that this request could support students in focusing on the relationships between two consecutive figures of the sequence, fostering an important *shift of attention* (Mason, 1989): from how to draw one figure of the sequence to how to modify one figure to obtain the following one. In this way an initial objectification of the RS could be attained. Request (2) introduces two new elements to the task: the need to describe the sequence up to a non-specific figure ("as many figures as she wants"), and the need to address the explanation to someone who cannot see the sequence itself. We hypothesized that these two elements, in tune with what happened in Radford's study

(2003), could create an effective joint labour (Radford, 2016) in the group, leading students to transitions from factual to contextual or symbolic generalizations.

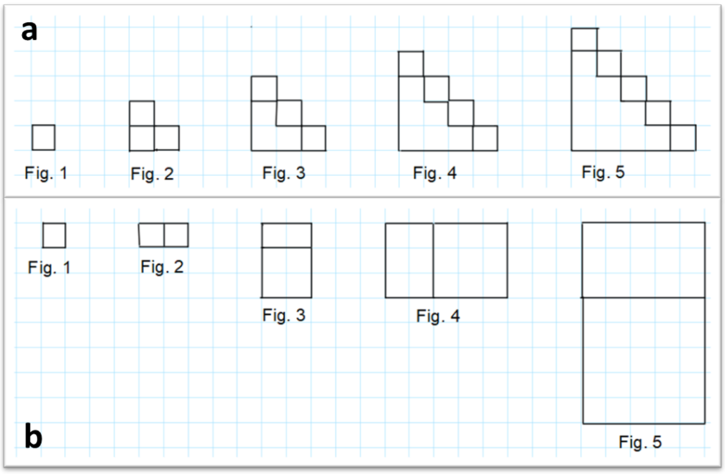

Figure 1: The two sequences of figures, presented as given to the students.

The two sequences were intentionally created with a difference. In the first one the connection between the shape of each figure and its number in the sequence is quite evident: the horizontal and vertical sides of the n-th figure are composed by n squares each. Thus, any figure could be potentially described only in terms of its numerical position in the sequence. In the second sequence, instead, this is not easily possible (it would require a closed form formula for the Fibonacci sequence). We thought that, for this reason, at least in this second case, students would have felt the necessity to describe any figure of the sequence after the first one in terms of the previous one.

CASE ANALYSES

Episode I - Factual generalization and the emergence of a contextual generalization

In this episode a group of 4 male students (G, V, O and P) of grade 12th is facing the second task (Figure 1.b), request (1). After a brief observation, G says:

1 G: It takes the previous figure as the little part, let's say…then it puts a square with area (*inaudible*). And it draws a square on the side.

The sentence is not grasped by the others and an exploratory phase follows, during which the students work on their own without coming to an agreement. After about a minute, while students are discussing and trying to describe the sequence (e.g., "It follows the factorial function, 2x1, 2. 3x2, 6…indeed figure 3 is 6"), G suggests and begins to sketch figure 5 on the board, then extends it to create figure 6 (Figure 2). Then V intervenes:

2 V: What are you trying to draw there?

3 G: The figure… the next one [Figure 6].

4 V: Thus, what do you do? You draw again figure 5… but add a side, right?

5 G: You add the square on the bigger side.

6 P: Ok, I understand.

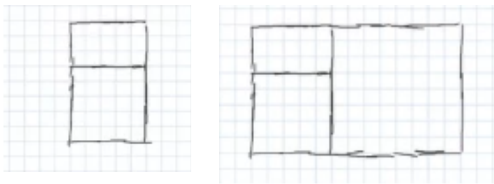

Figure 2: G's drawing of figure 5 and figure 6 on the shared board.

Some other interventions follow, aimed at further clarifying the construction of figure 6, when G is preparing to draw figure 7.

7 G: I think this can be done with the square… wait, I try to put a square over (G writes the label '7', then he inserts a rectangular shape over the drawing of figure 6 (see Figure 3.a)).

8 O: Yes, it's faster. (Meanwhile, G drags the rectangular shape below the label '7' (see Figure 3.b)).

9 P: Thus, what is the criterion?

10 O: You take the bigger side and construct a square over it. That's enough!

After this speech, P and V agree with O and G. Meanwhile, G inserts a square shape with the upper side corresponding to the bigger down side of figure 6, then he drags it under the rectangular shape to create figure 7 (Figure 3.c). Finally the group, answering to request (2), writes on the text document "You take the bigger side and construct a square over it, and so on for every next figure".

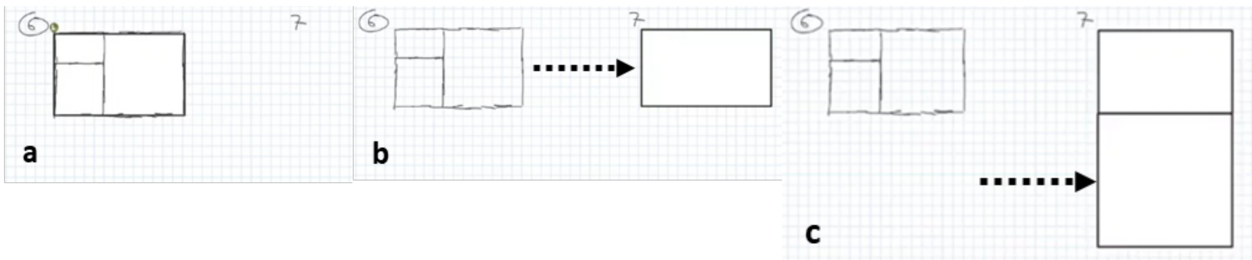

Figure 3: The drawing of figure 7 using figure 6 on the shared board. The arrows indicate the direction of the dragging of the shapes.

The episode shows an initial misalignment between the students in the group: G describes the sequence at a quasi-contextual level of generalization (line 1), highlighted by the expression "previous figure", but the other students do not understand and need to further explore the task. Later, G sketches the figures on the shared board, using figure 5 to draw figure 6 (Figure 2) and then figure 6 to draw figure 7 (Figure 3). Here a factual level of generalization arises, testified by an iterated cycle of actions (trace a figure, re-trace it, extend it to create the next one), where the focus is on the singular link between two consecutive figures (first on the link between figures 5 and 6, then on the link between figures 6 and 7). However, these actions, supported by the joint labour consisting of stimulus questions aimed at the generalization (lines 4 and 9), seem to make a new perceptual field emerge and induce a shared understanding. The transition from a figure to the subsequent one seems to progressively be grasped by students as a placeholder for each transition between consecutive figures in the sequence, towards a contextual level of generalization of the RS. This is testified by line 10 and the final solution written by the group, where the reference to previous and next figures is implicit, but a shift of attention can be highlighted. Indeed, the RS is generically expressed for all the figures in the sequence. Students display satisfaction that the brief description at line 10 is sufficient for a reader to draw the sequence, starting from the first figure ("That's enough!"). Indeed, they finally add "and so on for all the next figures" only because V says that the sentence in line 11 "is too skimpy".

Episode II – Toward a symbolic generalization.

A group of 4 students, three females (K, V and R) and a male (A), all of grade 13th, are facing the first task (Figure 1a). After V draws figure 6 and then, independently, figure 7 on the shared board through her graphic tablet, answering to request (1), the researcher provides request (2). The group tries to describe the sequence of figures in a non-recursive way, facing some difficulties (K: "it is easy to do, but difficult to say"). Then A tries to support the joint labour of the group, speaking to V:

1 A: Let us think about what you did before [when drawing figures 6 and 7].

2 V: I considered the previous figure and then I added those little square.

After one minute of group discussion on describing the diagonal of squares, V says:

3 V: If you think the next figure is the previous figure plus as many little squares as the number of the figure is.

4 A: I'm not sure.

5 V: For example, figure 3 is an L plus three little squares, figure 4 is figure 3 plus four little squares, figure 5 is figure 4 plus five little squares *(V rhythmically swings her head from left to right, according to the cadence of the phrase, (Figure 4))*… Do you understand?

Figure 4: V's head swinging at line 5.

6 A: Ah, ok…yes, ok.

7 R: Thus, you say that figure n will be figure n-1 plus n little squares.

After this excerpt, the group refines the final sentence and writes "figure n-1 is the contour of figure n with n little squares added" as shared solution in the text document. The episode shows the joint labour of the group, which is the key element through which all the students reach an agreement about the generalization of the RS. Initially students see the sequence in different ways, including non-recursive ones. Then V, stimulated by A's metacognitive intervention aimed at reconstructing the actions she performed (line 1), describes recursively the sequence, making explicit the connection between the drawings of figure 6 and figure 7 (line 2), albeit before she apparently drew independently the two figures. The description is at a contextual level of generalization (line 3), revealed by the use of generic and locative terms ("the figure", "the next figure") and references to space and time ("previous, next"). The RS is expressed for any transition between a figure and the next one. The need of sharing knowledge and the doubts of A (line 4) induce V to use the factual level of generalization (line 5), highlighted by the sentences with the same structure referring to iterative actions and by the body language. Although the group passes to a lower level of generalization, they now regard the sequence with a new perspective, oriented by the goal of "thinking about what [they] did before" (line 1). Then they reach an agreement, going beyond the contextual level. In line 7, as well as in the final solution, the RS is generalized at a symbolic level: the description of the sequence is impersonal, with reference to the generic number of the figures. Moreover, there are no references to space and time nor to individual actions or perceptions of the sequence.

CONCLUDING REMARKS

In this paper we interpreted the objectification of the RS as a process of generalization and, building on the Radford's model (2001), we elaborated three levels of generalization. This theoretical lens has been used to design tasks involving geometric recursive sequences and to analyze the processes activated in secondary school students' collaborative activities with these tasks. The analysis allowed us to identify different levels reached by students in grasping the RS and their transition between these levels. Typically, during the collaborative activities, students passed from lower levels of generalization of the RS to higher ones. In particular, some groups reached a symbolic level of generalization of the RS. A key role in these transitions has been played by some elements of the educational path: the chosen sequences, "easy to do, but difficult to say", together with the interaction at distance, preventing some forms of communication, made the tasks more challenging and induced students to make their thoughts explicit. The students' meta-reflection on the actions performed has been also crucial in supporting the development of an effective joint labour (Radford, 2016) within the groups and favouring a shift of attention (Mason, 1989) in viewing the RS. Our analysis also highlights some transitions from higher levels of generalization of the RS to lower ones. In both the episodes, there is an initial misalignment between the students about the way the sequence is viewed. The students who reach a contextual or quasi-contextual level of generalization feel the need to use a factual generalization to foster their colleagues' understanding. In episode I, the description of the RS at the factual level of generalization is made by means of the drawings, done through "copy-and-paste" techniques; instead, in episode II, the factual level emerges through linguistic means of objectification and the body language. This suggests that the joint labour could support an evolution of the students' objectification of the RS towards higher levels of generalization. However, students seem to rely on lower levels of generalization, namely the factual level, for communicative needs, and to reflect on their own actions.

In the future, we plan to extend the study on large scale to investigate these aspects. Moreover, further research is needed to deepen how the objectification of the RS could be exploited to foster the objectification of the induction step towards a meaningful learning of mathematical induction.

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THE EFFECT OF SECONDARY MATHEMATICS ON FUTURE CHOICE IN STEM PROFESSIONS

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Reducing leakage from STEM to non-STEM professions is important, mainly due to the great demand for quality manpower in STEM fields. This study aims to characterize learners who have the potential to drop out of STEM fields, as well as examining various pathways in which dropout occurs. Using big-data analysis based on 534,590 records retrieved from the CBS in Israel for several points in time over one and a half decades, we identified eight pathways to choosing a profession from secondary school to graduating a bachelor's degree, and characterized learners in each pathway based on educational characteristics. Findings reveal three dominant pathways of which one reflects a leakage from STEM in secondary school to non-STEM in higher education. Further, advanced secondary math is the best indicator of completing a STEM degree.

INTRODUCTION AND THEORETICAL BACKGROUND

Despite the growing demand for experts in STEM fields, many studies show a decline in the choice of STEM professionals throughout lifespan. This trend is referred to as the Leaky Pipeline Metaphor (LPM: NRC, 1986). The LPM describes the phenomenon of practitioners dropping out of various STEM fields throughout lifespan, starting from secondary school when the number of potential practitioners is relatively high, continuing into and graduating from STEM studies in higher education, and eventually working in STEM fields (Witteveen & Attewell, 2020) when the number of practitioners in STEM professions is alarmingly low (OECD, 2019).

Dropouts during higher education studies are not limited to STEM subjects. Yet, the LPM only accounts for dropouts from STEM fields, rather than the movement from non-STEM into STEM fields (Witteveen & Attewell, 2020). In addition, the leaking is not linear, so that students who drop out of STEM studies at some point in lifespan, may return later on to the field (Lykkegaard & Ulriksen, 2019). Therefore, along with efforts to hold the leak among STEM students, it would be worthwhile to make efforts to increase the number of streams from non-STEM fields to STEM fields.

Secondary mathematics have been found to provide a good foundation for later STEM studies, and entry into higher education (Kohen & Nitzan, 2021a; Sadler, Sonnert, Hazari, & Tai, 2014). The PCAST report (2012) indicates that a lack of substantial math skills often prevents students from choosing STEM fields for study, as mathematics is regarded as a fundamental subject for all other sciences (Li, 2013). Also, choosing STEM major in secondary school serves to develop and promote students' aspirations for future studies and careers in STEM fields (Holmes, Gore, Smith, & Lloyd, 2018).

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The present study explores diverse pathways for choosing a profession, by examination the choice of STEM and non-STEM fields as a profession during significant periods in lifespan, starting from secondary school, first year of higher education, and graduating a bachelor's degree. The data for this study that was retrieved from the Central Bureau of Statistics (CBS) in Israel enabled mapping the characteristics of learners in each pathway according to educational variables that had been found to influence profession choices, in particular mathematics and science studies and achievements in these subjects in secondary school.

Theoretical framework – The integrative four-phases model for career choice

The integrative model suggested by Reinhold, Holzberger, and Seidel (2018) is a four-phases model that expresses the operational of goals for choosing a future career according to significant periods in life. The first phase, the wishing phase, allows interests in a diverse career area with little commitment, characterizing students in elementary and junior high school, who are not required to choose any field of specialization in studies. The second phase, the planning phase, represents a growing commitment to a particular career field, based on ability and performance, characterizing students in secondary school who are required to select a major subject for study. The third phase, the action phase represents the actions taken to realize the chosen career, characterizing students in higher education studying for an academic certificate. Finally, the fourth phase, the pursuing phase, represents the attainment and persevering in the chosen career, characterizing the employed in the labour market.

The nature of the data at our disposal, indicating an actual objective choice, did not allow an examination of the wishing phase. In Israel, STEM education is mandatory before secondary school and is studied as a general subject. Therefore, our focus is on the last three phases that reflect actual choices towards a career, namely planning, action, and pursuing.

The phases in this study are defined as follows: The *planning* phase refers to choosing STEM as a major in secondary school, which was found to play a critical role in the likelihood of students to reject, persist or enter STEM fields (Engberg & Wolniak, 2010). The *action* phase is defined by choosing STEM as a major in first year in higher education, which is a critical crossroad towards choosing a career profession (Witteveen & Attewell, 2019). The *pursuing* phase is defined by obtaining a bachelor's degree in a STEM field, which indicates attainment and persevering in the field as a future occupation, as STEM graduates are more likely to work in STEM professions (Kohen & Nitzan, 2021b).

Secondary school mathematics and future STEM choice for study and career

Succeeding in mathematics during secondary school is considered an important factor in developing and promoting student's confidence in their ability to pursue a STEM career and is a good predictor of future STEM academic success and career (Kohen & Nitzan, 2021a; Holmes et al., 2018). A longitudinal study revealed that those who excelled in mathematics in secondary school were twice as likely to be employed in STEM professions than those with low mathematics achievements (Anlezark, Lim, Semo, & Nguyen, 2008). Also, secondary school STEM studies lay the foundation for further STEM studies in higher education (Lichtenberger & George-Jackson, 2013), and an interest in majoring in STEM in secondary school is affected by future employment orientations. Thus, students who wish to pursue a STEM career may start to take an interest in these fields in secondary school, so they can better prepare themselves for these studies in higher education.

AIM AND RESEARCH QUESTIONS

The aim of the present study is to characterize different pathways according to important stages in the lifespan and to examine educational data that identifies the learners in the various pathways. Accordingly, the research questions are:

1. What are the possible pathways to STEM and non-STEM bachelor's degrees, starting from secondary school through higher-education and graduation?
2. What are the characteristics of each pathway and how do they differ based on various educational variables?
3. Over different stages of life, what are the characteristics that best predict STEM choice?

METHODOLOGY

Secondary mathematics and STEM major in Israel. At 10th grade, Israeli students are required to choose a major subject, usually at an advanced level. There is also a division into three levels of mathematics, each with different levels of depth and topics covered. The basic level, that is the minimum required for obtaining a matriculation certificate, requires skills that are mainly applied techniques. The standard level provides a solid foundation of skills and knowledge of mathematics. The advanced level is the highest level, when emphasis is on developing mathematical-scientific thinking, designed to direct students towards STEM studies.

Participants

A base population of 534,590 Israeli secondary school students were sampled for this study. Data was obtained from the CBS in Israel, using systematic sampling that contains all secondary school population who graduated secondary school over one and a half decade, in the years 2001, 2006, 2011, 2015, and 2017.

Observed Variables

The CBS data allows to track student educational choices and achievements from secondary school to graduates and employees. The codebook that guided the analysis was comprised of educational data, including level of secondary mathematics, type of science major, and the level of success in mathematics and science major in secondary school. Based on the matriculation exam in math and science, this study defined success as a dichotomous variable, representing excellence when the score ranges between 91 to 100, or not excellent if it falls below 90.

The definition of the STEM values, which have been validated by experts, are as follows: *STEM subjects in secondary school* include the areas of physics, chemistry, biology, and computer science; *STEM subjects in higher education* refers to the following subjects: mathematics, statistics and computer sciences; physical sciences; biological sciences; agriculture; medicine; or engineering and architecture. The remaining subjects were defined as non-STEM values.

RESULTS

Pathways to STEM or non-STEM degree

In order to identify pathways for STEM and non-STEM bachelor's degrees, we created a three-tiered tree. Based on frequencies analysis, the first tier represents the major subject that was studied in secondary school, namely STEM or non-STEM. The second-tier divides each of the first two branches according to the choice made in first year of higher education. This tier was obtained using a Chi-Square test which combined descriptive distribution of STEM or non-STEM subjects studied in secondary school and first-year in higher education. The third tier builds on the previous two, and represents eight pathways, from secondary school to obtaining a bachelor's degree. For that we ran a split-file based on the type of major studied in secondary school, followed by a Chi-Square test based on the choice made in first year of higher education and the completion of a bachelor's degree. Each pathway was given a number from one to eight, so that a higher number indicates persistence in choosing STEM over the years. The internal ranking between these pathways was performed according to the degree of persistence and selection sequence in STEM throughout the three stages of life examined. For example, learners in pathway #8 were ranked at the highest grade, since they persevered in the STEM professions in all three stages examined and completed a STEM degree. Figure 1 presents visually the three-tiered tree of pathways toward STEM or non-STEM degrees.

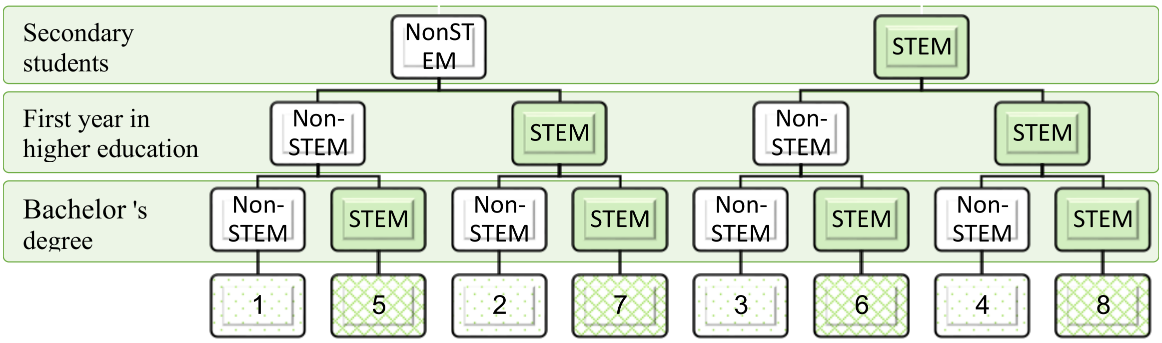

Figure 1: The three-tier tree, representing eight pathways toward STEM or non-STEM profession.

There is a similar distribution of STEM or non-STEM choices in secondary school, with STEM fields being favoured (57.2%). In higher education, almost 43% of those who studied STEM in secondary school choose a STEM field in their first year, as opposed to less than 13% of those who studied non-STEM. The distribution of graduates revealed an ongoing impact of the choices made at school. About 85% of STEM graduates, studied STEM major in secondary school compared to only 15% who studied non-STEM major. When focusing on absolute numbers, the three-tiered tree indicates three dominant pathways. Pathway #1 reflects about 35% (*N*=83,984) of total students, who persist in choosing a non-STEM field from secondary school throughout bachelor's degree graduation. Pathway #8 reflects the 22% persistent students (*N*=53,132) who followed a STEM path from secondary school to bachelor's degree graduation. Finally, there is pathway #3 which reflects about 33% of students (*N*=77,088), who studied STEM in secondary school, but chose a non-STEM track in higher education. As these three pathways reflect most of the students in the study sample who completed a bachelor's degree, at this point we base our findings on these three dominant pathways.

The characteristics of the dominant pathways.

For mapping the characteristics for each of the three dominant pathways, Chi-Square tests were conducted. Additionally, a Kruskal-Wallis analysis of variance was performed for analysing the statistical differences between the different pathways, in relation to each of the educational characteristics.

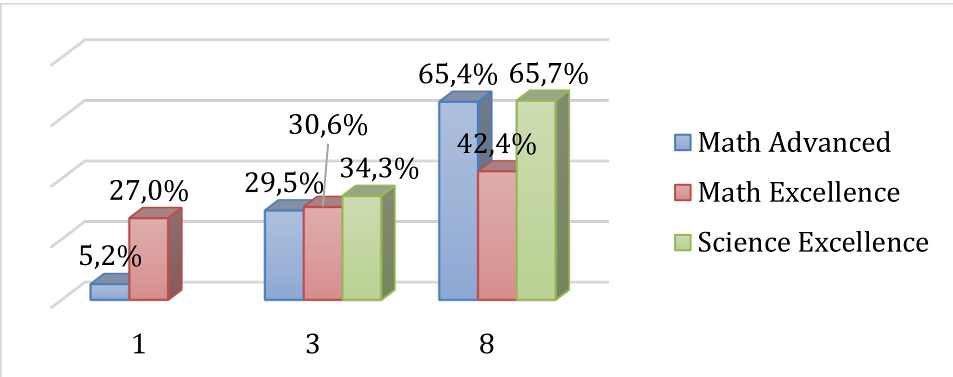

Figure 2: The distribution of the characteristics of the dominant pathways.

Kruskal-Wallis test revealed significant differences between the three dominant pathways in most characteristics. Most students who persist in the STEM path (#8) studied advanced mathematics and excelled in mathematics in secondary school compared to students in the non-STEM path (#1). In path #3, no significant difference was found in the distribution of these variables. It appears that there is another salient feature in path #3 which impacts the transition from STEM studies in secondary school to non-STEM studies in higher education.

Distribution by major field in secondary school. Figure 3 presents the distribution of the types of secondary STEM major for pathways #3 & #8, which are the ones that started in STEM studies in secondary school. Results revealed a statistically significant difference between these pathways in the distribution by all STEM majors. Those who studied physics and computer science, are more likely to follow path #8, whereas those who studied biology and chemistry are more likely to follow path #3.

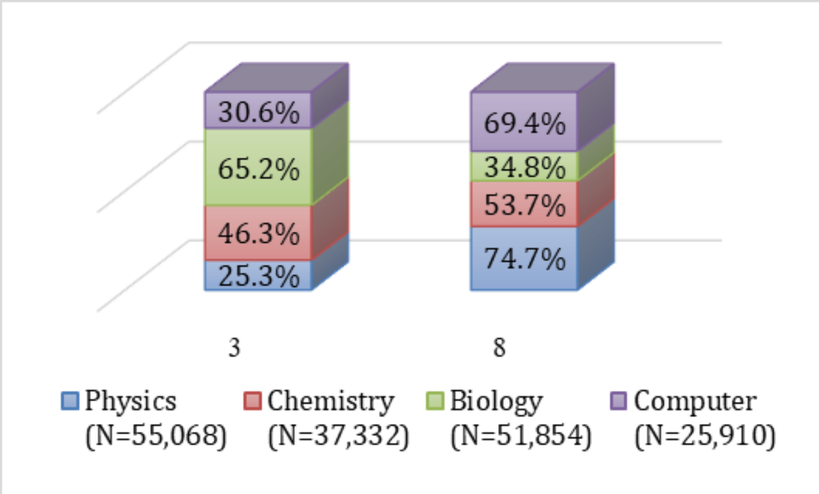

Figure 3: The distribution by STEM major type in secondary school.

Predicting the completion of a STEM bachelor's degree

A logistic regression revealed that all the investigated educational characteristics predict the likelihood of pursuing and completing a STEM undergraduate degree. The most influencing variable is the math level, meaning that students who study advanced mathematics are more likely to graduate STEM as a major in bachelor's degree, compared to students who study non-advanced mathematics in secondary school. Science major success level was found to be the second in predicting completion of a STEM undergraduate degree. As for the types of STEM majors, physics was found to be the best predictor of completing a STEM undergraduate degree, while biology was the least likely (see Table 1).

Characteristic	Predictor	B	Wald χ^2	Odds Ratio
Math level	Advanced compared non-Advanced	.56***	789.44	1.75
Math success	Excellence compared to non- Excellence	.11***	34.20	1.12
Science major success	Excellence compared to non- Excellence	.30***	183.62	1.36
Science major type	Physics	.21***	1867.21	1.24
	Chemistry	.10***	543.45	1.11
	Biology	.06***	153.40	1.06
	Computer science	.15***	965.95	1.16

Table 1: Regression findings for predicting the completion of a bachelor's degree in STEM.

DISCUSSION

Through a big data analysis, this study presents a three-tier tree which recognizes various pathways that lead to a STEM or non-STEM bachelor's degree, of which three were found to be the most dominant, reflecting the largest number of leaners who completed a bachelor's degree. The most significant finding is the recognition of path #3, that is a learner who started STEM in secondary school and moved to non-STEM in higher education. First, and contrary to the assumption underlying Reinhold et al. (2018) integrative model, the transition from the wishing phase to the planning phase is not the critical stage to choosing a specific career, as this study suggests that the critical transition occurs between the planning phase and the action phase. That is, the biggest leak of STEM learners to non-STEM fields occurs in the transition between secondary school to higher education. Further, this study shed some light on the characteristics of learners who persevere in STEM studies and those who drop out of STEM studies immediately after secondary school. We can point to a combination of characteristics in accordance with path #3 that do not encourage continued selection in STEM fields in higher education, for example students who did not study advanced mathematics and did not excel in mathematics and in the science major in secondary school, as well as students who studied biology or chemistry in secondary school. Therefore, and in accordance with the regression analysis, in order to encourage STEM choice in higher education and the completion of a STEM bachelor's degree, we might be focusing on increasing the percentage of students studying physics and computer sciences in secondary school, as well as those who study advanced mathematics. Finally, the identification of diverse pathways and characterization of learners in each path, develops new avenues through which the choice of STEM subjects can be preserved.

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WHAT HAPPENS OVER TIME TO STUDENTS IDENTIFIED AS BEING AT RISK OF FALLING BEHIND IN NUMERACY?

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This study followed 191 students in 10 schools over three years, investigating what happens over time with students identified as at risk of falling behind in numeracy. The study uses data from a formative, national numeracy mapping test. The results show that students' test scores often varies from one year to the next. Only 10 students were consistently identified as being at risk. A score pattern analysis revealed that they showed little progress from grade 1 to grade 2 in number concept and counting skills. This improved in grade 3, but they remained behind their peers in conceptual understanding and calculation skills. From a response to intervention perspective, these students would likely have benefitted from teaching interventions in grade 1 to familiarise them with the number line.

INTRODUCTION

Previous research has shown that many students who struggle with mathematics begin to fall behind in the lower grades (Scherer et al., 2016). Risk factors include weak conceptual understanding and rigid counting strategies. Seethaler and Fuchs (2010) stated that assessment activities should have a response-to-intervention focus, for instance when screening students in early grades to detect risk of developing mathematical learning disabilities. Following a 2006 educational policy on early intervention, the Norwegian Directorate for Education and Training developed national formative mapping tests in numeracy for primary grades 1, 2, and 3 (Nortvedt, 2018). The second generation of mapping tests and support materials was released in 2014 and used until 2021; the same assessments were used for several years to ensure that teachers and schools were familiar with the test content and supporting material.As the assessment is formative—to identify students at risk of falling behind and to plan teaching interventions for them—the assessment data is not reported to the national government but rather retained by local schools. Nevertheless, analysis of sample data collected nation-wide between 2014 and 2017 revealed that comparable numbers of students were identified as being at risk each year and that no test inflation was seen, even though the content was familiar to the schools (Nortvedt, 2018; Nortvedt et al., 2020). This may indicate that the tests are robust, but it may also mean that schools failed to use the test outcomes to plan interventions for identified students.

This paper reports from a study that followed 191 stude pnts from 10 schools from 2018 to 2020 to investigate what happens over time to students taking the mapping test. As the final data collection was in the 2020 school year, the outcomes will also be discussed in light of the potential influence of the COVID-19 pandemic on

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mathematics teaching in early grades. The research questions for the current study are

- What happens over time to students identified as at risk in grades 1 and 2?
- What developments in conceptual understanding and calculation skills can be seen amongst students consistently identified as being at risk?

PRIOR RESEARCH

The five basic components of counting (one-to-one correspondence, stable order, order irrelevance, cardinal principle, and abstraction principle; Gelman & Galistel, 1978) are the foundation of counting and provide students with experience that helps them to develop a mental number line (Dehaene, 2001). A well-developed mental number line allows students to count up and down, skip-count, and perform arithmetic operations. According to Aunola et al. (2004), students' counting skills are a good indicator of later achievement in mathematics; they might also be seen as the starting point of number sense, which comprises knowledge and understanding of numbers and quantity as well as mathematical concepts and symbols (Jordan et al., 2007). Students move on from the core knowledge of number to understanding relationships between numbers and to operating with them. Good number sense is the foundation for later mathematical achievement (Seethaler & Fuchs, 2010). Students at risk of falling behind in numeracy often struggle more when learning to count and demonstrate weaker conceptual understanding than typically developing students (Scherer et al., 2016). Their progression from counting strategies to addition and subtraction strategies for performing calculation problems is also slower, and many at-risk students keep using simple 'count-all' strategies, even in higher grades. Moreover, they often use rigid counting strategies when counting and make more errors than other students make.

Children come to school with different prior knowledge; while some already possess competences typically taught at school, others are still learning to count. The aim of schooling is to provide all students with the best possible opportunities to learn mathematics (Scherer et al., 2016), and as such, identifying students who are at risk of falling behind and mapping what they can do is an important starting point for classroom intervention that can improve these students' learning. In a meta-analysis of 34 studies on interventions in kindergarten and early primary school, Nelson and McMaster (2019) found larger treatment effects for interventions that included counting with one-to-one correspondence and that were eight weeks or shorter. Moreover, interventions were more effective for students at the highest risk of falling behind, but low socioeconomic status within a family could reduce the efficacy of the interventions. Aunola et al. (2004) found that children who enter kindergarten with low performance in basic number skills continue to perform below their peers in later school years; implementing evidence-based interventions in schools and documenting the responses to them are therefore vital (Seethaler & Fuchs, 2010), keeping in mind that learning is a social activity (Scherer et al., 2016).

In Norway, all assessment at the primary school level is formative. Education is inclusive, and the Education Act states that teaching should be adapted so that each student can reach their potential (Lovdata, 2006). Previous research has shown that Norwegian teachers tend to 'wait-and-see' when students experience difficulties (Haug, 2014). Special education is mainly delivered in lower secondary education, suggesting that interventions in earlier grades are not successful. In follow-up research on the mapping tests (Nortvedt, 2018; Nortvedt et al., 2021), teacher interviews revealed a range of different teaching interventions targeting individuals or groups of students and illuminated the challenges of interpreting assessment data, meaning that the interventions might not be sufficiently adapted to individual students and thus not effective in improving their learning. This interpretation is supported by the lack of inflation in the test results. In March 2020, due to the pandemic, Norwegian schoolchildren were sent home and teaching was moved from classroom to screen. However, Blikstad-Balas et al. (2022) found that, despite adequate access to infrastructure, the engagement of low-achieving students during remote instruction diminished more than that of their peers. Furthermore, strategies to equalise learning opportunities during remote learning were not implemented.

METHODS

This study followed 191 students from 10 schools for three years, collecting data on task levels from the grade 1–3 mapping tests. The assessment data was used to analyse what happened over time to the participating students.

Sample

The 10 schools were sampled in Oslo and surrounding cities. Invitation letters to students and parents were distributed by the schools, and parental consent was received from 435 homes in 2018. Parental consent and participation rates differed between the schools, and an analysis of the test results (and comparing to the 2014 – 2017 data sets) suggested that parental consent was likely not provided for the lowest performing students in most of the 10 schools. Due to the COVID-19 pandemic, many schools did not manage to administer the mapping tests in spring 2020. A number of students also changed schools between 2018 and 2020, so the 2020 sample is smaller than the 2018 sample, resulting in only 191 students with data from all three test administrations.

The mapping tests

The tests are paper-based and administered by class teachers. Each test has two parts and includes a break. The test booklet is organised around groups of tasks with the same format and content, printed on the same page. For each test page, the teacher gives an oral instruction that explains the visual example (see Figure 1) printed on the top of the page showing students what the following 2 – 8 tasks ask of them.

Table 1 displays the number of tasks and cut-off scores for the three grade-level tests. The cut-off scores were set in 2014 to identify the lowest scoring 20% of the student population at a national level. To maximize information about these students, the tests were designed to have a ceiling effect, meaning that even average ability students typically solved nearly all the test tasks correctly, while students close to the cut-off score typically solved 75% of the tasks correctly.

Figure 1: Instructional example and test task for 'neighbouring numbers'.

According to national guidelines, the schools annually had a window of approximately four weeks to conduct the mapping test, starting from mid-March and adjusting for Easter. In 2020, due to the pandemic, schools closed one week into the test window, which the Norwegian Directorate for Education and Training extended several times to accommodate schools, with the latest deadline being June 7—two weeks before the summer holiday. Schools only started opening again in May under strict regulations, making it challenging to administer the mapping test. Some schools, for example, only had outdoor teaching and learning activities due to local regulations.

Analytical approaches

Data on task level for all students for each year was entered into a database. Score patterns were investigated using mean scores. Because of the ceiling effect, analysis of above-average students found very little variation and should be interpreted with care. In addition, more qualitatively oriented analysis was applied to identify test tasks that most of the at-risk group could solve confidently (8 out of 10 students had a correct response) or to some extent (6 out of 10). This resulted in a description of what students close to the cut-off score could typically do while keeping in mind that, on average, a student near the cut-off score solved 75% of the tasks correctly. The qualitative data analysis sought to identify patterns in the students' responses to test tasks in order to investigate their progress in conceptual understanding, counting, and calculation skills.

RESULTS AND DISCUSSION

Table 1 displays the test results from students identified by the mapping test as being at risk in each of the three years and shows that, in grade 1, 10% of the participating students were identified as being at risk; these students scored an average 35 points out of 50 (SD = 3.57), which was not far below the cut-off score.

Test grade	Max score	Cut-off Score	Students identified as at risk N (%)	Score range	Mean score	SD
1	50	39	19 (10)	26–39	35	3.57
2	55	41	26 (14)	12–41	34	6.63
3	72	59	47 (25)	16–59	50	9.73

Table 1: Max score, cut-off score, number of identified students, score range, and mean scores for students identified by the mapping test as being at risk in grades 1–3.

In grade 2, nearly 14% of the students were identified as being at risk. Based on the average score and the variation in scores, the students identified in grade 2 were, on average, more diverse and more 'struggling' than those in grade 1 because the average score is further from the cut-off score and the standard deviation is larger. These results support the hypothesis that teaching interventions were not sufficiently targeted to individual students.

In 2020, the COVID-19 pandemic sent Norwegian schoolchildren home and into digital learning. Table 1 shows that, although the students had 'normal' teaching until the beginning of the test window, nearly twice as many students were identified as being at risk at the time they took the test.This might be expected if the schools usually took the mapping test close to the end of the test window and used the weeks before it to practice, but due to the lack of inflation in test scores observed in previous years (Nortvedt, 2018), this seems unlikely. A more likely interpretation is that the test results are a consequence of two months of home-schooling combined with subsequent restrictions on teaching activities, which affected the lowest achieving students more than their peers, as suggested by Blikstad-Balas et al. (2022). The results might also indicate that, for many students, continuity and close follow-up by a class teacher are necessary not only for further learning, but also to maintain what the students have previously learned.

Score patterns over time

Table 2 shows the score patterns over time for the 191 students; 70.2% of the students had test results above the cut-off score in all three years (a–a–a), while a small proportion consistently scored below the cut-off score (b–b–b).

Students scoring below the cut-off in grade 1		Students scoring above the cut-off in grade 1	
Results grades 1–2–3	N (%)	Results grades 1–2–3	N (%)
b–b–b	10 (5.2)	a–b–b	12 (6.3)
b–b–a	1 (0.5)	a–b–a	3 (1.6)
b–a–b	6 (3.1)	a–a–b	23 (12.0)
b–a–a	2 (1.0)	a–a–a	134 (70.2)

Table 2: Score patterns over time for students above (a) and below (b) the mapping test cut-off scores in grades 1, 2, and 3.

Comparing grades 1 and 2, it is clear that, while nearly one third of the students identified as at risk in grade 1 experienced positive development, moving from below the cut-off in grade 1 to above it in grade 2, there was also a group of students—15 in total (or 7.9% of the total sample)—who scored above the cut-off in grade 1 but were identified as being at risk in grade 2. This shows that, within the sample, more students fell behind moving from grade 1 to grade 2 than experienced improved performance. A comparison of grades 2 and 3 shows that fewer of the participating at-risk students improved their attainment than the large number of students now identified as being at risk in grade 3. This might be due to the effects of home schooling, as suggested by Blikstad-Balas et al. (2022). Variations in test scores are more likely for students close to the cut-off score than amongst those who solve nearly all the test tasks correctly. The students identified as being at risk or close to the cut-off score struggle more to use their conceptual understanding, and counting and calculation skills to solve the test tasks than their peers. At-risk students most likely have a less developed mental number line and weaker number sense. As counting skills and number sense predict later success (Aunola et al., 2004; Seethaler & Fuchs, 2010), such outcomes might be expected.

Table 2 shows that a small group of students (5.2%) were consistently identified as being at risk. In grade 1, this group scored an average of 33 points out of 50, similar to other students identified as at risk in grade 1. However, in grade 2, they averaged 29 points out of 55, and in grade 3, 44 out of 72. The weaker average in grade 2 than in grade 1 suggests that interventions based on grade 1 test results did not accelerate learning and it is likely that the interventions did not build on principles for formative assessment and building on what the students had demonstrated that they could.

Identifying what students consistently at risk know and can do

Teaching is more effective when it builds upon and extends prior knowledge, therefore identifying what an at-risk student knows and can do is important. Based on the grade 1 test tasks that at least eight of the 10 consistently at-risk students solved correctly, they had mastered counting with a one-to-one correspondence when the number of objects was small and showed understanding of concepts such as ‘like mange’ (equal amount), ‘til sammen’ (all together) and ‘mest’ (largest group) . In addition, they demonstrated knowledge of the number line up to 20, although they were more secure in the 1–10 range. They also showed that they could solve simple addition and subtraction problems (e.g., 6 + 2 or 8 – 3), both contextually and for given problems. Six of the 10 students could use a criterion to count (e.g., count only the squares), showing some understanding of what Gelman and Galistel (1987) termed the abstraction principle.

In grade 2, the same students showed that they could confidently do similar tasks and displayed knowledge of concepts like ‘nærmeste tall’ (nearest number) and ‘halvparten’ (half). However, many of the tasks in the grade 2 mapping test require a more secure mental number line and more solid knowledge of the base-10 system. The students identified as being at risk in both grades 1 and 2 were most likely still developing this knowledge; for instance, they could sort numbers below 100 in the correct order but took longer than other students and made more mistakes, such as confusing 14 and 41. Importantly, they still demonstrated that they could do simple addition and subtraction, but only six of the 10 consistently at-risk students could solve tasks such as 6 + 9 or 8 + 8. The students scoring below the cut-offs in both grades 1 and 2 thus showed slow development in the year between the assessments, possibly because learning how to count (e.g., counting in groups and skip-counting) took longer than for other students (as judged by test score patterns) and consequently so did development of a mental number line.

By the time of the grade 3 mapping test, these students had developed, although they still lagged behind; they could count in groups (see Figure 2) and knew the number line up to 200 (e.g., by sorting numbers by magnitude; see Figure 2) but still took longer than other students. They also showed that they could work with addition and subtraction problems that have answers close to 20 and are therefore countable (e.g., 12 + 5 or 8 + 13). It can be argued that the paper-based mapping test does not reveal what strategies the students used, but based on previous research (Scherer et al., 2016), it is likely they used counting-all or counting-on strategies.

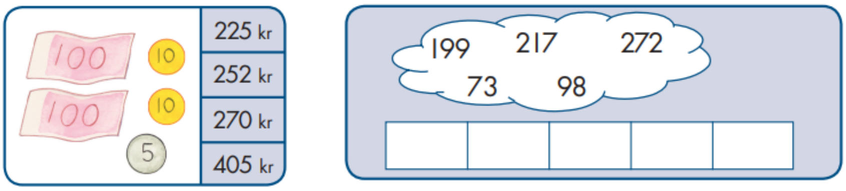

Figure 2: Grade 3 test task, group counting (left) and sorting numbers (right).

CONCLUDING DISCUSSION

Analysis of the mapping test data revealed that, within the 191-student sample, the number of students identified as being at risk grew over time, although interpretation of the transition from grade 2 to grade 3 should account for the possible effects of home-schooling during the pandemic. Nevertheless, the current study supports Blikstad-Balas et al. (2022), who found that sufficient measures were not taken to support struggling students during home schooling. Moreover, the growing number of identified students and the analysis of group averages indicate that teaching interventions might not have been sufficiently targeted to individual students, and slow progress in conceptual understanding, counting and the calculation skills of students consistently identified as at risk indicates that the grade 1 interventions likely did not target the students' counting.

Nelson and McMaster (2019) found that extended interventions that targeted counting principles were the most effective. Most of the students identified as consistently at-risk mastered one-to-one counting on the mapping test, but only for small numbers; interventions targeting the five basic principles might have boosted counting skills, but one size may not fit all, and the interventions would have needed to be targeted. In addition, as these students had weaker mental number lines than their peers in grade 1, working in the 0–100 range might have been helpful for some. By grade 3, the 10 identified students showed awareness of the number line and used it to successfully sort numbers up to 200; they also demonstrated better mastery of group counting. Taken together, this indicates that, by March in grade 3, the students had mastered skills measured on the grade 2 test.

The test data does not reveal the quality of the content of the teaching interventions taking place in the 10 schools, but prior research (Nortvedt, 2018; Nortvedt et al., 2020) indicates that teachers find it challenging to interpret assessment data and plan interventions. This does not stop them from engaging students in activities, but a next step should be to work with teachers and schools to evaluate teaching interventions and the extent to which they are targeted to the needs of the students.

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HOW CAN A SETTING INFLUENCE ONE'S REFLECTION?

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The scientific literature acknowledges the significance and benefits of reflection to teachers' practice and offers a variety of tools and environments for reflection-based professional development. In this paper, we analyze mathematics teachers' reflection in three different settings, using six categories of reflection we previously developed. We examine the unique opportunities for reflection that each setting offers and how it may cater for teachers' different needs.

INTRODUCTION AND THEORETICAL BACKGROUND

Reflection can be characterized as a process of looking at past, present, and future experiences in a detailed, analytical, and careful way, while considering plans, intensions, and behaviours, in order to gain insights about the self, about decisions and about actions (Karsenty & Arcavi, 2017). Reflection is a key component of professional development of mathematics teachers (Brown & Coles, 2012; Karsenty & Arcavi, 2017), since it can enhance awareness to teaching practices and to their underlying beliefs, thus enabling decision-making to become more deliberate (Finlay, 2008; Karsenty & Arcavi, 2017). Reflection may become a mechanism of knowledge development as well as a trigger for processes of change (e.g., Karsenty et al., 2015; Karsenty & Arcavi, 2017; Schwarts & Karsenty, 2020).

Despite the potential benefits of reflection, several studies indicate that teachers struggle to conduct productive reflective process, and may even be reluctant to engage in it altogether (Finlay, 2008; Korthagen, 2014; Lyons, 2010). Several explanations for this are suggested: First, definitions and models of reflection may be seen as somewhat abstract and unclear (Brown & Coles, 2012; Finlay, 2008; Lyons, 2010). Second, the importance of reflection and its potential value are not always fully recognized and appreciated (Finlay, 2008; Lyons, 2010). Third, reflection may sometimes be perceived as criticism, and as such can incite negative emotional reactions (Finlay, 2008; Korthagen, 2014). Finally, reflective processes require time, resources, and support, which are not always available within the intensive environment of the teaching profession (Finlay, 2008; Korthagen, 2014). However, the literature indicates that given careful guidance and appropriate tools, reflection can be productively learned and enacted (Finlay, 2008; Lyons, 2010).

