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Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

VOLUME 4
Research Reports (Si – Z)
Oral Communications, Poster Presentations

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STRATEGY USE IN NUMBER LINE TASKS OF STUDENTS WITH MATHEMATICAL DIFFICULTIES: AN EYE-TRACKING STUDY

Anna Lisa Simon and Maike Schindler

University of Cologne

The number line is an important mathematical tool, especially in primary education. Previous research suggests that students with mathematical difficulties (MD) tend to have difficulties in empty number line tasks, but little is known about marked number lines. The aim of this study was to investigate if students with MD differ in their strategy use from students without MD in marked number line tasks. In our empirical study with fifth-grade students with and without MD (each n=20), we used eye tracking (ET), the recording of eye movements to gain insights into students' strategies. Based on ET video data, we inductively developed a category system of student strategies using qualitative content analysis. Our data analysis revealed significant differences: Students with MD used counting strategies more often—and less direct locating.

INTRODUCTION

Students with mathematical difficulties (MD) are characterized by difficulties in their learning of mathematics, essentially in basic arithmetic at primary school level (e.g., Moser Opitz et al., 2017; Scherer et al., 2016). Fundamental to learning mathematics is the development of the concept of numbers. This involves several aspects, including an ordinal understanding of numbers (Fuson, 1988). To develop this understanding, tools are often used where numbers are arranged linearly, such as the number line (Diezmann & Lowrie, 2007). On number lines, in addition to the ordinal arrangement of numbers, the relational interpretation of numbers to each other is important (Schulz & Wartha, 2021). Number line tasks are commonly used to investigate the learning of mathematics and the development of mathematical skills, and students' performance on number line tasks has been shown to correlate with their overall mathematical achievement (Schneider et al., 2018). Previous research has indicated that students with MD have difficulties in number line tasks, such as less accurate locating of numbers (Landerl et al., 2017). Further studies using eye tracking (ET) as research method to investigate students' work on empty number line tasks have indicated that ET is insightful for analyzing students' strategy use on these number lines and that students with MD show less flexible strategy use than students without MD (e.g., van Viersen et al., 2013; van't Noordende et al., 2016). In this paper, we investigate students' strategy use on a marked number line. We present students' strategies as well as differences in the use of strategies between students with and without MD.

NUMBER LINE

The number line is one of the most important tools in mathematics teaching and learning, especially at primary level (Diezmann & Lowrie, 2007). Different types of

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number lines can be distinguished: For example, the empty number line, where only the start and end points are labelled, and the marked number line with more hatch marks. There are many possibilities for the presentation of marked number lines: They can be fully or partially marked with hatch marks and labelled with numbers. Hatch marks (even without being labelled with numbers) can be visual reference points. Depending on the different ranges of numbers that marked number lines can represent, there are different markings and scales, so that distances on the number line must be interpreted differently (Schulz & Wartha, 2021). Number lines are used to develop and deepen an ordinal understanding of numbers (Diezmann & Lowrie, 2007). This involves understanding numbers as ranks or positions on the number line. In order to use the number line adequately, it is necessary to understand that all natural numbers have unique positions—even if these are not always visibly marked—and that all numbers are equidistant to each other (Schulz & Wartha, 2021). Furthermore, the relational interpretation of the given structuring features (markings and labelled numbers) on the number line is a way to make numbers accessible (Schulz & Wartha, 2021), that is, numbers have to be interpreted in relation to other numbers.

Difficulties with number line tasks at preschool age are predictive of later mathematical difficulties (e.g., Bull et al., 2021). School-age students' performance in number line tasks has been shown to correlate with general mathematical achievement (for a meta-analysis, see Schneider et al., 2018). Research on number lines is therefore important with respect to mathematical skills and thus for research on MD.

MATHEMATICAL DIFFICULTIES

Students with MD show difficulties in understanding basic arithmetic concepts (e.g., Moser Opitz et al., 2017; Scherer et al., 2016). MD can involve both a conceptual level—for example, in understanding the decimal system and place values—and a procedural level—for example, in the flexible use of calculation strategies (e.g., Moser Opitz et al., 2017; Scherer et al., 2016). MD are also associated with students having difficulties using adequate strategies when working on different mathematical tasks (for quantity recognition, e.g., Schindler et al., 2019). Difficulties, which occur in primary school, can become manifest over the course of the school years (Scherer et al., 2016) and are also observed on secondary school level (Moser Opitz et al., 2017).

Students with MD also tend to have difficulties in number line tasks: For example, they are often less accurate in locating numbers than students without MD (Landerl et al., 2017). Furthermore, previous research on students' strategies in empty number line tasks indicates that students with MD use strategies for locating numbers on the number line less adaptively than students without MD (van Viersen et al., 2013; van't Noordende et al., 2016).

These results, along with the preceding findings that performance in number line tasks correlates with general mathematical achievement, point to the importance of exploring in more detail how students with MD handle number line tasks.

EYE TRACKING

ET—the recording of eye movements (Holmqvist et al., 2011)—is becoming increasingly important in mathematics education research (Lilienthal & Schindler, 2019; Strohmaier et al., 2020). ET is used as a research method to study cognitive processes, and several studies have shown the potential of ET to provide insights into students' strategies in mathematical tasks in different domains. ET provides an opportunity to reveal differences in strategy use between students with and without MD: For example, in an ET study on quantity recognition by Schindler et al. (2019), students with MD tended to use counting strategies frequently over all tasks, whereas students without MD adapted their strategies more often to the tasks. There are also ET studies investigating number line tasks that have examined students' strategies and possible differences in strategy use between students with and without MD (e.g., van Viersen et al., 2013; van't Noordende et al., 2016). These studies have shown that ET is valuable for analyzing number line tasks, also for students with MD. ET appears to have added value in number line tasks: A study by Simon and Schindler (2020) has indicated that ET can provide more detailed insights into students' strategies than thinking aloud protocols, especially for students with MD. These findings were all to be found on empty number lines. To date, little is known about how students use the marked number line. A study by Simon et al. (2022) has indicated the potential that ET provides for gaining insights into strategies for marked number line tasks. However, to the best of our knowledge, there are no ET studies yet examining the strategy use of students with MD in marked number line tasks.

The aim of this study is to investigate if students with MD differ from students without MD in their use of strategies to locate numbers on the marked number line. We ask the following research question: *Do students with and without MD differ in their use of strategies in marked number line tasks?*

THIS STUDY

Participants. A total of 165 fifth graders from a German comprehensive school worked on the number line tasks. Before conducting the ET study, we administered a standardized arithmetic test, HRT (Haffner et al., 2005), to all students to diagnose MD. According to the HRT, students with a PR≤10 are considered to have MD. Students with a PR>25 are considered not to have MD. Students with a PR in between are "at risk" for MD (Haffner et al., 2005). Based on the results of the HRT, we selected 40 students: 20 students with MD (12 girls; mean age: 10.11 years, SD: 0.7 years) with the lowest scores in HRT (mean t-value: 29.5, SD: 1.8) and 20 students without MD (8 girls; mean age: 10.6 years, SD: 0.7 years) with the highest scores in HRT (mean t-value: 53.9, SD: 4.9). We did so to represent students at the lower and upper ends of the performance spectrums.

Tasks and procedure. Students in fifth-grade, in the transition phase from primary to secondary school, may still have difficulties with the number line (Rodriguez et al., 2001), which was also evident in our piloting of the tasks. Since we addressed students with MD, we decided to use tasks with a low difficulty level with number lines ranging from 0 to 100 (e.g., Department of Education, 2013). The study took place in individual sessions in a quiet room at school. We used two different number line tasks: In the "position-to-number-task" (PN) (Figure 1), we showed the students different positions (red cross) on the number line (numbers: 80, 40, 60; in that order) and asked them to name the corresponding numbers.

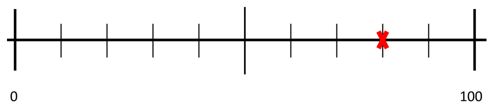

Figure 1: Position-to-number-task

In the "number-to-position-task" (NP), the number line did not have a red cross, but the students were asked to place presented symbolic numbers (70, 30, 90; in that order) on the number line. The numbers were displayed on the screen in the upper left corner before the number line appeared. Students were instructed to read the number aloud, to be sure that the number was perceived correctly. After that the number line appeared. Students were instructed to point at the place of the target number and to fixate this place with their eyes. Before each type of tasks, there was a practice task for the students. In between the tasks, the students were instructed to fixate a star displayed on the screen in the upper left corner, so that for all tasks and students the gazes started from the same place. We recorded students' oral responses with an audio-recorder. The students were not given feedback to their answers.

ET device. We used the Tobii Pro X3-120 eye tracker (120 Hz, binocular, infrared) to record the students' eye movements. This eye tracker was attached to a 24" full HD computer screen on which the tasks were displayed. Students were seated approximately 50 cm from the screen. The accuracy for our ET data was 0.8°.

Data analysis. For our data analysis, we used gaze-overlaid videos (eye gazes represented as a semi-transparent dot) provided by Tobii Pro Lab software. We analyzed the data inductively, following Mayring's (2014) qualitative content analysis. First, we *described* student's eye movements in the video. Then, we *paraphrased* the elements relevant to student strategies. Last, we *developed* the *categories*, that is, we inductively assigned categories with corresponding descriptions, and then *revised* the *categories*. From the content analysis, the following six categories of strategies emerged that the students used to locate numbers on the marked number line:

1. Direct locating: Students located numbers without looking at reference points (e.g., starting point or endpoint), that is, their gazes went immediately to the correct position.

2. Starting point use and counting: Students counted the given marks. They looked at the marks one by one—from the starting point to the target position or vice versa.

3. Midpoint use and direct locating: Students looked at the midpoint of the number line and located numbers directly from this reference point in one step. The midpoint strategy was used for numbers located to the left or right of the midpoint.

4. Midpoint use and counting: Students located numbers by looking at the midpoint of the number line and counting marks starting from that reference point.

5. Endpoint use and direct locating: Students looked at the endpoint of the number line and located numbers directly from this reference point, that is, they looked at the endpoint, and located the number from there in one step.

6. Endpoint use and counting: Students located numbers by looking at the endpoint of the number line and counting marks starting from the endpoint.

We made an a priori estimation of which strategies can be expected for each task (Table 1). For tasks with numbers larger than 50 (i.e., 60, 70, 80, 90), strategies with *endpoint use* are applicable. For tasks with numbers smaller than 50 (i.e., 30, 40), using the endpoint is theoretically possible, but unlikely being used here. For the numbers 40 and 60, when the strategy is *midpoint use*, there is no difference between *direct locating* and *counting*. The same applies to the number 90 and *endpoint use*. Therefore, these gazes are categorized as *midpoint* or *endpoint use* and *direct locating*.

		Number					
		30	40	60	70	80	90
Startpoint	Direct	x	x	x	x	x	x
	Counting	x	x	x	x	x	x
Midpoint	Direct	x	x	x	x	x	x
	Counting	x			x	x	x
Endpoint	Direct			x	x	x	x
	Counting			x	x	x	

Table 1: A priori estimation of the expected strategies
("x" indicates expected strategies for the respective number)

Based on the category system, one rater experienced in analyzing gaze-overlaid videos (first author of the paper) coded all data and another rater (second author) coded 20% of the ET videos independently. We calculated the interrater reliability using Cohen's kappa. The interrater agreement was 0.83, which is considered to be almost perfect.

Statistical analysis. To analyze differences in the use of strategies between students with and without MD, we carried out chi-square tests using SPSS 28. Effect sizes were calculated using *Cramér's V*. A total of 240 tasks were included in the analyses (60 tasks per group for PN; 60 tasks per group for NP). There was no data loss.

RESULTS

In the following, we answer the research question: *Do students with and without MD differ in their use of strategies in marked number line tasks?*

A chi-square test with the accumulated strategies for all tasks together (Figure 2) revealed significant differences, with small effect size, in the distribution of strategies between the students with and without MD: χ^2 (5) = 16.97, p = .005, V = .27.

In detail, cell tests for differences between the two groups revealed that students without MD used strategy *direct locating* (i.e., strategy 1) significantly more often (small effect size) than students with MD (χ^2 (1) = 14.65, p < .001, V = .25).

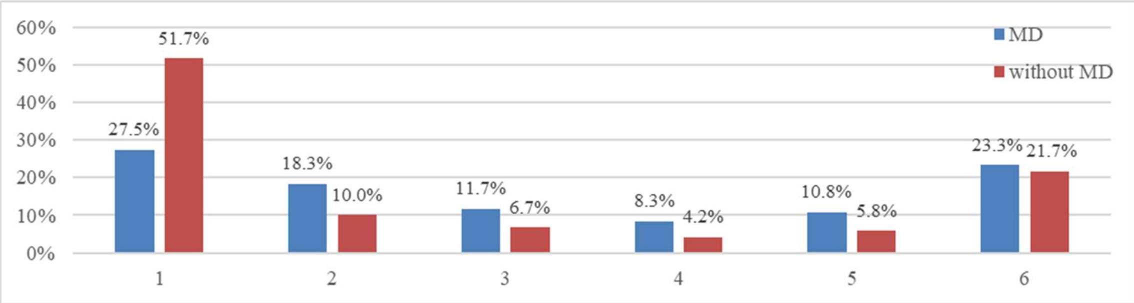

Figure 2: Distribution of student strategies over all tasks

A comparison of the summarized strategies that can be classified as *counting* (i.e., strategies 2, 4, 6) and the summarized strategies that can be labelled as *direct* (i.e., strategies 1, 3, 5) (Figure 3) showed that students with MD used *counting* strategies significantly more often, whereas they used direct strategies significantly less often than students without MD (χ^2 (1) = 4.92, p = .027, V = .14) (small effect size).

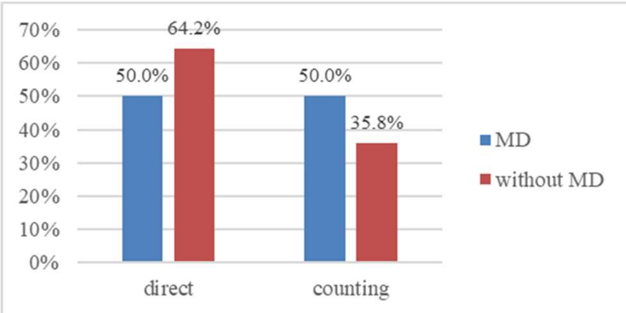

Figure 3: Distribution of students' counting and direct strategies over all tasks

DISCUSSION

The aim of this study was to investigate if students with MD differ from students without MD in their use of strategies to locate numbers on the marked number line. To pursue this aim, we qualitatively analyzed ET videos and developed a category system. We then investigated students' strategy use based on the qualitative ET data.

Our analyses indicate that strategy use differed significantly between students with and without MD. This relates to findings on different strategy use on the empty number line of students with MD compared to students without MD (e.g., van Viersen et al., 2013; van't Noordende et al., 2016). While students without MD used direct locating more often, students with MD counted more often. This is consistent with findings in previous studies on quantity recognition: Students with MD predominantly used counting strategies, whereas students without MD more often used structures and direct strategies (Schindler et al., 2019; Schindler et al., 2020).

Even though effect sizes are small, our findings indicate that students with MD differ in strategy use from students without MD. This is interesting given that the tasks were supposedly easy for the age of the students (e.g., Department of Education, 2013). This implies that locating numbers on a number line needs to be partially supported at the beginning of secondary school—even for this range of numbers. For students with MD, ways of accessing numbers on a number line should be fostered: For example, the relational interpretation of given structuring features should be addressed (Schulz & Wartha, 2021) as well as different reference points and their use for locating numbers.

As mentioned earlier, the number line is an important tool in mathematics education that can contribute to the development of the concept of numbers (Diezmann & Lowrie, 2007; Schulz & Wartha, 2021). Our study provides insight into student strategies when working on marked number line tasks. Our analyses of ET video data revealed students' strategies on the marked number line that had not been previously reported in this form. In the future, further research could examine if even more clear differences between students with and without MD are evident in more advanced tasks. A possible limitation of our study is the relatively small number of tasks. Further research should investigate if our results can be generalized to a larger set of tasks.

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EHNANCING REFLECTION ON THE CRITICAL ATTRIBUTES OF THE FIGURES: THE HEIGHT CHALLENGE GAME

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Geometrical concepts are characterized by the intertwining of theoretical and perceptual aspects, where examples play a crucial role from an educational perspective. We focus on the concept of height of triangles and propose an inquiry-game activity within GeoGebra, with the goal of prompting students to consider non-prototypical cases of triangles and enhancing their reflection on the critical attributes of this concept. After outlining a theoretical framework on the role of examples in geometrical reasoning, we present the Height Challenge Game and report some excerpt from a grade 7 classroom discussion. An emerging model for the role of different kind of figures at play in the game is proposed.

INTRODUCTION AND THEORETICAL FRAMEWORK

In most Italian primary school textbooks, the height of the triangles is exemplified mainly on equilateral or isosceles triangles. The choice to use these types of triangles can lead students to mistakenly think that the height divides the angle at the vertex in half and that its foot coincides with the midpoint of the base. In addition, in textbooks the base-height configuration is often represented following the horizontal-vertical orientation of the page, which may lead students to think that the base and the height of the triangle (and of other figures) must always be horizontal or vertical. In other words, elements not contained within the definition of height because they do not characterize the concept, are assumed as *critical attributes*, namely properties that an example of a concept must have in order to be considered as such (Hershkowitz & Vinner, 1983; Hershkowitz, 1987). The student's unexperienced eye is therefore deceived by accidental elements observable in the examples.

This kind of errors can be linked to spontaneous processes of categorization of things, people or events implemented in everyday life. In the field of cognitive psychology, the studies of Eleonor Rosch (1999) have highlighted the presence of two dimensions within the structure of the categorical system. In a *vertical dimension* the relationships between categories are established hierarchically on the basis of a three-level taxonomy: *superordinate*, *basic*, *subordinate*. For example, polygons - triangles - isosceles triangles represent the superordinate, basic and subordinate levels respectively. The *horizontal dimension* provides for an organization within the same category, which revolves around prototypes, namely "the clearest cases of category membership defined operationally by people's judgments of goodness of membership in a category" (Rosch, 1999, p.196). The relationship between formal-theorical-conceptual aspects and figural-diagrammatic-material ones has been at the core of

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research in geometry education. Fishbein (1993) introduced the expression of *figural concept* to indicate that the objects of geometrical reasoning “reflect spatial properties (shape, position, size), and at the same time possess conceptual qualities - such as ideality, abstractness, generality, perfection” (p.143). Building a figural concept means pursuing “the integration of conceptual and figural properties into a unitary mental structure, with the predominance of conceptual constraints over figural ones”. (Fischbein 1993, p.156).

Referring more in general to mathematics, Tall and Vinner distinguished between a student’s *personal conceptual image*, that is “[the] cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes […] built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (Tall & Vinner, 1981, p. 152) and the *formal definition of the concept*, which is established by the community of mathematicians within a certain theory. Hershkowitz (1987) spoke of *misconception* to indicate a conceptual image that is either partial or contains elements in conflict with the formal definition of the concept. In the intertwine between the conceptual and the figural dimensions, the use of prototypes represents a delicate element for learning geometry, since “popular examples” are often taken as prototypes of a concept (ibid.).

Applying this conceptual framework, we can hypothesize that when the height prototype for a student coincides with the height of the isosceles triangle, i.e. with an example of height belonging to a subordinate level, the student will show difficulties in recognizing and/or producing heights of geometric figures that belong to basic or higher level. It is possible that initially students will be able to operate correctly with this prototype at a subordinate level (therefore on isosceles and equilateral triangles), but later, when they will encounter examples of the same concept belonging to superordinate or basic levels (non-isosceles or non-equilateral triangles, quadrilaterals), the prototype at their disposal will lead them to make mistakes. Hershkowitz (1987) uses the expression *prototype effect* to characterize students’ productions in which it is recognizable the selection of non-critical attributes and their extension to all the examples of a given mathematical object. The examples that come to mind to students depend strongly on the situations they encountered, so it is important that students meet different typologies of examples of a given concept. Referring to this variation, Watson and Mason (2005) stress the importance of developing students’ rich *examples spaces*, indicating a metaphorical space that contains some examples and excludes others, in which different examples play different roles in structuring students’ sense-making. It is of particular relevance that students experience, besides prototypical examples, also non-prototypical ones. In the next paragraph we will present an activity that we purposefully designed to this aim.

THE HEIGHT CHALLENGE GAME

In order to foster the enlargement of students’ example space related to triangles and enhance their reflection on critical attributes of the height concept, we designed the

Height Challenge Game (HCG). The HCG is an inquiring-game activity in Geogebra, based on challenges between two players. The HCG may be found at https://www.geogebra.org/m/xnjuu2zc. In the challenge, students act on a given geometrical figure and each student plays a role: falsifier or verifier. The falsifier is the first to play: he "messes up" the triangle by dragging its vertices and chooses a vertex from which the verifier must position the height. This choice is made by clicking on one of "Vertex" check boxes: as a result, a segment is added to the figure as a candidate for the height of the triangle that starts from the chosen vertex. For example, in Figure 1a the falsifier has chosen vertex A, and the segment AE has appeared. E is the candidate foot of the height and can be dragged on the line passing through the base BC. Basing on visual perception, the verifier must drag the point E on the straight line in order to place the height in the correct position. Once the move is finished, in order to check if the height has been positioned correctly, students click on the "Test" check box, which displays the height segment built with the GeoGebra commands (in Fig. 1b, Test A is selected). If the two segments visually coincide, the verifier wins a point, otherwise the falsifier wins a point.

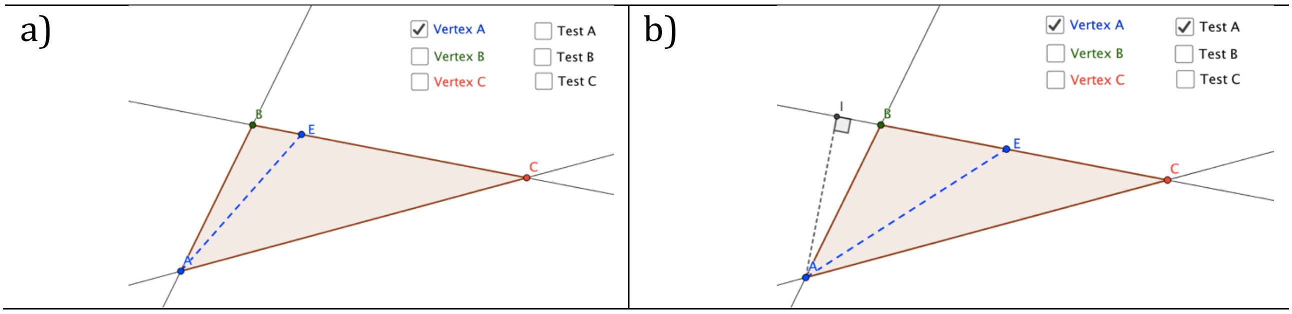

Figure 1: a) Effect of the "Vertex A" button. b) Effect of the "Test A" button

Inquiring-game activities are based on Hintikka's game theoretical semantic (1998). They have been used in previous research at secondary school level to investigate the effect of inquiring-game approach to geometry learning (Soldano & Arzarello, 2017, Soldano & Sabena, 2019). Differently from the previous inquiring-game activities, in the HCG an instrumented feedback is introduced (by means of the Test check box), to support students to visualize the correctness of the verifier's moves. We experimented the HCG in three 7th grade classes, where the height concept had been presented to students by their teachers in the previous year. In the experimented design, students are asked to play at least 4 times, exchanging their roles. At the end of the game, the students answer some questions contained in a written sheet. The first question (*When you were playing the verifier role, what did you pay attention to in order to place the height in the correct position?*) focuses students' attention on critical attributes of the height. The second one (*When you were playing the falsifier role, what geometric characteristics did the most difficult figures proposed to the verifier have?*) focuses on the falsifier's move and the possible use of non-prototypical configurations in the game. Finally, the third question (*When you played as a verifier, did you always manage to reach the goal? If yes, explain how you did it, otherwise explain why you* *did not succeed*) intends to investigate the formative role of the activity and see if, thanks to the feedback of the computer, students become aware of any errors and misconceptions. These questions were the ground of a collective discussion, in which we were present as participant observers, in collaboration with the classroom teacher.

EXCERPTS FROM THE CLASS DISCUSSION

R1 and R2 introduce the activity, videotape a couple of students while playing the game and the class discussion. The focuses of questions contained in the written sheet were proposed in the class discussion, of which we will analyse some extracts. We will consider words, gestures and written signs produced in the discussion as semiotic productions through which mathematical thinking evolves in the learning path of students, according to a multimodal perspective (Arzarello, 2006). Words, gestures and representations in GeoGebra will be analysed in order to investigate the critical attributes identified by students for the height concept and to have access to the students' personal conceptual image.

The discussion opens with a challenge on the IWB between two volunteer students. After the challenge, researcher R1 prompts students to reflect on their own processes:

1 R1: What did you watch in order to reach the goal as verifiers?

2 S1: Making the angles of 90 degree, that is, in order to make the height of a triangle you have to look at the angles of 90°

3 R1: Ok, come and show us which angles you were looking at. Can you point them out on the IWB?

4 S1: I was looking at this, the blue point, and that point ... [*he is indicating first a vertex and then one of the two angles that the height forms with the base, Figure 2a-b*]

5 T: Who wants to say something?

6 S2: I was looking… I was looking at the side on which to place the height point [*while walking towards the IWB, he raises the right arm horizontally, Figure 2c*]. I look at the side below and I look that the height is 90° [*he is raising his left forearm perpendicularly, Figure 2d*]

7 R1: So, can you tell us exactly where [you look]?

8 S2: I look at this here, this side below [*indicating the base, Figure 2e*] and I see that the height line is 90° [*placing his hand vertically, Figure 2f*]

In answering R1's question, S1 recalls "angles of 90°" (line 2) as a critical attribute for the height concept. He then indicates the vertex from which to trace the height and one of the right angles that the height forms with the base (line 4, Fig. 2a-b). The teacher gives the floor to another student, S2, who expresses the critical attributes in words and through gestures produced while walking towards the IWB.

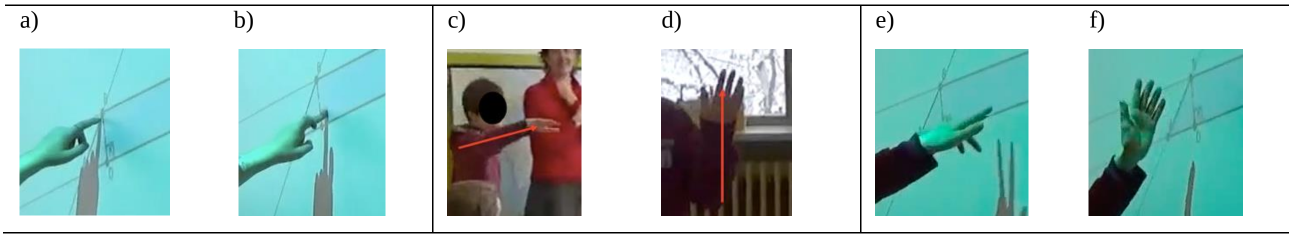

Figure 2: Students’ gestures during the discussion.

Whereas S1 mentioned first the vertex of the triangle and then the right angle, S2 is first referring to the base of the triangle. As a matter of fact, he is first lifting the right arm horizontally and then, while lifting the left forearm vertically, he says that he looks for the height to be at 90° (line 6). The combination of the vertical and horizontal gestures makes the property of perpendicularity between base and height—which is actually a critical attribute for the height concept—visible. The discussion continues:

9	R2:	And how can you conclude whether “it's 90” or “it is not”?
10	S2:	You can see it
11	S1:	You have to see if the two ... the height and the base are more or less at 90° ... if they are ... that ... [*he traces the base and height in the air with the two arms and then places his left hand perpendicular to the surface of the desk, Figure 3a-b*]
12	S3:	Well yes, but we see that it is like this
13	R2:	What do you mean? What is it that you see?
14	S3:	You can see that it is an angle of 90° because it is like this [*he places his palms perpendicular to each other, Figure 3c-d-e*]
15	S4:	It’s L-shaped
16	S2:	If I have in mind what a 90° angle does look like [*gestures with the two hands, as Figure 3f-g*]

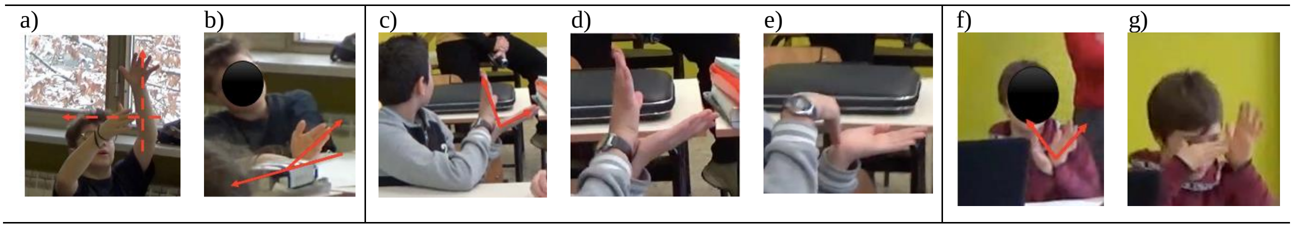

Figure 3: Students’ gestures relating to a 90° angle.

To explain how to visually establish whether an angle measures 90°, the students produce gestural configurations with two hands/arms representing the perpendicular sides of angles. If we observe closely the pictures reported in Figure 3, we may identify cases in which hands or arms are placed in the horizontal-vertical orientations (Fig. 3 a, b, e, g), according to the prototypical stance, and cases in which hands or arms are arranged in a rotated position (Fig. 3 c, d, f)—let us remember that the triangle shown at the IBW has no horizontal or vertical sides. S2 and S3 shift fluently between one case and the other, indicating that their conceptual

images are not anchored to a prototypical representation and that their examples space include both prototypical and non-prototypical examples of height.
Then R2 direct students' attention to reflect on mistakes made while playing:

17 R2: During the game, did anyone find a case where they got it wrong? Where you say, I was really wrong, I put the height just somewhere else. [*S5 raises his hand*]. Please, can you [*S5*] show us what it happened? What kind of mistake [you made]? And then what did you do?

18 S5: [*He gets up and walks towards the IWB*] I put it in this way [*he moves F very close to B, Fig. 4a*]

19 S1: No, he did place it in the center [*S5 drags F to the midpoint of AB, Fig. 4b*]

a) b)

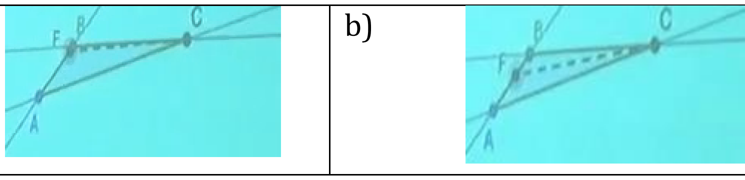

Figure 4: IWB figure proposed by S5 to discuss his mistake

S5's mistake consisted in attributing to the height a non-critical attribute, namely to have an extreme in the midpoint of a triangle side: this is true for isosceles cases, which are so often used in Italian school textbooks (and by many teachers) even if they belong to the subordinate category of triangles and not to the basic category. Such a mistake may be interpreted with reference to the prototype effect, in which a non-critical attribute for the concept of height has been erroneously attributed to it. During the game, the feedback from the computer allowed S5 to notice the inconsistency between where he expected the height to be (basing on his personal conceptual image), and where it was placed by the software. When prompted to reflect on his own mistake, the student brings it back to the attention of the whole class. Possibly, he has not yet realized *what* kind of mistake he did, as we may see in Fig. 4a, but his fellow S1 (challenger in the game) remembers precisely the mistake and is able to offer a description in words (line 19) which helps S5 to reproduce it (Fig. 4b) and may contribute to make him to reflect further.

The inconsistency between a student's incorrect personal conceptual image and the correct concept image can be perceived not only thanks to the feedback of the computer but also to the careful observation of the moves made by the challenger in the game:

20 R1: And this maybe happened at the beginning of the challenges? Or…

21 S1: Yes at the beginning, the second challenge

22 S3: I was always wrong!

23 R2: Who says that he was always wrong, then what did he do? Did they stop then making mistakes?

24 S3: I lost both challenges and then later, that is when we finished everything, when answering the questions on the sheet I understood how to do it

25 T: Did you then understand what was wrong?

26 S3: Yes, because I saw how he made them and then I understood how to put them

29 R2: But he also explained his trick to you or not ... you looked at

30 S3: No, I just looked at his moves

During the game, the careful observation of the moves made by a more experienced opponent takes on a formative role in understanding how to put the height and therefore can evolve the personal conceptual image of the student. The challenge partner is an opponent in the game but an ally in learning.

CONCLUSION

Identifying the critical attributes for the height concept and applying it in different examples (including non-prototypical ones) is crucial for winning the Height Challenge Game. From our observation, the students' engagement in the game and in reflecting on their thinking processes by means of the worksheet questions and during the discussion allows the teacher (and the students themselves) getting precious information on their personal conceptual images of the height of triangles. In addition, the game may enact an evolution of students' conceptual images, by means of the dialectical relationships between different kind of figures, as schematized in Figure 5.

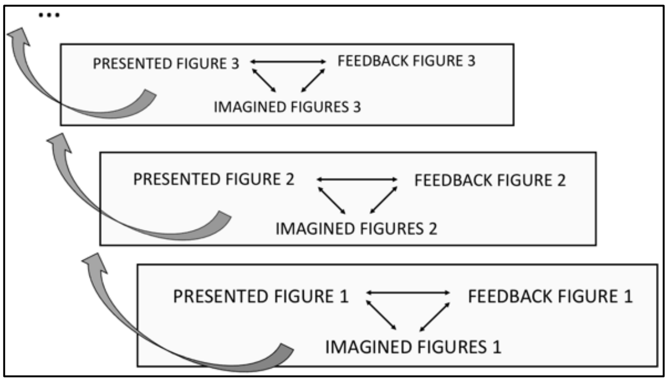

Figure 5: Evolution of the dialectical relationships between the different figures.

At the beginning of the game, the students have some personal conceptual images of height, which include the figures they imagine (Imagined Figures). Playing the verifier role, the figure imagined by the verifier is made visible through his move (Presented Figure). This move may show—to an expert's eye—possible students' misconceptions relative to the critical attributes of the geometric concept at stake. Thanks to the computer feedback, students are helped to notice these possible inconsistencies between the Imagined, Presented and Feedback Figure. Observing the inconsistencies offers the opportunity for a first evolution of the personal conceptual image of height. If such an evolution has occurred, then this will have effects in subsequent challenges where new Imagined, Presented and Feedback figures will be produced. The cycle repeats itself while playing. As the game is played by two students and the role are exchanged, it is likely that Imagined Figures of different kinds are proposed in the game, thus contributing to enlarging the students' example space. For the students who already have the correct conceptual image of height, the game plays a consolidating role. As part and parcel of our design and theoretical stance, the teacher's role is crucial for guiding the students in becoming aware of their personal concept image and for comparing it with the formal concept definition. From our observations, the HCG may offer rich material from which such a discussion may be orchestrated. In this paper we reported some brief excepts from the first experimentation, and further data will help us to validate the proposed model.

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PRESCHOOL TEACHERS' SELECTION OF PICTURE BOOKS FOR MATHEMATICS INSTRUCTION: AN INTERVIEW STUDY

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Various studies pointed to the potential of picture book reading (PBR) for supporting preschoolers' mathematical development. The features of picture books vary greatly, and these features contribute to the effectiveness of PBR. It is therefore important to adequately select picture books for mathematics instruction. We analyzed the general and mathematical PB features preschool teachers take into account when selecting picture books for this aim. Interviews with 66 preschool teachers indicated that they rate general features and features related to basic mathematical skills as most important, and explain their importance based on instructional goals rather than preschooler characteristics. Our results point to the need for professional development initiatives on the selection of picture books for mathematics instruction.

INTRODUCTION

Mathematics Instruction: Picture Books

Preschoolers' mathematical competencies are foundational for their later academic achievement (e.g., Anderson & Phillips, 2017; Watts et al., 2018). Preschool mathematics instruction can support the acquisition of these competencies. Recent studies point to the potential of picture book reading (PBR) activities for enhancing preschoolers' mathematical development (e.g., Purpura et al., 2017). Studies in the literacy domain showed that both general (e.g., tactile manipulations) and domain-specific (e.g., visual representations of words) picture book features contribute to the effectiveness of PBR activities for preschoolers' early language development (e.g., Chiong & DeLoache, 2012; Flack et al., 2018). Contrasting the literacy domain, studies on picture book features and their role in the selection of picture books and in the effectiveness of PBR activities in the domain of mathematics are almost non-existent. The limited number of studies on picture book features in the domain of mathematics revealed that picture books written with and without an explicit mathematical aim vary greatly in content and structural features (Ward et al., 2017; Splinter et al., Submitted). Moreover, according to Ward et al. (2017), they frequently contain features that are assumed to stimulate mathematical development (e.g., presence of Arabic numerals) but also features that may hinder this development (e.g., randomly presented items to

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 19-26). PME.

be counted). It is therefore important that preschool teachers carefully select picture books for mathematics instruction, taking into account the picture book features.

Preschool Teachers' Selection of Picture Books

Current insights into preschool teachers' selection of picture books for mathematics instruction are scarce. Only three studies addressed this timely issue, thereby focusing on a limited range of picture book features. Pentimonti et al. (2011) and Stites et al. (2020) analyzed preschool teachers' selection of picture books in view of mathematical content. They found that preschool teachers infrequently selected picture books that include explicit mathematical content. Moreover, these teachers reported that powerful preschool mathematics instruction should include activities different from PBR. Cooper et al. (2018) studied which criteria pre-service preschool teachers take into account when selecting a picture book for mathematics instruction. Their findings indicate that pre-service preschool teachers select picture books on the basis of the mathematical topic to be taught and their lecturer's recommendations, rather than the quality of the picture book.

Current Study

Given the large variety in mathematical picture books features, and Ward et al. (2017)'s assumptions about the learning-supportive versus hindering role of these features for preschoolers' mathematical development, the careful selection of picture books in view of their features is crucial. Systematic analyses of preschool teachers' selection of picture books along a wide range of both general (e.g., tactility) and domain-specific (i.e., mathematical) features are currently lacking. We aimed to deepen current understanding of this topic by systematically analyzing preschool teachers' selection of picture books for mathematics instruction, with special attention for the features they rate as most important and why.

METHOD

Participants were 66 preschool teachers (M_{age} = 41y, $M_{\text{teaching experience}}$ = 9y), providing instruction to 2.5- to 4-year-olds (n = 22), 4- to 5-year-olds (n = 26), and 5- to 6-year-olds (n = 18). Participants were first offered an online questionnaire on their demographical and occupational information. They next participated in an online interview on their use and selection of picture books for stimulating preschoolers' mathematical development.

Teachers were first interviewed about their use of picture books for mathematics instruction. Next, they were offered a series of questions focusing on their selection of picture books for mathematics instruction. In a first question teachers were offered a list of 22 picture book features (17 domain-specific and 5 general). They had to rank them into four categories (i.e., very important, fairly important, slightly important, and not important). Afterwards they were asked to rank the features within each category from most to least important so that a complete ranking of all features was made. Scores ranged from 1 up to 22, with 1 indicating least important and 22 most important. Picture book features were selected on the basis of previous studies (i.e., Splinter et al., submitted; Ward et al., 2017) and are shown in Table 1.

#	Domain-specific features	#	General features
1	Presence of numerosities 1-10	18	Possibilities for tactile manipulation
2	Presence of numerosities >10	19	Presence of story
3	Presence of the number zero	20	Having a good length or duration to read
4	Presence of Arabic numerals	21	Being child-oriented
5	Presence of number words	22	Presence of a theme that fits to the theme the teacher wants to use for the classes
6	Presence of an ascending counting format		
7	Presence of a descending counting format		
8	Presence of 1-1 correspondence		
9	Presence of cardinality		
10	Presence of ordinality		
11	Presence of arithmetical operations		
12	Presence of comparisons between quantities		
13	Presence of decomposition of sets		
14	Presence of items to be counted that have the same size		
15	Presence of items to be counted that are linearly arranged		
16	Presence of items to be counted are distinct		
17	Presence of items to be counted that are presented solely		

Table 1: Domain-Specific and General Features used in the Ranking Question

The second series of questions consisted of 14 forced-choice comparisons that required teachers to choose between two versions of a picture book. The two versions differed in one of the domain-specific features included in the ranking question. In ten comparisons the feature was present in one version versus absent in the other version (e.g., number zero is present versus absent). Four features did not allow such an operationalization; for these features, teachers had to compare two versions of the presence of the same feature (e.g., set arrangement: linearly versus randomly arranged). The 14 forced-choice comparisons are shown in Table 2. The teachers were asked to select the picture book version they would use for their preschool mathematics instruction and explain their choice.

Teachers' preferences were scored dichotomously, with 1 indicating version 1 and 2 indicating version 2. Their explanations for this preference were coded in three steps. First, every answer was checked for including only a reference to a feature, without any explanation (F), only an explanation, without any reference to a feature (E), or both a reference to a feature and an explanation (FE). Second, when a feature was mentioned (i.e., F and FE), we scored whether it was the manipulated feature (1), another feature (2), or unclear (3). Third, responses that included the manipulated feature and an explanation (i.e., FE, 1) were coded in terms of whether they referred to their preschoolers' characteristics (i.e., age or other characteristics), instruction (i.e., learning goals or other instructional possibilities), or other (personal reasons, reference to another domain than mathematics, unclear responses). Inter-rater reliability (9 teachers) was sufficient for all three steps (*Cohen's kappa* = 0.75-0.86).

Comparison	Version 1	Version 2
1. Number range[b]	Numerosities >10 present	Numerosities 1-10 present
2. Presence of zero[a]	Zero not present	Zero present
3. Presence of Arabic[a] numerals	Arabic numerals not present	Arabic numerals present
4. Counting format[b]	Ascending counting format	Descending counting format
5. Presence of arithmetical operations[a]	Arithmetical operations not present	Arithmetical operations present
6. Presence of 1-to-1 correspondence[a]	1-to-1 correspondence present	1-to-1 correspondence not present
7. Presence of cardinality[a]	Cardinality present	Cardinality not present
8. Presence of ordinality[a]	Ordinality not present	Ordinality present
9. Presence of decomposition[a]	Decomposition present	Decomposition not present
10. Presence of comparisons[a]	Comparisons not present	Comparisons present
11. Set size[b]	Items of same size	Items of discrepant sizes
12. Set arrangement[b]	Linearly arranged items	Randomly arranged items
13. Set differentiation[b]	Overlapping items	Distinct items
14. Presence of distractors[a]	Distractors not present	Distractors present

[a] Comparison between the presence versus absence of the feature.
[b] Comparison between two versions of the presence of the same feature.

Table 2: Forced-choice Comparisons of the Domain-Specific Features

RESULTS

Use of Picture Books

We analyzed the occurrence and frequency of picture book use for mathematics instruction. All teachers reported to use picture books in their instruction, but not all of them did so for mathematics: 8 teachers (12.1%) did not use picture books in view of preschoolers' mathematical development. Although 43 teachers (65.2%) reported to use picture books on a daily basis, only 4 (6.1%) used them for mathematics instruction every day. When asked which picture books they most frequently used for mathematics instruction, teachers spontaneously referred to picture books written with as well as without an explicit mathematical aim.

Selection: Picture book features

To deepen our insights into the selection process, we analyzed teachers' responses to the ranking question and the forced-choice comparisons.

When analyzing the teachers' rankings of the 22 features, we found that four general features and one domain-specific feature were part of the five features for picture book selection that were ranked as most important: (1) the picture book is child-oriented (M = 19.44, SD = 4.4), (2) includes a story (M = 17.2, SD = 4.6), (3) presents a fitting theme (M = 16.8, SD = 5.7), (4) has a proper length or duration to read (M = 15.0, SD = 5.4). The most important domain-specific feature was (5) the inclusion of the numerosities 1-10 (M = 16.9, SD = 4.2). The seven features ranked least important included six features related to more complex mathematical contents or the visual representation of the content, i.e., (1) includes the number zero (M = 7.4, SD = 4.8), (2) includes numerosities larger than 10 (M = 6.4, SD = 4.3), (3) includes arithmetical operations (M = 4.9, SD = 4.8), (4) has no distractors (M = 7.5, SD = 5.3), (5) has items to be counted that are linearly arranged (M = 7.0, SD = 5.3), and (6) includes items to be counted that have the same size (M = 6.9, SD = 4.1). The inclusion of tactile manipulations (M = 8.2, SD = 6.5) was also scored as not important (7). All other domain-specific features were ranked as slightly to fairly important (M = 8.4-14.8).

Regarding the forced-choice comparisons, we calculated the percentage of teachers that preferred version 1 versus version 2 per comparison. For most features, teachers clearly preferred one version over the other: 96% preferred the presence of Arabic numerals, 91% the presence of an ascending counting format, 88% the presence of ordinality, 85% the presence of decomposition and the presence of distinctly presented items, 83% the presence of numerosities 1-10, and 82% the presence of distractors. In addition, 74% preferred the presence of cardinality, 71% the presence of linearly arranged items, and 70% the presence of comparisons. For four features there was no overall agreement, i.e., 61% preferred the presence of zero, 60%, one-to-one correspondence, 63% items with the same set size, and 55% arithmetical operations.

Selection: Explanations

To better understand teachers' selections, we analyzed their explanations for their choices in the forced-choice comparisons. The results are presented in Table 3. Overall, for each of the 14 domain-specific features, teachers referred to preschooler characteristics and/or instruction in their explanation. But they explained their preferences more frequently on the basis of instructional goals compared to preschooler characteristics. Only for the feature "number range presented" teachers more frequently referred to preschooler characteristics to than instructional goals.

Furthermore, most teachers generally selected the picture book version with the feature assumed most learning-supportive by Ward et al. (2017; e.g., presence of Arabic numerals, presence of cardinality, presence of comparisons). Teachers explained these choices often in terms of instructional goals. A substantial number of teachers also preferred features that are assumed not to support children's mathematical development (Ward et al., 2017), i.e., the presence of items with discrepant sizes, randomly arranged items, and the inclusion of distractors. Teachers' explanations for these choices often referred to the potential of these more complex contents for stimulating their preschoolers' mathematical development.

	n[a]	Preschoolers		Instruction		Other
Comparison		Age	Other	Learning goals	Other	
1. Numerosities >10 present	8	50%	38%	0%	25%	13%
Numerosities 1-10 present	48	42%	19%	35%	23%	13%
2. Zero not present	24	30%	13%	50%	17%	4%
Zero present	37	3%	14%	49%	24%	16%
3. Arabic numerals not present	3	0%	100%	33%	33%	33%
Arabic numerals present	55	9%	24%	47%	26%	6%
4. Ascending counting format	47	30%	21%	32%	28%	6%
Descending counting format	6	0.%	17%	50%	50%	0%
5. Arith. operations not present	27	26%	7%	52%	19%	4%
Arith. operations present	27	7%	37%	15%	63%	4%
6. 1-1 correspondence present	25	0%	8%	60%	40%	0%
1-1 correspondence not present	14	14%	14%	7%	57%	29%
7. Cardinality present	35	6%	40%	34%	46%	11%
Cardinality not present	8	13%	38%	50%	50%	25%
8. Ordinality not present	4	0%	0%	0%	100%	0%
Ordinality present	36	3%	19%	47%	47%	11%
9. Decomposition present	21	5%	19%	29%	67%	5%
Decomposition not present	4	50%	25%	0%	25%	0%
10. Comparisons not present	9	33%	11%	22%	22%	44%
Comparisons present	23	9%	4%	52%	35%	4%
11. Items have the same size	35	26%	14%	6%	63%	0%
Items have discrepant sizes	17	18%	0%	53%	41%	0%
12. Items are linearly arranged	38	5%	26%	3%	63%	5%
Items are randomly arranged	18	28%	17%	17%	50%	6%
13. Items are overlapping	7	14.%	14%	0%	71%	43%
Items are distinct	31	23%	10%	7%	74%	13%
14. Distractors present	50	16%	12%	2%	76%	22%
Distractors not present	11	9%	36%	0%	91%	0%

[a] Number of teachers whose answers were labelled as FE, 1.

Table 3: Explanations for Forced-Choice Comparisons (in Percentages)

DISCUSSION

An increasing number of studies points to the potential of PBR activities for preschoolers' mathematical development. Studies in the domain of literacy revealed that picture book features contribute to the effectiveness of these activities. Given the large variety in picture book features and the scarcity of current insights into teachers' selection of picture books in view of these features in the domain of mathematics, we systematically analyzed preschool teachers' selection of picture books for mathematics instruction on the basis of general and domain-specific picture book features as well as their rationales for these choices.

Our results showed that all teachers use picture books for instructional purposes, and most of them even on a daily basis, but that these findings do not apply when focusing on the domain of mathematics. About 10% of the teachers reported not to use picture books for mathematics instruction and only a minority of the teachers who used picture books for mathematics instruction did so on a daily basis. Teachers referred to both picture books with explicit mathematical content and picture books without explicit mathematical content for the latter purpose. Our findings show higher percentages of picture book use for mathematics instruction compared to previous studies, which is probably due to the fact that these studies focused on mathematical content as indicator of this use (Pentimonti et al., 2011; Stites et al., 2020). Furthermore, when selecting picture books for mathematics instruction, teachers rated general (i.e., not domain-specific, mathematical) picture book features most important, and features related to more advanced mathematical concepts or visual representations of mathematical content least important. Although teachers generally preferred picture books with features that are assumed learning-supportive by Ward et al. (2017), they also selected picture books with features that these researchers assume to hinder preschoolers' mathematical development. Teachers' explanations for the latter choices clearly pointed to the potential of these (assumed hindering) features for engaging in complex mathematical thought during instruction. Hence, an important next step is to investigate whether and to what extent teachers effectively make use of these features in their mathematics instruction. The findings of those studies can significantly add to, and change, current assumptions about the beneficial role of specific picture book features for children's mathematical development.