This study aims to contribute to the existing knowledge on reflection, and how it can be supported. The focal point of the study is probing mathematics teachers' reflections as they are carried out in "real life", in three different settings, each with its inherent features. These settings are: (1) a professional development course (described below); (2) weekly reflective journals; and (3) stimulated-recall interviews where teachers

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watch their own videotaped lessons. In our analysis, we sought to characterize the reflections conducted by teachers in these different settings, in order to learn about the opportunities for reflection provided to teachers in each such setting. The ensuing research questions was: *What opportunities for reflection do different settings provide to mathematics teachers?*

THEORETICAL FRAMEWORK

In a previous work (Nurick et al., accepted), we unpacked the concept of reflection into six main categories (see Table 1), based on the existing literature, as well as on our own inductive analysis. These categories relate to actions that mathematics teachers perform when they reflect on their teaching practices.

Category (action)	**description**
Analysis of a situation	Analyzing and examining reasons for what happened; considering goals that stand at the basis of decisions and actions; cogitating broad aspects, issues, and contexts; evaluating the situation and the teacher's actions
Consideration of alternatives, doubts, or dilemmas	Pondering alternative actions, practices or perspectives and their possible applications; deliberating on certain issues; referring to dilemmas of practice
Re-orientation	Arriving at new insights as a result of the analysis, as realized either in “thinking forward”, i.e., referring to possible future actions, or in a change in the teacher's perspective (e.g., beliefs or perceptions)
Consideration of beliefs	Considering, reviewing, or questioning beliefs regarding mathematics, mathematics teaching and teachers’ roles
Addressing emotions	Confronting feelings emerging in certain situations
Addressing challenges of teaching	Elaborating various challenges that arise during teaching and analyzing them

Table 1: Categories of reflection (Nurick et al., accepted).

METHODOLOGY

Data collection

The study is defined as a collective case study (Yin, 2009) and comprises 11 cases of secondary school mathematics teachers. For each of the teachers, data was collected from three settings designed for stimulating teachers’ reflection on the mathematics teaching practice:

VIDEO-LM professional development (PD) meetings: Each of the 11 teachers participated in one of seven PD courses offered in Israel in 2015-2016 by a large project named VIDEO-LM (Viewing, Investigating and Discussing Environments of Learning Mathematics). The project aims at enhancing reflection skills and mathematical knowledge for teaching. Courses consist of 30 hours, divided into 7-10 sessions. In each session teachers watch a videotaped mathematics lesson taught by a different, usually unknown, teacher. The lesson serves as a basis for a peer discussion, directed and guided by a facilitator, who relies on a “six-lens framework” to observe and reflect on the mathematics, the lesson goals, the tasks, the classroom interactions, the teacher's dilemmas, and his/her manifested beliefs (for details, see Karsenty & Arcavi, 2017). For the purpose of the study, sessions in all seven courses were videotaped, and all excerpts in which the 11 teachers (the study subjects) talked were transcribed.

Weekly Reflective Journals (RJ): The 11 teachers wrote personal journals on a weekly basis during five months. In these journals, the teachers were asked to write about the most significant event which happened to them during the week, either while preparing for class or during the teaching itself. They were requested to relate to the reason the event was significant for them. There was no additional guidance or instructions.

Stimulated-Recall Interviews (SRIs), based on a videotaped lesson: One lesson (of the teacher’s choice) was filmed for each of the teachers. After some time, individual interviews were held with each teacher, where s/he watched the videotaped lesson with first author. The interviews were unstructured, and the only instruction for the teachers was that they are invited to stop the video whenever they see a “matter of interest” which they want to talk about. All the interviews were videotaped and transcribed.

Data analysis

The goal of the analysis was to identify opportunities for reflection that each setting offers to mathematics teachers. The analysis was done in several phases, while looking both across the 11 cases studies and across the three settings:

(1) <u>Defining units of analysis</u>: Expressions of teachers were divided into segments, with a different definition of "segment" for each setting: in the PD it was a turn or a sequence of turns where the teacher talked; in the RJ we took each weekly journal as one segment; in the SRI it was a sequence of turns where the teacher stopped the video and talked about a specific subject.

(2) <u>Coding</u> of the segments according to the six categories of reflection (see Table 1): Each segment was analyzed to identify which categories of reflection it alludes to.

(3) <u>Identifying patterns</u>: For each teacher we characterized the reflective process, relying on the coding as well as on repeated reading of the data.

(4) <u>Identifying opportunities for reflection</u> in each of the three settings: For each setting, searching for recurring patterns across the different cases helped us to point out the opportunities for reflection it may offer.

RESULTS

In this report we focus on opportunities for reflection identified in each of the settings. Due to space limitations, we present only some of the opportunities found, and demonstrate them using the case of Sam. At the time of the study, Sam had five years of experience. He taught in an urban junior-high school in a low socio-economic area.

Setting #1: VIDEO-LM PD – exposure to teaching practices and teachers' ideas as a catalyst for reflection

15 segments where Sam talked in the VIDEO-LM PD were analyzed. In 12 of the segments, a similar pattern was identified: Sam noticed a situation in the videotaped lesson, or an issue raised by another teacher. This led Sam to a reflection where he offered an alternative action and analyzed it. To exemplify this pattern, we describe a section from the third PD meeting of the course, where the teachers watched an introductory lesson to the topic of "growth and decay problems".

At the beginning of the videotaped lesson, the teacher presented two questions: one relating to the increasing price of a painting and the other to the decreasing price of a used car. The PD participants ascribed the following possible goal to the filmed teacher's choice of questions: to emphasize that in both growth and decay problems there is always a factor by which one multiplies to obtain a sequence of values. When the teachers wondered if the presented questions are appropriate for this goal, Sam said:

Sam: I think what Josh [another teacher in the PD] was trying to say, is that it would perhaps be better to present the same question. He [the filmed teacher] used a painting for the first question and a car for the second. [It's better to ask] the same question, let's say about money or a question of prices going up or down, but within the same context. […].

Facilitator: You would have used something with money [or] something with bacteria […] Why? What does it enable?

Sam: Because I think that here [in the lesson] they can… you can never know, but if you would have asked the students what is the difference between the questions, some might have not said that 'this is growth and this is decay', but that 'here it's a painting and here it's a car'.

Facilitator: But as Aaron [another teacher in the PD] says, the context here has strength, because a painting of a famous artist, you expect, especially if you present it like that, that its value will increase. A car, you would expect that…

Sam: Okay, as a later stage I would of course present questions from different contexts, to demonstrate it's the same. But at first I think I would show the same context. Then, maybe yes, expose them to a variety of questions,

Sam's articulations in this segment are representative of how he often demonstrated the following reflective actions: he referred to a situation that he identified, either in the videotaped lesson or in contributions made by another teacher in the PD, and *offered an alternative action*; he *analyzed the situation*, while explaining his goals and considerations (e.g., presenting the concept of a growth factor while avoiding surplus "noise"); he *referred to students' possible mistakes*, attending to assumptions on what is easy or difficult for them. In other segments he also related to *students' emotional challenges*. In addition, the last utterance was categorized as *re-orientation*, since after the facilitator' comment about the potential strength of the context, Sam expressed a certain shift in his perspective. However, the re-orientation category was not common in other segments. Sam usually did not refer to his *beliefs*. In the PD, he tended to present successful events and practices from his classroom, using a decisive tone to present his ideas, which hints to the need to save face. Nevertheless, comments made by the facilitator or by other teachers led Sam to rephrase and clarify his stance.

Setting #2: Reflective journals – a personal arena for a focused and deep analysis

Sam wrote 15 weekly reflective journals. Here too, he usually related to positive events, however unlike in the PD he also elaborated his beliefs and goals while revealing challenges and dilemmas he faced. Sam wrote relatively long journals (220 words per journal, on average). His writing was fluent, and it seemed he devoted time and thought to it.

Of the 15 journals Sam wrote, 12 related to situations in lower-level classes he taught. He often began with *addressing general mathematical-pedagogical challenges* (e.g. "Once again, I realized how difficult it is for struggling students to deeply understand the meaning of mathematical rules, concepts and definitions", RJ#6), or with alluding to specific student mistakes (e.g., "I gave the students a task, to collect like terms in the expression $13m + b + m + 4 + 1 + 3b - 3m$ […] a common solution was $17m + 4b + 5$", RJ#3). Sam *analyzed the situations* in a detailed way: he evaluated his actions and considered his goals, while referring to different aspects, especially to affective aspects of students learning (e.g., "students with low self-image in mathematics get frustrated easily, every little change takes them out of balance", RJ#6). Sometimes, Sam included a mathematical analysis, for example when he analyzed the different roles of arithmetic symbols, or when he wrote on the nature of mathematics as a discipline. Sam also *considered his beliefs* toward teaching mathematics to struggling students, and how he views their characteristics and the ensuing teacher's role (e.g., "giving students such challenges can change their attitudes towards the subject and can also develop their confidence to cope with unfamiliar exercises", RJ#4).

Unlike his articulations in the PD meetings, in his journals Sam hardly *considered alternative actions*. When he did so, it was usually as a contrast to a preferred action he already took. For example, in his first journal Sam wrote: "formulas should not be taught in a technical way, we should explain the rationale behind them, even to struggling students". On some occasions, Sam *expressed intense emotions* (e.g., "I felt that as a teacher, I sometimes lapse in teaching these concepts briefly, skipping quickly to the next topic", RJ#6). Regarding the *re-orientation* category, in some journals a change in Sam's perception could be identified, sometimes through his choice of words (e.g., "I learned"), or when he noted he was impressed with a new method that he tried in his class for the first time. Overall, Sam tended to write in a manner that can be interpreted as decisive and self-assured.

Setting #3: SRIs – a unique opportunity for in-depth self-observation

The SRI enabled Sam to talk at length about concrete situations he identified in his videotaped lesson, while analyzing them and connecting them to his goals and beliefs. Sam's SRI was focused, at his request, on the first part of a lesson in an advanced level 8th grade class. The subject of the lesson was the meaning of intersection points of linear graphs, by means of a realistic problem Sam posed. Students were asked to compare two optional destinations for vacation, Thailand and London, each with fixed expenses (e.g., air fare) and expenses depending on the length of stay (e.g., hotel, food). Based on different representations of the problem (tables, graphs, etc.), the class discussed various questions such as what is the meaning of one graph being higher than the other in different domains; what is the meaning of the intersection point of the graphs; where should one fly, based on how many days of vacation can be taken, etc.

The SRI was divided into 14 segments. Sam usually began with *analysis of the situations* he identified in the lesson: He considered the goals of his actions ("This is an important point […] I want the students to be accurate") and evaluated consequences of his actions ("in the first task, I gave the students some anchor, and then in the second task they immediately knew what to do"). He also *considered his beliefs* in detail. Sam's *consideration of broad aspects* was salient in the SRI. He related to both mathematical-pedagogical and interpersonal aspects ("the subject of linear functions is considered to be not easy, but when it is well-structured then it is interesting and relevant"). He also mentioned ways the socio-economical background of his students influences his decisions, for example to deliberately use high register words, in order to enrich students that he knows are not exposed to such words in their homes. However, in the SRI Sam hardly analyzed the mathematical content. Interestingly, unlike in the other settings, in the SRI Sam *addressed emotions the situations evoke* in him, both positive emotions of satisfaction and less pleasant emotions, for instance:

> Often, we teachers […] provide the answer ourselves […] and this is in my view my biggest problem. It is hard for me [to wait for students' answers] for two reasons. I want to cover the content, but also, I am afraid to let them… Maybe I don't trust them enough.

Regarding the categories of *considering alternative actions* and *re-orientation*, different and sometimes opposing patterns were identified within Sam's articulations in the SRI. Sam related to mathematical-pedagogical actions and choices in a confident way, defending them against other alternatives, but when relating to generic issues of teaching, he was less confident and sincerely considered alternatives. Searching for evidence for *re-orientation*, it was hard on the one hand to identify changes in Sam's perceptions or to trace thoughts about different future actions. On the other hand, the SRI revealed instances in which Sam seemed surprised by students' answers and behaviors which he missed noticing while he was teaching, and his reaction conveyed a shift in his view. For example: "wow, this student, I'm shocked […] I suddenly look at him in a completely different way, he has a learning disability […] and his answers are great". Overall, Sam's talk in the SRI was not always coherent or linked to situations in the video. Although we identified various categories of reflection, we also characterized Sam's articulations as sometimes tending to take the form of explanations and even self-justifications, at the expense of learning-oriented analysis.

CONCLUSIONS AND DISCUSSION

In this paper, we analyzed reflections of a mathematics teacher on his practice in three different settings. Sam's example is representative of what we learned about the three settings and the different opportunities for reflection they offer, as we elaborate below.

In the PD meetings, participants observed and discussed teaching practices enacted in videotaped lessons, a setting which enabled them to listen and consider ideas and comments of each other. This exposure stimulated and promoted teachers' reflection, based on the co-analysis of alternative practices, goals, actions, and more. In line with previous work (Karsenty & Arcavi, 2017; Karsenty et al., 2015; Schwarts & Karsenty, 2020), we suggest that a video-based PD setting offers a combination of peer discussions, a vivid object for analysis (the videotaped lesson), and guidance provided by a facilitator, which allows for deep reflection. Nonetheless, PD meetings do not always allow for personal reflections to arise. Issues such as the need to save face and the balance of power relations in the group can inhibit some teachers' reflection.

Journal writing offers teachers an intimate arena for a focused and deep personal analysis, where they can candidly write about specific and focused situations. In line with previous findings (e.g., Hiemstra, 2001), we found that the affordances of a journal include the possibility to freely express challenges, beliefs, and emotions and to inspect oneself critically, something that may be harder to do in the social environment of a PD. However, journal writing lacks external stimulus, guidance, and peer interaction, and thus some teachers will not fully utilize its benefits.

The SRI also provides a personal setting, where teachers can watch an authentic representation of their own teaching, examine situations, analyze, and evaluate them. The detailed depiction of one's own actions as displayed in a video, helps teachers to notice situations, including those that were overlooked in "real time". Nonetheless, self-watching stirs emotions, and some teachers tend to criticize themselves, or alternatively justify their actions, instead of productively examining their practice.

The results of this study reveal that beyond the importance of any reflection process per se, the settings in which the reflection takes place and their specific affordances (or limitations) make a difference. Thus, attempting to support teachers in learning to reflect, either by providing guidance or by offering various tools (Finlay, 2008; Lyons, 2010), must consider not only the inherent complexities of the processes, but also the possible different contexts in which to enact them. Rather than implying that there is a preferred setting, we point to the need to consider the characteristics of each of the three settings, and how they (and possibly others) may complement each other.

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HOW ARE PROOF AND PROVING CONCEPTUALIZED IN MATHEMATICS CURRICULUM DOCUMENTS IN THE USA AND JAPAN?

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Only a few international comparative studies have reported on proof and proving in curriculum documents. This report proposes a method of comparing the meaning of proof-related words in two specific countries' curriculum documents (the USA and Japan) through quantitative and interpretative analyses. Using a text mining approach to explore text data, we found that the co-occurrence network of the words "proof" and "prove" in curriculum documents from the two countries is quite different. In the USA, the word "proof" is concerned with justification and "prove" is used as a general process, while in Japan "proof" is more related to discovery and "prove" is more associated with specific theorems.

INTRODUCTION

Although the universality of mathematics is widely recognized, mathematics educators also acknowledge that it is situated differently in different countries' educational systems. This is also the case for proof and proving in a mathematics curriculum. However, only a few international research studies have reported on the role of proof in curricula (Reid, Jones, & Even, 2019). An international comparative study on proof and proving is promising but challenging because educational, linguistic, and cultural conditions vary according to country (Reid, 2015).

Currently, proof and proving are mentioned in the official curriculum documents of many countries. However, there are still debates among researchers over what constitutes proof, even after repeated discussions over the last three decades (e.g., Mariotti, 2006; Stylianides, Bieda, & Morselli, 2016). How are proof and proving conceptualized in curriculum documents? What are the specificities of their meanings in documents from different countries? How can we compare and analyze them? To address these questions, this study proposes a method of comparison based on the text mining approach, which allows us to analyze the co-occurrence of words in documents, both quantitatively and qualitatively.

The present study is a part of an ongoing international research project for comparative studies on argumentation and proof. In this report, we present a case study by focusing on documents from two specific countries: *Principles and Standards for School Mathematics* (NCTM, 2000; hereafter called *Standards*) in the USA and *Teaching Guide of the Course of Study: Mathematics* (MEXT, 2008; hereafter called *CoS*) in Japan. *Standards* is one of the most well-known and influential curricular documents

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 267-274). PME.

in the world. Though it has strongly influenced curricula in the USA and elsewhere, it is not the national curriculum in the USA. *CoS* is less known outside Japan, but it is an elaboration of the national curriculum. Therefore, while the two documents are different, they play similar roles in their contexts, and so are comparable.

THEORETICAL BACKGROUND AND LITERATURE REVIEW

Although there are limited international comparative studies on curriculum regarding proof and proving (except for Hemmi, Lepik, & Viholainen, 2013), several comparative studies have been conducted on mathematics textbooks (e.g., Miyakawa, 2017) and classrooms (e.g., Knipping, 2004). To gain a better understanding of how proof and proving are conceptualized in different countries' curricula, it is important for researchers to develop a methodological approach to compare and analyze them. To do so, it is reasonable to pay more attention to linguistic aspects related to proof and proving, although some other aspects such as "structure" and "function" can also be considered (Miyakawa & Shinno, 2021). While the approach developed in the Lexicon project (e.g., Clarke, Mesiti, Cao, & Novotná, 2017) in which the methodology focuses on general pedagogical vocabulary used by teachers is promising, we require a more particular approach to investigate how proof-related words are used in curricula.

Some previous studies have considered linguistic issues involving cultural elements, which may affect the nature of proof (e.g., Balacheff, 1987; Sekiguchi, 2002). For example, according to Sekiguchi (2002), "argumentation" is a culturally dependent notion and its meaning in Japanese is not equivalent to that in English or any other Western languages. However, even the term "culture" is often ambiguous, which sometimes represents an obstacle to international communications in our research field.

How then to compare the meanings of words across different languages and cultures? As Wittgenstein reminds us, "the meaning of a word is its use in the language" (Philosophical Investigations, §43). Hence, our study adopts a text mining approach that allows us to analyze proof-related words through quantitative comparisons of their *use* between different countries. Although it pays little direct attention to cultural issues, the results may create an opportunity for discussion among researchers, which may bring new insights into proof and proving from a cultural perspective.

METHODOLOGICAL CONSIDERATIONS

Text mining approach

The methodological approach adopted in our study employs text mining, specifically co-occurrence network analysis. This approach interprets the meaning of a word from its occurrence with other words, that is, co-occurrence relations. Since the meaning of a given word may vary from country to country, we cannot determine the "true" meaning of the word. However, a word's use in any given text can be interpreted quantitatively by its co-occurrence network in that text. The advantage here is that we can avoid possible ambiguities due to the linguistic nuance in each country's language, since it makes use of linguistic networks to characterize the usage of the word within the document.

Using this approach, we can compare and analyze the commonalities and specificities in the co-occurrences of proof-related words in curricular documents of both countries. In short, if the co-occurrence of a particular word in different languages is similar, then we can interpret that the word has a similar meaning. If not, the word can be interpreted as having a different meaning.

Data set

For the USA, *Standards* (NCTM, 2000) is used for the analysis. Although the actual mathematics curriculum varies from state to state, *Standards* has influenced the curriculum in most states. The recently published *Common Core State Standards for Mathematics* (CCSSI, 2010) is also influential nationally and internationally, but it contains fewer explanations about mathematical contents and processes than those in *Standards*. Therefore, we chose *Standards* for our quantitative text analysis due to the abundance of data in *Standards* for our analysis.

The words "proof" and "prove" often appear in the content standard "Geometry" and the process standard "Reasoning and Proof". The section analyzed in this paper is all text in the overview (Chapter 3) and standards for grades 6-8 part (Chapter 6) (See Table 1).

Standards	*CoS*
Overview of the Standards for mathematics education	Section 1.1: Objectives of Mathematics
	Section 2: Content
- Geometry	
- Reasoning and Proof	
Standard for Grades 6-8	Section 1.2: Objectives for Each Grade
- Geometry	Section 3: Contents of Each Grade
- Reasoning and Proof	- Geometrical Figures
	- Mathematical Activities
Total 1,090 sentences (in English)	Total 930 sentences (in Japanese)

Table 1: Contrast of the data.

For Japan, the national curriculum provided by the Ministry of Education (MEXT) consists of a small number of pages for mathematics and has no additional explanations about the objectives and contents. The document we analysed, *CoS*, is the teaching guide to the curriculum, which contains a greater number of pages with a detailed description of the objectives and contents. In practice, Japanese teachers use both publications as curriculum sources. We used the *Teaching Guide of the Course of*

Study: Mathematics (Grade 7-9), published in 2008 (MEXT, 2008) for our analysis. Although the latest *CoS* was published in 2017, we used the 2008 version because this version has an English translation (Isoda, 2010), which made identifying corresponding words easier. The Sections corresponding to the parts of *Standards* that were analyzed are shown in Table 1.

Text analysis software

In this report, we utilized *KH Coder* (Higuchi, 2016, 2017), which can be applied to both Japanese and English. One of its advantages is that it allows easy visualization of the results, thus helping us to perform an exploratory study. The procedure with KH Coder can be summarized into the following four steps: data preparation, pre-processing, visualizing, and exploring the co-occurrence network chart. KH Coder performs pre-processing and visualizing steps automatically, and exploring the displayed chart is an important step for us to understand and re-interpret the meaning of the words. Because the latter process takes place qualitatively, this is considered a mixed method study. The four steps are as follows.

Data Preparation. KH Coder can analyze text format data using sentences as the unit of analysis. Text files were prepared from *Standards* and *CoS* and anything that could not be identified as a sentence in the text was not included in the data. For example, section headings with no periods or words within the figures were not included.

Pre-processing. Pre-processing consists of morphological analysis and word counting of the text files. For English, the *Stanford POS Tagger* software was used to tokenize sentences into words and identify the part of speech. The stop words function in KH Coder identified common words that could occur in any text, such as articles and forms of the verb "to be," and these were omitted. For Japanese, *Chasen* software was used for morphological analysis. *Chasen* could not distinguish between a noun (e.g., 証明; *shōmei*, proof) and a nominal verb (e.g., 証明スル; *shōmei-suru*, prove), so the latter was manually specified as one word so that it could be counted separately.

Visualizing. The "Word Association" command was used to determine which words were closely associated with specific words. The command, under the condition "a specific word (e.g., proof, prove) must appear," searched for sentences satisfying the condition, and listed the words that occur with a particularly high probability. The results were displayed in the co-occurrence network chart and analyzed visually.

Exploring. Based on the co-occurrence of words centered around "proof" and "prove," their meanings were interpreted. In the co-occurrence network chart, words with similar appearance patterns (i.e., with high degrees of co-occurrence) are connected by edges. Thicker edges correspond to stronger co-occurrence. If words are not connected with edges, there is no strong co-occurrence. The number of edges drawn on the chart can be increased to the number at which a focused word can be interpreted by its co-occurrence. The color of each node represents sub-graphs, which means that the same color belongs to the same group. Edges between words belonging to different sub-graphs are represented by dotted lines. By considering words with strong co-occurrences and sub-graphs, the meaning of a particular word in the text can be interpreted.

RESULTS

The co-occurrence network of frequently occurring words in the *Standards* is shown in Figure 1.

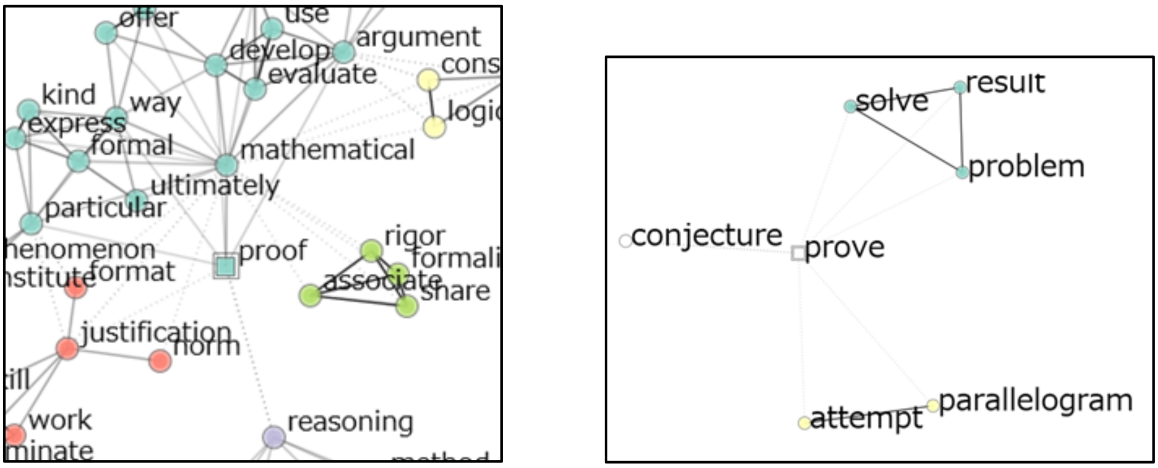

Figure 1: Extract of the co-occurrence network chart for “proof” (left; 164 edges) and “prove” (right; 10 edges) in *Standards.*

The word “proof” occurs 30 times and is strongly associated with the words “mathematical,” “develop,” “argument,” “way,” “particular,” “justification,” and “reasoning.” The proof in the text can be re-interpreted in three ways using connected words: (1) developing mathematical arguments in a particular way, (2) justification, and (3) reasoning. While “proof” is associated with a rich set of words, the word “prove” appears only 4 times. It is surprising that the noun form occurs more often than verb form. This may suggest that the process standard “proof” is not always understood as a process. “Prove” is used in the phrases such as “to prove conjectures” (NCTM, 2000, p.42), or “to solve problems and to prove their results” (ibid., p.43). “Prove” can be re-interpreted as a process that targets the conjectures about a figure and the results of problem-solving. Of course, due to the small amount of data, the interpretation may be biased.

In the *CoS*, the noun 証明 (proof) appears 40 times and the nominal verb 証明スル (prove) 12 times. The numerical tendency is the same as that of *Standards*. The co-occurrence network for these words in *CoS* is shown in Figure 2. The Japanese in the figure is translated into the corresponding English word, with reference to Isoda (2010).

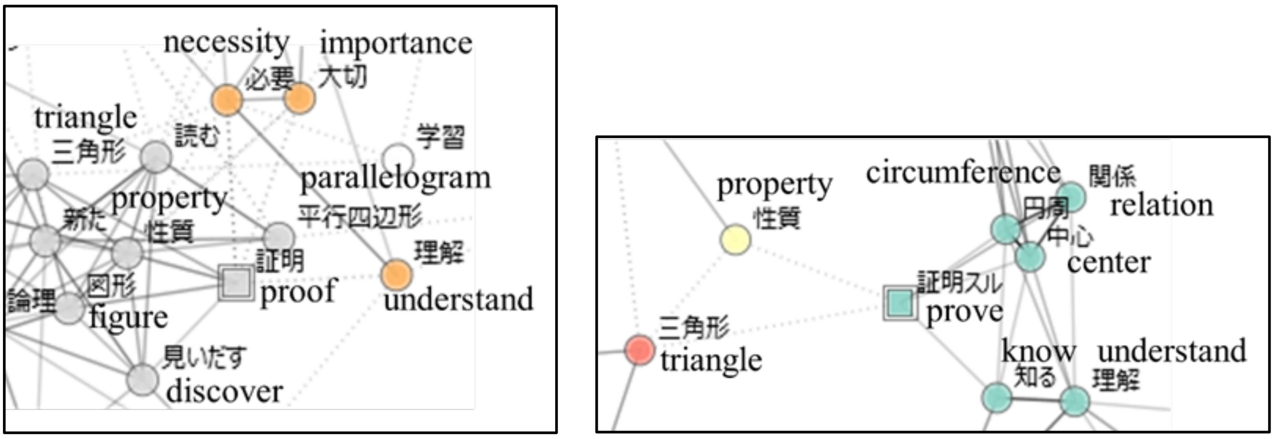

Figure 2: Extract of the co-occurrence network chart for 証明 (proof) (left; 150 edges) and 証明スル (prove) (right; 323 edges) in the *CoS.*

The word "proof" is relatively strongly associated with the words "property," "figure," "triangle," "parallelogram," "discover," "understand," "necessity," and "importance." The proof in the text can be re-interpreted in two ways: (1) to discover the properties of figures such as triangles and parallelograms and (2) something whose necessity and importance are supposed to be understood. Moreover, it is interesting to note that proof in the text is strongly associated with "discovery," not "justification." The description of the *CoS* emphasizes discovering new properties through reading proofs. On the contrary, it is found that the word "prove" is associated with words about the "inscribed angle theorem," such as "center," "circumference," "relation," and to "properties of triangles." Given that it did not co-occur with "discover," it is thus a different conceptualization from "proof." From the associated words, "prove" can be re-interpreted as a process that targets the properties of specific geometrical figures.

DISCUSSION AND CONCLUSION

The results show that the co-occurrence of "proof" and "prove" in curriculum documents in the USA and Japan is quite different. In *Standards*, the word "proof" is strongly associated with the development of a mathematical argument. This conceptualization is close to the definition by Stylianides (2007), who describes proof as a mathematical argument. In the *CoS*, it is associated with the understanding of the properties of specific geometrical figures. The former is more concerned with justification, whereas the latter is more concerned with discovery. Additionally, the word "prove" in *Standards* can be re-interpreted as a general reasoning process, and in the *CoS* as a process that associates with specific theorems. In this way, the text mining approach to the comparison allowed us to better understand the conceptualization of proof and proving in each document, since we could not get such an insight from a superficial comparison of the original texts, which describe the meaning of "proof" in each document as follows.

- "A formal way of expressing particular kinds of reasoning and justification […] arguments consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p.56).
- "A proof is a series of statements starting with the 'hypothesis' and leading to the 'conclusion,' supported by the ideas that have already been accepted as true" (MEXT, 2008, p.115; translation by Isoda, 2010, p.181).

This suggests that the conceptualization of "proof" and "prove" in the texts is different and that curriculum developers in both countries may use the terms in different ways. To be sure, what we have articulated in this paper is only one reasonable interpretation of the meaning in the specific texts, not the "true" meaning. However, it is very important to consider the possible influence of cultural differences when conducting and utilizing international comparative studies. Since the intended curriculum influences the implemented and attained curriculum, it is necessary to examine whether these differences are also found at other curriculum levels (textbooks or classrooms).

The text analysis approach to the usages of words in curriculum documents can be applied to other related words, such as "reasoning" or "argumentation," in other countries, although a certain amount of text is required. Furthermore, it allows us to understand how certain words are conceptualized in documents based on the linguistic culture of the country. Articulating how proof and proving are conceptualized in different curricula using the same methodology is important for further development of international comparative research.

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STUDENTS' EXPLORATION OF TANGIBLE GEOMETRIC MODELS: FOCUS ON SHIFTS OF ATTENTION

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This empirical study applies the analytical apparatus of Mason's shifts of attention theory to investigate why and how using physical models of different scales can facilitate learning of (spatial) geometry. In the presented case study, six high school students learned the properties of icosahedron by constructing and exploring physical models. Shifts in the focus and structures of attention were associated with multimodal perception and collaborative physical actions of students with and through the models. Models of different scales landed students different affordances for exploration, facilitating noticing of invariant scale-free features of a geometric object and influencing the dynamic of student collaboration.

INTRODUCTION AND THEORETICAL FRAMEWORK

According to Goldenberg et al. (1998), geometry is "an ideal intellectual territory within which to perform experiments, develop visually based reasoning styles, learn to search for invariants, and use these and other reasoning styles to spawn constructive arguments" (p. 5). This claim concurs both with Freudenthal's view on geometry as "one of the best opportunities which exist to learn how to mathematize reality" (Freudenthal, 1973, p. 407) and with tenets of embodied design for mathematics instruction (Abrahamson et al., 2020) supporting primacy of students' enactment of conceptually oriented movement forms and gradual formalization of gestures and actions in disciplinary formats. Embodied learning is rooted in an ecological approach in cognitive psychology (Gibson, 1986/2015), capitalizing on organism-environment relations. In particular, Gibson conceived perception as an active, embodied process in which we notice optical invariances of the object under the movement of the source of light, movement of the observer, movement of an observer's head, and manipulations and local transformations of the object itself. Students facing tasks in realistic 3D contexts can be introduced to the language of geometry, its objects and constructions (Doorman et al., 2020). They conduct mathematical modeling of their experiential world and then are invited to use informal strategies (horizontal mathematization) and further develop them into normative forms and practices of mathematics (vertical mathematization) (Gravemeijer, 1998). Several scholars suggested that mathematical modeling of geometric figures should take into account four distinct perceptual systems of the figure(s): (a) as physical navigation of macrospace (objects more than 50 times the size of an individual); (b) as capturing an object in mesospace (0.5 to 50 times); (c) as constructions of small objects in microspace (less than 0.5 times); and (d) as descriptions and manipulations of small objects in microspace (e.g. Herbst et al., 2017). Still, why and how physical models of different scales can facilitate learning of

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(spatial) geometry remains an open question. This empirical study seeks to provide an answer using the analytic apparatus of Mason's shifts of attention theory.

Learning as shifts of attention

Mason (2010) claims that learning is a transformation of attention involving "shifts in the form as well as the focus of attention" (p. 24). Thus, to characterize learning, Mason considers *what* is attended to and *how* the objects are attended to. Per Mason (2008), there are five different forms or *structures of attention.* One may *hold the wholes* without focusing on particularities or *discern details* among the rest of the elements of the attended object. From there, one may *recognize relationships* between discerned elements and even *perceive properties* by actively searching for additional elements fitting the relationship. The ultimate structure of attention is *reasoning based on perceived properties*. The shifts in attention structures are not necessarily sequential, and one may return to *holding the whole* to reassess the situation.

In Mason's works (2008, 2010), these theoretical constructs were suggested for use in teachers' education. In more recent studies, the theory of shifts of attention was applied as an analytical framework to study students' problem-solving efforts (Palatnik & Koichu, 2015) and assess individual changes in children's communication and conceptualization of arithmetical tasks (Voutsina et al., 2019). Palatnik and Sigler (2019) suggested that shifts in form and focus of attention can also be applied to analyzing geometric tasks and activities, particularly when introducing an auxiliary element is necessary. The current study expands the application of shifts of attention as an analytical framework for investigating spatial geometry learning. In this report, the analytical lens of shifts of attention is applied to collaborative geometric activity in which students explore tangible models of a geometric object on different scales.

Research questions

When students study 3D geometrical objects by exploring physical models, which shifts of attention do they experience? What role do physical features of the models (i.e., their relative size and their orientation in space) play in the process of student exploration?

METHOD

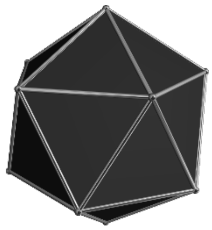
Context

This study is a part of a research project *Learning Geometry as Negotiating Perspectival Complementarities* studying activities that foster conceptually productive discursive and pragmatic tension between differing perspectives on sensorial features of shared displays of geometric objects (Benally et al., 2022). A distinctive feature of the empirical context of the current study is that students explore the same geometric objects at different scales. In one of the tasks, students are given a 2D diagram and written instruction (see Figure 1) to construct an icosahedron—a polyhedron whose exterior is composed of twenty equilateral triangular faces. They have to build relatively small as well as human-scale models using wooden rods and silicone joints.

Once both models are built, students are asked questions concerning the icosahedron's properties, for example, "How many vertices and edges does an icosahedron have?", "How many parallel edges?", "If the icosahedron were standing on its triangular base and filled halfway up with water, what would be the water's surface shape?"

Your team has to construct two three-dimensional models (one large, one small) of a geometric solid, a polyhedron.
The polyhedron has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converges at each vertex.



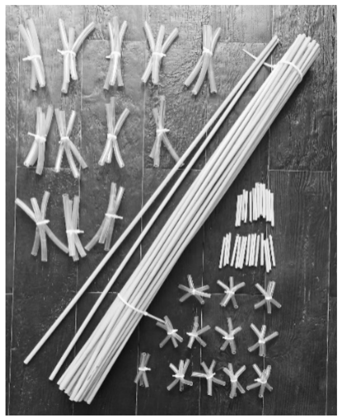


Figure 1: The icosahedron construction task and materials.

Data collection and analysis

The case presented in the paper provides an account of an outdoor implementation of the activity with a group of six tenth-grade students. This activity was a part of an enrichment program for the students at the beginning of their first year in the new high school. This case was chosen from the data collected (14 cases)for two reasons: First, the way this group constructed the models and answered the questions was typical of this activity. Second, the students were more verbal than other groups, making indications of their attention shifts more distinguishable.

The activity was video-audio recorded. To analyze the data, we combined multimodal analysis of students' interactions (Abrahamson et al., 2020) with microgenetic analysis of shifts of attention (Voutsina et al., 2019) in the following way. We prepared a complete transcription of the activity, overlaid with a description of students' actions, gestures, and movements. The resulting protocol was divided into episodes (i.e., construction of the large-scale model, construction of the hand-held model, the answer to the first question, etc.) In each episode, we looked for the indicators of the shifts in *focus* (what is attended to) and *structures of attention* (how it is attended to). Marking the objects directly mentioned in the conversation, the direction of the gestures and gaze (where available) helped us identify the focus of attention. To identify shifts in structures of attention, we, following Gibson's approach, interpreted changes in students' movement in space, manipulations with and local transformations of the object itself (for instance, change in the model's orientation). Particular attention was paid to the actions, gestures, and utterances that preceded students' advancements in the task. At the subsequent analysis stage, we compared how students interacted within the team and with models in different episodes. Due to the page limitation, we focus here on two episodes: finding the number of vertices and the number of edges.

FINDINGS

a b c d

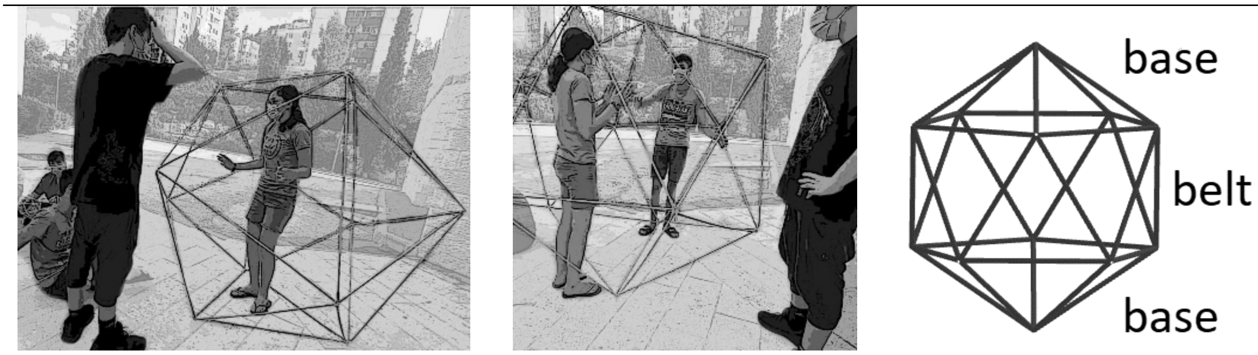
Figure 2: (a) Students discuss a small-scale model resting on a triangular face; (b) the student examines a small-scale model while holding it on a vertex; (c-d) different points of view on the same model.

Having constructed both the large and small models, the students used the small model to find the number of vertices (Participants are referred to by the color of their t-shirts; transcription translated from Hebrew by the author).

Grey	[holds a model on its vertex, starts counting] One, two, three, four, five. [touches an upper vertex, touches two additional vertices]
Black	How many vertices? [Reaches out for the model and touches it]. We already counted (them). It is a number of joints.
Yellow	[takes the model, starts to count by touching silicon joints] One, two.
Grey	Twelve. Times five. Sixty.
Yellow	How (it can be) twelve times five? How (it can be) sixty? [looks at the model].
Grey	[tries to take a model from Yellow] Ah, vertices… Twelve. Put it (the model) like this [tries to orient the model on the vertex]
Yellow	Give it (the model) to me for a moment. I know what I'm doing [takes the model away from Gray].
Grey	But, but…It's… Ohhh…
Yellow	[starts counting the joints from two facing her]. One, two. [continues counting] One, two, three…ten. It is twelve! [puts a model on a floor to write an answer]
Grey	[takes a model and tilts it on a vertex] Look [addressing Yellow] at it this way. [Starts counting from a lower basis] One, two…
Yellow	There are twelve!!!

In this episode, Gray and Yellow answer the question by counting the silicon joints of the small-scale model. The small size of the model allowed the students to group around it. The model became the *focus* of their joint attention. The model's size also enables students to simultaneously grasp most of its features, *holding the whole*. Both students physically touched the joints while counting vertices which helped them to *discern* these relevant *details* of the model (separating it from edges and faces). Both students were successful in counting 12 vertices. However, the ways of counting were qualitatively different. By orienting the model in a particular way (Figure 2 b, d), Grey *recognized a relationship* between several groups of vertices of the icosahedron. Grey's persistence to explain his point of view and his frustration when he was denied the explanation can indicate that he *perceived* this orientation as *the property* of an icosahedron, and it served as the *base for his reasoning*. Yellow was also successful in her attempt to count the vertices, which she separated into two groups of two and ten and did not see the value in the alternative orientation of the model in space. The next episode will demonstrate that Gray's unappreciated know-how of holding an icosahedron on its vertex will help answer how many edges an icosahedron has while Yellow's attempt will fail.

Yellow	How many edges are there?
Black	Okay, that's tricky because they're shared. (i.e., each edge is shared by two triangles).
Blue	I'll put a finger [on the first edge, to Yellow help her monitor the count].
Orange	You just count the sticks.
Yellow	I'll go to the big one (i.e., the large-scale model).
Black	The big one is just nicer.



a b c

Figure 3: (a) students' problem-solving attempt inside and outside a human-scale model (standing on a triangular face); (b) having tilted the structure onto a vertex, the students soon arrive at a critical breakthrough; (c) partition of an icosahedron.

Three of the six students rose and walked over to the nearby large-scale model. This larger model is advantageous for counting because its edges are more perceptually distinct. A model's greater size, while availing perceptual acuteness, may come with a price that its figural elements in question (the to-be-counted edges) are never all in one's arm's reach—you cannot directly touch or gesture to each edge as you tally it. Thus, using structures of attention terminology, greater size afforded students easier *holding the wholes* while impairing *discerning detail* by a sense of touch. To overcome this, Yellow entered inside the model, where all edges are within her reach (Figure 3a).

Still, when you are inside an object, part of it is always behind you, so you might lose track of your count. Indeed, Yellow's initial attempts to count failed.

The excerpt below demonstrates two phases of student solution. During the first phase, Yellow attempted to use some of the icosahedron properties that the team discovered during construction, yet again she failed to develop a systematic approach. During the second, Grey received an opportunity to demonstrate that his strategy of putting the model on a vertex has an advantage. In seconds, his physical action facilitated the restructuring of Yellow's attention leading her to a correct solution.

Yellow	There are five from each vertex. One should be subtracted. Then there are four. Two should be subtracted here. It's three. It doesn't work that way…3, 4, 5 [sits on the floor, inside the model, frustrated]. I can't count this. [Stands up]. How many sticks did we use [during the construction stage]? Three and another three, and another three, and another three, it's 12…
Grey	Let's do it as we did with (inaudible) [Stands up]
Yellow	[referring to triangular faces] …another three, 15, another three…
Black	We need a formula for this…
Gray	I'm tilting it. [starts tilting the model]
Yellow	No, no, no, no! eighteen…No! Why?
Gray	To make it like this (standing on the vertex). It will be easier to count like that [holds the model on the vertex] (Figure 3b). 1, 2, 3, 4, 5 [counts the edges diverging from the upper base vertex by pointing at them]; 1, 2, 3, 4, 5 [counts the edges diverging from the lower base vertex]
Yellow	[turns inside the model and counts the edges of a lower base pentagon by pointing at them] 1, 2, 3, 4, 5.
Grey	Look, the base is ten.
Yellow	[counts the middle section] 1, 2, 3, 4... Where did I start? (to Grey) Put your hand here. [continues to count silently] ... [raises arms to the upper base] ten, [lowers arms to the lower base] ten, [makes a circular breaststroke movement with both hands indicating a middle part] ten, ...thirty!

At the beginning of the episode, Yellow *discerned* relevant *details* of the model and even *recognized* the *relationship* between them: five edges meeting at the vertex, three edges forming each of the triangular faces. Each of these relationships has the potential to become *the property* leading students to a correct solution. However, these properties were not useful for Yellow's approach of direct counting. While standing inside the model and reassessing the situation, she cannot *hold the whole*. From a mathematical point of view, it does not matter how the icosahedron is positioned in space—the polyhedron's mathematical properties remain the same. However, in a material gravitational world, the model lay on one of its triangular faces, making it difficult to perceive certain structural symmetries.

When Grey tilted the model onto a vertex (Figure 3b), he restructured Yellow's attention. Previously this action enabled Grey to count the vertices, and now it helped Yellow to perceive an icosahedron as tripartite: two opposing "bases" and a connecting "belt" (Figure 3c). Grey also gave Yellow a hand (literally) in counting edges in a "belt." New structures of attention facilitated counting, and three aggregating gestures summarized the *perceived property* that there are ten edges in each of three groups.

DISCUSSION

The first research question raised by this study was on shifts in focus and form of students' attention when studying 3D geometrical objects by exploring physical models. By moving in space, changing points of view, and modifying a physical object (Gibson, 2015), the students experienced shifts in focus (small and large model, three distinct parts of the model, vertices, edges, groups of edges) and structures of attention. All five theoretical structures of attention and shifts between them (Mason, 2008) were documented in two episodes. Note that shifts in the structures of attention were associated with vision and touch, proprioception, and physical actions of students with and through the models. For instance, tilting the model on its vertex allowed students to structure their seeing of the icosahedron into three visible sets. We reported this case as indicative since this action helped students answer questions about vertices and edges or explain their solution to their peers in all the cases we possess.

The second question was on the role of physical features of the models in the process of student exploration. Models of different scales landed students different affordances (Gibson, 2015) for inquiry. For instance, in most cases, at least one student entered a human-scale model to examine the features of the polyhedron from within (as Yellow did). The activity enabled students to ground conceptions of the geometric figure simultaneously as objects in mesospace and macrospace (c.f. Herbst et al., 2017), providing more opportunities for possible shifts in focus and structures of attention and thus learning (Mason, 2008, 2010). Each model served as a physical attractor with different affordances for and constraints on the action; accordingly, students reorganized around these affordances and constraints. For instance, the small-scale model centered the group's multimodal interactions, but its size could not accommodate students' form of inquiry; the apparent availability of a larger model catalyzed splitting the group, which, in turn, juggled students' social roles.

The case study findings highlight the pedagogical potential of using different scales 3D models in spatial geometry instruction. First, the students experienced construction and informal exploration of polyhedron models producing a multitude of perspectives and collaborative insights on their features. Their efforts combined collaborative actions, gestures (indexing and iconic), and speech to indicate and highlight models' properties. The fluency with which students moved from one model to another—both physically and inferentially—suggests they noticed invariant scale-free features of a geometric object. Then, students' shifts of attention were multimodally grounded in their senses and converged to a gradual disciplinary formalization of the polyhedron's concept (c.f. Abrahamson et al.,2020, embodied design for mathematics instruction).

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THE MTE: MANAGING THE PROFESSIONAL EMPOWERMENT OF PROSPECTIVE PRIMARY TEACHERS

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The knowledge and practices of primary mathematics teacher educators in the period of initial training have the same impact on prospective primary teachers as teachers' knowledge does on their students' learning. The aim of this study is to contribute to characterising educators' knowledge, particularly with respect to developing their students' abilities and professional identity. Based on the observation of a training session and an interview, we recorded instances of a primary teacher educator's knowledge, which we then analysed. The results indicate that not only the construction of professional knowledge, but also the development of teaching abilities and the sense of belonging to a community of teaching practices, become key to the process of professional empowerment of the prospective primary teachers.

THEORETICAL BACKGROUND

Mathematics teacher educators' (MTEs) knowledge is one of the major challenges in mathematics education research today (Chapman, 2021). Understanding and characterising MTEs' knowledge means recognising their role as agents of change in the learning of prospective primary teachers (PPTs), in the same way as had traditionally been recognised in terms of mathematics teachers and their students (Jaworski, 2008).

Research into MTEs' knowledge is based on, at least, mathematics teachers' knowledge (MTK) and knowledge of mathematics teachers' education (KMTed) (Chapman, 2021). MTEs' knowledge should also contemplate the primary teacher's knowledge as part of the content to be imparted/constructed in the course of the training. Consequently, many approaches to educators' knowledge have been couched as extensions to the teacher's knowledge (Castro-Superfine, *et al.*, 2020; Perks & Prestage, 2008; Zaslavsky & Leikin, 2004), adapted to the primary training context.

Likewise, teacher educator knowledge in the domain of primary mathematics should consider aspect related to how to teach teachers' knowledge (KMTed). In this regard, the work of Chick and Beswick (2018) and their characterisation of the pedagogical content knowledge of primary teacher educators suggests that it is a question of the educator's meta-knowledge of how to teach content knowledge for teaching mathematics.

Our views on the knowledge of MTEs are consistent with many previous studies. On the one hand, we consider that there are many points of contact between the educator's knowledge and the teacher's knowledge, although there are areas of divergence, too. The differences revolve around the depth of understanding of mathematical content,

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and the connections between the mathematical content and pedagogical content knowledge which the educator might have (Escudero-Ávila, Montes & Contreras, 2021).

Nevertheless, bearing in mind the observations of Ponte (2012) regarding the structuring of primary teacher training, the educator's knowledge should also include elements of knowledge of the teaching as a profession, professional practices, and professional identity, along with such knowledge as will allow them to help the PPTs gain access to these elements. Likewise, these areas of knowledge should be combined at the two levels at which the educator discourse is focused – that of the initial training classroom and the PPTs, and that of the future mathematics classroom and the students (Jaworski & Huang, 2014). Although the educator's knowledge should be understood as *multidimensional*, *complex* and *indivisible* (Escudero-Ávila *et al*., 2021), the analytical advantages of establishing different categories of knowledge leads us to consider the structure and content of the MTSK model (Carrillo *et al.*, 2018) as the inspiration for the conceptualisation of the educator's knowledge.

Our study aims to contribute to the characterisation of the educator's knowledge, and to explore the connections between the elements of this knowledge, foregrounding how this promotes the development of PPTs' abilities and teaching identity during the initial training. To do so, we focus on analysing which aspects of the MTEs' knowledge enables them to manage sessions of primary teachers' initial training which promotes that the PPTs learn how to act as teachers, and to recognise themselves within the teaching community. In the next section, we describe the methodological aspects of the study.

ANALYTICAL APPROACH

The study took the form of a case study (Bassey, 1999), in which an expert informant was selected (a mathematics education researcher working in the field of teachers' knowledge and professional development, with more than thirty years as an educator, whose work is widely respected within the academic community), henceforth referred to as Lucas. Several distinct training sessions were observed and video-recorded, and the foundations of these sessions were then discussed with him.

The excerpts discussed in this study correspond to the evidence of knowledge identified in the course of a session in which the definition of polygon was constructed with the PPTs, as part of the course content on the methodology of geometry in the Degree in Primary Education at a Spanish university. Once the class extracts had been selected, we identified points providing evidence of different kinds of knowledge according to the method of content analysis (Krippendorff, 1980). The accumulation of these focal points developed our understanding of the educator's knowledge, and each was matched against the analytical structure by Escudero-Ávila *et al*. (2021). The process of discussing these episodes by a group of experts, and contrasting the evidence obtained according to different data gathering tools (Baxter y Lederman, 2001) enabled us to triangulate and validate the various elements of the educator's knowledge. This in turn helped us to understand how Lucas managed the learning of different professional practices related to task planning, and how he promoted certain values which constitute a teaching profile, and supported the development of the PPTs' professional identity.

ANALYSIS AND DISCUSSION

Lucas presented this session to the PPTs a class in which the main objective was tackling the mathematical practice of defining, specifically in this case the construction of the definition of a polygon. The session was structured around a set of geometric shapes which formed the basis of a discussion about the mathematical elements which make up the definition of a polygon (Figure 1).

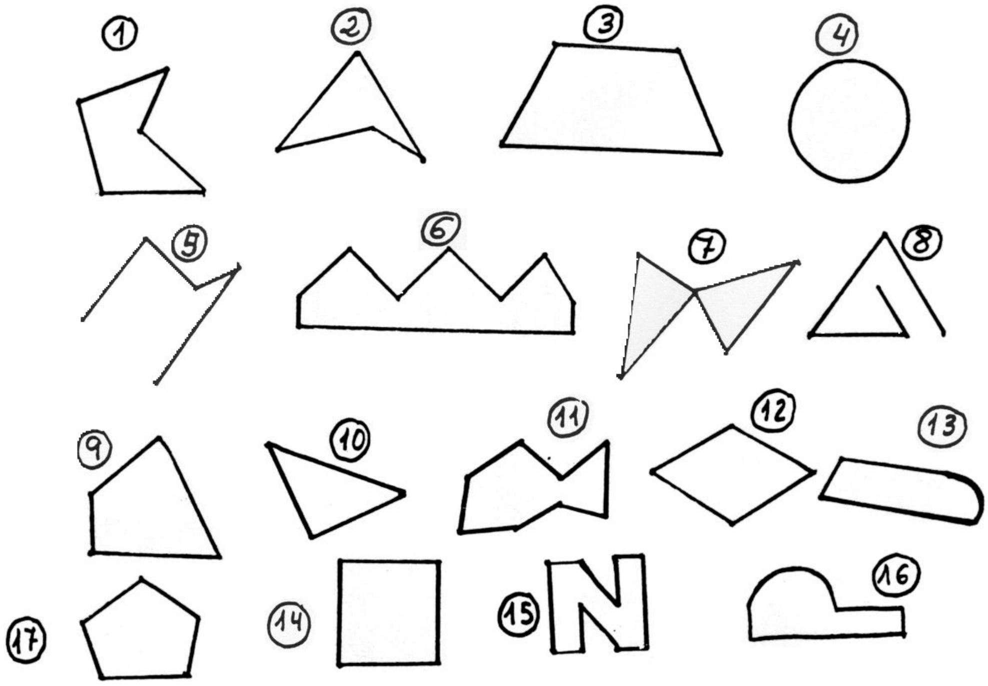

Figure 1: A selection of examples for constructing the definition of a polygon.

Taking the perspective on exemplification offered by Watson and Chick (2011), Lucas' handling of the selection of examples provides indications of his mathematical knowledge regarding the definition of polygons and the practice of defining, especially in the way he guides discussion of the necessary conditions and leads his students to inductively construct a definition of a polygon.

Lucas presents the construction of mathematically satisfactory definitions as a social activity, that is, as a collaborative endeavour. The approach reflects his beliefs about what mathematics is, which, although not the direct focus of this study, can be transmitted over the course of the initial training programme as a source for configuring the students' teaching profile:

Lucas: Let's see if we can remember. What are the arguments underlying what we've agreed on so far? What might be revealed by including or excluding the circle in the set of polygons? That's one of the things we're going to look at today. If we decide it's a polygon, what consequences follow on from that? ¿OK?

[…]

What else do we need to think about? Is there just one way of defining a polygon? Is there a correct way? Or are there various possibilities?

The discussion about the consequences of including certain shapes inn the set of polygons further illustrates Lucas' understanding of the interdependence of the mathematical results which they are hierarchically constructing. The fact that Lucas highlights these mathematical relationships not only illustrates his conceptualisation of mathematics as a connective network linking together concepts, procedures and practices, but also represents an opportunity to empower the PPTs to become actively engaged with mathematical knowledge. The ability to recognise these relationships and to be able to construct new ones by changing the underlying premises, is indicative of deep mathematical knowledge. It is this kind of knowledge, in particular with respect to the knowledge of topics (KoT) and knowledge of practices in mathematics (KPM) subdomains (Carrillo, *et al*., 2018), which mark the difference here between the primary teacher educator and primary education teachers, as other episodes across the full study also suggest.

Showing awareness that the PPTs are immersed in a process of reformulating their mathematical knowledge during the course, Lucas draws on exemplification and analysis of geometric definitions in order to lay the foundations of this new way of understanding mathematical content. Through the principle of isomorphism (Ponte & Chapman, 2008) and the premises of modelling (Rojas, *et al*., 2021), by which the teaching they receive in their initial training serves as a model for their own teaching when they enter the profession, the PPTs can transform their experience into content for their primary education training.

Lucas' discourse also shows evidence of his knowledge of how to teach content related to planning activities for classroom use, bridging between the context of initial training and that of primary teaching (Jaworski & Huang, 2014)

Lucas: Let's take a moment to reflect now about the wealth, when it comes to making distinctions, the wealth that each step we took in the elimination process could give rise, yeah? Thinking about how we now understand those steps. And how the act of defining, agreeing properties together, leads us to a shared definition of a polygon, as opposed to a definition that's imposed, where we don't understand why those particular criteria have been imposed. And it's just the same when you come to teach it, because your pupils can also go through this process, which is such a richer experience.

Lucas makes it very clear that the PPTs are perfectly capable of planning primary lessons which are fully consistent with what they have experienced themselves in their initial training. This excerpt brings to the fore how Lucas articulates various elements of his knowledge in order to mould the task, which at the same time serves as a model for teaching in the primary context. What he does can be interpreted as putting into effect his knowledge of teaching pedagogical knowledge for the teaching of mathematics.

In the extract below, when some of the PPTs encounter difficulties with the task, Lucas makes reference to different approaches to defining a polygon. In doing so, he demonstrates his knowledge of traditional modes of teaching mathematical concepts, where in the past deductive methods predominated:

Student: Me, at least the way I see it, if we can't compare it with anything, if we're not sure about what a polygon is, then how can we say if it's a polygon or not?

Lucas: Well, you're negating the main thing. What you're saying is: "Unless I have a definition, how can I know what a polygon is?" Well, OK, let me turn that back on you, because what you're essentially saying is that the only way to define something is to go from the general to the particular. And I want to question that. Maybe it's because that's what you're used to. I want to know if it is possible, based on different particular situations, to try and construct a general concept.

The activity of constructing the definition of a polygon, and the considerations of what to take into account when teaching the topic, is brought to a close by Lucas' reflections on the teaching materials used at this level, specifically the textbooks. His comments illuminate his perspective on the teacher's professional identity, locating them as an expert in the teaching of mathematics, capacitated to make analytically based decisions on the material they use.

Lucas: What would be nice now would be to have a look at the textbooks, which is the third part of the activity, and see if the definition of a polygon which they give in the textbooks is the same as ours, and see if our process has been a richer or less rich experience than what the book offers, which almost certainly the complete opposite. So I suggest that's what you do. We can definitely find it in the books for third year primary. I recommend you have a look at how polygons are introduced, how they introduce them.

The work on defining a polygon was complemented by the video recording of a primary lesson in which the teacher carried out an activity similar to that the PPTs had experienced. As a follow-up, they were asked to think about which aspects of the teacher's performance had stood out, what suggestions they might make for areas of improvement, how effective they considered the examples used, and what interventions or responses by the pupils had struck them.

The design and execution of this kind of activity by the educator as part of the initial training course illustrate his knowledge of the professional practice of planning tasks for teacher education which become the principal focus of the session. The teacher educator leads a discussion phase covering different elements of professional knowledge, but its essential function is to provide a model of how the PPTs might proceed in their own future classes, which they then subsequently discuss. The discussion of the examples deployed by the teacher in the video places the focus on the role of exemplification as a teaching tool, in particular the choice of examples, their degree of transparency, and the educational potential of the example space. In this regard, the educational practice considered in this phase provided evidence of another area of the educator's knowledge.

CONCLUSIONS

The conclusions to be drawn from this study underline the importance of the interconnections between the trainer's knowledge of primary education teaching practices, the means of transmitting these to the PPTs and professional empowerment in education. The teaching of such educational practices as task design and exemplification represents one of the distinguishing elements of initial primary training with respect to other university domains involving the teaching of mathematical content. Recognising the specificity of content involved in primary teacher training, and the importance of the educator's knowledge in developing this, represents an advance in improving the teaching and learning of mathematics at various educational levels. It is here where the chief contribution of this study lies, and in this regard it fits alongside the developments in characterising educators' knowledge proposed by Escudero-Ávila *et al*. (2021).

The knowledge required by MTEs to be able to manage the learning of these practices following the practices of isomorphism (Ponte & Chapman, 2008) or modelling (Rojas, *et al.*, 2021), can be interpreted from the perspective of practical wisdom described by Perks and Prestage (2008), but requires an exhaustive revision so as to understand the different ways MTEs deliver the training content to the PPTs. Progress along these lines would see an improvement in programmes for training educators. Analysis of video-recorded lesson, such as that described in this study, and roleplaying activities in the training course, which we also saw in or wider study, amplify the modes available to MTEs for teaching to teach mathematics, and rest on both knowledge of training content and pedagogical content knowledge for training.

The development of teachers' professional identity can take place in parallel with work on professional practices in initial training course. The inclusion of content dealing with the day to day concerns of teachers directs the PPTs' awareness towards their professional future, and encourages them to see themselves as teachers rather than students. Nevertheless, occasions when the educator expresses their knowledge of professional values or attitudes, and beliefs about mathematics and its teaching and learning, confirm the presence of content related to the configuration of teachers' professional identity in the initial training course. Further studies are needed in this vein to help us systematically identify the signals that recurrently appear in studies of educators' knowledge. The configuration of teachers' professional identity and the development of professional profiles constitute one of the major milestones in the initial training of primary teachers.

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A DISCUSSION ABOUT THE SEMANTIC CONGRUENCE OF A TRANSLATION: AN OPPORTUNITY TO PROMOTE ALGEBRAIC THINKING AT ELEMENTARY SCHOOL

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In this study, we analyze how 9-10 years old students represent and refer to indeterminate quantities when posing equations associated with arithmetic word problems. The analysis focuses on the semantic congruence of the expressions proposed by them and on the dialogue they held during the translation process. The results show that arithmetic word problems allow the indeterminate to become an object of thought for students, who represent it in multiple ways and refer to it when proposing equations that represent the structure of each problem. Another finding highlights that reflection on the interpretation of the equations favors the identification of two meanings associated with indeterminate quantities, namely, unknown and variable.

School algebra should be understood as a topic related to other mathematical contents (Kaput, 2008). One of the main aspects of school algebra in elementary school is to develop algebraic thinking, which allows children to interact with indeterminate quantities: unknowns, variables, parameters or generalized numbers (Radford, 2018). In this context, our interest is in establishing relationships between the resolution of arithmetic word problems (AWP, hereafter) and their translation using algebraic notation. We seek that elementary school students agree on the meaning of indeterminate quantities and how to represent them using letters or another symbol proposed by them. The role of AWP is important because the context favors the students to give meaning and support to indeterminate quantities (Janßen & Radford, 2015). Some previous studies have reported that 9–10-year-old children solve AWP and treat them in a general way. However, they cannot express them with algebraic notation (Fritzlar & Karpinski-Siebold, 2018).

Therefore, in this work we describe how 9-10 years old students represent and refer to indeterminate quantities when posing equations associated with AWP. For this aim, we focus on how students translate AWP into algebraic notation.

ALGEBRAIC THINKING AND EQUATIONS

One of the areas of content to algebraic thinking includes equivalence, expressions, equations, and inequalities (Kaput, 2008). This area seeks to develop a relational understanding of the equal sign, as well as reason with expressions, establish the equivalence between different expressions in general terms, and pose and solve equations and inequalities (Blanton et al., 2011).

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 291-298). PME.

In this work, we focus on the equations, understanding that an equation is a mathematical sentence that involves an equal sign to show that two algebraic or numeric expressions are equivalent (Blanton et al., 2011). Radford (2021) pointed out that using an equation to reason about the representation and communication of the relationship between quantities is a cornerstone of algebra. In addition, many problems are better solved if the equation is first written to represent the problem statement. He highlighted that developing an understanding of how equations can be written to represent problems at elementary school can build a foundation for later learning formal algebra.

ARITHMETIC WORD PPROBLEMS AND ITS TRANSLATION

AWP contain information that is presented exclusively through natural language, and to solve them it is necessary to apply one or more elementary mathematical operations. In turn, these problems can be represented using different representations, therefore, their interpretation and solution can lead to several translations carried out by the solver.

Regarding the translation of natural language to algebraic symbolism, several authors focus mainly on courses after primary education. For instance, Castro et al. (2021) have shown that to be successful in translations, students must identify the variables involved, the relationships between them, and the syntax of the symbolic representation. One of the difficulties that students face is to understand the meaning of algebraic notation, since this type of representation is considered opaque for them. They use to have difficulties to visualize the advantages of algebraic notation.

About the translation from one representation to another, Duval (2006) pointed out that two representations are congruent when the following three conditions are met: (a) semantic correspondence between the significant units that constitute them; (b) semantic univocity, i.e. each initial significant unit of output corresponds to one and only one significant elementary unit of the input record; and (c) the order within the organization of the significant output units is maintained in the arrival representation. When one of these criteria is no longer met, the representations are not congruent with each other. However, this author added that two expressions can be referentially equivalent without being semantically congruent. Semantic congruence allows us to see the degree of transparency of the relationship between two representations.

METHODOLOGY

The data that we present here comes from a Summer School that took place virtually for two weeks and was attended by pupils who had just finished the 4th year of elementary school.

Participants

The Summer School had 21 participants, who attended their previous school year online. The students belonged to two different schools that are part of the same Educational Foundation that serves children and young people from low-income sectors. The students were selected with the help of their regular math teachers under the following three criteria: (a) gender parity (10 girls and 11 boys); (b) willingness to work during the summer; and (c) different paces of learning.

Design

The Summer School was organized in 10 sessions following the areas of content to algebraic thinking (Blanton et al., 2011; Kaput, 2008): *generalized arithmetic* (sessions 2, 3 and 4), *equivalence, expressions*, and *equations* (sessions 5 and 6), and *functional thinking* (sessions 7, 8, and 9). In the first and last sessions, the students' knowledge was assessed. Sessions 2-9 followed a similar structure, organized into three parts: (a) small groups (4-5 students), in which the aim was for the students to dialogue and collaborate with each other in the search for regularities, conjectures, and solutions to the problems presented; (b) whole group, where each group presented their findings and two teachers led the discussion so that the pupils synthesized their ideas; and (c) medium-sized groups (10-11 students), in which the objective was to transfer what had been discussed to another similar situation or to delve into a finding from the previous parts on the problems presented. Each part favored the installation of spaces for cooperation, confrontation, and discussion of ideas.

In this paper we focus on session 6, however, we will describe in general terms what was done previously, without considering the initial assessment. In session 2, students expressed their general ideas through natural language by arguing what happens when odd and even numbers are added. Then, in sessions 3 and 4 they discussed the relational meaning of the equal sign. Here, for the first time the letter is introduced as a representation for generalizing arithmetic properties (for example, the commutativity of addition). In this first encounter with the letter, the students concluded that it could represent any number (generalized number). In session 5, they were proposed an adaptation of the cards and envelopes problem described in Janßen and Radford (2015). The students were asked to express and solve the equation involved in the problem using manipulatives and pictorial representations. However, it became clear that it was necessary to deepen the approach to equations. Although they were able to represent them with manipulative material, with drawings and even with letters, doubts remained about the meaning of the indeterminate.

Data selection: 6th Session

In the sixth session, we pursued two learning objectives: (a) to represent everyday situations with algebraic notation, and (b) to use the letter as a representation of an indeterminate quantity. We analyze the result obtained in the first two parts (small group and whole group), due to the extension of this work. In these parts, we proposed two AWP, which were represented with natural language and the students had to translate and represent them with algebraic notation. The first AWP involved an unknown, had a unique solution, and it involves the structure $y+15=20$. The other AWP implicated two unknowns whose value could not be determined due to the lack of data in the statement and it involves the structure $y=20+b$. In Figure 1 the AWP and representations that the teacher raised for each situation are presented.

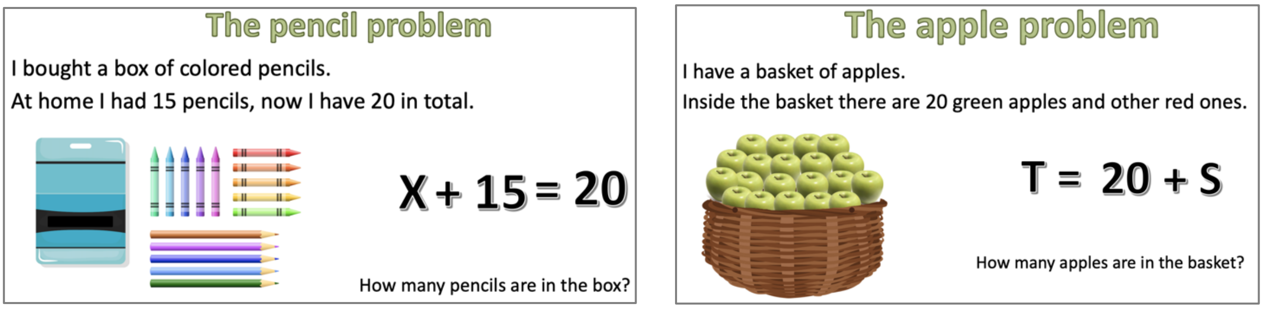

Figure 1: Representation of AWP introduced by the teacher.

The problems were presented following a similar structure. Each problem was presented orally and in writing, projected on the screen. Firstly, the students were asked to "tell the story" (i.e., to represent verbal sentences) with mathematical symbols. In the first instance, the undetermined quantities could be represented as they consider pertinent, then the possibility of representing them with letters was mentioned by the teacher and discussed within the group, as had been done in previous sessions. After reaching agreement on the representation of the problems in a group discussion, students solved each equation and discussed what the value of each unknown was.

Analysis

We analyze the students' written and oral responses to the two AWP presented. We focus on semantic congruence. Following the ideas of Duval's (2006) proposal, we describe the existing correspondence between the AWP and the algebraic expressions that they propose. Specifically, we focus our attention on: (a) semantic correspondence between semantic units, (b) semantic univocity between representations; and (c) organization of relations between representational units. The three previous elements allowed us to indicate the presence or absence of semantic congruence.

Students were labeled as S_i, where i = 1, ..., 19.

RESULTS

In this section, we present the responses of the 19 students who participated in the session (two students were absent). Firstly, they worked on “the pencil problem” and then “the apple problem”. The main results obtained are presented below, following our research objective: to analyze how students represent and refer to indeterminate quantities when proposing equations associated with AWP.

The pencil problem

Here, students had to identify how many pencils were in the box and discuss the value of the unknown. Broadly speaking, of the 19 students who solved this problem, 10 of them used the letter to represent the problem with an equation, three used a question mark “?”, four students made drawings, one focused on a specific calculation and another student did not respond.

Concerning the congruence, four students came up with an equation consistent with the pencil problem. Eight students raised referentially equivalent equations, but not congruent since they do not meet the organization of relations between representational unit criteria. In this case, the structure represented by these students was 15+y=20. One student proposed the equation 15+a=35 and another wrote the calculation 15+5=20. In these last cases only semantic univocity is observed.

In terms of representing the unknown quantity (number of pencils in the box), 10 students used letters (a, b, and x) and another three used a question mark "?". Similarly, the 13 students refer to the letters or the symbol "?" as the number of pencils in the box that they do not know, i.e., they interpreted them as an indeterminate number. For example, S_3 proposes the equation b+15=20, he said: "I thought: he says that he bought a box, but you do not know how many pencils that box has, and that represents b. 15 in the house. I used b, but any letter can be any number." This last sentence highlights that he accepts that the representations of his classmates are also correct, even when different letters are used.

Regarding the equivalence of the equations of the form y+15=20 and 15+y=20, students were asked if they represented the same thing, although the order of the addends was different. S_8 pointed out: "if we change their place, it does not alter the result due to the commutative property." S_{15} also mentioned the commutative property to justify the equivalence between expressions.

Regarding the value represented by the unknown, at the end of the discussion of this problem, all the children agreed that the only possible answer was 5, and this was mainly verified by replacing the letter by the number or by subtracting, as shown in Figure 2.

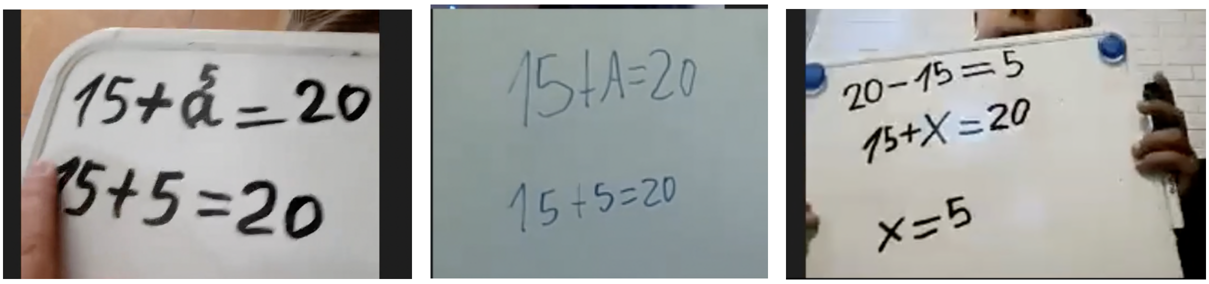

Figure 2: Representation and solution of the problem made by three schoolchildren.

The apple problem

This problem involves the structure y=20+b, and the idea was for the children to identify how to represent the total number of apples and discuss the value of the unknowns. Of the 19 students who solved this problem, 13 used the letter to represent the problem with an equation, three made drawings, one focused on a specific calculation, and two did not respond.

In the verbal statement of this problem, it was first mentioned that there were apples in a basket and then it was described what kind of apples they were, so the children were expected to represent the total of apples to the left of the equal sign to maintain the order of the statement. However, all the students represented it to their right, therefore, no expression was semantically congruent with the problem as the organization of relations between representational unit criteria was not met.

Three students proposed an equation of the form $y+20=b$ and seven students wrote an equation of the form $20+y=b$. Like in the previous problem, they recognized that these equations are equivalent and alluded to the commutative property of addition (explicitly or only mentioning that the order of the addends does not change the result). Two students represented the statement as $20+x=x$. This last equation does not meet the univocity criterion, since x represents the number of red apples and the total number of apples in the basket as well. In a discussion, S_6 said that it cannot be possible and, referring to indeterminate quantities as if they were known, pointed out: "when you add the letter it will give you a result and it cannot be because they will give different numbers". S_9 added "it must be another symbol otherwise the result would be the same as the red [apples]. Instead of x in the sum, you change it to a". This discussion was ended by S_8, who suggested that there were five red apples to show that the result would be a quantity other than five.

One student answered, *"20 + x="* and pointed out that he cannot complete the expression because he did not yet know how many red apples there were, therefore, he cannot write any. Another student wrote $20+n=40$, in this case he interpreted that there were 20 apples of each type, so his answer did not correspond to the statement. Something similar happened with another student, who answered: 10+10=20.

On the way to represent indeterminate quantities, as in the previous problem, the students recognize that they can use different letters. However, seven students used a question mark "?" to refer to the total number of apples in the basket. For example, S_6 said that "?" represents "the result that we do not know." Another example is S_{15}'s answer shown in Figure 3. She said: "I put the green ones with a black marker and the red one like the red apples. But we do not know how many red apples there are, and we do not know how much it gives us".

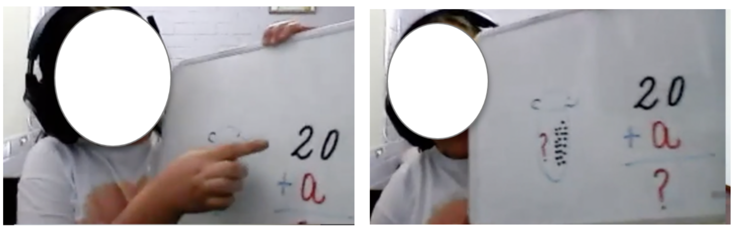

Figure 3: translation of a word problem into mathematical language.

Finally, after representing the problem statement, the student agreed that they cannot know how many apples were in the basket, therefore, they cannot know the value of the unknown quantities. S_{18} said that there was not enough information. S_8 said that "we are missing a clue. Example, how much would it give or how much was the result". To which S_7 and S_6 respond by pointing out that the number of red apples could also be said, for example 20 or 19. S_5, without referring to a certain quantity, argued: "it has to be greater than 21 for T, because it says 'others', so it has to be more than 1 [referring to the number of red apples]. In this case the letter can have different values".

DISCUSSION AND CONCLUSIONS

The objective of this work was to analyze the translation process of AWP to algebraic expressions. To achieve this, we focused on analyzing the way in which the students referred to and represented indeterminate quantities when posing equations associated with verbal sentences. The children identified the indeterminate quantities involved in the different contexts and represented them with letters or a question mark ("?"). In the process of establishing semantic congruence between natural language and symbolism, students expanded the meaning associated with indeterminate quantities by interpreting them as: (a) unknowns, which is evidenced when they had enough information in the statement and they interpreted the indeterminate as an unknown with a fixed value; and (b) variables, which is observed when they did not have enough information in the statement and argued that the value of the indeterminate depended on the values assigned to any of the quantities that they did not know.

Previous studies have already highlighted the importance of relying on everyday experiences, appropriate for the age, and possible to be approached by different students using their own natural intuition to develop algebraic thinking in the first school years (Molina & Castro, 2021). In this study, it is observed that family contexts and lower linguistic complexity helped children to visualize the relationship between each of the significant units expressed both in natural language and in symbolic language (semantic correspondence between semantic units). Just thinking about whether the expressions told the story mentioned in the problem motivated them and helped them discuss the relevance of using certain symbols in their equations. For example, they agreed that the same letter could not be used to represent different things in the problem (semantic univocity between representations) or that the order in which the addends are written does not have the same order in the problem statement (same organization of relations between representational unit), but it is mathematically correct due to the commutativity of addition.

The indeterminate, by becoming an object of thought of the pupils, gave the additive problems an algebraic character. Our proposal is to motivate students to understand the problem and express it algebraically before seeking its solution. Once they accept and understand how to correctly translate each statement, the next step would be to focus on the resolution strategies. In previous research, it can be observed that students of similar ages solve arithmetic-algebraic problems. However, they cannot express them with algebraic symbols (Fritzlar & Karpinski-Siebold, 2018).

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INTRODUCING MULTIPLES AND SUBMULTIPLES TO PRE-PRIMARY CHILDREN

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University of Ioannina, Greece

We present part of an ongoing topic-specific design research aiming at enhancing pre-primary students' multiplicative reasoning. We focus on an activity introducing vocabulary for multiples and submultiples with a view to enable children to express multiplicative relations verbally. We present the rationale and the design of the activity and findings of its first enactment with eight kindergarten children. The children appropriated the intended terms for multiples and, to a lesser extent, for submultiples as well, and used them to express multiplicative relations. The affordances and limitations of the activity are discussed, with a view to re-design it.

THEORETICAL FRAMEWORK

Research evidence indicates that there are early competences pertaining to multiplicative reasoning. Young children can trace multiplicative/proportional relations at a rudimentary level (McCrink & Spelke, 2016; Mix, Huttenlocher, & Levine, 2002); and can handle simple multiplicative situations involving discrete as well as continuous quantities (Hunting & Davis, 1991). These competences are limited; but may be enhanced if children are exposed to relevant informal and formal experiences (Hunting & Davis, 1991; Van den Heuvel-Panhuizen & Elia, 2020).

Such evidence has not been fully exploited in early childhood education even though learning objectives regarding multiplicative reasoning are included in mathematics curricula. For example, an analysis of a Greek mathematics curriculum (K-2) (Vamvakoussi & Kaldrimidou, 2018) has shown that learning objectives pertaining to additive reasoning were far more and were allocated far more teaching time, than the ones for multiplicative reasoning; and the latter referred far more to discrete, than to continuous, quantities. Moreover, the vocabulary necessary to express multiplicative relations was very limited. In kindergarten, in particular, all explicit learning objectives pertaining to multiplicative reasoning were placed in the context of discrete quantities, referring to the "equal groups" structure and the three problems that stem from this situation (corresponding to multiplication, partitive and quotitive division); and no term for multiplicative relations was mentioned, not even the word "half".

It is true that multiplicative situations that involve continuous quantities are more challenging. For example, in fair-sharing situations with discrete quantities, strategies like "dealing" one by one are accessible to young children. Such relatively simple strategies are not available for continuous quantities. Nevertheless, children appear to grasp underlying principles in multiplicative situations (e.g., "more recipients, smaller share") simultaneously for discrete and continuous quantities (Kornillaki & Nunes,

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 299-306). PME.

2005). For different, albeit complementary, reasons, several researchers suggest that discrete and continuous quantities should be treated similarly. For example, Steffe (2013) argues that the construction of both types of quantity is based on the same mental operation, namely unitizing; and that counting schemes can be qualified as measuring schemes. Sophian (2004) argues that the role of the unit in counting is more similar to the role of unit in measuring than typically assumed. Vamvakoussi and Kaldrimidou (2018) stress that three fundamental multiplicative operations—iteration of a quantity, equi-partitioning, and measuring with different units—are the same for both types of quantity.

On the other hand, linguistic tools are important so that children can identify the same multiplicative relation in different contexts and be able to organize their formal or informal experiences of multiplicative situations (Hunting & Davis, 1991). Indeed, there is evidence that knowing simple words for multiples (e.g., "double") in the first grade is associated with higher multiplicative/proportional reasoning competences in the second grade (Vanluydt, Supply, Verschaffel, & Van Dooren, 2021).

This paper presents part of an ongoing topic-specific design research (Gravemeijer & Prediger, 2019) aiming at developing a program of activities to support multiplicative reasoning at the first years of instruction. The three pillars of the program are a) addressing discrete and continuous quantities in a unified manner, b) providing experiences that pertain to three fundamental multiplicative operations (equi-partitioning, iteration of a unit and counting with various units (i.e., composite, fractional), and c) introducing terms for multiples and submultiples. The program has already been enacted once with kindergarten children (Pitta, Kaldrimidou, & Vamvakoussi, 2021), and has been redesigned, based on the findings. This paper focuses on a new activity intended as the introductory one in the current version of the program. This activity introduces vocabulary for multiples and submultiples with the purpose of describing multiplicative relations among the natural numbers 1-10—represented as continuous quantities—and the unit (1). The activity aims at capitalizing on children's experiences with the sequence of natural numbers, as cardinal and ordinal numbers, and at grounding the intended multiplicative relations on measurement. Our local hypothesis (for this activity) was that the regularities underlying the natural number sequence as well as the production of Greek words for multiples and submultiples would support students to learn and produce such terms; and that multiplicative comparison via measurement would help them assign meaning to the terms.

It should be noted that Greek words for multiples are produced with a prefix that is the main part of the corresponding number word, and an invariant suffix (-"(a)plasio"). For example, "pente" is the Greek word for "five" – "pentaplasio" is the word for "quintuple". Although there are slight variations in some of the terms, the regularity is quite salient. On the other hand, the words for submultiples are produced by the word for "one" ("ena") and the corresponding ordinal number (similarly to English), also for

1/2 (unlike English). Although 1/2 can be verbalized as "one second" in Greek, a different word, similar to "half" ("miso") is widely used.

METHOD

Design of the activity

The activity is story-based. The characters are imaginary creatures that play a sports game (two teams, "Pam's team" and "Pom's team"). The "players" are wooden cylinders with the same diameter but different height. Each team has a "captain" (Pam and Pom) representing the unit (4cm and 3cm high, respectively); and nine more players representing the numbers 2-10 with corresponding heights (multiples of the units). Each player has a paper "team shirt", bearing the syllable "PAM" and a void to be filled by the players' number (---PAM). Several copies of units are available.

A key part of the activity is naming the players: The 1st player is introduced as "Pam" and the children are asked to fill "the captain's number" (1) in her shirt. The 2nd player is introduced as "Pam-Pam", the 3rd as "Pam-Pam-Pam" and so on. The children are asked to a) to call the players by their names, clapping their hands with every "Pam", b) write their numbers on their team shirts, and c) predict the name of subsequent players from the third one on. As the names become longer and more difficult to say, the nickname of each player is introduced which is synthesized by the part of the number words that serves as prefix for the corresponding multiple (hereafter symbolized by *n*) and the suffix "Pam" (*2*-Pam, *3*-Pam, etc.). Then children are asked to compare the players with Pam (e.g., "how many Pams does it take to measure *3*-Pam?"), using the available materials. The relationship is described symmetrically (e.g., "It takes 3 Pams to measure *3*-Pam"/ "Pam is one out of the three parts of *3*-Pam". Then the terms for multiples and submultiples are also introduced symmetrically: The captain and the player verbalize their relationship (e.g., "I am the *3*-multiple of you" / "I am the one third of you"). In the process, the task for the children varies. One of the following is given and the rest are asked: a) the player as a physical object, b) the full name of the player, c) the players' nickname, and d) the number on its team shirt. We note that if (a) is given, then children need to measure its height by the unit (length of Pam); and if (b), (c), or (d) is given, then children need to iterate the unit to find (a). A constant task across all variations is to find and express verbally the relation between the "player" and the "captain", which is eventually explicitly asked using the word "relation". The activity is repeated with "Pom's team" (same multiplicative relations, different unit).

Participants

The participants were eight children (five girls), Greek native speakers with a mean age of 5 years and 9 months (ages varying from 5 years and 7 months to 6 years and 4 months). All children were students of a private Kindergarten in Ioannina. The children and their parents consented to their participation in the study.

Procedure

The children were tested via individual interviews before the intervention in order to investigate their prior knowledge on terms related to multiples and submultiples. They were familiar only with the term "half" but had no means available to explain or show what "half" means.

Our initial intention was to implement the entire program of activities, but this was not possible due to the pandemic. The introductory activity was implemented in May 2021, when schools opened after the lockdown. The children participated in groups (two groups, 4 members each) in the course of four weeks (one session per week for each group, about 45' each).

A retention test was conducted for every child individually, after the summer break (approximately four months later). The retention test was framed in the same context as the activity. Pam's team was presented to the child along with their team shirts, at random order. The researcher gave one player to the child who was asked to recognize the player and find her team shirt (Task A); verify/explain their answer (Task B); and express the multiplicative relation between the player's and the captain's heights (Task C). In total, 3 players were used—*2*-, *3*-, and *4*-Pam—addressing the relations 2:1/ 1:2, 3:1 /1:3, and 4:1/1:4, respectively; and corresponding to three trials for the tasks A, B, and C (A_k, B_k, and C_k, k=1, 2, 3).

RESULTS

During the intervention

The children indeed employed the natural number sequence to predict the name of the "next player" (e.g., Kostas below); they also started naming players beyond the given ones (e.g., Penny below):

> Kostas: Miss, I know who this is. It's *4*-Pam. You see, these are one-two-three [*pointing to the players already on the table*] and then comes four. So, this one is four.
>
> Penny: Let's get a really big one, *24*-Pam.

They also employed the recursive rule underlying the natural number sequence to predict the height of the "next player". For example, Aris selected the 5th player and explained how:

> Aris: I counted. The previous was the fourth one, so now I need her plus another one. To reach the fifth one [*illustrates placing a copy of the unit on top of 4-Pam*]
>
> Researcher: How many Pams do you need to get from *5*-Pam to *6-Pam?*
>
> Aris: One. Each time we must put one like this [*pointing to a copy of the unit*] on top.

With the players' nicknames at hand, and after the introduction of the term for "double" (*2*-multiple), the children quite readily adjusted the players' nicknames to terms for multiples:

Researcher: We said that *2*-Pam is the *2*-multiple of Pam. Could you tell me what is the relation of *3*-Pam to Pam?

Penny- Kostas: She is the *3*-multiple! [*together*].

On the other hand, the terms for submultiples proved quite challenging. The main obstacle was the over-use of the word "half". For example:

Researcher: Now, could you tell me what is the relation of Pam to *3*-Pam?

Lora- Kostas: She is the one-half of her [t*ogether*]

Researcher: One half? But we needed three pieces to make *3*-Pam.

Penny: She is the three-half of her.

We reasoned that this obstacle was probably due to our decision to use the word "half" for the very first submultiple, although it is very different from the subsequent ones, because we knew it was more familiar to the children. In light of this realization, 1/2 was re-expressed as "one second", and the relevance of the sequence of ordinal numbers was brought to the children's attention. This was indeed helpful for the children; still, multiples remained easier to produce and use than submultiples, as illustrated in the following excerpt:

Researcher: So, what is the relation of *5*-Pam, the taller one, to Pam?

Aris: She is the *5*-multiple.

Researcher: And how about Pam? What's her relation to *5*-Pam?

Aris: One fourth.

Researcher: How many Pams do they fit in *5*-Pam?.

Soti: I know, I know! It's one fifth.

During the intervention the children gradually started to use the new terms more accurately, connecting them to the two fundamental operations (measurement and iteration of a quantity) to respond to the tasks and justify their answers. In the following examples, Lora verbalizes and explains the relation of *4*-Pam to Pam; Kostas gets the team shirt of *8*-Pam and explains how he is going to find this player and how she is related to Pam; and Nikos verbalizes and explains the relation of Pam to *3*-Pam:

Lora: *4*-Pam is the *4*-multiple of Pam because you need to stack four Pams to make *4*-Pam, Pam-Pam-Pam-Pam [*illustrating with the materials*].

Kostas: I'll build her first. I don't know how tall she's going to be. I need eight like these [*points to the copies of the unit, then stacks eight units*]. It's the *8*-multiple. It's eight Pams.

Nikos: We need three Pams to measure the big one. This one [*pointing to Pam*] is one out of the three pieces of that one [*pointing to 3-Pam*]. She is the one third of her.

Retention test

In task A, a response was coded as 1 if the child had recognized the player, otherwise as 0. In task B, a response was coded as 1 if the child had implemented a valid strategy to verify or explain her/his answer, otherwise as 0. In task C, we first examined whether the child used or not (coded as 1/0, respectively) the terms for multiples and submultiples to describe the intended relations. Then we examined how the terms were explained, or the relations described (in case the terms were not used).