Although future studies are needed to deepen our understanding of teachers' use and selection of picture books for mathematics instruction, our findings significantly complement previous work on the topic. Also, our study points to the need for further professional development initiatives for preschool teachers related to the use and selection of picture books for mathematics instruction.

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STUDENTS' KNOWLEDGE ABOUT PROOF AND HANDLING PROOF

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Research has highlighted that students' have problems regarding mathematical proof. In part, these have been connected to deficits in their knowledge about proof and handling proof. However, empirical data on students' knowledge about proof and handling proof throughout secondary education is so far missing. Further, it is unclear if there is a connection between concept- and action-oriented knowledge about proof and handling proof. To address these gaps, an empirical study was conducted, investigating the knowledge about proof and handling proof concerning proof principles of N = 456 students in grade 8 to 11. Results indicate that (i) only concept-oriented knowledge significantly increases throughout secondary education and (ii) there is only little relation between concept- and action-oriented knowledge.

INTRODUCTION

Proofs play a central role in mathematics as a proving scientific discipline (Mariotti, 2006) and are thus an important part of mathematics education. Although there is a particular focus on proofs at the university level, proof and handling proofs are also established and important goals in secondary mathematics education (e.g., CCSSI, 2010). Thus, learners in secondary school are expected to build up an individual understanding of proof (cf. Sporn et al., 2021) throughout secondary education. One aspect of this individual understanding of proof is the availability of sufficient *knowledge about proof and handling proof*. Such knowledge about proof and handling proof has been investigated in prior research and includes, for example, knowledge about valid and invalid arguments in mathematical proofs or about different proof methods. In particular, it was found that learners often have deficits regarding knowledge about proof and handling proof, which was identified as a possible explanation for difficulties in handling proof (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000; Heinze & Reiss, 2003; Reiss et al., 2001). Accordingly, this knowledge plays an important role in the context of proof and is of great interest for research. However, empirical data on students' knowledge about proof and handling proof throughout secondary education is so far missing. In this context, knowledge about proof and handling proof can be focused (i) in relation to proofs or generic action situations in general without reference to a specific action situation (cf. Andersen, 2018), so-called *concept-oriented knowledge*, and (ii) in relation to a specific mathematical action situation, for example the construction or validation of a specific proof (cf. Healy & Hoyles, 1998; Heinze & Reiss, 2003), so-called *action-oriented knowledge* (cf. Sporn et al., 2021). Furthermore, it is not clear how concept- and action-oriented knowledge about proof and handling proof are related. While it makes sense

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that both are closely related and go hand in hand, it is also conceivable that concept-oriented knowledge does not automatically transfer to action-oriented knowledge and vice versa.

Data regarding both of these aspects, that is knowledge about proof and handling proof throughout secondary education and the relation of concept- and action-oriented knowledge, are relevant to better understand how students' reported difficulties with proof can be addressed and how specific learning opportunities in secondary education can be structured and designed. This paper thus presents results of an empirical study examining concept- and action-oriented knowledge about proof and handling proof by learners in grade 8 to 11 from German secondary schools.

THEORETICAL BACKGROUND

Mathematical Proof in Secondary Education

That proofs and proof-related activities (e.g., constructing or validating proofs) play a role in secondary mathematics education is reflected in their implementation in standard documents worldwide (e.g., CCSSI, 2010). Research on mathematical proof has repeatedly shown that learners at different stages of their mathematics education often have severe problems with mathematical proof. For example, Healy & Hoyles (2000) showed that even high attaining students have problems to correctly validate proofs and suggest that difficulties can (at least partially) be explained by different proof schemes (Harel & Sowder, 1998). Insufficient and differing knowledge about acceptance criteria for validating mathematical proofs (Sommerhoff & Ufer, 2019) and insufficient methodological knowledge (Heinze & Reiss, 2003) were highlighted as further explanations for learners' difficulties in this context. Overall, studies suggest that learners form a certain individual understanding of proof throughout their school mathematics education, which in specific ways can both facilitate and impede an exploration of mathematical proof (Sporn et al., 2021).

Knowledge about Proof and Handling Proof

Varying socio-mathematical norms in different mathematical communities and settings can lead to the lack of a universal acceptance of a correct proof (Inglis et al., 2013). Thus, specifying exactly when a general proof is to be considered a correct mathematical proof is highly non-trivial (cf. Sommerhoff & Ufer, 2019). However, from a *disciplinary perspective* on proofs and handling proof (i.e., focusing on the discipline of mathematics as a whole), criteria that an (ideal) mathematical proof must fulfil to be considered as correct can be defined. Such criteria, which are valid independently of a specific proof, can be summarized as *proof principles* (Besides these proof principles, proof methods, proof functions, and proof presentation can also be described from a disciplinary perspective; see Sporn et al., 2021).

From an *individual-psychological perspective* on proofs and handling proof (i.e., focusing on individual learners, cf. Sporn et al., 2021), knowing these proof principles can be seen as an important facet of individuals' *knowledge about proof and handling* *proof*, one aspect of an individual's understanding of proof. For example, Reiss et al. (2001) analyzed secondary students' proofs, suggesting that their knowledge about proof and handling proof concerning proof principles was insufficient, which was an obstacle for their construction of proofs. Heinze und Reiss (2003) give further indications for the importance of knowledge about proof and handling proof concerning proof principles for success in dealing with mathematical proofs by focusing on three important principles that they combine under the umbrella term "methodological knowledge".

While learners' *knowledge about proof and handling proof concerning proof principles* can (at least partially) explain their difficulties with proofs, prior research indicates that it is not only necessary to distinguish between knowledge and non-knowledge. It is additionally needed to distinguish whether (i) this knowledge about proof and handling proof is required in general contexts, meaning without reference to a specific proof and proving activity (e.g., Andersen, 2018) or whether (ii) this knowledge is required in a specific mathematical action situation, for example, in the context of the construction or validation of a specific proof (e.g., Healy & Hoyles, 1998; Heinze & Reiss, 2003). Here we refer to (i) *concept-oriented* and (ii) *action-oriented knowledge about proof and handling proof* (cf. Sporn et al., 2021). For example, concept-oriented knowledge about proof and handling proof concerning proof principles can be assessed using the statement *"Mathematical proofs that use the statement to be proved as a premise are particularly elegant."* and thus without reference to a specific proof and proving activity. In contrast, Figure 1 shows an example item, highlighting how knowledge about proof and handling proof concerning proof principles can be assessed with an action-oriented focus in the context of proof validation.

Ben has to prove the following proposition:
The sum of three consecutive natural numbers is divisible by 3.

Ben's purported proof:
I know this from school. Our textbook contained a proof that this is valid for every natural number. There, it was shown that: $3 + 4 + 5 = 3 + 3 + 1 + 3 + 2 = 3 \cdot 3 + 3$
This proves the proposition.

Is Ben's purported proof a valid mathematical proof?

- ○ Yes
- ○ No

Figure 1: Example item for action-oriented knowledge concerning proof principles.

In order to achieve the proof related goals in secondary education, it appears necessary that learners acquire sufficient knowledge about proof and handling proof (both concept- and action-oriented), otherwise problems may arise (cf. Healy & Hoyles, 1998; Heinze & Reiss, 2003). Further, German secondary education curricula include proof and handling proof in most grades, thus providing corresponding learning opportunities. Assuming that these learning opportunities occur in appropriate quality and quantity and that learners use them properly, students' knowledge about proof and handling proof can be expected to increase throughout their secondary school education.

RESEARCH QUESTIONS

The present paper sets out to gain a better understanding of (i) students' individual knowledge about proof and handling proof throughout secondary school education as well as (ii) about the connection of concept- and action-oriented knowledge about proof and handling proof. For this, a quasi-longitudinal study investigating concept- and action-oriented knowledge about proof and handling proof concerning proof principles of students in grades 8 to 11 is presented. The following research questions were focused:

> (RQ1) How can students' concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles be characterized throughout their secondary school education?
>
> (RQ2) How do students' concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles relate to each other in different grades?

Regarding RQ1, it was expected that students' knowledge concerning proof principles – both concept- and action-oriented – would increase (mostly monotonically) during the course of schooling, as mathematical proof is a learning goal throughout secondary education and according learning opportunities exist throughout multiple grades. Regarding RQ2, it was expected that concept-oriented knowledge about proof and handling proof serve as a basis for action-oriented knowledge about proof and handling proof, resulting in an at least medium correlation between concept- and action-oriented knowledge.

METHOD

To answer these questions, $N = 456$ (183 m, 263 f, 10 d) students in grade 8 to 11 ($N_8 = 139$; $N_9 = 122$; $N_{10} = 72$; $N_{11} = 123$) from eight German secondary schools were surveyed in an online study at the end of the school year 2020/2021. As part of the survey, students were questioned regarding various aspects of their individual understanding of proof and regarding more general information about the participants, for example demographic data. In this paper, we only consider the demographic data as well as data on concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles.

To assess concept-oriented knowledge about proof and handling proof concerning proof principles, students were asked to evaluate 18 statements (see example item above) on a 6-point Likert scale ("Not true at all" to "Totally true"). Each item was evaluated based on the disciplinary perspective on proof and handling proof (i.e., regarding an ideal concept of mathematics/proof). Internal consistency was acceptable with $\alpha = .75$. Action-oriented knowledge about proof and handling proof concerning proof principles was assessed using a task format focusing on the activity of proof validation, based on the tasks used by Healy & Hoyles (2000). Students were presented six purported proofs, all of which contained different errors regarding proof principles. They were asked to judge each purported proof as to whether it is a valid mathematical proof (dichotomous answer). Figure 1 shows an example item with a proof that includes higher authority as an invalid argument. The six purported proofs addressed different mathematical contents, which were known to all participants at the time of the survey. The internal consistency was poor, with $\alpha = .60$. The 18 statements corresponding to concept-oriented knowledge were combined to a mean score (M_{coK}; possible values for each statement ranged from 1 to 6). For action-oriented knowledge, the judgements of the 6 purported proof were combined to a mean score (M_{aoK}; possible values for each proof: 0 or 1). To allow a better comparison with concept-oriented knowledge, M_{aoK} was rescaled to a possible range from 1 to 6. To answer RQ1, we conducted two independent linear regressions with either M_{coK} or M_{aoK} as dependent variables and grade as independent variable. To answer RQ2, correlations between M_{coK} and M_{aoK} were calculated for each grade as well as based on the whole sample.

RESULTS

Table 1 shows descriptive statistics for concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles for each grade. Data indicate that concept-oriented knowledge increases with grade, while action-oriented knowledge remains mostly stable across grades with its lowest value at the end of 10th grade. Overall, the standard deviations for action-oriented knowledge appear to be higher than for concept-oriented knowledge for all grades.

Grade	**Concept-oriented Knowledge**		**Action-oriented Knowledge**	
	M_{coK}	*SD*	M_{aoK}	*SD*
8	3.69	0.58	3.24	1.55
9	3.75	0.59	3.44	1.84
10	3.91	0.65	3.04	1.61
11	3.99	0.67	3.12	1.62

Note: Possible values ranged from 1 to 6.

Table 1: Descriptive statistics for concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles for grade 8 to 11.

Students' Concept-oriented and Action-oriented Knowledge about Proof and Handling Proof Concerning Proof Principles (RQ1)

The linear regression for concept-oriented knowledge showed a significant positive impact of grade ($\beta = .193$, $p < .001$), while the regression for action-oriented knowledge showed an insignificant impact ($\beta = -.048$, $p = .393$). Similarly, the regression model for concept-oriented knowledge is significantly better than a Null-Model ($p_{\text{coK}} < .001$) while the model for action-oriented knowledge does not differ significantly ($p_{\text{aoK}} = .393$). However, the variance explained by grade is rather small in both linear models ($R^2_{\text{coK}} = .037$, $R^2_{\text{aoK}} = .002$).

Relation of Students' Concept-oriented and Action-oriented Knowledge about Proof and Handling Proof Concerning Proof Principles (RQ2)

Correlations between M_{coK} and M_{aoK} in total showed a significant positive, weak correlation (*r* = .16, *p* = .004). Table 2 shows corresponding correlations for each grade. Here, only for 9th grade a significant positive, weak correlation was found, while the correlations are insignificant for all other classes.

Grade	**8**	**9**	**10**	**11**	**Total**
$r_{\mathbf{Pearson}}$ **(p)**	.14 (.190)	**.22 (.044)**	.12 (.399)	.20 (.051)	**.16 (.004)**

Note: Significant results are highlighted in bold.

Table 2: Correlations between concept-oriented and action-oriented knowledge about proof and handling proof concerning proof principles

DISCUSSION & OUTLOOK

The present study investigated students' knowledge about proof and handling proof concerning proof principles by learners in grade 8 to 11, distinguishing between concept- and action-oriented foci on this knowledge. Descriptive data reveal medium mean scores for students‘ knowledge about proof and handling proof, both for the concept- and action-oriented focus. While this does not appear too detrimental, the problems pointed out in previous research (e.g., Healy & Hoyles, 2000) suggest that this amount of knowledge is not sufficient to handle proofs sufficiently well. Further, the results show that students' concept-oriented knowledge about proof and handling proof concerning proof principles increases throughout secondary education (RQ1). It thus seems, that the existing learning opportunities during this period have a positive effect and are used by learners at least in some way. However, with an increase of about half a standard deviation in four years of mathematical schooling, this effect does not seem particularly large. Thus, the quality and quantity of learning opportunities regarding mathematical proof need to be further investigated. The expected significant increase of students' action-oriented knowledge about proof and handling proof concerning proof principles throughout secondary education could not be confirmed (RQ1). One reason for this unexpected finding and difference to concept-oriented knowledge may be that learning opportunities for mathematical proof in secondary education do not include sufficient opportunities for students to engage in proofs themselves (Hemmi, 2006) or possibly require a rather imitating style of reasoning (Lithner, 2008). Another reason may be, that the result is an artefact of the items used to measure action-oriented knowledge in this study, which focus on proof validation. While validating proofs as an action situation may be easier than constructing proofs, it may be an activity that occurs less explicit in school and thus participants were less familiar with it.

Although the expected positive relation between concept- and action-oriented knowledge about proof and handling proof concerning proof principles could be confirmed (RQ2), the correlation is quite weak and about the size that would be expected for most types of cognitive variables. The result may indicate that while concept-oriented knowledge about proof and handling proof is available, the corresponding knowledge is not available in specific action situations or can only insufficiently be used. The result can be backed up by an observation by Healy and Hoyles (1998): Students knew in principle that an empirical proof was not sufficient to construct an acceptable proof, that is they had the corresponding concept-oriented knowledge about proof and handling proof. Still, many students constructed empirical proofs when put in a corresponding action situation, as they had no alternative means to construct an acceptable proof themselves. From a research perspective, the fact that concept- and action-oriented knowledge about proof and handling proof only show a weak correlation highlights the relevance of distinguishing between both foci and confirms the assumption by Sporn et al. (2021).

While the results appear plausible and give first empirical data on students' knowledge about proof and handling proof in grade 8 to 11, the presented study also has some limitations that should be considered: First, data refers to a quasi-longitudinal study. While this allows for a good overview, a longitudinal study may help to characterize the individual development of knowledge about proof and handling proof. Second, the low Cronbach's α of the action-oriented knowledge about proof and handling proof may limit the reliability of the results and require future research. However, the scale for action-oriented knowledge about proof and handling proof concerning proof principles includes only 6 items which refer to different mathematical topics and different proof principles, so that a particularly high Cronbach's α was not expected. Third, all interpretations of the results that are connected to learning opportunities rely solely on standards and curricula in Germany. However, we did not gather specific data about the actual learning opportunities of the participants during their secondary education. Future research is needed in this regard, especially focusing on quantity, quality and variance of these opportunity in the different grades and between teachers and schools.

Results show that students acquire an increasing amount of concept-oriented knowledge about proof and handling proof concerning proof principles throughout their secondary education, which should be celebrated. However, results from this study in conjunction with prior research indicating students' difficulties (e.g., Healy & Hoyles, 1998; Heinze & Reiss, 2003) also highlight that existing learning opportunities should be improved so that (i) more concept-oriented knowledge about proof and handling proof is acquired and (ii) also action-oriented knowledge is acquired, a current deficit clearly pointed out by the results. Moreover, comparisons between the examined grades suggest that it may be particularly preferable to provide support between the end of 9th grade and the end of 10th grade to prevent a drop-off in action-oriented knowledge about proof and handling proof at this point.

Overall, results give a first and important empirical impression of students' knowledge about proof and handling proof throughout secondary education. However, future research should more explicitly consider (i) a real (and not quasi-) longitudinal development of students' knowledge, (ii) include further aspects of knowledge about proof and handling proof (such as proof methods, proof functions, and proof presentation), and (iii) also include further aspects that may be relevant in the development in knowledge about proof and handling proof (such as beliefs, and learning orientations). Combining these points will lead to a more comprehensive overview that can be basis for further support and curricular adjustments.

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“WRITING THE WORLD” THROUGH WRITING: PRIORITIES FOR COMPOSITION RESEARCH IN SOCIAL JUSTICE MATHEMATICS

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This qualitative review of four internationally-distributed curricular accounts of social justice mathematics highlights ways in which the activity of writing is under-developed if students are to “write the world” with mathematics. Writing is defined as students’ linguistic expression of at least one sentence. A review of 63 lesson plans, lesson memoirs, or curricular planning guides suggests the pressing need for research on composition processes associated with students’ action steps subsequent to a social justice mathematics activity; on creative or fictional writing; and for research debate on scaffolding the act of interpreting mathematical work in justice terms.

INTRODUCTION

“Write just as you speak,” Freire advised his students (Freire & Macedo, 1987, pp. 48, 50). Indeed, social justice mathematics classrooms usually are portrayed as lively places filled with speaking, negotiation, and purposeful mathematical investigation (e.g., Gutstein & Peterson, 2005). However, the modalities of reading and writing have their most explicit development in social justice mathematics as theoretical constructs, metaphors for agentic processes in which students “read the mathematical word” in order to “write the world with mathematics” (Gutstein, 2016). Though transformational, this model belies the complexity of social justice mathematics writing. Literal writing is fundamental to the social justice mathematics project, yet is barely reported in mathematics education scholarship.

Students write mathematics in multiple forms such as equations, graphs, diagrams, explanations of methods, autobiographies, recasting many received discourses to make mathematics intelligible, for self or for others (Barwell, 2018). Furthering Barwell’s position, in social justice mathematics classrooms, students write not just for self and for others but *with* others and *about* others, too. This significant responsibility of telling others’ stories is intended to help students change the world. The act of writing is therefore both prominent and erased in social justice mathematics, relying on a kind of “theory hope,” (Fish, 1989, p. 342) that if we do valuable things in the classroom, valuable things for humans are likely to happen later. Mathematics education scholarship could productively deconstruct the metaphor of writing into operational terms to better inform pedagogical action.

As a beginning point, this paper offers a qualitative review of stances towards writing in several internationally-distributed social justice mathematics curricular accounts.

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We use a focused definition of writing: students' linguistic expression in the form of a sentence or a broader rhetorical form. When one writes a numerical answer or an equation, the social meaning is unspecified, but writing a sentence requires taking a moral or agentic stand on how the world works (van Leeuwen, 2008). Our research question, "In what ways are acts of writing and their consequences under-developed in social justice mathematics curricular accounts?" allows us to suggest priorities for future research on social justice mathematics composition.

LITERATURE REVIEWS

Within the comparatively rare research reports of social justice mathematics learning (Kokka, 2017), several dimensions are especially relevant to issues of writing. Students' learning may differ based on aspects of their social identities that are either privileged or marginalized and on whether the teacher shares life experiences with the students (Esmonde, 2014; Brantlinger, 2013). Students occupying privileged positions may find their stereotypes of marginalized people reinforced (Esmonde, 2014), or they may learn to conduct "display" of justice orientations rather than to critique the systems that provide them with advantages (Larnell et al., 2016). Social justice mathematics can produce feelings of sadness for marginalized students, who may come to prefer traditional math content and pedagogies (Brantlinger, 2013; Kokka, 2017). These complexities of social justice mathematics learning, and by extension, writing are amplified in superdiverse classrooms that bring together students of varied cultural histories or economic conditions. What it means to write for self, for others, with others and about others involves hidden potentials and vulnerabilities that have barely been addressed in mathematics education research.

Composition studies offer three interrelated orientations to researching and teaching writing that provide accessible entry points for social justice mathematics scholars: text-oriented, writer-oriented, and reader-oriented approaches (Hyland, 2015). Text-oriented approaches focus on the structure of writing rather than the writers' activity, but nonetheless involve social considerations such as genre, as when a student writes a persuasive, math-focused letter to a newspaper editor. Writer-focused approaches attend to authors' creative expression, to cognitive problem-solving processes, or to the autobiographical or material situation that influences text production. An ethnographic or interview-based study of intersectional identities of social justice mathematics writers could grow from this research tradition. Finally, reader-oriented approaches engage the sociality of writing through attention to communities of writers, intertextuality, and roles of power and ideology in writing processes.

METHODS

To understand the breadth of uses of writing in social justice mathematics lessons, we reviewed four, formally-published, internationally distributed resources spanning primary through secondary grade levels:

High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice (Berry III et al., 2020)

Rethinking Mathematics: Teaching Social Justice by the Numbers (Gutstein & Peterson, 2005)

"Mathematics," in Textbooks for Sustainable Development: A Guide to Embedding (Wagner et al., 2017)

Teaching About Gender Diversity: Teacher-Tested Lesson Plans for K-12 Classrooms (Woolley & Airton, 2020).

Our review of these accounts is neither comparative nor evaluative. Each is a successful and valuable contribution to social justice mathematics pedagogy. Instead, we reviewed them to compile a wide range of representations of writing in social justice mathematics activities. Within these sources, we reviewed 63 chapters, lesson memoirs, lesson plans, and for Berry III et al. (2020), publicly-available worksheets connected to many of the chapters. Data saturation was judged when an additional review of cross-disciplinary lessons in Gutstein and Peterson (2005) and a random selection of 20% of the lessons in Stocker (2017) did not generate new types of writing. A limitation of the method is that we can comment only on the lessons as presented. Focused attention towards writing might have occurred without being reported for the mathematics education audience. Although these four publications comprise a relatively small sample of formally and informally available curricular sources for social justice mathematics, the review was sufficiently detailed to generate useful perspectives on writing in social justice mathematics curriculum.

Using the sentence-level definition of student writing, we excluded activities that appeared to result only in a calculation, a graph, an equation or related mathematical representations, or a list of words. We excluded cases in which students spoke and the teacher wrote some of their words on the classroom board. We wanted to understand cases in which writing is under-imagined or taken for granted in the curricular documents, and so we included cases of "implicit writing," in which we were fairly certain that students needed to write in order to accomplish the task. Implicit writing often happened when students created a formal presentation for powerful stakeholders. We believed that in these cases, students would likely spend time preparing their presentation using written sentences.

Initially, we used provisional codes drawn from research on mathematics writing, contextualized mathematics, and our practitioner knowledge of social justice mathematics teaching and curriculum (Barwell, 2018; Miles et al., 2020). Each of the 63 accounts were read by at least two research team members, with coding differences resolved through discussion and consensus. The full team met regularly to refine the coding process through constant comparison and to review each other's analytical memos. Final coding collected information on types of writing, authorship groups, audience, purpose, and whether the writing was explicit or implicit.

FINDINGS

The four sources varied substantially in their attention to writing, doubtless due to their varied form and purpose: highly detailed and scaffolded lesson plans, generalized lesson plans, memoirs of particular class sessions, and curricular design suggestions. Our analysis identified 151 cases of social justice mathematics writing (Table 1).

Primary type of writing	**Subtype of writing**	***Rethinking Math***	***Social Injustice***	***Sust. Develop.***	***Gender Diversity***
Reflective writing	Individual reflection	23	25	2	3
	Interpret/ explain math	7	4	6	1
	Scaffold interpretation (including worksheets)	3	14		4
Group writing (not for action)		2	1		
Creative or fictional			1		
Public-facing writing	Persuasive or informative	9	2	1	1
	Promoting action	5	25		
Other		3	9		
Cases of writing		**52**	**81**	**9**	**9**

Table 1: Major types of writing in social justice mathematics lessons

Reflective writing was the most common form of social justice writing. This could involve activities such as journaling or writing briefly to externalize knowledge of a social situation; sometimes this writing was shared with others and sometimes it was only available to the student author. The reflective writing reported in these sources rarely (only once) focused solely on mathematics—otherwise, it always involved social knowledge. Two other forms of social justice mathematics writing are also common: writing that scaffolds or guides students towards expressing a social judgement, primarily within Berry III et al (2020) worksheets, and writing that promotes action–sometimes action at the level of the classroom itself, but usually at a greater distance such as the school or the community. Writing to promote action often involves implicit collaborative group writing. Rare forms of social justice mathematics writing include collaborative group writing that is not intended to promote action, and writing that is creative or fictional. Of the many types of social justice mathematics writing identified in Table 1, we focus our current discussion to highlight pressing research priorities.

DISCUSSION

Our goal in this paper is to articulate priorities for future research rather than to complete the analysis of any of these types of writing. Our discussion is framed by the conviction that writing and the teaching of writing are fundamentally social processes (Lee, 2000):

> the discourses in which we teach—in which we and our students write, speak, and represent our "selves," experiences, and our teaching—are already political, already historically and socially situated (p. 37).

The reader-focused tradition within composition studies is the most clearly committed to this social analysis, but depending on the grain-size of the writing and particular epistemological issues arising in socially-focused mathematics writing, the other traditions of text- or writer-orientation may also prove useful to future studies of social justice mathematics writing (Hyland, 2015).

Text-oriented writing research could frame a needed debate on the role of interpretive scaffolding in social justice mathematics lessons. Scaffolding from mathematical activity to social interpretation, the moment in which we say or write a sentence about what a particular kind of mathematical work means for humans, is an epistemological singularity. In this moment, discourses of mathematical factuality must interact with the contingencies of the experiences and justice orientations of all the immediately involved authors. The following worksheet question is an example of explicit scaffolding, to be completed by student groups, on the crisis of racial discrimination in police traffic stops in the United States that all too often lead to extra-judicial killings of Black people:

> In this study, ____ percent of the people stopped were Black, while ____ percent were white. In the previous activity, we learned that looking at only Black and white people in Oakland, ____ percent are Black and percent are white. From this I can conclude / wonder / observe (choose one) ____ because____ (Raygoza & Gorrin, 2020).

It was the last sentence that caused this activity to be coded as "writing" for this project. There is a brief mismatch of a group activity phrased as "I can conclude / wonder / observe", so that substantial and hidden justice negotiation needs to take place to complete this text-focused task. The prompt invites perspectives in a relatively open, non-impositional manner. We do not criticize the form of this prompt, but we call attention to the fact that the process of social interpretation is, literally, a blank, and that this blank is writ large in social justice mathematics research. As researchers, we do not know what it means to pass from the discourses of percents to the discourses of justice. We feel that scaffolding justice interpretation through cloze sentences or a sequence of brief guiding questions is likely to be a field for rich debate.

An exceedingly rare type of social justice mathematics writing is any type of writing that deviates from a factual, argumentative or analytical frame—creative writing as fiction, poetry, or brief but carefully composed messages such as the caption of a meme. Hyland categorizes writing that is expressive or that draws upon the authors' autobiographies within writer-focused research traditions, but social process approaches are also relevant (Lee, 2000). The single case identified through our coding approach asked student groups to write a scenario after learning about types of social resistance:

> Task each group with crafting a scenario (approximately one paragraph long) about a student or students engaging in one of the four types of resistance. The story can be entirely fictional, or it can be inspired by their own or other classmates' or schoolmates' stories (Raygoza, 2020, p. 77).

Raygoza provides some emotional space for students to share personal knowledge outside of a factual or autobiographical frame. Fictional and expressive writing can provide more flexible and explicit merging of social voices and perspectives than argumentative or declarative frames, allowing students more genuinely to "write just as they speak;" they may engage a wider range of student talents in social justice mathematics classes; and they may produce public-facing messages that could move audiences toward action. Staats (2014) provides additional examples of students' creative writing that engages their in-class mathematical learning.

Among the classes of writing in Table 1, writing with the goal of promoting social action is arguably the closest step towards "writing the world with mathematics." However, this vitally important aspect of social justice mathematics writing displayed several under-elaborated pedagogical dimensions. First, students seem to present the social messages of their mathematical work without presenting the mathematical reasoning itself. Explaining mathematical reasoning was required in only eight of the 30 cases of writing for action. This poses a pedagogical question: in presenting mathematically-informed calls for action, should students be prepared to explain their mathematical work or even to "teach" the mathematics to their audience?

Writing for social action is underspecified in other important ways, too. Half the time, writing for action was coded as "implicit" (14 out of 30 cases)—high stakes activities such as presenting to community or political figures seemed to require preparatory writing, but neither the prompt nor the activity of writing were described. Frequently, in 21 of these 30 cases, writing for action was conducted by groups of students rather than by individuals. Some of the persuasive or informative cases of writing involves groups of students presenting their findings to their class, but apart from these, mentions of collaborative writing groups were rare. This perhaps represents lost opportunities to practice collaborative writing before composing action steps for others. Measured against the potential learning vulnerabilities posed by social justice mathematics (Brantlinger, 2013; Esmonde, 2014; Larnell et al., 2016; Kokka, 2017), there is a significant research need to understand and to support more closely these communities of social justice mathematics writers as they negotiate written action steps. Future research on these aspects of mathematical writing for social action could draw upon reader-focused composition studies of collaborative writing groups and of writers' engagements with power and ideologies.

CONCLUSION

Torre and Fine document a young Black man's public presentation of quantiative analysis of racial inequality in school suspension data—a de facto form of racial educational segregation— to a series of school principals and other public stakeholders (2008). Some received his presentation openly, and many others resisted his analysis, diputing the validity of the data. Notably, the researchers were present and documented the entirety of this action step. The young man felt supported or legitimized by the "power of the aggregate," knowing that quantitative evidence upheld his position (2008, p. 413). We can only imagine that his classroom and research community was also a supportive aggregate throughout his action experiences. This story, reported from outside the field of mathematics education, speaks to the intense need for care, attention and support for students as they develop and deliver writing that might have the potential to "write the world with mathematics" (Gutstein, 2016) through stories written for self, others, with others and about others. Future social justice mathematics research will do well to bring the trope of writing into pedagogically-guided reality in social justice mathematics classrooms.

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LINEAR ALGEBRA PROOFS: WAYS OF UNDERSTANDING AND WAYS OF THINKING IN THE FORMAL WORLD

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Research on teaching and learning of proofs in linear algebra is scarce. To help fill this gap, we interviewed six students taking a second linear algebra course and examined some of their coursework that they handed in. In this paper, we examined students' ways of understanding and ways of thinking (Harel, 2008) in the formal world (Tall, 2008) of mathematical thinking from their statements as they unpacked a particular linear algebra proof. The results show that students were able to unpack and adjust the proof formally in a second course and reacted positively.

BACKGROUND AND THEORETICAL PERSPECTIVES

Linear algebra is an important topic for many mathematics majors. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors summarized some advances in many areas of linear algebra education (e.g., span, linear independence, eigenvectors, and eigenvalues). These studies highlight students' thought processes and difficulties while making sense of these concepts. The authors also identified areas needing more research and revealed some gaps in the literature. For example, research on how students make sense of linear algebra proofs is scarce. Research on topics in second courses of linear algebra, which contain more abstract content, is also desperately needed. The Linear Algebra Curriculum Study Group (LACSG) recommended that "at least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 45). The LACSG 2.0 recommends that mathematics departments offer a variety of second courses (e.g., numerical linear algebra) and include wider topics (Stewart et al., 2022).

Recognizing the wealth of studies in proof in mathematics education literature, in this paper, we focus our attention explicitly on linear algebra proofs (e.g., Stewart & Thomas, 2019; Britton & Henderson, 2009; Hannah, 2017; Uhlig, 2002; Malek & Movshovitz-Hadar, 2011). Stewart and Thomas (2019) aimed to uncover linear algebra students' perceptions of proofs in a first course. The results revealed that many students expressed their need for understanding. Both Hannah (2017) and Britton and Henderson (2009) agreed that the number of new definitions which linear algebra students must learn to begin writing proofs is overwhelming and makes learning proofs more difficult. Malek and Movshovitz- Hadar (2011) employed one-on-one workshops to determine the effect of using their Transparent Pseudo Proofs (TPPs) in teaching first-year linear algebra proofs. Their results showed that, for non-algorithmic proofs, students who learned using the TPPs wrote more in-depth and satisfactory answers

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 43-50). PME.

than students who learned proofs traditionally. For algorithmic proofs, both groups of students performed equally. Likewise, Uhlig (2002) developed a novel approach compared to the traditional Definition, Lemma, Proof, Theorem, Proof, Corollary (DLPTPC) to teach linear algebra proofs. His technique includes asking the following questions: "What happens if? Why does it happen? How do different cases occur? What is true here?" (p. 338). He believed after exploring these questions deeply for one specific subject, we can collect the gained knowledge in 'Theorems'. "Such a WWHWT sequence of presentation quickly leads students to understand, construct, reason through, enjoy, and actually demand 'salient point' type proofs" (p. 338).

As part of the framework of three worlds of mathematical thinking, Tall (2008) asserted that the formal world of mathematical thinking, which is based on formal definitions and proofs, "reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set theoretical definitions" (p. 7). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, proving, justifying, and problem solving, which are not necessarily unique to mathematics. Harel (2008) also defined the notion of a way of understanding as "a particular cognitive product of a mental act carried out by an individual" (p. 269), and a way of thinking as "a cognitive characteristic of a mental act" (p. 269). In Harel's (2008, p. 269) view:

> when analyzing students' mathematical behavior in terms of ways of understanding and ways of thinking, one begins with, and fixes, a mental act under consideration, looks at a class of its products (i.e., ways of understanding associated with it), and attempts to determine common cognitive properties among these ways of understanding. Any property found is a way of thinking associated with the mental act.

Harel asserts that the ability to reason abstractly, generalize, structure, visualize, and reason logically comes under the umbrella of ways of thinking. In terms of proofs, Harel (2008) claims that many students depend on the authority of the teacher or the textbook, namely the "*authoritative proof scheme*" (p. 271), others may rely on examples and visual tools, namely "*empirical proof scheme*" (p. 271). In his view, "proof schemes are ways of thinking associated with the proving act" (p. 271), and a proof is a way of understanding. Employing the above theories, the overarching research question for this project is: What are the ways of understanding and ways of thinking necessary for grasping linear algebra proofs in the formal world?

METHOD

This case study is part of a larger study on linear algebra proofs. The first named author was teaching a second course which was highly theoretical and proof-based, and selected the textbook, *Linear Algebra Done Right* by Sheldon Axler (2015) for this course. Abstract Linear Algebra course is the only second course in linear algebra offered at this mathematics department. The course is also slash-listed, meaning that graduate students can also take it since many do not have an adequate background in linear algebra and often benefit from taking this course. A second course in linear algebra usually attracts mathematics majors primarily. However, because of the increasing importance of linear algebra in business and industry, some computer science, meteorology, and physics majors (to name a few) also take this course. All students had taken a first course in linear algebra and at least one more advanced course, such as abstract algebra or analysis.

<table>
<tr><th>Theorem</th><th>Theorem Description</th><th>Proof</th></tr>
<tr><td>Pythagorean (p. 170)</td><td>Suppose u and v are orthogonal vectors in V. Then $||u+v||^2 = ||u||^2 + ||v||^2$.</td><td>Proof We have
$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2,\end{aligned}$$
as desired.</td></tr>
<tr><td>Complex Spectral
(p. 218)</td><td>Suppose **F** = C and $T \in L(V)$. Then the following are equivalent:
(a) T is normal.
(b) V has an orthonormal basis consisting of eigenvectors of T.
(c) T has a diagonal matrix with respect to some orthonormal basis of V.</td><td>Proof First suppose (c) holds, so T has a diagonal matrix with respect to some orthonormal basis of V. The matrix of T^* (with respect to the same basis) is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix. Any two diagonal matrices commute; thus T commutes with T^*, which means that T is normal. In other words, (a) holds.
Now suppose (a) holds, so T is normal. By Schur's Theorem (6.38), there is an orthonormal basis $e_1, \dots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus we can write
7.25 $$\mathcal{M}(T, (e_1, \dots, e_n)) = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$
We will show that this matrix is actually a diagonal matrix.
We see from the matrix above that
$$\|Te_1\|^2 = |a_{1,1}|^2$$
and
$$\|T^*e_1\|^2 = |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2.$$
Because T is normal, $\|Te_1\| = \|T^*e_1\|$ (see 7.20). Thus the two equations above imply that all entries in the first row of the matrix in 7.25, except possibly the first entry $a_{1,1}$, equal 0.
Now from 7.25 we see that
$$\|Te_2\|^2 = |a_{2,2}|^2$$
(because $a_{1,2} = 0$, as we showed in the paragraph above) and
$$\|T^*e_2\|^2 = |a_{2,2}|^2 + |a_{2,3}|^2 + \cdots + |a_{2,n}|^2.$$
Because T is normal, $\|Te_2\| = \|T^*e_2\|$. Thus the two equations above imply that all entries in the second row of the matrix in 7.25, except possibly the diagonal entry $a_{2,2}$, equal 0.
Continuing in this fashion, we see that all the nondiagonal entries in the matrix 7.25 equal 0. Thus (c) holds. ■</td></tr>
</table>

Table 1: The Pythagorean and Complex Spectral theorems and proofs (Axler, 2015).

The course covered the following topics: Vector spaces and their properties (including special Vector Spaces such as Isomorphic Vector Spaces and Invertibility), subspaces, span, and linear independence, bases, dimension, linear maps, polynomials, eigenvalues, eigenvectors, and invariant subspaces, inner product spaces/ operators on inner product spaces (The Spectral Theorem, Self-Adjoint, and Normal Operators, etc.), and trace and determinant. The course was taught as a mixture of lectures and tasks assigned in groups. The lecturer (first named author) engaged the students in a variety of activities, including evaluating proofs for clarity, elegance, and other criteria. On occasions, students were given pieces of a proof on paper to reassemble. Students also came to the front of the class and presented their own proofs or explained an existing one. Some homework assignments included unpacking a proof in their own words and sometimes coming up with different proofs and presenting them to the class. The interviews with six volunteers from this course took about 40-45 minutes. They were audio-recorded and later transcribed. A sample of the interview questions was: Which of the following proofs are convincing to you and why (Pythagorean theorem; Gram-Schmidt procedure; Characterization of Isometries, Complex and Real Spectral Theorems)? What is the purpose of the proofs in linear algebra? Describe the nature of the proofs in linear algebra. Is there a difference between linear algebra proofs and abstract algebra or real analysis proofs? How can we best teach linear algebra proof to enhance your learning experiences? Open coding by Strauss and Corbin (1998) was performed to analyse the data. In this paper, we analyse students' responses to the question of which of the given five proofs was most convincing and briefly show a glimpse into students' responses on the nature of linear algebra proofs. We also examined students' responses to a homework that asked them to: Study the proof of the Complex Spectral Theorem and fill out the gaps (missing steps or theorems).

RESULTS

Five out of six students mentioned that the proof of the Pythagorean Theorem (see Table 1) was the most convincing. Their common reason was that it "follows from definition", "used properties of inner products", "really simple and easy to understand", "no words only symbols", "the proof looks clean", "it's familiar. You know what the start and end are going to be". For example, Student 1 (S1) said:

> S1: I felt like it followed directly from the definition; only needed like one or two definitions to work through that proof, that's what made it more convincing to me, no words, all just equals, equals, equals, which is a pretty clear contrast from spectral theorems, where it's a lot of explaining.

Student 4 (S4) found the proof for the Gram-Schmidt theorem more convincing. Among the five proofs presented, students also made remarks about the Complex Spectral Theorem and found the proof convincing.

Normally, to prove three equivalent statements, we show (a) implies (b) implies (c) implies (a). To prove the Complex Spectral Theorem, Axler (2015) stated that the equivalence of (b) and (c) is easy and only focused on proving (c) implies (a) and (a) implies (c). To prove (c) implies (a), following the definitions, the proof naturally emerged (Table 1). However, the other direction, (a) implies (c), required more work. Student 1 (S1) did not hand in his homework. During the interview, he mentioned: "my understanding of them was not as great as it could have been… it's not that I distrusted, didn't trust the proofs, but I was that it was less like obvious or implicit, I guess".

Student 2 (S2) seemed to find the proof complete and wrote, "Are there any holes in this proof…? It looks pretty complete to me." However, during the interview, when she was shown this proof again, she showed some concerns:

> S2: I also think is a little confusing. I get lost personally where we show that the matrix is the diagonal matrix. I think that that's something I understand from experience with other, like, I understand why that's true, but not because of the way that the proof is presented here.

It seems that S2 had some conflicts with the way proof was written. We noticed that Student 3 (S3) translated everything symbolically for the proof of (c) implies to (a) (see figure 1). The theorem and its proof appeared after all the definitions and their symbolic representations were established and explained in the textbook prior to the proof. Here is the statement of the main definition: "An operator on an inner product space is called normal if it commutes its adjoint" (Axler, 2015, p. 212). Namely, T is normal if $TT^{*}=T^{*}T$.

During the interview, Student 3 (S3) mentioned:

S3: I like his complex spectral. I think the only reason why I can read through it so quickly now is just because I have it practically memorized. But I like the way he goes through it. It's very satisfying proof because it uses these two not exactly like, these two different ways of looking at this transformation or, and it provides a lot of, I guess, inside of like really why that ends up working side of, I'd almost say that this one my second favourite one of the bunch. It uses Schur's theorem, which isn't super intuitive, right off the top of your head.

Student 4 suggested some changes to (a) implies (c) part of the proof to make it clearer. She mentioned: "It would have been better to reaffirm that T^{*} is conjugate transpose matrix of T. She also wrote: "It would have been nice to see an 'updated' M(T) after…so that we can visualize that". During the interview, she said:

S4: I really like the arguments that's given in the complex spectral theorem proof, um, as to particularly why there is a diagonal matrix with respect to the Schur's, the normal basis that I, I found very clever but also relatively easy to follow. Um, and so I was, um, I guess that's the main part of the proof. The proof of a and b is it's straightforward. But then that last, the last part of the proof, I, I think I was able to follow it and, and understand how we got the diagonal matrix based on, on those basic assumptions. I thought that was pretty cool.

She added that "I'm better at remembering like the symbols that go with it because the words to me are easier to put on. So, I like ones with symbols and words". In some ways, Student 5 went ahead and performed some of the ideas that Student 4 suggested (figure 1, RHS). He unpacked part of Axler's (c) implies (a) section by displaying the matrices. He showed both M(T) and $M(T^{*})$ matrices which were similar to the work shown by S3 (LHS), who wrote: $M(T)\,M(T^{*}) = M(T^{*})\,M(T)$. Student 5 also tried to visualize the vectors by expanding them. He did not make any comments about this proof during his interview. Similarly, Student 6 (S6) wrote and included all the definitions and theorems that were mentioned but not shown in the text.

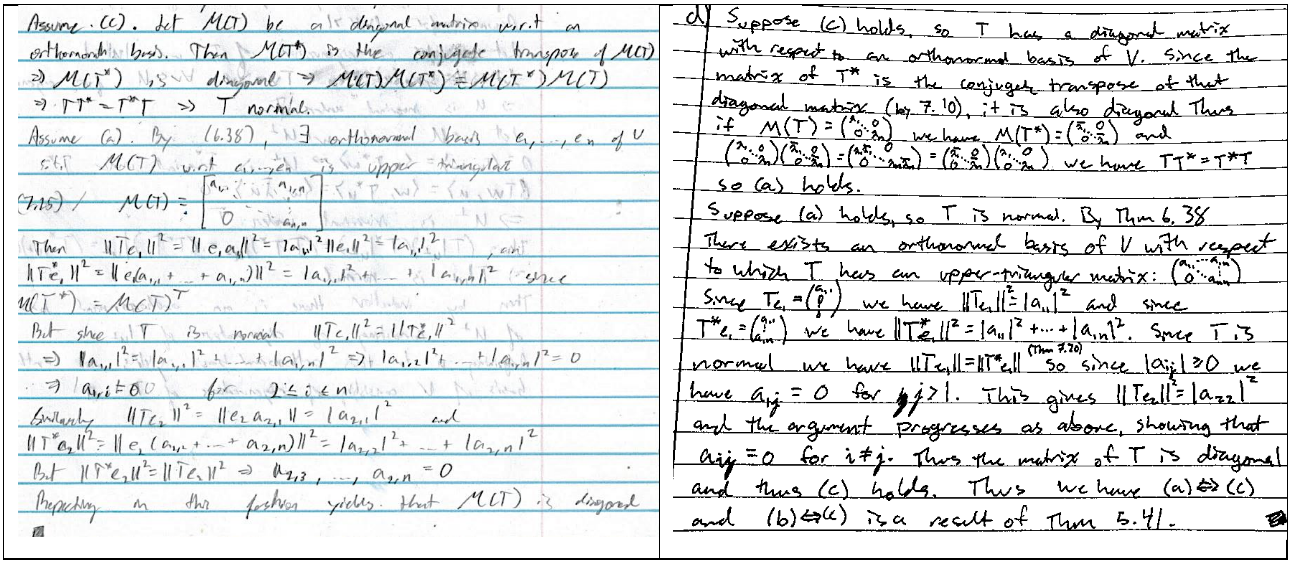

Figure 1: S3 (LHS) and S5's (RHS) works on Complex Spectral Theorem.

The analysis of the data also revealed that students believed that linear algebra proofs are different from other pure mathematics subjects such as analysis, algebra, and topology. The most common themes among their responses were: Connections to other concepts, many definitions, and theorems, unique, self-contained subjects and definitions, and conceptually difficult. For example, Student 3 noted the structure of linear algebra proofs and the ability to progressively reach the destination without worrying about small things along the way.

S3: It's like a lot of structure with a lot of linear algebra stuff. I enjoyed like being able to really not have to worry about the fact that or not having to worry about any type of convergence or doing epsilon delta proofs like you would an analysis. They'd get kind of messy, and you're just trying to almost like the little, the little thing that makes everything fall. It didn't feel the same with linear algebra. It felt much more like you're progressively getting to your destination rather than how can I find the one little key that or one little like modification to this Delta or Epsilon to make this work.

DISCUSSION AND CONCLUDING REMARKS

Results of this small case study revealed that students' ways of understanding and ways of thinking of dealing with the proof of the Complex Spectral Theorem had some common characteristics. All four students (S3, S4, S5, & S6) were content in working in the formal world and presented their arguments in the most general form. Their responses during the interviews indicated that they enjoyed the proof, and it was satisfying to learn it. There was no sign that they accepted the proof because they blindly trusted the textbook. However, both S1 and S2 appeared to place the responsibility of understanding a proof, in this case, the act of analysing a proof, on themselves. S1 mentioned that he still trusted the proofs even though his "understanding of them was not as great as it could have been". S2 also claimed that she did not find any holes in the proof, but she later said that it confused her. These conflicts seem to indicate that they simply trust the proof and they do not consider the book or teacher at fault for their confusion. According to Harel (2008, p. 286):

> …a way of understanding should not be treated by teachers as an absolute universal entity shared by all students, for it is inevitable that each individual student is likely to process an idiosyncratic way of understanding that depends on her or his experience and background.

Results also showed that the proof of the Pythagorean Theorem was most convincing for most students in this study. Their ways of thinking on this were almost identical. This was unexpected from the researchers' standpoint since we spent a considerable amount of time on the Complex Spectral Theorem in class, and in some ways, it is the climax of many fun previous results.

Although the literature reveals some insights on students' thought processes in first courses in linear algebra, studies on second courses and unravelling students' ways of understanding and ways of thinking in the formal world need more careful attention.

While employing Harel's (2008) framework in this study showed a glimpse into students' ways of understanding and ways of thinking, it is not the aim of this small study to make any concrete conclusions. From the teaching point of view, we believe that evaluating and constantly reviewing proofs in groups during the class helped the students to think critically about any written proofs. In Harel's view, "the goal of the teacher should be to promote interactions among students so that their necessarily different ways of understanding become compatible with each other and with that of the mathematical community" (p. 286). As Tall (2008, p. 15) asserts:

> …as mathematicians we begin to appreciate the purity and logic of the formal approach, but as human beings we should recognise the cognitive journey through embodiment and symbolism that enabled us to reach this viewpoint and helps us sustain it.

As we continue this new terrain of research, we plan to develop the theoretical framework further. Our future work will also include collecting more data from students and making recommendations for teaching linear algebra proof in seconds courses.

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ANALYSING TENSIONS FACED BY PRE-SERVICE MATHEMATICS TEACHERS ENGAGING IN DIGITAL FABRICATION

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This article analyses tensions pre-service teachers faced during the last workshop of participation in a series of professional development workshops on digital fabrication for creating manipulatives for mathematics education. Using the concept of community of practice and the notion of tension as a theoretical framework and a qualitative research design including video recording with 5 pre-service teachers, we identified three tension categories: technological, pedagogical, and workshop design. The results indicate that tensions are often caused by conflictual decisions between the workshop designers' and students' objectives as well as difficulties in digital fabrication and its application in teaching mathematics.

INTRODUCTION

Mathematics teachers often face tensions in the teaching profession (Nipper et al., 2011). Tensions are viewed as dichotomous forces that shape the experiences of mathematics teachers and affect both their practice and professional development (Rouleau & Liljedahl, 2017). For example, tensions may occur in the context of interaction with teachers and peers, digital resources being used, or when learning the mathematics subject matter. Tensions also appear when teachers adopt new elements in their practice, such as an innovative technology or a new approach to learning. Tensions may emerge when expectation and experience do not coincide, e.g., if a pre-service teacher (PST) expects one thing and experiences another (Nipper et al., 2011).