In the case of multiples, we identified three types of explanation that were coded as follows: M1-Repeated Addition (e.g., "*3*-Pam is one Pam, and another one, and another one"); M2-Measurement (e.g., "This is *2*-Pam, because it contains 2 Pams" or "*2*-Pam is two Pams"); M3: Multiplicative comparison (e.g., "She is three times as Pam"). In the case of submultiples, three types of explanation were identified and coded as follows: S1- Part/Whole Relation (e.g., "She is half, because she is one of the other's two pieces"); S2-Measurement (e.g., "One fourth! Because you can fit four Pams into *4*-Pam"); S3 -Combination of S1 and S2 (e.g., "*2*-Pam contains two pieces, and one of them is half").

Table 1 presents the responses per child, across the three trials A_k, B_k, and $C_{k,}$ of the tasks, corresponding to the relations examined. $C_{k.1}$ and $C_{k2.}$ refer to multiples and submultiples, respectively; and are assigned two codes each, one for response, and one for type of explanation.

As can be noticed in Table 1, one child (Heleni) was unwilling to respond to any of the tasks; she related that she just wanted "to play with Pam's team". All the remaining children identified the given players (A_k); used a valid strategy (i.e., measuring or repeating a quantity) to verify their answers (B_k); and used all terms for multiples ($C_{k.1}$). In addition, all children explained adequately the terms for multiples ($C_{k.2}$). Two used systematically the same type of explanation (M1 for Kostas and M3 for Peny). With the exception of Soti, who never used M3 explanations, the remaining children moved from M1 or M2 to M3, for "bigger" multiples.

With respect to submultiples, there was more variation among the children. All used a term in $C_{1.2}$, and all preferred the word "half". Only three, however, explained their answer. Overall, four children used all terms correctly, but only two of them (Aris and Yianna) also explained all their responses (via S3). Nikos did not provide any explanation, while Peny explained "one third" and "one fourth", but not "half".

Lora used the terms "half" and "one third" but only explained the second term. Finally, Kostas and Soti did not use any term other than "half", which Kostas also explained. It is worth noting that these three children constructed their own terms for the ones they missed, mis-using the word "half". For example, Lora said "half of fours" instead of "one fourth".

Child	Relation: 1:2/ 2:1						Relation: 1:3/ 3:1						Relation: 1:4/ 4:1					
	A_1	B_1	C_1				A_2	B_2	C_2				A_3	B_3	C_3			
			$C_{1.1}$		$C_{1.2}$				$C_{2.1}$		$C_{2.2}$				$C_{3.1}$		$C_{3.2}$	
Nikos	1	1	1	M1	1	-	1	1	1	M3	1	-	1	1	1	M3	1	-
Lora	1	1	1	M1	1	-	1	1	1	M3	1	S2	1	1	1	M3	0	-
Kostas	1	1	1	M1	1	S1	1	1	1	M1	0	-	1	1	1	M1	0	-
Penny	1	1	1	M3	1	-	1	1	1	M3	1	S2	1	1	1	M3	1	S2
Aris	1	1	1	M1	1	S3	1	1	1	M3	1	S3	1	1	1	M3	1	S3
Soti	1	1	1	M1	1	-	1	1	1	M2	0	-	1	1	1	M2	0	-
Yianna	1	1	1	M2	1	S3	1	1	1	M3	1	S3	1	1	1	M3	1	S3
Heleni	0	0	0	-	0	-	0	0	0	-	0	-	0	0	0	-	0	-

Table 1: Responses and types of explanation per child in the retention test.

CONCLUSIONS-DISCUSSION

The first enactment of the sequence indicated that—as we expected—the children employed their knowledge end experiences of the sequence of natural numbers, as cardinal and as ordinal numbers; and they also discerned the regularities in the production of the Greek words for multiples to learn and produce such terms. The introduction of the terms for submultiples was challenging. We attributed the difficulty to our (rather unfortunate) decision to harvest the children's informal knowledge of the term "half" to introduce the very first term. This could explain the misuse of "half" in the production of subsequent terms, which persisted for some children up until the moment of the retention test. It has been suggested that the word "half" as well as the particular relation are privileged but may become an obstacle in the long run (Hunting & Davis, 1991). Using from the beginning and systematically the alternative term ("one second" in Greek) that is compatible with the subsequent ones may prove helpful. We had such indications during the intervention, and we will take them into consideration in re-designing the activity.

A promising finding is that the children appropriated the intended multiplicative operations and employed them to deal with the tasks, and to explain their answers. The retention test showed that these competences were retained for multiples, with the observation that "larger" multiples triggered more elaborate explanations, which will be take into consideration in re-designing the activity. On the other hand, in the case of submultiples, there were many limitations, and differences among the children, with respect to the use of terms and also the explanations. We note that the use of terms for submultiples did not necessarily imply that an explanation was provided; however, a relation was never described, if the appropriate term had not been used. This is an indication of the supporting role of relevant vocabulary for multiplicative reasoning, consistent with the findings by Vanluydt and colleagues (2021).

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TEACHERS' PD UPTAKE: HOW VISUAL REPRESENTATIONS IMPACTED MATHEMATICS TEACHING

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This study provides an in-depth examination of two teachers who participated in a mathematics professional development project that focused on linguistically responsive teaching and what the teachers took up and used in their classrooms 3-4 years after their participation in the project. Survey, interview and classroom video data were analysed in order to explore the ways in which the teachers' learning from the PD endured over time. Results indicate that the teachers remembered, continued to use and hone their use of visual representations as a strategy to provide access to English learners. These strategies and implementation use were aligned to the goals and intention of the PD. Furthermore, they extended and transferred this knowledge to other content areas and to remote teaching settings.

INTRODUCTION

High-quality professional development (PD) in mathematics education is considered the key to improvement of students' mathematics learning. One commonly accepted trajectory of teaching and learning suggests that gains in teacher knowledge can lead to changes in instruction, which in turn has a positive impact on student learning (e.g., Yoon et al., 2007). Current trends related to research on mathematics PD have started to show more evidence of change, mostly incremental changes in teachers' knowledge and instruction, and somewhat less in terms of the more distal, student achievement (Jacobs, et al., 2019). We posit that teacher learning and implementation of new ideas and strategies takes time. This paper examines the instructional changes related to the goals and intentions of a US nationally funded PD that focused on linguistically responsive teaching (LRT) 3-4 years after participating in the project. Specifically, the research question that guided this work is: What specified and intentional instructional LRT practices did the teachers take up and use 3-4 years post PD experience?

THEORETICAL FRAMEWORK

Mathematical Knowledge for Teaching – Visual Representations (MKT-VR)

Ball and Bass (2000) were the first to coin and define *mathematical knowledge for teaching* (MKT), the requisite knowledge to effectively teach mathematics to K-12 students. This knowledge is complex and includes both content and pedagogical knowledge and provides the field with baseline knowledge to focus on in teacher preparation programs and professional development experiences for teachers (Jacob et al.,2017). DePiper and Driscoll (2018) were inspired to think about the need to further theorize the MKT constructs and created the MKT-VR theoretical framework. They define visual representations (VRs) as graphic creations such as diagrams or drawings

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that illustrate quantities, quantitative relationships or geometric relationships. Using models or representations is an important component of doing mathematics and especially important for English learners, a cornerstone for LRT (DePiper et al., 2021). VRs support students to make sense of problems by identifying quantities and the relationships between quantities in order to use VRs to reason with and ultimately justify mathematical solutions (Ng & Lee, 2009). However, this knowledge is complex and requires challenging skills including a strong grasp of the content, anticipation of students' thinking, and selecting the most appropriate VRs for particular purposes.

OVERVIEW OF TADD PROJECT

This paper highlights a project that is part of a large three-year impact study, *Taking a Deep Dive* (TaDD), that collects qualitative data from three large U.S. National Science Foundation PD projects in order understand what teachers take up and use and the factors that influence uptake 3-4 years post PD experience.

Visual Access to Mathematics (VAM)

The VAM PD, the focus PD of this paper, is a "60-hour blended, face to face and online course to build teachers' knowledge of and self-efficacy about LRT strategies to strengthen English Learners (Els) problem solving and discourse in middle grades" (De Piper et al., 2021 p. 491). The goals and intentions of VAM were to cultivate in teachers the fluent use of representations, anticipation of students' strategies, the ability to interpret and construct various mathematical solutions, and to reason with and across representations. Teachers learned how to strategically select and align VRs with their instructional goals, anticipate student thinking and misconceptions, and then implement lessons using these strategies in their classrooms. Once implemented they would share experiences and student work, and collaboratively and independently reflect on the teaching cycle in the VAM PD online workshops.

In particular, VAM focused on two VRs, the double number line (DNL) and tape diagrams. Both VRs are effective tools that have the potential to foster students' understanding of proportional reasoning and reinforce students' conceptual understanding of rational numbers (DePiper & Driscoll, 2018). The DNL is a representation that uses a pair of parallel lines to represent equivalent ratios. Tape diagrams, also referred to as bar diagrams, are rectangular representations that illustrate number relationships. Both diagrams represent quantities and the relationships between quantities, allow students that think more additively to "see" multiplicative relationships and the relationships between quantities with the representation. VAM focused on problem solving with rational number tasks that were easily represented on a DNL or tape diagram and actually many different representations. Subsequently these VRs were used as a communication tool to show and explain students' mathematical thinking in a very concrete and conceptual manner. We selected VAM as a case because the PD illustrates evidence of high uptake and at the same time provides evidence that teachers continued to hone their craft, modify and expand their use of VRs using different mathematical problems, domains and contexts.

Participants and Settings

This paper focuses on two teachers, Kimberly and Rachel (pseudonyms), 3-4 years post participation in the VAM PD to address the research question: *What specified and intentional instructional LRT practices did teachers take up and use 3-4 years post the PD experience?* We purposefully selected two teachers that showed high levels of uptake in the surveys we initially administered (n = 66) across all three projects. We wanted to dig deeper to better understand the factors that influenced their high levels of uptake. They were also chosen because they used and modified strategies and VRs learned in the PD while teaching remotely during the COVID 19 pandemic. Although the intent of this project was not to study remote teaching and learning, the pandemic changed the nature of the classroom data we collected and allowed us to explore the ways in which participating teachers used and modified representations in the online setting. Both teachers taught middle school mathematics in the northeast US and both used online platforms to teach synchronous mathematics lessons to their students in the data collected in this study.

Data Collection and Analysis

Multiple data sources were collected in order to comprehensively understand of the participants' uptake and enable triangulation of findings (Cresswell & Poth, 2018). These data sources included survey data, interviews, and videotapes of classroom instruction. The survey included questions that asked participants to reflect on their PD experience and characterize their past and/or current use of the PD content, pedagogy and materials. The survey included both Likert scale questions, in which participants responded to statements on a scale of 1-10, as well as follow up questions that allowed the participants to explain and provide more details about their numeric responses.

Following the completion of the survey, 17 case study teachers from across the three projects were asked to videotape their classroom once a month and identify clips in which they believed that they were using content, pedagogy and/or resources from the PD they participated in. Kimberly and Rachel were interviewed twice during spring and fall 2021 school year by the TaDD project research team for approximately one hour each. The first part of these interviews included questions aimed to understand the teachers' experiences with the PD, what they remembered related to the goals and intentions of the PD and what strategies, content and resources they used from the PD in the past and continue to use currently in their classrooms. The second part of these interviews followed a *think aloud* protocol, where teachers watched video clips they selected and described their perceived uptake and implementation of content, pedagogy, and resources from the PD. These interviews were recorded on Zoom and transcribed.

At least two of the research team members reviewed and took detailed notes on the survey, interview and video data several times to create a profile for each teacher. These profiles were analysed and coded for segments that related to participants' use of representations using the MKT-VR theoretical framework. Sample codes included use of DNL, use of tape diagrams, alignment of DNL/tape diagrams with teaching goals, and use of VRs to represent quantitative relationships. The members of the research team then met to discuss what they noticed, and identified salient themes and patterns that emerged. The three themes that emerged were related to (a) knowledge and uptake of new VRs, (b) use of VRs in remote settings and (c) application of VRs to other content areas. Findings from all the varied sources were validated through a triangulation process. For example, data from the individual teacher's surveys, interviews and classroom videos were matched for convergence and divergence. Following this, narratives were written for each of the three themes and reviewed for consistency and alignment across the data sources.

FINDINGS

Knowledge and uptake of new visual representations

Survey Data. Kimberly and Rachel both perceived high levels of uptake in terms of content, pedagogy and resources in their initial survey responses. They mentioned that they learned new VRs in VAM, in particular the DNL and rectangular tape diagrams. Rachel mentioned that she saw a DNL prior to VAM and wrote:

> I never really understood the purpose of them until I saw how many different ways they can be used to represent a situation and solve problems.

Kimberly reported that she enjoyed the DNL activities and shared the applet with her students. From both teachers indicated on their surveys that they implemented the VRs and perceived them as relevant and helpful teaching strategies.

Interview Data. Prior to participating in the PD, neither participant was familiar with using these VRs to solve problems and had not used them in their classrooms. In her interview, Kimberly said:

> I remember when the Common Core Standards came out and we were like, ‘What’s a tape diagram? I don't know what a double number line is. How can I figure out how to use these tools in my classroom?

Similarly, Rachel mentioned in her interview:

> I didn’t know a lot of the representations that they were teaching us [in VAM]. I had been teaching middle school math for 8 years and I had never used a DNL. I was solving these problems using proportions or equations and I never knew this thing existed.

Prior to the PD, Rachel mentioned that she used equations to solve proportions, as she did not know about the VRs options. This aligns with the research indicating that when teachers are unfamiliar with how to use these tools they typically rely on algorithmic thinking to solve these types of ratio and proportional reasoning problems (Orrill & Brown, 2012).

After the PD, Kimberly and Rachel began to incorporate these representations in their teaching and continue to use them. Interview data reflected their uptake of the tape diagram and DNL in their classrooms. For example, Rachel noted in her interview:

> I started to take the DNL and completely change the way I teach ratio and proportion and percent and I started to use the tape diagrams and the DNL for everything. I still am using the materials from VAM for those units.

<u>Video Data</u>. The classroom video data aligns with the perceptions they shared with us during the interview and on their surveys about their use of the representations in their classrooms. Video clips include examples of them modelling the creation of DNLs and tape diagrams, asking students to examine different tape diagrams and DNLs and having students create their own tape diagrams and DNLs. When talking about one of her video clips, Rachel discussed her use of the DNL in a unit rate lesson and how she would never have thought to use it before VAM. She mentioned that her goal in the lesson was to help students make connections between what was going on in the context of the problem and the visual representation of it.

The two teachers also strategically chose VRs and used them to promote quantitative reasoning. They encouraged their students to use the diagrams to reason about the relationships among quantities in the problems. The teachers also articulated why they chose to use certain representations. In discussing one of her classroom clips in which she displayed tape diagrams to represent different percent amounts, Kimberly noted:

> In VAM, we used a lot of tape diagrams which I don't think I had used as much beforehand. So that was a newer model to me. I think a lot of teachers want to go back to the pie, you know the fraction pie, but I think the tape diagram leads us to the DNL which is so useful with percent so I definitely took that away-- starting with that tape diagram talking about percent and fractions and leading them to the DNL later on.

Both teachers demonstrate this strategic use of representations throughout their video clips and explained how their choice of representations in their lessons align to their instructional goals in the interview data.

Using visual representations in a remote setting

<u>Video Data</u>. Kimberly and Rachel adapted their use of VRs to their remote settings. In some instances, they had students draw VRs on their papers and hold them up to the camera. In other situations, they used online platforms such as PearDeck to display teacher created representations as well as to allow students to create and display representations to the class. One feature of these online tools that they noted was to the ability to give immediate feedback to the students about their representations. The teachers saw the students' diagrams in real time and gave feedback on the labelling or accuracy of the representation. The video of the two teachers showed their use of these tools to strategically select students' representations to share with the class, which allowed students to communicate their mathematical thinking.

<u>Interview Data</u>. Both participants noted in the discussion of their video clips is that one of the ideas that resonated with them from the VAM PD was the use of VRs as communication tools. Kimberly mentioned:

> I think one thing I took away was getting kids to talk about their models or look at each other's models because I don't think we always did that before. We had them draw their models and then that was it. But I think one thing I really like to do is have kids draw a model, and then have somebody else look at it and try to see what connections that student made. That was one thing we definitely did during the PD that I continue to do.

In one of Kimberly's classroom videos, she demonstrated how she used representations as a communication tool by having students sometimes hold up the VRs they drew and at other times, displaying one of the VRs on the slides they had created from the online PearDeck slides. Then, another student explained the diagram that was displayed. Similarly, in one of Rachel's videos, she selected and displayed two tape diagrams students created during the lesson. She then asked the class to compare them and discuss what they noticed and what they would change about the representations. In the interview, both teachers noted how the online tools made it easy to select and display students' representations.

The two teachers also noted some challenges that occurred in the remote setting. One was related to creating classroom records of work. Kimberly mentioned:

> I usually annotate a representation as a student is explaining in the classroom and leave that up and that's harder to do in the electronic world as they sort of disappear as we go slide to slide.

They also discussed the challenges of group work in an online setting and the ways in which they need to adapt tasks and use technology to address these challenges.

Application of representations to other content areas

<u>Interview Data</u>. Although the VAM PD focused on proportional reasoning, interview and video data from both participants indicated that they applied what they learned about VRs to other content areas. Kimberly noted:

> I like to have visual representations in everything we do. The VAM PD focused on ratio/proportion but I have taken it though the whole curriculum. They didn't give us models for other topics but when I see them I know what they should look like. You leave with this idea of this is what a good model looks like.

Kimberly added that she now uses mobiles to teach algebra because they allow students to develop a visual understanding of the quantities in an algebraic equation.

<u>Video Data</u>. Similarly, a video clip from Rachel demonstrated how she uses tape diagrams in her algebra unit. She noted:

> I have never taught tape diagrams with expressions and equations until this year. But it was a really interesting way to think about not just the distributive property (to show them how you can chunk things up differently and still get the same value) but in terms of solving equations thinking of what you can get rid of immediately and what you have left.

Years after the PD, she incorporates what she learned about VRs in new ways and applies these ideas to new content areas. While the VAM PD did not focus representations in these content areas, the teachers credited the PD with their extension of representations throughout their other units.

DISCUSSION AND IMPLICATIONS

This in-depth examination of teachers' classroom practices 3-4 years after participating in the VAM PD showed that their uptake of VRs continued in ways that connected to the goals and intentions of the VAM PD. The two teachers explain and demonstrate the ways in which they select and use specific representations, in particular tape diagrams and DNLs, in their classrooms. Classroom video data supports their survey and interview data and illustrates the ways in which these teachers use VRs to foster an understanding of important mathematical ideas related to ratio and proportional reasoning. The teachers also use representations to assist student' communication and explanation of their mathematical thinking. This knowledge and use of representations aligns to the goals and intentions of the PD developers.

These findings provide insight into the ways in which teachers continue to take up ideas that they learned in PD, years after their participation and the ways in which they adapt and apply them to novel contexts. In the case of these two teachers studied, knowledge related to VRs endured over time and is evidenced in their practice. Furthermore, while the initial PD did not focus on remote settings or content areas outside of proportional reasoning, these participants were able to transfer and extend their knowledge and use of VRs to these contexts. They also use online tools to provide immediate feedback on the accuracy and labelling of students' representations as well as to strategically select students' representations to share with the class. They incorporate the online tools to allow students to communicate their thinking related to the representations they create and to allow other students to comment and unpack the diagrams that their classmates create. This adaptation to remote learning aligns with the initial goals of the PD of having teachers strategically select VRs and use them as communication tools.

From our case studies presented here, it appears that learning strategies focused on LRT were transformative. Specifically, VRs added to teachers' pedagogical toolboxes in ways that cut across mathematical domains, types of teaching (online to face to face) and uses for different learners. We hypothesize that this type of flexibility that allows teachers to hone their craft over the four years and show high levels of uptake. Additionally, this PD was optional and teachers that elected to participate were teaching in schools with high numbers of Els so the strategies that they picked up were important for their context in order to reach and be successful with students learning English. These predictors of uptake need further examination to identify the other possible predictors of high levels of impact. All classroom video included in the analysis were of remote instruction and we are still unclear how this translates to in-person classroom practice. Our next steps are to continue this work as teachers transition back to in-person teaching. We also plan to conduct cross-case analyses in order to understand the similarities and differences that may exist across different PD projects. While our paper reports on a small-scale case study, it contributes to how we conceive and theorize about teacher learning and the importance of recognizing that learning happens over time. Over the past two decades, the studies on teacher PD that show incremental change are possibly the seeds which call for further exploration, as a pre- post randomized control study might not provide enough time for planning, implementation and reflection to support teacher learning.

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INITIATING A PROJECT FOR LANGUAGE-AND-LEARNER RESPONSIVENESS IN MATHEMATICS CONTENT TEACHING

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Although our community has come to know that language is an important resource for mathematics teaching and learning, there is a less fine-grained understanding of how developmental work with mathematics teachers can be designed to support content teacher talk that is language-and-learner responsive. In this report, we first discuss our theoretical framework with the tools of naming and lexicalization, interpreted as uses in classroom teacher talk of content-related word names and explanatory sentences with the potential to reduce specific learning difficulties. We then change the emphasis to explore challenges of thinking a version of the framework and tools for work with mathematics teachers in order to inform their decisions as to how and why selected names and/into explanations can be particularly responsive in content teaching.

OUR RESEARCH AND DEVELOPMENTAL PROJECT

The path from research findings and theories about mathematics teaching and language to practical proposals of developmental work with mathematics teachers on content teaching, or even vice versa, is not straightforward. Any formulation of how to traverse this path is problematic because theoretical tools may not be directly applicable or easily understandable in the developmental site. In the domain of mathematics teacher development, the common aspiration of producing research that can impact on the professional learning and knowledge of teachers, and ultimately on content teaching practices in classrooms, is nonetheless essential and remains a driving force (Adler, 2021). As we share this aspiration, we continue to draw on sociocultural approaches to language and mathematics (e.g., Pournara, Adler, Pillay, & Hodgen, 2015; Planas, Morgan, & Schütte, 2021) in our pursuit of content teaching that is language-and-learner responsive, or grounded on the provision of talk in the interaction with learners that is mathematically focused and responds to learning demands and challenges.

In this report, we present two theoretical tools in construction —*naming* and *lexicalization*—, and examine their potential and nuances for use in developmental work on language-and-learner responsive content teaching with mathematics teachers. These tools and the framework they conform were implemented implicitly rather than explicitly in a pilot intervention study with two secondary school teachers on the exploration of mathematical languages at the levels of word names and explanatory sentences for the teaching of algebraic concepts (Planas, 2019, 2021). At its actual stage of conceptualization, the framework and the tools reveal strong theoretical and practical interest. Whilst it is relatively uncomplicated to identify the potential of the

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 315-322). PME.

theoretical tools for reflection on specialized meaning making in language, it is not clear-cut the process towards specifying how to introduce them to teachers in ways that are not too highly conceptual to be practical for them, and that enhance content mathematics teaching aimed at the reduction of school learners' challenges.

Following this introduction, the report is structured to discuss two questions. In the first section, we discuss: 1) How do the theoretical tools of naming and lexicalization relate to the study of content teacher talk? In the second section, we discuss: 2) How can they be reinterpreted into developmental tools for work towards language-and-learner responsiveness in mathematics content teaching? We finish with some remarks about possibilities of continuing the refinement and expansion of the framework.

THEORETICAL TOOLS FOR THE STUDY OF TEACHER TALK

Our framework for the study of teacher content talk started to unravel backed up by intensive revision of literature on language and mathematics teaching and specifically grounded on Halliday's functional grammar (1985). Without diminishing the importance of nonverbal and paralinguistic tools in language, the intention was to increase the understanding of mathematical meaning making enhanced at the levels of words and sentences in classroom teacher talk. Earlier field research on mathematical meaning making in classroom talk (e.g. Pimm, 1987; Schleppegrell, 2007) already suggested the study of dense noun phrases, being and having verbs, conjunctions with technical meaning or logical connectors, which all fit into our focus on words and sentences. Today, analyses of mathematics teacher talk often privilege the study of conversational patterns and communicational moves and, when mathematical content specificity is also addressed, words and sentences tend to be studied in general terms and subsumed to, instead of interacting with, the broader discourse level. Instances of words and sentences are often illustrated and said to be mathematically and pedagogically relevant but the criteria of relevance are not detailed or focused on.

The interconnected distinction in Halliday (1985) between the linguistic forms in a language and their functions to produce situated meaning expresses the diverse ways by which words, sentences, and discourses in a language system and an interactional situation are lexically elaborated to communicate meaning (Morgan, 2021). Alongside the study of discourses or larger language units over isolated words, words into sentences, and sentences, meaning making crucially develops at granular linguistic levels. In this regard, the experiences of teachers and learners in classroom content teaching and learning are subject to the complexity of using words into/and sentences to communicate some meanings considered as (more) appropriate amongst all those possibly lexicalized —i.e., encoded with precise meaning— in the interaction and the language system. In Halliday (1978, p. 195), a register is precisely, "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings." In the mathematics classroom, the forms used to encode meanings within a mathematical content register may also bring with them less appropriate or unintended meanings. In order to address this complexity, we examined two tools in language; one at the level of the set of words or lexicon in a language system, the other at the level of further lexical elaboration into sentences:

- *Naming* or giving word names from mathematical content registers
- *Lexicalization* or giving sentences with encoded explanations of mathematical content meaning

In this version of the tools, responsiveness in teacher talk is basically a function of content specificity through the use of content-related names and explanations. In the current more advanced version, responsiveness in teacher talk is a function of content and learning specificity through the use of content-related names and explanations aimed at supporting content learning challenges (see Figure 1). The refinement to strengthen the emphasis on learner responsiveness has therefore led to:

- *Naming* or giving word names from mathematical content registers oriented to reduce content learning challenges
- *Lexicalization* or giving sentences with encoded explanations of mathematical content meaning oriented to reduce content learning challenges

Language-and-learner responsiveness in this way emphasizes the learning goal without losing the focus on curricular content demands. It connects mathematical meaning making in content teaching to mathematical meaning misunderstood or overlooked by learners. If we think of the teaching of fractions, for example, language-and-learner responsive names and explanations would address and challenge field-documented learning misunderstandings such as the common belief that the parts of the continuous whole are equal-shape (Darrough, 2015). Equal-size and (non)equal-shape would be instances of naming within the fraction register, and the equal-size parts of a continuous whole are not always equal-shape would be an instance of lexicalization. If we consider the teaching of angles, in a lesson with dynamic software where secondary school learners keep referring to angles as static bounded regions only (Mitchelmore & White, 2000), the rotation about a point also makes an angle, would be an instance of lexicalization including important names.

Language responsiveness and learner responsiveness are then different phenomena in content teacher talk with special connection between them. Language responsiveness in content teaching exists as soon as the language of mathematics is made explicit and public at the levels of words, sentences, and discourse, although it does not necessarily address the needs or demands of learners in content learning. Learner responsiveness therefore involves language responsiveness, but the converse cannot be argued.

CONTENT REGISTER	*Theoretical refinement*
Words Sentences *Form-function*	<u>Studying language responsiveness</u> Giving names from the content register Giving explanations of mathematical content meanings
Words Sentences	<u>Studying language-and-learner responsiveness</u> Giving names from the content register aimed at **reducing learning challenges** Giving explanations of mathematical content meanings aimed at **reducing learning challenges**

Figure 1: Successive versions of the theoretical framework.

TOWARDS A DEVELOPMENTAL VERSION OF THE FRAMEWORK

The reinterpretation of research tools for use in developmental practice implies shifts in meaning. This is the case with the reinterpretation in Adler (2021, p. 83) of "naming" —very close in meaning to our first version of naming— as *word use* in the teaching version of the Mathematics Discourse in Instruction frame. We rethink naming and lexicalization in mutually supportive ways, rather than treated separately, as *word names into/and explanatory sentences* (see Figure 2), whose communication in teacher talk can prevent or diminish learning challenges shown to be persistent across school ages, individual learners and classroom settings. Field research has actually documented numerous *reasoning biases* or tendencies of school learners to confirm and retain meanings, experiences and beliefs that do not conform or that enter in negative conflict with mathematical content. We have already mentioned biases in the thinking of: the fraction parts of the continuous whole as equal-shape (Darrington, 2014), and the angle as static bounded region only (Mitchelmore & White, 2000).

We assume that reasoning biases remain behind important content learning difficulties, and accordingly propose work on noticing processes (e.g., ZDM issue edited by Dindyal, Schack, Choy, & Sherin, 2021) with mathematics teachers towards:

- Knowing common *reasoning biases* of school learners, and considering their importance in mathematics content learning.
- Identifying, interpreting, and deciding on *names into/and explanations* for mathematics content teaching aimed at reducing biased reasoning.

In the progressive thinking of how to make operative the theoretical framework and tools (see Figure 2), the issue of how to produce knowledge-based *names into/and explanations* is crucial. The amount of mathematical meanings associated to each curricular content is enormous, and hence in the work with teachers some criteria must be given for the effective selection of some names and explanations over others.

Otherwise, the framework tools may remain too open to be fully useful or manageable. Although conditions posed to the choice of words and sentences can generally be read as limitations to creative teaching, conditions regarding the content learning challenges to be addressed positively relate to teacher talk of higher language-and-learner responsiveness. The attention to particular reasoning biases can especially help teachers to gain knowledge-based autonomy and to produce content-related names into/and explanations aimed at reducing or preventing the biases in play.

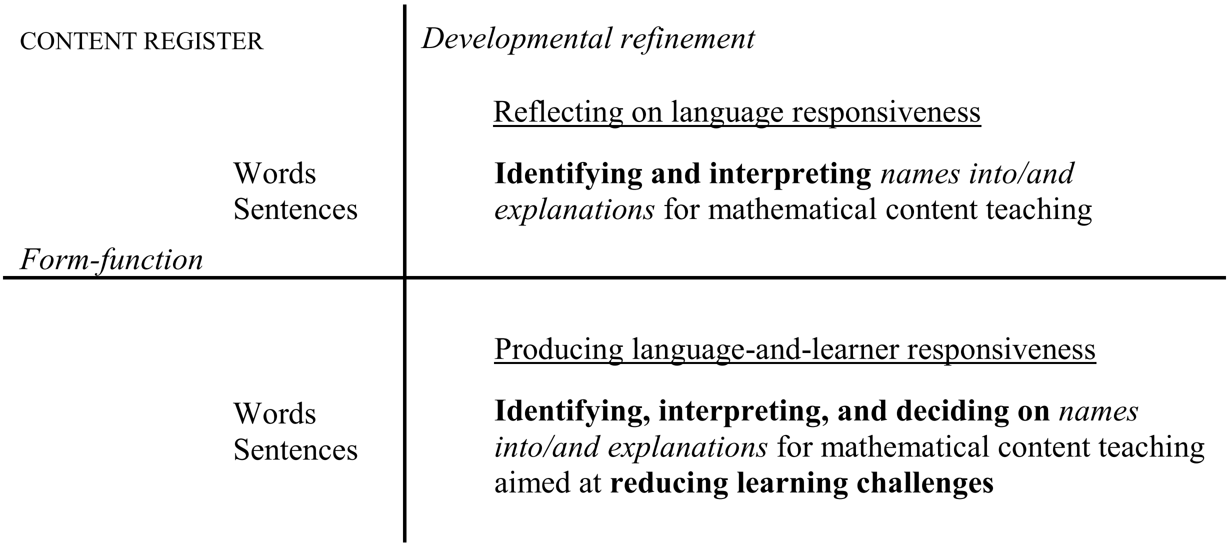

Figure 2: Successive versions of the developmental framework.

We cannot totally anticipate, accurately predict, or make a definite distinction of content teacher talk that will be learner-and-language responsive over the diverse interactional situations of a classroom lesson. Nonetheless, the curricular context and field-based knowledge can help to distinguish words and sentences which are expected to be responsive with respect to specific content learning demands and challenges.

In the upper secondary school classroom, for example, *the angle in between these lines measures one hundred and eighty degrees* is highly language-responsive, compared to *the angle in between these lines 'is' one hundred and eighty degrees*, or to *this is one hundred and eighty* (see Table 1). This explanatory sentence and the specialized names included, however, do not meet the particular challenge around the persistence of the static angle bias, compared to *the rotation from this line to this other line is half of a whole turn* —or to *the rotation about a point also makes an angle*—. Learner responsiveness makes these sentences qualitatively different (see Table 1). While all words and/into sentences in teacher content talk cannot be 'equally' responsive regarding particular registers and learning challenges, there must be some words and/into sentences offering opportunities for listening to specialized names and to explanations of mathematical meanings whose learning is possibly hindered by reasoning biases documented in field research as common and pervasive.

<table>
<tr><td colspan="4">Teaching angles in the upper secondary school – Static angle bias</td></tr>
<tr><td>Quality</td><td>Low</td><td>Medium</td><td>High</td></tr>
<tr><td>Language responsiveness</td><td>This is one hundred and eighty</td><td>The angle in between these lines is one hundred and eighty degrees</td><td>The angle in between these lines measures one hundred and eighty degrees</td></tr>
<tr><td>Learner responsiveness</td><td colspan="2">The angle in between these lines measures one hundred and eighty degrees</td><td>The rotation from this line to this other line is half of a whole turn</td></tr>
</table>

Table 1: Examples of variability of responsiveness in teacher talk.

Although incorrect reasoning biases in a content domain are persistent in nature, and preventing, reducing or even eliminating them require the adoption of multiple directions, teachers need to develop the ability to identify, interpret, and decide on classroom talk that refers to, for example, the dynamic meaning for angle, or to the meaning of equal sizes of unequal shape for the fractional parts of a continuous whole.

One more example for developmental work would be the presentation to teachers of the equiprobability bias reasoning (Green, 1982), or the tendency of secondary school learners —but also younger and older learners— to believe that every process in which randomness is implied corresponds to a fair distribution, with equal probabilities for any possible outcome. Once the equiprobability bias was introduced and discussed, teachers would be able to notice that the probabilistic meaning of all the outcomes of an event being equally likely is not obvious or intuitive, or that semantic everyday associations operate in and interfere with the learners' thinking such as the physical meaning of equally likely or *physically equal*. The practice with them could then move towards identifying, interpreting, and deciding on talk for the communication of the probabilistic meanings encoded into names such as *equally likely* and into its distinction from *nonequally likely* in situations in which either A or B can occur, but one of them can be *most/more or least/less likely.* High responsive explanations to be considered would be: *They are all possible but five is the most likely outcome when you roll the die with the five painted twice*. In the project context of different intervention studies, we are engaged and making good progress in the production of materials (on fractions, angles, and probability teaching) for primary and secondary school teachers to gain knowledge on learners' specific reasoning biases, and professional noticing abilities at the levels of words and sentences within mathematical content registers.

MORE REFINEMENT, POSSIBLE EXTENSION

It is common to describe when the use of certain theoretical and practical constructs began in the literature, and then to draw on them, as if they were finished products, to conduct our investigations. In this report, we have addressed a framework in the middle of its conceptualization in research and developmental work with mathematics teachers on language-and-learner responsive content talk in teaching. We have argued that to facilitate work with teachers it is necessary to clearly outline criteria for identifying, interpreting, and taking decisions on language-and-learner responsive languages of content teaching. When preparing and conducting developmental tasks around the teaching of a mathematical content and showing or asking for specialized word names and/into explanatory sentences, we thus need to provide criteria as to why these names and/into sentences can support the school learning of the content and meet learners' demands. We have proposed presenting to teachers well-documented content reasoning biases in order to guide their processes of noticing talk for content teaching.

Our theoretical and developmental project with mathematics teachers towards language-and-learner responsive content teaching remains unfinished in many respects. The realization of teacher talk, from the perspectives of explicit content teaching and reduction or prevention of learners' biased reasoning, requires further refinement and expansion work. The current framework integrates mathematics teaching that is responsive of mathematical content learning and mathematical language teaching with the sentence level linked to content-related explanations. Yet, this level can additionally be linked to examples or variations of content-related elements so that the following third tool in language is being examined:

- *Exemplification* or giving sentences with encoded variations of content-related elements oriented to reduce content learning challenges.

Furthermore, our project is grounded on the broader sociocultural interpretation of teacher talk as discourse, and hence on views that primarily focus on words and sentences once they are put to use or thought for use in situated communication. Rather than highlighted sporadically, the attention to word names and/into explanatory (and exemplifying) sentences should be blended and embedded in developmental work on mathematical discourse practices. While the tools of naming and lexicalization refer to discrete resources in the language system, our attention to these tools is shaped by social understandings of mathematical meaning making through participation in discourses that offer sustained opportunities of doing and talking mathematics. Regardless of strategic developmental orientations and analytical research decisions, there is not indeed a linear order in classroom practice from words to sentences, and from sentences to discourse, since mathematical meaning making is constructed and negotiated on a synergetic continuum across all levels of language.

By presenting the above-mentioned possibilities of extending the framework and of continuing the refinement of the theoretical and developmental tools, we hope to inspire other researchers to re-evaluate the importance of language-and-learner responsiveness in teacher talk, and perhaps to establish connections with their own frameworks for professional development on mathematical content teaching.

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MATHEMATICS TEACHER EDUCATORS' WORK TO FOSTER AN INQUIRY COMMUNITY

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For several years, the study of mathematics teacher educators (MTEs) has been considered central in research. The present study is focused, in particular, on MTE expertise in generating documents for the work with teachers, during a professional development (PD) program. We aim to analyse how MTEs generate documents coherently with the goal they have set for the PD program itself. In the case study presented in this paper, MTEs' goal is building an inquiry community with the teachers. The results illustrate what kind of resources are involved in MTEs' documentational genesis and how MTEs' choices in their documentation work are connected with their goal for the PD program.

INTRODUCTION AND THEORETICAL FRAMEWORK

Working Group 3 (WG3) at PME44 conference was on the theme "Conceptualizing the expertise of the mathematics teacher educators" (Helliwell, & Chorney, 2021), in continuity with the WG on the same topic at PME 43 (Helliwell, & Chorney, 2019), testifying the centrality of mathematics teacher educators' (MTEs) role in current research. Two of the aims of WG3 at PME44 were: 1) "to formulate approaches and research questions around MTE expertise"; 2) "to explore and develop potential methodologies that support these approaches and research questions." (Helliwell, & Chorney, 2021, p. 97).

The study presented in this report is framed on these aims, which give space for new fields of investigation. The authors' research group has been involved for several years in the study of teacher professional development, introducing theoretical models for the analysis of teachers' and MTEs' practices as part of a community evolution process (e.g. Robutti, 2020; Robutti et al., 2021). The report is focused on MTE expertise in generating documents for teacher professional development (PD), analysed with the Documentational Approach to Didactics (DAD: Gueudet & Trouche, 2009; 2010; 2012). Specifically, we study MTE expertise in generating documents consistent with the goal they set for the PD program, through a case study involving a group of researchers in mathematics education. The researchers (including the authors) have the role of MTEs and have the task of designing and implementing a teacher PD program. We identify the relationships between their documentation work (Gueudet & Trouche, 2009; 2010; 2012) and their goal of building an inquiry community (Jaworski, 2006; 2008) with the teachers participating in the PD program.

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 323-330). PME.

Inquiry communities

Jaworski (2006, 2008) introduces the term “inquiry community”, referring to didacticians (researchers, who are also teacher educators) and teachers working together, exploring and developing mathematics teaching-learning in classrooms. Jaworski’s vision of inquiry communities is based on the concept of co-learning inquiry, that means people learning together through inquiry. Inquiry, in this context, is intended both at class level and at teachers and didacticians level, while exploring how to use inquiry-based tasks with their students (Jaworski, 2006). The idea of inquiry community is built on Wenger’s (1998) idea of communities of practice, whose members experience engagement, imagination and alignment of shared practices. In this approach, alignment means engaging in forms of practice and ways of being, in order to conform to expectations and to the “normal desirable state”. The difference with respect to communities of practice is that, in an inquiry community, the “normal desirable state” is continuously challenged, with a questioning attitude (Jaworski, 2008), called critical alignment (Jaworski, 2006). This means bringing a critical attitude to alignment - questioning, exploring and seeking alternatives – that renders possible to develop and change the normal state.

In our study, MTEs’ goal is to build an inquiry community with teachers, by promoting a questioning attitude and by prompting critical alignment, resulting from a process of co-learning inquiry. To achieve this goal, MTEs engage teachers in an inquiry cycle (plan, act and observe, reflect and analyse, feedback), which led to a continuous process of reconceptualization and redesign (Jaworski, 2008) of teaching materials for their students, based on inquiry-based tasks. Besides contributing to the design of teaching materials, MTEs also have to design materials to be used with teachers during the meetings of the PD program. We analyze this latter design work with the theoretical lenses of DAD, because this framework allows to highlight exactly the aspects that interest us, related in particular to the resources on which MTEs relies.

Documentational Approach to Didactics (DAD)

DAD framework (Gueudet and Trouche, 2009; 2010; 2012) focuses on teachers using different kinds of resources to prepare their lessons and to support students’ learning. It is framed on the instrumental approach (Rabardel, 2002), which distinguishes between artifact (only object) and instrument (involving also the subject). DAD introduces a parallel distinction between resources and documents: a document consists of a set of resources and subject’s utilization schemes for a particular class of situations. Documents may be material, or psychological entities, like instruments in the sense of Rabardel. Documentational genesis is the process by which documents are generated (Gueudet & Trouche, 2009), starting from resources and introducing utilization schemes, namely classes of situations (in which resources are used), rules of action (stable elements in the way the resources are used) and operational invariants (which are part of the set of beliefs and knowledge of the teacher).

Usually, DAD is used to investigate teachers' work and growth via understanding changes in their documentation work. However, there are also other possible studies: Kieran et al. (2013) apply DAD framework to researchers' documentational genesis: the documents generated by researchers are directly designed for the students and not for the work with teachers; Psycharis & Kalogeria (2018) analyse the documentation work of trainee teacher educators, who were themselves teachers (and not researchers). Their learning was expected to be developed through their engagement in designing resources for teachers during a PD program.

Aiming to broaden the horizons of these previous studies, and remaining situated in continuity with previous PME reports, we analyse the documentation work of MTEs, who are researchers in mathematics education, when they design documents for teachers' PD. Our research question, therefore, is:

How do MTEs generate documents to foster the building of an inquiry community with teachers?

Answering this question will allow us to have a deeper insight in MTEs' expertise in performing their documentation work consistently with the goal of the PD program.

METHODOLOGY

The context

The PD program is part of the Turin University project Scuole Secondarie Potenziate in Matematica (SSPM, https://frida.unito.it/wn_pages/tmContenuto.php/456_matematica-teorie-e-applicazioni/45/), which is part of the national project Liceo Matematico (https://www.liceomatematico.it/torino/). Through an agreement with the Mathematics Dept. of the University, the schools involved in the project provide additional mathematics hours to the students, taught by their mathematics teachers, who attend a PD program (30 hours per year), held by mathematics education researchers.

We examine here the community of lower secondary school (grades 6-8) mathematics teachers, created in 2017: they are 17 teachers, who have attended the program from the start. The MTEs are academics and two of them coincide with the authors. The data collected refer to the fourth year of teachers' attendance, when the program was held online, due to Covid-19 pandemic restrictions. The program for the teachers consists in: ten 2-hour meetings of PD – one per month - and additional work online through a platform in, and of 33 hours of classroom implementations, in charge of teachers. The meetings had a fixed structure: a first moment in a common session, in which the MTEs presented the activities to the teachers, followed by a moment in which the teachers worked in groups on the activities, and, in the end, a collective discussion orchestrated by the MTEs.

Data collection

All the analysed data are retrievable on the web platform (Moodle) used for the asynchronous interactions between teachers and MTEs, during the PD program.

In particular, we collected:

1. The ten “activity sheets” given to teachers during the meetings. They include ideas for mathematics tasks for students, whose design must be implemented and reflection questions for the teachers.
2. All the slides projected by the MTEs during the meetings, to introduce the activities.
3. The transcripts of the video-recordings of all the interactions, included collective discussions, occurred when teachers and MTEs were altogether in the main session of the on-line meeting (in the separated sub-sessions it was not possible to record, due to technical limitations).
4. Teachers’ protocols, provided in response to MTEs’ requests. They include the design of tasks for students, teachers’ answers to reflection questions and reports on classroom experimentations.
5. Teachers’ answers to a written questionnaire about their beliefs and practices, administered during the first meeting.
6. Transcripts of teachers’ semi-structured interviews, conducted by an educator (one of the authors), remotely via a web platform.

The written questionnaire, mentioned in point 5., consisted of 25 questions: 23 open questions, a multiple choice and a Likert Scale.

Data analysis

As in DAD is analysed the reflective investigation of teachers’ documentation work, in this study we base our analysis on the reflections of the MTEs, scrutinizing their documentation work. Since DAD points out the importance of an active involvement of teachers, because they have access to their documentation work and they can make visible some hidden resources (Gueudet & Trouche, 2012), here too we rely on the MTEs’ reflective attitude towards their own documentational genesis.

The first two types of data enlisted above (points 1. and 2.) can be considered the “material part” of documents generated by the MTEs, for the work during the PD program. In addition, we analysed them also as resources for the generation of new documents, by MTEs, to be used in the subsequent meetings of the PD program. For all the documents, we identified the utilization scheme, composed by the class of situations, the rules of action and the operational invariants. Besides that, we identified the main resources on which the MTEs relied for their documentational genesis, proceeding with a backward analysis and having an overall look at the development of the entire year of the PD program.

The data enlisted in (points 3.; 4.; 5. and 6.) were analysed following the principles of qualitative thematic analysis (Braun & Clarke, 2006), with an inductive approach. Our aim, related to our research question, was to identify the different kinds of resources, embedded in these data, that were involved in the MTEs' documentation work and to study how they contributed to their documentational genesis.

In the end, we traced a connection between MTEs' documentation work and their goal of building an inquiry community with teachers, explaining the motivations at the basis of their choices in the process of documentational genesis. Particularly, we identified MTEs' attempts to promote critical alignment (Jaworski, 2006) in the teachers, through a process of co-learning inquiry, involving a critical attitude.

At every stage, the authors worked, at first, individually, especially focusing on the part of the documentational genesis in which they had been more involved. In a second moment, they met together to share and discuss the results of their analysis.

RESULTS AND DISCUSSION

We will present an example of a document generated by the MTEs', identifying the main resources on which it is based and its utilization scheme. Other examples can be presented during the conference. We will trace, in this document, evidence of connections with the MTEs' goal of building an inquiry community with teachers.

Document for the 4th meeting

In this example, we will show a MTEs' document, generated for the 4th meeting of the PD program. On that occasion, the MTEs presented some slides to the teachers, to prompt a reflection moment and a collective discussion. During the previous meetings, the teachers had been asked to propose task designs for their students and to report on the implementations of their classroom activities, based on what they had designed. The 4th meeting started with a slideshow, presented by one of the authors, which had the aim of triggering a collective discussion on the teachers' task design proposals and reports of classroom activities. We consider the slides as the "material part" of the document we are describing, associated with the utilization scheme that we will illustrate below.

The first slide shows the distribution of the answers to the Likert Scale question of the preliminary questionnaire (point 5.), administered to teachers during the first meeting. In this question, teachers were asked to express with a score from 1 to 6 how much they feel that certain tasks are central to the role of the mathematics teacher. The image that emerges from the answers is that of teachers who have to promote student centred activities, in which students are engaged in creative processes. The items with the highest scores are, in fact, those most in line with what is required of a teacher in laboratory, inquiry-based activities: "To create situations in which students have to make decisions and choices" (5,5 points), "To promote freedom of thought and creativity" (5,47 points) and "To promote students' awareness and critical sense" (5,44 points).

The second slide, presented during the 4th meeting, contains four questions for teachers, based on the answers showed in the first slide:

"How and to what extent have awareness and critical sense been promoted with this activity?"; "How and to what extent have freedom of thought and curiosity been promoted?"; "How and to what extent did students have to make decisions and choices?"; "Could the previous aspects have been further promoted? How?"

The MTEs' aim was to prompt a collective reflection, making a comparison between the reports, made by teachers, about the implementation of the activities they had designed for their students during the previous meetings and their answers to the Likert Scale question of the preliminary questionnaire. There was, in fact, an evident (at least for MTEs) inconsistency between what teachers had declared in the questionnaire and their task designs and the content of their classroom activities reports. These last, in fact, never mentioned opportunities for students to make choices, to exercise their freedom of thought, creativity, or their awareness and critical sense.

The utilization scheme of the document, whose material part is constituted by the slideshow described above, is the following:

Class of situations: collective discussion and reflection with teachers on their task design and classroom reports.

Rules of action: 1) Show a slide with the teachers' answers to a question of the preliminary questionnaire, about the teacher's role. 2) Ask teachers to reflect on the consistency between their answers to the questionnaire, the task design they proposed and the reports of the activities they carried out in their classrooms.

Operational invariants: 1) Teachers should be confronted with possible inconsistencies between their answers in the preliminary questionnaire and their practices. 2) The collective discussion should address possible issues, which prevented teachers from enacting practices coherent with what they had declared in the questionnaire.

Resources

Resource 1. Answers to the Likert Scale question of the preliminary questionnaire (point 5.), provided by the teachers participating in the PD program. These answers were used by the MTEs, during the 4th meeting of the PD program, as a stimulus for the collective discussion, because they appeared in contrast with the reports of the classroom activities, made by the same teachers during the meetings of the PD program.

Resource 2 – Reports of classroom activities (point 4.). In many reports about the implementation of the activities, designed by teachers in the first meetings, it emerged that teachers guided their students a lot, to try to lead them to find the solution of the proposed tasks. For example, in the reports related to the task design and the implementation in the classroom of an activity, intended for a VI grade class, there was no evidence of peer discussion, sharing and comparison of conjectures or wrong discoveries. A teacher reported a justification of her task design, saying:

> Lucia: In grade VI, […] the questions must be made explicit as clearly as possible and, above all, they must be progressive. In grade VII you can also skip a question and make sure that, in order to answer another question, the students must have already answered the underlying one. Instead, in grade VI no, in my opinion they must be guided step-by-step to the solution […].

Based on the reports obtained during the first three meetings, the MTEs felt the need to deeper investigate the teachers' reluctance to engage both low and high-achieving students in activities, which promote higher order thinking, creativity, critical sense, freedom of thought and awareness. So, they generated the document for the fourth meeting, presented above, with this aim.

Other resources. There are, of course, other resources, in addition to those detailed above, that contributed to the MTEs' documentational genesis. Among them, we can list*:* transcripts of the collective discussions among teachers and MTEs, notes of the meetings among MTEs on the design of the PD program, literature in the field of mathematics education, national and international meetings with scholars who work in the field of teacher professional development.

Building an inquiry community

The documentational genesis we described in the previous section is connected with the MTEs' goal of building an inquiry community among MTEs and teachers. This connection is testified by their effort to prompt collective discussions and reflections among teachers and MTEs, to promote a questioning attitude. The MTEs designed documents aimed to highlight possible issues, which can hinder the implementation of the inquiry-based approach by the teachers in their classrooms. These documents are thought to foster a co-learning inquiry process, in which teachers and MTEs try to address the emerging issues and to implement an inquiry cycle. The teachers, in fact, are requested to reflect on their task design and on their classroom implementations, in order to make improvements and redesign their teaching materials.

With this study, we obtained an insight in the expertise of MTEs, who are also researchers, interested in generating documents based on teachers' feedback and on the interactions during the PD program. Such an insight could also have an impact on MTEs' own professional development, which could be object of further research.

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STAND.OUT.ERRORS: A STARTING POINT TO ADDRESS MATHEMATICS LEARNING LOSSES POST COVID-19

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The COVID-19 pandemic resulted in school closures and loss of learning and teaching across the world, and the impact will be most severe in low-income countries and communities. Research on impact of similar severe disruptions is unanimous that learning of mathematics is impacted most but recommendations for recovery are vague. The notion of stand.out.errors is proposed as a starting point for mathematical recovery. This notion is defined, operationalised and applied to the test responses of 376 Grade 7 learners. The stand.out.errors in four test items are discussed and recommendations for addressing errors are made.

INTRODUCTION

In the last quarter of 2021, after several months of school closures, learners in low-income communities in South Africa (SA) typically got five hours of mathematics lessons in a two-week period. In some developing countries like Uganda and the Philippines, public schools did not open at all in 2021. What do schools, teachers and learners do to recover and catch up the lost learning time and forgotten mathematics in such circumstances? While some important principles can be learned from research on the impacts of previous extended disruptions to schooling, the findings lack specificity for school mathematics. I propose the notion of stand.out.errors as a starting point to guide teachers and policy makers in focusing initial attempts to address learning loss in mathematics, with particular focus on learners from low-income communities. The empirical data on which this paper is based comes from the pilot implementation of a mathematics test involving multiple-choice questions (MCQ) designed to provide diagnostic information on learners' mathematical knowledge as they transition from primary to secondary school in the midst of the COVID-19 pandemic. The research question guiding the larger study was: What mathematics do Grade 7 learners bring to secondary school? This paper is guided by the question: What is the potential in the notion of stand.out.errors to inform Grade 8recovery mathematics programmes?

BACKGROUND AND LITERATURE REVIEW

Severe disruptions to schooling come in various forms, including natural disasters (e.g. floods, earthquakes), man-made disasters (e.g. war, nuclear explosions) and disease (e.g. Ebola, Polio, COVID-19). The impact of such disruptions on schooling are immense, not just in terms of teaching and learning but because schools provide services such as meals for learners in low-income communities. Research on the impact of such disruption is seldom published in peer-reviewed academic journals. Typically, it appears in reports, blogs and other web-based publications, reported by international

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 331-338). PME.

aid agencies (e.g. UNESCO, USAID, World Bank) and non-profits (e.g. UKFIET, RTI). The research is mostly quantitative and based on relatively large samples. Nevertheless, the findings are important to consider in a response to the COVID-19 pandemic which led to school closures in over 190 countries, affecting more than 90% of school-going learners (UNESCO, 2020).

The research is unanimous that extended school disruptions lead to learning loss and negative psycho-social effects for all learners, but this is greatest for learners from low-income families (e.g. World Bank, 2020). Consequently, interventions to address learning loss must pay particular attention to schooling for those in poor communities. A review of research on the impact of severe disruptions on schooling in economically poor countries such as Indonesia (Rush, 2018), Nepal (Mu et al., 2016), Pakistan (Andrabi, Daniels & Das, 2020), Sierra Leone (Powers & Azzi-Huck, 2016), and Rwanda (Thomas, 2010) provides important insights to inform responses to the impact of COVID-19, as outlined below.

Severe disruptions typically have a greater impact on secondary schools than primary schools (Rush, 2018) but learning loss is greater for younger children and these effects accumulate over time. Consequently, they may be unable to learn new content at later stages because of their earlier gaps (Das, Daniels & Andrabi, 2020). On the other hand, younger learners will have more years in school with potential to benefit more from intervention programmes. Research conducted four years after the Pakistan earthquakes in 2005 showed that, although learners in the most severely affected areas had only missed three months of school, they were a 1.5 years behind their peers living in areas less affected by the disaster (Andrabi, Daniels & Das, 2020).

Research on school subjects generally focuses only on performance in literacy/ language/reading and numeracy/mathematics, where it is unanimous that disruptions have greater impact on performance in numeracy/mathematics (e.g. World Bank, 2020). Since the research is generally reported by aid agencies and their affiliates, recommendations relating to mathematics teaching and learning lack mathematical detail. However, the recommendations should not be ignored. They include: start where the learners are, based on their current levels of mathematical knowledges, not with the curriculum requirements; pay attention to core concepts first; pay deliberate attention to gaps in learners' knowledge; do not seek to cover large volumes of content which inevitably requires a fast pace (World Bank, 2020; Mu et al, 2016).

Of course, there is a wealth of mathematics-specific research reporting on learners' mathematical difficulties and errors which complements these broader findings from research on disruptions. However, at least in the SA context, the findings and recommendations from mathematics education research are not easily transferable to contexts requiring large-scale interventions to address learning loss and learner backlogs, particularly in contexts of limited resources which dominate the country. It is highly likely that this situation is found in many parts of the developing world.

The research reported here contributes to the knowledge gap by taking a more nuanced mathematical focus on learners' errors. It also seeks to provide practical starting points which are briefly discussed in the final section.

STAND.OUT.ERRORS: DEFINING AND OPERATIONALSING

A stand.out.error (SOE) is an incorrect response to an MCQ item which occurs much more frequently than other incorrect responses to that item. For example, a test item required learners to calculate $6 + 24 \div 3$. The most frequent response by far was 10. This SOE indicates that learners are not paying attention to *order of operations* but merely working from left to right. To operationalise the definition of SOE, we first used two criteria. An incorrect response, R, is an SOE if the:

1. Error frequency of R ≥ 20% AND
2. Error frequency of R ≥ 10 percentage points (pp) of the frequencies of other incorrect responses.

The definition was later expanded for the case where the frequency of two incorrect responses was at least 20% but they were not 10pp apart: incorrect responses R_1 and R_2 are both SOEs if the error frequency of R_1 and R_2 are both ≥ 20% (even if they are not 10pp apart nor 10pp more than the next most frequent response). Examples are provided below. This expanded definition was necessary when seeking trends in SOEs. Three (or more) incorrect responses with a frequency of 20% or more, is likely a sign of random guessing and so none of the responses is considered an SOE.

In order to identify SOEs, the learner sample is divided into five sub-groups (quintiles) of approximately equal size. A frequency count is generated for each quintile for each distractor on each MCQ test item. The above SOE criteria are then applied to the top four quintiles. The bottom quintile (in this case, scores below 28%) is excluded because previous analysis of their errors did not reveal trends as clearly as the other quintiles.

Identifying SOEs for individual quintiles is insufficient to address common errors. Trends in SOEs across quintiles have potential to reveal more detail. A 3-point SOE is a stand.out.error in three or more consecutive quintiles. A 2-point SOE applies to exactly two consecutive quintiles. It is possible to have more than one 3-point or 2-point SOE for an item but a 3-point SOE classification overrides a 2-point classification. Identifying and then addressing 2- and 3-point SOEs is important because it will impact a wider range of learners, not just the stronger or weaker groups. A comparison of SOEs for the whole group versus for consecutive quintiles is provided in Table 1. This shows that while the 3-point SOEs will be picked up in a simple analysis of the SOEs for the full group, many of the 2-point SOEs will not be identified.

METHODOLOGY

The pilot test instrument consisted of 66 MCQ items covering typical curriculum content of Grades 4 to 7 on: whole number properties and operations; rational number (fractions, decimals, percent, ratio and rate); patterns, functions and introductory algebra; measurement and geometry. The topics were weighted differently based on their relative importance in the primary school mathematics curriculum, with 65% of the items dedicated to the number topics (see Table 1). Items were selected/adapted from a range of sources including local, national and international assessments. Each item incorporated distractors that reflect typical errors and/or misconceptions identified by teachers and/or reported in the local and international literature such as conceptions of equality (Kieran, 1981), decimals (Steinle & Stacey, 2004), and the arithmetic-algebra transition (Kaput, 2008).

The items were trialled with Grades 7 and 8 learners in 16 schools. This paper focuses on an opportunistic sample where a school requested to test their incoming cohort for 2021 but administered the test during an orientation programme in December 2020, after schools had closed for the year-end holidays. This sample of 376 Grade 7 learners was drawn from more than 20 primary schools. The relatively large number of different schools reduces the teacher and school effects on the sample although learners had inevitably had varying opportunities to learn throughout primary school and during the COVID lockdown periods since some attended well-resourced schools with well-qualified teachers while others came from poorer areas, with larger class sizes and fewer resources. Nevertheless, the vast majority of learners in the sample would not have access to online learning at home and this would have negatively impacted their learning during the extended school closures in 2020.

Learners wrote their responses on a pre-prepared answer grid which was scanned and processed by an online learner management system. Accuracy checks of the scanned images showed that approximately 10% of answer sheets were not scanned with 100% accuracy and therefore required manual capture. Data cleaning and processing revealed that the mean percentage of blank responses was 3.5%. The mean percentage of "bad" responses, e.g. where learners selected more than one distractor or their choice of distractor was not clear, was 4.8%. Each correct response was given a score of 1.

FINDINGS AND DISCUSSION

With respect to overall test performance, the weighted mean score was 43.9%. The breakdown per topic is given in Table 1. The performance trends per topic reflect those of the other schools where the test was piloted.

The procedure to identify SOEs was completed as described above. There were 41 SOEs for the full sample (all quintiles) with the breakdown shown in Table 1. When separating quintiles, 31 3-point SOEs and 18 2-point SOEs were identified. While only two of the 3-point SOEs were not picked up by the whole group analysis, there were nine 2-point SOEs that were not picked up. This shows the value of disaggregating the group into quintiles, identifying SOEs per quintile and then across adjacent quintiles. Further detail of SOEs is given in the four examples below (see Tables 2-5), all of which involve number and number operations. Correct options are shown in bold.

Topic	# items	% correct	Full sample SOEs	3-point SOEs	2-point SOEs
Whole number & operations	23	45.9	14	10	4
Rational number	20	40.9	13	11	6
Measurement	8	35.4	9	8	1
Patterns, functions, algebra	10	49.5	4	2	5
Geometry	5	49.4	1	0	2
TOTAL	66		41	31	18

Table 1: Summary of performance per topic and SOEs on MCQ test.

Item 48 tested learners' understanding of the equal sign as an equivalence operator (Kieran, 1981, see Table 2). Surprisingly, this issue is not explicitly mentioned in SA maths curriculum documents. The incorrect answers reflect the following reasoning: A: right-to- left reasoning, operational view (7–3=4); B: operating on all numbers (5+4-7=2); D: left-to-right reasoning, operational view (5+4=9). This item illustrates a case where there is no SOE for the full group but the disaggregated scores reveal a 2-point SOE (see bold frequencies in Table 2). In quintiles 2 and 3, more than a third of learners appear to hold an operational view of the equal sign given their choice of D. In addition, more than 20% of learners in quintile 2 operated on all three numbers in the equation thus also reflecting a lack of understanding of the notion of equality.

Item 48: $5 + 4 = \square - 7$

	A 3	B 2	**C** **16**	D 9
Quint 2 (28.1-35.0%)	7.9	23.7	25.0	**35.5**
Quint 3 (35.1-45.0%)	8.0	13.3	41.3	**33.3**
Quint 4 (45.1-59.9%)	3.7	9.8	68.3	14.6
Quint 5 (60.0-100.0%)	0.0	9.2	88.2	1.3
All quintiles	4.5	15.7	48.4	**24.2**

Table 2: Response frequencies and SOEs per quintile for Item 48.

The 2-point SOE classification (linked to D) suggests that approximately 60% of the sample need support to develop an appropriate understanding of the equal sign. By contrast, most learners in the top two quintiles are entering secondary school with an equivalence view of the equal sign.

Item 3 tested addition of decimals, introduced in Grade 6 in SA. It is well-known that learners apply whole number thinking when operating on decimals (Steinle & Stacey, 2004) and this sample was no different (Table 3). Response B reflects typical whole number thinking in adding significant digits, irrespective of their place value.

Responses C and D reflect partial awareness of place value. Response D mixes rules for multiplication of decimals since the distractor contains three decimal places, which is the number of decimal places in the two addends.

There is a 3-point SOE for quintiles 2-4, but no SOE for the full group because the percentage point difference between the frequency of B and D is marginally less than 10 pp. There are two SOEs for quintile 2 based on the expanded definition of SOEs. The anomaly where the frequency of B increases from quintile 2 to 3 (23.7% to 32.0%) may merely be a consequence of the high frequency of quintile 2 learners choosing D.

Item 3: $0.5 + 0.03 =$				
	A **0.53**	B 0.8	C 5.3	D 0.053
Quint 2 (28.1-35.0%)	35.5	**23.7**	13.2	**21.1**
Quint 3 (35.1-45.0%)	46.7	**32.0**	8.0	10.7
Quint 4 (45.1-59.9%)	61.0	**20.7**	2.4	9.8
Quint 5 (60.0-100.0%)	88.2	7.9	0.0	2.6
All quintiles	52.4	22.3	5.6	12.8

Table 3: Response frequencies and SOEs per quintile for Item 3.

Item 12 contains a 3-point SOE that extends through four quintiles but is not an SOE for the full group (Table 4). It is also contains two SOEs in quintiles 2 and 3. The item deals with squares and cubes. In the SA curriculum learners work only with these powers until Grade 7. Thereafter the exponential laws are introduced. The high frequency of response A may be partly due to learners expecting that the answer should be in exponential form. However, the choice of A provides an important insight into pre-conceptions learners bring about adding powers, i.e. "when you add powers with the same base, you add the exponents". Similarly, given the relatively high frequency of option C for the whole group, it is important to make teachers aware of this incorrect thinking, and thus to better prepare them to introduce exponential laws, paying careful attention to the base and the operation being performed on the powers.

Item 12: $2^3 + 2^2$				
	A 2^5	**B** **12** $(8 + 4)$	C 4^5	D 10 $(2 \times 3 + 2 \times 2)$
Quint 2 (28.1-35.0%)	**26.3**	31.6	**26.3**	9.2
Quint 3 (35.1-45.0%)	**21.3**	44.0	**22.7**	9.3
Quint 4 (45.1-59.9%)	**25.6**	54.9	7.3	6.1
Quint 5 (60.0-100.0%)	**21.1**	55.3	7.9	6.6

All quintiles	23.1	39.4	18.6	9.8

Table 4: Response frequencies and SOEs per quintile for Item 12.

Item 39 was one of several items dealing with generalisation of number properties and operations in preparation for algebra (Blanton, Stephens, Knuth, Gardiner, Isler & Kim, 2015). The 3-point SOE in response C shows strong evidence that most learners are not yet able to generalise division by 1 (Table 5).

Item 39: If ◆ is any whole number, then $\frac{◆}{1} =$

	A 0	**B** ◆	C 1	D Impossible to tell
Quint 2 (28.1-35.0%)	5.3	11.8	**56.6**	18.4
Quint 3 (35.1-45.0%)	9.3	17.3	**45.3**	24.0
Quint 4 (45.1-59.9%)	6.1	26.8	**41.5**	15.9
Quint 5 (60.0-100.0%)	1.3	51.3	**25.0**	13.2
All quintiles	7.2	25.5	**38.8**	17.8

Table 5: Response frequencies and SOEs per quintile for Item 39.

CONCLUSION

Whether or not we are comfortable with deficit discourses, we must acknowledge that the COVID-19 pandemic has wreaked havoc on education and will continue to do. In countries where learners may have missed an entire year of schooling, they are also likely to have forgotten some of what they had previously learned. Those with access to online learning, have a wealth of resources at their disposal. Learners from poor communities do not. For many mathematics teachers the problem may seem insurmountable, begging the question “where do I begin?” The notion of SOEs is a starting point to identify specific gaps in learners’ knowledge and evidence of incorrect thinking. Since these SOEs are manifested (at least to some extent) in learners’ answers to carefully-designed MCQs, we can identify specific concepts and procedures that are not fully understood, and ways of thinking that are partially or substantially wrong. In this study, a total of 49 SOEs were identified in learners’ test responses. This provides a list of priorities. The more detailed discussion of four items in this paper served to pinpoint in more detail what needs to be done to address learners’ errors. For example, teachers cannot assume that learners enter secondary school with an equivalance view of the equal sign, and yet many do. While previous research has already shown this (e.g. Knuth, Stephens, McNeil, & Alibali, 2006), the extent of the problem may be larger in the wake of the pandemic. A finding not reported in the literature suggests that learners’ incorrect pre-conceptions about operating with powers could underpin the difficulties they experience with exponents in over-generalising the exponential laws. The transition from arithmetic to algebra is one of the key challenges in the transition from primary to secondary school. The SOE reported here regarding item 39 provides one example of work that mathematics teachers need to do with regard to number and moving from a focus on calculations to a focus on structure.

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INTEGRATING THE AFFECTIVE DOMAIN WHEN INTERPRETING UNDERSTANDING IN MATHEMATICS: AN OPERATIONAL APPROACH

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We present here an integrative proposal allowing to interpret the different systems of the affective domain and its relationship with understanding when performing mathematical activity. A conceptualisation of the affective domain is advanced based on the emotional system, which serves as a central point of reference, and a functional perspective of understanding relating to the uses given to mathematical knowledge. We also provide a specific interpretive method and exemplify it with a practical case of a preservice elementary teacher engaged in solving a flat surface measurement task. To conclude, by incorporating affective phenomena into the interpretation, we found complementary reasons that accounted for the student's mathematical understanding.

INTRODUCTION

While progress has been made in research on the affective domain in Mathematics Education, a consensus has yet to be reached on its organisation, characterisation and assessment. Calls have been made for specific models to be elaborated that would include affective domain conceptualisations linked to the specific issues under study (Hannula, 2012; Schlöglmann, 2002). It is in this problematic context that we conducted the present study, which addresses understanding in mathematics. Focusing on the key role of emotions in the development of understanding in mathematics, we sought to elaborate an integrative framework of the various components of affect in mathematics, a major challenge recognised in our field of research.

We used a developing model that is based on the interpretation of students' mathematical understanding (Gallardo & Quintanilla, 2019; Quintanilla, 2019). At a theoretical level, we put forward a dialectical approach in which the affective domain has a systemic nature, in order to incorporate a number of consolidated results within a single common process. At the methodological level, we provide a specific qualitative method to observe and interpret students' mathematical understanding based on the affective dimension. To show the method's potential in practice, we implemented it in an empirical qualitative study with preservice elementary teachers who undertook the resolution of measurement tasks. By integrating the different components of the affective domain and the understanding of mathematical knowledge, we succeeded at characterising the student's distinctive cognitive and affective features within a single interpretative process as he performed mathematical activity in the classroom.

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THEORETICAL FRAMEWORK

We conceive the affective domain as an autonomous, dynamic, cyclical and closed meta-system that arises from the interactions among the related main systems it is made of: (a) belief system, (b) motivational and behavioural system, (c) emotional system, (d) attitudinal system, and (e) values and norms. Emotions are a central reference of the affective domain and are directly linked to the rest of the components, acting as mediators among them.

Every emotional experience starts with a first unconscious phase during which the person seeks – though a cognitive evaluation, conditioned by the context – to establish whether a certain object (physical or mental) or event (real, evoked or imaginary) may become an emotionally competent stimulus (ECS) (Damasio, 1994). This natural or acquired stimulus has the ability to trigger a particular *emotion.* Emotion is always the product of a value judgment made by the individual's cognitive system based on innate genetic patterns or socially and culturally learned patterns (Nussbaum, 2001). Beliefs, motivation, values and norms also contributes to this cognitive assessment of whether an object is an ECS (Di Martino & Signorini, 2019; Rouleau, 2019). In addition, emotions usually manifest themselves through *emotional responses* in the form of facial expressions, body language, tone of voice, and verbal locutions that can be recognised by outside observers (Ekman, 1993).

In a second phase of the emotional experience, the person's awareness of the different physiological changes provokes new thoughts relative to the object or initial situation that generated them and also relative to one's general physical status. A *feeling* thus appears as a mental representation of emotion (Damasio, 1994). Feelings prolong the impact and effects of emotions and, when evaluated cognitively, enable the generation of new emotions following a dynamic and cyclical process (meta-emotion). In addition, feelings predispose the subject to *action* by creating consciously adapted responses. Changes in emotional states also form stable affective patterns that are intertwined with cognition, creating contexts that are conducive to the individual taking action and in which beliefs act as systems of applied rules (Beswick, 2018; Goldin, 2004). Attitudes, on the other hand, constitute a tendency towards a specific type of action and contribute to shaping a person's identity. Again, the motivational system intervenes in the cognitive evaluations of this phase of the emotional experience, decisively influencing the subsequent course of action. In short, the specific actions associated with a given emotion in a particular context are triggered by a decision-making process link to the self-regulating and self-controlling nature of the emotion itself (Goldin, 2004; Lazarus & Lazarus, 1994). This facet allows to manage the emotion's *external representations* according to the interests of the person and the surrounding social and cultural norms, and finally, to generate the voluntary behaviours considered appropriate in each situation (Nussbaum, 2001).

From this perspective, the relationships between the different systems that compose the affective domain are established and transit through the emotional system. Thus, norms regulate emotional experience by influencing how people value objects within their social group (Nussbaum, 2001). This system of values and norms can modify attitudes and beliefs with the mediation of the emotional system. The attitudinal system, on the other hand, is based on repeated emotional experiences and it is these experiences that link beliefs to attitudes. Beliefs, in turn, are directly related to meta-affect, self-regulation, and motivation, processes that can also modify the belief system itself (Di Martino & Signorini, 2019; Goldin, 2004; Hannula, 2012). Finally, objectives and motivations, such as impulses to act, can lead to specific attitudes and beliefs, behaviour being the most reliable manifestation of motivation (Rouleau, 2019).

Regarding our approach to understanding in mathematics, we assume that individuals understand a mathematical knowledge in so far as they are able to use it, in any of its possible forms, in situations where the knowledge makes sense and contributes to a resolution. It is a functional view based on the uses of mathematical knowledge that students implement during their classroom activity. The actions deployed in the concrete situation, including the uses of mathematical knowledge, are directly related to a decision process in which emotions are key. The student's various accumulated emotional experiences, resulting from their experiences in the mathematics classroom, directly influence their future decisions, actions and uses in the classroom. We thus recognise the existence of mental processes that are strongly linked to the emotions underlying the decisions about the uses of mathematical knowledge and which explain the student’s understanding. Therefore, we characterise understanding in mathematics as an intellectual activity of an affective nature that enables the person to elaborate observable, adapted and contextualised responses, involving a recordable and interpretable usage of mathematical knowledge.

METHODOLOGY

In recent years, we have been developing an interpretive method which we call *the hermeneutic circle of understanding in mathematics* (Gallardo & Quintanilla, 2019). This method allows us to interpret simultaneously both the affective traces that accompany actions and motivate them, and the traces of understanding displayed by students when they solve problems. We applied our method in a qualitative study in which we interpreted the preservice elementary teachers’ understanding of measurement based on their various manifestations of affect.

Participants and context

The participants were 20 volunteer preservice teachers enrolled in their fourth year of their *Degree in Primary Education* at the University of Málaga. They studied the *Didactics of Measurement* subject during the second semester of the 2017-2018 academic year. The participants solved measurement tasks in their ordinary classroom, within the usual classroom schedule and together with the rest of their classmates. Our method was implemented over nine weeks, during the subject’s two weekly hours of practice.

Mathematical tasks

The selection was made based on each representative task of the different phases involved in the mathematical foundation of magnitude measurement (identification of magnitudes, conservation and comparison of quantities of magnitude, choice of measurement units, quantification and use of measurement instruments and arithmetisation). Non-equivalent tasks were used, the joint resolution of which would allow us to characterise the understanding of measurement. We illustrated the study using the records generated by one of the participating teachers (Antonio) when solving a task focused on the surface measurement of flat figures (Figure 1).

How large is the surface of the following figures?

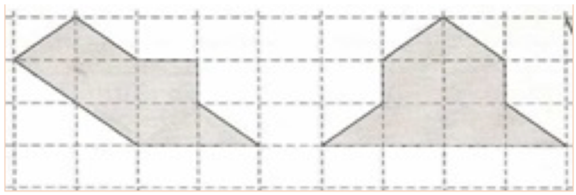


Figure 1: Flat Figure Surface Measurement Task.

Data collection

Each episode was conducted over three consecutive phases in which we used different data collection instruments.

Phase 1. Each participant was given the measurement tasks and a brief conversational interview took place. We addressed three main themes: (a) initial emotions when observing the tasks; (b) beliefs about mathematics and the students' personal relationship to mathematics; and (c) experiences with mathematics in the past. The interviews were recorded in audio (transcription as first written record).

Phase 2. The students were organised in pairs and solved the different tasks collaboratively. We sought to detect evidence of the interaction between the students' affective processes and their understanding as they use the mathematical knowledge. All the mathematical activity was recorded in audio and video (second written record).

Phase 3. The researcher shared her findings based on the results obtained in the previous phases and presented an interpretation of the student's performance in order to reach an agreement with him/her regarding the uses given to mathematical knowledge and their relationship with the affect experienced. Each interview was recorded in audio (third written record).

Data analysis and interpretation

The hermeneutic circle follows the various semiotic, phenomenon-epistemological and dialogical planes included in their interpretative trajectory.

We sought to identify the uses given to the mathematical knowledge and traces of understanding on the semiotic and phenomenon-epistemological planes. The following analyses served as a reference: (a) the *phenomenon-epistemological analysis* of the problem raised, in which we clarified the essential knowledge that could help to solve the problem; and (b) the *phenomenological analysis* of the student's emerging affective components during mathematical practice. We characterised these components using different representation systems that informed us of what was being communicated and how it was communicated: (i) the *verbal* system (tone of voice and locutions) and (ii) the *kinesthetic* system (facial and body expressions).

On the circle's dialogical plane, we compared the student's mathematical activity during the episode, we established relationships with the uses given to mathematical knowledge, and then structured the conclusions regarding the student's understanding. The search for a consensus also allowed us to contrast information relating to the different affective components displayed by the student during the episode's previous phases. The appropriation that occurred during the agreement-building with the other was expected to generate a transformative effect on the protagonists.

RESULTS

In Phase 1 of the study, students' beliefs and personal mathematical history began to emerge, and we identified a number of initial emotions.

Antonio: *(Low voice, nervous laughter)* I don't know how to solve them *(belief about what I should know)*. I should have worked on many of these things at school when I was a child *(belief about school teaching and learning*).

Interviewer: If you had to find a word for that feeling.

Antonio: Being unsure. Hesitations, quite a lot of doubts *(uncertainty)*. I think that the lack of usage ... makes me forget about it *(causal attribution)*. I don't understand why they don't explain this to me *(belief about teaching)*. I am very eager to understand certain things *(belief about oneself)*. I would say: I'm going to look for the answer, let's see if I find it *(perseverance).*

Antonio shows shame and, as he describes having doubts, he recognises feeling uncertainty. These emotions originate from the discrepancy between a belief about himself (he should know more about mathematics) and the recognition of not being able to immediately solve the tasks posed. He uses causal attributions as a justification (a characteristic component of the belief system) probably as a way of attempting to minimise the negative effects of his emotions. When describing his mathematical past, he is also showing beliefs about his own preferences and he is shaping a recognised attitude of perseverance.

During Phase 2, Antonio perceives the task as easy and becomes suspicious (belief). In addition, he manifests a limited understanding of the fraction as a measure. The fact of not regarding the triangle as a unit generates uncertainty (emotion):

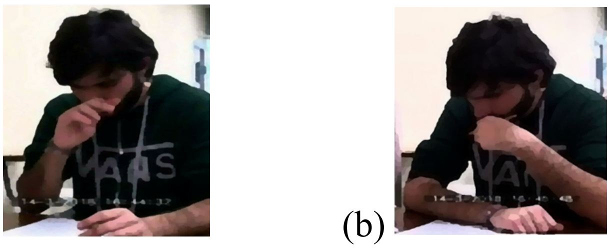
Antonio: Maybe you can't, let's see... There is always a catch in these exercises. *(In the first figure)* 1, 2, 3, 4, 5, 6, 7, 8... Exactly half is left over *(half a square or a right-angled triangle*). I don't know if it is linked to...

He modifies his strategy and uses knowledge of formulas, without paying much attention to its adequacy. New beliefs are involved that may favour this change (mathematics requires the use of sophisticated calculations):

Antonio: In Primary school, you don't know the Pythagorean theorem, but if I know this *(the cathetus)* and I know this *(the cathetus),* I know how much the diagonal measures *(hypotenuse).* Taking the area of this triangle, I add it to the area of the square and we obtain the surface. And if not, then you would have to calculate the surface of one of the triangles and since all triangles are the same, you simply add the surface. That would be another option.

His beliefs (numbers are needed to calculate measurements) and his understanding influence his emotions. He shows uncertainty and distress that are reflected through tensions (emotional responses in Figure 2). He abandons the first flat figure:

Antonio: With the Pythagorean theorem, you have to know the measurement to get an answer. We don't have the numbers here, so we can't give numbers. The quadrilateral in the first figure is formed with...



(a) (b)

Figure 2: (a) Tense body and face, continuously touches his nose; (b) Covers mouth with hand, hunched shoulders, clenched hands.

Finally, he manages to measure the second figure and, therefore, decides to that the task is solved with an expression of relief (emotion):

Antonio: In this case *(second flat figure),* you can make the square with the additional pieces, those that are left over, and yes, it is possible to calculate the surface. Here we could form six squares. It would be the same. That's it!

In Phase 3, we obtained complementary information which allowed us to confirm the relationships existing between Antonio's affective domain and his understanding.

1 I: Did you have doubts while you were doing the exercise?

2 A: Yes *(firm voice, sounds very certain*). Uncertainty and frustration, partly. Because I should be able to solve it and... why can't I? I find it hard to believe that things can be simple *(origin of a belief).*

3 I: You started talking about the Pythagorean theorem.

4 A: I wanted to apply the Pythagorean theorem if a measurement was given. We would multiply the triangle by two and we would obtain the surface of the square. Here the surface would be given in terms of numbers of squares.

5 I: How much would it be?

6 A: Here, six *(second flat figure).* And for this one *(first flat figure),* let's see: One, two, three, these would form two, four, five. No! Five and a half.

7	I:	Five and a half. That was the answer. What happened do you think?
8	A:	I like it when things are complicated *(muffled laugh).* Since there was no tangible measure, I didn't give an answer. Numbers were missing. I'm used to having to give a precise result... but it's not like that *(timid laughs).* But I've still got it there... I don't know if one day I'll lose it...
9	I:	But you told me that your perspective was changing...
10	A:	I have to change the way I see things. I have the mathematical knowledge. What I need to understand is how I can really teach that knowledge.

We found evidence of the relationship between the uncertainty and frustration that Antonio felt and his beliefs about himself and the social context in which he was immersed (his own expectations and that of others about what he should be able to do) (1-2, 8-10). Moreover, he again displayed the emotional responses of tension and nervousness (8). The strategy he planned based on the use of the Pythagorean theorem was conditioned by his beliefs (mathematics consists of rules, formulas and complex procedures) (3-4). Subsequently, because he could not apply the theorem due to missing numerical data, he came back to resorting to his initial knowledge, and managed to solve the task (5-7). Finally, as he became aware of his mathematical performance during the task, Antonio encountered fresh motivation regarding his future teaching practice: He opened up to a possible modification of his own previous beliefs manifested in Phase 1 (the learning of mathematics depends on the teacher's actions). Above all, he was aware of the influence of the affective domain on his understanding (I have the knowledge, but I must change the way I see things) (9-10).

DISCUSSION AND CONCLUSION

A mathematical situation was evaluated by Antonio's cognitive system, based on his belief system about mathematics, about his teaching and learning, about himself and about the context. These beliefs were incompatible with the reality of the context of the task, thus generating different emotions. His facial and body expressions during the episode provided emotional evidence of his uncertainty, frustration, distress and relief. These emotions, in turn, generated emotional responses (tension and blockage), and the latter determined his decisions of action. Antonio's affective system thus intervened in his mathematical practice, conditioning the uses given to mathematical knowledge and providing reasons for his understanding of measurement.

The configuration of theoretical frameworks that help to understand the role of affect in mathematical learning is an ongoing objective in the field of Mathematics Education (Goldin, 2004; Hannula, 2012). Another goal is to define procedures that allow to interpret the acknowledged relationship between affective domain and cognition in mathematics (Schlöglmann, 2002). The specific contribution of our study is an approach that enables exploring the understanding of mathematical knowledge through different components of affect. Identifying the relationship between affective domain and understanding allows to obtain a more accurate assessment of the students' actual mathematical understanding. Such an assessment will help us in the future to guide students' affective responses towards learning with understanding.

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DRAWING INSTRUCTIONS, STRATEGIC KNOWLEDGE, STRATEGY-BASED MOTIVATION, AND STUDENTS' USE OF DRAWINGS

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Although self-generated drawing is a powerful strategy in the domain of geometry, students lack spontaneous use of the drawing strategy. In the current study, we investigated instructional, cognitive, and motivational predictors of students' drawing use. We first assessed strategic knowledge about drawing and strategy-based motivation in 132 students in Grades 9 and 10. Then, students were randomly assigned to solve geometry modelling problems either with or without drawing instructions. Students with drawing instructions constructed more drawings than students without drawing instructions. Strategic knowledge about drawing, self-efficacy expectations, and perceived costs predicted drawing use while intramathematical abilities were controlled for. Utility value did not predict drawing use in the current study.

INTRODUCTION

Self-generated drawing is considered a powerful strategy for finding a solution to a geometry modelling problem. Although 13- to 15-year-old students are familiar with the strategy of self-generated drawing, many of them do not spontaneously use the drawing strategy (Uesaka et al., 2007). One way to increase students' use of drawings is to explicitly instruct them to make a drawing before solving a modelling problem. Previous research has indicated that a notable proportion of students still do not make a drawing even when instructed to do so (De Bock et al., 1998). Explanations for students' lack of drawing use include strategy-based cognitive and motivational factors. In the current study, we investigated how drawing instructions, strategic knowledge about drawing, and strategy-based motivation (self-efficacy expectations, utility value, and perceived costs) predict students' use of drawings to solve geometry modelling problems.

THEORETICAL BACKGROUND

The use of learner-generated drawings to solve geometry modelling problems

Past research has repeatedly shown that students experience diverse difficulties when solving modelling problems (e.g., Galbraith & Stillman, 2006). Modelling problems are ill-defined mathematical problems with a connection to reality that, amongst other functions, allow students to make realistic assumptions and apply different mathematical solution methods. An exemplary modelling problem is presented in Figure 1.

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Cable car

The municipality of Engelsberg needs to replace the steel rope of holding the cable car. One meter of the steel rope costs 9 €. How much will the new steel rope cost? The following data on the cable car are available:

Model:	Engelsberg cable car
Bottom station:	1,023 m above sea level
Top station:	1,605 m above sea level
Horizontal difference:	1,041 m
Transportation capacity:	585 passengers per hour
Driving speed:	9 m/s

Figure 1: Exemplary modelling problem *Cable car*.

One way to help students overcome difficulties in the modelling process is to instruct them to use powerful strategies, such as self-generated drawing (Galbraith & Stillman, 2006). The strategy of self-generated drawing describes the process of constructing a structurally analogous representation of the modelling problem on paper and to use it as a problem-solving aid (Van Meter & Firetto, 2013). From a theoretical perspective, the drawings that are used to solve modelling problems can be classified as situational or mathematical drawings (see Figure 2).

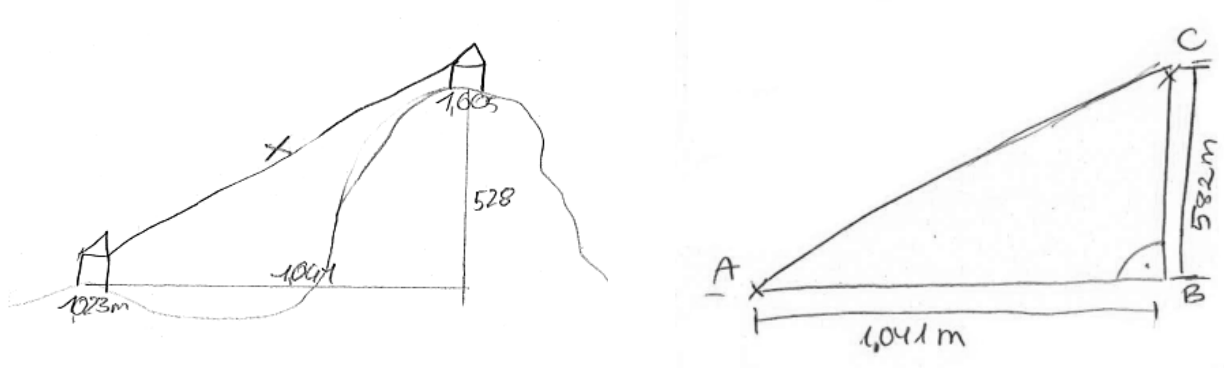

Figure 2: Exemplary situational (left) and mathematical (right) drawings by students.

The Cognitive Theory of Drawing Construction (Van Meter & Firetto, 2013) supports the assumption that making a drawing can help students work through the modelling process because it promotes the construction of mental models. A previous study confirmed that situational and mathematical drawings are powerful types of drawings that can help students solve a geometry modelling problem because students who made a more accurate situational or mathematical drawing solved the geometry modelling problem better than peers with a less accurate drawing (Rellensmann et al., in press). For example, a student can benefit from making a drawing for the modelling problem *Cable car* because the drawing may help them understand the relationship between the top and bottom stations or may help them figure out that the information they are looking for can be modelled as the hypotenuse of a right-angled triangle.

Drawing instructions

Although self-generated drawing is a powerful problem-solving strategy, previous research has shown that students often do not make drawings spontaneously (De Bock et al., 1998; Uesaka et al., 2007). For example, Uesaka and Manolo (2012) reported that 38% to 70% of students made spontaneous use of the drawing strategy to solve geometry word problems. One instructional approach that can be used to increase students' use of drawings is to explicitly ask them to make a drawing. Still, notable proportions of students do not follow such instructions (De Bock et al., 1998). Explanations for why students do not use drawings consist of deficient strategic knowledge about drawing (Rellensmann et al., in press) and a lack of strategy-based motivation (Uesaka & Manalo, 2012).

Strategic knowledge about drawing

Strategic knowledge about drawing (SKD) is specific strategic knowledge (Borkowski et al., 2000). It includes knowledge about the characteristics of an accurate drawing for solving a modelling problem, including the adequate representation of relevant objects and their relationships and complete labelling with relevant numbers (Rellensmann et al., 2020). According to the theoretical model proposed by Borkowski et al. (2000), SKD is an important precondition for the use of the drawing strategy. A recent study showed that improving students' SKD with strategy training resulted in more accurate drawings (Rellensmann et al., in press). Whether SKD predicts students' use of drawings has not yet been investigated.

Strategy-based motivation

Strategy-based motivation (SBM) is motivation that derives from the characteristics of strategies and their use. Referring to the expectancy-value theory of motivation (Eccles & Wigfield, 2020), SBM stems from specific manifestations of expectancies and value appraisals and can explain strategy-related decisions (e.g., the use of the drawing strategy). In the current study, we examined whether drawing-related SBM (self-efficacy expectations, utility value, and perceived costs) would predict students' use of drawings.

Regarding the drawing strategy, self-efficacy expectations comprise a student's confidence in being able to construct accurate drawings to solve modelling problems. A student with high self-efficacy expectations would give an affirmative answer to the question "Are you confident that you can make a very good drawing for any modelling problem?" Previous studies have found that self-efficacy expectations are positively related to drawing use (Uesaka et al., 2007) and drawing accuracy (Schukajlow et al., 2021). One explanation is that students who have more confidence in their abilities to generate accurate drawings set higher goals and engage in deeper learning processes compared with students who have less confidence in their drawing abilities. To date, no studies have investigated whether self-efficacy expectations affect drawing use.