The aim of this study is to advance knowledge on tensions that emerge when digital fabrication (DF) is introduced in a series of teacher education workshops. DF is "the process of translating a digital design developed on a computer into a physical object" (Berry et al., 2010, p. 168). Our argument for exploring tensions is that DF and its potential impact on learning cannot be studied without providing deeper insights into the role of tensions as driving forces or obstacles in teacher professional development.

THEORETICAL FRAMEWORK

In line with the research literature (Olanoff, et al., 2021; Goos, 2004), this study uses the concept of community of practice (CoP) in a pre-service teacher education context (Wenger, 1998). CoPs are characterized by tensions, conflicts, challenges, or disagreements, or what is referred to as dualities. Tensions refer to overlapping yet conflicting activities and can be forms of participation in a shared practice that is characterized by mutual engagement and joint enterprise (Wenger, 1998). As such

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tensions are a useful analytical principle to characterize participant behaviour in a CoP (Wenger, 1998).

However, there exists little research addressing what tensions emerge in a mathematics teacher education using DF as an instrument for learning and how to manage them. Moreover, most studies on DF take an individual approach to learning based on constructionism (Stigberg, 2022), which does not consider social-cultural aspects of learning, such as a group of PSTs sharing a common learning goal directed toward DF practices, in line with Wenger's ideas of joint enterprise and mutual engagement (Wenger 1998). Given this background, our research question is: What tensions are faced by pre-service teachers engaging in DF?

To address this question, the notion of "tension" in this study is not used in its everyday sense, but in a technical and specialized sense, making categorization of tensions possible. The notion of tension is used in the literature to define "difficulties, internal or external to the community, or unstable conditions, oscillating between two different and competing states that the community should address to ensure community survival over time" (Braccini, et al., 2017, p. 151). In teacher professional development, three types of emerging tensions have been identified (Nipper et al., 2011). The first type concerns the pedagogy used by the teacher. The second concerns the mathematical content, and the third one relates to digital technologies being integrated in the teaching process. Similar types of tension could be current in PSTs when they are introduced to DF.

In terms of tension categorizations in a DF context, some indicators are thus necessary to identify them as tensions. Tensions are not the same as dialectical contradictions, which are treated as inherent aspects of an activity system (Fredriksen & Hadjerrouit, 2019; Harnseth, 2008), but we argue that tensions in a CoP can be identified using similar indicators. Accordingly, tensions are defined and expressed as forces pulling in opposite directions, imbalance of participation or divergent objectives among participants, disagreement among participants, disagreement between a participant and a pedagogical approach, or imbalance in participant-tool interactions. Moreover, tensions cannot be observed directly, they can only be identified through their visible manifestations that would qualify them as tensions in terms of linguistic expressions, behaviour, actions, or signs (Fredriksen & Hadjerrouit, 2019). In other words, disturbances, disagreements, or imbalance are the visible manifestations of underlying tensions. They become recognized when participants express them in words and actions.

METHODOLOGY

Context of the study

We (two authors, together with one additional teacher educator and one DF expert) conducted four workshops, each four hours long., The participants were five PSTs, divided into two groups (group A and Group B) in their fourth year of their master's degree. The workshops aimed to introduce DF technology for creating manipulatives for mathematics teaching.

Previous research highlights that it is challenging to introduce DF through a purely design thinking approach. Georgiev et al. (2018) report that students' design results depend largely on a skill threshold as well as their previous experiences. We decided to apply a "Use-Modify-Create" as a scaffolding approach in our study (Lee et al., 2011). During the first workshop we started with finding existing manipulatives at DF platforms and learning how to fabricate those using a 3D printer or laser cutter. During the upcoming two workshops, the PSTs learned how to modify manipulatives using 2D and 3D modelling applications to appropriate them to their teaching context. In the final workshop, the goal was to create their own manipulatives and ideate on how to apply them in teaching, using the tools and techniques they learned in the previous workshops. All four workshops included activities for exploring DF technologies to create manipulatives, reflecting about mathematical concepts reified by manipulatives, and planning classroom activities.

In this study, we focus on the fourth workshop. In this workshop, the participants were asked to reflect on didactical questions about their manipulative they had started to develop in the previous workshop and to write their reflections on our common sharing platform. After doing so, they were instructed to continue to adapt their manipulatives. Group A worked with manipulatives reifying angles, and group B worked with fractions and created a container for the parts as well. Finally, they presented their manipulative in a simulated classroom context. We have chosen to focus our analysis of tensions emerging in the last workshop because we believe that the participants have acquired the best overview of technologies and how to use manipulatives in school. In the last workshop, they also work with the most amount of freedom to create their manipulatives, and by doing so, we can learn from analyzing the emerging tensions.

Data collection and analysis methods

Our approach to data analysis draws on Stouraitis et al. (2017), and Fredriksen and Hadjerrouit (2019). We use similar indicators to analyse students' expressions, views, discussions, decisions, actions, expectations, or/and choices to reveal tensions. An indicator of a tension could be based on similar criteria described in the theoretical framework on identifying tensions, such as:

a) Disagreements between the teacher and students, which may indicate a tension between learning objectives.
b) Disagreement between a participant and a method or approach to concretize a mathematical concept.
c) Imbalance in students-tool (or digital tool) interactions, challenging them when using the tool.
d) Students' disagreement on pedagogical approach and imposed mathematical learning objectives.

We use codes to describe each identified tension which relates to its manifestation (e.g., students' difficulties/ disagreements, task), and characteristics of the tension (e.g., a tension between a student and a DF method, between peers, or between students and technology). In the process of data analysis, similar tensions are classified into tension categories (TC). Accordingly, a TC is understood as a set of concrete detected tensions similar to such a degree that they can be subsumed under the same category (Fredriksen & Hadjerrouit, 2019; Stouraitis et al., 2017).

The coding of TCs was performed in several stages using an inductive-deductive approach. Firstly, we identify a set of tensions through the analysis of the video-data from the last workshop through an inductive open coding approach. Secondly, we discussed our coding and our differences and agreed on TCs. In the third final stage, we use our theoretical framework to interpret the results achieved in the second stage through a deductive approach, which lead to the identification of the TCs.

We use two data collection instruments. The first is the video recording to identify a set of tensions. Then, after the workshops, we conducted semi-structured interviews with the PSTs to clarify parts of the video data that helped us understand the rationale of their actions. Summarizing, while the classification of TCs is the result of our theoretically informed approach based on an inductive approach, the identification of tensions is a result of our efforts to interpret the TCs based on our theoretical framework through a deductive approach.

RESULTS

From the video analysis of the recordings, we could find three TCs that appear when introducing DF to PSTs: workshop design tensions, technological tensions, and pedagogical tensions. The results are chronologically presented and divided into three main sections as they emerged in the workshop.

Workshop design tensions

At the beginning of the last workshop, the students' task was to reflect on and share their work from the previous workshop by discussing and answering questions on our sharing platform. Analysis from the video recordings reveals a tension between doing the given task and modelling the manipulative they had started with the previous workshop. Group B doubted if they should do the given task or not. An example from the discussion (our translation from Norwegian to English):

Student 1: Shall we answer those things first, or?

Student 2: Hehe, maybe we can.

Student 1: Or shall we not.

At the same time, one of us teachers came to the group, unaware of the ongoing discussion, to see how they had started their work. Since one of the group members did not participate in the previous workshop, the teachers reminded the group to inform the other about the last workshop. As a result, the group decided not to do the reflection task and moved directly to show the modelled manipulative and continued to work with it. Video recordings from Group A did not reveal any sign of considering the reflection task. One of the group members had finished the manipulative at home and was eager to print it. The other group members continued where they had ended the last workshop. They found creating manipulatives more motivating than writing their thoughts about didactical considerations. They agree on the importance of the didactical reflections but prioritized modelling the manipulative.

Technological tensions

Almost right from the beginning of the workshop, the PSTs focused on modelling the manipulative. Therefore, tensions between students and DF technology were dominant. Tensions appeared in all phases of the design and production process in particular 3D modelling and preparing for printing. Participants often struggled with finding ways to model the manipulative they had in mind in 3D and using the functionality offered by the 3D modelling software. Furthermore, they had difficulties in defining the correct level of support for printing the 3D object. Tensions were often resolved through collaboration within the team and with the instructors.

Before printing, participants further edited their manipulatives, broke them in pieces and negotiated object aspects e.g., the size of the manipulative taking printing time from the slicing program explicitly into account. They tried different sizes, qualities, and designs to reduce the printing time to manage a finished manipulative during the workshop. One PST argued that a laser cutter has greater potential for school because it is much faster than a 3D printer.

Overall, participants’ focus was not the same. As mentioned earlier, one participant had a nearly finished manipulative, so his main focus was preparing, printing, and assembling it. The other two participants in Group A were mainly concerned with 3D modelling the manipulative they had in mind. One of them finished modelling and attempted printing during the workshop, while the other did not attempt printing (probably due to lack of time).

There was significant interaction between participants who exchanged 3D modelling and printing knowledge with each other. One participant was most advanced in 3D modelling and assisted the other two extensively. There was also a significant exchange between participants in the group.

Pedagogical tensions

Lastly, the PSTs were instructed to share their work and demonstrate how the manipulative could be used in a classroom context. We classified tensions emerging between the student and pedagogical approaches as pedagogical. The PSTs struggled to connect the reified concept to the curriculum, e.g., they concluded that the angles are not explicitly mentioned in the curriculum. After approximately a 2 min, 30 seconds search for an appropriate competence goal in the curriculum, the group turns to the manipulative and does not attempt to relate to the curriculum anymore and focus on the manipulative's affordances. PSTs discovered different mathematical concepts that could be reified with their created manipulative, e.g., displaying different divisions of a straight 180 degrees angle, measuring angles, or demonstrating concepts of acute, right, and obtuse angles. Still, the PSTs struggled to agree on a concept and a way to present it to students. Group B planned to teach an activity with fractions, which did not depend on the manipulative they made. Instead, the manipulative was used as a supplement and a catalyst for cooperation and discussion. One participant mentioned that dynamic mathematical software, e.g., Geogebra, could be used instead. Group B was spending the majority of the time on creating a container for the fraction parts, arguing that a box for all components is essential in a school context. However, when doing so, time spent on the box was prioritized. Neither the mathematical concept nor planning for a demonstration on how the manipulative could be used in school was prioritized.

DISCUSSION AND CONCLUSIONS

Our research question was: What tensions are faced by pre-service teachers engaging in DF? Drawing on Wenger's community of practice and the notion of tension as a theoretical framework, the aim of this study was to make a first step in identifying and analysing the tensions that emerged when PSTs engaged in DF in a series of workshops. Even though this is a work in progress, and that it is limited to one workshop of four hours, the findings of this study help to advance current knowledge in this field both from a theoretical and practical point of view.

Firstly, the theoretical framework provided useful guidance for operationalising the notion of tension, making a categorisation of tensions possible, and using indicators to identify tensions through their visible manifestations by means of linguistic expressions, behaviour, actions, or signs in a CoP setting. The framework thus provides a foundation for identifying and analysing tensions in new ways, in contrast to constructionist approaches to DF, which focus on individual construction of knowledge. Secondly, we identified three TCs: workshop design tensions, technological tensions, and pedagogical tensions in a setting and methodology that models a series of workshops that involve 5 PSTs divided in two groups, participating in DF activities as an instrument for learning mathematics didactics, focusing on manipulatives. Those tensions emerge and are closely related to each other. We could not motivate the PSTs to do the first task where they should reflect on how to reify the manipulative in a classroom context. By not doing so, we argue that pedagogical tensions emerged from what was coded as a workshop design tension. Thirdly, the introduction of DF presents many challenges. The DF technological tools were challenging for PSTs for many reasons, which relate to the lack of knowledge on how to apply DF technologies and to the minimum training spent on DF technologies and prior knowledge in using manipulatives in a school context. Given that PSTs had already participated in three workshops, the challenges associated with correctly modelling and printing 3D manipulatives appear substantial but possible to resolve within a community of teacher educators. Good foundations in both mathematics and DF technologies and their interaction are important for applying in DF in teacher education. It is important to research how the teaching and learning environment needs to be designed to ensure a smooth integration of DF and mathematics. Based on our analysis of tensions from this study, we will design new workshops for in-service teachers, and hope to produce changes DF practices, and as a consequence, changes in learning mathematics.

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SOCIOECONOMIC STATUS AND WORD PROBLEM SOLVING IN PISA: THE ROLE OF MATHEMATICAL CONTENT AREAS

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Mathematics performance and socioeconomic status (SES) are positively related, but the reasons are not well understood. Moreover, the strength of the relationship in large-scale assessments like PISA differs between countries, for example, between Finland and Germany. In the PISA studies, mathematical word problems are used, which cover four mathematical content areas. In the present study, we reanalyzed data from PISA 2003 to 2018 to investigate whether word problems in these content areas were related differently to SES across the two countries. The results suggest that the relationship can be attributed to different content areas in both countries. This emphasizes the importance of considering item characteristics when addressing the relationship between SES and mathematical word problem solving.

INTRODUCTION

It is a common observation that students' mathematics achievement is related to their *socioeconomic status* (SES; e.g., Martins & Veiga, 2010). For example, this is apparent in the *Programme for International Student Assessment* (PISA) studies, which show that such a relationship is found across all participating OECD countries (e.g., OECD, 2013). In 2012, when mathematics was the focus of PISA most recently, the achievement gap in mathematics between the top and bottom quarter of students with regard to SES was 90 points, which is equivalent to more than two years of schooling (OECD, 2013).

Although a positive relationship between mathematics achievement and SES is found universally, its extent differs substantially between countries. For example, in 2012, Finland and Estonia were the only participating European OECD countries that combined a high performance in mathematics and low relationship between mathematics achievement and SES (OECD, 2013). On the other hand, students in Germany and Belgium also performed above the OECD average in mathematics, but in combination with a relationship between mathematics achievement and SES that was higher than the OECD average.

Several studies have investigated possible causes for the relationship between mathematics achievement and SES (see Hopfenbeck et al., 2018, for an overview in the context of PISA). Gustafsson et al. (2018) show that SES at school level is a relevant predictor of the relationship. On the classroom level, Yang Hansen and Striethold (2018) analyzed the role of opportunities to learn but found no substantial

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evidence that they were a relevant mediator between SES and mathematics achievement. On an individual level, Prediger et al. (2018) showed that language proficiency explained a substantial amount of the relationship in a high-stakes mathematics test.

These and other studies show that the relationship between mathematics achievement and SES is probably multicausal in nature and rooted on various levels, ranging from the educational system to individual learning processes. However, previous research has not yet focused on the level of the mathematics tasks. In large-scale assessment studies like PISA, the tasks that are used to assess mathematical achievement can be considered *complex word problems* (Strohmaier et al., 2021). These tasks typically embed a mathematical problem in a realistic context by enriching it with additional text and visual representations. Complex word problems include plenty of task characteristics that might cause an influence of SES. For example, the use of academic language features could provide additional challenges for students that are less exposed to this register in their everyday life (Prediger et al., 2018). The context in which a mathematical task is embedded might also provide students with specific advantages and disadvantages with regard to their social background (e.g., Carraher et al., 1985), but also the mathematical content of a task might be a relevant factor. Everyday arithmetic abilities may be considered the foundation for any further mathematical abilities and therefore teachers, students, and parents might regard them as the highest priority for students from all backgrounds. According to the PISA framework, the content area *Quantity* "may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, our world" (OECD 2019, p. 85). On the other hand, more academic mathematical areas like formal proofs, statistics, or functional thinking might be regarded to be more relevant in academic contexts and to serve a propaedeutic role in preparing for higher secondary and tertiary education, which might be of higher priority in high-SES families and schools. Importantly, these differences between content areas might differ between countries and educational systems, depending on school systems, prevailing beliefs, and values. For example, they might be reinforced by a multi-track school system like in Germany, where curricula might emphasize different contents in different school tracks (Skopek & Passaretta, 2021). To our knowledge, no previous research has yet investigated whether performance in mathematical content areas is differentially related to SES.

The present study

In the present study, we investigated whether the content areas of the mathematical word problems that were used in the PISA studies between 2003 to 2018 can offer a more differentiated view on the relationship between SES and mathematics performance and the differences between countries. To this end, we chose Finland and Germany as examples for European countries with a relationship between mathematics achievement and SES that was lower and higher than the OECD average, respectively.

In PISA, mathematics tasks are categorized into four *content areas* (also referred to as *content categories*, *content ideas* or *overarching ideas* in the PISA assessment cycles, OECD, 2019): *Quantity, Space and Shape, Change and Relationships,* and *Uncertainty and Data*. While previous research has looked for explanations for the role of SES in learning mathematics on the institutional, classroom, and student level, we add to these findings by taking into account the content area as a task characteristic.

In line with previous analyses of the PISA datasets (e.g., OECD, 2013), we first analyzed the relationship between SES and performance in the four content areas without control variables. However, because language proficiency has shown to explain a substantial part of the relationship between SES and mathematics achievement in previous studies (Prediger et al., 2018), we included reading abilities as a measure of language proficiency as a control variable in a second step. Accordingly, we posed the following research question:

For which content areas does the relationship between SES and mathematical word problem solving in Finish and German students differ from the OECD average (with and without reading abilities as control variable)?

Overall, this approach aims at contributing to the question what causes mathematical tasks to be systematically more difficult for low-SES students, and might ultimately provide ways how to tackle the issue of social disadvantages in learning mathematics.

METHODS

Sample

This study is a secondary data analysis of the datasets from the six assessment cycles of PISA 2003 to 2018 which are publicly available via the PISA website (OECD, 2021a). No additional data were collected. Details of the sampling procedure and the number of participants are available in the technical reports by the OECD (e.g., OECD, 2021b). Across all assessment cycles between 2003 and 2018, the average number of participants was M = 5248 (SD = 609) in Germany, M = 6113 (SD = 1278) in Finland, and M = 268732 (SD = 28759) across all OECD countries. All participants were 15-year-old students.

Instruments

In the PISA studies, SES was operationalized by the *PISA index of economic, social and cultural status* (ESCS). It is a composite score based on the *highest parental occupation* (HISEI), the *parental education* (PARED) and *home possessions* (HOMEPOS), including *books in the home* (OECD, 2021b).

Throughout the six studies between 2003 and 2018, a pool of about 180 mathematical word problems have been developed to assess *mathematical literacy*. These items are embedded in a functional, real-world context and cover a variety of mathematical processes within the four content areas (OECD, 2019).

Reading abilities (*reading literacy* in PISA*)* were assessed with tasks that cover a range of different texts, processes, and scenarios. Reading literacy thus reflects a functional perspective on language abilities, situated in real-world contexts (OECD, 2019).

Data and Analyses

Even though only about 70 of the mathematics items used in PISA have been made publicly available, information about their content areas is available in the technical reports (e.g., OECD, 2021b). Because of the rotated study design in PISA, each mathematics word problem was only solved by a subset of students each year. The average number of student solutions per item per year was M = 1234 (SD = 304) in Germany, M = 2713 (SD = 689) in Finland, and M = 63044 (SD = 21585) across all OECD countries. Students' individual item solutions are available in the raw data (OECD, 2021a) and were recoded according to the technical reports as *correct* or *incorrect*.

In a first step, for each item in each assessment cycle, we conducted two logistic regressions with the item solution (incorrect/correct) as dependent variable, ESCS as independent variable (Model 1 and 2), and with reading literacy as a control variable (only Model 2). ESCS was z-standardized by the OECD sample for ease of interpretation. Consistent with the methodology used in the PISA studies, a *balanced repeated replication* (BRR) procedure was followed in order to account for the nested structure of the data (OECD, 2009). This first step was done for the Finnish subsample, the German subsample, and for the sample of all students from OECD countries. It resulted in a total of 2340 separate item analyses (390 items x 2 models x 3 subsamples).

In the second step, the regression coefficients for SES from the 2340 separate logistic regressions were then synthesized in six separate random-effects meta-analyses (Lipsey & Wilson, 2000) for Model 1 and Model 2, for each of the three subsamples. *Content area* was included as a factor (*Quantity*: 103 items, *Space and Shape*: 94 items, *Change and Relationship*: 98 items, *Uncertainty and Data*: 95 items). Coefficients were weighted based on their inverse error variance. Because most items were included in more than one assessment cycle and their coefficients might be nested, a random effect for *item* was included. Furthermore, because analyses within one assessment cycle were based on the same sample of students and therefore might be correlated, another random effect for *assessment cycle* was included. To test for statistical differences between countries, a meta-regression with dummy variables for Finland and Germany was conducted for each content area.

Finally, the reported coefficients and confidence intervals were transformed to odds ratios (*OR*s) for ease of interpretation. *OR*s indicate the multiplicative change in the solution odds (correct solution probability/incorrect solution probability) per unit change of the predictor variable. For example, an odds ratio of 1.50 means that participants who had a z-standardized ESCS of 1 (one standard deviation above OECD average) had 1.5-times or 50% higher odds of solving this task correctly than a participant with average ESCS.

We chose this two-step approach over using a comprehensive statistical model because of the complex data structure. Combining the datasets into one single logistic regression model would cause several methodological issues, including the multiple levels of nesting, missing datapoints, and weighting procedures. Meta-analysis is considered an appropriate approach to synthesize regression coefficients, provided that the regression models are sufficiently comparable (Lipsey & Wilson, 2000). The *R* packages *intsvy* (Caro & Biecek, 2017) and *metafor* (Viechtbauer, 2010) were used for analyses.

RESULTS

Results are given in Table 1. Model 1 gives the *OR*s for z-standardized ESCS on the solution probability of mathematical word problems in the four content areas. It shows that across OECD countries, students with a one standard deviation higher ESCS had 57% to 72% higher odds of correctly solving each word problem than a student with average ESCS. For Finish students, the *OR*s were significantly lower than for the OECD sample (but still significantly higher than 1) for all content areas except for *Space and Shape*, where there was no significant difference. For German students, the *OR*s were significantly higher than for the OECD sample for all content areas except *Quantity*, for which it did not differ significantly.

	Quantity		Space and Shape		Change and Relationships		Uncertainty and Data	
	OR	*CI*	*OR*	*CI*	*OR*	*CI*	*OR*	*CI*
Model 1								
OECD	1.57	1.49 - 1.65	1.60	1.52 - 1.67	1.72	1.64 - 1.80	1.59	1.50 - 1.69
Finland	**1.50**	1.15 - 1.56	1.57	1.46 - 1.69	**1.60**	1.50 - 1.71	**1.53**	1.44 - 1.61
Germany	1.60	1.51 - 1.69	**1.69**	1.58 - 1.80	**1.82**	1.70 - 1.94	**1.71**	1.61 - 1.83
Model 2								
OECD	1.14	1.10 - 1.19	1.13	1.09 - 1.18	1.19	1.15 - 1.23	1.16	1.11 - 1.21
Finland	1.15	1.11 - 1.20	**1.19**	1.14 - 1.24	1.18	1.14 - 1.23	1.18	1.13 - 1.23
Germany	**1.11**	1.09 - 1.14	**1.19**	1.14 - 1.23	1.21	1.16 - 1.27	**1.21**	1.17 - 1.26

Table 1: Odds ratios and 95% confidence intervals of z-standardized ESCS on mathematical word problem solution probability by content area. Model 2 controls for reading abilities. *OR*s printed in **bold** differ significantly ($p < .01$) from the *OR*s of the OECD sample.

In Model 2, reading abilities were included as a control variable. This substantially decreased the *OR*s for ESCS in all three samples, but still resulted in *OR*s that were significantly higher than 1, meaning that a higher ESCS was still associated with higher solution odds. Across the OECD, students with a one standard deviation higher ESCS had 13% to 19% higher odds of solving each word problem correctly compared to students with an average ESCS. For Finish students, there were no significant differences for three of the four content areas, but a higher *OR* for *Space and Shape* compared to the OECD. For German students, the *OR* was significantly lower compared to the OECD sample for *Quantity*, greater for *Space and Shape* as well as for *Uncertainty and Data*, and not significantly different for *Change and Relationships.*

DISCUSSION

Considering content areas of mathematical word problems that were used in PISA revealed a new perspective on the relationship between mathematics achievement and SES. With regard to our research question, we found that the relationship between SES and performance in the content areas differed between Finland and Germany. Without controlling for reading abilities, Finish students' mathematics performance was less influenced by SES in all content areas, although this difference was not significant for the content area *Space and Shape*. This is consistent with coarser analyses of the same data that show the overall relationship between SES and mathematics achievement is weaker in Finland compared to the OECD average (OECD, 2013). However, when controlling for reading abilities, Finland was no longer below the OECD average, and *Space and Shape* was now significantly stronger related to SES than across the OECD. Thus, it seems that while the role of SES was smaller in Finland in general, it was not explained by language proficiency to the same extent as in other OECD countries.

For Germany, the relationship between mathematics achievement and SES was higher than the OECD average before and after controlling for reading abilities for the content areas *Space and Shape* and *Uncertainty and Data.* In contrast, the relationship between SES and *Change and Relationships* was significantly different from the OECD average before, but not after controlling for reading abilities. *Quantity* even had a significantly lower relationship to SES than the OECD average when controlling for reading abilities. Across all content areas, this shows that language proficiency plays an important role in explaining why the relationship between mathematics achievement and SES is higher in Germany than across the OECD. In fact, when this role of language is taken into account, the particularly high relationship between SES and mathematics in Germany was limited to the content areas *Space and Shape* and *Uncertainty and Data.*

This seems to be an interesting starting point for investigating the processes through which the association between SES and mathematics emerges. For example, future analyses could investigate whether the different school tracks in Germany, which are associated with differences in SES (Skopek & Passaretta, 2021), offer different opportunities to learn mathematical content in the areas of *Space and Shap*e and

Uncertainty and Data, or whether the contents of these word problems offer any advantage or disadvantage for particular groups of students with regard to their SES.

The fact that the content areas of mathematical word problems were differently associated with SES in Finland and Germany raises the question where such country differences are rooted. A possible topic for future research might be the subjective importance of mathematical content areas, and whether content that is believed to be important for everyday life might be considered more relevant for all students than more theoretical, academic concepts. Similarly, including interest as a possible mediator between SES and mathematics might help to understand their relationship.

Overall, our analyses further reiterate the role that reading abilities play when investigating SES in the context of mathematics achievement (see also Prediger et al., 2018). While SES remained a positive predictor of word problem performance, controlling for reading abilities decreased the influence on SES substantially across content areas in the OECD, from 57% to 72% higher odds of a correct solution per standard deviation in SES to 13% to 19% higher odds after controlling for reading abilities.

Reanalyzing existing datasets from studies like PISA offers the benefit of a large amount of available data, but naturally, comes with limitations. The majority of items has not yet been fully published, and only their content area was available as a task characteristic. In order to investigate in detail which task characteristics play a role for the relationship with SES, studies are needed that specifically address and manipulate specific facets of mathematical word problems. At the same time, the interaction with student characteristics like interests and beliefs might provide additional insights into the role of SES in learning mathematics. Judging from the present study, a detailed look on role of the mathematical content might be an informative starting point, while taking interactions with language proficiency into account.

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STORYTELLING AS A PEDAGOGICAL STRATEGY FOR CULTURALLY RESPONSIVE MATHEMATICS TEACHING

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This paper reports an intervention study with 60 Primary three students in Hong Kong mathematics classrooms in which the language of instruction (English) was not the first language for neither the teachers nor their students. The purpose of this study is to investigate the effectiveness of storytelling to develop fraction language and concepts based on the theory of culturally responsive teaching. The pre-test scores of the intervention group was significantly lower than the control group, yet post-test scores showed that both groups performed comparably well. These initial findings suggested that integrating storytelling with a culturally responsive teaching approach can reduce the achievement gap between culturally and linguistically diverse students in rural settings and their peers in urban districts.

INTRODUCTION

The number of ethnic minority (hereafter, "EM") students across all levels of public schooling in Hong Kong has increased by 10% between 2015 and 2020 (Legislative Council of Hong Kong Special Administrative Region, 2020). As such, mathematics classrooms are becoming more culturally diverse in a society which has remained hitherto culturally homogenous, and mathematics pedagogy has become more complex. In particular, EM students in Hong Kong have encountered many difficulties when learning mathematics (Tse & Hui, 2012). After all, for example, they value different aspects of mathematics learning compared to their ethnic Chinese peers (Sum et al., submitted). For Hong Kong, this is an emerging issue not just for the EM students but also for their ethnic Chinese peers in their classrooms. The quality of mathematics learning notwithstanding, a student's performance in mathematics assessments is also a function of this learning in class, and such performance has important implications on a student's future in Hong Kong, such as for entry to universities.

Fractions is one of the most difficult topics in school mathematics (Lortie-Forguesa et al., 2015). For instance, the part-whole concept is conceptualised differently in different languages, with the Asian analytic conceptualisation (thinking from whole to part) being different from Western synthetic conceptualisation (thinking from part to whole) (Leung, 2016). The differences concern not only the reading order, but also the articulation corresponding to the analytical way of thinking from whole to part, in which the part-whole relation is an integral part of the linguistic term. Constructing

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meanings of fraction concepts requires the use of concise language as a thinking tool for expressing the abstract relation. In this paper, we report the preliminary results of an ongoing study investigating the effectiveness of employing storytelling as a pedagogical strategy to support students' development of fraction concepts.

LITERATURE REVIEW & THEORETICAL BACKGROUND

Storytelling

Storytelling can build vocabulary, conceptual/content knowledge and improve comprehensions of English-language learners (Hickman et al., 2004) while fostering cultural awareness in multilingual educational settings (Hernández-Castillo & Pujol-Valls, 2018). Storytelling improves students' mathematics achievement (Orr, 1997), and promotes high level thinking and problem solving skills (Hong, 1996). Storytelling as a pedagogical strategy in mathematics can be adopted in several ways, such as providing a meaningful context that is experientially appropriate for students, preparing and developing concept and skill, and posing problems to foster thinking and reasoning (Welchman-Tischler, 1992). It remains unknown, however, if storytelling will be useful in multilingual settings involving Asian analytic and Western synthetic conceptualisations.

Culturally responsive teaching

Culturally responsive teaching (CRT) attempts to boost the academic achievement of students from diverse cultural and linguistic backgrounds by "using the cultural knowledge, prior experiences, frames of references and performances styles of ethnically diverse students to making learning encounters more relevant and effective for them" (Gay, 2018, p. 36). In the context of mathematics education, "all students can be successful in mathematics when their understanding of it is linked to meaningful cultural referents, and when the instruction assumes that all students are capable of mastering the subject matter" (Ladson-Billings, 1995, p. 141). CRT aims to make mathematics learning more relevant to students' cultural backgrounds and experiences, including their respective frames of references. Our study is largely guided (theoretically and practically) by the work of Gay (2018) and others who have developed the ideas of CRT (e.g., Aguirre & Zavala, 2013). The following research questions have thus guided our study:

RQ1: Does storytelling intervention support students to develop language-related conceptualisations of fraction in ways which lead to improvement in achievement?

RQ2: How does such intervention develop students' fraction language and concept in a classroom oriented towards culturally responsive teaching?

Due to space constraints in this paper, the results of the qualitative analysis will not be reported here. As such, only RQ1 will be responded to in this paper.

METHOD

Participants

The research design comprised a pre-test, an intervention, and a post-test. A total of 186 Primary three students from three schools in Hong Kong participated in this study, of whom 60 were involved in the intervention. The language of instruction (English) was not the first language for both the teachers and their students. The intervention was conducted in two classrooms at a village school in Hong Kong, located in a catchment area for EM communities with lower socioeconomic status. Among the 60 students, there were 28 ethnic Chinese (n=23 were born in Hong Kong, n=4 were born in Mainland China, n=1 was born in Korea) and 32 EM students (n=19 were born in Hong Kong, and of which 9 were second generation, whom was born in HK with at least one of the parents also born in HK). The ethnicities of the EM students include Black, Filipino, Korean, Nepalese, and Pakistani. The students in the intervention group speak different languages at home, namely, Cantonese (n=29), English (n=8), Nepali (n=17), Urdu (n=6).

Storytelling intervention and data collection

Two teachers worked collaboratively with the research team, whereby the designed reading activities and mathematical tasks were enacted with students. As such, data consisted of classroom observations, teacher-developed artefacts, teacher interviews, field notes, students' products and other artefacts. Two picture books were selected: *Breakfast Around the World* written by Ye-shil Kim, and *Charlie Piechart and the Case of the Missing Pizza Slice* by Marilyn Sadler. Both stories were set on everyday context situations with rich mathematics vocabulary that provokes students to consider partitioning, fair share and iterating. Our focus was on developing students' understanding and use of fraction vocabulary (e.g., half, quarter, one-fourth, equal share, numerator, denominator), and their associated fractional representations and symbols. The intervention took place in two classrooms each encompassing six 20-minute class sessions. All lessons were video-recorded and transcribed. As each story was read aloud, teachers stopped at pre-planned points and posed questions for discussion, which was built on the theory of CRT. This provided opportunities for conversational interaction, which in turn developed students' mathematical language and content knowledge. Teachers also highlighted how fraction language was used in the story and explained their meaning using illustrations. Students were also asked to repeat the fraction words in English to provide a phonological representation of the words, and to respond to questions in other contexts to make the words a part of their working vocabulary.

Data analysis

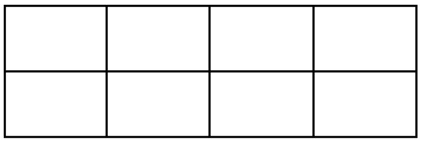
We assessed the students' mathematical performance prior to the intervention in early March 2021, and after the intervention two months later. Both the 13-item pre- and post-tests had the same content, although the context or presentation may be different (see Figure 1), and the numbers used in 7 items were modified (see Figure 2). Items include reading and writing fractions representing a part of one whole and one group/set, dividing the whole into a number of equal parts, specifying and/or drawing fractions in part-whole and part-group models, recognising the concept of equivalent

Pre-test: I ate $\frac{3}{8}$ of a pizza. My sister ate $\frac{5}{8}$. Who ate more?

Post-test: I ate $\frac{5}{8}$ of a pizza. My sister ate $\frac{3}{8}$. Who ate more?

Figure 1: Changing the context of the same question for the pre- and post-tests.

Pre-test: Shade $\frac{3}{8}$ of the whole figure.



Post-test: Shade $\frac{3}{12}$ of the whole figure.

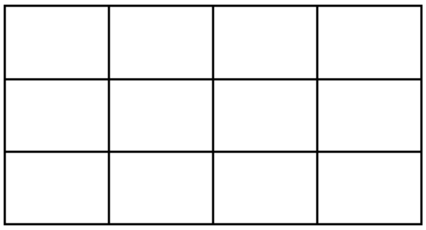


Figure 2: Modifying the numerals used in pre-test and post-test items.

fractions, comparing fractions with the same denominator, and solving problems represented in diagrams. Students were told that they could provide their answers in any language. With Cronbach's alphas at .823 and .803 in the pre- and post-tests respectively, the tests showed a satisfactory internal consistency.

A series of paired-samples *t*-tests were conducted to compare the pre- and post-test scores for intervention and control groups at a significance level of .05. The effect size (ES) of the difference in group means were measured by Cohen's *d* for *t* tests to show the extent to which students improved their achievement in the fraction test across time. In general, $d < 0.2$ counts as small effect size, and $d > 0.8$ counts as large effect size. A 2×2×2 repeated measures ANOVA was conducted to investigate the effectiveness of intervention across time between Chinese and EM students, with Time (pre-test, post-test) as the within-subject independent variable and Group (Intervention, Control) and Ethnicity (Chinese, EM) as the between-subject independent variables, and response accuracy rate as dependent variable.

FINDINGS AND DISCUSSION

As shown in Table 1, results of the paired-samples *t*-tests revealed a significant difference in response accuracy rate for all students between pre- and post-tests ($ps < .001$). In addition, the effect sizes (larger than 1) indicate students' huge progression in the fraction test from pre-test to post-test.

	Pre-test		Post-test			
School	Mean	SD	Mean	SD	*t*	Cohen's *d*
Total (n=186)	35.65%	24.08%	68.28%	21.18%	17.50***	1.28
Intervention (n=60)	25.38%	17.76%	68.46%	21.97%	16.33***	2.11
Chinese (n=28)	29.95%	20.33%	74.73%	21.94%	11.55***	2.18
EM (n=32)	21.39%	14.32%	62.98%	20.81%	11.44***	2.02
Control (n=126)	40.54%	25.19%	68.19%	20.88%	11.88***	1.06
Chinese (n=67)	44.55%	27.56%	71.76%	21.27%	8.08***	.99
EM (n=59)	35.98%	21.54%	64.15%	19.82%	8.79***	1.15

Table 1: Comparisons between Pre-test and Post-test scores.

Since all students, with or without intervention received, demonstrated significant improvement, we further investigated the effectiveness of intervention in relation to time and ethnicity in a repeated measures ANOVA. The analysis showed significant main effect for Time, $F(1, 182) = 336.90$, $p < .001$, $\eta_p^2 = .65$, suggesting that all students had a higher response accuracy rate in the post-test (M = 68.28%, SD = 21.18%) than in the pre-test (M = 35.65%, SD = 24.08%). Significant main effects were also found for Group, $F(1, 182) = 5.82$, $p = .017$, $\eta_p^2 = .03$, in which students in the Control group (M = 54.37%, SD = 19.09%) generally had a higher response accuracy rate than those in the Intervention group (M = 46.92%, SD = 17.17%), and for Ethnicity, $F(1, 182) = 10.31$, $p = .002$, $\eta_p^2 = .05$, in which the overall response accuracy rate of ethnic Chinese students (M = 56.44%, SD = 19.93%) was higher than that of

EM students (M = 47.30%, SD = 16.32%). The significant main effect for Group where students in the Control group scored higher than those in the Intervention group might be counterintuitive because we expected that students who received intervention would perform better in the fraction test than those who were taught in traditional classrooms. It is possible that the overall higher response accuracy rate of students in the Control group is attributed to the obvious discrepancy between both groups' prior knowledge and experiences as demonstrated in the pre-test. It is also plausible that the ratios of ethnic Chinese to EM students in both groups were different, i.e., more ethnic Chinese in the Control group while more EM students in the Intervention group, thereby the achievement gap between ethnic Chinese and EM students masked the intervention effect.

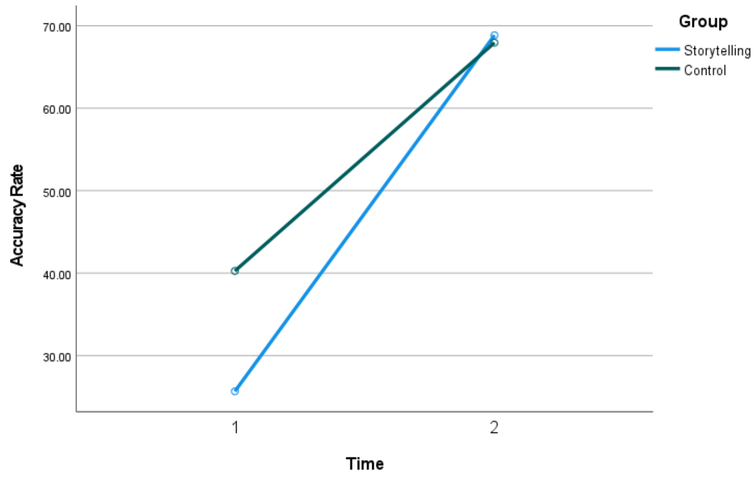


Figure 3: Comparison of students' response accuracy rate between the Intervention (storytelling) and Control Groups at pre-test and post-test.

Given these possibilities, the interaction effects of Time × Group, Time × Ethnicity, and Group × Ethnicity were examined. Among these, only the interaction effect of Time × Group was statistically significant, $F(1, 182) = 16.11$, $p < .001$, $\eta_p^2 = .08$. The response accuracy rate of students in the Intervention group at pre-test (M = 25.38%, SD = 17.76%) was significantly lower than that of students in the Control group at pre-test (M = 40.54%, SD = 25.19%), $p < .001$, while the response accuracy rates of both groups at post-test were comparable (Intervention: M = 68.46%, SD = 21.97%; Control: M = 68.19%, SD = 20.88%), $p = ns$. This result indicates that even though students in the Intervention group might have less prior knowledge and background in comparison to those students in the Control group, they had caught up to similar knowledge levels as students in the Control group, reducing the achievement gap after experiencing the intervention (Figure 3). In addition, the non-significant interaction effects of Time × Ethnicity, $F(1, 182) = .08$, $p = .772$, $\eta_p^2 < .001$, and Group × Ethnicity, $F(1, 182) = .13$, $p = .717$, $\eta_p^2 = .001$, both suggested that Ethnicity did not moderate the associations between Time and accuracy rate and between Group and accuracy rate. In other words, ethnic Chinese students performed better than EM students in the fraction tests across time and group.

The intervention has several prominent aspects: CRT embraces diversity and allows students to connect mathematics with relevant/authentic situations in their respective lives, making learning meaningful and memorable. Storytelling provides language support for English language learners. While textbook language consists of academic language, usually with fragmented single word utterances, the language used in the stories is presented in relevant context of conversational language. Fraction language and concepts are developed through extensive listening and language scaffold (such as gestures, revoicing, and facial/oral expressions), with additional support including illustrations and images in the picture books. The multimodality of picture books provides better access for students to engage more fully in the mathematical meanings they encountered, and serves as resources in supporting students' understanding, and fills the gaps of missing prior knowledge and experiences. In our study, teachers posed contextual problems based on the stories to engage students in authentic communication, and created opportunities for them to use fraction words and phrases to communicate their reasoning. Teachers also used problem situations to make explicit connections with students' cultural frames of references, making teaching more culturally responsive.

CONCLUSIONS

Culturally and linguistically diverse students in Hong Kong rural areas with lower socioeconomic status face many barriers to realising their (mathematics) learning potential. Our initial findings here suggest that integrating storytelling with culturally responsive teaching could be a useful intervention, helping the EM students to at least not fall further behind in their mathematics learning. Further qualitative analysis of data collected from artefacts, observations and interviews is currently being conducted to confirm, interpret further, and extend these initial findings.

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UNPACKING OF MATHEMATICAL KNOWLEDGE IN LECTURING: THE MATHEMATICAL PRACTICE OF DECOMPRESSING

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This paper examines university mathematics lecturing using the theoretical frame of unpacking of mathematical knowledge. Unpacking is defined as the micro-level decompressing of mathematical knowledge in the practice of lecturing. Data from lecture observations and interviews of four research mathematicians are analysed. Results suggest that unpacking of mathematical knowledge is done for different mathematical and pedagogical purposes in lectures.

INTRODUCTION

Ball and Bass (2000, p. 98) explains the notion of decompression of teacher's mathematical knowledge as "*decompression is the ability to deconstruct one's own mathematical knowledge into less polished and final form where the elemental components are accessible and visible.*" They draw our attention to the highly compressed nature of mathematical knowledge (Ball et al., 2008) and its microscopic elemental components. According to Ball and Bass (2000), mathematics is a discipline in which compression of knowledge is central. The challenge for teachers or lecturers is to decompress their mathematical knowledge so it is appropriate for the specific lecturing and learning context. Decompressing of mathematical knowledge assists in understanding practice through closely examining the context and using it to better student mathematical thinking (McCrory et al., 2012). This paper examines university mathematicians' mathematical knowledge in lecturing using the theoretical lens of decompressing (McCrory et al., 2012; Wasserman, 2015). Decompressing is '*unpacking a* topic's *mathematical complexity to make it more comprehensible*' (Wasserman, 2015, p. 77). The main research question addressed here is: what are the specific reasons for which university mathematicians decompress and unpack their mathematical knowledge in the practice of lecturing?

THEORETICAL PERSPECTIVES

Mathematical knowledge – situated perspective

This research takes the situated perspective and argues that mathematical knowledge in lecturing is situated in the context of lecturing. Teaching context has played an important role in our understanding of the different aspects of mathematical knowledge in lecturing (Elbaz, 1983; Petrou & Goulding, 2011). This is important because it is mostly in teaching contexts where a lecturer's

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mathematical knowledge unfolds and finds expression. According to Putnam and Borko (2000, p. 6): "*The classroom is a powerful environment for shaping and constraining how practising teachers think and act.*" Hence a situated perspective is useful in understanding the nuances of mathematical knowledge in lecturing. The key practices of teaching framework developed by McCrory et al. (2012) is used to explain the decompressing of mathematical knowledge in lecturing. Decompression comes into play in addressing student misunderstandings and questions, in designing lessons and in the choice and design of learning tasks.

Wasserman (2015), building on the work of McCrory et al. (2012) gave a more detailed analysis of teachers' use of mathematical knowledge in teaching. The notion of decompressing was explained by focusing on the mathematical complexity of concepts using the micro and macro levels of complexity. The mathematical complexity within micro-level mathematics necessitates explicit unpacking to make ideas more accessible for students. Wasserman (2015) considers 'unpacking' synonymous with 'micro-level decompressing' which refers to the unpacking of mathematical complexity at the local neighbourhood of the particular mathematical concept.

Unpacking as a micro-level decompressing in the practice of lecturing

Mathematics is considered a discipline of complexities inherent in its abstract nature (Wasserman, 2015). The practice of lecturing needs explicit decompressing of this mathematical complexity. Hodgen (2011, p. 34) views the power of mathematics in its packed nature as "*one aspect of the power of mathematics lies in this "packed" and abbreviated nature ... The essence of teacher knowledge involves an explicit recognition of this – "unpacking" the mathematical ideas ...*". Ball and Bass (2000) maintain that unpacking is a critical dimension in the practice of teaching as it involves working backward from the compressed mathematical understanding to unpack its constituent elements that in a way assist student understanding of mathematical concepts. They identify unpacking as the '*hallmark of expert knowledge*' and explain (p. 98): *"The teaching of mathematics entails work with the microscopic elements of mathematical knowledge, elements that were invisible..."* This paper considers university research mathematicians' unpacking of mathematical knowledge in lecturing.

METHODOLOGY

The data collected from four research mathematicians (RM1, RM2, RM3 and RM4) in a university mathematics department are presented. The research mathematicians had more than 15 years of teaching experience and three of them had Post-doctoral qualifications in mathematics. None of them had any formal teaching qualifications. Research mathematicians were observed in class and were followed up with post-observation interviews. The classes observed were in undergraduate mathematics including calculus, linear algebra, statistics and applied mathematics.

The lectures were video-recorded and interviews were audio-recorded. The interviews lasted for 30 minutes. Semi-structured post-observation interviews focused on the details of mathematical knowledge observed in the lectures. The interview question was; 'I have observed this moment in your lecture; can you recall and tell me why it was done that way?' The observational data and interviews were transcribed. Using inductive thematic analysis (Braun & Clark, 2006), data were categorised into themes and were analysed corresponding to the theme of unpacking of mathematical knowledge. It is presented in episodes of mathematicians' verbatim quotes from lectures and post-observation interviews.

RESULTS AND DISCUSSION

Unpacking is the pedagogical practice of intentionally highlighting mathematical complexity in the local neighbourhood of the mathematical content. Unpacking reveals how mathematicians' mathematical knowledge unfolds in lectures. The ability to unpack compressed knowledge leads to a better student understanding of mathematical concepts (Wasserman, 2015). Unpacking mathematical knowledge is done with different purposes in the lectures.

Addressing student confusion

In the following lecture observation and post-observation interview, RM1 addressed student confusion related to Laplace and Fourier transforms.

> So just keep in mind that Laplace transforms of derivatives are with respect to x where x is a parameter, just the derivative of F with respect to x of the Laplace transform. The same thing showed up with Fourier transforms, but it was the opposite thing, the Laplace transform of a time derivative was the time derivative of the Laplace transform. And derivatives of Laplace transforms don't show up, so without any confusion, we can just write this is an ordinary derivative because we don't have any derivatives with respect to s. [RM 1, Lecture 1, Stage 3 course]

In the post-observation interview, RM1 said,

> One thing I have noticed and other people who have taught the course mentioned it as well: We had a discussion about *Laplace transforms* and *Fourier transforms* and it seems that one of the problems students have is which transform to apply to various problems, so I mentioned and I wanted to stress again in today's lecture, I followed up on that ... came up with my own observations and discussions with a colleague. [RM1, Post-observation interview 1]

RM1 unpacked mathematical knowledge to address student confusion by comparing the similarities between Laplace and Fourier transforms. RM1 made use of his knowledge of content and students (Ball et al., 2008) in addressing student misconceptions. He mentions that his own reflections and discussions with colleagues assisted in identifying and addressing student misconceptions.

Introducing complexity

The context of this example is a discussion about how very few rational numbers there really are on the number line. The discussion point was: by rolling a dice, if we choose a number between 0 and 1 at random, what is the chance of it's being a rational number? The lecture theatre was set up as a debating place and most of the students were engaged in thinking and coming up with arguments.

RM2: What is the chance of getting a rational number by making a random selection between 0 and 1?

S1: Zero

RM2: Why? Why is our chance of getting a rational number zero?

S1: Either you will have to roll the dice for an infinite number of zeroes.

RM2: That is half the answer. Half the answer is either you will have to roll the dice for an infinite number of zeroes eventually which is never going to happen, or what is the other option? Not zeroes yet.

S1: You have to keep on the same pattern.

S2: And actually the same repeating.

RM2: Absolutely (gives two chocolate fishes). That's right, either you will always have to keep on rolling [the dice] zero, zero, zero, zero which is never going to happen or you will have to keep on rolling 3, 3, 3, 3, 3, 3 or you will have to keep on rolling 1, 7, 1, 7, 1, 7, 1, 7 for ever. So the chance that you will end up with a rational number is what?

S1: Is zero.

RM2: That means if you take a random number between 0 and 1 and if you choose randomly, you will never ever pick a rational number. But we have just shown that there is an infinite number of rational numbers and an infinite number of irrational numbers. But now you have shown you can never ever pick up a rational number [randomly]. This means that you might have an infinite number of them, but actually, you almost have none of them at all. For that you will all agree is a kind of weird; everybody agrees that it is a kind of weird? That means that you have different sizes of infinity. And some infinities are really really small infinities and other infinities are really really big infinities. So just saying it's infinite doesn't mean anything because it could be really big or really small. Does everybody agree with this? [RM2, Lecture 1, Stage 1 course]

RM2 introduced complexity and asked '*isn't this weird*' and unpacked complexity. He encouraged student interaction and created an ongoing debate during the lecture. To make the elemental properties visible (Ball & Bass, 2000) and to create a 'brainstorming' in the classroom environment, RM2 used mathematical as well as pedagogical knowledge. In unpacking, he points to the complexity using the dice example and scaffolds by experimenting. He complicates the situation by introducing the concept of infinity and the different sizes of infinity, creating cognitive dissonance, engaging students in deep thinking mode. The aim is to show students how complex the concepts are compared with how they first appear (Wasserman, 2015).

Planting a concern

The context for the example was proving the theorem: Let X be a discrete random variable with PGF $G_x(s)$. Then: $E(X) = G_x'(1)$.

> The gradient of G_x at s=1, that is, in fact the mean of the distribution. So for the Poisson distribution with parameter λ= 4, can you tell me what the gradient will be? What is the Mean of the Poisson 4 distribution is the parameter, λ, in this case it is 4. At s=1, E(X) = 4 for $X \sim P_o(4)$. We have to prove $E(X) = G_x'(1)$. Notice that this is a sketch proof because we are doing something that we need to be a little bit aware of whether we ought to be allowed to do them. Or at least we ought to have a little uncertainty in our minds if it is allowed or not. It is allowed, but there is actually quite a lot of deep theory behind why it is allowed. Let me show you what it is. So we have got $G_x(s) = \sum_{x=0}^{\infty} S^x P_x$ by definition. Differentiate: $G_x'(s) = \sum_{x=0}^{\infty} xS^{x-1} P_x$. This is a bit where we should be a little cautious about. Is it okay for me to sum and differentiate each term one at a time? Let's have a look at what I get if I do it. P_x is just a constant. What is a constant? Something, which does not depend on s. s, is the argument of the function. When we differentiate it, we are differentiating with respect to s. So anything that does not depend on s is just a constant. So why am I little bit uncertain about whether I'm allowed to do that or not? The answer is it is a subtle point. But the point is, actually we have got an infinite sum [which] is actually an infinite limit. And when we take the derivative, we are taking actually an infinitesimal limit and is it oaky just kind of swap those two things around? One is going off to infinity and the other is going off to zero. Are we sure that are we allowed to do that? What happens if one thing goes off to infinity faster than the other thing goes off to zero or is there any concern about that? The answer is … we are always allowed to do that for a probability generating function. So all I want to do is to plant in your minds the concern that you might not always be able to do that. When you have got an infinite sum, you might not always be able to switch it around with the derivative and differentiate term by term. That is why we call this just a sketch proof. But in this context, we are allowed to do it and you are always allowed to do this with power series. That's why the power series are special. Just want to make that point [RM3, Lecture 1, stage 3 statistics].

RM3 introduced mathematical complexity and attempted to *plant a mathematical concern in students' minds*.

> It was basically a teaching strategy...I just pointed out that concern. I think the students will learn more from pointing out the concern than they would from me going through all the details because the details just too swamps, they turn off, they stop and they have forgotten even why we were doing this whereas if you just point out, then that's more likely to stick [pointing to student comprehension] [RM3, Post-observation interview].

To unpack a mathematical concept, the mathematician may use different pedagogical strategies and might try to create pedagogical opportunity to have student involvement. RM3 opened up a pedagogical opportunity (Leathamv et al., 2015) to talk about a mathematically complicated idea and an explanation of why they can do something in this case but may not in others. This pedagogical opportunity is used to draw students' attention to the complexity.

Mathematical symbols and notations

RM4 unpacked a potential student confusion which he had earlier pointed out in the case of a set of vectors.

Let $X \subseteq \mathbb{R}^n$. Then

$$X^{\perp} := \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{x} = 0 \text{ for all } \boldsymbol{x} \in X\}.$$

We pronounce this "X perp". If X is a subspace of $\mathbb{R}^n$ then we call $X^{\perp}$ the **orthogonal complement** to X.
If $\boldsymbol{a} \in \mathbb{R}^n$ then we write $\boldsymbol{a}^{\perp}$ instead of $\{\boldsymbol{a}\}^{\perp}$.

> You have to be careful; this notation is potentially confusing because when we think about perp [see figure 3], the perp is really an operation that we apply to sets, not to single vectors, okay. So you can get confused between sets and the vectors and this is a confusion that I have talked about for basis as a set of vectors. A set of vectors is not the same thing as the single vector. Even the set with one vector in it is not the same as the vector without the perp. But this is just a notation that we use because it is convenient, it is potentially confusing, but helps us write the stuff down. [RM4, Lecture 1, Stage 1 course]

The post-observation interview with RM4 confirmed the unpacking was done to address the student misconception. This was done compared to what they learned previously.

> And this particular piece of notation where you write a *perp* without the brackets, that's really confusing because *perp* is an operation applied to sets, so writing '*a*' without the brackets and then the *perp* is a kind of reinforcing the confusion that students might have between vectors and sets of vectors. And so I thought it was necessary to point that out. [RM4, Post-observation interview 1]

The mathematical knowledge of anticipating potential student confusion and addressing it adaptively is a critical characteristic of the practice of unpacking. The knowledge of different mathematical symbols and notations and students' difficulties of understanding and misunderstanding is part of a mathematicians' mathematical knowledge (McCrory et al., 2012). It is the knowledge of the mathematical content and identifying similar situations of potential confusion in the mathematical content that enabled RM4 to raise this point.

Helping students notice

The context was to show that for any subset X of R^n, X perp is a subspace of R^n. RM4 explained the first statement followed by the second.

> *Show that for any subset X of* $\mathbb{R}^n$, $X^\perp$ *is a subspace of* $\mathbb{R}^n$. *Also show that if X is a subspace and* $X = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ *then*
>
> $$X^\perp = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}^\perp = \bigcap\{\boldsymbol{u}_j^\perp : 1 \leq j \leq m\}.$$

Show that if *X* is a subspace and *X* is *spanned* by some vectors, then we get this funny stuff down here, what is it saying? I will tell you what that is saying, that is saying if you know that *X* is spanned by some vectors, $\{u_1, u_2, ..., u_m\}$ and you want to check whether *v* is in *X perp*, you only need to look at the inner product of *v* with *u*'*s*, that's enough, you don't have to look at the inner product of *v* with all other vectors in the subspace of *X*. So this is a very useful thing. So the last statement means if *X* is a subspace and X=*Span* $\{u_1, u_2, ..., u_m\}$ of some vector, then check that some vector *v* is in *X* perp, so according to definition, you have to check that *v* is orthogonal to all of the vectors in *X*, and this result says that no, actually you just need to check that it is orthogonal to the *u*'*s*. That is $v.u_1$, $v.u_2$ ... $v.u_m = 0$. So that's a very useful result. [RM4, Lecture 1, Stage 2 course]

RM4 unpacked this exercise by first explaining the mathematical meaning of the statement concisely and telling students what they should notice. He drew students' attention to the important concept rather than just explaining plainly what the mathematical statement meant. The usefulness is highlighted in relation to the previous result.

CONCLUSION

The situated nature of mathematical knowledge in lecturing explains the relationship between the context of lecturing and the unfolding of mathematical knowledge. The episodes represented mathematicians decompressing their mathematical knowledge in lecturing and it was analysed using the framework of unpacking of mathematical knowledge. The data showed that unpacking in lecturing involves breaking a mathematical concept into its components parts and examining the mathematical complexity associated with the concepts. The aim was to make the concept accessible and comprehensible for student learning. The unpacking of mathematical knowledge in this study was directed to addressing student misconceptions/confusions (RM1, RM4), to brainstorm the ideas by revealing further complexity (RM2), to plant a concern and scaffold it (RM3), to draw student attention to mathematical symbols and notations and to help students notice the usefulness of mathematical statements (RM4) in relation to a previous result. Hence, unpacking is a critical dimension of mathematical knowledge in the practice of lecturing (Ball et al., 2008). Mathematicians' use it contextually and to assist in student comprehension, as evidenced in this research. This has implications for identifying and understanding practice-based approaches to teacher knowledge.

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FRAMEWORK FOR ANALYSING SECONDARY MATHEMATICS TEACHERS' DEVELOPMENT OF GEOMETRIC REASONING

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Developing geometric reasoning is an important aim of school mathematics. Some influential theoretical frameworks have been used to map progress in geometric reasoning among learners and teachers internationally. We propose an alternate analytical framework, suitable for the context of our work, to analyse secondary mathematics teachers' development of geometric reasoning as they participate in content-focused professional development. We show how the emerging framework uses conceptual categories drawn from the existing literature on geometric reasoning and empirical levels mapped to analyse teachers' developing reasoning. We conclude by anticipating ways in which the framework can be used and extended.

INTRODUCTION

In the South African context, geometry is considered a difficult domain, with low learner and teacher achievement (Bowie, 2013). But the challenges posed by geometry education have also been reported internationally. In a review of research on geometry presented in the PME conferences from 2005 to 2015, Jones and Tzekaki (2016) observed low understanding of geometry subject matter among students and teachers. Research investigating teachers' geometric reasoning has reported teachers' struggles in (a) defining an angle (Silfverberg & Joutsenlahti, 2014), slope or gradient (Mudaly & Moore-Russo, 2011), quadrilaterals using necessary and sufficient conditions (Brunheira & da Ponte, 2016), and in (b) connecting properties with definitions to reason deductively (Chiang & Stacey, 2015).

Our interest is in developing geometric reasoning (henceforth GR) among secondary teachers who identify as being weak in geometry. We define GR as an understanding of the theoretical status of the different geometric attributes (GAs) such as definitions, properties, axioms, and theorems relevant to a geometric object; and organising them in a deductive argument. In the research reported in this paper, we worked with a unique group of teachers, who identified themselves as needing support in basic geometrical ideas, that is, GAs of lines, angles, and triangles. Most of these teachers work in under-resourced schools that cater for learners from relatively poor socio-economic backgrounds, with little access to digital technology for teaching and learning. An initial analysis of teachers' difficulties suggested that they (a) struggled to use the necessary and sufficient conditions when constructing definitions; (b) were aware of the propositions but could not easily recall them correctly or sequence them to solve a problem; and (c) faced difficulty in writing simple numerical proofs. A short research based professional development course (PD) was offered to support these

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teachers' GR. The research reported here contributes to the existing literature on developing GR by disaggregating levels appropriate for analysing teachers' basic understanding of GR. The research questions we address are:

1. What are the key elements of a trajectory for developing GR?
2. How can we use this trajectory to map progress in teachers' GR?

EXISTING FRAMEWORKS ON GEOMETRIC REASONING

Several influential frameworks have been used in geometry research to map learners' and teachers' GR, for example, Van Hiele levels (Crowley, 1987), Fischbein's theory of figural concepts (1993), Kuzniak's Geometric spaces (2018), and de Villiers (1995). The key ideas from the existing frameworks, relevant to the reported research, are discussed here.

In extensive reviews of the literature in geometry education, Jones and Tzekaki (2016) and Sinclair et al. (2016) have noted the use of Van Hiele levels to analyse teachers' and learners' reasoning. Despite criticisms about the discreteness of these levels, they are useful in identifying shifts in attention from visual to discursive reasoning. de Villiers (1995) concluded that most learners entering high school in South Africa are at the early Van Hiele levels and Bowie (2013) reported similar results for pre-service teachers' knowledge of definitions. Van Hiele levels are a hierarchical and fine-grained classification of the processes involved in visualisation noted by Duval, namely, (a) visual identification, (b) visual identification with discursive elements such as properties, and (c) discursive tools supporting visualisation. The link between the visual and discursive is key to developing GR. In the theory of figural concepts, Fischbein (1993) acknowledges the tensions that learners face between the conceptual and figural components of a geometric figure. The "turn" in drawing attention from visual aspects to the conceptual constraints of the geometric figure is significant for developing deductive reasoning. Kuzniak (2018) identifies reasoning based on perception, experimentation, and deduction as key to school geometric space.

The existing frameworks on GR drew our attention to the links between the visual and the discursive, and the need to move from perceptual to deductive reasoning. However, these frameworks were insufficient for mapping (a) the low levels of GR that our teachers brought to the PD setting; and (b) the growth in their GR through the PD. We learnt from the existing frameworks and our work in GR that it is important to:

1. identify a trajectory for developing GR, where progressive differentiation is characterised and supported through relevant tasks;
2. acknowledge the interplay of visual and/or verbal information along with the inferred information about the geometric figure in focus;
3. experience a geometric figure in multiple orientations and complexity;
4. focus on definitions in terms of necessary and sufficient properties; and

5. pay explicit attention to organising relevant properties to formulate an argument.

These learnings form the core of the analytical framework discussed below.

ANALYTICAL FRAMEWORK

We adapted the analytical framework, Mathematical Discourse in Instruction (MDI, see Adler & Ronda, 2015), which was initially developed for algebra and functions, and we also used the aforementioned frameworks on geometric reasoning to develop a working framework for investigating and supporting secondary teachers' GR.

Located within a socio-cultural orientation, MDI assumes that teaching and learning is goal directed, towards an *object of learning* (OoL) which could be a mathematical concept, process, or capability. The OoL is mediated through *exemplification* and accompanying *explanatory talk*. The selection and sequencing of example sets in and across lessons, along with the accompanying tasks, constitutes exemplification. Explanatory talk includes the naming of mathematical objects using symbols and words, and the criteria for their legitimation. These criteria are divided into non-mathematical (visual, positional, or everyday) and mathematical (local, partially, or fully general). *Learner participation* in mathematical activity is promoted through systematically varying examples and the accompanying tasks; linking of representations; and an explicit focus on the accompanying discursive tools.

From the geometric education research, we draw on Duval's (1998) characterisation of GR in terms of its *form* which includes verbal (identified as discursive and conceptual in the existing literature) and visual components, and the *organisation* of propositions (or attributes) to form a logical argument. He suggests that reasoning through propositions distinguishes naïve apprehensions from mathematical behaviour in geometry. GR is defined as understanding the theoretical status of propositions and sequencing them to form arguments to reach conclusions. We designed the PD to promote and support this kind of GR among teachers with explicit attention to its form and organisation, using the mediational tools identified in MDI. The conceptual categories for Promoting Geometric Reasoning (PGR) emerged from the networking of MDI with the topic-specific literature on GR (see Figure 1).

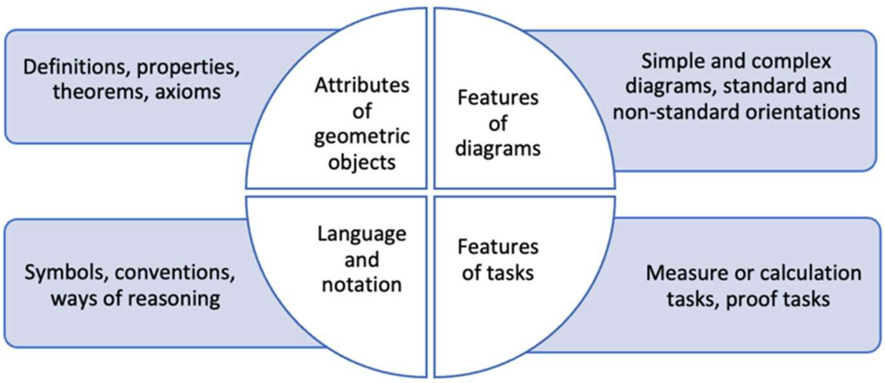


Figure 1: Conceptual categories in PGR

METHODOLOGY

We worked with a group of 10 junior secondary maths teachers (teaching Grades 8 and 9) from eight schools in Johannesburg, South Africa. This group is relatively unique in that the teachers identified themselves as needing support to improve their knowledge of basic geometric ideas, that is, attributes of angles, lines, and triangles. In the South African school curriculum, while geometry learning is compulsory, it has not received as much attention in teaching and research as arithmetic, algebra, and functions.

We designed a short PD course dealing with lines, angles, and triangles. Due to the COVID-19 pandemic, the implementation involved five online and four in-person sessions over three months in mid-2021. The course began with a pre-assessment, with simpler tasks, to identify teachers' challenges. We identified that teachers struggled to use relevant GAs to solve problems and formulate an argument. The course tasks were designed to promote progression in complexity and hence demand, for instance, moving from simple figures in standard orientation to complex figures in multiple orientations, and from numerical measures to proof tasks. Teachers worked on these tasks individually and in groups during the course. We use 10 teachers' responses to nineteen tasks of which six tasks were offered in the form of pre-assessment, ten during the course, and three towards the end to map their developing GR.

RESULTS

The results section is organised around the two research questions and the use of the analytical framework to capture teachers' developing reasoning.

Elements of our framework on geometric reasoning

We argue that three elements are important in mapping and promoting GR. These are (a) identification of relevant GA based on how the given information is interpreted; (b) the accuracy of mathematical statements and reasons, and their structure as a mathematical argument; and (c) connections that are made between the given verbal and visual information. We explain these elements using a task (see Figure 2) and its accompanying explanation.

You will use algebra to answer the questions.

a) Determine the value of x. Give reasons.
b) Determine the value of y. Give reasons.
c) Fill in the sizes of all the 8 angles on the diagram.

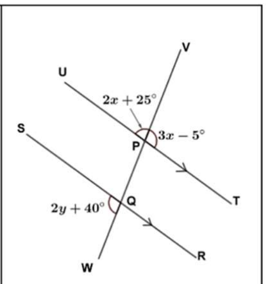


Figure 2: Algebra Task (T19)

The geometric figure includes a pair of parallel lines intersected by a transversal. Three angle measures are given as algebraic expressions. To begin solving the task, angles formed by the straight line UPT or alternate angles can be used. The properties of corresponding angles and vertically opposite angles can be used to find the measures of other angles. An awareness of these properties (angles formed and relations between them) of the geometric figures in focus (parallel lines intersected by a transversal) constitute the GAs of this task. One of the common errors among the teachers was overgeneralising these properties for a pair of non-parallel lines cut by a transversal. Thus, in teachers' justification, it was important to take note of the notation of parallel lines in the diagram, and to distinguish these properties and relations from those between angles formed by a pair of non-parallel lines and a transversal. Reading and interpreting information given in the diagram and as verbal statements is a part of the process of reasoning, we refer to this as "connecting verbal and visual". This task requires formulating and solving algebraic equations to find the measure of each angle along with accompanying justifications using GAs. We refer to forming algebraic equations as "mathematical statements" and the justification as "reasons". A complete argument contains a series of mathematical statements and reasons organised so that they either refer to the attributes of the geometric object or follow logically from one another. In summary, the three elements of GR are:

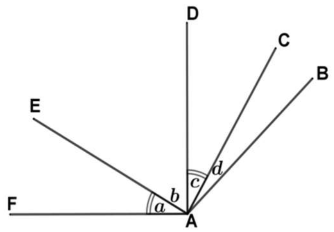
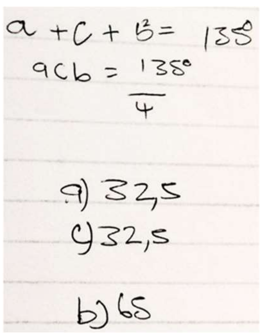
Geometric attributes: Properties of geometric figure in focus

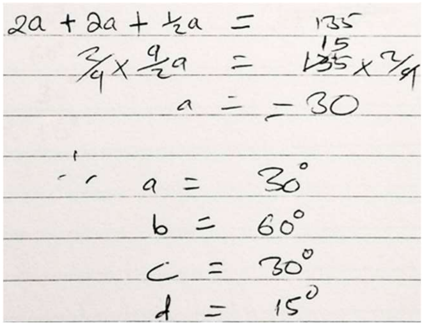
Nature and quality of reasoning: Mathematical statement, reasons, flow of argument

Visual-Verbal connect: Relation between figural and verbal aspects

Framework to map teachers' developing geometric reasoning

We analysed teachers' responses to the tasks posed during the course using the three elements of GR. Analysis of two teachers' responses is presented in Table 1.

T7: In the given diagram, angles *a* and *c* are equal, *b* is twice *a* and *d* is half of *c*. All the angles add up to 135°. a) Find the values of *a, b, c*, and *d*. b) Name all the right angles in the figure. 	*Task analysis* Geometric figure with angles in different orientation. Relation between angle measure is given as verbal statements. Visual supports the position of angles.
Ben's response 	*Response analysis* Identifies angles but misses relation between angle measures. Difficulty in connecting verbal and visual information. Incorrect mathematical statement with no reasons.

<table>
<tr>
<td>Adam's response</td>
<td>
</td>
<td>Identifies all relevant GA based on given information.

Uses given verbal and visual information to infer GA.

Formulates correct mathematical statements without explicit reasons.</td>
</tr>
</table>

Table 1: Analysis of two teachers' responses to T7

Since our interest is in tracking development of teachers' GR, we need a framework to capture levels in their reasoning. In Table 2, we present the analytical framework with progressive levels to differentiate teachers' developing reasoning.

Geometric attributes	*Nature and quality of reasoning*	*Visual-Verbal connect*
L1: Does not identify relevant geometric attributes. L2: Identify some of the relevant GA based on given information. L3: Identify all relevant GA based on given information. L4: Use relevant GA to solve the task.	L1: Does not formulate mathematical statements (MS). L2: Formulates incorrect MS. L3: Formulates correct MS. L4: Partial reasoning using MS and reasons. L5: Complete reasoning with MS and reasons. L6: Uses MS and reasons in a deductive argument.	L1: Uses only visual clues to identify GA. L2: Struggles to connect visual and verbal information. L3: Uses given verbal information and interprets visual notations to infer GA.

Table 2: Analytical framework - Levels in Geometric Reasoning

Using the analytical framework, we map levels in two teachers' (Adam and Tony) developing GR. Table 3 is a summary of the levels for numerical measure, algebraic measure, and proof tasks in early (T1, T7, T8) and late PD sessions (T14, T17, T18).

	Numerical measure		Algebraic measure		Proof	
	T1	T17	T7	T18	T8	T14
Adam	L2,3,3	L4,5,3	L2-3,3,2	L4,5,3	L3,3,3	L4,6,3
Tony	L2,0,2	L4,4,3	L2-3,3,2	L3,4,3	L2,4,2	L4,3,3

Table 3: Summary of two teachers' responses to six tasks

Table 3 shows noticeable changes in teachers' GR. The qualitative changes in teachers' reasoning become evident through the description of these levels. We are currently in the process of investigating the progress in each teacher's reasoning.

CONCLUSIONS AND DISCUSSION

The paper proposes an analytical framework to analyse secondary teachers' development of GR, who are weak in their geometry knowledge. The analytical framework for mapping the development of teachers' GR has been developed using the existing literature and empirical work with teachers. The categories of GR are informed by the research literature in geometry education. The descriptive levels within each category are empirically derived from teachers' responses to tasks. While the levels might vary depending on the contexts where the framework is used, the potential of the framework lies in offering conceptual categories, drawn from a synthesis of research on GR, which are fairly generalisable.

At the beginning of the paper, we raised two research questions about the development of a framework and its use in capturing developing teachers' GR. We have shown that such a framework can be developed using the existing literature and empirical work with teachers, with descriptive levels of progressive GR. Further, we have indicated how the framework can be used to map teachers' changing GR. We anticipate that this framework can be used to analyse learners' and teachers' reasoning for complex tasks and for mapping development of GR.

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COMPUTATIONAL THINKING IN DENMARK FROM AN ANTHROPOLOGICAL THEORY OF THE DIDACTIC PERSPECTIVE

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In this paper, we study the external didactical transposition of programming and computational thinking (PCT) into the Danish compulsory school mathematics curriculum. Taking an anthropological theory of the didactic approach, we focus on the nature of mathematical and PCT knowledge being transposed into the new curriculum as prescribed praxeologies. Two main critiques arise. First, PCT knowledge being transposed into mathematics is broad and has no immediate relation to definitions from the literature. Second, the new competence and subject-matter areas are barely juxtaposed to those from mathematics. As a consequence, the responsibility of developing meaningful integrations in concrete teaching—the internal didactic transposition—lies entirely on teachers and material developers.

INTRODUCTION AND BACKGROUND

In 2006, Jeannette Wing revived the concept of computational thinking (CT) initially introduced by Papert (1980), and argued that it is a "fundamental skill for everyone, not just for computer scientists" (Wing, 2006, p. 33). Wing's (2006) paper sparked a new wave of interest, which has led to research in programming and computational thinking (PCT) teaching and learning and revisions of national curricula to include this (Bocconi et al., 2016). In many countries, these revisions have consisted of adding PCT elements to the mathematics curriculum. In the literature, there is consensus that mathematics and CT share a focus on logical structures and modelling (e.g., Gadanidis et al. 2017), and that there are potential educational synergies in combining the two areas (Benton et al., 2017). Still, there are not yet conclusive findings in terms of how we establish such synergetic relations between mathematics and PCT. In part, this knowledge gap is due to the fact that PCT is still an ambiguous term, defined and practiced in a diversity of ways (Shute et al., 2017). While PCT is closely related to the scientific domain of computer science, it sometimes also draws on other fields, such as sociology and philosophy (see, e.g., Helenius & Misfeldt, 2021). In turn, the specific PCT content that is being implemented at a curricular level varies substantially. In this paper, we tackle the nature of PCT and mathematical knowledge in the interplay between scholarly notions and knowledge aimed to be taught, focusing on the particular case of Denmark. While several countries (e.g. Sweden, England and Norway) have already implemented PCT in their respective compulsory school

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national curricula, the Danish Ministry of Education (UVM) took a more cautious approach. It consisted of trying out a new subject called Technology Comprehension (TC) at 46 schools, and then evaluating its outcomes under two implementation strategies (UVM, 2021). These ideas were to inform a future decision for mainstream implementation. Whereas the first strategy regarded TC as a subject in its own right, the second strategy inserted TC competence areas and learning goals in existing subjects, including Danish (language 1), mathematics, arts, physics/chemistry, science, craft and design, and social science. Both approaches were supported by a newly drafted curriculum, including mathematics (UVM, 2019). In order to address the interrelation between out-of-school types of knowledge and those depicted in the Danish curriculum, we activate appropriate concepts from the didactic transposition (Brousseau, 2006) and the anthropological theory of the didactic (ATD). Below, we elaborate on the theoretical underpinnings and describe the empirical foundation for our analysis to formulate our research questions.

THEORETICAL ELEMENTS FROM ATD

ATD has been developed as an epistemological perspective to the teaching and learning of mathematics (Chevallard & Sensevy, 2014). It focuses on the nature of mathematical knowledge regarded as a human activity, and how it is disseminated and taught (Bosch & Gascón, 2006). According to Chevallard and Sensevy (2014), knowledge to be taught can be modelled as a *praxeology*, which consists of two main building blocks: *praxis* and *logos*. The praxis block includes the type of *tasks*, i.e., the concrete challenges to be confronted by students, and specific *techniques*, which describe how the tasks should be handled. The logos block consists of technology and theory. *Technology* covers the discursive knowledge that builds the language needed to talk about the tasks and techniques. *Theory* is defined as the logical and conceptual frames that explain and justify the technological components, and relate these to other areas with their respective theories. Although ATD focuses on knowledge as being experienced in human activity, we are investigating knowledge as depicted in the curriculum, which we might label as *prescribed praxeologies*. The second element of interest is the transition between these domains of knowledge and the knowledge that ought to be taught in a school setting. In the language of ATD, this process is called *external didactical transposition* (Bosch & Gascón, 2006). In contrast, the *internal didactical transposition* takes place from the knowledge meant to be taught (e.g. curriculum) to that factually being taught (i.e. teaching practices). Typically, ATD regards the point of departure for the external didactical transposition to be scholarly mathematical knowledge and the role of mathematical knowledge and skills in society and everyday life. Analyses commonly regard praxeologies as situated within a discipline's subfield, such as 'algebra' within mathematics. However, as Helenius and Misfeldt (2021) point out, the integration of PCT into mathematics requires the mobilization of knowledge from several domains. This diversity calls for specific attention to the level of coordination between the elements of the didactical transposition coming from the different sources. Such coordination can happen on the level of external and on the level of internal didactical transposition (Schmidt & Winsløw, 2021).

RESEARCH QUESTIONS

We first aim to characterize the diversity and structure of the fields of knowledge—mathematical or otherwise—involved in this new Danish TC curriculum, leading to the first research question:

- What domains of knowledge make up the new PCT-related praxeologies in the Danish mathematics curriculum?

In the above-described openness to the domains of knowledge, we formulate our second research question:

- To what extent is the interplay between knowledge from different domains (mathematics versus TC) contained in the external didactical transposition, and which elements are left to the internal didactical transposition?

DATA: THE DANISH TC CURRICULUM

The Danish exploratory project to implement TC at the 46 schools began with developing a curriculum for TC as a subject in its own right. This curriculum included four competence areas, namely: digital empowerment; digital design and design processes; computational thinking; and technological agency. Further, each competence area was defined by 3–5 subject matter areas presented as pairs of skillset and knowledge. For example, Table 2 displays those included in mathematics.

As our primary source of data, we draw on the official mathematics goals overview with integrated TC published by UVM. This document includes a general declaration and description of competence goals for mathematics in Denmark's compulsory K-9 education. These descriptions are organized into four competence areas: (1) mathematical competencies (see Niss & Højgaard, 2019); and subject-matter areas, (2) numbers and algebra, (3) geometry and measure, and (4) probability and statistics. This experimental version of the curriculum then adds TC as a fifth competence area, including elements of PCT (see Table 1).

Although the exploratory program planned to experiment with two implementation strategies (as an independent subject and integrated into others), it was decided that both should address the same curriculum components. Hence, in order to integrate TC in existing subjects, the individual competence areas of the curriculum for TC, as a subject in its own right, were to be distributed among the subjects in which TC should be integrated. In the case of mathematics, six TC components were integrated into mathematics: digital design and design processes; modelling; programming; data, algorithms and structures; user studies and redesign; computer systems (see Table 1).

Mathematical competencies	Description of six competencies: Problem treatment; Modelling; Reasoning and thinking; Representation and symbol treatment; Communication; Aids and tools
Algebra and numbers	Description of five areas: Numbers; Calculation strategies; Equations; Formulas and algebraic expressions; Functions
Geometry and measurements	Description of three areas: Geometric properties and relationships; Geometric sketching; Placement and movements; Measurement
Statistics and probability	Description of two areas: Statistics; Probability
Technology comprehension	Description of six areas of comprehension: Digital design and design processes; Modelling; Programming; Data, algorithms, and structures; User studies and redesign; Computer systems

Table 1: TC added to the description of the mathematics curriculum for Danish K-9.

The objective of TC in relation to the mathematics curriculum reads: “The student can act with judgment concerning the use of digital technologies in working with open problems from the surrounding world” (UVM, 2019, p. 4, our translation from Danish). Table 2 illustrates this for the six components of TC.

TC components	**Skillset**	**Knowledge**
Digital design and design processes	The student can design digital artefacts through an iterative design process that will benefit the individual, the community and society	The student has knowledge about complex problem solving and iterative design processes
Modelling	The student can construct and act on digital models of the real world and assess the range of the model	The student has knowledge about how models of the real world can be used to describe and treat this
Programming	The student can modify and construct programs for solving a given task	The student has knowledge about methods for stepwise development of programs
Data, algorithms, and structures	The student can recognize and utilize patterns in structuring of data and algorithms with a departure point in specific problems	The student has knowledge about patterns in structuring data and algorithms

User studies and redesign	The student can plan and carry out investigations of users' perspectives and applications of digital artefacts	The student has knowledge about users' perspective and application of digital artefacts
Computer systems	The student can assess different computer systems' possibilities and limitations	The student has knowledge about how number systems, encryption mechanisms, and network protocols affect the basic construction and mode of operation of computers and networks

Table 2: The six TC components related to the mathematics curriculum for Danish K-9 (our translation from Danish).

ANALYSIS

While the new TC competence and subject matter areas are added to the curriculum, they are not explicitly related to existing mathematical competencies and subject matter areas. We may model this situation as a didactical transposition following, for instance, the example of Schmidt and Winsløw (2021). We see that the external didactical transpositions from TC and mathematics are only superficially coordinated by juxtaposing the curricular learning objectives and providing examples of teaching materials. This situation resembles the model in Figure 1.

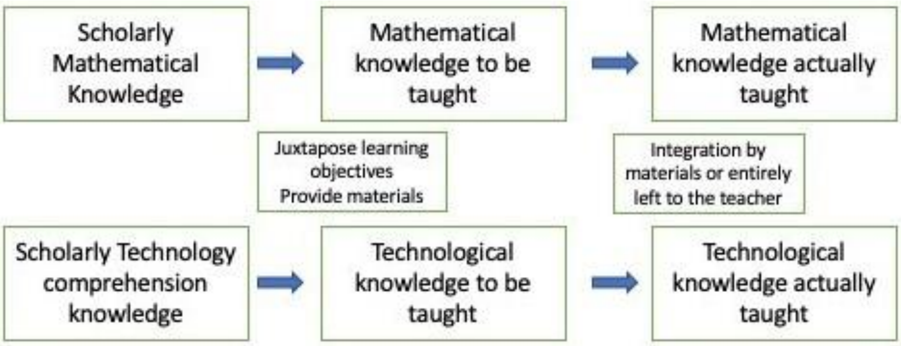

Figure 1: Superposed didactical transposition in the Danish exploratory integration of TC and mathematics. The external didactical transposition is weakly coordinated

The inclusion results in a curriculum consisting of juxtaposed components from TC and mathematics, thereby placing the responsibility of developing meaningful integrations in concrete teaching on the shoulders of teachers and, in part, on curriculum material developers. Given the existing level of PCT skills among the participating Danish mathematics teachers, it seems a quite unrealistic and unreasonable requirement (UVM, 2021). This problem has been acknowledged and addressed in the Danish exploratory TC project by developing rather targeted teaching sequences available to the public in tekforsøget.dk/forlob. In a sense, this combination still leaves the teacher with most of the formal responsibility to interpret the interplay between mathematics and TC. Moreover, it does so without providing the teachers with any real influence on the matter. The specific content from TC that is suggested integrated into mathematics is undoubtedly relevant to the subject of mathematics. Still, if we look at Table 2, it is not entirely clear how this relation can be articulated and promoted in actual student activities. All six objectives clearly can be related to mathematics but in a myriad of different ways.

Digital design and design processes as well as *user studies and redesign* are described completely generic with no specific relations to mathematical praxeologies. For *modelling* and *programming* the case is almost similar, even though these objectives are more closely related to PCT, and thus perhaps easier to interpret in a mathematical direction. The two components, *data, algorithms, and structures* and *computer systems*, do have some relations to mathematics in the short description provided. Data, algorithms, and structures contain elements of algorithms, data and patterns that are obviously relevant to a number of mathematical activities and insights. In relation to computer systems there are references to number systems and encryption that could support the development of relevant mathematical praxeology.

DISCUSSION

Above, we have seen how the external didactical transposition prescribes the integration of PCT and mathematics in the Danish exploratory subject TC. It seems clear that TC is the result of transpositions of several scientific disciplines and knowledge domains, not only computer science. We also notice that the external didactical transposition does not form or prescribe new mathematical praxeologies as such. Rather, independent TC praxeologies are suggested to the mathematical curriculum by juxtaposition, leaving the teachers without guidance.

Evidently, this situation is not ideal, but could it have been avoided? Let us take a closer look at how this problem has been handled in other countries that have tried to integrate PCT into mathematics education. We can see that the UK has chosen not to formally integrate PCT and mathematics, but instead has an independent Computing subject. On the other hand, Sweden has made a more specific and detailed integration of PCT into algebra, among other mathematics subject areas (Bråting & Kilhamn, 2021; Helenius & Misfeldt, 2021). In England, the computing subject is formed by transposing a university-level computer science topic. The Danish TC curriculum is formed from a much broader range of topics, including computer science, sociology (e.g., democracy and surveillance), and philosophy (e.g., ethics). The didactic transposition in the two countries, however surprisingly, share the feature that none of them prescribes specific praxeologies that explicitly include both mathematics and PCT. For England this is not surprise, since the computing subject shares no structural overlap with the mathematics curriculum. However, it is remarkable that the didactical transposition of PCT and TC into the Danish curriculum merely consists of juxtaposing new praxeologies from TC to the existing mathematics curriculum. The task of finding meaningful ways of connecting mathematics and PCT is, in both cases, a matter to be handled in the internal didactical transposition, i.e., as mentioned above, it is left to curriculum material producers and teachers. Nevertheless, the difference is that the Danish exploratory topic requires this integration to occur, whereas the English subject does not. The Danish case is an example of a required yet unsupported, superposed didactical transposition, whereas the English case is neither supported nor required. Both approaches are rather different from how computer programming is embedded in the Swedish mathematics curriculum, where, for example, algorithms have been embedded in problem-solving and algebra (see Helenius & Misfeldt, 2021).

Our first research question aimed at understanding what domains of knowledge make up the new PCT praxeologies in the Danish mathematics curriculum. This inquiry is actually not easy to answer with an outset in the core curriculum document. The juxtaposed TC components open a wide range of possibilities yet give little clear direction for practitioners. Nevertheless, the areas *data, algorithms, and structures*, *programming*, and *modelling* are all parts of what Wing (2006), Shute et al. (2017), and Weintrop et al. (2016) refer to as CT. Hence, these areas are unquestionably in play. The lack of clarity regarding the first research question, in a sense, answers the second one. Here we asked to what extent the interplay between knowledge from different domains (mathematics versus TC) are contained in the external didactical transposition and which elements are left to the internal didactical transposition. The answer seems to be that the majority of the work is left to the practitioners in the internal didactical transposition.

We thus see this as a case of a superposed didactical transposition supporting transdisciplinary education by integrating knowledge from different domains into coherent educational scenarios. This makes it difficult, although necessary, to be specific in the external transposition.

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DIALOGIC-DIALECTIC MATHEMATICAL ARGUMENTATION

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The importance of mathematical argumentation for learning is undeniable. Yet there seems to be an underlying complexity constituting a good mathematical argumentation, particularly in the context of classroom-based group work, where social goals as well as disciplinary ones are at play. This paper hopes to provide a better understanding on how we can study learners' behaviour during collective argumentation to inform the teaching and learning of argumentation in mathematics. In particular, it highlights how the potential and value of two seemingly distinct perspectives of understanding argumentation - dialogic and dialectic - can be used in concert for a more comprehensive understanding of mathematics argumentation.

INTRODUCTION

Mathematical proofs, as an essential aspect of the work that mathematicians typically engage in, help humans *understand*, *explore* and *think* about mathematics. Yet, more fundamentally, it can be argued that the underlying central processes they engage in are *mathematical argumentation* and *reasoning* (Schwarz et al., 2010). In the context of mathematics classrooms, it has been noted that learners are more likely to be engaging in argumentation (and reasoning) than in proving based on formal logic (Krummheuer, 1995). Mathematical argumentation has also gained attention in the twentieth century due to the reconceptualization of mathematics learning that places greater emphasis on understanding how mathematical knowledge is constructed and why it makes sense; and the shift in focus towards discursive activities in recognition of the benefits of argumentation for learning (Schwarz et al., 2010). However, despite the increasing acknowledgement of its importance to learning, mathematical argumentation does not necessarily or naturally happen in all classrooms, possibly due to its underlying complexity and ambiguity (e.g., Krummheuer, 1995; Schwarz et al., 2010). Without clarity in the specifics (of the various aspects) of mathematical argumentation, it will be challenging for researchers to analyse argumentations; for teachers to design and orchestrate meaningful argumentation in mathematics classrooms; and for mathematics learners to know how to argue and to learn through argumentation.

This paper will discuss how mathematical argumentation has been framed or analysed from two seemingly distinct perspectives, namely dialectic and dialogic. I will also discuss the potential in a coexistence of both perspectives in the mathematical argumentation process.

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MATHEMATICAL ARGUMENTATION - THE TWO PERSPECTIVES

Although there is no common definition for mathematical argumentation in the field of mathematics education, there seems to be agreement that its overarching focus resides in the rational reasoning and meaning-making process that mathematics learners engage in collectively. Many different aspects of mathematical argumentation have been studied. The more common ones include the cognitive, meta-cognitive and social-cultural aspects of argumentation (Krummheuer, 1995; Schwarz et al., 2010). However, according to Schwarz (2009), since "argumentation functions in two ways" (p. 104), there have been two strands of research on (mathematical) argumentation, which adopt either a dialectical approach or a dialogical one. The former places emphasis on the rationality behind argumentation. It focuses on how learners make connections in order to provide reasons to support or refute the multiple different ideas or claims proposed, before arriving at an agreement or consensus. As a result, it tends to focus on the cognitive aspect of argumentation. The latter, i.e. the dialogic perspective, places emphasis on the rationality that is situated within social-cultural rules and orientations that facilitate the progress and development of the collective argumentation process (Schwarz & Shahar, 2017). It thus tends to focus on the social-cultural aspects of argumentation. Notably, each perspective has been important for understanding the specifics of a good mathematical argumentation process.

THE DIALECTIC PERSPECTIVE

In mathematics education, it seems that understanding argumentation from the dialectic perspective has been the primary focus of research, due to its strong association with proofs. With respect to this perspective, research generally attend to how the rationality of a good mathematical argumentation can be established. In particular, to understand what makes a good mathematical argumentation process, the research literature has highlighted two key elements, namely, the structure of the argument; and the types of reasoning used in supporting the argumentation process.

Structured by Toulmin's Model of Argument

Toulmin (2003) proposed that arguments generally follow a structure where claims (C) are made based on data (D), the foundation for the claims. But claims need to be supported by warrants (W), usually "general, hypothetical statements, which can act as bridges" (p. 91) between the data and the claims. In other words, the warrants provide plausible reasons to explain the validity and soundness of the claims. Nevertheless, the warrants may only support the claims for some conditions and not others, as such the claims need to be specified further with qualifiers (Q). For the conditions where the claims may not be valid, they will then be refuted with exceptions or rebuttals (R). Lastly, the appropriateness and applicability of the warrants to the claims may need to be defended and elaborated with the necessary backings (B), which usually take the form of statements that are more absolute in nature. Thus, according to Toulmin's model, a logically valid argument should contain all the above six elements with each serving a different function. A cogent visual representation for Toulmin's model of argument is seen in Figure 1. Toulmin's model is helpful for us to understand how the rationality of learners' arguments can be established and is often used in research to analyse and understand individual mathematics learners' arguments.

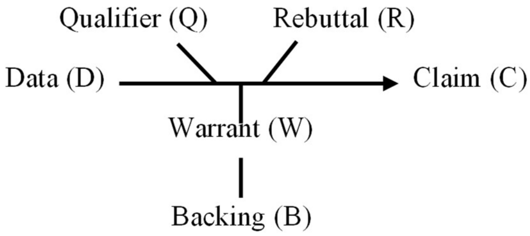

Figure 1: Toulmin's model of argument

Supported by Reasoning

Markedly, mathematics is often associated with deductive reasoning which preserves truths to establish valid conclusions from premises by following the rules of a well-specified logic (Meyer, 2010). However, non-deductive forms of reasoning have also been observed to be important and commonly used in mathematics. Above all, two types of non-deductive reasoning have been highlighted in research (Feeny & Heit, 2007). They are abductive reasoning, which seeks to provide the best explanations for the claims made from observations; and inductive reasoning, which infers generalizations from particular observations. While the order of the different forms of reasoning used may vary, a typical process will start with abductive reasoning to suggest a tentative explanatory hypothesis, followed using inductive reasoning to support or refute its plausibility. If the hypothesis is refuted, the search for another explanatory hypothesis will be required through abductive reasoning and this process will repeat until the best explanatory hypothesis, supported by inductive evidence, is formulated. After which, a proof can be pursued through deductive reasoning to establish the certainty of the conclusion (Meyer, 2010). As such, the type of reasoning that learners adopt in supporting their claims is likely to be dependent on the stage of argumentation and the context. The types of reasoning are also often used together with Toulmin's model to analyse individual learners' mathematical arguments.

THE DIALOGIC PERSPECTIVE

While research based on the dialectic perspective focuses on understanding the rationality of mathematical argumentation through the structure of their arguments, research taking a dialogic perspective aims to understand how the presence of multiple and differing perspectives enhances learning in the argumentation process. In particular, the dialogic perspective, which draws heavily on dialogism, emphasizes the importance of learners' interactions and their engagement with others' ideas and arguments where the focus is on how learners explore the relationship between diverse ideas without the need for a final synthesis or unification (Wegerif et al., 1999). Such a perspective seems to be in great contrast with the conventional expectations of mathematical argumentation, which draw on dialectical assumptions and focus on comparing or evaluating the validity and strength of the suggested arguments before converging and deciding unanimously on the best argument (Langer-Osuna & Avalos, 2015). This is probably also why research based on the dialogic perspective, which focuses on differences and learner interactions, seems to be less favoured as compared to those with a dialectic perspective.

Driven by Differences

Fundamentally, dialogism is based on differences (that are irreducible) rather than identity or unity, where knowledge is deemed to be formed through differences (Bakhtin, 1975/1981; Wegerif, 2008). In other words, object A is not known in and of itself, but in relation to object B (and other objects). It is often contrasted with the notion of dialecticism which emphasises the independent knowledge of A and B (through their respective properties) and then the logical synthesis that can arise by the overcoming of differences, towards uniformity and certainty. Dialogism and dialecticism might seem to be incompatible, in terms of their epistemological and ontological assumptions. Indeed, Bakhtin described dialecticism as being a dialogue in which different voices, perspectives, and emotions have been removed; and the "abstract concepts and judgements from living words and responses" extracted and presented as a single view (Bakhtin, 1986, p. 147, as cited in Wegerif, 2008).

In dialogism, meaning is not universal, but rather, a product of the different perspectives and contexts present. It cannot be constructed without the awareness of at least one other possible point of view, and this is how the dialogic approach differs from a monologic one. The consciousness of the multiple plausible ways of looking at something; the switch between different perspectives; and the context involved, is necessary for understanding to be developed (Wegerif, 2011). Meaning is also realised through dialogues which bring the differences or at least two perspectives together and situate them in a particular context. There is no need for agreement to be met to establish meaning. In fact, Bakhtin proposed that if common ground is achieved, the dialogue will be discontinued and there will be no further progress in meaning making as the presence of variation is the spark that opens up opportunities for perspectives to shift to allow for meaning to be made (Wegerif, 2011). Furthermore, the presence of differences does not necessarily imply that one perspective is correct or superior to their other. The various perspectives should be equally valued and accepted to reflect the multitude of meanings existing in the world such that "the world of meaning is essentially dialogic" (Wegerif, 2008, p. 349).

Augmented with the Other

Another key aspect of Bakhtin's dialogism is the mutual influence or dependency between the two (or more) participants present in the dialogue (Baktin, 1975/1981; Wegerif, 2008, 2011). It highlights their interrelatedness where any utterance is dependent on both parties - the speaker; and to whom it is directed at, the addressee.

The speaker must take into consideration who the addressee is before formulating the utterance as the message should be conveyed in a manner appropriate for the addressee, in terms of the language and style and keeping in mind how the addressee may react to it. Similarly, the addressee will also respond based on a projected view of the speaker. As such, they define each other mutually, where each utterance contains the voices of self and the other; and with each utterance, both the self and the other are being constructed and reconstructed. Eventually, the meaning and understanding constructed is not ascribed to any individual but an integration of the ideas put forth by both.

As a result of the mutual dependency, a metaphorical dialogic space (including position and time) that encompasses both the speaker and the addressee is generated, beyond a fixed and simple connection between them and their ideas (Wegerif, 2008, 2011). It is a shared and generative space where the two can relate to each other dynamically. Their positions in the space can change as the dialogue progresses (and where time changes too). They may take on each other's perspectives, or they may gradually or totally shift their initial positions as new insights emerge from their interaction.

EXPLICATING THE DIALECTIC AND DIALOGIC PERSPECTIVES

Although the dialectic and dialogic perspectives have been used extensively in their own respect within mathematics research, recent research evidence seems to be leaning towards a joint approach in analysing argumentation. Individually, each perspective may no longer be sufficient in providing a comprehensive understanding of mathematical argumentation and the conditions for it to happen, particularly in the case of pair-wise or group activity in the classroom. As such, I want to explore how the two perspectives possibly interact and complement each other during argumentation. I will use a short paradigmatic example of collective mathematical argumentation to illustrate some instances of the dialectic and dialogic perspectives. This example is situated in the context of an undergraduate geometry course where students were tasked to use a dynamic geometry program to form a conjecture and construct a proof for the following problem:

> Construct a circle with centre O and a fixed point Y in the interior of the circle. Let AB be any chord of the circle that passes through the point Y, when is the product of AY and YB a maximum?

Alex: Oh right. Hmm, will the maximum be when the chord is the diameter, so that the two parts have longer length?

Luke: Oh that's possible, since diameter is the longest possible chord, so we will be multiplying two biggest numbers! Let's drag the chord until it passes through the centre and see the length.

Alex: Yes, yes that's what I was thinking. See, the length of this part is 8.5, that is bigger than the lengths of both parts of that chord we started with that doesn't pass through the centre.

Nora: Hmm, wait, wait this doesn't sound quite right. Yes I agree that this part is definitely longer but the other part is shorter, which will cause the product

	to be smaller, isn't it? Your claim is only valid if both lengths are longer but it's not the case.
Luke:	Ah, I think Nora has a point there. Let's calculate the product, here, 8.5 is longer but the other part is shorter at 3.9, so the product is about, hmm, about 33? For the other chord, can you drag it back? The two lengths are 7.2 and 4.6 which is also 30 something, [pause] about 33.
Alex:	Huh, really? That's not what I was expecting.
Luke:	Ya, the products are so similar. Let's try another one and see what's the product. [Drags point *A*.] How about this, 4.1 and 8.1? [pause] The product is also 33!
Nora:	Does that mean that the product is a constant? There is no maximum?
Alex:	I don't quite believe this. Can we move that point *Y* and see what happens? What if the fixed point *Y* is nearer to the edge, the circumference of the circle instead of the centre? I am still trying to test my conjecture about the diameter.