Drawing-related value appraisals comprise utility value (Barron & Hulleman, 2015), that is, a student's belief that the activity of making a drawing is helpful for achieving their goals (e.g., solving the modelling problem). Previous empirical findings indicate that utility value predicts strategy use only when students have free choice of strategies: In studies on the spontaneous use of drawings, utility value positively predicted drawing use (Blomberg et al., 2020; Uesaka et al., 2007), whereas utility value did not predict drawing use when students were instructed to make a drawing (Schukajlow et al., 2021).

Another component in expectancy-value theories is the component of perceived costs (Eccles & Wigfield, 2020). Perceived costs of drawing comprise a student's belief about the amount of time and effort they need to invest to make a drawing. Previous research found negative relationships between the objective costs of drawing and spontaneous drawing use (Uesaka & Manalo, 2012) and negative relationships between perceived costs and drawing accuracy when students were instructed to make a drawing (Schukajlow et al., 2021). To date, it is an open question whether the perceived costs associated with making a drawing impede students' use of drawings.

RESEARCH QUESTION AND HYPOTHESES

In the current study, we investigated the following research question: Do drawing instructions, SKD, and SBM (self-efficacy, utility value, and perceived costs) predict students' use of drawings while mathematical abilities are controlled for? We expected that drawing instructions, SKD, self-efficacy expectations, and utility value would positively affect students' use of drawings, whereas perceived costs would negatively affect students' use of drawings.

METHOD

Procedure and participants

Participants were 132 students (45% female, 15–16 years old) in Grades 9 and 10 and in the middle achievement track of two German secondary schools. Students were randomly assigned to one of two groups: instructions to make a situational or mathematical drawing for each modelling problem ($n = 91$) and no instructions to make a drawing ($n = 41$). We aggregated students with situational and mathematical drawing instructions into the group *with drawing instructions* because the analyses did not reveal any differences between the groups with different drawing instructions. Data were collected on two different occasions to reduce the possibility that students in the control condition would be inadvertently prompted by the questionnaire to generate drawings. On the first data collection date, students worked on the test of intramathematical abilities, the strategic knowledge test about drawing, and the strategy-based motivation questionnaire. On the second data collection date, students were asked to solve eight modelling problems with or without drawing instructions.

Measuring instruments

Intramathematical abilities. To control for students' intramathematical abilities, we asked students to solve mathematical tasks without a connection to reality (10 items). For example, students were asked to set up an equation that fit a right-angled triangle or to solve a quadratic equation. Students' solutions were scored 0 (incorrect solution) or 1 (correct solution).

Strategic knowledge about drawing. The SKD scale (16 items) was developed and pilot tested in previous studies (Rellensmann et al., 2020). To solve an item from the SKD scale, students were asked to use a Likert scale to rate how helpful three situational drawings and three mathematical drawings were for solving a word problem. The three drawings that were provided differed in their accuracy. Students' evaluations of the three drawings were scored from 3 to 0 with respect to their accuracy.

Strategy-based motivation. To answer the strategy-based motivation questionnaire, students rated statements about themselves and their strategy-based motivation on a 5-point Likert scale. The items formed scales representing self-efficacy expectations (e.g., "I am confident that I can make a very good drawing for any word problem," 4 items), utility value (e.g., "I believe that it is important to make a drawing because making a drawing can help me solve a difficult word problem," 4 items), and costs (e.g., "I have to put forth a lot of effort to make a drawing for a difficult word problem," 3 items).

Drawing instructions. On the second data collection date, students worked on eight geometry modelling problems (see Figure 1). Students' group assignment was dummy coded: 0 (without drawing instructions) or 1 (with drawing instructions).

Drawing use. For each of the eight modelling problems, a student's use of a drawing was coded. When the student did not make a drawing, a code of 0 was given. When the student made a drawing, a code of 1 was given.

Interrater reliabilities (Fleiss' $\varkappa > .84$) and scale reliabilities (Cronbach's $\alpha > .64$) were satisfactory for all scales.

RESULTS

Correlations, means, and standard deviations for the investigated variables are presented in Figure 3. All correlations were in the expected directions, as SKD, self-efficacy expectations, and utility value were positively related to the use of drawings, and costs were negatively related to the use of drawings. Across the eight modelling problems, on average, 33% and 21% of the students made a drawing for a modelling problem in the groups with and without drawing instructions, respectively.

As we found notable correlations between the SBM components (e.g., $r = -.41$ between self-efficacy expectations and perseived costs), we computed multiple regression analyses with self-efficacy expectations, utility value, and perceived costs as simultanous predictors (Model 1) or separate predictors of students' use of drawings (Models 2a-c) (Figure 4).

	Intramathematical abilities (1)	Strategic knowledge about drawing (2)	Self-efficacy expectation (3)	Utility value (4)	Cost (5)	Use of drawings (6)
(1)	1	.10	.21*	-.10	-.23*	.31**
(2)		1	.05	-.04	-.13	.23*
(3)			1	.36**	-.41**	.30**
(4)				1	-.25**	.14
(5)					1	-.31**
(6)						1
M	0.26	1.97	3.25	2.62	2.65	0.27
SD	0.24	0.46	0.75	0.85	0.84	0.25

Note. * $p < .05$, ** $p < .01$, p two-tailed.

Figure 3: Correlations, means, and standard deviations for the investigated variables.

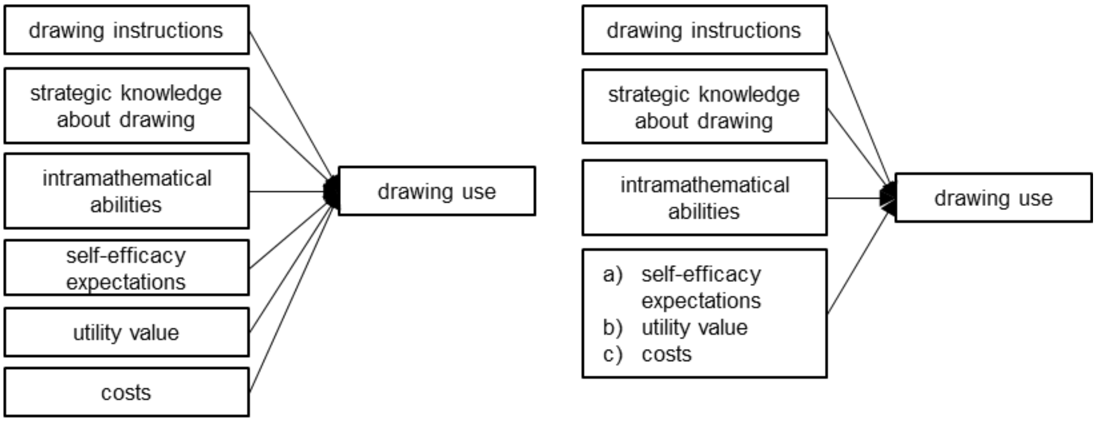

Figure 4: Model 1 with simultaneous SBM predictors (left) and Models 2a-c with separate SBM predictors (right).

In Model 1 with simultaneous predictors, we found that drawing instructions ($\beta = .19$, $p < .05$), SKD ($\beta = .19$, $p < .05$), and self-efficacy expectations ($\beta = .28$, $p < .01$) predicted students' use of drawings while intramathematical abilities were controlled for ($\beta = .17$, $p < .05$). Perceived costs ($\beta = -.11$, $p > .05$) and utility value ($\beta = .01$, $p > .05$) did not predict students' use of drawings.

In Models 2a, 2b, and 2c, we computed regression models with drawing instructions and SKD as predictors and intramathematical abilities as a covariate. We also entered self-efficacy expectations, utility value, and costs as separate predictors one at a time. In line with our hypotheses, self-efficacy expectations ($\beta = .31$, $p < .01$) and perceived costs ($\beta = -.22$, $p < .05$) were significant predictors of students' drawing use. Contrary to our hypothesis, utility value did not predict drawing use ($\beta = .04$, $p = .67$).

DISCUSSION

In line with previous research, we found that large proportions of students lacked spontaneous drawing use or did not follow drawing instructions to solve geometry modelling problems. This study contributes to previous research on drawing use, as we identified instructional, cognitive, and motivational predictors of drawing use while controlling for students' intramathematical abilities. First, we found that drawing instructions are an instructional means for overcoming students' lack of spontaneous drawing use. Thus, teachers might explicitly ask students to make a drawing to solve a modelling problem to give students more experience with the drawing strategy.

Second, as hypothesized in the model by Borkowski et al. (2000), we found that students with good SKD used drawings more often than students with lower SKD. This finding adds to previous research that showed that SKD is an important prerequisite for drawing accuracy (Rellensmann et al., 2020). Thus, teachers should aim to create opportunities for students to develop their SKD. Strategy training, which can increase students' SKD, involves instructional elements (e.g., comparing drawings of varying accuracy) that can be used to promote students' SKD (Rellensmann et al., in press).

Third, we found support for Borkowski et al.'s (2000) hypothesis that SBM affects strategy use. Our results extend Borkowski et al.'s (2000) model by indicating which components of SBM are particularly important for strategy use. As hypothesized, we found that students with high self-efficacy expectations used drawings more often than peers with lower self-efficacy expectations. Also, students who perceived drawing as too cost-intensive (i.e., taking too much time and effort) did not use drawings as much as students who perceived drawing to be less cost-intensive. Due to the strong correlation between self-efficacy expectations and costs, the effects of costs were no longer statistically significant when self-efficacy expectations were simultaneously considered in the regression model. Thus, self-efficacy expectations were found to be the stronger predictor of students' drawing use. Contrary to our expectations, utility value did not predict drawing use in the current study. One explanation is that utility value is powerful in educational settings that give students a choice between different strategies (e.g., Uesaka & Manalo, 2012). Thus, the current findings suggest that the promotion of SBM will help students make use of the drawing strategy. Further, prior research has demonstrated ways to enhance SBM in educational settings (Eccles & Wigfield, 2020). One way is for teachers to scaffold students' drawing construction (Zhang & Fiorella, 2019) to facilitate a mastery experience, thus enhancing students' self-efficacy expectations and reducing the perceived costs associated with drawing.

In the current study, we investigated relationships between strategy instructions, strategic knowledge, strategy-based motivation, and strategy use for self-generated drawing. Further research should confirm the relationships that were hypothesized in Borkowski et al.'s (2000) model for other strategies (e.g., backward or forward strategies).

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MATHEMATICS AS GENDERED? VIEWS FROM PALESTINIAN/ARAB ISRAELI HIGH SCHOOLERS

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In light of a gender-gap in school mathematics favoring girls among Palestinian/Arab Israelis, this study explores 147 high school students' views of mathematics with respect to gender. We administered a "Who and Mathematics" survey used in a prior study. Findings include gender-neutrality about teacher interactions, parent expectations, and future employment. Negative relations with mathematics were attributed to boys. A broader set of narratives were attributed to girls, including items about caring or worrying about success and finding mathematics interesting. These results challenge an assumption that the social construction of mathematics as masculine is universal and present a picture of how gender shapes and is shaped by mathematics education for Palestinian/Arab Israelis.

INTRODUCTION

Israel presents an interesting case with respect to gender and mathematics achievement. On the one hand, across schools separated by language of instruction, Israel posts large differences between its Jewish majority (in Hebrew-speaking schools) and Palestinian/Arab Israeli (P/AI) minority (in Arabic-speaking schools) (Rapp, 2015). These achievement differences can be interpreted as part of a legacy of inequitable access and opportunity. In Israel's Hebrew-speaking schools, there is a persistent gender gap in mathematics and science participation that favors boys (Friedman-Sokuler & Justman, 2020). In those schools, which represent Israel's majority, boys are slightly over-represented in advanced mathematics and heavily over-represented in physics and computer science coursework. In contrast, however, in Israel's Arabic-speaking schools, girls tend to outperform boys–on state and international mathematics tests and at all school levels– and are over-represented in advanced mathematics, physics, and computer science coursework (Pinson et al., 2020). Despite their eligibility, however, few Palestinian/Arab Israeli women continue to higher education in STEM fields (Fuchs, 2018); instead, teacher education is the most common undergraduate path of study (Arar & Haj-Yahia, 2016).

Prior research has pointed to various potential explanations for an absence of a gender gap favoring boys in mathematics achievement and participation at K-12 levels. For one, Palestinian/Arab Israeli girls are said to have limited opportunities for success, outside of school, as girls or women (Nasser & Birenbaum, 2005). Furthermore, the structure of Israel's separate school system has provided a narrower range of kinds of advanced course offerings to Arabic-speaking schools (Ayalon, 2002), positioning mathematics as a singular option for excellence. A third explanation, of particular

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interest for this study, is that the social gendering of mathematics as masculine common elsewhere and among Israel's majority (Markovits & Forgasz, 2017) might not extend to Palestinian/Arab Israelis. Our goal here is to explore this conjecture, by surveying high school students among Palestinian/Arab Israelis about gender and mathematics.

PRIOR RESEARCH

Guided by prior research that deemphasizes any biological differences between sexes, our perspective is, instead, a social one: We view gender as a social construct that is performed in different ways across individuals and across contexts (Butler, 1990). In mathematics education, participation and success can be understood in terms of gendered stories about mathematics in circulation in a society: these stories are seen to shape classroom interactions, between teacher and students, and among students, and to inform students' identity development with respect to mathematics (Mendick, 2005). One explanation for gender-based differences in mathematics achievement and participation is in terms of the stories or narratives that are told and retold that construct mathematics as a masculine domain (Leyva, 2017). Many interrelated narratives comprise binary but unequal oppositions -- like analytic/emotional, objective/subjective, innately able/ hard-working, confident/lacking confidence-- wherein one in every pair is more valued *and* socially associated with masculinity (Mendick, 2005).

One set of related studies in the literature explores the gendering of mathematics through a focus on *teachers*. For example, Fenemma et al. (1990) and later, Tiedemann (2000) and Sarouphim and Chartnouy (2017), in the United States, Germany, and Lebanon respectively, point to a tendency among teachers to attribute boy's success to innate ability and girls' success to effort. These studies speculate that these narratives shape teachers' instruction and ways of relating with students in ways that benefit boys. A parallel set of studies explores *children and students.* Children's adherence to gendered stories about mathematics in which mathematics is masculinized has been shown among students at all levels, from college to elementary school and even in pre-school contexts and across a wide range of geographical contexts (e.g., Cvencek et al., 2015; Passolunghi et al., 2014).

In the Israeli context specifically, recent findings include that elementary school boys tend to more highly rate their own achievement in mathematics than girls rate theirs (Markovits & Forgasz, 2017). Whereas students tended to indicate that girls and boys are equally good at math, when shown two photographs, the majority indicated that the image perceived to be a "man" was more likely to use mathematics at work than the image perceived to be a "woman" (Forgasz & Markovits, 2018). These studies suggest that children do not adhere to explicit gendered attributions of ability in mathematics but that their sense of who participates in mathematics as adults remains gendered. However, these recent studies of Israeli school-children included only students in Hebrew-speaking schools.

In terms of Palestinian/Arab Israeli school students, the population of interest in this study, the field's knowledge rests on Forgasz and Mittelberg's (2008) comparative survey of 9th graders. In that study, Forgasz and Mittelberg compared responses of Palestinian/Arab Israelis with Jewish Israelis and with Australians and found that the Palestinian/Arab Israeli students' responses were *less* consistent with dominant narratives about gender and mathematics. An absence of a masculinazation of mathematics is a potential explanation for the success and participation of Palestinian/Arab Israeli girls and women in school mathematics. Forgasz and Mittelberg's (2008) findings are surprising, however, since the Palestinian/Arab Israeli society, in general, is considered traditional about gender roles, at least with respect to caretaking, food preparation, and childrearing (Haj-Yahya et al., 2018). Nearly 15 years later, in light of broader changes towards gender equity around the world, we are curious to repeat Forgasz and Mittelberg's survey, with a larger number of participants. Our research interest is not in intra-group comparisons, but rather, to explore Palestinian/Arab Israeli high school students' views of gender and mathematics.

METHODS

We utilized the survey "Who and Mathematics" from Forgasz and Mittelberg (2008) with our own translation of the items into Arabic. The survey comprises 30 Likert-scale items; each indicates a statement–for example, "Give up when they find a math problem too difficult" (item 4), with an associated prompt to select one from among: 1 (boys definitely more likely than girls), 2 (boys probably more likely than girls); 3 (no difference between boys and girls); 4 (girls probably more likely than boys); 5 (girls definitely more likely than boys). We administered the survey to students at four Arabic-speaking schools in Israel, in an electronic format. Participants include 147 people: 72 ninth-graders (41 identify as girls, 31 as boys) and 75 eleventh-graders (35 identify as girls and 40 as boys). We chose high-school because we assume that students at this age group are highly reflective about teachers, parents, and future employment. We chose two different age groups within high school as a way to determine if students' responses at each grade level are different.

For each item, we computed averages and standard deviations of responses. We compared the responses of girls with boys, and compared the responses of 9th graders with 11th graders, using t-tests. There were few instances of differences across these subgroups: girls and boys answered only two items differently (items 4, 10) and 9th and 11th graders answered only three items differently (items 18, 23, 25). We aggregated the responses of all 147 students. Next, using t-tests, we checked to see which items had averages statistically close to 3 (attributed to neither boys nor girls), which items had averages significantly less than 3 (attributed to boys), and which items had averages significantly greater than 3 (attributed to girls).

RESULTS

Narratives Not Assigned to Gender

On 18 of the 30 items, the average scores were not statistically distinguishable from the value *3*, meaning that these items were not attributed on average to any gender. For 10 of these items, which we show in Table 1, our results confirm the results of Forgasz and Mittelberg and, furthermore, challenge the gendered stories predicted by the literature. Included in this set are finding mathematics easy or difficult (items 18, 27), liking challenging mathematics problems (item 11), enjoy math or it’s their favorite subject (items 6, 1), and multiple items about interactions with teachers (items 3, 12, 25).

Item		Predicted by Literature, as reported by F&M	F&M (2008) Nd = no difference	Mean	Std dev
1	Math is their favorite subject	M	nd	2.891	1.148
2	Think it's important to understand math	F	nd	3.163	1.034
3	Are asked more questions by your math teacher	M	nd	3.054	1.103
6	Enjoy mathematics	M	nd	3.177	1.127
11	Like challenging mathematics problems	M	nd	2.932	1.121
12	Are encouraged to do well by their math teacher	M	nd	3.054	1.032
18	Find mathematics easy	M	nd	2.939	1.093
20	Need help in mathematics	F	nd	2.871	1.195
25	Mathematics teachers spend more time with them	M	nd	2.905	1.009
27	Find mathematics difficult	F	nd	2.857	1.123

Table 1: Items Attributed Neutrally, In Agreement with Forgasz & Mittelberg (F&M, 2008).

In addition to the ten items in Table 1, we found an additional eight neutrally designated items on average, which we share in Table 2. Here we have items about parental expectations (items 9, 19), teacher expectations (item 13), and the meaning of mathematics for one’s future (items 10, 14). These items were found to be gendered in previous literature and found to be gendered in Forgasz and Mittelberg’s study, but in our study, were found to be neutral.

Gendered Narratives

The other 12 items produced results statistically different from the value *3*, meaning that they were attributed on average to girls or to boys. There were five items that were attributed on average to boys, which we show in Table 3. In four of the five cases, these results confirm the earlier results from Forgasz and Mittelberg. All of these are negative statements – about negative relations with mathematics [consider mathematics to be boring (item 26), giving up in response to difficulty (item 4), being not good at mathematics (item 23)] or about negative classroom behaviors [distracting others (item 16), teasing boys who are good at mathematics (item 21)].

Item		Predicted	F&M, 2008	Mean	Std dev
9	Parents would be disappointed if they don't do well in math	M	M	3	1.104
10	Need math to maximize future employment opportunities	M	M	2.959	1.042
13	Math teachers think they will do well	M	F	3.014	1.135
14	Think mathematics will be important in their adult life	M	M	2.98	1.101
15	Expect to do well in mathematics	M	F	2.98	1.101
17	get the wrong answers in mathematics	F	M	2.932	0.956
19	Parents think it is important for them to study mathematics	M	M	3.048	0.968
30	Tease girls if they are good at mathematics	M	M	2.898	1.065

Table 2: Items Attributed Neutrally, In Disagreement with Forgasz & Mittelberg (F&M, 2008).

Item		Pred	F&M 2008	Mean	SD	t	p
4	Give up when they find a math problem too difficult	F	nd	2.769	1.147	-2.445	0.016
16	Distract other students from their mathematics work	M	M	2.408	1.145	-6.265	<.001
21	Tease boys if they are good at mathematics	M	M	2.639	1.027	-4.258	< .001
23	Are not good at mathematics	F	M	2.741	1.073	-2.92	0.004
26	Consider mathematics to be boring	F	M	2.605	1.101	-4.343	< .001

Table 3: Items Attributed to Boys.

Finally, participants on average attributed seven items to girls, which we show in Table 4. Six of these items had previously been found by Forgasz and Mittelberg to be gender neutral. Some of these items have traditionally been associated with boys, about the link between preparation and success (item 8), liking using computers to do mathematics (item 24), and finding mathematics to be interesting (item 29). Other items that have traditionally been found to be gender neutral that pertain to success and diligence [care about doing well in math (item 7), worry if they do not do well in math (item 22)] here were attributed to girls. We summarize results and how they confirm or challenge Forgasz and Mittelberg's results in Table 5.

Item		Pred	F&M 2008	Mean	SD	t	p
5	Have to work hard in math to do well	F	F	3.293	1.21	2.94	0.004
7	Care about doing well in math	nd	nd	3.184	1.05	2.13	0.035
8	Think they did not study hard enough if they did not do well in math	M	nd	3.279	1.18	2.89	0.004
22	Worry if they do not do well in mathematics	nd	nd	3.293	1.16	3.06	0.003
24	Like using computers to work on mathematics problems	M	nd	3.197	1.08	2.22	0.028
28	Get on with their work in class	F	nd	3.224	1.12	2.43	0.016
29	Think mathematics is interesting	M	nd	3.231	1.06	2.65	0.009

Table 4: Items Attributed to Girls.

Outcome	Forgasz & Mittelberg 2008	Present study
Girls or boys: No difference	1, 2, 3, 6, 11, 12, 18, 20, 25, 27 4, 7, 8, 22, 24, 28, 29	1,2,3, 6, 11, 12, 18, 20, 25, 27 9, 10, 13, 14, 15, 17, 19, 30
"Boys more likely"	16, 21, 23, 26, 9, 10, 14, 17, 19, 30	16, 21, 23, 26, 4
"Girls more likely"	5, 13, 15	5, 7, 8, 22, 24, 28, 29

Italicized items are common, Underlined items are reversed from prior literature.

Table 5: Comparison of Current Results with Previous Results.

DISCUSSION

We note how among the 12 items found to be gendered, the direction of the gender stereotype in half of them reverses the predictions based on findings in the literature pertaining to Western contexts about masculinization of mathematics. In some cases, (items 4, 23, 26), the students attributed items to boys about negative relations with mathematics, about their finding it boring or being not good at or giving up in the face of challenge. In contrast, the students attributed to girls (items 2, 24, 29) interest in mathematics, enjoying working with technology, and an expectation that their own hard work will produce success. This is and of itself is significant because it shows how the masculinization of mathematics that is endemic in some cultures or geographies does not seem to be universal.

Our results correspond to Forgasz and Mittelberg's results with respect to 15 of the 30 items. We had expected that over the passage of 15 years, more items would shift to the gender neutral category, because of increasing gender equity, but this was not the case. Some of the items that had previously been found to be gendered (9,10,13, 14, 17, 19, 30), in this study then fell into the neutral category – these include parental or teacher expectations (9, 13, 19) and use of mathematics later in life (10, 14). The fact that these are considered neutral by high school students likely reflect recent changes in society especially in terms of more opportunities in higher education and employment for Palestinian/Arab Israeli women.

However, there were seven items that had been previously found by Forgasz and Mittelberg to be neutral, but in this study were attributed to a gender group, and mostly, to girls (7, 8, 22, 24, 28, 29). We note that many of these items pertain to success or "doing well"– caring or worrying about success and diligence with classwork – along with attributing to girls the finding of mathematics as interesting. Unlike the previous study, here more items were assigned to girls. These items communicate views of a *positive* alignment between girls and mathematics and their interest in or commitment to success, accomplishments, or recognition. Our findings are limited by the survey methodology which does not allow for insights into why students answered these items in these ways or how they might make sense of connections across the items. Further studies can complement these findings using additional qualitative methods.

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SINGLE UNIT COUNTING – AN IMPEDIMENT FOR ARITHMETIC LEARNING

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In this paper we direct attention to the single unit counting strategy that is observed to be limiting students' opportunities to develop their arithmetic skills. We describe what impediments single unit counting may entail when encountering novel subtraction tasks, and how these impediments can be explained. From a sample of 121 interviews of students aged 7-8 we have chosen nine who were using single unit counting as their dominating arithmetic strategy. An analysis based on variation theory reveals that the impediments are related to the students not experiencing numbers as composed units and thereby lack in discerning number relations necessary to handle multi-digit subtraction. Educational implications are discussed grounded in the theoretically driven findings.

INTRODUCTION

In the large body of research on young children's arithmetic development, many scholars have described strategies for arithmetic problem solving (e.g., Fuson, 1992; Baroody & Purpura, 2017 for overviews), ending up in recommendations for how to teach arithmetic strategies (e.g., Baroody, 1987; Torbeyns et al., 2004). What seems to be lacking is however a critical view on the observed strategies, whether the observed trajectory in fact reflects a powerful path to proficient arithmetic problem solving skills, which likely have implications for teaching practices. In this paper we aim to raise attention to particularly one of the basic strategies observed in early arithmetic development and in teaching – single unit counting – and raise some issues on what implications students' single unit counting strategies may have in a long-term perspective. The specific research questions are: 1) what impediments may single unit counting as the dominating arithmetic strategy entail when encountering novel subtraction tasks, and 2) how can these impediments be explained? To answer these questions, we analysed task-based interviews with school beginners (age 7-8 years in Swedish schools).

RESEARCH ON EARLY ARITHMETIC STRATEGIES

There are many studies describing the trajectory of arithmetic skills development, but in this paper, we direct specific attention to single unit counting. Counting strategies are ways to keep track of counted units, either by raising one finger at a time (representing the single units counted) or by making markers, one for each counted unit. These are commonly observed among young children (Laski et al., 2014) and

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some researchers find this to be a normal step in the trajectory of learning arithmetic (Fuson, 1988). Furthermore, counting by ones may also entail double counting that is keeping track of the sequence of words that become the entities to be counted, a strategy which Steffe (2004) interprets as the student having constructed a scheme of the number sequence which in turn bridges to strategic arithmetic reasoning and is thereby considered a higher level of functioning.

Single unit counting strategies (such as "counting all" and "counting on") may solve simple arithmetic tasks but do not *per se* support students' recognising a part-whole structure of an arithmetic task, which becomes necessary to solve more advanced tasks, for example multi-digit subtraction. The shortcomings of single unit counting as the dominating strategy among young students were shown in Neuman's (1987) studies and later in studies by Ellemor-Collins and Wright (2009), in which students who rely on single unit counting strategies did not develop efficient arithmetic skills. Research influenced by cognitive science conclude that students are "forced" to develop more advanced strategies when the number range exceeds 10 and concrete units like fingers can no longer support their keeping track (Carpenter & Moser, 1982). However, there seems to be a need for educational interventions for some students to learn to discern the relation between and within numbers and thus make use of more efficient strategies than single unit counting. What is it then that these students lack in order to develop and broaden their repertoire of arithmetic strategies? This becomes a critical issue in educational research and practice, because studies from teaching interventions show that learnt single unit counting strategies are not easily abandoned by students (Cheng, 2012).

When arithmetic strategies' limitations are discussed in the literature, this is mainly in relation to students having mathematics difficulties (e.g., Ostad, 1998; Geary et al., 2004). Not surprisingly, these studies show that students who rely on counting strategies have difficulties solving novel problems because they lack in conceptual understanding of arithmetic. From the research literature, we see that students are observed making use of single unit counting strategies but often abandon these for more powerful strategies. Nevertheless, research also shows that this is not true for all students and normally developing students may prefer the cumbersome strategies even when encountering larger number ranges (Ellemor-Collins & Wright, 2009). It surprises us that not more attention is directed towards the single unit counting strategy and the limitation it may entail for students' developing arithmetic skills.

METHODS

This is a study of young students solving arithmetic tasks, as part of a larger project focusing on early arithmetic teaching and learning. Teachers and students from five elementary schools participate in the project. The classes were selected due to their teacher's interest in participating in a practice-based research project where their teaching was target for development and study. To follow any learning progress among the students in these classes, each student was asked to participate in task-based interviews at three occasions. The students' legal guardians were asked for written consent, which included the option of participating in video-recorded interview or only audio-recorded. This ended up in 121 students participating in the interviews at three occasions.

Procedure

The interview guide consists of arithmetic tasks given orally or on paper. To increase verbal reasoning and reflections among the students, no manipulatives were given to the students. The interviews were conducted individually at the students' own schools, by researchers trained in interviewing young students. The interview occasions were done at the beginning of Grade 1 (Interview I), at the end of Grade 1 (Interview II) and at the end of Grade 2 (Interview III).

For the purpose of this particular analysis, we selected subtraction tasks from the interviews that were the same in all three interviews: **10–6=_, 15–7=_, 24–_=15**, 14–_=6, a new subtraction task given in Interview II: **32–25=_**and new ones in Interview III: **57–38=_,** 83–7=_, 204–193=_, 204–12=_ (**bold** = oral context based tasks, e.g., "you have ten candies and eat six of them, how many are left?", normal text = written tasks with only numerals).

Analysis

All students participating in the interviews were coded for strategy use, either *Counting* or *Structuring.* If a student was coded as structuring, it meant the student reasoned her way to an answer to a particular task by experiencing numbers in the task as parts and whole, dealing with larger units than one (e.g., task 15–7=_ "I take five from the seven, that makes ten and then two more and I have eight left). For the purpose of our research interest, we selected those students who at the *first interview only made use of a single unit counting strategy* when attempting to solve the three subtraction tasks. We chose subtraction as target tasks, since these are more likely to induce counting-strategies if a student does not have a repertoire of number facts or know how to apply the complement principle (thus experiences numbers' part-whole relations and being able to use structuring strategies). This ended up in a sample of 39 students. These were followed through the second and third interview, resulting in three groups: students who abandoned the single unit counting strategy and approached subtraction tasks in the later interviews by structuring numbers (N=26), students who expressed a mix of structuring and single unit counting strategies (N=4) and students who remained using single unit counting as the main strategy in all three interviews (N=9). The nine students who remained using single unit counting through all three interviews are chosen for further analysis in this paper.

To answer RQ1, we did a qualitative analysis of the nine students who remained using single unit counting throughout all three interviews, to find out how they encountered novel tasks. To answer RQ2, principles from variation theory (Marton, 2015) were used as analytical tools. The theory states that powerful strategies stems from powerful ways of experiencing, which presupposes the discernment of critical aspects of what is learned (the object of learning). From a variation theory perspective, learning difficulties, e.g. to solve a subtraction task like 83–7= is explained in terms of not (yet) having discerned certain aspects of the task, the numbers involved and relations between and within them. This way of analysing students' responses to arithmetic tasks ends up in categories that reveal qualitative different ways of experiencing numbers in a task. The answer to our research questions is thereby shown in such categories, where those aspects of numbers that a student discerns constitute his or her way of knowing and thus what strategies he or she is able to execute in completing the arithmetic task. The categories found among the nine single counting strategy users in our sample are presented in the following.

RESULTS

In the first and second interview we observe the students solving the subtraction tasks by counting down in ones and using fingers to keep track of counted numbers, or if the numbers exceeded the student's fingers, using other objects (e.g. sheets of paper). The strategy counting single units was thereby considered strong in the selected group of students for our further analysis. When analysing the students' ways of solving novel tasks in the larger number range in interview III, certain problems emerged that direct attention to aspects that seem to become critical for these students to discern in order to develop arithmetic skills that allow them to try to solve novel tasks in a larger number range. All of the nine students were primarily counting down in ones, but encountered difficulties when the subtrahend (to be counted down) was larger than they managed to keep track with their fingers. Thus, they had to make use of some other strategy to complete the task, usually operating with numbers in similar positions (similar to a written algorithm line-up). The strategies these students apply in their completing the tasks may bring an answer to the task, sometimes even correct ones, but as a recurring strategy we here aim to interpret how such encounters may become an impediment in the students' development of arithmetic skills.

Cannot create a composite unit of single entities

Our observations show that the students rely on counting single units as a primary strategy. In novel tasks where the number range is larger, they operate the task as a "counting down" act on the number sequence.

Task: 204–12=_

Jonas: 203, 204 (folds down one finger for each said counting word) Wait. 203, 202, 201 (stops) 200 (stops) 199 (stops) eeh, 198, 196, 194, 193 (hesitates) 192, 191 (still folding one finger for each counting word).

This observation of the student Jonas is typical. The students operate on the number sequence, but need to keep track with their fingers. What stands out is that each number (counting word) appears as a single unit and particularly bridging hundreds (or tens) does not indicate a benchmark to them. When experiencing numbers as single units in this way, ten or hundred do not mean a composed set of "ones" and thus becomes one number just like any other number in the long line of numbers in a sequence. This way of experiencing numbers makes the counting sequence an important asset to apply the single unit counting act on, which we in the observation above can see becomes an obstacle when counting "backwards" while having to keep track of the number of counted (spoken) counting words. When the number of counted single units (the subtrahend) is large, this entails a severe challenge, because of the difficulties to keep track of counted units.

Task: 57 – 38 =_

George:	(unfolds one finger at a time on his right hand, then on his left hand and on the right hand again) Thirty. It's thirty.
Interviewer:	Did you count up or down?
George:	Up, no down, down, down from 57. To 25, I think.

The student George encounters a problem when it becomes necessary to keep track of single units and does not experience any benchmark in the counting sequence that could indicate larger units to relate to. The same student George responds to the task 204–193=_ by saying: "Wait, this one is impossible. It's too difficult". His response indicates that the strategy he executed in earlier subtraction tasks would not be helpful in solving the subtraction task with such a large subtrahend (that is, counting down 193 single units). He does not either try to solve the task by any other strategy.

Number relations – What to add and what to subtract

Students who realize they cannot execute the "counting down in single units" strategy when encountering the subtraction tasks may turn to another strategy based on an algorithmic-like approach. This means, the students are operating on the numbers based on their position, reminding of written calculations. However, when executing this strategy mentally, our observations reveal that these students do not necessarily experience multi-digit numbers as composed units, but rather operate on the numbers as if they were single units. We can see expressions of this way of experiencing numbers when students complete the task 83–7=_ by first operating on the three and the seven, then realizing the eight should also be part of the operation, for example as one of the students, Vera, starting with "three plus seven", then continuing saying "eighty… eight-hundred-ten, no, eighty, eight-hundred-one". This way of reasoning indicates difficulties in experiencing how numbers relate to each other and particularly how ones and tens, as well as hundreds relate. Below is another example of a similar way of experiencing numbers that frequently appear in our sample when encountering larger number ranges.

Task: 204–193=_

Jenny:	(unfolds index finger, folds it again) It's one hundred ninety one. Because you take the four minus three, and then the zero minus nine makes nine and then two minus one, that's one.

The student Jenny also seems to experience numbers as single units that are to be treated as individual entities rather than composed units of tens or hundreds. The student Vera seems though to experience some sense of value difference between numbers, since she claims the result of her operation cannot become "more" as in eight hundred one. Nevertheless, the relation of ones and tens are not discerned by her. The same way of experiencing numbers is observed in how the student Jenny attempts to solve 204–193=_ by subtracting each number as "taking the smaller from the larger" and disregarding any meaning of the positions that the numerals are presented – the numbers are not related to one another as would be necessary to experience the idea of the base ten system and positioning of numbers. This way of experiencing numbers induces that, what is part and what is whole are not discerned. The students seem to attend to some kind of algorithmic-like strategy but they do not experience numbers' relations within the task, such as how digits in a multi-digit number represent tens or hundreds.

DISCUSSION

The conclusion we draw from the analysis above, is that single counting units becomes an impediment for these students when encountering multi-digit subtraction tasks, which confirms Ellemor-Collins and Wright's (2009) as well as Neuman's (1987) observations. We add to these observations that these students' ways of completing subtraction tasks may be explained by their way of experiencing numbers and the meaning assigned to numbers and their relations. When students are experiencing numbers as single entities rather than composed units, they are not discerning the relations between parts and whole within numbers and thereby not relations between numbers either. That is, ten is not seen as a benchmark either. Tens and hundreds are merely experienced as single numbers in a long line of numbers and do not represent composed units, which is why ten is not taken as a benchmark to help structure their problem solving.

To recognize and make use of number structure builds on the student experiencing numbers as composite sets that can be decomposed, and that there are numerical relations between and within numbers. For example, in subtraction the subtrahend can be decomposed into two parts in order to bridge the nearest ten (e.g., 83–7=_ , 7 is decomposed and 80 is a benchmark, 83–3=80, 80–4=76). Number relations do not appear when counting single units, for instance when keeping track of counted units on the fingers or by making markers, because number relations and experiencing units larger than one are not needed to solve the task. To prevent un-developable strategies among students and support conceptually founded knowledge, some researchers advocate that a structural approach to arithmetic problem solving, which primarily directs attention towards relationships between numbers in a task (Venkat et al., 2019) and making use of part-whole relations rather than single unit counting strategies, should be emphasised already in the early years (Brissiaud, 1992; Davydov, 1982; Neuman, 1987; Polotskaia & Savard, 2018). In following reports, we will do analyses of the teaching conducted between the interviews, to find possible keys for how teaching may influence arithmetic development that apparently is necessary for the students in our sample.

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ARAB AND JEWISH MATHEMATICS TEACHERS' ENDORSED PEDAGOGICAL NARRATIVES AND REPORTED PRACTICES

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Our aim in this study was to adjust a qualitative discursive framework on teachers' pedagogical discourse to develop a survey tool that would enable a quantitative examination of teachers' endorsed pedagogical narratives, or beliefs, and their reported practices. Furthermore, we aimed to use the survey tool to assess differences in beliefs and reported practices between Arab (N=28) and Jewish (N=45) teachers in TEAMS (Teaching Exploratively for All Mathematics Students) professional learning communities. The survey was developed around three sub-scales: Exploratory pedagogical discourse (EPD), Delivery pedagogical discourse (DPD), and mathematical literacy. Findings reveal satisfactory reliability measures of the three subscales, as well as interesting differences between the Arab and Jewish teachers.

BACKGROUND

In recent years, efforts have been made to integrate explorative teaching practices for the development of 21st century mathematical skills (e.g. Heyd-Metzuyanim, Nachlieli, Weingarden, & Baor, 2020; Seif, 2019). Studies show (e.g., Eilam, 2003) that some teachers have more difficulty than others in implementing these student-centered teaching practices. For example, in Israel, the norm in Arab schools is to value obedience and respect for adult authority. Thus, previous studies (e.g. Seif, 2019) have raised conjectures that the Arab teacher population may need extra efforts in professional development and other supports to introduce student-centred teaching practices into the classroom. Our goal in this study was to better understand the differences between Hebrew-speaking (Jewish) and Arab teachers' pedagogical discourses as they relate to explorative, student-centered instruction in mathematics.

Explorative teaching practices are important to minimize students' reliance on external authority, strengthen their mathematical reasoning, and acquaint them with the norms for establishing mathematical arguments (Schoenfeld, 2014). These practices include promoting students' authority through productive classroom discussions where students raise mathematical ideas and the teacher, as well as other students, discuss these ideas (Heyd-Metzuyanim et al., 2020). Teaching for explorative participation has some overlap with practices that promote mathematical literacy (OECD, 2018). These teaching practices involve engaging students in problem solving, using a variety of means to represent mathematical concepts, and encouraging communication about mathematics (Gleason, Livers, & Zelkowski, 2017). There are reasons to believe that student-centred instruction would promote mathematical literacy, as it promotes

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conceptual understanding and students' agency in general (e.g. Hwang, Choi, Bae, & Shin, 2018).

Despite their importance, explorative teaching practices are difficult to establish classrooms (Resnick et al., 2015). In particular, Arab classrooms have been known to be more teacher-centered than Jewish classrooms (Eilam, 2003). Because of their cultural differences, collectivist norms in Arab schools translate to more authoritarian teachers, discouraging students from argumentation, or expression of their own opinions (Eilam, 2003; Hwang et al., 2018).

One of the concerning issues among researchers and educators is the difficulty to instigate changes to teachers' practices (e.g. Kennedy, 2016). One reason for the difficulty is teachers' beliefs or the Pedagogical Discourses they align with (Heyd-Metzuyanim, Greeno, & Munter, 2018). Previous studies (Heyd-Metzuyanim et al., 2018; Heyd-Metzuyanim & Shabtay, 2019) have demonstrated that although teachers are often very enthusiastic about adopting explorative teaching practices, they eventually do not implement them in their classrooms. Heyd-Metzuyanim and Shabtay (2019) explain this phenomenon by pointing to "misalignments" between the teachers' individual pedagogical discourse (beliefs) and the socially constructed Explorative Pedagogical Discourse which they try to adopt. These misalignments may explain why teachers often report implementing explorative practices while observations (done by researchers or teacher-educators) reveal otherwise, (Heyd-Metzuyanim et al., 2018). To delineate between Pedagogical Discourses that support traditional vs. "reform" instructional practices, Heyd-Metzuyanim and Shabtay (2019) used the terms Delivery vs. Exploration Pedagogical Discourses (DPD vs. EPD). Explorative Pedagogical Discourse (EPD) values students' struggle, agency, and conceptual understanding. By contrast, Delivery Pedagogical Discourse (DPD), values the teacher's "delivery" of the knowledge, and students' accurate application of procedures (Heyd-Metzuyanim & Shabtay, 2019).

While studies of beliefs, for example, show that teachers' beliefs differ from the behaviours they engage in (e.g. Beswick, 2018) a discursive theory of pedagogical discourses offers a more nuanced view of these differences. Thus, differences may be a result of how teachers perceive or frame their own practice, in contrast to how the researchers frame their actions in the classroom. Following this logic, we will refer to what others often refer to as "teachers' beliefs" as "endorsed pedagogical narratives". In doing so, we align with the common definition of beliefs as "anything that the individual regards as true" (e.g. Beswick, 2018; p. 3) yet foreground the fact that these beliefs originate from culturally produced narratives.

So far, studies adopting the discursive view of teaching practices have been based on qualitative methods such as interviews and classroom discourse analysis. Yet such methods are limited if one wishes to examine the differences between groups of teachers as they align with a certain Pedagogical Discourse. We thus designed a survey tool that would enable studying the alignment with EPD/DPD and ML practices at scale. Our question in this research was thus: Can the EPD/DPD discourses be detected through a Likert-style survey? And are the ML narratives distinct from EPD/DPD? Are there differences in beliefs and reported practices between Arabic and Hebrew speaking teachers?

METHOD

TEAMS21- Teaching Exploratively for All Mathematics Students, is an in-service professional development program for teachers and teacher-leaders that includes professional learning communities (PLCs) focused on explorative teaching practices. At the time of the study, the project included five PLCs that were of mostly Hebrew speaking (Jewish) teachers and two PLCs that were of Arab-speaking teachers. The PLCs are led by teacher-leaders who participate in a mixed (Jewish and Arab) leaders' PLC.

We designed a survey that was administered to the teachers in the TEAMS21 PLCS. 28 Arabic-speaking and 45 Hebrew-speaking teachers answered the survey during one of the PLC meetings, which was held at the middle of the first year of the project.

Development of the Instrument

Thirty-one items were designed to assess teachers' endorsement of pedagogical narratives (beliefs) and reported practices associated with EPD, DPD and ML. Some of the items used in the instrument were developed for this study and other items were selected from previous survey instruments, as described below.

Items measuring teachers' endorsement of narratives (beliefs)

The scale of endorsed pedagogical narratives contained 15 items on a Likert scale of 1-5 from "completely disagree" to "completely agree". These items began with "I think it is important that in mathematics lessons…" or with "please mark the extent to which you agree with the following statements". All the items related to EPD and DPD were adapted from Stein and her colleagues' survey (2017). This survey was picked as a basis for our instrument since it relied on a theoretical framework close to that underlying our TEAMS project. Thus, Stein et al. (2017) searched for teachers' beliefs about giving students opportunities to struggle with meaningful tasks, along with the teacher giving explicit attention to concepts in classroom discussions, two major themes in our TEAMS project (Heyd-Metzuyanim et al., 2020). Items related to EPD (7 items) were, for example, "(It is important that…) The students will use different strategies for solving the same problem", "I will facilitate students' connecting of ideas in order that they arrive at their own explanations of a general mathematical principle", "Students will work on cognitively challenging tasks with minimal direction from the teacher". In general, all these items aligned with messages and instructional practices that were promoted in the TEAMS21 PLCs.

Items related to DPD (4 items) were chosen to reflect practices that were de-valued in the TEAMS21 PLCs, and that were thought to be the opposite of explorative teaching. These included the valuation of the teacher explaining a mathematical idea (thus reducing struggle for the students), e.g. “The students will receive an explanation from me before they investigate the idea”. Also thought to be valued in the DPD were statements valuing drill and practice, for example “Students will have the opportunity to develop conceptual understanding through repeated practice of the same algorithm, applied to different problems”.