From the dialectic perspective, it can be observed that the group of learners were engaged in abductive reasoning. Alex was the first to offer a conjecture, that the maximum product occurs when the chord *AB* is the diameter. His conjecture seems to be based on abductive reasoning as a warrant that the longer the chord, the longer the two segments that make up the chord was provided to back his claim. However, there was no data from the problem to support his claim, which is characteristic of abductive reasoning. His claim was accepted as a possibility by Luke who further elaborated on the claim that the diameter is the longest chord and provided a backing that the product should be larger when the two lengths are longer. Luke also initiated the use of the dynamic geometry program to find supporting data. Alex then expanded on his argument as he thought that he found suitable data (that the length of one part was longer than that for the initial chord) to support his claim. But this was quickly countered by Nora who highlighted that the length of the other part was actually shorter than the corresponding segment for the initial chord. As a result, the tentative claim by Alex was refuted as the data did not satisfy the assumption that the two lengths *AY* and *YB* will be the longest with a longest chord (i.e. the diameter) to start with. Using Toulmin's model, this sequence of argumentation, supported by abductive reasoning, can be represented graphically as shown in Figure 2. As the conjecture was proposed in the form of a question, it seemed to suggest the uncertainty of its validity. As such, a qualifier was necessary to illustrate the learners' degree of confidence.

From the dialogic perspective, it can be observed that differences in the learners' ideas were evident throughout the argumentation process and contributed much to the development and refinement of their arguments. For instance, if Nora had not challenged the validity of Alex's initial conjecture, the group may have embarked on

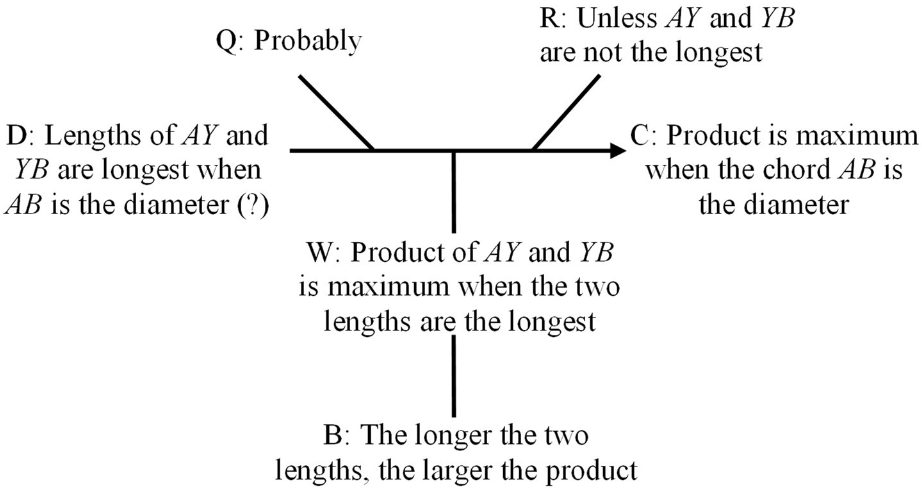

Figure 2: Alex's conjecture based on abductive argumentation

an attempt to prove an erroneous conjecture. It also led the group to discover another hypothesis to the problem which was rather different from the one before. The group shifted from a conjecture of having a possible maximum product to another stating a likely absence of a maximum as the product is a constant. This switch was made possible due to the presence of differing ideas which prompted a search for a better conjecture. Interestingly, despite the challenge from Nora and her supporting data from the dynamic geometry program, it was not convincing enough for Alex to switch his point of view on the problem. He appeared to be able to understand the alternative conjecture but was reluctant to shift his position and continued to make attempts to find evidence to support his own conjecture.

DISCUSSION AND CONCLUSION

Based on this short example, the dialectic and dialogic moments seem to occur almost successively. Rather than argue that the example must be read either in one way or the other, I will instead consider how each approach related to specific functions. While the learners had been observed to be able to articulate their ideas or claims with reasons or evidence, i.e. behaved dialectically, they were also able to see other perspectives and relate to each other's ideas and build on them, i.e. behaved dialogically. In particular, the two components appeared to complement each other at times. On the one hand, the more the learners questioned or challenged others' arguments, the more they had to be articulate in their reasoning and explanations in order to respond or counter-challenge the other. On the other hand, with an increase in the amount and depth of reasoning to present different ideas, the learners were more likely to be able to understand and engage with the differing views. Moreover, there appears to be moments when the dialogic aspect is more prominent as compared to the dialectic and vice versa, depending on the stage and function of the argumentation process. Specifically, the dialogic aspect seems to be more apparent when the two different conjectures were suggested and brought together to open the discussion while the dialectic may be more prevalent when the learners were trying to agree at a best conjecture and when there was more homogeneity in their ideas and arguments (e.g. between Alex and Luke at the beginning of the argumentation).

Hence, there is potential in using both the dialectic and dialogic perspectives in concert to better understand the complexity of mathematical argumentation in a collective setting. However, the potential and value of such an approach may not have been fully explicated through only one example. More authentic examples may need to be examined using this approach to increase its validity in understanding this phenomenon.

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ENHANCING PROSPECTIVE MATHEMATICS TEACHERS' NOTICING SKILLS THROUGH ONLINE LABORATORY SCHOOL ACTIVITIES

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This study investigated how prospective mathematics teachers' (PMT) noticing skills, (i.e., attending, interpretation, and decision-making) were influenced through online laboratory school (OLS) activities. OLS provided PMTs opportunities for online fieldwork and work with students. The activities included lesson planning with peers under the supervision of academicians and experienced teachers, teaching, reflection and getting feedback. PMTs' reflections on a video-taped lesson served as the pre-post assessment of the intervention. Quantitative analyses of data indicated PMTs showed statistically significant improvement in both interpretation and decision-making. Attending, on the other hand, was improved but not in a statistically significant way.

INTRODUCTION

Noticing skills in general are related to how teachers view certain situations related to teaching and engage in reasoning in order to make appropriate instructional decisions (Sherin & van Es, 2009). In recent years, teachers' noticing skills have been found to be critical in order to build instruction according to students' needs (Meschede et al., 2017; Sherin et al., 2011). For teacher education programs, the goal is to raise teachers who can be responsive to students' mathematical thinking (National Council of Teachers of Mathematics, 2014; Sherin et al., 2011).

Noticing has been conceptualized in different ways. While some scholars studied noticing as selective attention and knowledge-based reasoning (interpretation) (van Es & Sherin, 2002), others included aspects of decision-making in relation to what teachers attended to and made sense of (Barnhart & van Es, 2015). Decision-making or deciding to respond are considered as one of the most difficult aspects of noticing for teachers (Jacobs et al., 2010; Schack et al., 2013). Therefore, in this study, we conceptualize noticing as attending to what is noteworthy in the classroom, interpreting classroom events and decision-making (deciding how to respond). In order to prepare prospective teachers for noticing the complexity of teaching (attending, interpretation and decision-making), it is important to both study and support noticing by considering complex environments of teaching (Stockero et al., 2017).

There have been different types of interventions in order to support prospective teachers in noticing student thinking, most of which utilized video content (Santagata et al., 2021). Some researchers focused on designing programs where PMTs view and analyze video clips of student thinking in a specific mathematics content area (Jacobs

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et al., 2010; Shack et al., 2013), others included videos of whole-class instruction and opportunities to analyze student thinking observed in the classroom (Stockero et al., 2017; Ulusoy & Çakıroğlu, 2020) and by way of using structured frameworks such as lesson analysis (Santagata & Angelici, 2010). In order to support PMTs' noticing skills, in general PMTs are required to reflect on and analyze video cases as a group or individually. There are few interventions that provide opportunities for PMTs to be actively engaged in actual teaching practices in the context of noticing studies (Santagata et al., 2021). In this study, the aim is to fill this gap by providing opportunities for PMTs' active participation in teaching practices.

This study is part of a larger project on PMTs' professional growth in the context of Online Laboratory School (OLS). OLS is an online school which provided opportunities of fieldwork for prospective teachers in a private university during the COVID-19 pandemic. The quality of internship practices, and lack of cooperation between school mentors and teacher educators became problematic during this time (Özüdoğru, 2020). Similar to original laboratory schools (Mayhew & Edwards, 2007), OLS was founded so that prospective teachers could learn from core practices of teaching and under close monitoring and guidance of the teacher educators at a time where mentor teachers in internship schools had difficulties in conducting online teaching. In this school prospective teachers had opportunities to collectively plan lessons using a student-centered approach (by the support of teacher educators), engage in effective online teaching activities (DiPietro et al., 2008), and also reflection.

Purpose

The purpose of the study is to investigate the influence of OLS activities in prospective mathematics teachers' noticing skills. In particular, the study considers changes related to prospective teachers' skills of attending to significant events, interpretation and deciding to respond on the basis of interpretation as a result of participating in OLS activities.

METHODS

This study is a quantitative single group pre-post-test design. We quantify qualitative analyses in investigating the influence of OLS activities on PMTs' noticing skills.

Context

In the context of the OLS, prospective teachers were actively involved in online teaching activities under the close supervision of seven supervisors for duration of eight weeks. The OLS included 15 online middle school mathematics classes. For eight weeks, 23 PMTs were involved in planning, teaching, and reflecting (see Fig. 1 for details). Following the observed lesson, a reflection meeting took place with the PMTs and a supervisor. Additionally, a weekly meeting was held during which all PMTs and supervisors gathered and discussed significant aspects related to the implemented lessons. All meetings and classroom sessions were video-recorded. In order to support prospective teachers' noticing skills, seminars on using online tools for teaching and conducting lesson analysis (Santagata & Angelici, 2010) were also provided. The participants of the study were 3rd and 4th year prospective teachers who engaged in OLS activities and volunteered to submit both pre-and post-assessment (*n*=19).

Figure 1: Weekly Lesson Plan Preparation Process in the OLS

Data Collection

Pre-assessment was conducted at the start of the program while post-assessment was conducted at the end of the OLS activities. PMTs' noticing skills were assessed by a video-based assessment designed by researchers. The video is a separate record of a prospective teacher who taught a lesson on fractions. The specific lesson was selected by the researchers as there were opportunities 1) to observe different types of student thinking, 2) to reflect on the teacher's instructional decisions and student learning, and 3) to provide alternative suggestions as the instructional practices had room for improvement. The 40-minute video lesson was divided into five segments in order to assess noticing in a detailed way. PMTs were asked to write what they noticed in the given segment, why they chose to focus on the specific moment, and provide alternative instructional decisions based on their professional judgment.

Data Analysis

Video assessment included three noticing components of each video segment: attending to significant events, interpretation, and decision-making. PMTs' responses were first analyzed qualitatively by considering previous frameworks (Barnhardt & van Es, 2015; Jacobs et al., 2010; van Es, 2011): consistency with the events in the video, providing mathematical detail and evidence as well as depth of interpretation and appropriateness of suggestions. Each dimension was coded by two separate researchers as "emerging evidence" (coded as 1), "medium evidence" (coded as 2) and "robust evidence" (coded as 3). Inter-rater reliability was found satisfactory.

With regards to the attending dimension, emerging evidence suggested describing general aspects of the video excerpt, while medium evidence suggested PMTs provided some mathematical details about teaching, and robust evidence referred to including sophisticated details about student understanding or teaching (identified by researchers). In the interpretation dimension, emerging evidence suggested only describing events while medium-evidence suggested making claims and providing evidence without providing mathematical details. On the other hand, robust evidence suggested establishing a relationship between student understanding and teacher actions, and providing evidence for the claims. With regards to the decision-making dimension, emerging-evidence indicated no suggestion or providing very general suggestions not in line with the video excerpt as identified by researchers. Medium evidence indicated providing suggestions with some mathematical detail but not considering student understanding. On the other hand, robust evidence indicated appropriate suggestions which were in line with the mathematics and student understanding evident in the video.

Pre-and post-assessment scores were determined by adding scores in each of the five parts of the assessment. The difference between pre-post assessment scores were analyzed by using the Wilcoxon-Signed rank test (IBM SPSS, 2012).

RESULTS AND DISCUSSION

The post-assessment included more evidence, mathematical details, consistency with events in the video as well as pedagogically appropriate suggestions for instructional decision-making in line with the observed student understanding in the video. Quantitative analyses revealed that PMTs' noticing skills changed positively. In particular, comparing the pre- and post-assessment scores revealed that the change in dimensions of interpretation ($Z = 92.00$, $p = .012 < .05$) and deciding to respond ($Z = 92.00$, $p = .012 < .05$) was statistically significant. On the other hand, the change in dimension of attending was not found to be statistically significant ($Z = 57.00$, $p = .42$).

While each PMT did not improve in all three dimensions, we provide in-depth evidence for one PMT's growth in noticing as observed in the analyses of the responses to the assessment. The paragraph below is how researchers described and identified significant aspects of one video segment. The following parts demonstrate how the PMT's noticing skills improved between pre-and post-assessment considering the researchers' commentary regarding the video assessment.

Researcher commentary regarding the video segment:

> *In this video excerpt, PMT aimed to introduce unit fractions, discussed the definition of it with some examples and showed unit and proper fractions on the numberline. She provided a chocolate bar example and focused on counting equally partitioned quantities; one piece of chocolate bar out of the whole, which was 9 equal pieces. She made one ninth (1/9) and later defined it as the unit fraction. After discussing this example on the chocolate bar, she moved to showing this unit fraction of 1/9 on the number line. She made the discussion of*

proper fractions that they were always between 0 and 1 and 0-1 interval on the numberline should have been equally divided for interpreting fractions.

Comparing PMT X's response to the pre- and post-assessment:

> *The reason for choosing those parts was because the teacher navigates based on her questions or answers the students gave. Instead of giving the correct information, she tried to get students' attention by intentionally giving incorrect information in parts that must be emphasized. She tried to show that unit fractions are in fact proper fractions and therefore they are located in the interval between 0 and 1.* (PMT X, Part3, Pre, Code:1)

Qualitative analysis of the pre-assessment response was categorized as "emerging evidence" on attending and interpretation as there are not enough mathematical details, no focus on individual or whole group student understanding failed to provide a suggestion as instructional decision-making. Therefore, the response was coded as 1 for all dimensions.

Below is the same teacher candidate's post-assessment response to the same video excerpt:

> *In fact, the thing Student E said is correct, but its reason should have been investigated. His answer could have been examined further with questions like 'How did you understand that it was a unit fraction? What does unit fraction mean? What are the criteria for being a unit fraction?' If our answer is to get "1/9", then the question might have been asked as 'What was the amount of chocolate that Meryem ate?' If the question were asked in this way, students' misconceptions might have emerged. Maybe, students will respond as 1 by thinking about whole numbers instead of fractions…. It was nice that the teacher questioned the reason for number line's being constructed between 0 and 1 and students could explain the reason. It was told that proper fractions and unit fractions are located between 0 and 1. It was nice to partition the number line into parts with different sizes for getting student awareness. I think it got attention from the students and it was good to check whether students internalized the equal part situation.* (PMT X, Part3, Post, Code:3)

In the post-assessment, PMT X made a quality discussion focusing on 'equal' partitioning. Comparing pre-post assessment responses, it is evident that PMT X provided pedagogically and mathematically appropriate and specific instructional suggestions (decision-making), focused on student understanding and mathematics in the video segment (attending), and focused on the relationship between teacher actions and student understanding in the post-assessment and provided evidence for the claims (interpretation). This post-assessment response was considered as robust-evidence and coded as 3 for all three dimensions of noticing.

In summary, the results indicated that noticing skills of PMTs could be improved by way of OLS activities, including lesson analysis practices. Incorporating lesson analysis is a frequently used method to enhance teacher noticing (Santagata & Angelici, 2010). Different from previous interventions, our study combines lesson analysis with online planning, teaching and reflecting activities in the context of building learning communities (Wenger, 1998) of prospective teachers.

In line with previous literature, the results of this study demonstrated that PMTs' noticing skills could be supported with carefully designed interventions (Barnhard & van Es, 2015; Schack et al., 2013; Stockero et al., 2017; Ulusoy & Çakıroğlu, 2020). In previous studies, it is acknowledged that decision-making is particularly difficult for both PMTs and teachers (Jacobs et al., 2010; Shack et al., 2013). A combination of being involved in lesson planning cycles, online teaching, and reflection individually and as a group (Fig. 1) may be associated with enhanced decision-making skills of PMTs as a result of the intervention. Unlike previous studies (Santagata et al., 2021), PMTs in our study were actively involved in teaching rather than solely viewing and reflecting on video content. Similar to previous studies (Ulusoy & Çakıroğlu), interpretation of PMTs were enhanced as a result of the intervention. This may be explained by providing reflection opportunities for PMTs both in lesson planning process and after lesson implementation as well as getting familiar with lesson analysis. PMTs had a chance to analyze relationships between teaching and student learning both individually and as a group during reflection meetings.

Existing literature has not focused much on teachers' noticing skills in middle schools or online teaching. In general, the interventions did not involve active teaching practices. The present study contributes to studying teacher noticing in an online teaching context as well as teaching middle schoolers actively. Future interventions similar to OLS may investigate PMT noticing in different contexts and with a larger number of participants. In this study, the dimension of attending was improved, but not in a statistically significant way. There is a need for further investigation about why attending scores did not improve as much as the other dimensions. This may be due to the nature of video assessment. There is still much to know about the design and incorporation of types of videos in the context of noticing studies (Santagata et al., 2021). Alternatively, it may be the result of close pre-post assessment scores in this dimension. Future studies may also investigate further whether PMTs' noticing skills of video content is related to their performance in online teaching or not, particularly their decision-making skills during teaching.

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STUDYING MATHEMATICS TEACHERS' DESIGN OF TASKS INSPIRED BY AUTHENTIC PRACTICES

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This paper reports a case study research aiming to explore the potential of authentic workplace situations in mathematics teaching in upper secondary classes. For this purpose, seven teachers participated in a group aiming to connect the teaching and learning of mathematics with the marine navy context (ship navigation). We use the notion of the mathematical working space to compare the tasks designed by two of the teachers for their lessons inspired by authentic ship navigation practices. The results indicate substantial differences of the two designed working spaces in terms of their semiotic, instrumental and discursive dimensions.

INTRODUCTION

In recent years there has been consensus among researchers about the shift in mathematics teaching for the 21st century to promote making real-world connections (Gravemeijer et al., 2017). Therefore, a discussion has emerged among researchers about the potentiality of using authentic workplace tasks in mathematics classrooms by suggesting the idea of using authentic practices as a source of inspiration for designing educational materials (Dierdorp et al., 2011). Many researchers find the above idea very promising since authentic workplace practices are rich and meaningful and offer students' chances for inquiry activities; engage students with challenging problem-solving practices and support students' development of mathematical reasoning skills (e.g., Dierdorp et al., 2011; Wake, 2015). However, workplace research informs about the complexity of identifying mathematics in professional practice. Researchers in this field argue that school and workplace mathematics are different practices with different goals, types of tools, and genres of mathematical language and community rules while workplace mathematics is black-boxed in professionals' routine tasks (Williams & Wake, 2007). Hence, to reach a modus vivendi between authenticity and classroom mathematics teaching seems to be a challenge for teachers who should provide students' proper familiarization with workplace tools and discourse and at the same time engage students in a rich mathematical activity (Nicol, 2002). Research has highlighted the need for more research on how a workplace context orients a working space putting under investigation how teachers might introduce this working space in their teaching as a mathematical working space (Kuzniak et al., 2016). The aim of the reported study is to contribute in this direction by exploring the potential of naval navigation as a context for mathematics learning in secondary schools. Our focus is on the choices made and

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the challenges faced by secondary teachers when they are engaged in designing tasks for their mathematics classrooms inspired by authentic ship navigation practices.

THEORETICAL FRAMEWORK

Mathematical working spaces (MWS) offers a framework relevant for the study of mathematical work in an educational context. Under this perspective, mathematical work is understood as an intellectual work of production, the development of which is oriented and supported by mathematics.

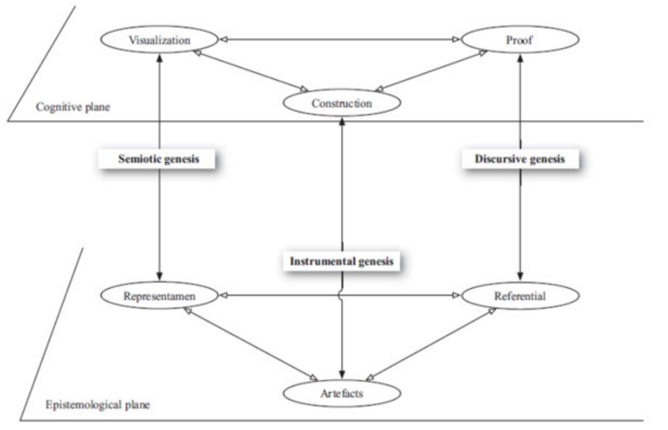

Figure 1: The MWS diagram

According to Kuzniak et al. (2016), an MWS consists of two planes, the epistemological and the cognitive (Figure 1). The epistemological plane is related to the mathematical content and the tasks that will take place, while the cognitive is related to the students' way of thinking and the reasoning followed during their work on a task. The two levels are each made up of three components. More specifically, epistemological plane consists of: a set of distinct objects (representamen); a set of objects such as drawing instruments or software (artefacts); and a theoretical referential framework consisting of definitions, properties and theorems. The cognitive level is composed of the following three components: visualisation, which corresponds to the creation, manipulation and interpretation of the symbols of the specific representation of each field of work; the construction, that refers to the form of reasoning that depends on the tools used and the relevant techniques; and the proof that emerges through the creation of mathematical arguments and validation. Under the lens of MWS, the mathematical work of an individual is evolving through intertwined generative developments (i.e., geneses) between the epistemological plane and the cognitive plane defining three dimensions: semiotic, instrumental and discursive (Figure 1). The semiotic dimension concerns the use of algebraic symbols, geometric representations, graphs, diagrams, etc. That is, it creates connections between verbal, conceptual, functional expressions and geometric constructions with symbolic expressions. The instrumental dimension refers to treating objects as tools and using them for the necessary mathematical and non-mathematical constructions and is used to explain how artefacts are transformed into learning tools through the interaction of teachers and students. Finally, the discursive dimension refers to processes of justification and proof and concerns the production of mathematical meanings. In this paper, we use MWS as a tool to analyze tasks designed by teachers for engaging their students in ship navigation activities in the classroom. In terms of the MWS theory, we analyze how the *suitable MWS* is shaped by teachers' design choices. Our focus is on how teachers conceive authentic practices and adapt them for their lessons and how the dimensions of mathematical work defined by MWS are considered in their designs.

METHODOLOGY

A group was set up for the study consisting of seven mathematics teachers working in different schools, two researchers, and one researcher/teacher who conducted the research (Vroutsis et al., 2018). The main goal of the group was to inform the teachers about the context of the workplace, and to design and implement authentic tasks inspired by naval navigation in their classrooms. The group was supportive in providing feedback on task implementation. The researcher/teacher had experience of ship navigation; he studied official navigation textbooks and collaborated with a professional captain in order to get more familiar with the workplace. There were four group meetings over three months. The researcher/teacher acquainted the group with the context of the workplace, the naval chart, the captain's authentic tools, and the original measurements used for fixing the ship's position through relevant professionals' videos.

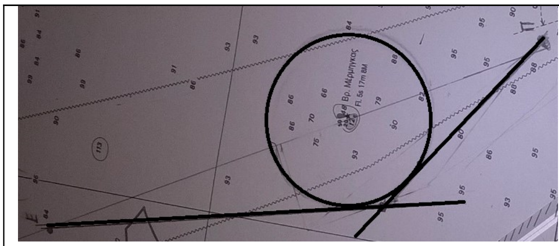

Figure 2: Avoid Obstacle

The teachers were introduced to the following authentic tools and measurements: nautical chart, nautical divider (its legs end in sharp edges and are longer than the legs of a common divider); parallel rulers (two connected rulers moving in parallel lines); compass rose (protractor integrated to the nautical chart); bearing (the clockwise angle between the direction of an object and that of true north); range (the distance between two objects). In addition, the teachers had the opportunity to experiment with the above elements of the workplace through small tasks given to them (e.g., "Plot and determine the course from the port of Syros to the port of Naxos through the use of bearing"). Finally, the researcher/teacher introduced the group to the authentic practices of the captain provoking discussion about the mathematical content that is black boxed in these practices. Later the teachers designed and implemented tasks inspired by the aforementioned professional's authentic practices. In this paper, we focus on two teachers' (A and B) designs. Both teachers had a master's degree in mathematics education while Teacher A had experience in the workplace as he had served in the Navy. The tasks were implemented in two general secondary education schools (grade 9 classes, teacher B; grade 10 classes, teacher A). We analyse the tasks of the two teachers inspired by the authentic practice "Avoid Obstacle". Professional captains apply this practise to avoid an obstacle in ship's course. They consider one imaginary circle around the obstacle (safety distance). The new route consists of two tangent lines in the circle, one from the starting point and one from the destination (Figure 2).

The collected data consisted of: transcriptions of recording of the group meetings; teachers' personal notes; teachers' resources and materials (lesson plans, worksheets); semi-structured interviews of the two teachers after the implementation. The analysis was performed in two phases. Initially, each subtask in the worksheet presenting the main task was analysed in the light of the MWS through the triplet of dimensions (semiotic, instrumental, discursive) and their elements (e.g., visualisation, construction). In the next phase, teachers' design choices that emerged from the analysis of worksheets were cross-analysed with the teachers' interviews so as to synthesize design choices and underlying reasons.

RESULTS

Both teachers noted the importance of familiarizing students with the workplace. Teacher A said, *"I had no particular problem communicating the tasks to the students as in the first two hours of the implementation the students had acquired the skills to handle the new elements brought from the workplace."* Thus, reaching the task "Avoid Obstacle" the students had been acquainted to the basic elements of the workplace, needed to get involved in the main tasks, through small activities. Table 1 lists the task of teacher A. With bold writing in the left column we have located the quotes of the task that we focus on based on the three dimensions of the MWS. In the left column, we quote the corresponding dimension of the suitable MWS in which we include it.

We want to go from Kavia Bay (**southwest in Kea island**) to Mavriana Bay (**southwest in Kithnos island**). For **safety reasons** throughout the course, we should not approach Cape Tamelos **less than two nautical miles.**	*Semiotic* (workplace terminology and restriction, visualizing the distance between two landscapes on the nautical chart').
Q1: **Locate** the forbidden **area on the map**.	*Instrumental* (plotting the desired area as a circle - *construction*). *Discursive* (reasoning with use of geometric locus).
Q2: **Map the route** that you consider to be the shortest possible length.	*Instrumental* (plotting tangents - *construction*). *Discursive* (reasoning on the shortest length).

Q3: **Record the courses zl_1, zl_2** that you used.	*Semiotic* (interpreting symbolic authentic notation, using bearings to determine zl - *visualisation*). *Instrumental* (using authentic artefacts, parallel ruler and compass rose). *Discursive* (reasoning on ship's course though).
Q4: Find **the total length** of the route.	*Semiotic* (interpreting symbolic authentic notation, visualizing the segment as distance of two landscapes). *Instrumental* (using authentic artefacts, nautical divider). *Discursive* (reasoning on ship's course length).
Q5: **Record the coordinates** of the point S at which **a change of course** takes place	*Semiotic* (interpreting symbolic authentic notation, geographical coordinates - *visualisation*). *Instrumental* (using authentic artefacts, position fixing on the nautical chart).
Q6: The captain of the ship decided to turn when he has **the proper bearing** of Cape Tamelos. **Can you figure** out what **this bearing**? **How many degrees** will the ship turn at point S?	*Semiotic* (interpreting symbolic authentic notation, ship turn, angle - *visualisation*). *Instrumental* (using authentic artefacts, parallel ruler and compass rose; parallel line displacement). *Discursive* (calculating ship's turn through subtraction of bearings).
Q7: We consider an alternative route in which we start with the path zl_1 that you calculated before, **continue one nautical mile** after the point of change of course of the previous route and then turn. **Calculate the new path zl_2** that we must now follow to move at the place of destination.	*Instrumental* (using authentic artefacts, nautical divider and parallel ruler). *Discursive* (calculating ship's course through the use of bearing zl).
Q8: Can you **use your knowledge of Geometry to show that** the new path is necessarily longer than the original?	*Discursive* (reasoning through mathematical proof).
Q9: Confirm the previous one by **recording the length** of the new route.	*Instrumental* (using authentic artefacts, measuring on the map with nautical divider). *Discursive* (justifying on Q8).

Table 1: Teacher's A suitable MWS

We present in the same way the analysis for teacher's B task in the Table 2. It is obvious that teacher's B task is shorter. Also, the quotes in parentheses in the left column are explanations that the teacher himself had added to help the students.

You are **in the role of "cadet"** who helps the captain set the course and steer your boat properly. You have in your hands a text of the sailor, which probably describes the path to a forgotten chest. The text says: "From where we left the key of the chest to go to Cape Cyclops (Serifos Island) you have to travel **at least 12.27 n.m.,** while from Agios Dimitrios (Kithnos Island) you will travel **at least 11.4 n.m."** Don't waste time**. Where are you heading?**	*Semiotic* (visualizing a situation of treasure hunting, using authentic data and marine measurement units). *Instrumental* (using authentic artefacts, position fixing on the nautical chart; making specific measurements with nautical divider). *Discursive* (using authentic measurements to identify position fixing through two ranges).
Q1: What will be **your course? (Zλ =….)**	*Semiotic* (interpreting symbolic authentic notation, using bearings to determine zl - visualisation). *Instrumental* (using authentic artefacts, parallel ruler and compass rose, parallel line displacement). *Discursive* (calculating with authentic measures the course bearing zl).
Q2: **Plot the route** from the Baths of Kythnos to the chest position, **avoiding the dangerous waters** and **determine it with zl** = … (the "angle" of the course according to the compass rose of the area).	*Semiotic* (interpreting symbolic authentic notation range, circle radius). *Instrumental* (plotting tangents, use of parallel ruler and compass rose, parallel line displacement - *construction*). *Discursive* (developing mathematical model to overcome a problematic situation).

Table 2: Teacher's B suitable MWS

By recording similarities and differences between the tasks, at the semiotic level we note that both teachers built a narration in order to engage the students into the tasks more effectively. Teacher A presented the tasks through a story about a ship's voyage and the dangers it faces *"I wanted to give students a complete story so they would make sense of their involvement."* Teacher B reported on a treasure hunt, through which he introduced the tasks, giving students the role of a professional *"I gave them a realistic scenario to challenge them and get them into the role of the professional."*

On the other hand, although both give students two tasks, teacher A breaks them into individual small sub tasks while teacher B does not follow the same approach. Explaining his choice, teacher A talks about his anxiety for the students to complete the task and that is why he chose to "guide" them in this way.

> Teacher A: I owe it to my anxiety to complete the task. That way I could guide the students, when needed. On the other hand, I had the option to skip questions that turned out to be insignificant and save time.

Another issue in which the two teachers present differences is their view on the authenticity of the tasks they gave to the students. Both teachers used workplace terminology and jargon. In addition, the measurements given to the students were realistic, and the tasks required the students to handle authentic tools and to interpret the professional's measurements. However, teacher A speaks clearly of a dominant authentic framework while teacher B speaks of realistic rather than authentic tasks.

> Teacher A: The framework is highly authentic (nautical chart, authentic tools and practices), even workplace restrictions affect students' mathematical activity. The context is dominant; overall, the application had clearly authentic character.

> Teacher B: I have doubts about the authenticity, because I do not know the workplace context very well. A professional may have recognized situations, in the tasks that were either incompatible with reality or "ideal". On the other hand, the scenario is realistic, as is the data given to the students and the context itself puts the students in the role of the professional.

At the instrumental level both teachers used authentic tools in their tasks, with which the students plotted ship courses, bearings and distances on the nautical chart. The above constructions had also mathematical meaning for the students, for example, they treated the distance as a radius of a circle and the course of the ship as tangent to a circle. However, teacher A seems to seek, sometimes explicitly, to connect the authentic elements with the mathematical concepts hidden in them (e.g., “How many degrees will the ship turn”; “Calculate the new path zl_2”).

The above discrimination is clearly visible in the discursive level of the MWS targeted through the tasks, which is ultimately the element that differentiates the approaches of the two teachers. Teacher A emphasized the importance of students’ engagement with school mathematics in the new context and prioritised validation within mathematics. This is also manifested in his words (interview).

> Teacher A: I seek students to apply in a new context different from school mathematics the geometric properties hidden in the authentic practical ... Yes, it was my intention to ask for geometric proof and mathematical validation.

On the other hand, teacher B described clearly in his interview his choice to engage students in mathematical exploration within the authentic context.

> Teacher B: I seek for students to build strategies to try them, possibly reject them or adapt them ... It is more in the direction of solving a problem. Although I do not deny the role of math teacher, I would like students to explore their solutions through the new context of the workplace and less with the use of school mathematics.

CONCLUSIONS

We analysed two experienced teachers’ tasks inspired from an authentic ship navigation practise so as to address the design of their suitable MWS. Comparing the two suitable MWS designed by the teachers, the analysis indicated similarities and differences. As regards the similarities, both MWS were based on tasks that: involved explicitly elements of the ship navigation practice; were based on stories related to the authentic situation; were implemented on the nautical chart; and the values of the measurement data given to students were realistic. However, the analysis brought to the fore distinct differences. Teacher B chose to introduce the authentic context through a game-like task (treasure hunting), similar to the "imaginative" tasks reported by Nicol and Crespo (2005), without reference to any obvious mathematical content or aim. The task offered space for students to work with the original measurements and tools, explore, and develop strategies to discover the solution. In the initial questions, teacher A used an authentic story to engage students in the situation providing them the role of a captain who faces an authentic problem (danger). In the next parts of the worksheet the authentic context fades and the tasks become quite guided asking mainly for calculations and mathematical validation. A comparative look at the dimensions of the MWS framework indicates that both suitable MWS are characterised by rich instrumental and semiotic dimensions in terms of mathematics and workplace signs (e.g., terminology) and instruments (e.g., artefacts, measurements). Teachers' choices in the discursive dimension determine the balance between authenticity and school mathematics in the designed tasks and reveal differences. While Teacher B provides space for students to explore the problem and validate it either through mathematics or workplace, Teacher A guides students towards a solution targeting validation within mathematics. Further analysis including classroom data is expected to allow us getting a deeper understanding of how the suitable MWS is transformed in actual teaching and its potential for exploring further what makes authentic tasks meaningful for students.

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PERSPECTIVES OF PROFESSORS IN MATHEMATICS EDUCATION ON FRUIT SALAD ALGEBRA – A COMPARISON BETWEEN TAIWAN AND GERMANY

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This study investigated intercultural differences in experts' perspectives on the appropriateness of the use of mathematical representations. A total of 19 Taiwanese and 12 German experts (professors in mathematics education) participated in an online survey to evaluate the use of representations in a text vignette illustrating a classroom situation in which the teacher used fruit salad algebra. Through content analysis, this study revealed that the majority of German experts criticized the inappropriateness of the representations for causing letter as object misconceptions, whereas the perspectives of Taiwanese experts were dominated by practical concerns related to whether the use of the representations was a waste of time or could help students successfully perform mathematical operations on the symbols.

INTRODUCTION

The use of representations, as both something to be taught and something to aid learning (Cai & Wang, 2006), has long been a key issue in mathematics education. However, are perspectives regarding the appropriateness of the use of representations universal across cultures? Teachers' perspectives on good or proper mathematics teaching have been investigated in cross-cultural studies, and the findings have revealed that the definitions of good or proper mathematics teaching vary widely between cultures (e.g., Bryan et al., 2007). As those responsible for training teachers, the perspectives held by professors in mathematics education usually reflect the features of expected instructional quality in their cultures. Therefore, this study examines how perspectives of professors (experts) in our international mathematics education community on the appropriateness of representations used in mathematics educational contexts can vary between different cultures, with Taiwan and Germany representing an East Asian and a Western country. Specifically, we focus on how experts in mathematics education in these two countries evaluate the use of representations in a classroom situation where the teacher uses fruit salad algebra.

THEORETICAL BACKGROUND

Appropriate use of representations

In mathematics, representations cannot be understood in isolation; they are embedded in a wider structure where meanings and conventions are established and where they complement and richly relate to one another. The appropriate use of representations

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can help students successfully construct mathematical concepts and procedures, whereas their inappropriate use hinders student learning and gives rise to misconceptions (Goldin & Shteingold, 2001).

One well-known student misconception caused by the improper use of representations (so-called "fruit salad algebra") is the *letter as object* misconception, (Küchemann, 1981). An example of fruit salad algebra is the use of images of apples (as the referent of a) and bananas (as the referent of b) to represent $2a + 3b$ (Chick, 2009), where a letter is reinforced to be regarded as an object rather than as an unknown or variable. Küchemann (1981) indicated that this reduction in the meaning of the letters from something abstract to something concrete and "real" allowed many students, who had problems with variables, to successfully deal with symbols. However, this reduction hindered subsequent learning when it became essential to substitute numbers for letters, to execute further operations (e.g., multiplication), or to form a relationship between variables. One famous example is students representing the statement *six times as many students as professors* as 6S = P rather than 6P = S in the mathematical problem formulated by Clement (1982). In Chick's (2009) study, few teachers were aware that fruit salad algebra is inappropriate in the first place, and more than 70% of that study's teachers indicated interest in using such a representation model in the future. Is this a problem of teacher education? Or can this phenomenon also be seen among experts in mathematics education? Despite the long existing (Western) literature on the problems of fruit salad algebra, it is not clear whether a critical stance on such use of representations in algebra can be seen as a consensus among scholars in our intercultural mathematics education community.

Differences in mathematics education in East Asian and the Western cultures

Different identities in mathematics education deeply rooted in East Asian and the Western cultures have been identified in the literature. For example, Leung (2001) pointed out the dichotomy of emphasizing the final product versus the learning process in these cultures. The emphasis on the product in East Asia aligns with the findings of Pratt et al. (1999) that the aim of learning is to get foundational knowledge including factual knowledge, principles, and procedures, which is usually the content in the examinations. A series of studies empirically investigated common as well as distinct characteristics of effective teaching in different cultures (e.g., Bryan et al., 2007; Wang & Cai, 2007). East Asian and Western teachers were reported to emphasise the structure vs pragmatic aspects of mathematics, respectively. Teachers from both cultures valued learning processes that start from the concrete and move on to the abstract. Nonetheless, teachers from East Asia viewed concrete representations as only an initial crutch the student uses to gain facility with abstract mathematical concepts and skills while Western teachers viewed them as a thinking tool that students may continuously use.

Studies have further investigated the features of ideal mathematics pedagogy in East Asia after decades of Western influence on education (e.g., Hsieh et al., 2017). The findings revealed that some of the factors endorsed by teachers and students are rooted in traditional Chinese educational culture (e.g., conceptual connection and meaning in handling teaching materials) and some are influenced by Western culture (e.g., concrete representations and those grounded in everyday life). In addition, some factors rooted in traditional culture were looked upon less favourably (e.g., an emphasis on speed and challenge in problem solving).

Studies comparing expert perspectives of characteristics on good mathematics teaching in different cultures are scarce. The Teacher Education and Development Study in Mathematics (TEDS-M) reported some similarities and differences between cultures not only for teachers but also for professors. For example, both Taiwanese and German professors who trained mathematics teachers valued active-learning approaches more than teacher-directed approaches in the teaching and learning of mathematics. However, regarding the nature of mathematics, approximately 60% of Taiwanese educators believed equally in the primacy of both, the process-of-inquiry and the rules-and-procedures aspect, and approximately 40% believed that the process-of-inquiry aspect is the more important of the two. In Germany, most educators believed in the primacy of the process-of-inquiry aspect (Wang & Hsieh, 2014).

RESEARCH METHOD

Instrument

This research report is part of the findings of a Taiwanese-German cooperative research project (TaiGer noticing) to explore the perspectives of experts on characteristics of good mathematics teaching, and teacher professional noticing (Dreher et al., 2021).

The teacher T introduces *combining like terms* in the unit *linear equations in two unknowns.* T takes out picture cards and puts them on the blackboard (as shown on the right).

T: Today we will learn what like terms are and what to do with them by using a real-life example. These are the candies I just bought. If you would sort them, how would you do that?

S: I would put the lollipops together and put the gummy bears together.

T: Very good. The candies that look alike are of the same sort and can be combined. And it's the same with combining like terms. Now, we have $3x+2y+2x+x+y$ [writing on the blackboard]. Each term of this expression corresponds to a candy card. So, we combine the like terms corresponding to the lollipops, this makes $6x$. We also combine the like terms corresponding to the the gummy bears, we get $3y$ [moves the candy cards as shown]. That is, we add the coefficients of like terms, $3 + 2 + 1 = 6$ and $2 + 1 = 3$.

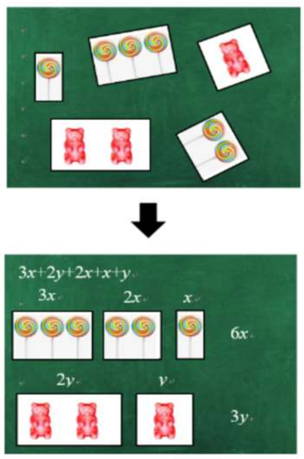

Figure 1: Vignette on the use of representations

The research team designed 18 vignettes, in which classroom situations and something not meeting our expectations of good teaching were depicted (breach-experiment), to elicit the experts' perspectives on a certain aspect of instructional quality and the teachers' noticing. In this report, we focus on a vignette representing a classroom situation where the teacher uses fruit salad algebra (Figure 1). The team members from both countries agreed that the visual representation used by the teacher in the vignette was inappropriate; the picture cards used to represent the concept of combining like terms were unable to properly represent the nature of variables. The experts were prompted to evaluate the situation by the open-ended question ‘Please evaluate the teacher’s use of representations in this situation and give reasons for your answer.’

Sample and data collection

The survey was conducted online in the native language of the survey respondents. We recruited professors who are (1) active in mathematics education research and (2) active in preparing future secondary mathematics teachers. We aimed for a sample of 15 experts in each country. With the assumption of a participation rate of at least 50%, a random sample of 30 professors out of the full list of those meet these criteria was contacted in Germany. The criteria yielded a list of only 32 professors in Taiwan and thus all of them were contacted. In the end, 19 professors from 10 universities in Taiwan and 12 professors from 10 universities in Germany worked on the targeted vignette. Some experts had experience as school teachers (Taiwan: 14, Germany: 12) and some had experience as researchers in mathematics (Taiwan: 5, Germany: 4).

Data analysis

All the responses were translated into English, and the materials for coding were presented in both English and the native languages of the experts to facilitate the process of coding and discussion (for details see Dreher et al., 2021). The experts’ evaluations were analysed mainly regarding two aspects, which were (1) whether they see the teacher’s use of representations was inappropriate and (2) in which way it was inappropriate. With regard to the second, the expert responses were coded using a combination of top-down and bottom-up processes. The inappropriateness of the use of representations as seen by the research team (the picture cards used to represent the concept of combining like terms were unable to properly represent the nature of variables) made up one code, and other codes were determined through inductive analysis of the responses. The research team members analysed the responses in a first round to inductively extract possible codes within each culture, and then we determined the final coding scheme in a cross-cultural discussion. In addition to the original reason for inappropriateness, one major other reason emerged in the coding: some experts criticized that the representations used were limited in its functions and possibilities for extension. In the second round, first within-country coding was performed. All the responses were coded by two coders, and the discrepancies were resolved through discussion. Subsequently, a discussion was held between the Taiwanese and German researchers to reach a consensus.

RESULTS

In total, 63% of the Taiwanese experts and 92% of the German experts mentioned that the teacher’s use of representations, as displayed in Figure 1, was *inappropriate* (Table 1). The Taiwanese and German experts provided different reasons why. Among those evaluating the use of representations negatively, most of the German experts (73%) indicated the inappropriate use originally set by the research team, whereas only 33% of Taiwanese experts indicated this inappropriateness. These experts mentioned that using cards depicting candies to represent variables as displayed in Figure 1 is not appropriate and could cause *letter as object* misconceptions in students. The following answers are typical examples from both cultures.

TW2: This is an old issue; although the teacher was teaching the combination of like terms, did the picture actually represent an object or a number? The teacher did not clearly know about the misconception that his approach could have possibly caused.

GER3: […] there is the impressive example in which interviewees have translated the situation 'For each professor there are six students' into the equation P = 6S. Behind this is the substantial misconception that a variable like "P" is an abbreviation for an object like "professor". In fact, however, variables do not stand for objects, but for NUMBERS of objects. Exactly this misconception is however provoked by the situation described: "x stands for lollipop", so "3x stands for 3 lollipops". Insofar, the object "lollipop" is a very bad representation of the variable x. […]

	Taiwan (N=19)	Germany (N=12)
Evaluated the use of representations as being inappropriate	12(63%)	11(92%)
Reasons for inappropriateness		
Indicated the inappropriateness of representing variables by objects	4(33%)	8(73%)
Indicated that the representations used were limited in its extension or functions	9(75%)	3(27%)

Table 1: Frequencies and percentages of experts in each case

Among those evaluating the use of representations negatively, most of the Taiwanese experts (75%) indicated that the use of the representations was limited. Their views fell mainly into three categories. Specifically, three experts said that the representation limits the coefficients to only natural numbers (TW5). Two experts mentioned that the representation did not suggest that you cannot combine xs and ys in terms of addition because lollipops and gummy bears can, in real life, be combined as candies (TW15). All of these concerns focused on whether students could proceed the operations correctly when doing future tasks. Another three Taiwanese experts mentioned that the students were cognitively mature enough to grasp the concepts without this (basic) card representation, which may indicate that the extra effort of making or using cards was not necessary (TW19). Three German experts also indicated limitations of the representation with one falling into each of the three categories.

TW5: Teacher T analogised different terms to the different categories regarding lollipops and gummy bears and used the comparison between symbolic representations and picture cards (connecting different representations and contexts) to establish the meaning of the addition of the terms. This was helpful for understanding and applying the concept of combining like terms. However, difficulties would occur when directly applying such representations to certain special cases, such as the subtraction of terms with negative coefficients or with positive coefficients that are not integers. […]

TW15: 1. Using the representations of lollipops and gummy bears was well intended, but the meaning of the lollipops and gummy bears corresponding to x and y was not dealt with in the lesson. 2. The students could also have said that a total of nine candies could be eaten without distinguishing between lollipops and gummy bears. […]

TW19: If the lesson was for students aged 12 years and older, I personally think that this representation is probably excessive. If the students have already achieved the cognitive competence for [understanding] *variables* (a symbol representing a value), the teacher can directly manipulate the symbol. I would guess that if students had not yet acquired the ability to manage the multiplication, addition, and subtraction of abstract symbols, it would be futile to switch to using representations of cards and flower counters. […]

The perspectives of the Taiwanese experts revealed that, regarding the use of the representations, they did not address the nature of the letters for being unknowns or variables as much as the German experts did. However, given the high content knowledge of Taiwanese experts, we doubt whether they missed the point that the representation does not properly represent the nature of variables. So even if Taiwanese experts understood the *letter-as-object* misconception, they were more concerned with 'practical' factors, such as whether students could perform the operations correctly in the future and also with the efficient use of time in the class. These may reflect educational beliefs rooted in traditional Chinese culture: Students are expected to grasp the body of mathematics knowledge correctly and perform well in examinations, and thus, whether the representations can help them succeed in solving problems in the future is essential (Leung, 2001). Mathematics instruction should be fast paced to cover the demanding curriculum, and sufficient exercises should be provided for examination preparation; wasting time doing something viewed as unnecessary is thus undesirable.

Notably, as many as 68% of the 19 Taiwanese experts mentioned some positive aspects of the use of the representation, whereas only 2 German experts did so. Some of their perspectives aligned with the function of the reduction indicated by Küchemann (1981): They mentioned that the use of cards was natural and reasonable for student cognition regarding the analogy between the categorisation of concrete objects in real life and the operations performed on the symbols and that this helped students quickly learn how to perform the operations (TW1 and TW13). This phenomenon again reflected the "practical" issue aforementioned. The experts saw the advantages of quickly obtaining the skills to perform operations correctly even if they may understand the potential disadvantages of causing misunderstanding or hindering future learning. Some experts mentioned that the graphical representation was helpful because of its characteristics of being visualizable and manipulable (TW7). The perspectives of these experts are in line with findings regarding the Western influence on the East Asian use of concrete representations (Hsieh et al., 2017; Wang & Cai, 2007).

TW1: The teacher introduced like terms by classifying real-life objects (candy) as a representation, and the students could immediately relate to the situation.

TW13: Appropriate. [The teacher] could use concrete objects to represent the abstract *x*. However […]

TW7: The use of manipulative iconic representations to link to symbolic representations was good […]

CONCLUSION AND DISCUSSION

This study has its limitation that the findings based on the expert evaluation on one vignette regarding the use representation may not be generalizable. However, using this vignette representing a classroom situation that used fruit salad algebra, this research report illustrated how experts' perspectives on the appropriateness of the use of representations can be different in East Asian and Western cultures. The large majority of German experts paid attention to whether the representation reflects the nature of the mathematical concept. From their perspectives, an appropriate representation of letters should reflect the nature of unknowns or variables, and they indicated that using cards of candies to represent algebraic terms could result in a "letter as object" misconception raised by Küchemann (1981). Although Taiwanese experts may recognise that the "letter as object" misconception could arise from the use of cards depicting candies, what dominated most of their perspectives on the appropriateness of the representation use were practical issues in mathematics instruction. Some experts considered the representations to be improper because the use of cards did not match student cognition and resulted in an inefficient use of time in mathematics class, and some indicated that the use of cards depicting candy would lead students to only perceive coefficients as natural (rather than rational or real) numbers and fail to grasp the illicitness of combining different letters. These could result in students being unsuccessful regarding the operations on symbols. Hsieh et al. (2017) explored teachers' perspectives on effective mathematics teaching using a questionnaire without specific teaching situations and revealed that Taiwanese teachers focused on conceptual understanding and meaningful learning over skill preparation and performance in evaluation. The findings of Taiwanese experts' perspectives in this research report seem to be in contrast to that of Hsieh et al. This phenomenon may indicate a gap between conceptions aroused without a specific classroom situation and the dominated perspectives revealed in situational practice.