For measuring teachers endorsed narratives about ML, we chose three items from the CLES (Constructivist Learning Environment Survey, Johnson and McClure (2004)) that fit narratives promoted in our TEAMS21 project around mathematical literacy. For example, “Students learn about the world outside of the school”. One item assessing narratives about ML (in the negative form) was devised for this study “There is no need to give literacy context when teaching the required procedure”. Narratives in positive and negative form (e.g. valuing and de-valuing ML) were devised to increase reliability of the sub-scales.

Items measuring teachers' reported practices

The scale measuring teachers’ reported practices included 16 items asking teachers to testify if they enact a certain practice on a Likert scale of 1(Never) to 4 (In any lesson or almost any lesson). These items began with the question “How often do you do the following in your mathematics classroom?”. Five items related to practices associated with the EPD, four of them were adapted from the TIMSS Math Teacher Questionnaire, for example “I ask students to decide their own problem-solving procedures” and “I encourage students to express their ideas in class”. One item was adapted from Stein and colleagues’ (2017) survey: “(I) answer a student’s questions with more questions rather than just providing the correct answer”.

For measuring reported practices related to the DPD, we devised five items. Two of them were adapted from the TIMSS Questionnaire, e.g. “I solve problems with the whole class with direct guidance from me”, and three items were devised for this study. These were related to valuing drill and practice, for example “I make sure to practice procedures until they become automatic”.

For measuring reported practices related to ML, we devised six items. Three of them were adapted from Johnson and McClure (2004), including, “I teach interesting things about the world outside of school”, “I start teaching a new topic with problems pertaining to the world outside of school “, and “I teach how mathematics can be a part of students’ out-of-school lives”. Two ML items were adapted from the TIMSS Math Teacher Questionnaire, such as “I connect what was learned in the lesson to students’ daily lives”.

Validation process

The content of the survey was validated through consultation with two content experts whose previous research made use of the concepts of EPD and DPD. This process was done to verify that the items aligned with narratives previously elicited in qualitative studies of teachers’ pedagogical discourses (e.g. Heyd-Metzuyanim & Shabtay, 2019;

Nachlieli & Heyd-Metzuyanim, 2021). After that, we interviewed two teachers (an Arabic-speaking teacher and a Hebrew-speaking teacher) to ensure that the items were understood. Thereafter, a re-validation process was conducted to clarify the content of the items that were not well-understood.

Data analysis

We conducted two types of statistical analysis. First, we used Exploratory Factor Analysis (EFA) to determine the internal structure of the instrument. Because the data had a theoretical structure of multiple interdependent factors, we analyzed the data using a maximum-likelihood extraction of three factors (EPD, DPD, ML). Two exploratory factor analyses were conducted, one for the beliefs items and one for the practices items. We checked whether the data generated by the instrument were consistent with the theoretical constructs hypothesized by the content experts.

The EFA process yielded three factors for the instrument, forming subscales. We then analyzed the interrater reliability on the subscales obtained through the EFA to answer the first RQ, whether the Discourses could be detected, in other words, whether statements theoretically hypothesized to cohere with each other indeed do. In the second step, we conducted independent group means t-tests to compare Arabic and Hebrew speaking teachers. This test was used to examine the relationship between teachers' ethnicity and beliefs along the three factors found in the EFA (EPD, DPD, ML beliefs) and between ethnicity and reported practices related to EPD, DPD and ML.

FINDINGS

Regarding endorsed pedagogical narratives (beliefs), the exploratory factor analysis revealed three dimensions that explained a total of 51.78 percent of the common variance. This was assessed after excluding four items that had low loadings (<.30) on the extracted factor, or whose loadings appeared on more than one factor. These removed items included four that were originally thought to relate to DPD endorsed narratives: “Students are not ready for inquiry problems until they have acquired the necessary basic mathematics“, “(It is important that) Students perform the procedures accurately“, “In general, students' errors can be characterized as a lack of practice of the procedure used”; and “(It is important that) students practice using a procedure enough for them to be able to apply it to more complex problems”. Thus, the final result of the factor analysis consisted of 15 items. One factor consisted of 7 items, all designed initially to measure EPD-aligned endorsed narratives. This subscale had a sufficiently high reliability (Cronbach's alpha = 0.77) (Lance, Butts, & Michels, 2006). The second factor consisted of 4 items that were originally designed to assess DPD-aligned endorsed narratives. This subscale also had reasonable reliability (Cronbach's alpha = 0.653). The third factor consisted of 4 items assessing endorsed narratives about ML. This subscale had reasonable reliability (Cronbach's alpha = 0.686).

Regarding reported practices items, the exploratory factor analysis yielded three dimensions, which altogether explained 55.46 percent of the common variance. This was assessed after excluding one item that was originally designed to measure EPD-related practices: "(I) link new content to students' prior knowledge". As a result, the final solution of the factor analysis consisted of 16 items. The first factor consisted of 5 items designed to measure reported EPD practices (Cronbach's alpha = 0.720) and was labelled EPDP. The second factor consisted of 5 items designed to measure DPD reported practices (DPDP, Cronbach's alpha = 0.654). The third factor consisted of 6 items designed to measure practices associated with ML (MLP, Cronbach's alpha = 0.873).

To answer the second RQ, relating to the differences between Arabic and Hebrew speaking teachers, we used a t-test for independent group means. Figure 1 shows the means of the respondents' agreement with the endorsed pedagogical narratives and reported practices associated with EPD, DPD, and ML.

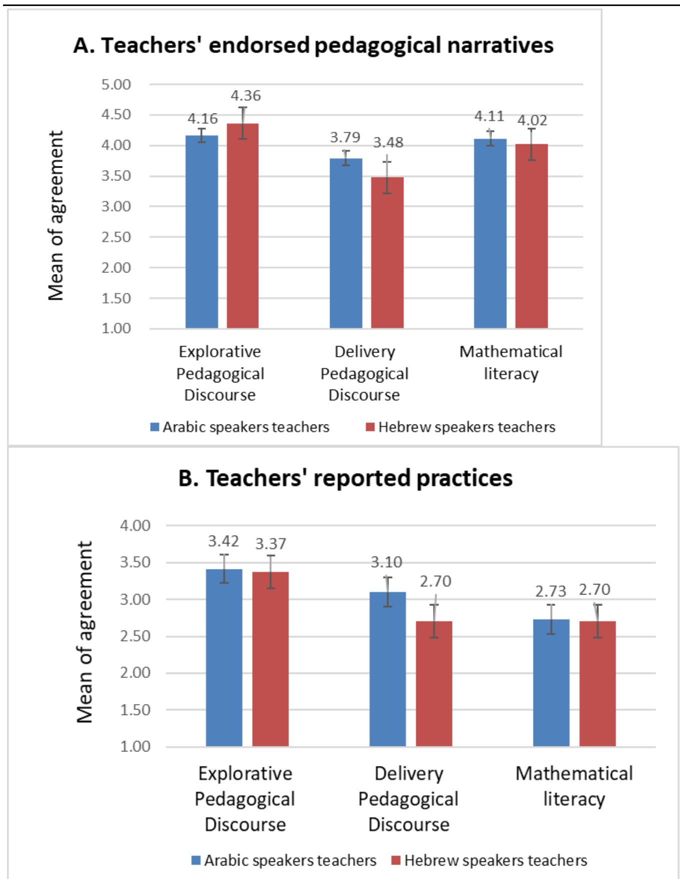

Figure 1. Distribution of Arabic (N=28) and Hebrew (N=45) speaking teachers' answers on: A. a Likert-scale (1-5) regarding EPD, DPD, and ML endorsed narratives. B. a Likert-scale (1-4) regarding EPD, DPD, and ML reported practices.

We found no significant differences between Arabic and Hebrew speaking teachers in EPD and ML endorsed narratives. In contrast, differences were found between the groups with relation to DPD endorsed narrative. Thus the mean of agreement with the statements of the DPD was higher among Arabic speaking teachers than among

Hebrew speaking teachers ($t(71)=0.309$ $p < 0.05$). Similarly, there were no significant differences between Arabic and Hebrew speaking teachers in the reported EPD and ML practices, but a significant difference was found in the means of DPD reported practices. Thus, reported practices around DPD were higher among the Arabic speaking teachers than among the Hebrew speaking teachers ($t(71)=1.99$ $p < 0.01$).

DISCUSSION AND CONCLUSIONS

The survey designed for this study appears to be a reliable tool for examining endorsed narratives and reported practices aligned with the EPD, DPD, and ML. The narratives belonging to each sub-scale showed sufficient reliability, indicating that these statements indeed cohere with each other, and may thus be a useful, albeit reductionist, tool of measuring teachers' pedagogical discourses.

An interesting implication of the independence of the EPD and DPD scales (as seen in them forming separate factors) is that these are, in fact, not opposite views. Thus, teachers can agree with narratives aligned with the EPD (such as giving students opportunities to struggle or encouraging multiple solutions for a problem) and at the same time, agree with narratives aligned with the DPD (valuing, for example, drill and practice of procedures). It thus may be that the traditional dichotomy between "teacher-centered" and "reform" instruction may need to be re-examined. Similarly, pedagogical narratives and reported practices related to ML were independent of the EPD or DPD (although some of them correlated well with EPD). Thus, it makes sense to continue measuring beliefs about ML independently from other pedagogical narratives.

Finally, our findings agree, in part, with previous studies (e.g., Eilam 2003) about the Israeli teacher population, which have shown that Jewish teachers believe more in student-centred instruction than Arab teachers. However, importantly, they show that the difference can only be found in relation to valuing of DPD related practices. Given that EPD and DPD may be separate Discourses, it is important to notice that the fact a teacher values drilling procedures, does not necessarily mean that they do not value discourse-rich instruction.

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STRATEGIES PROPOSED BY PRESERVICE TEACHERS TO FOSTER THEIR STUDENTS' CREATIVITY

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In this study, we aim to analyse the preservice teachers' perspectives on creativity, in particular, if they consider that creativity can be developed, and their strategies to foster students' creativity. The participants are 43 preservice teachers who were taking a master's degree to become teachers of secondary school. The master's program did not include a specific training in creativity. They answered a questionnaire about creativity and then three of them were interviewed. We did a content analysis of their answers. Most of the preservice teachers think that creativity can (and should) be developed in the mathematics classroom. They suggest different strategies to foster students' creativity that agree with literature, but solving open-ended problems stands out among the rest of strategies.

INTRODUCTION

The research interest in creativity has increased in the last decades (Joklitschke et al., 2018). At the same time, the view of creativity as an ability that can be developed and that is related to the learning processes, instead of being just an innate and uncommon ability, has spread among researchers and education professionals (Beghetto & Kaufman, 2007). In particular, creativity is associated with mathematical processes, such as problem posing and problem solving (Mann, 2006). Therefore, mathematics teaching and learning processes should include the development of students' creativity in order to be meaningful for them; however, this rarely happens in practice (Luria et al., 2017). Yazgan-Sağ and Emre-Akdoğan (2016) highlight that teachers should be aware of the importance of creativity and how to enhance it in their lessons.

This study is part of a research about the secondary school preservice teachers' perspectives on creativity and its enhancement in mathematics classroom, when they are not specifically trained to develop their students' creativity. The context of this research is a master's program in teaching in secondary school (specialization of mathematics), which does not include a specific training in creativity. In this paper, we aim to answer these questions: 1) Do preservice teachers think that creativity can (and should) be developed at school? 2) Which strategies do they propose to enhance students' creativity?

THEORETICAL FRAMEWORK

In this section, we briefly review some previous research on strategies to enhance students' creativity in the mathematics classroom and research on the conceptions of creativity of preservice and in-service teachers.

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Strategies to enhance creativity

There are several research on how to foster students' creativity. In particular, in the mathematics classroom, Silver (1997) suggests that the students' work should be more similar to the work of a mathematician, including problems solved and posed by the students. Some authors focus on the characteristics of the problems; for example, Sitorus and Masrayati (2016), with Realistic Mathematics Education; or Chamberlin and Moon (2005), with model-eliciting activities. Moreover, according to some authors, creativity can be related to processes such as visualization, exploration, making conjectures or argumentation that can be enhanced with the use of physical (Siew & Chong, 2014) and virtual manipulatives (Yildiz et al., 2017). Other strategies to foster creativity at school are enhancing the students' interaction, since verbalizing ideas is useful to structure them and generate new connections (Fischer, 2004; Levenson, 2011), and using an informational evaluation (Amabile and Pillemer, 2012).

Teachers' conceptions of creativity

In order to provide the students with the opportunity of developing their creativity, first, teachers should be aware of the importance of developing students' creativity, as Yazgan-Sağ and Emre-Akdoğan (2016) suggest. They compared the answers of four prospective teachers of mathematics and one of their educators about the actions that define a creative teacher. The prospective teachers highlighted the characteristics of the activities that the creative teacher would propose (use of different resources, real-life or open-ended problems, etc.), whereas the teacher educator gave more relevance to the thinking processes and explained that a creative teacher enables students to solve problems autonomously. Another example of research with preservice teachers is the study of Vanegas and Giménez (2018). They included a specific training in creativity for early childhood preservice teachers. We also found some research on the conceptions of creativity of in-service teachers (e.g., Cheng, 2010; Leikin et al., 2013, Lev-Zamir & Leikin, 2013). Cheng (2010) did an action research with seventy-five primary school teachers. They had to implement creative teaching in their schools and then they explained the tensions and dilemmas that they had experienced. Leikin et al. (2013) reported the results of a questionnaire answered by 1089 teachers from six countries. The questions were about the characterization of a creative student and a creative teacher, the relation between creativity in mathematics and culture and participants' general view of creativity. Lev-Zamir and Leikin (2013) studied the differences between the teachers' declarative conceptions of creativity and their conceptions-in-action.

METHODOLOGY

We used a qualitative methodology, based on the interpretation of the preservice teachers' answers to a questionnaire and some interviews about their perspectives on creativity and its development at school.

Context and participants

Forty-three preservice teachers, who were taking a master's program in teaching in secondary school (specialization of mathematics) in the year 2017-2018, voluntarily answered the questionnaire. Afterwards, three of them (P1, P2 and P3) were interviewed. The questionnaire was implemented after the period of teaching practice of the master's program; therefore, the participants already had some teaching experience, though scarce. At the end of the master's program, the preservice teachers present a master's final project (MFP), where they reflect on their teaching practice, assess it and propose some improvements for the learning sequence that they implemented. The MFP of the three preservice teachers that participated in the interviews, at the end of the course, were considered in the design of their interviews.

The questionnaire and the interviews

We adapted a questionnaire used in previous research (Seckel et al., 2019) to the specific context of the master's program. The questionnaire has 26 Likert questions and 5 open-ended questions. The Likert questions used a 5-point scale and were structured in the following topics: characteristics of creativity and creative thinking; the elements of a creative process; characteristics of a creative student; characteristics of a creative teacher; elements to enhance mathematical creativity in the classroom; and the impact of enhancing creativity in the classroom. The open-ended questions were about: the characteristics of a mathematical activity that enhances students' creativity; general strategies to foster creativity in the classroom; an example of activity; the importance that designing these activities should have within the teachers' work; and whether this topic was present in the master's program and how. The last question was posed to check our supposition that the master's program did not include a specific training in creativity, although some ideas about the enhancement of creativity could have been commented in the sessions.

At the end of the course, we interviewed three preservice teachers that had previously answered the questionnaire. The interview was semi-structured and had two parts. In the first part, we asked the participants about their definition of creativity, especially in school context, the characteristics of a student's creative work, the importance of fostering creativity and the difficulties to do it. In the second part of the interview, we used some comments related to creativity that had been previously identified in the MFP of each participant. We asked the preservice teachers to explain their comments and whether they considered that they could foster students' creativity with the tasks that they had proposed in the MFP. We also asked them about other strategies to enhance creativity. In this work, we focus on whether participants think that creativity is an innate ability or can be developed, and the strategies that they propose to enhance students' creativity in the classroom.

Analysis of the participants' answers

In order to analyse the answers to the questionnaire, first, we considered the questions separately. With the Likert questions, we calculated the percentages of answers in each level of the scale. Then, we did a content analysis (Miles & Huberman, 1994) of the answers to the open-ended questions. We generated categories of answers in the first, second and fourth open-ended questions. The answers to the third question were compared to the answers in the first and second questions. In the fifth question, we could identify some subjects of the master's program where, according to the preservice teachers, some ideas about creativity and creative work in secondary school had been introduced. In a second phase of analysis, we compared each participant's answers to the questionnaire, organizing the questions using the dimensions of the didactic suitability criteria (DSC) of the Onto-Semiotic Approach (Breda et al., 2017; Godino, 2013). The DSC enabled us to consider the different dimensions of a teaching and learning process (epistemic, cognitive, mediational, affective, interactional and ecological) and classify the strategies to foster creativity that the preservice teachers propose, based on these dimensions. The DSC were used similarly in the research of Seckel et al. (2019). Moreover, the comparison of each participant's answers was useful for detecting possible inconsistencies between the answers to the Likert questions and the answers to the open-ended questions. In these cases, the answers were not considered in the report of the results.

Then, we did a content analysis of the three interviews. Regarding the strategies that the preservice teachers proposed to enhance creativity, we used the DSC again to classify the strategies. The interviews complemented the results of the questionnaire.

RESULTS AND DISCUSSION

Based on the results of the questionnaire, most of the preservice teachers (69.7%) consider that creativity can be developed, though some of them (11.6%) disagree with this view (Table 1). In addition, seven participants agree with both statements A.1.1 (creativity is an innate ability) and A.1.2. (creativity can be developed). On the other hand, most of the preservice teachers do not think that creative thinking is a consequence of exceptional moments of inspirations, but they do not relate it to a thoughtful analysis of a problem either. Similar results are reported by Seckel et al. (2019), whose participants were in-service teachers.

A.1. What characterizes creativity and creative thinking?	1 %	2 %	3 %	4 %	5 %
1. Creativity is an innate ability or quality.	11.6	25.6	30.2	20.9	11.6
2. Creativity is a quality that can be developed, trained, etc.	0.0	11.6	18.6	30.2	39.5
3. Creative thinking is a consequence of exceptional moments of inspiration.	14.0	39.5	25.6	16.3	4.7

4. Creative thinking is associated with a long and thoughtful process of study of a problem.	18.6	20.9	30.2	27.9	2.3

Table 1: Percentages of answers to some Likert questions of the questionnaire. (In this scale, 1 = strongly disagree, 2 = disagree, 3 = not agree nor disagree, 4 = agree, and 5 = strongly agree).

In general, preservice teachers consider that enhancing creativity in the mathematics classroom is something positive (Vanegas & Giménez, 2018), because it makes the students like the subject more (33 participants, 76.7%), learn more (33 participants, 76.7%) and work more (28 participants, 65.1%). Some of the participants (11.6%) think that it is important to develop students' creativity because this prepares them better for their future jobs and society.

Regarding the strategies that the preservice teachers propose to enhance students' creativity, most of them are related to the epistemic dimension. They suggest to enhance creativity through activities that are rich in mathematical processes (Mann, 2006; Chamberlin & Moon, 2005; Silver, 1997), especially solving problems and open-ended tasks (30 participants mention it in the questionnaire). Some participants also mention that the tasks should be contextualized (Sitorus & Masrayati, 2016), include interdisciplinary connections, and let the students pose their own mathematical questions and make conjectures.

Considering the cognitive dimension, some preservice teachers explain that the tasks should be adapted to the diversity of students' mathematical level. For example, P24 says that the task should be "affordable for the different learning rhythms", with "clear and short objectives" and "motivating". Moreover, some participants mention that the task should be challenging for the students. Regarding the assessment of students' learning, P34 suggests that the teacher "assesses aspects that are not usually assessed", without specifying which aspects should be considered. In the interviews, we asked preservice teachers if the assessment could enhance students' creativity. P3 thinks that the assessment process determines a lot if students are more or less creative. He explains that the fact that students get responsible with their work and their learning process leads to foster more creative answers or processes and suggests the use of personal learning journals. P2 says that the assessment can foster creativity, but he does not know how to do it. On the other hand, P1 responds that the assessment cannot foster creativity, but affect it negatively:

P1: For me, one of the main problems is that students study just for the final result, not for the knowledge. Then, the assessment does not foster creativity, it fosters competitiveness and willingness to have a better mark, but not creativity. (…)

Interviewer: Okey. But assessment is a broad concept. (…) There is also the formative assessment and other things…

P1: Exactly. Well, it would not foster it [creativity] either. (…) Could it consider it [creativity]? Yes, it could, but it does not foster it [creativity]. (…) No, because then you would try to force it and creativity should be something that appears spontaneously.

We could observe a relation between the interviewed preservice teachers' ideas about assessment and their ideas about students' responsibility, that may also explain their reflections on whether assessment can foster creativity or not. P3 relates responsibility to creativity; whereas, P2 and P1 do not relate these terms and P1 justifies that the students' responsibility is studying what they are told to. As Amabile and Pillemer (2012) highlight, using the assessment as an extrinsic motivation does not usually foster the development of creativity; in contrast, an informational assessment can help to enhance students' creativity.

Nineteen participants suggest the use of manipulatives (mediational dimension) to enhance students' creativity, since manipulatives can help them to visualize a mathematical object and its properties (Siew & Chong, 2014; Yildiz et al., 2017). For instance, P14 explains that the use of physical manipulatives or learning and knowledge technologies "allow for experimentation to discover". In the interviews, P2 and P3 say that the use of manipulatives can also motivate the students and this boosts their creativity (affective dimension). Indeed, in the questionnaire, most of the preservice teachers (65.1%) agree both that motivation boosts creativity and that enhancing creativity makes the students like the subject more, indicating a possible positive feedback between creativity and motivation. Cheng (2010) identifies a similar relation between creativity and motivation (though expressed in negative terms), based on the answers of in-service teachers: when students are not used to work creatively in the classroom, they have more difficulties to respond to tasks that aim to enhance their creativity, then they maintain their previous learning habits and are less motivated; at the same time, if they have a low motivation, they tend to participate less in the classroom, which hinders the development of creativity.

Thirteen preservice teachers propose enhancing students' participation in the classroom as a strategy to foster the development of their creativity (Sitorus & Masrayati, 2016). Other strategies of the participants related to the interactional dimension are working in little groups (Fischer, 2004; Levenson, 2011), giving freedom to the students and fostering students' autonomy. In the interviews, P2 and P3 mention that the work in groups may enhance creativity; however, P1 does not think the same way. For him, when the teacher proposes an activity to work in groups, the students divide the work and there is not a real interaction between them.

Finally, most of the preservice teachers (93%) consider that the teacher's attitude affects the enhancement of students' creativity. In addition, eight participants indicate in the open-ended questions that the teacher should have an open and positive attitude (Levenson, 2011; Sitorus & Masrayati, 2016), three participants say that the teacher should have an open but critical attitude, and two participants think that the teacher should like the activities that they implement. The latter aspect is also identified in previous research with in-service teachers (Cheng, 2010; Leikin et al., 2013).

CONCLUSIONS

Although the preservice teachers did not receive a specific training in creativity, most of them assume that students' creativity can be developed at school and that it is something positive. They propose several strategies to foster creativity that are coherent with literature. In particular, most of them associate the enhancement of creativity in the mathematics classroom with problem solving. However, less than half of the preservice teachers explain an activity to enhance students' creativity in the third open-ended question of the questionnaire. This may suggest that the strategies that they recognize in theory are not incorporated into their teaching practice. In this sense, Lev-Zamir and Leikin (2013) also detect a gap between teachers' declarative conceptions of creativity in mathematics teaching and their conceptions-in-action.

Moreover, we observe that some strategies, such as working in little groups and using the assessment to foster creativity, are not so often assumed among the participants. Assessment seems to be a key aspect, since depending on how it is designed it could enhance or hinder students' creativity (Amabile & Pillemer, 2012). These results may be useful to design a specific training in creativity for preservice teachers that strengthens their skills to design tasks that foster students' creativity and focuses on those aspects that they usually find more difficult to manage in the classroom.

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MODELLING PROPORTIONAL REASONING SKILLS IN LEVELS WITHIN A DIGITAL SETTING

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Proportional reasoning skills are central for fraction learning which, in turn, is an important aim in secondary mathematics education. In an effort to better understand students' proportional reasoning and its relation to fractions, previous research has used paper-based tests and succeeded in modelling proportional reasoning skills in levels. However, so far these findings cannot be used by teachers because of the high effort involved with paper-based testing. In this report, we explore the applicability of digital methods to address this issue with data from fifth graders (n = 93) in a mode study. Despite of a need to adapt the instrument for the digital assessment, we observe similar levels based in IRT modelling as in the paper-based settings. The findings inform the further development of digital assessments to support fraction learning.

MOTIVATION

It is well known that dealing with fractions causes problems for several students far beyond secondary education (Mazzocco & Devlin, 2008; van Dooren et al., 2015). Further, previous cross-sectional (DeWolf et al., 2015) and longitudinal (Siegler et al., 2012) research provided evidence for the relevance of fraction learning for later success in mathematics. At the same time, research showed a broad range of different mathematics-specific skills to predict the learning of fractions (e.g., Hansen et al., 2015; Schadl, 2020). Specifically, proportional reasoning skills have been found to be relevant for later fraction learning (e.g., regarding conceptual knowledge: Hansen et al., 2015, additionally regarding procedural knowledge: Schadl, 2020). Typically, these studies used paper-pencil based large scale assessments and showed the covariation of relevant prerequisite skills with targeted outcome skills. Using item-response-theory (IRT) modelling approaches, Schadl (2020) could further shed light on the nature of different proportional reasoning skills and describe innovative level models. These models are suited to describe the relations between proportional reasoning skills and fraction knowledge beyond a "more is better". So, for example depending on the nature of proportional relations successfully solved, students seem to be more or less successful when dealing with fractions later on.

Proportional reasoning, its assessment and the role of numerical structure

Proportional reasoning skills are typically assessed with missing value tasks with three given quantities and an unknown one of the following form (van Dooren et al., 2009).

Format: In a shop, a packs with pencils cost b euro. A teacher buys c packs. How much does she have to pay? (Schadl, 2020; van Dooren et al., 2009)

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 387-394). PME.

Such missing value tasks contain either proportional or non-proportional relations of quantities. For example, considering the sample task, the quantities have to be related proportionally. To solve this situation, the number of packs can be compared with the price (comparison a and b in the sample task). This relation is called the *external ratio* (van Dooren et al., 2009) as two different quantities are compared. If the solution uses a comparison within one quantity (comparison a and c in the sample task), it builds on the *internal ratio*.

The difficulty of missing value tasks seems to vary depending on whether these ratios are natural or non-natural (termed simply "rational" for the rest of this report). Thus, according to extensive international research (e.g., Fernández et al., 2011; van Dooren et al., 2009), missing value tasks whose internal and external ratio is natural (NN-structure), as well as those with a rational external ratio and a natural internal ratio (QN-structure) seem to be easiest for most students. In contrast, missing value tasks with a natural external ratio and a rational internal ratio (NQ-structure), as well as those with both ratios in the rational range (QQ-structure) prove to be more difficult for most students (ibid.). With the given QQ-example, a student using the external ratio could argue $6 \cdot 4/3 = 8$, so $9 \cdot 4/3 = 12$, whereas using the internal ratio leads to $6 \cdot 3/2 = 9$, so $8 \cdot 3/2 = 12$.

NN-version: In a shop, 6 packs with pencils cost 12 euro. A teacher buys 24 packs. How much does she have to pay?

QQ-version: In a shop, 6 packs with pencils cost 8 euro. A teacher buys 9 packs. How much does she have to pay?

Modelling proportional reasoning skills in levels within a paper-based setting

As explained, missing value tasks differ in respect to numerical structure and this task feature proved to be relevant for the task difficulty. Furthermore, the contexts in which missing value tasks are embedded, are more or less familiar to students and hence, also impact task difficulty. So, despite of using the same task structure, missing value tasks with different numerical and context characteristics can be used to map differences between students' proportional reasoning skills (e.g., van Dooren et al., 2009).

Schadl (2020) has recently used these findings to design a paper-based instrument that systematically varies task features and developed a level model for students' proportional reasoning skills using IRT (N = 784, grades 4-6). This model allows to assign students to well-described proficiency-levels. Students on the lowest level can solve missing value tasks in a particularly familiar context (shopping context) with a numerical structure of at least one natural ratio. On the second level, tasks may show different contexts (e.g., sports or mixtures of juices and colours) including NN- and NQ-structures. From the third level on, proportional reasoning skills can be flexibly applied in different contexts with QN-structures. On the highest fourth level, students are expected in addition to solve tasks with QQ-structures. In contrast to the previous results by van Dooren and colleagues (2009), in the more recent German study, students seem to focus more on external ratios as missing value tasks with QN-structures have proven to be more difficult for most students than NQ-structures.

These level models allow Schadl (2020) to describe the relations between the learning prerequisite of proportional reasoning skills and different facets of fraction knowledge beyond a "more is better" as students' assessment results can be linked to the kinds of tasks a student is (not) able to solve. This could, for example, allow to identify students at risk for learning fractions before they start with learning fractions. Hence, results from this assessment could support teachers to identify difficulties of students timely and foster them adequately. Unfortunately, the paper-based testing involves high effort and is hence not suited for everyday use in practice. Digital methods could possibly solve the problem, but so far there is a lack of appropriate digital tools for monitoring students' learning progress in the context of fraction learning. In this paper, we address this gap with regard to a proportional reasoning assessment.

THE PRESENT STUDY

This report presents a digital adaption of the proportional reasoning test of Schadl (2020) and investigates whether results based on the adapted instrument show similar characteristics as the paper-based version. We specified the following two research questions:

Research question 1 (RQ1): Is it possible to model proportional reasoning skills assessed with an adapted digital instrument with IRT methods?

Research question 2 (RQ2): Which levels for proportional reasoning skills can be modelled within the digital setting and to what extent do they replicate the levels from the paper-based setting?

Despite of necessary adaptions (see below), we expected the numerical structure and the type of contexts to be relevant for task difficulty. In detail, we supposed missing value tasks with NN- and NQ-structures to characterize lower levels compared to those with QN- and QQ-structures. Regarding contexts, we assumed shopping contexts to be more familiar to students and hence, to characterize easier tasks.

METHOD

Study design and procedures

We digitally assessed proportional reasoning skills of fifth graders approximately four months after the start of the school year. We carried out the study in a whole-class setting in the computer room of the schools. Students worked about 30 minutes on the instrument including 14 tasks with differing task features (numerical structure, contexts). Tasks were ordered at random and administered through the online platform Levumi (Gebhardt et al., 2016).

Sample

The sample consisted of 93 fifth graders (45.2% female) who attended four classes preparing for higher education (Gymnasium) in Germany. Students' participation was voluntarily and needed informed consent.

Instrument

We used a digital adaption of the paper-based proportional reasoning test of Schadl (2020). The original test asked for open responses ("Write down your solution.") and needed manual coding. To reduce coding effort and avoid expected difficulties for students to enter their solutions, the adaption should have a closed format. We chose a multiple choice format and presented three solutions by fictitious students per missing value task. Students had to select the correct solution. To reduce the probability of guessing, we also offered the option "no solution is correct".

For example, for the QQ-version of the shopping context, the instruction read: "Antonia, Jonas and Lena have already solved the task. Who did it in the correct way? Tick the appropriate box." Antonia's solution focuses on the internal ratio. Regarding the packs of pencils, she calculates $6 \cdot 1{,}5 = 9$, and consequently $8 \cdot 1{,}5 = 12$ for the price to get the correct solution. Jonas calculates $6 + 8 = 14$ and afterwards $9 + 14 = 23$, representing an incorrect solution using an additive strategy. Lena focuses on the external ratio, but calculates $6 + 2 = 8$ and $9 + 2 = 11$. Hence, she also failed through additive strategies. Note that different from this specific example, we also presented correct solutions with additive strategies with some items.

For each task, we presented two solutions that could be related either to a multiplicative (see Antonia) or an additive strategy (see Lena). Further, the third solution could have been alternatively related to another strategy, including a rather unsystematic way of operating with the given numbers (see Jonas) that was derived from the previous paper-based study by Schadl (2020). Whether the presented students' solutions used the external (see Lena) or internal (see Antonia) ratio, was systematically varied, in particular for missing value tasks with NQ- and QN-structures. For example, regarding tasks with NQ-structures, solutions represented a multiplicative strategy based on the natural external ratios or alternatively an additive strategy based on the rational internal ratios. This systematic variation was based on previous findings (e.g., Fernández et al., 2011; van Dooren et al., 2009) indicating that students tend to use multiplicative strategies when focusing on natural ratios and additive strategies when focusing on rational ratios. So, we mirrored known student strategies.

In our digital instrument, we not only systematically varied the numerical structures as it is proposed by van Dooren and colleagues (2009), but also the contexts (without varying the text structure). So, for example regarding the shopping context, we replaced the shop by a bakery and the packs with pencils by breads, to obtain different shopping contexts. We did so not only for the shopping contexts, but also for the other ones like the sports contexts and juice mixtures.

The adapted digital instrument comprised 14 proportional missing value tasks in the newly developed multiple choice format. 4 items referred to the shopping context and 10 to other contexts. Among them, 2 items were constructed as distractors where all three presented students' solutions were incorrect. Each of the four numerical structures was given in three tasks, of whom one referred to the shopping and two to other contexts. Concerning the latter one, for one context the numbers of the three given quantities were multiples of ten and for the other one not, as it was also the case for all shopping contexts.

Analysis

To test whether the adapted instrument is indeed suited to capture proportional reasoning skills with the intended breadth, we used IRT and estimated person-parameters and item difficulties on a common scale. According to the dichotomous Rasch model (Rasch, 1960), a person v solves an item i with the probability of $f_i(\theta_v) = P(x_{vi} = 1|\theta_v, \sigma_i) = \frac{e^{\theta_v - \sigma_i}}{1+e^{\theta_v - \sigma_i}}$. The person-parameter θ_v models the person's ability with regard to a latent trait (here: proportional reasoning skills) and the item-parameter σ_i the difficulty of the item i (here: missing value task in multiple choice format). If the person-parameter is larger than the item-parameter, the probability to solve an item increases and approximates 1 in case of high skills. Vice versa, if the item-parameter is larger than the person-parameter, the probability to solve an item declines and approximates 0 in case of low skills. A Wright-map allows the joint representation of item- and person-parameters as item-person-map (Wilson, 2011). In this map, persons with low skills as well as less difficult items are plotted at the bottom and persons with high skills as well as more difficult items at the top. If Rasch-modelling supports the psychometric quality of the instrument, the estimations of item-parameters are used to qualify levels. The bookmark-method is a typical procedure to set meaningful borders for different skill levels using an ordered item booklet (Mitzel et al., 2001). This ordered booklet informs about the difficulty of the items and allows researchers to identify groups of items based on the knowledge about task features. Having identified groups of tasks with similar task contents and demands, they are, in turn, used to characterize the different levels. Rasch analysis was run with ConQuest 2.0.

RESULTS

In mean, the dataset holds answers from 93 students for each of the 14 items. The Rasch-model estimated to fit the data shows acceptable to good fit-indices as is indicated by the internal consistency (Cronbach's alpha = 0.72, WLE-reliability = 0.67), item discrimination ($.34 < r_{\text{point-biserial}} < .60$) and infits ranging from 0.80 to 1.09 (Bond & Fox, 2013; Field, 2014; Linacre, 2002). Further, as the item-parameters of two sub-samples (girls vs. boys were chosen as the required sample independence had to be checked) distributed largely roughly around the main diagonal in a bivariate scatter plot, we could assume Rasch-homogenous items. In sum, Rasch-modelling is possible and hence, RQ1 affirmed. Figure 1 summarizes the person- and item-parameter estimates as Wright-map according to numerical structure. Regarding RQ2, we next describe the digitally assessed proportional reasoning skills in levels. The lowest level is characterized by proportional reasoning in specific shopping contexts with either a natural internal or external ratio. On the second level, we model proportional reasoning with natural and rational ratios in probably less familiar shopping contexts as well as in different contexts (e.g., mixtures of juices or colours). On the third level, proportional reasoning becomes more flexible as it is characterized by the QN-structures, compared to the prior level including particularly NN- and NQ-structures. The highest level is modelled by proportional reasoning with rational ratios in different contexts. In total, we observed that proportional reasoning skills assessed with the adapted digital instrument led to very similar levels compared to the original paper-based assessment.

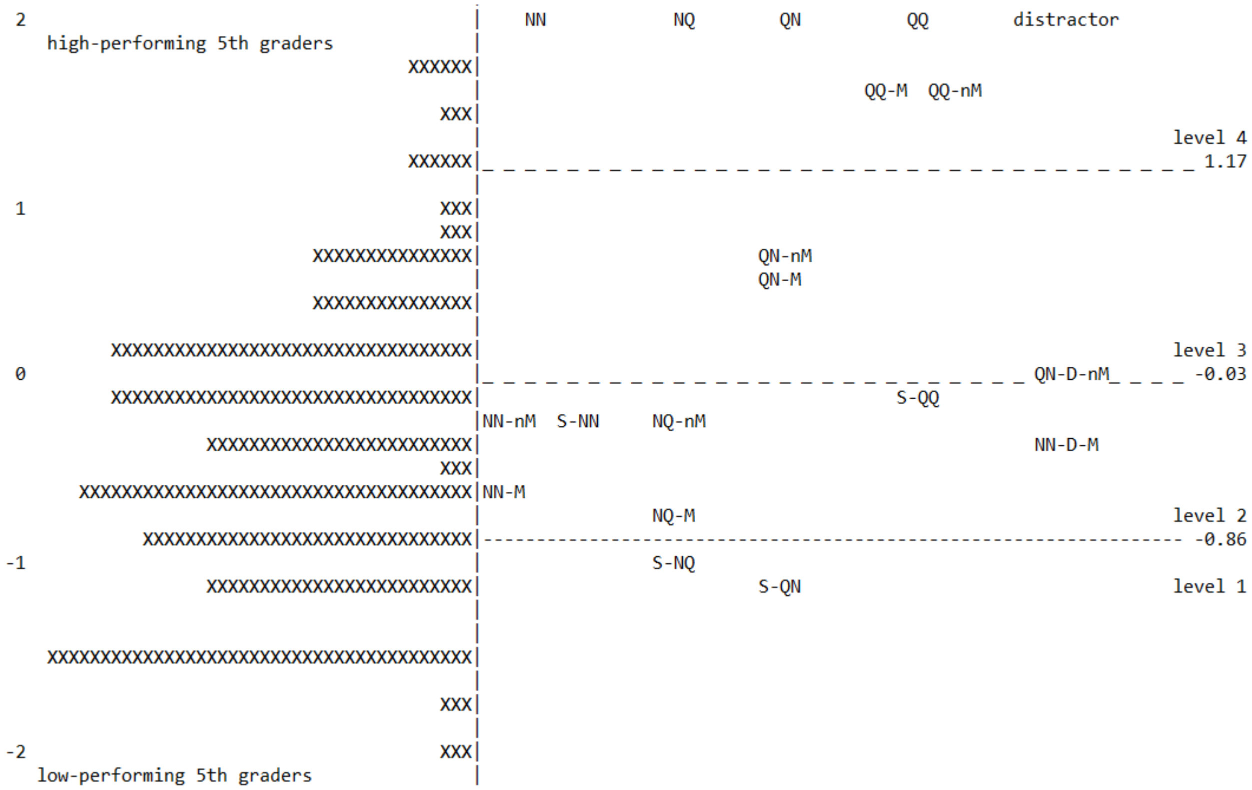

Figure 1: Wright-map of proportional reasoning skills. Items are plotted in columns according to their numerical structure. Item labels are constructed as follows: NN, NQ, QN and QQ indicates the numerical structure; infix “-D-” signifies a distractor; prefix “S-” signifies a shopping context; with regard to the other contexts, it is labelled whether the number triples were multiples of ten (M) or not (nM).

DISCUSSION

Fraction learning is a central topic in secondary mathematics education and research led to valuable insights into learning conditions such as that proportional reasoning skills predict fraction learning. However, such findings are still not available for most mathematics teachers in practical contexts because of the high effort involved with the paper-based settings. Hence, to address this issue, we explored the applicability of digital methods. We could not just digitalize the paper-based instrument as we expected students to struggle with entering their solutions for example. Thus, we developed a new closed format with multiple choice tasks students could deal with. Also, from the perspective of psychometric quality, our results indicate that the digital assessment can map the range of proportional reasoning skills (see RQ1).

Further, it was of our interest to study whether the levels of the paper-based setting could also be transferred to the digital setting. Even though further evidence is needed to ensure level characterization, we replicated largely the findings from previous paper-based research (Schadl, 2020), in particular with regard to the numerical structure. So, proportional reasoning with rational ratios revealed as being more difficult compared to natural ratios. In detail, lower proportional reasoning skills referred to dealing with shopping contexts and other ones including NN- and NQ-structures, whereas dealing with QN- and QQ-structures characterized higher proportional reasoning skills. However, regarding familiarity with contexts, it is not clear based on our data set, whether shopping contexts characterize a separate level or not. So, the existence of shopping contexts that emerge as particularly familiar and easy to solve for most students seems plausible (see items S-NQ and S-QN). Further, due to the fact, that the shopping context with two rational ratios (item S-QQ) was scaled on level 2, it is plausible that proportional reasoning is less difficult for most students in shopping contexts compared to other contexts. The item using a shopping context with two natural ratios (item S-NN) was – contrary to our expectations – scaled on level 2. Compared to the other tasks with shopping contexts, however, rather large numbers were used here (16, 48, 80) which could have led to the observed higher difficulty. But we cannot rule out the possibility that the closed response format of the digital test affects task difficulty and mitigates effects of context familiarity. Shopping context tasks did not largely differ in difficulty from tasks of other contexts including NN- and NQ-structures in our model. Whether the number triples a, b, c in missing value tasks are chosen as multiples of ten or not, seems in particular of relevance on a lower level. So, our results revealed the items NN-M and NQ-M as less difficult compared to the items NN-nM and NQ-nM, whereas this was not the case for the QN- and QQ-structures (see items QN-M and QN-nM respectively QQ-M and QQ-nM). Of course, this study has some limitations such as a small sample size. Further, as no common scaling was possible, level characterization was done based on a comparison with the original levels from the paper-based setting.

To conclude, the digital adaption of the instrument to assess proportional reasoning skills shows promising characteristics and is a step towards our goal of making the research results on fraction learning accessible for practical use by teachers through providing research-based support for learning assessment.

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EXPLORING DEFICIT-BASED AND STRENGTHS-BASED FRAMINGS IN NOTICING STUDENT MATHEMATICAL THINKING

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This study explores prospective teachers' framings in noticing students' mathematical thinking. A course was designed to engage prospective teachers in critical reflection of their framings and to encourage strengths-based framings when noticing students' mathematical thinking. Responses to noticing tasks during the first and last session of the course were analysed to identify what aspects prospective teachers pay attention to, what stances they adopt when interpreting, and what instructional moves they propose in responding to students' mathematical thinking. On this basis, prospective teachers' framings were characterised as deficit-based or strengths-based. The results show that prospective teachers shifted from deficit-based framings to strengths-based framings, and specific changes in prospective teachers' noticing are discussed.

INTRODUCTION

Research in teacher education over the last two decades has focused on an essential skill for teaching – the ability to pay attention to, interpret and respond to students' thinking – which has been termed 'teacher noticing' (for an overview, see Dindyal et al., 2021). One reason for this is that it captures teachers' moment-to-moment decision-making, which relies on teachers paying attention to what students are thinking and doing, and interpreting students' ideas to make informed decisions about how a lesson should proceed (Mason, 2002; Schoenfeld, 2011). This ability to notice students' thinking is central to the kind of instruction advocated, in particular by mathematics education reform initiatives that promote a student-centred, responsive approach to teaching (Franke et al., 2001).

Research shows that noticing matters for teaching and learning mathematics: when teachers pay close attention to the details of students' mathematical thinking, more opportunities for students' mathematical learning emerge (Santagata & Yeh, 2014). Research also shows that teachers can pay attention to the substance of students' mathematical thinking through targeted professional development (Santagata et al., 2021); however, the literature raises questions about what triggers changes in noticing. Some research suggests that changes in noticing are related to the specificity with which teachers see a phenomenon (van Es, 2011). Other research suggests that changes in noticing are related to how teachers frame or reframe the object of attention (Russ & Luna, 2013).

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The construct of framing is becoming increasingly important for understanding the nature of teachers' noticing and for designing teacher education and professional development programmes (Scheiner, 2021; Sherin & Russ, 2014). For example, research shows how framing can strongly influence teachers' noticing and actions in the classroom (Levin et al. 2009). Research also shows that teachers rely heavily on deficit-based framing, i.e. ways of thinking that portray students' mathematical thinking as deficits, inadequacies or failures (Louie et al., 2021). Teachers who use deficit-based framing often identify what students do not know or cannot do. However, deficit-based framing is a barrier to improving students' mathematical learning and can be detrimental to students' development of a positive mathematical identity (Aguirre et al., 2013). There are increasing calls for alternatives to deficit-based framing of students' mathematical thinking; one such alternative is strengths-based framing. Strengths-based framings are ways of viewing students' mathematical thinking as assets or resources rather than weaknesses or deficits (Crespo, 2000).

However, noticing students' mathematical strengths is a complex skill that needs to be learned in part because deficit-based framings are systematically embedded in mathematics education (Adiredja & Louie, 2020). Therefore, teachers need guided support to productively move away from deficit-based framings and embrace strengths-based framings to notice students' mathematical strengths. To this end, a teacher education course was designed to engage prospective mathematics teachers in critical reflection of their individual and collectively shared framings of students' mathematical thinking, and thus bring about a change in their orientations in noticing students' mathematical thinking.