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THE POTENTIAL OF MATHEMATICAL PICTURE BOOKS: A SYSTEMATIC ANALYSIS OF THEIR DOMAIN-SPECIFIC PICTURE BOOK FEATURES

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Studies in the domain of early literacy and recently also mathematics point to the potential of picture book reading (PBR) activities for children's early literacy resp. mathematical development. Contrasting the domain of early literacy, the contribution of picture book features to the effectiveness of PBR activities in the domain of mathematics is hardly studied. To increase our understanding of the topic, we systematically analysed the domain-specific features of 100 mathematical picture books. Our analyses pointed to the presence of features that previous studies assume to support and to hinder early mathematical development (Ward et al., 2017). These findings complement previous findings based on English mathematical picture books from the U.S. and offer important insights for future studies and educational practice.

THEORETICAL BACKGROUND

Cumulative evidence points to the effectiveness of shared picture book reading (PBR) activities for the development of young children's language and literacy competencies (Mol & Bus, 2011). This effectiveness of PBR activities was shown to be mediated by the content and structural features of the picture book shared with the child, as the occurrence and number of new words included in the book (e.g., Flack et al., 2018) or the presence of manipulative features (e.g., Chiong & DeLoache, 2015). Recent studies in the domain of early mathematics demonstrated the potential of PBR activities for stimulating young children's mathematical development as well (e.g., Purpura et al., 2017; Van den Heuvel-Panhuizen et al., 2016). Contrasting the domain of early literacy, the contribution of the domain-specific features of the picture book to this effectiveness was hardly studied. To the best of our knowledge, only Gibson et al. (2020) addressed this issue, focusing on the contribution of one specific feature, i.e., the numerosities offered in a researcher-developed picture book. Their findings indicated that picture books including smaller numerosities (i.e., 1-3) were more effective than picture books including larger numerosities (i.e., 4-6) for stimulating 2-4-year-olds' early number development.

To deepen our understanding of the contribution of picture book features to the effectiveness of PBR in the domain of mathematics, further research is needed. A critical first step is the systematic analysis of the broad range of domain-specific features of publicly available mathematical picture books in view of providing a comprehensive overview of the features of picture books used during PBR activities.

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This overview will not only be helpful in further scientific research on mathematical PBR. It will also inform picture book authors and designers, and preschool teachers and parents, about the features that need to be considered to adequately design respectively select picture books for stimulating young children's mathematical development.

Recently, Powell and Nurnberger-Haag (2015) and Ward et al. (2017) did a first attempt to analyze the domain-specific content and structural features of mathematical picture books in the U.S. Their analyses included a large sample of publicly available picture books written with the aim to enhance children's counting skills. Their findings revealed that these counting picture books contain both features these researchers assume to promote children's mathematical development, such as numbers in the 1-10 range presented in ascending counting format, and features that they assume to hinder this development, such as the presence of distractor items. Other domain-specific features that these researchers discussed as beneficial for children's early mathematical development, such as the presence of counting principles, were seldomly observed in the counting picture books included in their analyses. Although these two studies add to our understanding of the domain-specific features of mathematical picture books, they suffer from three weaknesses. First, these investigations focused on counting books. Counting is only one subdomain of numeracy and therefore the study covers only a small area within the domain of early mathematics. Second, they did not systematically analyse similarities and differences between books with or without a story context for the complete range of domain-specific features, whereas studies in the domain of early literacy pointed to associations with other domain-specific picture book features and to differences in the learning effectiveness for these two types of books (e.g., Price et al., 2009). Third, their sample only included mathematical picture books written in English and available in the U.S. and cannot be generalised to other languages and countries, as previous research has shown that the content and structural features of picture books tend to differ across languages and countries (e.g., Löwenhielm et al., 2017).

CURRENT STUDY

Taking into account both the strengths and the weaknesses of the studies of Powell and Nurnberger-Haag (2015) and Ward et al. (2017), we aimed to systematically analyse the domain-specific content and structural features of publicly available picture books that are written to stimulate children's counting skills as well as other important competencies within the domain of numeracy, i.e., numbers and arithmetical operations. Moreover, we aimed to analyse the features of picture books written in Dutch and available in a European country (i.e., Belgium, Flanders), thereby evaluating the generalisability of their findings. Finally, to deepen our insights into the similarities and differences in all domain-specific features between picture books with versus without a story context, we explicitly took the presence of a story context into account in our analyses.

METHOD

We included 100 mathematical picture books in the area of counting, number, and arithmetic, written in Dutch and available in public libraries in Belgium (Flanders). Taking into account both the studies of Powell and Nurnberger-Haag (2015) and Ward et al. (2017) and the preschool age range in the country, they further had to be written for children aged 2,5-5 years. We adapted the coding scheme of Ward et al. (2017) in view of our research goals, and coded (1) general book features, focusing on the presence of a story, (2) features of numbers, and (3) features of sets. The complete overview of the features of numbers and sets we scored are shown in Tables 1 and 2. We coded the features of numbers and sets for each picture (picture level), except for the features number range and presence of zero that were scored at the level of the book (i.e., book as a whole) and counting format, which was coded at both the book and picture level. Inter-rater reliability (10 books) for all coded features was strong to almost perfect (k = .83 - 1.00, p < .001).

RESULTS

First, our analysis revealed both similarities and differences in the picture books' features of numbers. About 50% of them presented the full number range 1-10, and about 25% a non-inclusive number range. Only a small minority of the picture books included numbers larger than 10 or the number zero. A substantive amount of books presented the numbers in an ascending order at the book level (e.g., number 1 on page 1, number 2 on page 2, etc.) and at the picture level (e.g., a page including the numbers 1, 2, 3, etc.). Descending counting formats, skip counting and non-sequential formats were far less frequent. Next, most books presented numbers in different representation types (i.e., as Arabic numeral, number word and set), and made links between these different representation types when possible. By contrast, counting principles, number relations and arithmetic operations occurred rather rarely.

	Percentage of books	Percentage books with ≥ 1 case	Percentage of pictures
Range			
1-5	3.0%		
1-10	54.0%		
0-10	6.0%		
1-12	3.0%		
1-30	1.0%		
1-50	1.0%		
Inclusive other (e.g., 0-9)	9.0%		
Non-inclusive (e.g., 1-9, 12)	23.0%		
Presence of zero	16.0%		
Number representation type			

Arabic numeral		79.0%	78.0%
Number word		88.0%	68.7%
Set		98.0%	79.9%
Links between number representations			
Sets and Arabic numerals[a]		93.0%	93.4%
Sets and number words[a]		91.0%	90.5%
Number words and Arabic numerals[a]		90.0%	88.9%
Counting format			
Ascending	69.0%	41.0%	24.5%
Descending	12.0%	10.0%	13.5%
Skip-counting	2.0%	2.0%	7.5%
Non-sequential	17.0%	34.0%	25.3%
Counting principles			
One-to-one correspondence		16.0%	19.5%
Constancy		1.0%	8.0%
Cardinality		8.0%	21.2%
Ordinality		5.0%	27.2%
Number relations			
Number and set comparison		3.0%	9.0%
Decomposition		13.0%	10.5%
Mathematical operations			
Addition		12.0%	16.3%
Subtraction		12.0%	21.4%
Multiplication		5.0%	6.0%
Division		5.0%	9.0%

[a] Links were scored when there was a possibility for a link (i.e., the two number representations occurred on the picture), which resulted in: sets and Arabic numerals n = 75, sets and number words n = 85, number words and Arabic numerals n = 61

Table 1: Frequency of features of numbers in mathematical picture books

Second, the picture books were highly similar in how they featured sets. Almost all books presented sets of real non-identical objects. The objects were mostly of similar item sizes and randomly arranged. Half of the books presented the sets at least once without any distractors, but most books added more than six distractor items to the sets on at least one page.

	Percentage books with ≥ 1 case	Percentage of pictures
Item type		
Real objects	98.0%	95.9%
Shapes	4.1%	35.8%
Non-tangible referents	21.4%	20.6%
Consistency of item type		

Identical items	9.3%	46.9%
Non-identical items	100.0%	98.6%
Item size		
Exact same size	10.3%	47.2%
Similar size	97.9%	77.5%
Discrepant size	80.4%	42.4%
Set arrangement		
Linear	94.8%	38.6%
Canonical pattern	48.5%	22.3%
Randomly	97.9%	72.7%
Set differentiation		
Distinct	99.0%	71.0%
Overlapping	88.7%	46.6%
Obstructed	34.0%	18.9%
Number of distractors		
No distractors	49.0%	32.7%
1 distractor	40.8%	23.1%
2-3 distractors	57.1%	20.8%
4-6 distractors	38.8%	18.2%
>6 distractors	88.8%	62.7%
Consistency between number representations		
Consistent[a]	100.0%	95.2%
Inconsistent[a]	36.2%	23.1%

[a] Consistency between number representations was only scored when there was a link present between sets and Arabic numerals or sets and number words, which resulted in $n = 94$ for this feature

Table 2: Frequency of features of sets in mathematical picture books ($n = 98$)

Finally, we analyzed the domain-specific features of mathematical picture books with versus without the presence of a story ($n = 73$ resp. $n = 27$) via chi-square analyses for categorical and Mann-Whitney U tests for continuous features. These analyses indicated that these two types of picture books differed on various picture book features, namely the number range presented, number representations offered and inclusion of mathematical operations (features of numbers) and item type, item size and presence of distractor items (features of sets). But after Holmes-Bonferroni correction for multiple comparisons, all these differences became insignificant.

DISCUSSION

We aimed to systematically analyse the domain-specific features of mathematical picture books written in Dutch and available in a European country, with special attention for the presence of a story context and its association with these features. Our findings generally replicate those from previous studies on English picture books in the U.S. First, most picture books presented numerosities in the 1-10 range in a complete and ascending counting format, and presented these numerosities in different representational forms, which previous studies consider beneficial to stimulate early mathematical development (Ward et al., 2017). However, other features that these researchers claim equally valuable as the presence of the number zero and counting principles were hardly observed, and – according to these researchers – non-beneficial features as the presence of distractor items were omnipresent, suggesting missed opportunities to enhance young children's mathematical development. We found no differences in the features of picture books with versus without the presence of a story context after correction for multiple comparisons. Together, our findings point to the generality of the findings of the limited number of previous studies. These findings have important implications for future studies and for educational practice. First, future researchers use them to carefully select adequate picture books for evaluating the effectiveness of PBR activities in the domain of mathematics in view of picture book features. Second, the systematic and comprehensive overview of domain-specific picture book features informs picture book authors and designers about the adequate design of such books in the domain of mathematics. Likewise, it offers preschool teachers and parents building blocks to carefully select picture books in view of enhancing their children's mathematical development.

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DIGITAL TECHNOLOGIES AND THE DEVELOPMENT OF THE DYNAMIC VIEW OF FUNCTIONAL THINKING

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Functional thinking can be characterised by three properties, namely the assignment aspect, the covariation aspect, and the object aspect. Furthermore, static and dynamic views can be distinguished. In this article, an empirical study is presented that investigates the development of the dynamic view when working with linear functions through the use of digital technologies in the context of suitably selected tasks. The results show prerequisites and necessities for the integration of digital technologies into a constructive teaching concept for the development of functional thinking.

FUNCTIONAL THINKING – STATICALLY AND DYNAMICALLY

At least since the Merano Reform of 1905, essentially initiated by Felix Klein (1849–1925), the goal of developing "*functional thinking*" has been emphasised for mathematics education (Krüger, 2019). For this concept, which was used somewhat diffusely for a long time, Vollrath proposed in 1989 a definition that at first seemed almost tautological:

> "Functional thinking is a way of thinking that is typical for dealing with functions." (S. 6)

By this turn of phrase, the thinking of a person that cannot be directly observed, which can only be inferred and interpreted from the actions or verbalisations performed, is brought back to the mathematical level. Thereby, functional thinking is closely linked to the mathematical concept of function and becomes accessible through the representation of and dealing with functions. In the frame of this approach, three aspects or characteristics of working with functions are distinguished (e.g., Vollrath 1989; Doorman et al. 2012):

- *Assignment aspect*: a function creates a relation between two variables.
- *Covariation aspect*: a *function* describes how changes in the independent variable affect the dependent variable.
- *Object aspect*: a function can be seen as a whole and therefore be dealt with as a mathematical object.

While the *assignment aspect* emphasizes the functional relationship f: x → y selectively by assigning each element x of the definition set to an element y of the values range, the *covariation concept* considers *changes* in the variable x and their effects on the variable y.

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These two aspects already allow two views of the concept of function and functional thinking: the *static* and the *dynamic view*. The *assignment aspect* represents the *static view* by referring to a (fixed) pair of values, while the *covariation aspect* emphasizes the *dynamic* view of functions and functional thinking by thinking of *changes* of values. These changes do not have to be carried out in reality, they can also be mentally applied to functional relationships.

The *object aspect* describes a function as a whole, e.g., as linear, quadratic or exponential function, as periodic or monotone function. It can be seen *statically* and *dynamically*. The *static view* is emphasized when functions are added, subtracted, multiplied or divided. The *dynamic view* means varying the object, e.g., varying parameters of a function equation and consequently also varying its representation, e.g., graphs or tables. Concerning linear functions, the change of the parameters m or b in the function equation $y = m \cdot x + b$ leads to a family of straight lines which can be represented either simultaneously or successively in time. The latter can give the impression of a moving straight line.

DIGITAL TECHNOLOGIES AND FUNCTIONAL THINKING

Digital technologies open up the possibility of creating different representations of functions in a simple way and to obtain dynamic representations. They especially support the development of the *covariation* and *dynamic view* of the *object aspect*. E.g., graphs of parameter-dependent functions can be represented as time-varying graphs. Tables can be successively subdivided by smaller step sizes and thus environments of interesting points can be viewed with a "numerical magnifying glass". However, the high cognitive demand on learners should not be underestimated with such technically simple but not always immediately comprehensible changes in representation (Dreher et al., 2013).

An answer to the question *how* digital technologies should be used to support the development of the dynamic view of functional thinking might be given by using the *operative principle*. According to Piaget, understanding of a concept begins with actions (e.g., Dubinsky & Harel 1992). An *action* is a repeatable manipulation of objects and hence cannot be viewed on its own but must be considered in relation to the objects, as well as the subject who is performing the action. If these actions are repeated and reflected upon, they can be interiorized as flexible mental processes, so-called *operations*. The development of these operations can be initiated by the *operative principle*, which might be—quite roughly—expressed by the characteristic question “What happens to, … if …” (for more details see Günster and Weigand, 2020).

Example: Given is a square with the side length as an independent and the perimeter of the square as the dependent variable. What happens if the side length is changed?

What happens if the square is substituted by a regular polygon?

With e.g. Geogebra and the use of a "slider" the side length can be changed dynamically and the impact on the perimeter can be viewed (e.g., in a graphical representation). Then, the square can be substituted by regular polygons, e.g. a regular triangle and a regular hexagon, with the same side length as the square in the starting condition. (Figure 1)

The dynamical view is emphasized on the one side by the aspect of co-variation concerning the variation of the side length, and on the other side by the object aspect when investigating and comparing different polygons.

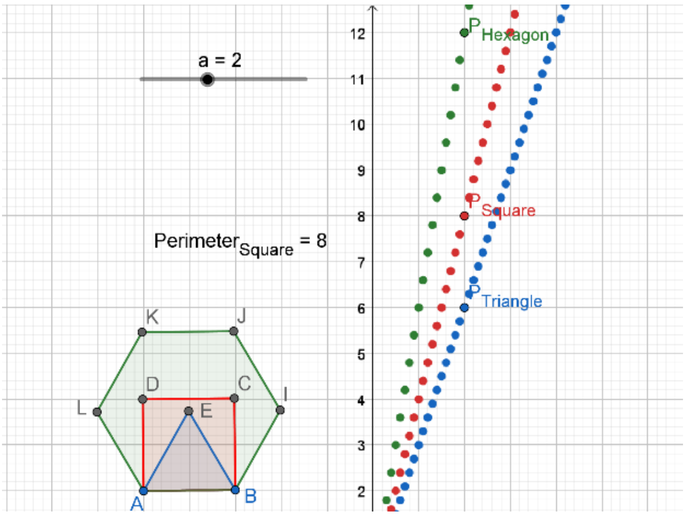

Figure 1. The relation between the side length and the perimeter of regular polygons

RESEARCH QUESTION

This study on the development of the dynamic view of functional thinking is integrated into an on-going larger research project that investigates the importance of the operational principle for the development of functional thinking (see Günster 2019). To this end, a model is developed and empirically tested according to which tasks for the use of tablet computers in grade 8 can be constructed and their use in lessons planned and implemented in order to develop functional thinking. In the context of this article, we focus on a specific

Research question: Does the use of digital technologies in problem solving situations with linear functions promote the development of the dynamic view of functional thinking, especially with regard to the covariation and the object aspect?

METHODOLOGY

To answer the research question five tablet classes at four German secondary schools (Gymnasium) were monitored for one entire school year. In the frame of the common curriculum contents the students were regularly given additional theory-based developed tasks, which emphasized the dynamic view of working with functions.

Participants

The 8th grade students used tablet computers for the whole school year (two of the schools used iPads, one Android-Tablets and one Microsoft Surface computers). The students attending the participating schools could choose, at the end of the 7th grade, whether they continued in a class using tablets or not. 5 classes, taught in the traditional way, without tablets, built the control group. The students were between the age of 14 and 15 and represented the full range of mathematical ability in grade 8.

Tests and Interviews

The quantitative part of the empirical study consisted of a pre- and a post-test that assessed the students' functional thinking abilities at the beginning and at the end of the school year. The qualitative part was built by 12 task-based interviews, 7 of which were conducted around mid-year and 5 of which were conducted at the end of the school year. In each case, two students were interviewed together to give them the opportunity to swap ideas about solving the task and to explain the solution to each other. The students were combined into pairs with homogeneous achievement levels. Each group was asked to solve two tasks using the tablet, which recorded the activities of the students with a screen-capture program. Additionally, the sessions were audiotaped and the combination of both audio and screen-capture was transcribed and translated afterwards.

Example of a test task

Given is a linear function with the equation $y = m \cdot x + b$. Explain how the graph of the function changes when b is varied.

Two strategies are conceivable for the answer. The static view argues about the position of the y-intercept of the graph, the dynamic view describes the shift of the graph as a whole or the movement of the y-intercept ("it shifts up or down"). A displacement of the graph as a whole can be assigned to the object aspect of functional thinking. This task was only part of the post-test.

Example of an interview task

*Given is the linear function f with $f(x) = m \cdot x + b$. If we add **c** to f(x) and multiply the sum with the factor **a**, then we get $g(x) = \boldsymbol{a} \cdot (f(x) + \boldsymbol{c})$. First describe the impact of **a** (for $c = 0$) and **c** (for $a = 1$) on the graph of the function g in relation to the graph of f. Then consider varying both parameters at the same time.*

The plotting and finding graphs of linear functions has been a central theme in the school year. This task was given during the end-year interviews. Its focus is on the object aspect, however, solving this task only referring to this aspect requires an advanced understanding of the relationship between graphs and their transformations. The task can also be solved by referring to the assignment and co-variation aspect.

Data analysis

A total of $n = 214$ responses is available for the post-test. $n_T = 101$ from students in the tablet classes and $n_C = 113$ from students in the control classes. For the analysis, the cases for which no answer was available or did not make sense were excluded and a chi-square test was carried out. There was no expected cell frequency smaller than 5.

RESULTS

The test task

The students of the tablet and the control classes solved the task with 56.4 % (tablet) and 54.0 % (control) with about the same success. However, there is a difference in the strategies used. The students in the tablet classes tended to argue using the dynamic view ($n_{T1} = 68$): 73.5 % dynamic and 26.5 % static view. In the control classes, on the other hand, the ratio is relatively balanced ($n_{K1} = 69$): 55.1 % dynamic and 44.9% static view. The difference is significant, $\chi^2(1) = 5.078, p = 0.024,\ \varphi = 0.193$. However, no advantage of either strategy is evident. With a solution rate of 85.2 % for the dynamic view and 88.2 % for the static view, they are almost equal.

Answers according to the dynamic view also tend to be accompanied by the object aspect ($n_D = 81$): 81.5 % argue in the sense of the object and 18.5 % of the assignment aspect. The static view and the assignment aspect are related ($n_S = 44$): 9.1 % object and 90.9 % assignment aspect. The type of view and the assigned functional aspect are thus strongly related, $\chi^2(1) = 60.639,\ p < 0.001,\ \varphi = 0.696$. Accordingly, the object aspect can be assigned more frequently to the answers of the students in the tablet classes ($n_{T2} = 62$): 67.7 % object and 32.3 % assignment aspect, while this seems to be balanced again in the control classes ($n_{K2} = 68$): 44.1 % object and 55.9 % assignment aspect. The correlation between group affiliation and used functional aspect is also significant, $\chi^2(1) = 7.325$, $p=0.007$, $\varphi=0.237$.

The interview task

All students created sliders during the interviews—some with the help of the interviewer—to assess the influence of the parameters ***a*** and ***c*** on the graph of a linear function with $f(x) = m \cdot x + t$. They were then able to apply the actions of adding as well as multiplying the parameters to the equation through the variation of the sliders. Some—supposed less experienced students—used the sliders by randomly changing the values, while others—experienced students—changed them in a systematic way by changing only one slider at a time, for specific values of the other ones (see Figure 2).

Students viewed the effects of those actions and evaluated them. As a result, many were able to describe the changes of the function with the graphical representation—"the line rotates around zero"—but had—as expected—difficulties interpreting this within the context and describing their reasoning in relation to the symbolic representation. With further exploration, students determined whether the y-intercept or gradient changed and how this is connected to the function equation itself. In the following is an excerpt from a transcript:

S11-2: ... the smaller **a** and the larger **c** the smaller the y-intercept and if you make ***c*** smaller then the y-intercept is larger [changes ***a*** and ***c*** accordingly]
S11-1: that means if **a** becomes negative, it does the exact opposite.
Int: and what happens when **a** is close to zero for example at 0.3?

6 S11-1: it moves less [**a** at 0.3 and varies **c**]
7 S11-2: yes, it's on a smaller range

The students realized that if ***a*** is a negative number, the effect of acting on ***c*** is reversed. Finally, the students were also able to recognize that ***a*** influences the effect of the variation of **c** in such a way, that for a small value of ***a*** the effect of varying ***c*** is more limited. The dynamic views in expressions like "the smaller … the larger" or "moves less" are obvious. Moreover, students achieved their results through engaging in an operative process by selecting specific values and evaluating the effects of their actions by viewing the graph as a whole.

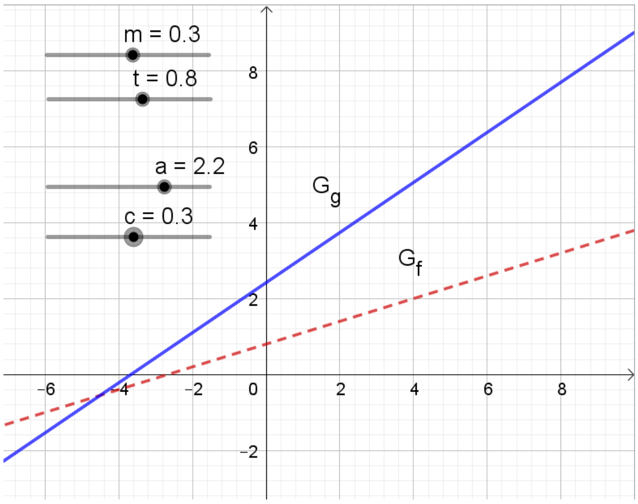

Figure **2**. The graph G_f of the function f with $f(x) = m\cdot x + t$ and the graph G_g with the function g with $g(x) = \boldsymbol{a} \cdot (f(x)+\boldsymbol{c})$

DISCUSSION

The general result of this study and the answer to the research question is that the use of the tablet in solving problems with linear functions supports the development of the *dynamic view* in connection with the *covariance aspect* and the *object aspect*. However, this positive and desired outcome comes with some limitations.

Although every student had access to a tablet throughout the entire school year, they developed very different skill sets when handling digital technologies. While some had already problems entering a function equation, others used the tablet as a tool, which means that they made use of the options presented by the applications such as sliders and could use the commands of GeoGebra adequately in the situation.

In working with functions dynamically, especially two possibilities can be distinguished: manipulating and reflecting (vom Hofe, 2004). The results of the interviews showed that learners have no problems in manipulating graphical representations. However, they seem to have difficulties in systematically changing suitable variables, which indicates a lack of understanding of the problem situation and prevents mathematical interpretation as well as reflection on the situation. The interviews also showed that learners have difficulties relating the changes, based on the manipulation of the graphical representation, to the symbolic representation. However, this problem is well known from many empirical studies (e.g. Nitsch et al., 2016).

CONCLUSIONS

Some conclusions can be drawn from the results for further work with digital technologies and the development of functional thinking.

- The operative *principle* in the sense of the characteristic question 'What happens to…, if…?' is a suitable tool for promoting the dynamic view of functional thinking, since it promotes the investigation of dependencies between actions and effects by varying variables. However, reflecting on the results of the variations is central to the process of understanding. This includes especially the development of the relation between graphic and symbolic representations.
- *Digital technologies* – especially sliders for parameters – are easy-to-handle tools for dynamizing the situation and drawing attention to the change of parameters and/or objects. However, the *reflection* of these manipulations is an indispensable prerequisite for understanding and this requires a basic mathematical knowledge (concerning the properties of linear functions).
- Digital technologies can support the development of the *dynamic view* of functional thinking. However, this does not happen on its own, but requires thoughtful guidance, especially through the selection of appropriate tasks (Leung and Baccaglini-Frank, 2017). However, it is even more important to highlight the specific importance of dynamic perspectives for solving specific (which?) problems. This is a present problem for mathematics education research (see Rolfes et al., 2020).
- The use of digital technologies in mathematics lessons has to be integrated into *learning* environment. This could be done by providing students with ready-made files containing corresponding features, i.e., sliders and defined functions, or the technical environment has to be constructed by the students. Both options carry dangers. On the one hand the lack of understanding the relationship between the representations and the mathematical objects 'behind', on the other hand the risk of failing at the first hurdle or spending too much time on the development of the supporting infrastructure. Presenting *'half-baked' files* might be middle way (Kynigos, 2007).
- The *difficulties of the technical handling* of digital technologies and the necessary longer-term processes of building up familiarity with the devices may not be underestimated.

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HOW DO TEACHERS ADOPT AUTONOMOUS PROBLEM SOLVING AS A CLASSROOM PRACTICE?

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Autonomous student problem solving is still rare in mathematics classes, and its incorporation in lessons requires profound knowledge and readiness of the teacher. In this paper we present a study conducted in the context of a professional development course aimed to enhance 12 teachers' readiness to create opportunities for students' autonomous problem solving. We characterize processes that the teacher-participants went through and report the extent to which the course achieved its goals. Based on qualitative and quantitative data, we offer a conceptual framework for characterizing the adoption processes and argue for the feasibility of the desirable change in the teacher readiness to adopt problem-solving instruction.

INTRODUCTION

Problem solving – engagement with a task for which the solver does not have a readily available solution path but has the required background for finding a solution (Schoenfeld, 1985) – is a desirable mathematical practice, whose implementation in a real classroom poses a genuine pedagogical challenge. Although keen attention had been paid to problem solving over the last decades (e.g., Felmer et al., 2019), most students do not actually experience problem solving in their mathematics classes (Schoenfeld, 2021).

One of the reasons for this state of affairs is that mathematics teachers, while usually devoted to enabling their students to attain success in task completion (Prediger, 2020), frequently tend to orchestrate the students' effort from the very beginning (e.g., Stein et al., 2008). However, they rarely manifest *readiness* (i.e., inclination supported by relevant knowledge) to let students experience Autonomous Problem Solving (APS), that is, to enable students to struggle with a problem on their own at least for a while. This is in spite of the fact that productive struggle is at the very heart of mathematical problem solving (Livy et al., 2018). In addition, effectively leading students towards problem-solving proficiency requires profound knowledge on the part of teacher. To this end, Chapman (2015) coined the term *Mathematical Problem Solving Knowledge for Teaching* (MPSKT), and suggested that MPSKT includes, among other types of knowledge, (i) content knowledge of problems and problem solving, (ii) knowledge of problem-solving instructional practices, and (iii) knowledge of students as problem solvers.

The importance of the first two of the mentioned types of teacher knowledge is (almost) self-evident. The importance of *teacher knowledge of students as problem solvers* can

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be deduced from the observation that the same task can be or not be a problem for its solver depending on the solver's prior knowledge (Schoenfeld, 1985), incentive to struggle with a task (Livy et al., 2018), and timing of approaching the task (Kilpatrick, 1985). An immediate corollary of this observation is that mathematics teachers should be prepared to anticipate how a task *might* be approached by their students prior to offering the task in a lesson. In other words, a teacher who wishes to create an opportunity for students' APS should first put herself in the student's shoes and try to imagine how various students would act on the task (Thompson, 1985; Stein et al., 2008).

It is broadly recognized that there is no 'recipe' for successful incorporation of APS in mathematics lessons. Moreover, the ability to understand what others are having difficulty with, what fascinates them, or how their prior experience shapes their interpretations of mathematical tasks is not at all simple. It is therefore essential to properly support teachers who wish to develop these rather unnatural skills (Ball & Forzani, 2011). Though MPSKT has received considerable research attention in recent decades (Chapmen, 2015), still little is known about processes of the development of teacher knowledge of students as problem solvers as a means for adopting APS as a classroom practice. The current study addresses this lacuna in the context of a professional developmental (PD) course for high-school mathematics teachers conducted in Israel in 2020-2021. Our research question is as follows: How does teachers' readiness to adopt APS as a classroom practice develop during their participation in a PD focused on knowledge of students as problem solvers?

CONCEPTUAL FRAMEWORK

The study draws on conceptual apparatuses from two theoretical sources. The first one is a model for challenging teachers' values, beliefs, and practices proposed by Swan (2011). Along with the above-mentioned framework of MPSKT (Chapman, 2015), this model provides a theoretical basis of the PD of interest. The second source is the Theory of Diffusion of Innovation (TDI) (Rogers, 2003) in its pedagogical version offered by Barker et al. (2015). This theory helps us to elaborate on the process of adoption of APS as a legitimate and desirable classroom practice.

Swan (2011) argued that changes in teachers' beliefs and teaching practices intertwine and motivate each other, and proposed a four-step developmental model describing stages of change as a result of a PD: (1) Recognizing existing values, beliefs and practices, (2) Analyzing discussion-based practices, (3) Suspending disbelief and adopting new practices, and (4) Reflecting on the experience. The third step is crucial as it alludes to temporarily setting aside one's doubts and fears, and denotes the emergent readiness to open-mindedly implement tasks offered by the PD facilitator in a classroom. The feasibility of this step for a teacher is stipulated by support from the PD facilitator as well as from the peers. Swan (2011) reported on the success of PD courses based on this model, but did not attend to the processes the teachers went through towards their emerging readiness to suspend disbelief and adopt the tasks offered in the PD.

Rogers (2003) defines *adoption* as a decision to make full use of an innovation as the best course of action available. TDI characterizes the processes, in which individuals decide whether or not to accept an innovation (APS as a legitimate classroom practice is the innovation in our case). Rogers distinguishes five stages of the process: knowledge of the innovation, persuasion, decision, implementation and confirmation. Barker et al. (2015) relied on the TDI in their exploration of how teachers adopt innovative teaching practices. They refined Rogers's model and focused on three stages of the adoption process, which they named *awareness*, *experimentation*, and *routinization*. Awareness of an innovative practice can develop as a result of an active search for solutions to a disturbing issue, and the search can occur either in a specially-designed setting (e.g., a PD) or in an informal setting. The *experimentation* stage can include thought experimentation when a teacher mentally applies a new idea to the present or anticipated future situation prior to deciding whether or not the idea is worth trying in real life. Learning from experiences of those who have already implemented the innovation is crucial at this stage. The *routinization* stage consists of an iterative series of implementation-related decisions made in a social context, relying on normative traditions, social cueing and emotional processes (Barker et al., 2015).

To summarize, the PD of interest was informed by Swan's (2011) four-stage model, and characterization of the adoption process in the course of the PD was initially informed by the *awareness-experimentation-routinization* framework. Figure 1 schematically presents the relationships between the stages of the PD and stages of the adoption process as an idealized pathway. In these terms, our study sought to elaborate, and possibly refine, the stages based on empirical evidence from the PD.

<table>
<tr><td>Stages of the PD</td><td>Recognizing existing values beliefs and practices</td><td colspan="2">Analyzing discussion-based practices</td><td colspan="2">Suspending disbelief and adopting new practices</td><td colspan="2">Reflecting on the experience</td></tr>
<tr><td rowspan="2">Stages of the adoption</td><td colspan="2" rowspan="2">Awareness</td><td colspan="4">Experimentation</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">Routinization</td></tr>
</table>

Figure 1: An idealized pathway of adoption of APS as a result of the PD

METHODOLOGY

The PD was led by the first author of this article. The participants were 12 high-school mathematics teachers of different teaching experience ranging from 1 to 28 years. The PD consisted of 10 three-hour Zoom meetings. The participants completed various written assignments related to planning, enacting and reflecting on APS-related activities. Two of these assignments are presented below in more detail.

Anticipating students' solutions for various tasks was a central theme of the PD. Every task that was raised in the PD meetings, either by the facilitator or by the teacher-participants, was used as an opportunity to imagine as many solutions as possible and to predict how different students might approach the task. The participants' anticipations were evoked in many ways. For example, a common activity was role playing, where the teachers acted as students with specific characteristics. The teachers' predictions were frequently challenged by peers in group discussions, or tested against available student artefacts such as a collection of videotaped lessons or students' written assignments.

All PD meetings were video-recorded and fully transcribed (120 pages in total). For the study presented in this paper we use the parts of the transcripts and the written assignments that included the participants' explicit assertions on APS-related matters. Eight out of 12 participants made such assertions, and the remaining four did not, though they might have been active and verbal when the discussions concerned mathematical aspects of the tasks. The selected parts were categorized using thematic analysis (Saldaña, 2021). The initial categorization was aligned with the awareness-experimentation-routinization framework, and then gradually refined.

Complementary data about the overall effects of the PD on the teachers' readiness to adopt APS as a classroom practice came from a specially-designed 20-item questionnaire (see endnote [i] for a link to the questionnaire). The questionnaire's design was inspired by Swan's (2011) approach to considering teachers' values, beliefs and practices as deeply interrelated. Accordingly, part A of the questionnaire concerns teachers' values and beliefs regarding APS, whereas part B focuses on ASP-related practices. The items were developed based on quotations from in-depth preliminary interviews with six experienced teachers. For example, one teacher's assertion "Students should not be alarmed by a task that they don't immediately know how to solve" inspired a Likert-scale question inviting respondents to use a 1-4 scale to indicate extent to which they agree with this assertion. The internal consistency (Cronbach's alpha) for 20 items was found to be .77 (n=60). High score in the questionnaire was interpreted as an indicator of a strong readiness to introduce APS in a classroom, and vice versa. All 12 PD participants filled in the questionnaire twice: at the second PD meeting and 1-3 months after the end of the PD.

RESULTS AND DISCUSSION

We begin by presenting and discussing the case of Keren (all the teachers' names are pseudonyms). Keren had six years of experience teaching mathematics for middle-track (second of three tracks) high-school students in a public school in the central part of Israel. Keren's case was chosen because it shows how challenging the adoption process can be. It also demonstrates the emergence of the new categories of analysis. We then outline the main findings for the entire group.

At the last PD meeting Keren summarized her journey as follows: “To be honest, at first it [APS] seemed to me a waste of time… [then I realized that] it is really nice, and when I tried it - it really made the lessons better.” We now describe how the change emerged in some detail. At the beginning of the PD, Keren expressed strong opposition to student APS, on the grounds that this activity is inappropriate for the students she worked with. In the first four meetings of the PD, which we associate with the first two of Swan’s stages (see Figure 1), Keren was exposed to the opinions of her peers, some of whom were positive towards problem-solving instruction. Along with the entire group, Keren gradually learned to anticipate student’s solutions for various challenging tasks by means of role playing and other means. However, up to the fifth meeting there were no signs that Keren began to modify her initial stance. In her words in meeting 4: “As long as I teach middle-track students, I do not see it [APS] in my classes… At the moment the whole story seems to me irrelevant.” In the terms of Barker et al. (2015), Keren was aware of the innovation but had not yet entered the experimentation stage.

The first PD assignment included thought experimentation. The participants were asked to reflect on their regular lessons during two consecutive weeks, in order to spot situations in which APS might have been incorporated. They then were asked to describe these opportunities and to think how the students might respond to an invitation for APS. Keren identified four opportunities for APS, and for two of them she described the difference between her expectations and what actually took place in the class. This exercise seemed to help Keren realize that her involvement in student work in a real lesson was probably too dominant. In Rogers’ terms, Keren identified an issue that needed to be resolved. She was still not ready to see in APS a feasible solution, but since then the tone of Keren’s assertion towards APS began to gradually change. For example, she said in meeting 6:

> “In order for it [a task] to be taken [by the students] as a problem, I must think a lot before the lesson… My head is simply not there at all! I just want them to know the material, they should pass the exams, that’s it. I don’t know. Maybe I... I need to think about it.”

We interpret this assertion as a sign that Keren moved from unconditional opposition to APS towards an inclination while remaining skeptical. Thus, the thought experiment, along with growing MPSKT, became for Keren a trigger that made the as-yet small change visible to us. The next change occurred after Keren succeeded to temporary suspend her disbelief (the third stage of Swan’s model) and enabled students to experience APS in one of her lessons. This was done in response to the second PD assignment. Keren was surprised to find that the students, who did not immediately solve the task, nevertheless valued APS and did not give up. She wrote in the assignment: “…It was evident that they were interested in trying and challenging themselves. This is something I could not think of [before].” Later, in the last PD meeting, Keren further shared her insights:

> “...there are many things [tasks] in the book [Keren’s textbook] that could be problems for my students if I would not direct them [the students] up there [to the solution] … If I would

always act like I did in that lesson, it would be great [...] So yes. Just a small change which for me personally was hard to make. I'm a little fixated, but slowly learning..."

We see in this assertion evidence for Keren's inclination to engage her students in APS in the future while recognizing that she is now equipped with some elements of MPSKT (Chapman, 2015). Namely, she developed ideas about where to look for appropriate tasks, about how to organize the instruction and what to expect from the students. Of special note is that the change occurred at the last of Swan's stages, a reflection on practice. This is in remarkable accordance with Swan's suggestion for how beliefs can be challenged and modified via practices. In his words, "…when teachers adopt new practices and reflect upon the often-surprising consequences, their beliefs are changed in profound ways (Swan, 2011, p. 70)."

To summarize the case, Keren improved her MPSKT with particular emphasis on the knowledge of students as problem solvers, and went quite far in the process of adoption of APS as a classroom practice: from opposition via skepticism to inclination to integrate APS-instruction activities, while progressing from the development of awareness of the merits of APS to experimentation with it. With somewhat less confidence we suggest that Keren is at the entrance to the routinization stage.

We will now overview the entire pool of the qualitative data. For only one of the 12 teacher-participants, the starting point could be associated with the *routinization* stage, as she asserted in the first meeting that she routinely allows her students to experience APS. Five participants started the PD at the *awareness* stage. These teachers manifested different types of awareness: awareness of the ASP practice while being skeptical about its merits, awareness of the gap between their wish to use APS and their actual practices, and awareness of some of the instructional methods by which APS can be facilitated. Two participants (including Keren) began the PD with strong opposition to APS. As mentioned, the remaining four participants did not articulate a clear stance regarding APS during the PD meetings and thus they are excluded from the summary.

Awareness
Experimentation
Routinization

	Rivi	Keren	Tamar	Ziv	Shira	Hadassa	Leah	Gila
opposition	1	4		2				
skepticism	3,4,5	5	2,5	10	5,6		4	
inclination	6,10	6	6,10	10	9,10	2,5	5	
readiness	10	10				5	6	1
intercession							5,9	4

Figure 2: Numbers of meetings in which the teachers shared their different stances

Figure 2 presents a summary of the teacher adoption processes, which are categorized by means of the refined awareness-experimentation-routinization model; most of the emergent categories were exemplified in the case of Keren. The remaining category –

intercession – denotes the fact that two teachers became active in convincing the other teacher, thus acting as agents of change (Rogers, 2003).

In good accordance with TDI (Rogers, 2003), the adoption processes of teacher-participants in our study varied while being affected by their beliefs and past experiences. Nevertheless, all cases in Figure 2 indicate clear progression of the PD participants along the stages of the adoption process. The quantitative data further supports this conclusion as a statistically significant change between the average scores of readiness to adopt APS as a classroom practice was found for the entire 12-teacher population. Namely, Wilcoxon test for paired samples showed a significant difference between the average score of the readiness in the pre-test ($\bar{x}$=2.94, SD=.33) and after the PD ($\bar{x}$=3.28, SD=.40, Z=-2.43, $p<.05$).

THE STUDY'S CONTRIBUTION

Our study suggests that even those teachers who, due to various reasons (see Chapman, 2015), do not see in APS a feasible classroom practice, can reconsider its merits and prepare for its implementation. The study also extends the range of use of Swan's (2011) model, while providing further support for its power. Indeed, the PD organized in accordance with Swan's model, and with special focus on the development of teacher knowledge of students as problem solvers, not only challenged the teachers' values, beliefs and practices, but also resulted in greater readiness of the teachers to adopt a classroom practice that was new for some of them. In addition, the study offers an elaboration and a refinement of Barker et al.'s (2015) conceptual framework for making sense of the stages of the processes of adoption of new teaching practices. We hope that the refined framework can be used in future research on students' autonomous problem solving as well as on additional classroom practices, which are desirable but known as difficult to implement.

We believe that the presented study conveys an optimistic message for problem solving instruction. However, further empirical research and further theorizing is needed in order to make problem solving a regular practice in mathematics classroom, as it was envisioned several decades ago (Schoenfeld, 1985, 2021).

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[i] The questionnaire is available at https://stwis.org/iarb2z

HOW PRIMARY SCHOOL STUDENTS PERFORM MULTIPLICATIVE STRUCTURE PROBLEMS WITH NATURAL AND RATIONAL NUMBERS

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This paper is part of a larger study focuses on a teaching experiment (pre-test, instruction, post-test) that aims to analyse primary school students' features on the transition from natural to rational numbers when solving multiplicative structure problems. Here, we analysed 61 6th graders responses to nine multiplicative structure problems with natural numbers and fractions (pre-test). We analysed students' performance and strategies. Results showed differences in students' performance considering the numerical set, indicating difficulties in identifying the problem structure's invariance. The most used strategy was the algorithm in both correct and incorrect answers. Results suggest that specific instruction is needed to help students focus on the problem structure invariance when the numerical set changes.

INTRODUCTION

In the 1980s, Bell et al. (1981) analysed 12-16-year-old students' difficulties when solving multiplicative structure problems with rational numbers. Some difficulties observed were related to the choice of the operation for solving the problem. Later, other studies (e.g., Fischbein et al., 1985; Levain, 1992) highlighted the existence of implicit models for operations (e.g., multiplication leads to a larger number and division leads to a smaller number) and students' difficulties in identifying the problem structure when the numerical set changed. These difficulties were also documented in the National Assessment of Educational Progress (NAEP) from the United States, where only 27% of 9-10-year-old students chose the correct answer in the multiple-choice question "Jim has 3/4 of a yard of string which he wishes to divide into pieces, each 1/8 of a yard long. How many pieces will he have?" (National Center for Education Statistics, 2003). These difficulties persisted in later grades since only 55% of 13-14-year-old students solved the question correctly.

More recent studies have focused on students' strategies used to solve multiplicative structure problems (e.g., Cheeseman & Downton, 2021; Downton, 2009; Empson & Levi, 2011; Hulbert et al., 2017; Ivars & Fernández, 2016; Mulligan, 1992). It has been shown that students' thinking evolves from strategies that do not lead to the correct answer to additive strategies, such as counting and repeated addition, that lead to correct answers. Later, multiplicative strategies such as the algorithm appear. Strategies leading to an incorrect answer also include the algorithm since students commonly identify an incorrect algorithm to use, which may be due to the lack of

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understanding of the situation and the relationship between the quantities involved (Hulbert et al., 2017).

Given these previous results, specific instruction in primary education focused on identifying the mathematical structure of the problem independently of the numerical set involved (natural or rational numbers) is necessary. With this regard, this paper is part of a larger study whose objective is to identify characteristics of the transition from natural to rational numbers when primary school students solve multiplicative structure problems. For this purpose, we have designed a teaching experiment that focuses primary school students' attention on identifying the mathematical structure of the problem independently of the numerical set involved. The teaching experiment consists of a pre-test, an instruction and a post-test. In this paper, we focus on the results of the pre-test. Its objective is to examine how sixth graders (11-12 years old) solve multiplicative structure problems with natural and rational numbers, and the strategies they use. This information will be used in the design of the instruction.

THEORETICAL BACKGROUND

Mathematically, we focus on the isomorphism of measures problems whose structure is a proportion between two measure spaces, each including two quantities (Vergnaud, 1981). In these problems, if one of the quantities is reduced to 1, three types of problems arise depending on which of the other three quantities is the unknown (Greer, 1992): (a) multiplication, whose unknown quantity is the total quantity; (b) partitive division, whose unknown quantity is the quantity per group; and (c) measurement division, whose unknown quantity is the number of groups.

Although different authors have pinpointed students' strategies to solve these kinds of problems (see above), we use the strategies identified by Empson and Levi (2011). These strategies develop from a basic way of thinking (*represents each group*) to a more sophisticated way of thinking (*multiplicative strategies*):

- Represents each group. Students represent all the quantities (symbolically or with drawings) and then count, add or subtract to get the answer. It includes *direct modeling* (the quantities are represented with a drawing) and *repeated addition* (the quantities are represented by mathematical symbols).
- Grouping and combining strategies. Students represent the "necessary" quantities, i.e., they group quantities additively until they get "friendlier amounts" to operate with (Empson & Levi, 2011, p. 57). Usually these "friendlier amounts" are natural numbers.
- Multiplicative strategies. Students form groupings, which are linked multiplicatively.

Considering the objective of this paper, the research questions are: How do sixth graders (11-12 years old) solve multiplication, partitive division and measurement division problems with natural numbers and fractions? What strategies do they use to solve these types of problems?

METHOD

Participants and instrument

The participants were 61 sixth graders (11-12 years old) from a Spanish primary school. According to the Spanish curriculum, the students had been introduced to multiplication and division algorithms with natural numbers in previous grades. They had also been introduced to the multiplication algorithm with fractions. Nevertheless, they had informal strategies to solve multiplication, partitive division and measurement division problems without the algorithm.

The pre-test consisted of nine problems (Table 1): three multiplication (M), three partitive division (PD) and three measurement division (MD) problems. Furthermore, considering our objective, we varied the numerical sets in each type of problem: a problem with natural numbers (N), a problem with a proper fraction (Q1) and a problem with two proper fractions (Q2). The pre-test was solved individually during a 50-minute session. The participants had to justify their answers, and they could not use electronic devices.

Characteristics of the problem	Statement	Structure
M-N	My grandmother uses 2 cups of flour when she makes a tray of cookies. If she wants to bake 8 trays of cookies, how many cups of flour will she need?	8 × 2 = 16
PD-N	We bought 20 yoghurts at the weekly shopping trip. If they came grouped in 5 packages with the same number of yoghurts, how many yoghurts are in each package?	5 × 4 = 20
MD-N	A baker made 24 *Easter cakes* and packed them in boxes of 4 cakes, how many boxes did he need?	6 × 4 = 24
M-Q1	At Marcos' birthday party there was lemon soda. If there were 3 bottles left at the end of the party and each bottle contained 2/3 of a litre, how many litres of lemon soda were left over?	3 × 2/3 = 2
PD-Q1	At Marcos' birthday party, there were 12 sandwiches left over. These sandwiches take up 3/4 of a tray. If all the sandwiches had the same size, how many sandwiches were there on the tray?	3/4 × 16 = 12
MD-Q1	My mother has made 2 litres of orange juice. If she has distributed the 2 litres in cups of 1/4 litres of capacity, how many cups has she filled?	8 × 1/4 = 2

M-Q2	Diego has picked oranges from his vegetable garden. To store them, he has used boxes of 3 kilos. If he finally counted 1/4 of a box, how many kilos of oranges has Diego picked?	$1/4 \times 3 = 3/4$
PD-Q2	Roberto has used 3/4 of a kilo of clay to make figures. If he has made 6 identical figures, how much clay has he used for each figure?	$6 \times 1/8 = 3/4$
MD-Q2	We have a bottle of 1/2 of a litre of perfume, and we want to distribute it in little bottles of 1/10 of a litre. How many little bottles do we need?	$5 \times 1/10 = 1/2$

Table 1: Problems of the pre-test

Analysis

The analysis was performed in two phases. In the first phase, we analysed the correctness of students' responses. For each problem, we coded with "1" the students' correct responses and with "0" the students' incorrect responses (Table 2). In the second phase, we focused on the students' strategies used. We initially based on the strategies proposed by Empson and Levi (2011), and then we performed an inductive analysis from our data that allowed us to refine these categories. In what follows, we describe the final category system, with some examples in Table 2:

- Representing each group. This category includes direct modelling and repeated addition/subtraction.
- Additive grouping and combining strategies. Students form groupings, which are linked additively.
- Multiplicative strategies. Students establish multiplicative relationships.
    - Multiplicative grouping and combining strategies. Students form groupings linked multiplicatively.
    - Algorithm. Students use the multiplication algorithm in multiplication problems and the division algorithm in division problems.
    - Inverse algorithm. Students use the multiplication algorithm in division problems, looking for the value whose product is the total quantity.
    - Equivalent fraction. Students look for an equivalent fraction whose number of parts is equivalent to the parts of the unit of measure (MD problem) or whose numerator is equivalent to the number of groups (PD problem).
- Unidentified strategies. Answers without explaining the procedure or procedures without sense.