The study presented here contributes to research on teacher noticing: it identifies teachers' framings and noticing practices in relation to students' mathematical thinking and investigates the nature and development of teachers' noticing of students' mathematical strengths. Specifically, the objectives of the study were: (a) the identification of a typology of deficit-based and strengths-based framings in noticing students' mathematical thinking, and (b) the characterization of how changes in framings promote changes in prospective teachers' attending, interpreting and responding to students' mathematical thinking.

THEORETICAL FRAMEWORK

The study presented here draws on research on teacher noticing and teacher framing. An extensive body of research in teacher education has focused on understanding teacher noticing (for a critical discussion, see Scheiner, 2016; for a recent review, see König et al., 2021). There is broad consensus that noticing consists of the ability to pay attention to noteworthy aspects of teaching, interpret what is observed, and decide how to respond (Jacobs et al., 2010; Kaiser et al., 2015; van Es & Sherin, 2002). Conceptualisations of noticing that include attending, interpreting and responding have been used to examine teachers' noticing with different foci, with particular attention to noticing students' mathematical thinking (Sherin et al., 2011). However, common approaches to teacher noticing focus on individual teachers and their internal mental processes, obscuring the fundamental ways in which noticing is shaped by historically and culturally constituted ways of structuring and organising experiences (Louie, 2018; Scheiner, 2021).

However, learning to recognise and interpret students' mathematical strengths requires acquiring tools and frameworks to figure out what to look for and how to characterise students' mathematical thinking. Such a perspective was articulated and applied in this study through the use of framing theory. Framings are understood here as culturally and historically constituted ways of organising and structuring experience (Goffman, 1974). They provide interpretive contexts that help participants in a given situation understand what tasks they are engaged in, what knowledge is relevant, and what behaviours are expected of them and others (Hammer et al., 2005).

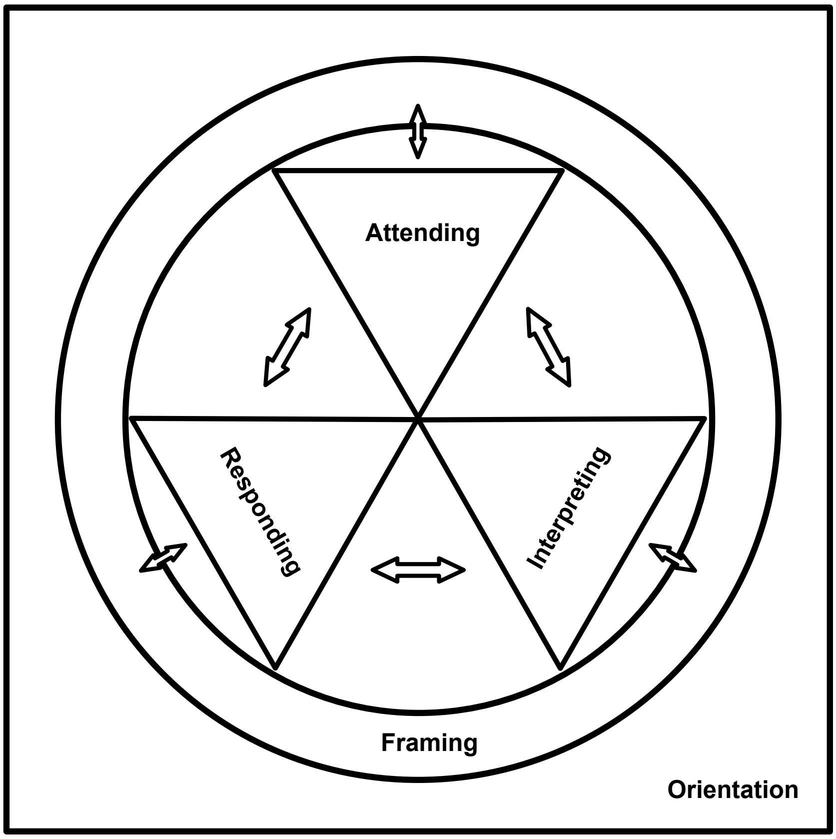


Figure 1: An integrated view of teacher noticing and teacher framing.

Following Levin et al. (2009) and Russ and Luna (2013), this study took an integrated view of framing and noticing (see Figure 1). That is, the three processes of attending, interpreting and responding are shaped by the ways teachers frame the object of attention, which in turn is constituted by broader orientations, such as orientations to deficits in students' mathematical understanding (see Scheiner, 2021). Accordingly, framing and noticing often reinforce each other. This makes it essential to critically reflect on framing and the ways in which it drives noticing.

RESEARCH DESIGN AND METHOD

Mathematics teacher education course

In this study, a teacher education course consisting of fourteen three-hour face-to-face meetings was designed to support prospective teachers notice students' mathematical strengths by encouraging more systematic reflection on their own and others' framings using methods of critical reflection (Brookfield, 1995; Liu, 2015). Specifically, carefully designed case studies of students' mathematical work were used (e.g., Scheiner & Pinto, 2019), and prospective teachers were asked to respond in writing to what they noticed about students' mathematical thinking. These written noticing responses were intended to help the prospective teachers reflect critically on their own and others' noticing as they thought about, talked about and looked at students' mathematical thinking. These reflections went far beyond reflecting on personal framings by encouraging the prospective teachers to consider framings of students' mathematical thinking represented in the literature and in critical writings that counter deficit-based framings of students' mathematical thinking, such as Smith et al. (1994). The prospective teachers then explored how they might use these new perspectives and ideas in their own framing of students' mathematical thinking.

Data collection

The study data were collected from nine prospective secondary mathematics teachers who participated in the mathematics teacher education course. The study data consisted of the prospective teachers' written responses to noticing tasks collected during the first and last sessions of the course. The noticing tasks were specifically designed to gain insight into the nature and development of prospective teachers' noticing of students' mathematical understanding of limits.

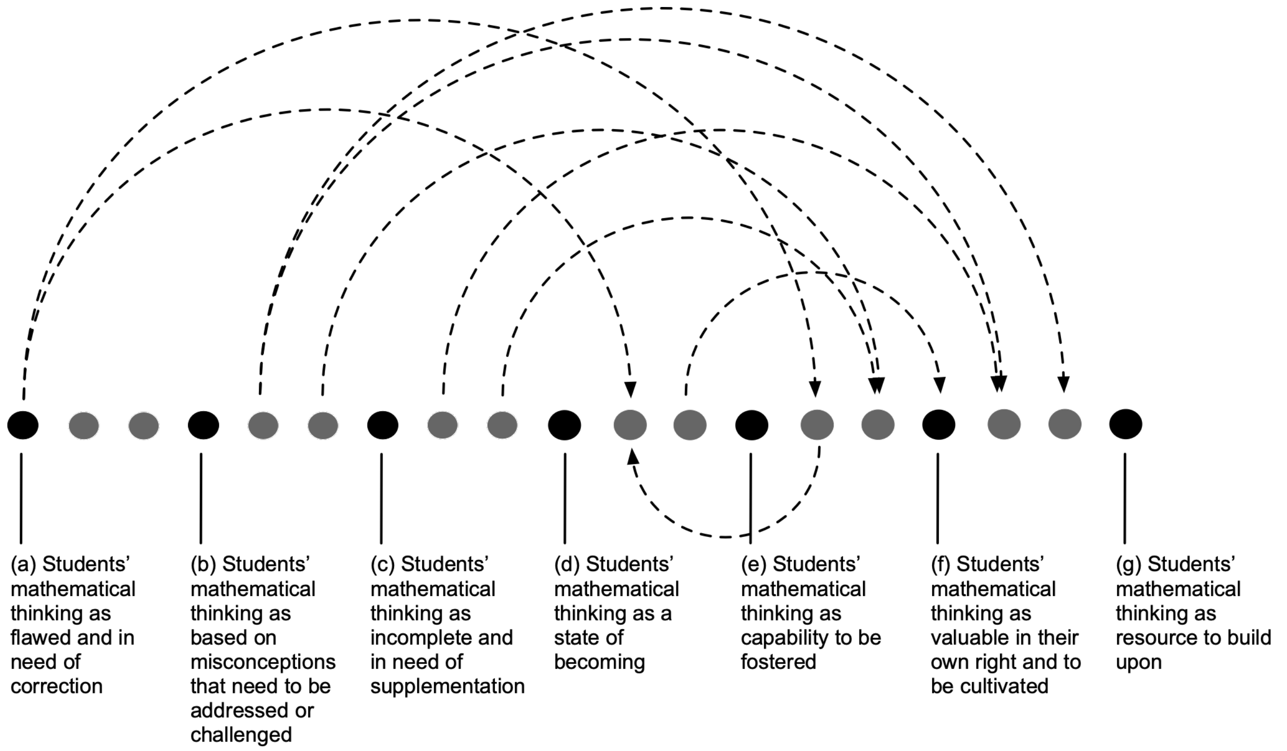
Similar to Jacobs et al. (2010), each of the tasks involved a series of noticing activities that focused the prospective teachers' attention on the particular student's reasoning ('What do you find noteworthy about the student's mathematical reasoning?'), their interpretation of the student's understanding ('What did you learn about the student's mathematical thinking and how can you interpret the student's understanding?'), and their response to the student's thinking ('Suppose you were the student's teacher, what and how would you respond to the student's mathematical thinking?'). The purpose of these noticing activities was to find out which aspects the prospective teachers highlighted as noteworthy, what their stances were in interpreting students' mathematical thinking, and what instructional moves they suggested in their responses to the students' thinking under consideration.

Data analysis

The analysis of the prospective teachers' written noticing responses was conducted in four phases. First, the written noticing responses were divided into three units: attending, interpreting and responding. Second, fine-grained analyses (see diSessa et al., 2016) were conducted at the level of each written noticing response unit to identify the aspects that the prospective teachers highlighted as noteworthy in their attention to students' mathematical reasoning, the stances they used in interpreting students' understanding, and the instructional moves they suggested in responding to students' thinking. Third, the identified aspects, stances and instructional moves were used to code the response units (attending, interpreting and responding units) of the prospective teachers at the beginning and end of the course. This process involved double coding of all attending, interpreting or responding units for the presence or absence of each of the aspects, stances or instructional moves. Inter-rater reliability was above 80% for all categories (aspects, stances and instructional moves) for all response units (attending, interpreting and responding). Disagreements were resolved by consensus. Fourth, the framings that the prospective teachers used in noticing students' mathematical thinking were derived based on the aspects, stances and instructional moves that the prospective teachers identified in their written noticing responses.

RESULTS AND DISCUSSION

In total, seven different framings of students' mathematical thinking were identified; three of these were deficit-based (a-c), one was uncommitted, i.e., neither deficit-based nor strengths-based (d), and three were strengths-based (e-g) (see Figure 2).



Note. Black coloured circles refer to different framings, the grey coloured circles in between refer to tendencies towards the respective framings. Each of the dashed lines refers to one of the prospective teachers' shifts in framing, the direction of which is indicated by the arrow.

Figure 2: Prospective teachers' shifts from deficit-based to strengths-based framings of students' mathematical thinking.

Analyses of the written noticing responses indicated that the prospective teachers purposefully shifted from deficit-based framings to strengths-based framings. Seven of the nine prospective teachers initially showed a strong tendency towards deficit-based framings in their written noticing responses, while the other two prospective teachers tended towards strengths-based framings at the beginning of the course. By the end of the course, all prospective teachers tended towards strengths-based framings in noticing students' mathematical thinking.

The shifts in the prospective teachers' framing (see Figure 2) promoted a mode of attention, interpretation and response that differed substantially from the way the prospective teachers had previously noticed students' mathematical thinking. First, the prospective teachers' attention shifted in terms of the aspects they highlighted in students' mathematical thinking. Not only did the prospective teachers shift from a general tendency to identify students' weaknesses to identifying students' strengths; they also paid less attention to students' weaknesses while their attention to students' strengths increased. Second, the stances that prospective teachers adopted when interpreting students' mathematical understanding changed. Prospective teachers moved from a general tendency to use deficit-based stances to strengths-based stances; but they also moved beyond simply evaluating or judging students' mathematical understanding to interpreting it as a phenomenon in its own right. Third, the instructional moves that the prospective teachers proposed in response to students' mathematical work also changed. In the beginning, the prospective teachers tended to propose instructional moves aimed at addressing or overcoming deficits and weaknesses that the prospective teachers had discovered in the students' mathematical thinking. In the end, however, the prospective teachers tended to propose instructional activities aimed at enriching, extending or building upon students' understanding.

Of course, it is very likely that the framings identified here and the ways in which they promoted changes in teachers' noticing are specific to the context of this study. Thus, the framings shown in Figure 2 are not necessarily representative but illustrate the many ways in which prospective teachers attend, interpret and respond to students' mathematical thinking.

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INTEREST AND PERFORMANCE IN SOLVING OPEN MODELLING PROBLEMS AND CLOSED REAL-WORLD PROBLEMS

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Modelling is an important part of mathematical learning. One characteristic feature of modelling problems is their openness. In this study, we investigated the relationship between interest and performance in solving open modelling problems and closed real-world problems. We used questionnaires and tests to assess the interest and performance of 143 ninth- and 10th-grade students at different achievement levels. We found that low-achieving students were more interested in solving open modelling problems than closed real-world problems. Also, prior individual interest in mathematics and performance were positively related to situational (task-specific) interest. These results contribute to interest theories by underlining the importance of types of real-world problems and achievement levels for situational interest.

INTRODUCTION

Modelling competencies are essential for mathematical learning. One important characteristic of modelling problems is their openness. In short, in our study, openness means that some important information is missing from the problem, and students must make assumptions about this information to solve the problem. Open problems can often be found in the real world, and thus, abilities to solve open modelling problems should be addressed in school. However, we do not know much about students' views on open modelling problems and their relationship to students' performance in solving this type of problem. We addressed students' interest as an important affective factor with high relevance for students' future educational choices (Hidi & Renninger, 2006) and examined differences in situational interest when solving open versus closed problems in high- and low-achieving students (i.e., students who attend middle- and low-track schools). We also analyzed how initial individual interest in mathematics and students' performance are related to situational interest in solving open modelling problems and closed real-world problems. We aimed to uncover the role of different kinds of mathematical problems (open modelling problems vs. closed real-world problems) in piquing students' situational interest. We seek to contribute to interest theories by clarifying how individual interest in mathematics and performance are related to situational interest in solving different types of mathematical problems.

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THEORETICAL BACKGROUND

Open modelling problems and closed real-world problems

To solve modelling problems, problem solvers must engage in the demanding transfer process between the real world and the mathematical world (Niss et al., 2007). Open modelling problems refer to problems with vague conditions. They do not include all the information needed to develop a solution, require problem solvers to make assumptions, and result in multiple solutions. Open modelling problems are examples of so-called ill-structured problems and rely on the model of ill-structured problem solving (Jonassen, 1997). Ill-structured problems are usually situated in a specific context in which one or more aspects of the problem situation are not specified, and the information needed to solve the problem is not completely provided in the problem. By contrast, well-structured problems provide all the information needed for a solution, and the problem solver just needs to select the relevant information from the task and link this information by using an appropriate mathematical procedure. In the past, a lot of research was carried out on closed real-world problems, whereas not much research on the affective and cognitive factors of modelling problems has focused on the openness of this type of problem.

Theoretical models of solution processes in mathematical modelling include, among other activities, understanding, structuring, simplifying, and idealizing a given situation (Blum & Leiss, 2007). Solving open problems requires problem solvers to notice missing information and make realistic assumptions about the situation described in the task and about the quantities that are missing (Krawitz et al., 2018). For example, while solving the Speaker problem (Figure 1), students need to notice that the information about the diameter of the speaker is missing and assume—by using the picture—that it might be one fourth of the height (about 5 cm).

Speaker

Maria bought the *Ultimate Ears BOOM* Speaker for 149.95 €. It has 360° sound with deep and precise bass. The speaker is 18.4 cm high.
Maria looks for a box with a cover for her speaker. On the web, she found a beautiful box. It is 14 cm wide, 10 cm high, and 14 cm deep.
Will the speaker fit in the box?

Figure 1: Open modelling problem “The Speaker”.

Simplifying and idealizing are much easier when solving closed real-world problems. For example, after understanding the real-word problem Pyramid (Figure 2), students can directly construct the real model and the mathematical model, calculate the mathematical result, and interpret it to answer the question. In mathematics classrooms, closed real-world problems are a lot more common than open modelling problems.

Pyramid

The Cheops pyramid was built about 4,500 years before Christ, and it is the highest pyramid in Egypt. The blueprints show that the square base of the Cheops pyramid has a length of 230 m. The original lateral edge of the pyramid was originally 219 m long.
Because the pyramid was used for a long time as a quarry, it is now only 138 m high.
How many meters less is the Cheops pyramid now than it was before?

Figure 2: Closed real-world problem "Pyramid".

Interest and performance

Interest is defined as a relation between a person and an object (e.g., mathematical problems). Students with high interest engage with their object of interest over time (Hidi & Renninger, 2006). Models of interest development assume that unstable situational interest (state) develops into stable individual interest (trait), with individual interest strongly predicting situational interest (Hidi & Renninger, 2006). Individuals with more interest in mathematics can be expected to engage more often and more deeply in solving mathematical problems, consequently achieving higher performance. Many empirical studies have indeed found that the relation between individual interest and performance ranges from small to medium, depending on performance tests (Heinze et al., 2005). Situational interest assessed during problem solving in mathematics was found to be positively related to performance in solving problems and to initial individual interest in a prior study (Nuutila et al., 2020). Furthermore, students' initial individual interest in mathematics contributed to students' engagement and situational interest while solving the problem (Nuutila et al., 2020).

Motivational constructs, including interest, can address different objects (e.g., learning, mathematics, or modelling competencies) (Schukajlow et al., 2017). The strengths of relations between motivation-related measures (e.g., situational and individual interest) and performance depends on the domain and the types of problems. Researchers have found differences in situational interest between different domains (e.g., writing vs. mathematics) and within the domain of mathematics (e.g., interest in analytic vs. numerical reasoning) (Ainley et al., 2009; Nuutila et al., 2020). Prior research indicated that students' motivation (self-efficacy and task value) for solving open modelling problems was lower than for "dressed up" word problems (i.e., problems that do not require assumptions to be made, offer a model of the situation, and are related to closed real-world problems) (Krawitz & Schukajlow, 2018). This result contradicted the assumption that realistic problems are more motivating for students and was explained by the high difficulty of open modelling problems and students' lack of confidence in solving this type of problem. Further, researchers found that the relations between individual interest, performance, and situational interest depend on students' prerequisites and the type of task (Ainley et al., 2009; Nuutila et al., 2020). One explanation for this phenomenon is an alignment between the objects of initial individual interest: performance and situational interest. If the tasks offered in the classroom do not meet students' expectancies, the relation between initial interest on the one hand and performance and situational interest on the other might be weak. For open problems, if students with high mathematical interest solve an unfamiliar open problem that does not include all the information needed to solve it, the relations of individual interest and performance to situational interest might be weaker than for familiar closed real-world problems. Another important factor for the development of students' individual interest is students' prior performance. Problems that are too difficult or too easy for students might have a negative impact on situational interest. Prior research has rarely analyzed the relations between individual interest, situational interest, and performance for students with different levels of prior performance, even though it is important to determine the role of individual prerequisites (e.g., performance in this study) for the validity of theoretical assumptions and to draw practical implications from interest theories for the teaching of mathematics.

PRESENT STUDY, RESEARCH QUESTIONS, AND EXPECATIONS

This study was carried out within the framework of the project *O*ffene *Mod*ellierungs*a*ufgaben in einem selbständikeitsorientierten Unterricht (OModA), in English, Open Modelling Problems in Self-Regulated Teaching, which is aimed at investigating cognitive, strategic, and affective conditions for the teaching and learning of open modelling problems. Our research questions and expectations were:

RQ 1: Does students' situational interest in open modelling problems differ from their interest in closed real-world problems for both high- and low-achieving students? Because open problems are more realistic than closed problems (Blum & Leiss, 2007; Jonassen, 1997; Krawitz et al., 2018), we expected higher situational interest in solving open problems in both high- and low-achieving students.

RQ 2: Is students' initial individual interest in mathematics and performance related to situational interest in open modelling problems and closed real-world problems for high- and low-achieving students? On the basis of theories of the development of interest and motivation (Hidi & Renninger, 2006; Schukajlow et al., 2017), we expected initial individual interest and performance to be positively related to situational interest for both types of problems and in high- and low-achieving students. We had no clear expectations of differences between high- and low-achieving students.

METHOD

Sample, procedure, and measures

One hundred forty-three ninth graders (51% female; mean age = 15.66 years) participated in the study. The school system in the region of the study is organized such that, after attending primary schools, most students continue their education in mixed-track schools (Gesamtschule) or in high-track schools (Gymnasium). In order to capture students with different performance levels, we asked 76 students from a mixed-track school (called low achievers in this study) and 67 students from three high-track schools (called high achievers in this study) to participate voluntary in our study. Students filled out a questionnaire on individual interest in mathematics and took a performance test that included both open modelling problems and closed real-world problems in a mixed order. Immediately after solving each problem, students responded to the situational interest questionnaire.

Individual interest was assessed with a well-validated scale from a prior study ranging from 1 = not at all true to 5 = completely true (Frenzel et al., 2012) (six items, e.g., “I am interested in mathematics”). Internal consistency (Cronbach’s α) was .84. Students’ performances in solving open modelling problems and closed real-world problems included six problems of each of the two problem types. An example of an open modelling problem is the Speaker problem (Figure 1), and an example of a closed real-world problem is the Pyramid problem (Figure 2). To analyze performance in solving open modelling problems, students’ solutions to these problems were scored 0 (wrong solution), 1 (no assumptions or unrealistic assumptions but otherwise accurate solution), or 2 (accurate solution under realistic assumptions). For performance in solving closed real-world problems, students were given a 0 for a wrong solution or a 1 for an accurate solution. The internal consistencies (Cronbach’s α) of the instruments were .714 (open problems) and .711 (closed problems). Situational interest was assessed by asking students directly after solving each problem about their interest in solving the problem: “It was interesting to solve this problem” (1 = not at all true, 5 = completely true). We built a scale for situational interest in solving open modelling problems (six items) and closed real-world problems (six items) by calculating the mean across the respective types of problems. The internal consistencies were .78 (interest in solving open problems) and .83 (interest in solving closed problems).

We used ANOVAs, *t* tests, and regression analyses to address the research questions. Less than 5% of the students had missing values, when they skipped a questionnaire or did not solve any problems on the test. Students with missing values were excluded from the analyses.

RESULTS

Preliminary analyses confirmed differences between students in the high-achieving and low-achieving groups in their performances in solving open problems, $M(SD)_{high\text{-}ach} = .62(.33)$, $M(SD)_{low\text{-}ach} = .30(.28)$, $t(141) = 6.260$, $p < .001$, and closed problems, $M(\mathrm{SD})_{high\text{-}ach} = .48(.28)$, $M(\mathrm{SD})_{low\text{-}ach} = .21(.21)$, $t(141) = 6.459$, $p < .001$.

RQ 1 was about the differences in students’ situational interest in solving open modelling problems and closed real-world problems (Table 1). A repeated-measures ANOVA with the factors type of problem (open vs. closed) and students’ achievement (high vs. low) revealed no difference in interest regarding open versus closed problems across the whole sample, $F(1, 137) = .232$, $p = .37$, $\eta^2 = .006$. However, there was an interaction between type of problem and students’ achievement, $F(1, 137) = 11.293$, p

< .001, $\eta^2 = .076$. Whereas high-achieving students had similar interest in solving open modelling problems and closed real-world problems with a slight tendency toward higher interest in closed problems, $t(66) = 1.825$, $p = .072$, Cohen's $d = 0.223$, low-achieving students were more interested in solving open modelling problems, $t(71) = 2.905$, $p = .003$, Cohen's $d = 0.342$.

Situational interest	High-achieving students *M* (*SD*)	Low-achieving students *M* (*SD*)
Interest in open modelling problems	2.98 (0.88)	3.01 (1.05)
Interest in closed real-world problems	3.12 (0.93)	2.79 (1.05)

Table 1: Means (standard deviations) for students' situational interest

RQ 2 was about the relations of prior individual interest, performance in solving problems, and situational interest (see the correlations in Table 2).

	Prior individual interest (1)	Performance open problems (2)	Interest open problems (3)	Performance closed problems (4)	Interest closed problems (5)
(1)	1	.235	.941**	.355**	.946**
(2)	.216	1	.158	.547**	.280*
(3)	.943**	.209	1	.291*	.783**
(4)	.271*	.772**	.235*	1	.372**
(5)	.946**	.208	.790**	.283*	1

* $p < .05$, two-tailed. ** $p < .01$, two-tailed.

Table 2: Pearson correlations for performance and interest in high-achieving students (above the diagonal) and low-achieving students (below the diagonal).

Prior individual interest in mathematics was strongly related to situational interest in solving open modelling problems and closed real-world problems. Students who were interested in mathematics were also interested in solving open and closed problems. The correlation was very high ($r > .9$) in high- and low-achieving students. Performance in solving open modelling problems was not related to situational interest in either achievement group, but performance in solving closed real-world problems was positively related to situational interest. Students who solved the real-world problems more accurately reported higher interest in this type of problem. When we included both individual interest in mathematics and performance as predictors of situational interest in solving real-world problems in a linear regression model, only individual interest remained statistically significant (high achievers: $\beta_{int} = .94$, $p < .001$, $\beta_{perf} = .026$, $p = .525$; low achievers: $\beta_{int} = .93$, $p < .001$, $\beta_{perf} = .041$, $p = .345$), indicating that individual interest was more important than performance for situational interest.

DISCUSSION

The goals of the present study were to identify the role of the type of problem (open modelling problems and closed real-world problems) for situational interest (i.e., task-specific interest) and to examine whether prior individual interest and performance were related to situational interest for high- and low-achieving students. In line with prior research (Krawitz & Schukajlow, 2018), the analysis revealed that high-achieving students reported similar interest in both types of problems with a slight tendency toward higher interest in closed problems. However, low-achieving students were more interested in solving open problems. This result is in line with theories of interest and modelling discussions, which assume that problems with a stronger connection to reality are more interesting for students, but why this was not the case for high-achieving students remains an open question. A possible explanation for the difference between high- and low-achieving students might be that low-achieving students did not notice that they needed to make assumptions to solve the open problems.

In line with interest theories (Hidi & Renninger, 2006), students' prior individual interest was found to be a strong predictor of situational interest. Interestingly, this finding held for traditional closed real-world problems and for less familiar open modelling problems. Students' performance was found to be related to situational interest for closed problems but not for open modelling problems. One reason for this result might be the differences in students' perceptions of the accuracy of their solutions for the two types of problems, which in turn might influence their situational interest. For example, some students might overlook the importance of the diameter of the speaker, calculate the diagonal of the box (see Figure 1), and assume that they developed the correct solution. The inaccurate perception of the correctness of a solution might decrease the relation between performance and situational interest in solving open problems in our study. A qualitative analysis of students' task processing and perceptions of the correctness of solutions to open problems is important to clarify this possibility. One important limitation of this study is that high- and low-achieving students can differ not only in their performance but also in other factors (e.g., learning materials distributed in the classroom) because they attend different types of schools.

The novel contribution of this study is that we addressed students' situational interest in open modelling problems. One theoretical implication of our study is the importance of individual interest in mathematics for the emergence of situational interest in different types of real-world problems and for students at different performance levels. Practical implications might be the possibility to evoke situational interest in low-achieving students by offering open modelling problems in the classroom.

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ASYNCHRONOUS MATHEMATICS PD: DESIGN AND FACILITATION FORMAT EFFECTS ON TEACHER LEARNING

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WestEd

In this paper, we share the design and effects on teacher learning of a set of two-hour online mathematics professional development modules adapted from face-to-face video-based materials. The modules are designed to be used in three facilitation formats: project staff- facilitated, district leader-facilitated, or structured independent. The modules aim to impact teachers' mathematical knowledge for teaching linear functions and effective mathematics teaching practices (MTPs; NCTM, 2014). Analysis of teacher learning, as related to evidence of the MTPs in teachers' written reflections, found teachers demonstrated learning of key MTPs, and in particular, there were not significant differences by facilitation format. Results and implications are discussed.

INTRODUCTION

Incorporating video within a professional learning environment offers great potential for mathematics teacher educators to support teachers in unpacking the relationships among pedagogical decisions and practices, students' thinking, and the disciplinary content (Borko et al., 2011). With video, teachers can observe and study the complexity of classroom life, reflect on their own instructional decisions, and integrate multiple domains of knowledge to solve problems of practice (Blomberg et al., 2013). Recent reviews of the literature on video use in professional development (PD) point to the value of video as a tool for improving instructional practice (Major & Watson, 2018).

As video technology and online video sharing have become more accessible and widespread, video-based PD is well-positioned to leverage the benefits of digital platforms (Teräs & Kartoglu, 2017). Online platforms can allow teachers access to professional learning resources that may not be available to them locally. Asynchronous PD allows participants flexible access to PD, with choice of schedule and location, and teachers report that the ability to access online PD anytime is very or extremely important (Parsons et al., 2019). Online PD may also be more scalable than comparable face-to-face PD and may have fewer monetary and logistical constraints (Killion, 2013). Asynchronous forms of online PD have resulted in positive findings related to teachers' attitude and self-efficacy (An, 2018) as well as high satisfaction and relatively high levels of information sharing (Yoon et al., 2020). In the research reported here, we investigate how asynchronous PD participation can support secondary mathematics teachers' mathematical knowledge for teaching.

THEORETICAL FRAMEWORK

Ball and colleagues have identified and elucidated "mathematical knowledge for teaching" (MKT) as the professional knowledge that mathematics teachers must have

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to do the mathematical work of teaching effectively (Ball & Bass, 2002). This conception of knowledge of mathematics for teaching is multifaceted and includes both content and pedagogical content knowledge. MKT includes a sophisticated understanding of effective instructional practices and student thinking related to specific mathematical content. Incorporating video within a professional learning environment supports opportunities for teachers to develop their MKT by designing opportunities for teachers to examine the relationships among pedagogical decisions and practices, students' thinking, and the disciplinary content (Bloomberg et al., 2013). Viewing video clips allows for the complexities of classroom practice to be stopped in time, unpacked, and thoughtfully analyzed, helping to bridge the theory-to-practice divide and support instructional reflection and improvement.

MODULE STRUCTURE AND DESIGN

The Video in the Middle (VIM) project is adapting a face-to-face video-based PD to online 40 two-hour modules asynchronous PD modules designed to expand teachers' MKT. The modules incorporate MKT as a design principle by creating multiple and varied experiences for teachers to examine and compare a variety of mathematical methods and representations, and to analyze the complex relations between content, pedagogy, and student thinking. The bite-sized modules offer flexibility by allowing mathematics educators the opportunity to design a variety of module sequences to fit their learning needs and have the potential to eliminate common roadblocks to participation such as scheduling difficulties and geographic distance.

Each module contains a common set of structured activities, where a video clip is at the center, or "in the middle," of professional learning as teachers take part in an online experience of mathematical problem solving, video analysis of classroom practice, and pedagogical reflection (Seago et al., 2018; Figure 1). This structure is intended to support teachers' professional learning related to mathematical knowledge for teaching (Ball & Bass, 2002) and NCTM's (2014) Mathematical Teaching practices (MTPs), a research-driven "core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (p. 9). The VIM modules emphasize six of the eight MTPs, as noted below:

1. Establishing mathematics goals to focus learning
2. Implementing tasks that promote reasoning & problem solving
3. Using and connecting mathematical representations
4. Facilitate meaningful mathematical discourse
5. Pose purposeful questions

8. Elicit and use evidence of student thinking.

Two additional design principles are also reflected: 1) All materials are rooted in the activities and materials of practice—authentic, unedited videos of classroom interactions, representing a *practice-based theory of professional learning* (Ball &

Cohen, 2002), and 2) there are multiple opportunities for teachers to access alternative perspectives from students, peers, mathematicians and educators, following the principle of *promoting multiple perspectives and accessing expert knowledge* (Herrington et al., 2010). While the overall module structure and these design principles may not be new to mathematics teacher PD, we seek to label this structure and investigate how it supports teacher learning in asynchronous teacher PD.

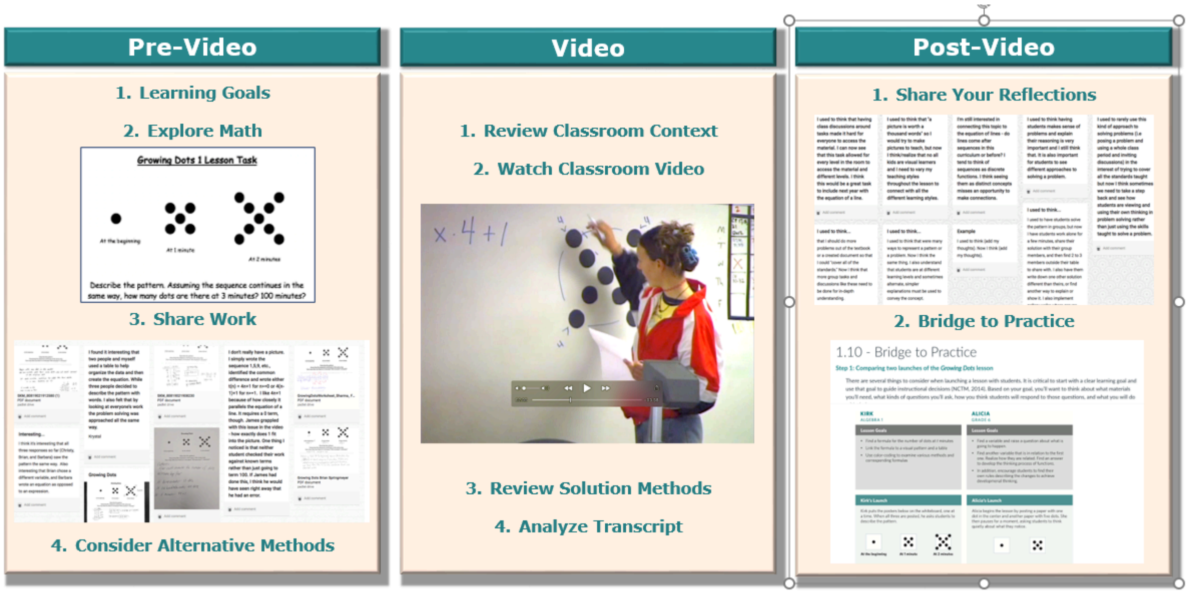

Figure 1: Video in the middle consistent set of activities.

METHODOLOGY

During Spring 2020, middle and high school teachers were recruited across California to participate in a pilot study to address the following research question: *How does VIM participation support teacher learning outcomes related to instructional practice, and how do they differ by facilitation format?*

Intervention. All teachers experienced the same four sequenced, two-hour modules for a total of eight hours of professional development over the course of eight weeks (February-March 2020). The four modules shared a common set of design principles, structure, and resources. Modules were offered in three formats: (1) project staff-facilitated, (2) district leader-facilitated, and (3) structured independent. Teachers in each of the two district leader-facilitated cohorts were all from the same district, while the other two groups included teachers from many different districts. The study intended to test if and how different facilitation formats impact teacher learning to meet the demand for scalable, high-quality PD (Koellner et al., in press). All three facilitation formats reflect what is known about effective teacher PD (Darling-Hammond et al., 2017) and particularly mathematics PD (Heck et al., 2019). Key features of effective PD were embedded in all conditions (Table 1).

PD feature	Project staff-facilitated	District leader-facilitated	Structured independent
Duration	Four sequenced two-hour modules (a total of eight hours) spread over the course of eight weeks; teachers compete one module per week.		Four sequenced two-hour modules (a total of eight hours); teachers work at their own pace and on their own time schedule.
Content focus	Each module is designed around mathematical content and pedagogical content knowledge goals.		
Coherence	Each module contains a “Bridge to Practice” activity at the end of the module that connects the module goals to instruction and their own teaching context.		
Active & Practice-Based	Participating teachers complete a mathematics task and share their work asynchronously with colleagues, then review a video of the mathematics task as a part of an instructional sequence in a classroom, write reflections on the classroom interactions, and then describe in writing their plan for integrating their learning into their own instructional practice.		
Collective Participation	Teachers share their solution methods and reflections on the classroom video with colleagues by posting them on an online discussion board. Teachers were asked to comment on other teachers’ solution methods and engage in dialogue on their written reflections.		
Expert Facilitation	The structure of each nodule and the sequence of the four VIM modules were designed by experts in mathematics content and pedagogy and reflected research on teacher learning, attention to student thinking, and the importance of teacher reflection.		
	A project team member with expertise in mathematics teaching and learning led teacher participation (e.g., encouraged teachers to complete modules, post on their work, and respond to journal reflections) and answered teacher questions during their experiences.	A member of the school district with expertise in mathematics teaching and with knowledge of school and district contexts and goals led teacher participation and answered teacher questions during their experiences.	While the participants in this condition did not have an additional facilitator directing their participation, the structure of each module and the pacing across modules was explained and detailed.

Table 1: How three facilitation formats reflect key features of mathematics PD.

Facilitator training. In January 2020, project and district facilitators participated in a 90-minute video-conference orientation with project staff, including an overview of the study and timeline, VIM module structure, and online tools. Facilitators also had access to a web-based facilitator guide and video tutorial demonstrating how to respond to participants.

Participants. Participating teachers taught middle school math, Algebra 1, or first-year high school math. Teachers in the district leader-facilitated condition were recruited by mathematics leaders from each of two school districts. Each of the two leaders then served as the facilitator for their district group. Additional teachers were recruited from districts across California and randomized into either the structured independent condition or the project staff-facilitated condition. Where multiple teachers were recruited from the same district, teachers were randomly split between the two conditions. Where single teachers were recruited from a site, singleton teachers were matched by similar site location or demographics; matched pairs were then randomized into the two conditions. Of the 68 teachers who began the study, 82% completed all or nearly all study activities across the four modules.

Measures. Multiple measures were used to gather impact data on teachers, including teachers’ pre-post analysis of student work, their work on the mathematics tasks, module reflections, and post-study interviews. The focus of this paper is the analysis of teachers' responses to two end-of-module reflection prompts: *What did you learn from this module? What new ideas do you intend to take/use from this professional learning?* Although the prompts were originally designed as a PD activity to support teacher learning and not a research measure, they offer insights into how teachers made sense of their learning and how the VIM modules supported teachers' MKT.

Analysis. 61 teachers (18 district-leader facilitated, 17 project-staff facilitated, 26 structured independent) responded to at least one of the eight prompts, resulting in 446 end-of-module reflections, and 54 to 59 teachers responding to each prompt. Responses were loaded into MAXQDA in order to organize and facilitate coding. Responses were coded using the MTPs (NCTM, 2014), as in addition to being a valueable set of mathematics teaching practices and skills, the MTPs offer a valuable framework for conceptualizing and identifying teachers' MKT growth and intended shifts in classroom practice. Coding for MTPs was as a means to identify evidence of and differences in teachers' MKT across conditions. Two coders, blind to teacher condition, coded responses in small batches of 10 to 15 teachers, adding details to the coding document and reaching consensus for coding of all responses.

RESULTS

Figure 2 presents the the percent of teachers in each facilitation format that showed evidence of MTPs in their responses, suggesting MKT growth by MTP. As shown in Figure 2, there is overall little difference in evidence of MTPs by teacher condition; for example, the percent of teachers that demonstrated evidence of MTP-4 (facilitate meaningful mathematical discourse), ranged from 30.8 to 41.2%, with 7 or 8 teachers per group respectively. Analyses using chi-square tests were completed when the chi-square test assumption of minimum number of expected values in all cells was met (MTP 2, 3, 4, 5, and 8). Results showed that the differences across groups were not statistically significant for these outcomes (e.g. MTP-2 (implementing tasks that promote problem-solving and reasoning), $X2 = 1.89, p = 0.39$; MTP-8, eliciting and using student thinking, $X2 = 3.38, p = 0.16$). As analyses demonstrate that the differences in MTPs were not significant, we can suggest that the differences by condition for MTPs are not significant at this time and further research is needed before more conclusions can be made.

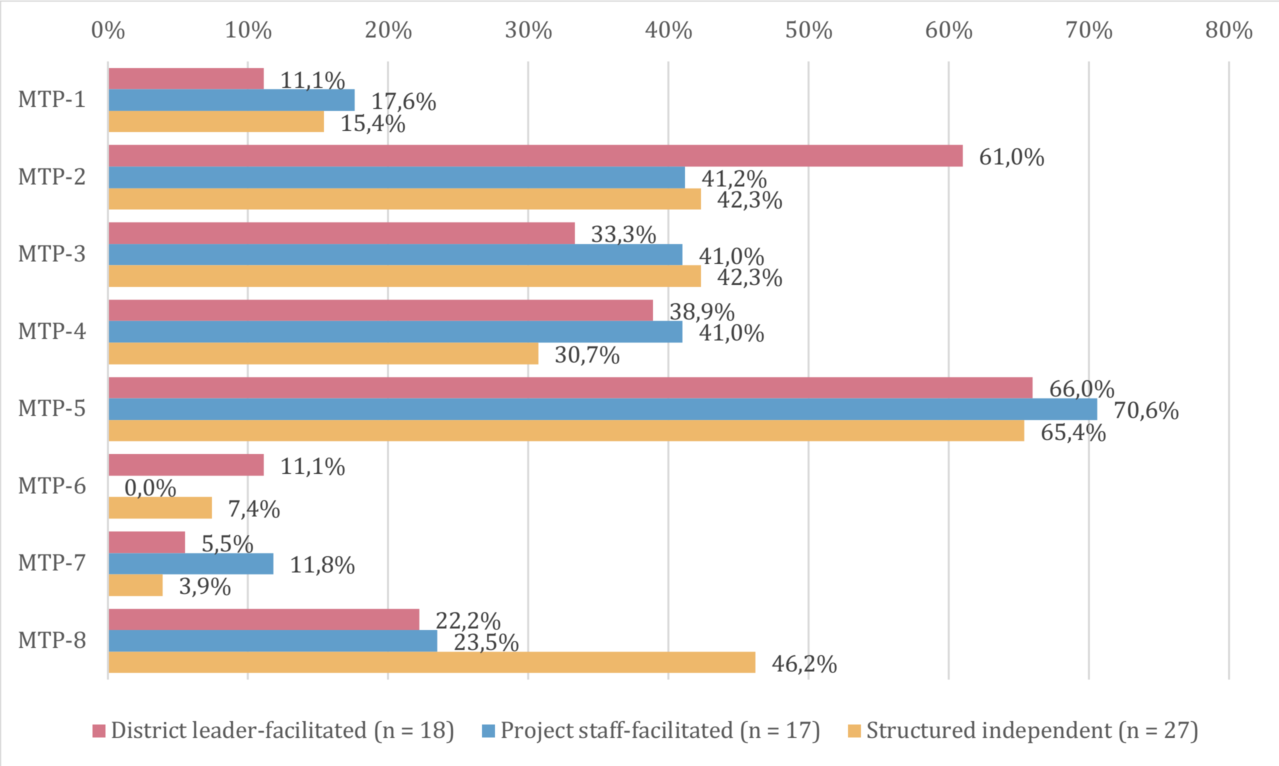

Figure 2. Evidence of MTPs in teachers' refelctions, percent by facilitation format.

DISCUSSION

Across conditions, teachers showed evidence of learning consistent with the VIM modules. Over 92% of the 61 participants who responded to the reflection questions gave at least one response indicating meaningful learning related to an MTP. This is notable given the open-ended nature of the prompts and that they were not written as a research instrument but rather as PD activities. High percentages of teachers across conditions showed evidence of learning related to MTP-2 (46.7%), MTP-3 (39.3%), MTP-5 (67.2%) and MTP-8 (32.8%), areas that were emphasized in the VIM modules.

The evidence of MTP-related learnings after VIM participation, as designed and hypothesized, emphasizes how the VIM modules supported all teachers *across condition,* and particularly statistical analyses do not show differences in evidence of MTPs by condition. That is, while the number of teachers who evidenced learning about a particular MTP did vary across facilitation formats, these differences were not statically significant and thus suggest that at this time there was no differential impact for one facilitation format over another.

CONCLUSION

High-quality professional learning is widely accepted as a core component of meaningful school reform (Borko et al., 2014); however, if schools and districts are to scale quality PD in a cost-effective and widely accessible manner, innovative tools and strategies that do not rely on individual providers spending extensive face-to-face time with small groups of teachers are needed (Cai et al., 2017).

These results support those of Heck et al. (2019), which suggest that the participation format of a PD experience is less critical than the presence of the key design features described in Table 1. As noted above, all three formats of VIM module facilitation were designed and structured following researched-based structure and design principles. The trends in analyses of the MTPs in teachers' written responses show promising preliminary evidence of teacher learning related to MTPs and emphasizes the strength of all three facilitation formats. This analysis also provides initial evidence of impact of independent, asynchronous PD, when it is well designed and structured. Responses also offer opportunities for further analyses of trends and additional themes, as teachers' responses from each condition were detailed, while varied.

There may be a bias towards face-to-face PD and localized PD contexts with an underlying assumption that they are more likely to lead to teacher learning than asynchronous PD. While local and face-to-face experiences may support teacher learning, it may be that they include key features of high-quality PD, and the format itself is less important. The preliminary findings we highlight in this paper suggest that future research is needed to study the relationship between PD design structures, PD format and context, and teacher learning of mathematical teaching practices and further understand the benefits of research-based, structured asynchronous PD.

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