- Blank answers.

Correctness code	Strategy category	Example: Student answer	Description
1	Multiplicative strategy: Equivalent fraction	$\frac{1}{2} = \frac{5}{10}$ En 5 botellitas	The student looks for an equivalent fraction, whose number of parts is equivalent to the parts of the unit of measure (1/10)
0	Algorithm	$\frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$ necesitamos $\frac{1}{10}$ botellitas.	The student uses a multiplication algorithm incorrectly

Table 2: Examples of the analysis performed (MD-Q2 problem)

RESULTS

Students' performances in each type of problem and numerical set

Table 3 shows the percentage of students' correct responses considering each type of problem and the numerical set.

	N	Q1	Q2	Total
M	91.8	68.8	44.3	68.3
PD	91.8	41.0	24.6	52.5
MD	98.4	47.5	49.2	65.0
Total	94.0	52.4	39.3	61.9

Table 3: Percentage of correct responses

Students provided more correct responses in multiplication problems (68.3%) than in division ones (58.8%). Furthermore, they provided more correct responses in measurement division than in partitive division problems (65% and 52.5%, respectively). According to the numerical sets involved, students were more successful in problems with natural numbers (94%) than with fractions (45.9%).

Students' strategies in each type of problem and numerical set

Table 4 shows the percentages of students' use of each strategy and the percentage of correct (C) and incorrect (I) responses in each strategy. More than 80% of students used the *algorithm* in the three types of problems with natural numbers. Regarding problems with fractions, the use of the *algorithm* was also the most representative strategy. Nevertheless, this was the strategy that also led to more incorrect responses.

In the multiplication problem with a proper fraction, *representing each group* was also used, while in the problem with two proper fractions students used *multiplicative grouping and combining strategies*. In the partitive division problem with a proper fraction, they used *multiplicative grouping and combining strategies,* and in the problem with two proper fractions, they used *equivalent fraction*. In measurement division problems with proper fractions, *multiplicative grouping and combining strategies*, the *inverse algorithm* and *representing each group* were observed.

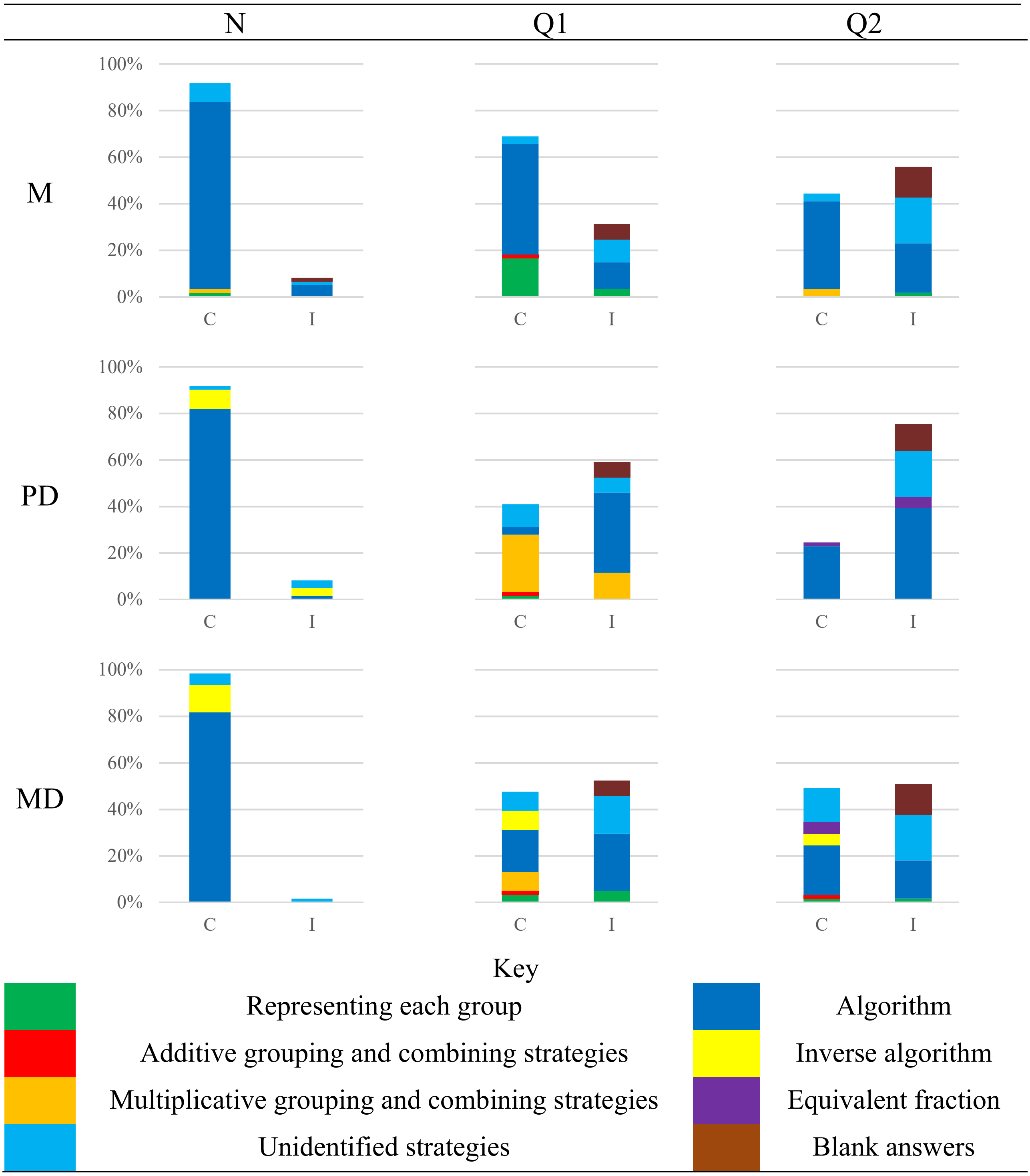

Table 4: Percentages of students' strategies and the type of answer

DISCUSSION AND CONCLUSION

In this paper, we examined how sixth graders solve multiplicative structure problems with natural and rational numbers, and the strategies they use. Our results show that students provided more correct responses in problems with natural numbers (94%); this percentage dropped remarkably when natural numbers were replaced by fractions (52.4% with one proper fraction and 39.3% with two proper fractions). Therefore, these results show that students did not recognise the invariance of the mathematical structure of the problem independently of the numerical set involved, showing the same difficulties obtained in previous studies some decades ago (Levain, 1992).

Regarding students' strategies, the use of the algorithm was the most common strategy in multiplicative structure problems with natural numbers and proper fractions. Nevertheless, in problems with fractions, the use of the algorithm led students to incorrect answers. This result was identified both in multiplication problems (in which students were introduced to the multiplication algorithm with fractions) and in division problems (in which students were not introduced to the division algorithm with fractions although they used the algorithm by converting the fractions to decimal numbers). Our results have also shown that students used other strategies different to the algorithm to solve multiplication and division problems with fractions and that these strategies allowed students to get correct responses to these problems.

Considering the results obtained in the pre-test, the next step in our research is to design the instruction aimed at focusing primary school students' attention on identifying the invariance of the problem structure when the numerical set changes. Theoretically, this instruction will be based on developing students' relational thinking, that is, on developing flexible strategies based on properties and relations between quantities (Empson & Levi, 2011) and on teaching with variation (e.g., Sun, 2019). In other words, we will provide a systematic variation of problems and quantities that will allow students to observe that the structure of the problem remains invariable.

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ORAL COMMUNICATIONS

ANALYSING GROUP INTERACTIONS OF STUDENTS SOLVING A MATHEMATICAL MODELLING TASK WITH THE AID OF DIGITAL TECHNOLOGIES

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This study, a part of an ongoing project, analyses how group interactions in mathematical modelling activities affect and are affected by digital technologies using Cultural-Historical Activity Theory (CHAT) as an analytical framework.

Digital technologies play an important role in mathematics teaching and learning, and an area where such technologies are used in complex problem solving is mathematical modelling. Research supports the assertion that digital technologies as mediating artefacts have an impact on students' modelling processes (Molina-Toro et al., 2019). Interactions generated by group activities in modelling affect and are affected by digital technologies. More specifically, GeoGebra and similar digital technologies as mediating artefacts have an impact on sense-making in group interactions as students solve mathematical tasks (Zengin, 2021).

I followed a qualitative approach in this study. The participants were three second year upper secondary school students, aged 16-17. Data was collected by means of video recordings and screen capture software. This source of data reveals the working processes of the students and their interactions with each other as they solve a mathematical modelling task in a group.

From a CHAT perspective, the student-to-student interactions are directed by the individual's engagement with the mediating artefacts, which influence the outcome of the activity. Preliminary empirical results show that a key factor to the modelling outcome resides in the interactions between the students generated by the digital technologies. Students personalize the problem as a result of the outcome from the use of digital technologies, which hinders the potential for sophisticated sense-making that could lead to a better outcome of the modelling activities. The features of the digital technologies used in the students' solution process depends on the representational choice of the dominant student in group interactions.

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STUDENTS' METAPHORS FOR IMPROPER FRACTIONS USED IN SCRIPTING

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Understanding the concept of fractions is important in the transition from primary to secondary school level. To understand fractions, students need to grasp the relations between a numerator and a denominator. The part-whole relations between the two numbers are commonly used to introduce fractions, but it has limitations in terms of constructing improper fraction concepts in that the parts cannot exceed the whole.

According to conceptual metaphor theory, metaphor is about what humans think and do, and the language individuals use is metaphorically structured in their minds. (Lakoff & Núñez, 2000). Metaphors represent individuals' conceptual structures between concepts, by understanding abstract concepts in terms of relatively concrete concepts. Lakoff and Núñez proposed four embodied metaphors for arithmetic, based on sensory-motor activities, with fractions described as the composition of a partitioned unit. I wonder, how do students understand improper fractions? Ahead of the question, how can I come to know students' uses of metaphors? This study is aimed at examining students' conceptions of improper fractions through their metaphors, by extending the metaphors by Lakoff and Núñez and proposing a way of using scripting as a novel research method for investigating metaphors.

The scripting I used in this study refers to creating a dialogue for an imagined play between imagined characters inspired by Zazkis et al. (2009). Through scriptwriting, I assume that students can naturally use their language which allows me to explore and analyse their written metaphors. The scripting task was conducted with 27 sixth grade (11-12 years old) Korean students. A prompt about improper fractions was provided, and students created scripts imagining the ensuing situation of the provided prompt. Of 27 students, 19 students think of improper fractions as process. In addition, improper fractions are metaphorically conceptualised as splitting, overflowing, and two independent numbers. Area models implying partitioning are commonly used for the conceptions. The results will be discussed in detail in the presentation. I expect the list of metaphors identified to be helpful for teaching and researching fraction concepts.

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UTILIZING COGNITIVE INTERVIEWS TO IMPROVE ITEMS THAT MEASURE MATHEMATICAL KNOWLEDGE FOR TEACHING COMMUNITY COLLEGE ALGEBRA

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Cognitive interviews are an important qualitative method used by researchers to discern interviewees' understanding of survey items (Willis, 2005). As part of a larger project that is developing an instrument to measure mathematical knowledge for teaching algebra at community colleges, we conducted cognitive interviews to provide feedback for the development of items for an instrument that will measure mathematical knowledge for teaching community college algebra (MKT-CCA). We hypothesized the measure of MKT-CCA consists of two main constructs, Tasks of Teaching (*Choosing Problems* and *Understanding Student Work*; Ko & Herbst, 2020) and Function Types (*linear*, *exponential*, and *rational functions*; foundational for mathematical work in later classes). We proposed items to assess the six resulting dimensions. The interviews were designed to determine whether the participants interpreted the items as intended and whether they used the anticipated knowledge.

A purposive sample of 12 College Algebra instructors was selected from a stratified sample of 1,386 instructors in 199 community colleges in the United States. Collectively the instructors responded to 36 items (6 per dimension); each item was answered by two instructors. Conrad and Blair's (1996) cognitive coding scheme guided the analysis. Each interviewee response per item was the unit of analysis. We found that most participants understood the intent of the items but in a few instances participants selected the correct answer without using anticipated knowledge and that some items focused on multiple mathematical ideas. In addition, the mathematical language used in some items was unclear or unfamiliar, and real-world contexts, both realistic and unrealistic, distracted respondents from focusing on the mathematical content of the item. In the presentation, additional results will be discussed.

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NOTICING STUDENTS' DIFFICULTIES WHILE WORKING ON MODELLING PROBLEMS

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When working on a mathematical modelling problem, students encounter a variety of difficulties that may relate both to working on the problem itself and to working together as a group. Every transition from one phase of the modelling process to the next holds potential barriers (Stillman, 2011). In order to support students in overcoming these barriers, teachers need to adaptively intervene. This requires that teachers notice students' difficulties to be able to react adequately. Noticing involves perceiving noteworthy aspects such as students' modelling specific difficulties, interpreting them based on knowledge, experience and beliefs and come to a decision (Jacobs et al., 2010). As noticing is a component of teachers' competence, we pose the following question: *How many and what types of difficulties do pre-service teachers perceive in students' modelling processes?*

Noticing competencies of 52 pre-service teachers, who took part in a modelling seminar, in which noticing in mathematical modelling had been fostered, were assessed with a video-based test before and after the seminar (Alwast & Vorhölter, 2022).

On average three out of six difficulties were perceived with a standard deviation of 1.1 in the pre-test and 1.4 in the post-test. 75% of the participants perceived that the students stopped working on the modelling problem after finding a mathematical solution without interpreting, validating or exposing their solution. On the other hand, only 29% perceived that two students lacked motivation and thereby had a negative impact on the group work. All in all, the results clearly indicate that pre-service teachers' perception of difficulties strongly depends on the type of difficulty. In the presentation, differences in pre-service teachers' perception will be discussed in more detail.

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THE PROFESSIONAL ACTIVITY OF A PROSPECTIVE TEACHER THROUGH THE DOCUMENTATIONAL APPROACH TO DIDACTICS: JOSÉ'S CASE

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One of the aims of the qualification education programs for prospective secondary school teachers (PSTs) is to prepare them to carry out the teaching professional activities: planning and implementing lessons, and reflecting on them. To do this, PSTs must learn to read, use and adapt curricular materials and resources in general. To study José's professional activity when planning a lesson, in his teaching internship period, course 2020-21, at the National Autonomous University of Nicaragua-León, we have adopted the *Documentational Approach to Didactics* (DAD) in order to understand José's professional development through the study of his interactions with the resources he used and designed in/for his teaching (Trouche et al., 2018).

These interactions are mediated by how teachers use, modify, adapt these resources (Instrumentalisation) and how their possibilities and limitations can influence the teaching-learning process of students (Instrumentation). *Teacher documentation work* is the engine of a *documentational genesis*, which jointly develops a new resource and a scheme of utilisation of this resource, giving rise to a document for a class of professional situations. A scheme of utilisation not only implies *invisible aspects,* the *operational invariants* (the cognitive structure that guides the action), but *observable aspects (uses*). Therefore, we ask ourselves the research question:

- What are the documents developed by José in his lesson planning?

The data of this study is a lesson planned by José, on the application of the trigonometric ratio tangent of the angle in problems of the environment, in which he used Official Curricular Resources and Cognitive Resources. Through the inferred mapping of the resource system that José used in the planning of the lesson, we obtained five documents and made the operational invariants explicit, such as "Introducing key questions to help students infer the rule of use of the tangent."

The DAD allowed us to know how José used the different resources at his disposal to plan a lesson and infer his mathematical knowledge through the instrumentation and instrumentalisation carried out of these, and beliefs through the operational invariants.

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KNOWLEDGE DEPLOYED BY PRESERVICE TEACHERS IN A TEACHING EXPERIMENT ABOUT REASONING AND PROVING

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Mathematics teachers must be able to programme and manage learning opportunities linked with reasoning and proving processes, so there is a need to design and implement effective tasks to revise and develop the knowledge of preservice teachers about them. This research aims to examine the mathematical knowledge about reasoning and proving processes deployed by elementary preservice teachers (EPT) during a teaching experiment about the inscribed angle theorem, focusing on conjecturing and proving. Mathematics Teacher's Specialised Knowledge (MTSK) model (Carrillo et al., 2018) is used for this purpose. This research also seeks potential relationships between this knowledge and the characteristics of the tasks.

The teaching experiment was carried out with 45 EPT, in groups of two or three. The experiment covered a session of 90 minutes. It had three stages: the first two dealt with the conjecture phase of the inscribed angle theorem (firstly drawing and measuring by hand, then using GeoGebra), and the third with the transition from conjecturing to proving and the proving phase. After every stage, questions on the conjecture certainty and if it is or is not enough proved were asked to obtain information of students' personal proof schemes (PS) (Harel & Sowder, 2007). Preliminarily, EPT who give evidence of an analytical PS, showing knowledge about what constitutes a proof, benefit from the characteristics of the experiment: GeoGebra facilitates them to establish the conjecture and gain certainty about it, and the third stage allows them to deploy the knowledge of properties needed to construct a proof, something they did not do before. EPT who show an inductive PS also establish the conjecture in the first two stages, but express that the conjecture is already proved. In the third stage, they deploy knowledge of properties, but the experiment seems not enough to generate the intellectual need of a deductive proof, showing the persistence of inductive PS.

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DESIGN OF VAN HIELE'S LEVEL 5 INDICATORS USING THE DELPHI METHODOLOGY

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The van Hiele model (van Hiele, 1986) establishes five levels of development of geometric thought, from level 1 (visual) to level 5 (rigor). Despite the fact that the van Hiele model has been deeply studied, there are few research works concerning the fifth level. Our goal is to describe this level through the construction and validation of a list of indicators for each of the processes involved in geometrical reasoning. One of the few works about the fifth van Hiele level is the thesis of Stephen D. Blair where he states that working with level 5-activities promotes the development of lower levels "…explorations involving taxicab and spherical geometry support students' understanding within Euclidean geometry" (Blair, 2004, p. 334).

We are following the Delphi research methodology (Hsu & Sandford, 2007) that allows us to collect information from a panel of experts to reach a consensus through a series of phases. In the first phase, we collect information from this panel (Geometry researchers) analysing narratives in which they describe the reasoning at this level. With this information, we designed a list of tentative indicators of the level. For example, two experts described the proof abilities of a person who reasons at level 5 as "...can compare proofs of the same result" and "...analyses proofs to decide if, by modifying them, they can proof a different result". Based on the previous sentences, we have constructed the indicator "a person at level 5 can compare proofs on the basis of criteria such as the possibility of using it to prove more general statements".

We have obtained a list of 19 indicators concerning definition (6), proof (5), classification (4) and identification (4). In the following phases, an extended panel of experts will rate these indicators until a consensus is reached. The final product of the iterative application of this method is expected to be a list of indicators of the fifth van Hiele level of reasoning for each process.

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LEARNING MATHEMATICS IN A MULTILINGUAL COUNTRY: STUDENTS' DIFFICULTIES AND TEACHERS' AWARENESS

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Algeria is a multilingual country. Arabic is the official language of school instruction; French is the official language of STEM instruction at the university (including textbooks in both cases). The oral Algerian Dialect is used in school and at the university for explanations and interactions, due to students' weak mastery of Arabic and French. In two previous studies, I investigated university students' difficulties to write a proof text in French for non-routine proof tasks, and the languages used by first year university mathematics students. The results suggested to search for possible reasons for students' difficulties, related to logical structures of the three languages and schoolteachers' level of awareness about them and their consequences on students' learning. For this research, I chose two specific constructs: 'code-switching' (Adler, 2001), and 'academic level' of language mastery (Cummins, 1979).

The comparison of Arabic, French and Dialect as concerns aspects of logical reasoning that are fundamental in mathematics revealed important logical-structural differences. For instance, French uses the same form: 'si…alors…' ('if…then…' in English) to express conditional reasoning in the different cases (actual, possible, past unrealized conditions); the modes of the verbs make the difference. Arabic and Dialect use different expressions in those different cases, with a different organization of the discourse. The semi-structured interviews to a representative sample of eight high school mathematics teachers put into evidence their lack of awareness about these logical-structural differences between the three languages and of their consequences on students' transition from high school to university in mathematics. All teachers confirmed that dialect is the current language for explanations and interaction with/among students, due to their weak mastery of Arabic, which impacts on the level of teaching and learning in high school. Findings revealed some reasons for students' difficulties that emerged in my previous studies: not only a serious problem of code switching (Adler, 2001) in the transition from high school to university due to the logical-structural differences between the written languages of mathematics instruction (Arabic and French), but also the lack of mastery of any language at the academic level (Cummins, 1979), which results in difficulties in non-routine mathematical activities.

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CONCEPTUAL BLENDING UNRAVELS COVARIATION

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Covariation is a form of reasoning that deals with creating new conceptual objects and mathematical meaning (Thompson & Carlson, 2017): when a person co-varies two varying quantities, she mentally forms a new object resulting from the two initial magnitudes. Hence the input knowledge at stake leads to the formation of a new object of knowledge. This cognitive process has many features in common with the mechanism of conceptual blending (Fauconnier & Turner, 2002): blending processes between different mental spaces of knowledge are a way in which people make sense of new information.

In this communication, we shed light on the connection between the cognitive processes of blending and covariational reasoning. We specifically describe how the conceptual blending framework can help in revealing and grasping forms of covariational reasoning. This seems to be an uncharted territory that could enlighten the unexpected forms of covariation emerging in students' reasoning, mainly when the input spaces of knowledge are mediated by several representations of the phenomenon under investigation.

The analyzed data consist of two episodes from two different teaching experiments conducted in a 10^{th} and 11^{th} grade in a scientific-oriented school in Italy. The tasks were specifically designed to enhance covariational reasoning: they were focused on the modelling of real phenomena (the motion of a ball rolling along an inclined plane and the temperature-humidity relationship) and involved the use of various technological supports. Commenting on these episodes, we will show how the blending of the information provided by the involved artefacts and representations reveals and sustains both first- and second-order forms of covariation (Bagossi, 2022) and enables us to grasp the complexity of the analysed phenomenon better.

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INVESTIGATION OF PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' UNDERSTANDING OF PROPOSITIONS' LOGICAL STRUCTURES

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Evaluating the truth and falsity of a simple proposition requires complex cognitive activity (Epp, 2003). Thus, one of the reasons that students have difficulty in the proving process is related to their understanding of the logical structure of the proposition to be proved (Moore, 1994). However, the understanding of a proposition's logical structure is not only necessary for proving process, but also the understanding of many mathematical discourses and the use of mathematical language properly.

The aim of this study is to investigate senior pre-service elementary mathematics teachers' understanding of propositions' logical structures. The study was designed qualitatively. Firstly, an inquiry was developed. The propositions in various logical structures but had only one quantifier (in mother-tongue or mathematical language) been given and asked to translate from mother-tongue to mathematical language and vice versa, to negate the propositions and to evaluate truth value of them in the inquiry. The inquiry was applied to 67 senior pre-service elementary mathematics teachers. After analysis of the data qualitatively,10 participants were selected by using criterion sampling. The clinical interviews were conducted with the participants about inquiry items. Obtained data was analysed qualitatively and were interpreted using APOS theory (Dubinsky, 1997). The results of the study showed that most of the participants had difficulties about understanding the logical structures of the propositions. Although the participants could understand the logical structure and also determine the truth value of the proposition correctly, it was seen that they could not negate the propositions. Some of the participants negated all the mathematical notions in the proposition without considering the meaning of the proposition. In addition, the participants could not use mathematical language properly. In conclusion, it can be said that the participants are mostly action or process level but not at the object level. Further results will be given in the presentation.

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ADAPTIVE TASKS—TEACHERS' DIFFERENTIATING VIEW ON SURFACE AND DEEP STRUCTURES

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INTRODUCTION

Tasks play a central role in math lessons. However, not all tasks in mathematics textbooks are suitable for differentiated teaching because of their low "differentiation potential", i.e., the implementation of specific adaptive task features in the deep structure of a task. We call a task with a high differentiation potential "adaptive task". Our aim is to examine differences in teachers' reasoning with task features when considering the differentiation potential of tasks.

TEACHERS' FOCUSSED TASK FEATURES

In a study with *N = 78* secondary mathematics teachers, we asked whether teachers were able to assess the differentiation potential of tasks—and we found that several different task features guide teachers in their reasoning (Bardy et al., 2021). Among these 23 features are those that focus on relevant features in the deep structure of tasks in terms of the differentiation potential, e.g., "openness", "accessibility", "goal differentiation", and "difficulty" (e.g., Sullivan, 1999), but also those that focus on features at the surface structure, e.g., "layout" or "presentation" (e.g., Hammer, 2016).

Given these broad variety of different task features, we aimed at extracting "prototype" of teachers that describe specific groups of teachers with similar focus when assessing the differential potential of tasks. A hierarchical cluster analysis revealed three types of teachers: with a broad focus on the surface and the deep structure of tasks (*41%*), with a focus on the deep structure of tasks in terms of the task content (*40%*), with a focus on the deep structure of tasks in terms of the differentiation potential (*19%*).

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THE ROLE OF THE ENJOYMENT IN MATHEMATICAL MODELLING ACTIVITIES

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According to Schukajlow et al. (2018), affective variables have been seldom investigated in the Mathematical Modelling (MM) literature. In the present work, we combine the MM cycle of Greefrath (2011) with the affect framework in Hannula (2012), focusing on the emotion enjoyment due to its link with achievement. We formulate two research questions: RQ1. Which affective factors (cognitive, emotional and motivational) linked to the emotion enjoyment are reported by students after the MM teaching experience? RQ2. Which strategies could be employed to foster students' enjoyment in MM?

To address RQ1 and RQ2, we qualitatively analysed the data collected from a MM activity performed by 18 students (grade 11). The teaching experience originated from a real-world problem posed by a stakeholder: the alderman to the culture wanted to know how young people use the library and what they would like to find there. This question was investigated by means of a survey which has been built and analysed by the students. During 8 meetings of 2 hours each, students were guided through the MM process by means of 5 activities. With respect to RQ1, we were able to identify (lack of) self-efficacy concerning the cognitive factor; value and (lack of) interest for the motivational factor; task enjoyment, boredom and anxiety for the emotional factor. Concerning RQ2, results confirm the importance of employing real-world problems to foster students' interest. Moreover, they also suggest that the design of teaching experiences in which the initial steps of the modelling cycle can be carried out with little mathematical language could enhance students' perception of self-efficacy, providing a motivational push also for the subsequent steps, where specific language must be employed. Finally, the use of digital and mathematical tools which are perceived by the students as belonging to the university and employment worlds can increase students' perception of value.

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SECONDARY SCHOOL STUDENTS' UNDERSTANDING OF SAMPLING VARIABILITY

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In this work, we analyse the progression with school year in the understanding of sampling, a key stochastic idea (Burrill & Biehler, 2011) since it brings a bridge between statistics and probability. We particularly draw on Noll and Shaughnessy (2012), who described idiosyncratic, additive, proportional and distributional reasoning of sampling. We continue our prior analysis (Batanero et al., 2020) with 234 high school students (17–18-year-olds), extending the sample with two additional groups of secondary students (157 students: 13–14-year-olds, and 145 students: 15-16-year-olds) to whom we proposed sampling tasks from a binomial population systematically changing the sample size and the value of the population proportion.

In this paper we compare the responses by the three groups in two tasks where the students were asked to estimate the number of heads in flipping four times 10 and 100 fair coins. For each task, the distribution of the ranges of the four values provided by each student was compared with the theoretical distribution of ranges in samples taken from the binomial population. Overall, the variability of the sample proportion was overestimated, and only a minority of students in the different samples achieved distributional sampling reasoning (Noll & Shaugnessy, 2012). However, the percentage of student providing excessive variability in task 1 (100 coins) decreased in the groups with older students ($F=14.5$; $df=2$, $p>.001$ in the Anova test), but the differences were not significant in task 2 (10 coins). These results suggest a better understanding of sampling variability in small than in big sample.

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HOW TEXTBOOKS IN DIFFERENT SCHOOL YEARS GIVE MEANING TO MATHEMATICAL SYMBOLS

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Research shows that natural language (for example English) is central to students' meaningful understanding of mathematical symbols. For example, natural language can be important as a starting point for the learning of the symbolic language (Caspi & Sfard, 2012). However, longitudinal studies are missing.

This ongoing study is part of a larger project concerning the role of natural language when learning the mathematical symbolic language, including how progression in the use of natural language can relate to advancements in the use and understanding of the symbolic language. Here we focus on textbooks, with the aim to answer the research question: How is meaning given to mathematical symbols through the use of natural language in mathematics textbooks in different school years?

In the analyses, we used a set of categories that describe different ways of using natural language to give meaning to symbols. Giving meaning to symbols can be done by specifying a referent to mathematical symbols (Hiebert, 1988). We delimit this study to situations when some (property of a) referent is specified through natural language. One of our categories includes situations when a symbol is defined or explained in very explicit language. For example, the category includes statements as "x is a number", which specifies a referent (number), or "13 is odd", which specifies a property (being odd) of the referent. Other categories describe situations when information is given in more implicit and indirect manners.

We analysed three different textbook series commonly used in Sweden, from school years 2, 5, and 8, and we randomly selected 20 pages from each of the nine textbooks. Analyses are ongoing, but preliminary results show the fraction of analysed situations that explicitly describe a (property of) a referent, in the different school years: 6.5% in year 2, 9.2% in year 5, and 7.2% in year 8, among a total of 1283 units of analysis.

There is no statistically significant difference between the school years, showing that the more explicit focus on the meaning of symbols is equally common across school years. Continued analyses will examine if there are any qualitative differences.

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TRANSITION FROM SECONDARY TO TERTIARY EDUCATION - THE ROLE OF LANGUAGE FOR STUDENTS' DIFFICULTIES

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The transition from secondary to tertiary mathematics education is difficult for many students (Bergsten et al., 2015). A common situation is that students who get good grades in upper secondary school have difficulties early in the university mathematics. Results from recent research (Bergsten et al., 2015) indicate that several explanations of learning difficulties in university mathematics are connected to the use of language. University teachers use a language with a high degree of formalism, especially concerning mathematical proofs (Bergsten et al., 2015). University students quickly realises that the teachers use a language they are relatively unfamiliar with (Gueudet, 2008) and they feel like foreigners who doesn't master the language.

The aim of this ongoing study is to characterise similarities and differences in the use of language in textbooks for upper secondary school and first semester university mathematics. Based on a framework by Morgan and Tang (2016) the analysis focus on two aspects: Specialisation and objectification. A specialised word has a well-defined meaning in school mathematics. Objectification can be of two types: reification (processes changes to objects), and alienation (free from human action).

Preliminary results indicate that the difference in specialisation is not very prominent. The use of specialised language is rather similar in the textbooks analysed so far (about 64% of all analysed words). However, a clear difference can be found in the use of alienation. The analysed university textbooks have a higher use of processes independent of human action (about 57%) compared to upper secondary school (11%).

Removal of human actors in mathematical texts could support the "disengagement from university mathematics" (Solomon & Croft, 2016, p. 275), and be one of the reasons that the transition from secondary to tertiary education is difficult.

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HIGH SCHOOL LEARNERS' BACKLOGS IN NUMBER

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There is significant concern about the mathematical backlogs that learners enter high school with in South Africa. For example, learners attending schools in poorer areas are three to four grade levels behind their expected grade level (Spaull & Kotze, 2015).

The research reported in this communication is linked to a larger project investigating key levers for helping early high school learners from disadvantaged backgrounds succeed at mathematics. In the component of this research reported here, we analysed learners' performance on tasks related to number to identify key aspects of number work requiring remediation. We draw on two data sets: 1) responses of a nationally representative sample of South African learners to the released items on number concepts from the grade 8 Trends in International Mathematics and Science Study (TIMSS); 2) responses of grade 8 learners in ten South African high schools in disadvantaged areas to multiple-choice items focusing on whole and rational number from a baseline test.

We used Yang and Lin's (2015) five components of number sense as a framework to analyse learners' performance. Overall, the analysis indicates weaknesses across all five components of number sense. For example, 1) learners had difficulty in recognising number size with 76% of the learners unable to identify the largest fraction from a list of common fractions with the same numerator, but different denominators; and 2) 86% of the learners lacked sufficient understanding of place value and operations to arrange the digits 1; 2; 3 and 4 to create two two-digit numbers with the smallest product. The analysis of learner performance also suggests lack of fluency and versatility in basic calculations. We argue that learners' difficulties working with the properties of and operations on number are likely to hinder their access to high school mathematics and thus require careful attention in programmes supporting early high school learners.

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HOW PROSPECTIVE PRIMARY SCHOOL TEACHERS INTERPRET PUPILS' SOLUTIONS TO A FAIR GAME TASK

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It is broadly accepted that teachers' mathematical knowledge is crucial to foster teaching competences related to the organization of mathematical content and the understanding of how students learn mathematics. Recent research suggests that many prospective teachers share common biases in probabilistic reasoning with their students (Batanero et al., 2016; Chernoff & Russel 2012; Prodromou, 2012). Characterising the components of teachers' knowledge of probability will enable the design of appropriate materials and effective activities for teacher education (Batanero et al., 2016).

The aim of this paper is to assess the competence of prospective primary school teachers to interpret pupils' solution to a fair game problem and to recognise proportional reasoning in their mathematical practices. This research was conducted with 116 prospective primary school teachers at a Spanish university. The written answers were analysed using content and descriptive analysis methods. To assess the cognitive facet of didactical-mathematical knowledge and competence, the participants analysed the correctness degree of different pupils' solutions to a probability problem concerning the fairness of a chance games, and identified the proportional reasoning involved or not, as a relevant mathematical element of pupils' mathematical thinking when solving this type of task. Our results reveal the difficulties in interpreting and justifying the correctness of pupils' solutions to a fair game problem, as well as in identifying proportional reasoning in their answers. The information gathered can be used in the design of training programmes focused on working on the facets of teachers' knowledge in order to guarantee an adequate teaching-learning process.

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PRESERVICE TEACHERS' ABILITY TO NOTICE CHILDREN'S ALGEBRAIC THINKING

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The professional noticing skill is an important aspect of teacher's practice and thus it is crucial that preservice teachers (PTs) have opportunities to develop it. Specifically, it is necessary to understand how teacher education can contribute to PTs' noticing skill of children's thinking as they analyze different classroom resources as students' written productions or videos (Magiera & Zambak, 2021; Zapatera-Llinares, 2019).

The present study aims to understand the development of PTs' ability to notice (to describe and to interpret) 4 to 10 years old children's algebraic thinking, through a teacher education course in an undergraduate program in elementary education. The study follows a qualitative methodology and focuses on a pair of PTs as participants. The methods of data collection were observation of the course's classes, with audio and video recordings, and collection of the PTs' written productions. The data analyses framework results from crossing the skills of describing and interpreting children's thinking with specific aspects of algebraic thinking. The results show that PT's ability to notice children's algebraic thinking evolved favourably throughout the teacher education course, with a positive impact of PTs' analysis of videos and children's written productions. Although PTs express some difficulties in the interpretation dimension, the results show that even without professional experience they can notice important aspects of children's algebraic thinking. The study also allows us to reflect on some limitations of the research on the noticing skill namely when it focuses on PTs' written analysis and thus on possible implications for teacher education.

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COMPARING LANGUAGE DEMAND IN INQUIRY-ORIENTED VS. LECTURE-BASED UNDERGRADUATE MATH COURSES

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Undergraduate math instructors are increasingly aware of the overall benefit of inquiry-oriented (IO) instruction. However, IO instruction, relative to lecturing, may not be equitable for certain student groups (e.g., gender inequities in Johnson et al., 2020). This work investigates a question related to equity for multilingual students: What are the language demands of IO undergraduate math courses, in comparison to lectures?

Grounded in a sociocultural perspective (e.g., Moschkovich, 2015), I developed an analytical framework that captures language demands at three levels: (a) lexical, (b) discursive (e.g., discourse practices), and (c) organizational (e.g., grouping). The framework was used to examine samples of IO and lecture-based differential equations classes. The IO class consisted of 8 students: 2 White men, 2 White women, 1 Latina woman, 1 Latino man, 1 Mexican-Thai woman, 1 Middle-Eastern man. The lecture was obtained from the MIT Open Courseware, for which no student data was available. Classroom talk from video recordings of three class sessions from each course was transcribed and analyzed using the constant comparative method, allowing for both a priori codes and emergent codes. The units of analysis for the lexical, discursive, and organizational levels were, respectively, words, utterances, and entire lessons.

Some of the main preliminary findings comparing the IO class to the lecture include:

- *Lexical:* IO course introduced fewer technical terms and made multiple meanings of terms more explicit, but asked students to create referential terms.
- *Discursive:* The expected ways of participating in IO course were more implicit and sophisticated, and required more negotiation.
- *Organizational:* IO course explored multiple representations more explicitly, but varied audiences more, inducing more sophisticated language use.

These findings indicate particular ways in which IO classes may pose different challenges than lectures, suggesting the need to explore corresponding linguistic supports in IO courses. In the presentation, further details and results will be discussed.

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LEARNING TRAJECTORIES AND MATHEMATICAL KNOWLEDGE PROFILES OF TEACHER CANDIDATES WHEN ENTERING TEACHER EDUCATION

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The selection mechanisms for admission to teacher education (TE) programs have increased in recent years, especially the assessment of disciplinary knowledge, as they are associated with greater achievement in measurements of international school tests (Ingvarson & Rowley, 2017). This is mainly due to the verification of the existence of different profiles of knowledge and mathematical ability, of the teacher candidates (TC) that are accepted in the TE programs, due to the diversity of learning and teaching trajectories that they have experienced in their school itinerary, or because of the learning opportunities of the ET processes themselves (Chandía et al., 2021).

However, the methodologies and results of such evaluations do not allow characterizing these profiles or identifying the possible learning trajectory that established them, referring only to the comparison of the subjects on a continuous scale of scores given the use of classical test theory or of item response theory. Thus, using diagnostic classification models and the discrete learning measurement models, an instrument was built and applied to 119 TCs that had just entered the TE process, addressing 5 disciplinary areas established in the teacher education standards of the TE quality assurance system of Chile.

The results confirm that the TCs have different profiles of mathematical knowledge by disciplinary area, defined by their learning trajectories when entering the TE process. Even more, similar states of knowledge show different learning trajectories, which may or may not respect epistemological or didactic positions. This enriches the processes of access of TE, that mainly order the TCs to select them, transforming into a resource for improving the desired initial profile of the TCs and thus achieving the exit profiles established by the TEs or by the TE's quality assurance policies.

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THE USE OF PATTERN BLOCKS IN DEVELOPING STUDENTS' HIGHER-ORDER THINKING

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Higher-order thinking is an essential soft skill and is recognized as one of the 21st century skills encompassing expert thinking and complex communication (Levy & Murnane, 2004). We need to cultivate student higher-order thinking skills so that they can utilize the new skills, models, and attitudes to face the significant challenges in their daily lives (Araya, 2021). Higher-order thinking in mathematics can be developed by learning through appropriate task sequences (Isoda, 2020). Pattern blocks is a hands-on activity that supports students to possess conceptual understanding through their abilities in solving problems independently (Inprasitha, 2011).

Our study involved 47 participants. They have engaged in the pattern block activity while they were studying in the non-degree program. Participants were faculty members, school principals, educational supervisors, novice teachers, in-service teachers, and graduate students. Data were collected using self-reflections. The quantitative findings revealed that the average scores of engagement in the virtual classroom are as follows: 1) cognitive presence 2) social presence and 3) teaching presence was strongly agreed. In general, most participants have positive perceptions toward the phenomena that occurred in the classroom in terms of cognitive, social, and teaching presence, ranging from a mean score of 4.44 to 4.65. Moreover, findings also showed that the participants' reflections toward pattern block activity are also positive as follows: 1) pattern blocks activity is active learning at the mean score of 4.59, and this pattern block activity support for developing higher-order thinking is found at the mean score of 4.46. On the other hand, qualitative findings revealed that the majority of the participants could solve the problem by themselves, they formed their conceptual understanding on how to extend the pattern block into the twofold, and they also had what they want to do next after they challenged in the pattern blocks problems.

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TEACHER KNOWLEDGE FOR TEACHING MATHEMATICAL PROBLEM SOLVING

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The importance of problem solving (PS) in mathematics education is well established regarding its central role in teaching and learning mathematics for deep understanding. With a PS habit-of-mind, students are also prepared for real problems and to thrive in today's world. To support students' learning of PS, teachers should hold appropriate knowledge for teaching PS from the perspective of PS as involving a problematic situation, a way of thinking, and a cyclic process (e.g., Polya, 1954). This presentation is based on the initial stage of a study that considers this knowledge in terms of PS research in the 21st century. This stage involved a survey of PME research reports (RR) on PS for 2000 to 2021. PME RR provide a snapshot of the international landscape of research on PS in the 21st century that could offer initial insights of this knowledge relevant to the field. The survey explored the types of knowledge the RR addressed in relation to teaching and learning PS and the relationship to mathematical PS knowledge for teaching (MPSKT). Chapman (2015) framework for MPSKT provided the theoretical baseline to frame the study. It consists of seven types or areas of knowledge teachers should hold to engage students in PS effectively and meaningfully.

The survey process included identification of relevant studies based on a systematic examination of PME RR titles and abstracts using keywords related to MPSKT (e.g., PS, problem posing, problems, modelling, beliefs, and metacognition) and exclusion of studies not focused on genuine PS and K-12 students or teachers. The final set of studies were examined to identify common themes. The findings consisted of eight themes for studies on learners and six themes for studies on teachers related to MPSKT. The themes for learners consisted of: instructional strategies, PS process, technology, assessment, problem posing, collaboration, metacognition, and emotion. They suggest teacher knowledge needed to understand students as problem solvers and ways to support and evaluate their learning of PS. The themes for teacher consisted of: problem posing, PS knowledge development, knowledge for teaching PS, beliefs, technology, and collaboration. While the findings aligned with Chapman's (2015) model of MPSKT, more importantly, they suggested ways to make visible subcategories of com -ponents of the model (e.g., collaboration, emotion) to provide a more comprehensive landscape of MPSKT that could be used to inform teacher education and research.

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THE CONSTRUCTION AND VALIDATION OF THE SCALE FOR EXAMINING SECONDARY SCHOOL STUDENTS' SELF-REGULATION OF MATHEMATICS LEARNING

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A new mathematics curricular guideline was initiated in 2019 in Taiwan to promote the next generation's mathematical literacy and self-regulation of mathematics learning (SRML). It is the first time that the math curriculum emphasizes developing students' SRML in Taiwan. However, the impact of the reformed curriculum on students' SRML is unclear because the term has not been well-defined and the lack of a valid and reliable instrument. Thus, this study constructed a scale to measure SRML and validated it to address the issue.

The conceptual framework of self-regulation of learning (National Academies of Sciences, Engineering, and Medicine, 2018) guided the construction of the SRML scale. The construction began with consulting math professors, curriculum guideline committee members, evaluation specialists, and school math teachers to define SRML. Then the items of the scale were developed based on the definition. Next, the experts were asked to rate the appropriateness of the items, and the content validity indices were applied to select items. Finally, 70 items passed the examination of content validity for further analysis.

Subsequently, the researcher administered the items to 865 secondary school students online for item analysis and to examine the construct validity and reliability. He received 861 valid responses and filtered these items by checking the item-remainder coefficient and the critical ratio. Items that passed the examinations were taken for conducting exploratory factor analysis (EFA). EFA suggested that the SRML scale consists of 64 items and four subscales: learning motivation, learning strategies, learning regulation, and learning resource management. In addition, the Cronbach alpha coefficient of each dimension of the subscales was above .70, which indicated that the SRML scale has acceptable internal consistency.

In sum, the scale is a valid and reliable instrument for examining the impact of implementing the reformed math curriculum on Taiwanese students' SRML. The process of constructing the scale help clarify what SRML means in the context of curricular reform. The process and product of the study expect to contribute to the implementation and improvement of the new math curriculum.

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IN-SERVICE MATHEMATICS TEACHERS' NOTICING OF EXEMPLARY LESSONS: AN EXPLORATORY STUDY IN CHINA

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Teacher noticing is regarded as a desirable ability in mathematics teachers' expertise in teaching. Research on teacher noticing mainly focus on pre-service teachers (PSTs) and relatively less on in-service teachers, and most studies are conducted in Western countries. There is a need to study in-service teachers' noticing in other cultures. We apply two frameworks for data collection, which are Learning to Notice (van Es, 2011) and FOCUS Framework adapting from Three-Point Framework (Yang & Ricks, 2012). Our questions are: 1) What do in-service teachers notice, and how do they respond to their noticing? 2) What is the influence of these two frameworks in teacher noticing?

17 in-service primary teachers in northeast China were invited to observe 10 exemplary primary mathematics lessons online. Their teaching experience ranged from 3 months to 27 years. Each teacher was required to choose at least one lesson to observe with one noticing framework. These lessons were taught by expert teachers in China. We used two coding methods to distinguish the feather of what and how teachers notice. One method was concerning agent, topic and stance (Van Es & Sherin, 2006); another concerned the four-level criteria of teacher noticing (Van Es, 2011). Results showed that teachers mainly noticed students' mathematical thinking and teachers' teaching pedagogies, which were two central elements of a lesson. Most teachers who chose the FOCUS framework had noticing levels in baseline and mixed. Compared to the FOCUS framework, teachers choosing Learning to Notice had a better performance in both breadth and level of noticing. It was found that higher-levels of teacher noticing mainly were found among teachers with more than 19 years of teaching. The findings can help us understand in-service teachers' noticing ability in China and enlightens us to explore the relationship between teachers noticing level and teaching experience.

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USING HEURISTICS AND 'TEACCH' MATERIALS IN PROBLEM SOLVING WITH STUDENTS WITH ASPERGER SYNDROM

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This research aims to contribute to inclusive education regarding the needs of children with Asperger Syndrom (AS), in terms of the alterations of central coherence and executive functions: difficulties to infer meanings from communicative situations, verbal comprehension, organisation, predictions, self-regulation and flexibility of thought (Bae et al., 2015). Specifically, it considers how a problem-solving approach (PS) can be adapted for students with AS, and trials classroom-based strategies that can be applied in any classroom. This approach allows for encouraging the organisation of data and cognitive flexibility through the incorporation of TEACCH material and different strategies along Polya's resolution phases (Liljedahl et al., 2011). This enables the teacher to design an environment conducive to diversity of thinking through the application of different heuristics, generalizable to other contexts.

A two-day PS workshop was organised for 16 participants (6-18 years old) diagnosed with AS, grouped according to age and abilities by the collaborating AS association psychologist. The problems were selected in terms of content and applicability of the heuristics, adding manipulatives, pictograms and visual organisers for supporting the PS. Both the materials and the problems were adapted reducing the amount of information and numerical relationships, after being tested with students not diagnosed with AS and with the same ages as the sample. The units of information, obtained from children's productions and video transcriptions, are categorized according to the heuristic employed and the AS characteristic showed.

Visual representation was the most frequently used heuristic at all stages, by applying tables and manipulatives for searching a pattern, trial and error and going backwards. Hence, we propose the deployment of TEACCH material and heuristics as a support for those characteristics associated with SA, or to cover specific needs. This study continues through long-term PS workshops in 2022, working with 4 children diagnosed with AS on the appliance of heuristics supported by TEACCH material.

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UNDERGRADUATE STUDENTS' UNDERSTANDINGS FOR DERIVATIVES IN MULTIVARIABLE CALCULUS

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Multivariable calculus is a generic extension of the single variable calculus. However, there are relatively few studies about how students understand and interpret the meanings of derivatives in multivariable calculus. This leads to the problem that we overlook the process of how multivariable calculus students extend their prior concepts through the course (Dorko & Weber, 2014). The concepts of derivatives in multivariable calculus can be generalized from those in single variable calculus, and generalization is an essential factor of mathematical thinking. Therefore, this study aims to analyze how students generalize the meanings of derivatives concepts in multivariable calculus from those in single variable calculus.

Eight undergraduate students who complete a course of multivariable calculus were selected, then given three interview tasks. Each student has participated in an individual interview. Tasks were based on (1) what are students' understandings for derivatives in single variable calculus, (2) those in multivariable calculus, and (3) how students relate and extend those two understandings. The interviews were analyzed based on actor-oriented generalization framework; generalizing actions (relating, searching, and extending), and reflection generalizations (identification, definition, and influence) (Ellis, 2007) with symbolic and geometric understandings.

Firstly, students understand the meanings for derivatives both symbolically and geometrically in single variable calculus, and symbolically in multivariable calculus, but go through difficulties in geometrically in multivariable calculus. Secondly, students only showed generalizing actions on relating the symbolic understandings, yet suffered on the other cases; relating geometric understandings, expanding both symbolic and geometric understandings. Finally, in students' modified responses, their reflection generalizations on identification were found in both symbolic and geometric understandings for derivatives in multivariable calculus weakly. This study provides a helpful perspective on multivariable calculus learning by analyzing how students generalize their prior knowledge. Thus it contributes to research on identifying the process of students' understanding multivariable calculus concepts.

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PATTERN OF MEDIATION BY AN EARLY CHILDHOOD TEACHER TO MOBILISE MATHEMATICAL KNOWLEDGE USING CARDBOARD PIECES

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The teacher has a major role as a mediator in the construction of meanings based on actions performed with artifacts. Mariotti (2009) found some patterns of teacher action that refer to teacher interventions that "share the common goal of promoting both the unfolding of the semiotic potential of the artifact and the co-construction of common signs" (p. 434).

We have been carrying out a qualitative and interpretive study with the video recording of a class of 5 years preschool teacher, following the first pair of categories of Mariotti (2009): *Ask to go back to the task* and *Focalize on certain aspects of the use of the artifact*. In this session, the artifact is made up of pieces of cardboard whose use is managed by a teacher to promote the construction of mathematical knowledge. These pieces are made up of different types of boxes, cardboard tubes and cardboard cutouts for packaging, all reused material.

For the teacher, one of the potentialities of the cardboard lies in the versatility to work the shapes, which leads her to guide the students so that they focus their attention on the characteristics of the different shapes of the cardboard pieces. In a situation of exploration of these pieces, a child names the squares and the teacher's insistence asking for square shapes awakens in another child the possibility of cutting the rectangular faces to obtain squares. Thus, a mathematical element emerges that involves the abstraction of decomposing a rectangular surface into squares by making parallel cuts to the smaller side of the rectangle.

The elements that characterise the way she orchestrates the process of instrumental genesis are the continuous exposure of the students to questions in a context of observation and manipulation. The pattern the teacher follows is a combination of the category "*Ask to go back to the task*", particularly motivated by a student's response, followed by the category "*Focus in certain aspects of the use of the artifact*", which encourages another student to mobilise complex mathematical knowledge.

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LEVERAGING THE ROLE OF MATHEMATICS TEACHING COORDINATORS TO LEAD EFFECTIVE PD FOR TEACHERS

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Effective, scalable, and sustainable professional development (PD) programs for teachers aim to improve teaching quality. Implementing PD in scale requires designated instructional materials, professional learning communities (PLCs) and systemic support, as well as the preparation of qualified PD leaders. The study relays on the effective PD model regarding the design features (Darling-Hammond et al., 2017), and the professional identity (PI) model regarding the teaching expertise (Beijaard et al., 2000), while elaborating it to suit PD leadership expertise.

The study explores a PD program that prepares mathematics teaching coordinators to apply effective PD at PLCs within their school teams. COVID-19 emerged during the PD program application, changing the program and teams' PD sessions from face-to-face (F2F) to online. We explored the PD program and school team's PD design before and during COVID-19 emergence, detecting effective PD features and orientation to support and demonstration of PD leadership expertise.

Participants were 29 coordinators and 86 teachers. Research tools include qualitative inquiry of the PD design and PD leadership expertise, through observations and reflective questionnaires in both samples, and a quantitative (pre-and-post) questionnaires to measure the coordinators' perceived PD leadership expertise.

Findings revealed that the program preserved diverse effective PD design targeted to support PD leadership expertise before and during the COVID-19 period. Coordinators applied mainly content focused PD in both periods, more intensely in the online period demonstrating didactical (32% vs. 17%) and subject matter (16% vs. 7%) expertise. In the F2F period they applied more expert support (23% vs. 11%), and collaboration (17% vs. 11%) as pedagogical expertise. Despite the program support and continuous team PD even during a crisis, the coordinators rated their subject matter ($T(28) = 2.30$, $p < .05$) and didactical expertise ($T(28) = 5.37$, $p < .001$) as lower compared to the program support. The study connects effective PD and teachers' PI theoretical models and suggests an effective and sustainable PD for teachers which is applicable in scale.

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TEACHERS' QUALITATIVE REPLICATION OF RESEARCH

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In preparation and professional development programs, teachers are often required to study academic educational research, whose findings purport to have implications for practice. However, teachers often do not appreciate the relevance of such studies for practice, and as a result may not see *how*, or even *why*, to "implement" their findings.

Labaree (2003) described cultural differences in the world-views of teachers and education researchers. We propose that these differences can frame the notion of *relevance* – for educational research to be perceived by teachers as relevant for practice, is must be situated in teachers' world views, which tend to be more normative than analytical, more personal than intellectual, more particular than universal and more experiential than theoretical (Labaree, 2003). We propose that qualitative replication of academic research by teachers can increase its relevance in this sense.

In a course on teaching and learning geometry, practicing teachers in a graduate program were required to design and conduct a small qualitative replication of research by Fischbein and Kedem (1982), who found that many high school students who accept the universal validity of a proof, still maintain an empirical attitude that relies on examples. The purpose of the replication activity was to encourage teachers to take an inquiry stance towards teaching, while shaping this inquiry in ways that are personally meaningful. Changes that teachers introduced at various stages of the research – preparation (mathematical problems used, selection and size of research population), data collection (emergent interview questions), analysis (what teachers noticed), conclusions, and perceived implications for practice – demonstrate how they culturally aligned the research along the four dimensions of Labaree's framework.

Evidence on how teachers, in their qualitative replications, can transform academic research into something personally relevant for their practice, and our theoretical framing of this relevance, contribute to an emerging line of research on replication and implementation of research findings in mathematics education (Jankvist et al., 2021).

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RACIALIZED MATHEMATICS NARRATIVES OF BLACK UNDERGRADUATE STUDENTS IN ONTARIO

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Across Canada, education has a long and persistent history of excluding Black people that has resulted in racial achievement gaps at almost all levels of education. Despite the challenges, more Black students are joining university in STEM programs. Research on racialized experiences of Black students indicates a general focus on the K-12 system and that race-related experiences are often associated with a personal deficit. The aim of this study is to explore the racialized mathematics experiences of Black undergraduate students in Ontario – Canada's most populous province. The study contributes to research by disrupting deficit-centred discourse around Black student experiences while amplifying Black student voices and validating their strengths. The study is guided by two questions: 1) How do Black undergraduate students describe their racialized mathematics learning experiences? And 2) What resilience strategies do Black undergraduate students of mathematics employ to navigate unfamiliar/challenging learning environments?

The research employs a narrative inquiry methodology (Clandinin, 2013), which involves autobiographical accounts of lived experiences and the identification of the meanings that participants attach to those experiences. Eight students from four different universities in Ontario were interviewed by means of one-on-one dialogic conversations shaped by semi-structured research questions that were meant to understand their lived experiences. The community cultural wealth model (Yosso, 2005) was employed as the theoretical framework to analyze the collected data. This model explores an asset-based perspective and holds that students have funds of knowledge and various forms of capital that they can draw upon to attain success in their academic endeavours despite existing challenges.

Preliminary analysis highlights participants' voices. For example: "There was this incident where a classmate tried to put me down. I don't know if being Black played into it. There was a question and I suggested, let's proceed this way and a colleague, unprovoked, just like blurt it out, you know, like your answer couldn't possibly be right because you're a woman, a Black woman." Findings reveal that Black undergraduate students draw upon different sorts of knowledge and capital to navigate their racialized mathematics learning experiences and to inform their resilience strategies.

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UNDERSTANDING COVID_19 PANDEMIC WITH STEM EDUCATION

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An integrated approach of STEM (Science, Technology, Engineering and Mathematics) education with a focus on the “M” of Mathematics (steM) is being defended in the literature to correspond to the real-world needs (Stohlmann, 2018).

This paper uses activity theory (Engeström, 2001) to discuss steM practices. In this regard, a case study of a teacher who participated in a Professional Development Programme (PDP), and implemented mathematical tasks related with the real-world scenario of COVID_19 pandemic is presented. Elisabete is a Mathematics and Natural Science teacher who developed steM tasks to raise awareness of her students regarding several scenarios of transmission of the virus. Activity theory is used to analyse data collected from participant observation and interviews during the workshops of the PDP (Costa et al., 2020), and Elisabete’s portfolio containing evidence of the work implemented in class. Based on different scenarios of transmission (*Object*), students (*Subject*) produced tables (*Tools and signals*) representing the increasing number of infections, and discussed the results in class (*Community*), mediated by the teacher. In this research, it is discussed how students understood the evolution of the pandemic and the need for measures to prevent it (*Outcome*). It is concluded that the steM approach in the mathematical class was more effective regarding the comprehension by the students of transmission of the virus. Although the teacher had already addressed the topic in the science class, she found that it was based on interdisciplinary mathematical tasks that students finally understood the scenario of transmission of the virus, and consequently the need for measures to prevent the transmission.

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"MISSIONARIES" NO MORE: TUTORS AS BENEFICIARIES IN A COMMUNITY-BASED TUTORING PROGRAM

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Informal learning spaces play an important role in the formation of students' relationships with mathematics. More often than not, struggling learners have found an opportunity to engage more successfully with mathematics when out of the classroom (Roberts et al. 2018; Saxe, 1988). Prepare2Nspire (P2N) is an example of an informal learning system that combines tutoring, mentoring, and culturally-responsive practices to directly cater to urban youth and their mathematics learning. When we think of informal tutoring environments, youth participants and their academic success are typically the focus. We argue that the undergraduate mentutor (mentor + tutor) also benefits from the interactions in these spaces, both academically and professionally. In addition, mentutors develop experiential knowledge that can in-turn transform the program for future participants. The research questions are:

- What key features of P2N impact mentutors' social and professional development?
- How can the experiential knowledge of mentutors be utilized to transform the program for future cohorts?

For this initial study, five mentutors participated in a singular semi-structured interview and narrative inquiry was utilized to better understand mentutors' situated identities and professional development while engaging with the program. The findings described the assets and personal impact of P2N as well as recommendations for future P2N cohorts. P2N provided mentutors with a number of tangible resources, including reliable meals, consistent pay checks, and graphing calculators. In addition, P2N was a scared communal space where mentutors formed long-lasting relationships with other college students, youth participants, and program directors. Finally, mentutors utilized their experiential knowledge to provide advice, grounded in authenticity and adaptability, for new undergraduate mentutors. These scholarly findings allow researchers the opportunity to reflect on mentutors' experiences, both before and after the COVID-19 pandemic, and reimagine informal learning systems as restorative spaces for both adult and youth participants.

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TAKING THE DIDACTIC TRIANGLE TO THE NEXT LEVEL: A 3D HEURISTIC MODEL FOR ANALYSING SUBJECT SPECIFIC APPROACHES TO MENTORING

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School based mentors have an influential role to play in the training and education of novice teachers. However, nationally and internationally support for mentors is sparse and rather generic, at the expense of subject specific support (Barrera-Pedemonte, 2016), while the mentoring issues discussed and reported in the literature are mostly of a generic nature, more concerned with general rather than with subject-specific teaching situations. Within the specific context of mathematics classrooms, subject specific mentoring has not received much attention neither in the UK, nor internationally. To fill this gap, the current study was carried out with the aim of tapping into and learning from experienced mathematics school mentors. The research questions about the knowledge for teaching mathematics that mentors foreground when supporting novice mathematics teachers. Guided by the 2D didactic triangle theoretical construct (Straesser, 2007), an interview protocol was developed to probe into school mentors' views of the kind of knowledge, skills and values they aim to instil in their mentees (novice teachers), and aspects of teaching and learning mathematics they bring to mentor-mentee conversations were also sought.

In this oral communication, we report on how the data collected from three experienced mathematics school mentors indicates that the Mentor-Mentee-Subject-Students relationships are quite complex, and not well captured when interactions were described using the 2D didactic triangle. It emerged that the new 3D adaptation is a more nuanced model, with potential to draw attention to all its components, such as 'facets' and 'edges' of interactions (e.g., Mentor-Subject-Mentee & Mentee-Subject), all of which interact in complex ways when mentors and mentees work together with a focus of the specific subject content. The potential such heuristic has as a tool to help to structure and ground mentor-mentees conversation in relation to the specific subject taught will be discussed and exemplified with data collected from interviews with three experienced mentors.

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A CONTRIBUTION TO NON-EXISTENT (AS YET) PHILOSOPHY OF CREATIVITY IN MATHEMATICS EDUCATION

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The paper initiates the discussion of the subdomain of Philosophy of Creativity in Mathematics Education (PoCME). Such a subdomain doesn't exist yet; to a degree this situation is the reflection of the fact that the general philosophy of creativity is also in the *statu nascendi* (Paul & Kaufman, 2014, p. 3). Philosophy of Creativity of any scientific domain is needed whenever the domain encounters fundamental unsolved problems, implicit ambiguities in research and practice. It analyses the concepts and methods of scientific theories, engages with the interpretation of their results and heuristically helps in formulation of new ones. It starts accordingly to Ernest (2018) with the critical examination of fundamental problems, two of which we discuss shortly here: What is creativity? What is the relationship between creativity and learning? We intend to address the issue of creativity and rationality in mathematics education.

The methodology of work here engages two sources: two recent volumes of philosophy of creativity of Paul and Kaufman (2013) and Gaut and Kieran (2018) to help identifying corresponding issues in Math Ed; as well as relevant research and practice within mathematics education, which bring the critical problems to the forefront, sometimes not "covered" by the general approaches (Czarnocha & Baker, 2021).

The problem of the different definition of creativity used in mathematics education is well known; we discuss two most often met approaches of Guilford's divergent thinking and of the stage theory of Wallas / Hadamard, show their difficulties in addressing creativity of all students in the context of curriculum, offer new definition of creativity based on the creativity of Aha! Moment and suggest that different student populations might be sensitive to different definitions (different aspects) of creativity.

The discussion of creativity and learning demonstrates that while we can always identify learning within the creative act, facilitation of creativity in learning requires special creative learning environment. We signal the issue as one of the critical problems in the quest to introduce creativity systematically into curriculum.

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TALKING ABOUT MATHEMATICS: HOW TEACHERS CREATE AND SHAPE DIALOGIC SPACES IN THEIR CLASSROOMS

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An increasing interest in dialogic teaching and learning has been largely linked to expectations of improvements in outcomes. Opportunities for student talk are considered valuable but the form and purpose of student dialogue is shaped by the nature of the teacher talk that precedes or accompanies it (Alexander, 2018). This may result in object-level or meta-level changes in classroom discourse (Sfard, 2015) which teachers influence as they use various repertoires and create a blend of dialogic forms within existing contextual frames, lessons and activities (Alexander, 2018).

In our research in England, 130 mathematics teachers in post-16 education are taking part in a nationwide randomised control trial of the use of a 'teaching for mastery approach', which provides a range of opportunities for classroom dialogue. Teachers develop the approach with the support of carefully designed lessons and professional development. Impact is analysed from student achievement and self-efficacy measures, whilst implementation is evaluated from teacher surveys and case studies. The case study data allows us to explore the research question of how teachers use opportunities in the lesson designs to develop dialogic spaces in their classrooms, and the reasons for 'in the moment' decisions as they teach.

Two contrasting cases are used to illustrate findings. Detailed observation notes and transcripts of teacher interviews are coded to Alexander's (2018) framework of repertoires and examined further using an iterative grounded approach. Early findings indicate that teachers adapt the opportunities for dialogue embedded into the lesson designs to develop dialogic spaces in different ways but retain a dominant form of teacher talk that orchestrates and influences outcomes. This analysis adds to understanding of how teachers interpret and use dialogic principles to create their own blend of dialogic spaces and indicates how carefully designed 'teaching for mastery' lessons might be used to bring about changes in teacher practice.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' SOLUTIONS TO AN OPEN-ENDED PROBLEM

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Problem-solving (PS) promotes higher order thinking skills, mathematical sense, ability to reason and to communicate mathematically. Moreover, PS enables to increase knowledge and stimulate mathematics learning (Cai & Lester, 2010). The relevance of this topic is also evidenced by the extensive research carried out (e.g., Liljedahl & Santos-Trigo, 2019). Studies on open-ended problems, those which can be solved in diverse ways and allow infinite solutions, show these tasks help improve mathematical understanding, argumentative, and decision making and problem-solving skills (Chan & Clarke, 2017). As activities that contribute to mathematical development, they must belong to future teachers training programs. In this context, we developed an open-ended PS activity with 16 prospective secondary mathematics teachers (PSMT). They had to find lines that intersect a given parabola at two points. The objective of the presented research is to analyse the PSMT's solution, identifying the strategy used, the kind of solutions PSMT look for and the types of representations they use.

Results show two main strategies used: search particular lines and search general conditions for all lines fulfilling the requirement. Most find particular lines and two approaches stand out: lines through two points on the parabola and parallel lines above the vertex. Focus on mathematical representations, those using graphic representation of the parabola are slightly higher than those who only use algebraic representation. Most used graphic representation to search particular lines and highlight that many of them make the same mistake with the vertex placing, presuming that it is on the ordinate axis. Among those trying to find general conditions, the use of algebraic representation predominates. None had a correct solution and most abandoned the process to give particular lines satisfying conditions found up to that moment.

Results suggest that PSMT have difficulties to find a general expression for the lines and they prefer to study particular cases. We continue analysing more characteristics of the PS process.

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INVESTIGATING A WEB-BASED TEACHER TRAINING ON PROBLEM SOLVING WITH A FOCUS ON INDIVIDUAL USAGE BEHAVIOUR

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Although problem solving is a core mathematical activity that is emphasized in national standards, it is still underrepresented in German mathematics classrooms. Therefore, problem solving is in the focus of many in-service teacher trainings. In some cases, the effectiveness of such teacher trainings concerning the development of teacher competences has been empirically investigated (e.g., Collet & Bruder, 2006). In most of these studies, varying competence growth was examined depending on different characteristics of the training offer. Offer-and-use-models, however, emphasize that investigating the mechanisms of effective teacher trainings requires to consider characteristics and the usage behaviour of different types of participants (Lipowsky & Rzejak, 2015).

Our research project aims to examine to what extent the characteristics of participants and their use of a modular training offer can predict their competence growth regarding problem-solving instruction. A web-based modular training offer was developed that consists of three chapters and for each of these chapters there are three types of offers with different degrees of situatedness: a) explanatory video, b) material package and c) classroom video. N=30 in-service teachers participated in our training over a period of seven months. For each chapter, the teachers could decide which type(s) of offer they choose according to their needs. Data about their usage behavior (quantity and quality of use) was collected by a training journal containing questions about how they used the different modules and what their take-home messages were. Initial results of a qualitative evaluation of the journals show that the assumed heterogeneity regarding the teachers' usage behavior can be confirmed. This does not only include differences in the quantity and quality of use of different modules, but also in the preferred order of use (e.g., from abstract to situated or vice versa). In the presentation, the design development and further results will be discussed in detail.

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SUPPORT MATHEMATICS TEACHERS' TECHNOLOGY TRAINING IN PORTUGUESE-SPEAKING AFRICAN COUNTRIES (PALOP)

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In our presentation, we will outline results from the implementation of a project aiming to support, by developing culture-sensitive professional development, mathematics teachers in PALOP in the integration of technology into their practices. Participants of this project were mathematics teachers from Cape Verde, Mozambique, and Angola. In PALOP Portuguese is an official language, among other local languages, being a factor of national unity. Also, since the independence of these states, there are strong cultural and cooperation relations with Portugal, particularly in education. Consequently, in the PALOP, mathematics curricula, teaching materials and textbooks have many affinities and have been influenced by the curricular options in Portugal, so it was essential to start from this context to develop this project.

Our vision of professional development of teachers is not common in PALOP. However, this project encourages teachers to develop their skills in mathematics and education in the context of their practices and in autonomous lifelong learning. Strategically, offered training courses promotes a systematic reflection on teachers' practices, encouraging research in the classroom, through the development of teaching experiences (TE), without neglecting the publication of the results of TE developed during training. GeoGebra is used in the project because: allow the teaching of various topics of school mathematics by multiple approaches; facilitate the b-learning training project; support a large global community of Portuguese-speaking teachers; it is a free software for educational use. In our research, the sources of qualitative data are diverse, namely the TE developed by the participants, therefore the models of progressive TPACK, considering the Rogers decision process and innovation model (Lyublinskaya & Kaplon-Schilis, 2022), was crucial to guide the research. In addition, training improvement uses an inventory of teachers' practices and knowledge in GeoGebra, applied before and after training courses, already in use since 2016. Thus, we will outline how participants promoted changes in their practices, using technology in their classrooms to promote mathematics learning, highlighting how the context was crucial to the project's achieved results.

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TEACHERS INCORPORATING 6-YEAROLDS' INPUT IN MATHEMATICS TEACHING

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The potential for student mathematics learning lies both in the teacher ability to ask questions and to follow up and incorporate student input into the teaching of a specific content (Murata, 2015). Swedish students are expected to engage in reasoning and collective problem solving in highly communicative teaching practice. To improve these learning situations, it is important to understand how teachers are making use of student input in teaching. In this study we seek to map and understand how teachers in preschool classes respond to and incorporate student input in mathematics teaching.

This paper reports on findings from a study focusing on mathematics teaching in preschool classes in Sweden (6-yearolds). The data consist of 145 observations (from 95 individual teachers) of mathematics teaching relating to whole numbers. The data for analysis consist of fieldnotes and was collected during fall 2021. The MPM-framework "Mediating Primary Mathematics" (Venkat & Askew, 2018) was used as an analytical tool to identify the ways teachers in preschool classes respond to and incorporate student input in their mathematics teaching. Following the four levels in the framework, the results show that in 61,4% of the observations, teachers give students very little opportunity to contribute with input beyond short responses, merely confirming or giving generally encouraging responses to the student. This is to be compared to 29,7% of the observations, where teachers take advantage of student input by incorporating these into the teaching situation to advance or verify students' mathematical reasoning. In the third largest group (8.3 % of the observations), the teachers pulled back or made no evaluation of the student input. Only in one case (0.7 % of the observations), the teacher took advantage of the student input and both advanced and explained it further to support learner progression.

The results raise questions about how teachers' ways of responding to and elaborating on students' input might influence students' opportunities for learning about numbers. In particular, when teachers advance, verify, and explain student input, significant connections and justifications for solution methods are highlighted.

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DYSCALCULIA IN THE SCIENTIFIC LITERATURE: A SYSTEMATIC REVIEW

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Dyscalculia is a learning disorder that affects the correct acquisition of arithmetic skills. Despite having an estimated prevalence of approximately 5% among the school population, it is an unknown disorder in society and it has received much less attention than others disorders, or learning disabilities, such as dyslexia. The objective of this research is to examine the main features of the most influential literature on dyscalculia.

We conducted a systematic literature review of the term *d*scalculia* focusing on titles and abstracts in the publications of the Web of Science and Scopus databases between 1970 and 2020. We followed the main lines of the systematic review methodology used by García-Peñalvo (2017). English and Spanish publications were included and publications with less than 20 citations and with adults were excluded. We obtained 502 documents in WoS and 1220 in Scopus and after four screenings in the review process, applying the inclusion and exclusion criteria and removing the duplicates, we retained 102 publications. We read and coded them by number of citations, bibliographic reference, topic, type of research and aim and contributions research. Paying attention to the topics developed by the studies, the results indicate that an evaluative or diagnostic tendency is the most dominant, where the researchers evaluate the mathematics skills of children with dyscalculia, who present a lower math performance than typical achievers. The second tendency is related with the neurological alterations involved in this disorder, followed by a theoretical tendency where the publications offer a review of the most relevant evidences of dyscalculia. Finally, in order, stand out studies about the comorbidity of dyscalculia with other disorders and learning disabilities, the intervention or treatment, its classification and the relation with math anxiety. To conclude, although the diagnosis of dyscalculia has evolved considerably, incorporating more and better detection techniques and instruments, there is still the need of more research on the intervention.

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MULTIMODAL RESOURCE USE IN DECISION MAKING FOR MAXIMISING EARNING

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Multimodal approaches drawing from literacy and conversation analysis have now become popular in studying mathematics classroom interaction (Farsani, 2015). Multimodality refers to complex repertoire of semiotic resources that people draw in different social spaces and involves language and other means of meaning making such as images, text, graphic symbols, three-dimensional forms, speech and gesture (Bezemer & Jewitt, 2010). Multimodality focuses on the relationships between visual and non-visual modes such as speech, intonation and paralinguistic features within a communicative event. It perceives that the full meaning of ‘the spoken language’ can only be derived from the relation of each of its parts to one another, namely, gestures, embodiment and nonverbal aspects of communication woven into social interaction, demonstrating a holistic understanding of language and language use.

Over the last two decades, researchers in mathematics education have attended to the role of nonverbal communication in traditional mathematics classrooms, but similar exploration of language and communication in out-of-school spaces is few. This presentation will focus on the role of ‘language’ and mathematical meaning-making practices in an out-of-school everyday context of trade and economic transaction in a low-income settlement. It will explore how a young teenage phone repairer Salman (pseudonym, a seventh grader in a government-run school in Mumbai, India) guesses his prospective client’s paying capacity through body language, gestures, dressing style, scent and speech model as nonverbal cues. He would make his decisions primarily based on deductive reasoning by ‘observing language’ his prospective clients use, empowering him to make better mathematical decision-making by optimising resources and maximising earnings. The community’s resource-rich work contexts accessible to its dwellers have made Salman as expert in hands skills as well as in dealing with customers using multimodal resources. These multimodal and hands-on mathematical decision-making practices are community-based resources available in sociocultural-historical contexts but are often hidden from school curricular domains even though these practices possess crucial pedagogic cues about what fosters mathematical meaning-making and through what kind of processes.

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RESEARCH-TO-PRACTICE-TO-COMMUNITY-ENGAGED RESEARCH: EMPOWERING COMMUNITY LEARNERS

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Equity-centered mathematics research and practice provides a powerful opportunity for educators to resist deficit constructions towards marginalized youth and contextualize mathematics for all learners. These critical practices should not only combat deficit framing but also create a dynamic relationship between community culture and school culture (Ladson-Billings, 1995). We argue that in order to transform the mathematical experiences of urban youth, educators must embrace community culture as a valuable knowledge source and utilize students' funds of knowledge (Civil & Bernier, 2006) to diversify teaching practices and curricula. The goals of this research project aim to: 1. provide an example of culturally responsive mathematics teaching and learning, 2. center the assets and communal histories of urban youth, and 3. intentionally embrace the collaborative networks that support student learning.

Community-engaged methods were utilized to create culturally responsive activities for Prepare2Nspire (P2N), an after-school mathematics tutoring/mentoring program for middle and high school students. Seven community members (e.g. parents, college students, and elders) were interviewed about their experiences in the predominately African American neighborhood which houses P2N. Themes from the semi-structured interviews included the history and assets of the neighborhood, community efforts to counteract injustice, and social determinants of community prosperity. These themes supported the creation of three mathematical activities: a. two data analysis activities representing various resources for youth and b. a series of mathematical tasks that can be completed using a coordinate plane constructed from local institutions mentioned in participant interviews. Students were observed during P2N tutoring sessions as they completed these activities in a community setting. This line of research explores the process of authentically connecting the traditionally rigid content area of mathematics with the beauty, brilliance, and histories of communities that are not always seen through an asset-based lens. The overall goal is to transform the mathematical experiences of urban youth while also strengthening students' understanding of their own cultural identities.

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VISION OF HIGH-QUALITY MATHEMATICS INSTRUCTION: WHAT SHOULD ELEMENTARY SCHOOL PRINCIPALS NOTICE?

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Principal leadership plays a significant role in improving teachers' mathematics instructional practices. This requires a vision of high-quality mathematics instruction– the discourse that principals use to describe ideal classroom practices that are not necessarily mastered yet (Munter, 2014). We hypothesize that for principals to develop their vision and then support their teachers' learning effectively, their noticing skills are key: to attend to specific aspects of mathematics instruction, interpret them using frames of reference that characterize high-quality practices, and discuss the teacher's instructional decisions (Sherin et al., 2011). Hence, we ask: How does the principals' noticing of student mathematical thinking evolve in the context of observing and debriefing fourth-grade mathematics lessons? How is the noticing of each principal related to their instructional vision of high-quality mathematics instruction?

We report on a research-practice partnership with six elementary school principals that aims to develop their instructional leadership practices, so they can support their teachers in creating more socially just mathematics classrooms for a predominantly Latinx population. Principals engaged in five monthly visits to a fourth-grade math classroom. We draw on four sources of data to examine principals' learning process: audio transcripts of the sessions, a noticing task at the end of each session, field notes, and interviews. A coding system informed by teacher noticing research was applied to the data, allowing emergent codes and then triangulating the data. Twenty percent of the data were double coded by independent researchers.

Initial findings suggest that principals' noticing became more sophisticated across visits from attending to general aspects of the lesson toward noticing specific elements of students' thinking and of instructional moves. The discussion rarely included the breakdown of conceptual underpinnings of mathematics ideas. Final interview data on participants' vision will be analyzed alongside responses to the noticing task to examine relationships between the two constructs. These results contribute to the scarce existing research on principals' noticing and on how noticing might contribute to effective instructional leadership.

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PRE-SERVICE TEACHER'S CONCEPTIONS ABOUT THE USE OF THE HISTORY OF MATHEMATICS AS A DIDACTIC RESOURCE

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Previous studies have pointed out the importance and advantages that the History of Mathematics can have as a didactic resource, not only to teach mathematical concepts, but also to show that mathematical knowledge is part of our culture. The role that History of Mathematics can play in Mathematics Education and how to integrate it into teacher education remains, however, an open question (Clark et al., 2018; Fauvel & van Maanen, 2000).

In order to find out what are the attitudes, knowledge and beliefs of trainee teachers about the use of the History of Mathematics as a teaching resource, a survey based on the one used by Alpaslan et al. (2013), has been carried out in two Spanish universities, among 2nd and 3rd course preservice teachers (N=141), also retrieving information about their initial training, their usual qualifications in mathematics and their knowledge of History of Mathematics.

Using the SPSS statistical program, lineal clustering, analysis of variance (ANOVA), correlation study and comparison of means has been done, seeking for relationships between previous studies, knowledge and the main conceptions about the use of mathematics as a didactic resource. Preliminary results show that most of the preservice teachers are prone to use the history of mathematics as a didactic resource (71%), and are aware of its advantages, as a way to understand mathematics in depth (73%), realizing that mathematics is a universal creation of different cultures (81%). But most of them do not know how to integrate it into their future classes (65%). Cluster and ANOVA analyses show that these general results are independent of their previous studies and even of their history of mathematics knowledge. These results can be considered when designing mathematical training of future teachers.

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THE DYNAMIC QUALITIES OF MATHEMATICAL MOVEMENT

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In mathematics education research, movement is increasingly linked with cognition. Within these studies students' movements are more than a means of communication, or an adjunct to understanding: movement provokes new mathematical knowing. Although a variety of methods for observing and analysing movement have arisen, these usually focus on the quantitative aspects of movement (how the body moves in space) with qualitative aspects (how the body feels to move) often omitted.

Movement analysis has a long history across many fields including the arts, industrial work studies, and computer interface technology. Laban's movement elements (Moore & Yamamoto, 2012) provides a well-established movement framework which classifies quantitative aspects of movement while also paying careful attention to the inherent, and intertwined, dynamic qualities of movement. By employing Laban's movement elements, this study explores how qualitative aspects of movement can provide further insights into students' emerging mathematical doing and knowing.

As part of a doctoral thesis, this study examines the full body movements of a group of four non-mathematics major bridging education students as they engage with a mathematical task. To encourage movement, the students are in large open room with no tables and chairs and presented with a modular arithmetic task. The task, throwing a ball around a group, is written on A3 paper posted on a vertical whiteboard with counters and marker pens. During a one-hour session the students' activities are recorded by three video cameras and the students' movements and verbalizations are transcribed. The fragments of the session chosen for micro-analysis represent discrete segments of the wider session and include a variety of students' movements.

In analysis of the students' dynamic movement qualities, from one fragment, a movement emerges as a sketch of a solution for a smaller problem. Another movement uses qualities to differentiate, then combine, two problematizations, demonstrating the students' evolving knowing in movement. This study argues that Laban's movement elements provide researchers a means to analyse the dynamic movement qualities embedded in any movement thus providing further access to, and information on, students' mathematical knowing.

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STUDYING EINSTELLUNG EFFECT THROUGH STUDENTS' GAZE BEHAVIOR

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The Einstellung effect refers to an individual's tendency to solve a set of problems using a certain complex method repeatedly despite the existence of more direct or simpler methods. In mathematics education, this can manifest in students' routine application of similar solving methods in new or different problem situations. Overcoming such is important for developing mathematical thinking. While the effect is well understood in the psychology of problem-solving (see Abramovich, 2018), we believe that eye-tracking research holds promise in determining the cognitive mechanisms that underpin the persistence of the effect.

Using a post-test two group research design, 17 senior high school students (ages 17-19) were asked to solve 10 algebraic problems adapted from Luchins' (1942) water jar problems. From the results, 13 students were identified to be affected by the Einstellung effect, while the remaining 4 were not. While the students solved, eye-movement indicators, such as "number of fixations" and "fixation duration" on stimuli and non-stimuli parts of the problem were captured using an eye-tracking device. These were compared between the two groups. Cross-analysis of interview and eye-tracking data based on students' solution method and order of viewing of the stimuli in the problems were also carried out.

Preliminary analysis based on 2,000 bootstrap samples using simple method showed a very strong positive correlation between the number of fixations and fixation durations among all participants, $r(168) = 0.97, p < 0.01$. Fixation durations on the white space of affected and not-affected group did not greatly differ, 95% CI [3.68, 5.11], [4.84, 7.78], but it was found that participants not affected by Einstellung demonstrated more fixations on the white space of the screen than those affected, 95% CI [0.56, 11.18]. This latter result suggests that deliberative thinking, as indicated by the students' fixation on non-stimuli areas, might aid in overcoming the Einstellung effect.

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FLEXIBLE THINKING WHEN WORKING WITH GEOGEBRA

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In the field of mathematical-related affect, one demand is the connection between affective constructs and cognition (Hannula et al., 2019). The contribution of GeoGebra to learning and problem solving has been widely informed and its potential to foster students' motivation and involvement is recognized. But the question remains open about how to capitalize on this initial motivation to build sound, positive mathematics attitudes, such as Flexible Thinking, tightly linked to mathematical activity and cognition. Flexible Thinking implies being able to change the direction of mental processes when the situation so requires. A student shows this attitude when: (a) it doesn't get blocked solving a task in a steady way, but considers alternatives and redirects its mental processes; and (b) it doesn´t change its mind without conviction, but understands it is the right way to proceed (Zaldívar et al., 2006).

To date, much research on mathematics-related affect relies on the participants´ self-reports where doing mathematics is simply a context. We intend to transcend this lack of mathematical specificity through a design-research study that addresses the “how” question. The study was carried out with 46 students, 14-15 years old, at a public school in Spain. Along two months (25 one-hour sessions), two sequences of geometrical tasks were implemented with the same methodological principles, except for the use of the GeoGebra, which was introduced in the second sequence to work on plane tessellations. Data were taken observationally, using (a) and (b) as indicators, and through the students’ productions and audio recordings while solving the tasks, which were analyzed by means of the software Atlas.ti. Quantitative results show that the percentages of sessions in which most of students (over 2/3 of the total) manifested Flexibility of Thought was 11.5% for paper and pencil tasks and 66.7% for GeoGebra tasks. Qualitative data show how, by shortening the time to do mosaic calculations and representations, GeoGebra enabled the students to try to solve tasks in different ways and think of alternatives. The feedback offered by the software after each action (interactivity) allowed the students to calibrate the rightness of their strategies, changing track if deemed necessary. The accuracy for calculations and representations (constructivity) made the students realize that errors were due to an inappropriate strategy or action, which led them to a justified change in their procedure.

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CarRace: A MULTIDISCIPLINAIRE ENGAGING PROJECT

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We need to find alternatives to the norm examinations which consist in performing algorithms and memory work (Wagner, 2021). We should use projects that develop a different group of competencies like self-learning, self-assessment, planning, dealing managing deadlines and challenging problems. It is also desirable that students became engaged with their work and mathematics (Skilling, 2021). This may be fostered by using multidisciplinary applied projects where students see the real use of mathematics and get new competencies as, for example, programming.

From 2017 to 2020, students of a graduation in Informatic Engineering and Multimedia had as complementary assessment a project named CarRace. It consisted in using parameterizations and programming to make a car perform a race on its own circuit. The research questions are: Is CarRace an engaging project? Do students perceive its interdisciplinarity as important? Does it help students understand the subject in a deeper way? The research methodology is a Survey Research - Successive Independent Studies. We used as a tool an anonymous survey available each year on Moodle to all students of the course. It was available to a total of 473 students, and it was answered by 183 of them.

The answers of the students to the question whether CarRace project made them pleased or displeased was that around 75% felt pleased or highly pleased. Around 50% of the students believe that the project made them understand better the subject. Very few disagree with "Make sense to me to use Python transversally in graduation, therefore also in curricular units of mathematics". Around 80% agreed that "It was interesting to see in CarRace an immediate application of Mathematics." Globally, nearly 80% of the students classified CarRace as a positive project.

As conclusion, the CarRace project is engaging to students, its interdisciplinarity is important and makes students understand the subject in a deeper manner. It develops unusual competencies. Therefore, it is considered a positive project and it is able, and recommendable to be used in other higher education institutes.

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STUDENT CHALLENGES IN ABSTRACT ALGEBRA: HOW DO INSTRUCTORS REACT AND RESPOND?

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We investigate the challenges that college students reported having in abstract algebra courses across the United States and how their instructors reacted and responded to those challenges. Because learning about students' thinking can be difficult in upper-division mathematics courses, instructors are often left to rely on their own experience of teaching these courses to anticipate student challenges and make plans for addressing them. In response to this issue, we implemented a classroom assessment technique (Angelo & Cross, 1993) for five abstract algebra instructors and their 64 students during Fall 2021. The technique was a feedback process from students to instructors about the challenges that they faced in the course, distributed to students via bi-weekly open-ended surveys. Every two weeks, we asked the students about the mathematical content they found challenging in their abstract algebra courses; we then showed each instructor their students' anonymous responses every few weeks and inquired their reactions as well as changes they planned to implement in their teaching.

Relying on the notion of the didactical contract by Brousseau (1984), we expected that upon learning about students' challenges, instructors would feel obligated to help students with learning the content and react and respond in ways that would assist students in overcoming those challenges. In our oral communication, we (1) show challenges with learning abstract algebra reported by students, highlighting those not mentioned in the literature before; and (2) report instructors' reactions and responses to students' challenges. While the instructors were often not surprised by their students' challenges, we found a wide range of reactions and responses, with some unique to individual instructors. Slowing down and reviewing the content were mentioned by more than one instructor and were the most common. We propose additional research regarding a more frequent regime of feedback.

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LOSING MOTIVATION – MATHEMATICS RELATED VALUES OF STUDENTS WHO DROPED OUT FROM UNIVERSITY

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Motivation is a relevant factor for study dropout in mathematics. Common conceptualizations frame this motivation within the situated-expectancy-value framework (Eccles & Wigfield, 2020). Accordingly, a dropout decision is related to the individuals' expectancy for success and the importance or value the individual attaches to the various options. Herby low values and expectancies, as well as decreases in those, are known as high risk factors for study dropout. Specifically, the individual hierarchy of values emerged from the specific social background is assumed as relevant, but underrepresented in research on study dropout from mathematics (Eccles & Wigfield, 2020). Understanding these hierarchies of values and their development in the social context, might help to prevent motivation-based dropout decisions.

Our research goal is to investigate how expectancies and individual hierarchies of values emerge in the social context of the transition to university mathematics. We draw on semi-structured interviews with seven students (4 female) that were originally conducted by the third author between 2016 and 2018 at a large public German university (Geisler, 2017). All students dropped out from university mathematics after having attended at least the first weeks and at most four semesters of their studies (proof-based, formal mathematics courses). We coded their interviews deductively in line with the situated-expectancy-value theory. Our findings suggest that especially attainment value, as well as psychological costs and expectancies may be highly influenced by the social context experienced during transition to university mathematics and may thus lead to dropout: Students that leak social references, may devalue the subject, even when performing extremely well and thus showing high expectancies. Low expectancies were associated with high psychological cost, while both were highly depending on social references as well. Moreover, values may have different dimensions, related to one's specific self-image as well as beliefs about mathematics. Addressing these dimensions in quantitative studies, may give clearer insights about student's dropout decisions. Value-interventions may as well focus on the overall social context, rather than the individual. Further results and implications well be given more detailed in the presentation.

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KNOWING STUDENTS' STRATEGIES BEFORE THE LESSON AND LEADING WHOLE-CLASS DISCUSSIONS

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Lesson study is a formative process, focused on students' learning, where a group of teachers works to plan a lesson in detail (Stigler & Hiebert, 1999). We report two lesson studies, aiming to analyze the strengths and the constraints of leading the whole-class discussion, knowing students' strategies during lesson planning.

The research follows a qualitative approach. Data collection includes recordings of sessions (Sx) and interviews, as well document collection. One lesson study involved two in-service teachers with several years of teaching experience and the facilitator was the first author. The other lesson study involved three prospective teachers without teaching experience and was facilitated by the second author and the teacher educator.

After analyzing students' strategies, the in-service teachers decided to explore in whole-class discussion a strategy that surprised them: "he really said 'half of 12.5%' ... he is not stuck to the formulas [of conditional probability], he knows how the numbers are related" (Sofia, S7). However, they considered that leading it right after the students' autonomous work is more productive since "the students immediately explained what they did... and it makes more sense for them" (Sofia, interview).

In the prospective teachers' case, Maria planned the lesson in detail intending to promote "student-teacher interactions" (S4). She considered that "since I already knew the students' strategies, ... nothing had surprised me" (S15). However, she recognized that "there weren't as many interactions as I intended to ... It was difficult to trigger interactions from students. ... Maybe I could pose more open-ended questions" (S15).

The results suggest that analyzing students' strategies can help the teachers to lead the whole-class discussion by reducing the complexity of the in-the-moment decisions they have to make considering students' work. However, both the in-service and the prospective teachers considered that the time gap between students' work and the whole-class discussion can lower their involvement in the discussion. Nevertheless, during the lesson study, the teachers could rethink and improve their teaching practices.

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MEANINGS OF LIMIT OF A FUNCTION ENRICHED BY MEANS OF REPRESENTATIONS AND MODES OF USE

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Some research on the conceptions of the limit in high school and university students shows that many of them have an intuitive notion of the limit, which they describe with terms such as tend, approach, reach, exceed and limit. Fernández-Plaza et al. (2013) described and interpreted the definitions of high school students on the concept of the finite limit of a function at a point in terms of structural aspects compiled and synthesized from previous research (e.g., Cornu, 2002). These aspects were adapted and validated in a reliable category system that allows the analysis of definitions of the limit concept in an orderly, systematic and replicable manner (González-Flores et al., 2021). The work highlighted the dual concept of object-process that students attributed to the limit. Using these categories, which are supported by a theoretical framework related to Frege's semiotic triangle based on reference, sign and sense, we analyse the enrichment of the meaning of limit of Calculus' students at the National University of Costa Rica not only when defining but also when representing and provide modes of use of this notion.

From representations and modes of use expressed by the students, the findings include that graphic representation is the most used by students and also is the representation that provides more information from all our categories of analysis. Though a minority of students provide vague applications for the concept of limit, almost all students offer several terms and modes of use for this concept. We emphasize that, although representations and modes of use are not usually required in a standard definition of limit, they are essential to utterly construct its meaning. We conclude that the use of different representation systems and the mention of applications, and terms and modes of use of the limit enrich the meaning that students express of this notion. Expressing ideas about the limit in other way than defining is nothing more than completing conceptions of the limit.

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COMPARISON PROBLEM SOLVING BY AUTISTIC STUDENTS

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Given the positive effects of inclusive education and the recent increase in the prevalence of people with autism spectrum disorder (ASD), students with this diagnosis are increasingly incorporated into mainstream educational programs (Whitby, 2013). In general, these students require additional support for mathematical learning. For this reason, several research studies try to go deeper into the teaching and learning of mathematics with ASD students.

We present a research work where the effectiveness of a modified schema-based instruction (MSBI) for teaching additive comparison problem solving is evaluated with three 7 to 9-year old students with ASD. Modifications of the traditional SBI that consider the cognitive characteristics of individuals with ASD have been carried out (e.g. Polo-Blanco et al., 2021), like problem-solving guidelines to help with planning deficits and visual support with pictograms to help with verbal comprehension deficits. A multi-baseline across students design was followed as a research methodology to demonstrate a functional relationship between the MSBI instruction and students' improvement. The results show that all three students acquired the skills taught, improved their performance from baseline, and maintained the skills four weeks after the end of the instruction. In addition, two of them generalized the acquired abilities to two-operation problems.

It is necessary to continue making contributions aimed at improving the mathematical skills of the population with ASD, as this promotes the transfer of skills to everyday situations and facilitates the search for employment in adult life. In short, the effects of an appropriate educational intervention could be translated into an increase in the levels of self-esteem, autonomy and development of people with ASD entering adulthood.

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A TEACHING EXPERIEMENT ON FRACTION DIVISION

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Division of fractions is among the most challenging concepts in elementary school mathematics. Teaching methods for this concept are typically based on procedures that students do not understand, namely the “invert and multiply” algorithm. This operation is often linked to sharing contexts, but the meaning “measure” of division is another way to think about this operation (Greer, 1992). The use of models and the common denominator algorithm makes it easier for student to understand this meaning of division and allows to verify how many times the divisor fits into the dividend.

This study aims to identify grade 5 students’ understandings of fraction division operation as a result of a teaching experiment and their knowledge at the end of the experiment. The work carried out emphasized the use of geometric models to promote students’ understanding in problem solving contexts. The participants were four grade 5 students from a public school in Portugal, who were studying for the first time the concept of fraction division. Data was gathered through students’ work on classroom tasks, an individual test applied after the teaching experiment and individual interviews. During the experiment, students were asked to represent the initial quantity (dividend) using geometric models and to verify how many times de divisor fits into that quantity. The transformations carried out in the models were symbolically recorded so that the students could discover the (common denominator) algorithm for themselves. The modelling also allowed to understand the meaning of the denominator and to interpret the remaining part, relating this operation to a whole number division. In the test, students demonstrated to conceptually understand this meaning of division: “I took this amount [$\frac{3}{4}$] several times [from 2 $\frac{1}{2}$]. Each time, it was a cookie she could bake” (Maria). Helena emphasizes the remaining part saying it could be $\frac{1}{3}$ of a cake because “each cake takes 3 pieces of these” or $\frac{1}{4}$ of a kg because “each model represents 1 kg and is divided into 4 parts”. Students reflected in the quantities and how they were related, and overcome the misconception that division always decreases.

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USING DIGITAL TECHNOLOGY TO FOSTER STUDENTS' MATHEMATICAL AND SELF-ASSESSMENT COMPETENCIES

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Formative assessment (Black & Wiliam, 2009) is regarded as a key component of effective classroom instruction. However, formative assessment is often lacking student-centered practices like self-assessment which can not only support students' learning outcomes but also their metacognition. Moreover, little is known how digital technology can support students' formative self-assessment. In the BASE-project, we aim at the development of a digital formative self-assessment-tool (BASE-tool) to foster students' basic arithmetic competencies as well as their self-assessment competencies (Thurm, 2021).

In a first design-based-research cycle a digital diagnostic number line task (represent the multiplication 6 x 4 on the number line) has been designed. The number line task includes three specific design elements: (1) dynamic and interactive sample-solutions, (2) adaptive feedback on task solutions and self-assessment and (3) task-specific self-assessment criteria, which are criteria that focus on specific aspects of a correct solution (e.g., the first factor corresponds to the number of my jumps, the result of 6x4 corresponds to the end point of my last jump). We are currently conducting a qualitative video study (thinking aloud) with eight fifth graders where we investigate if and how the interplay between the three design elements of the number line task foster students' multiplication and self-assessment competencies. First, we expect that the comparison of the students' own solution with the sample solution using the detailed conceptual self-assessment criteria will support students' self-assessment competencies. In particular, we assume that students can relate the different sources of information from the three design elements and understand that self-assessment goes beyond a simple "right/wrong" assessment. Furthermore, we expect that the dynamic and interactive representations in the sample solution in combination with the other design elements support students' multiplication competencies. In the presentation, the results of the video study will be presented and discussed.

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GRADE 2 STUDENTS' EXPERIENCES AND USE OF FORMATIVE FEEDBACK IN MATHEMATICS

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The purpose of this study is to better understand the conditions of formative feedback practice in mathematics, since students not always use feedback formatively (e.g., Hargreaves, 2012). To achieve the purpose, the following question is answered: How do primary students experience and use formative feedback in mathematics? The learning context is a potential factor for students' involvement in feedback (Winstone et al., 2017) and part of observing the context can involve observing norms, especially those regulating the interaction between students, teacher, and the subject, as these can be critical in students' involvement in feedback. Since the subject mathematics is in focus, the interaction is regulated by the socio-mathematical norms (Cobb & Yackel, 1996). During a lesson, 15 primary students (7- to 8-year-old) were provided with process-focused feedback, aiming to challenge them to come up with their own solution methods and explain the underlying mathematics as a means to support their further learning. The day after the feedback sessions, video-recorded stimulated recall sessions held with each student were used to examine experiences, responses, and possible conflicts in relation to the norms instigated by the feedback provided.

Most of the students appreciated feedback that focused on the process, instead of simply offering solution methods. However, due to norm conflicts between teacher and students, some of the students did not understand the purpose while others wanted the teacher to state the solution method. Hence, there were aspects that could act as barriers to getting students involved in formative feedback, which shows that it is important not only which norms are established, but also that establishment is done at an early stage. Both teachers and students need to understand and accept the norms, and potentially establish new norms, if the current ones are counterproductive.

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COMPARING THE USAGE OF PROOF-RELATED WORDS IN GERMAN AND JAPANESE MATHEMATICS TEXTBOOKS

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How can we better understand the linguistic specificities of proof and proving to be taught in different countries? Looking at proof in curricular documents (e.g., textbooks, national curricula) in a given country, one may notice that what is called a proof and what is taught as proof differs, within and across these documents. International comparisons are one way to investigate the diversity related to proof. In this presentation, we aim to analyse and compare the usage of proof-related words in German and Japanese mathematics textbooks. Although the educational systems of the two countries are different, 'proof' is introduced in middle schools in both countries.

We chose five series of textbooks for each country and explored some common chapters. As a result, *argumentieren* (argue), *begründen* (justify), *beweisen* (prove), *erklären* (explain) and *zeigen* (show) were identified as proof-related words in the German textbooks, as well as *説明する setsumei-suru* (explain), *証明する shōmei-suru* (prove) and *示す shimesu* (show) in the Japanese textbooks. Comparing different 'functions' (roles of proofs) of these terms (Miyakawa & Shinno, 2021) revealed that different functions can be attributed to different words in the textbooks. For example, both *beweisen* and *shōmei-suru* emphasize the function of verification, providing the logical value of a statement (Duval, 1991). Other words (*erklären*, *begründen*, and *setsumei-suru*) are associated with the function of explanation, gaining an understanding of the reason why a statement is true. In addition, both *erklären* and *setsumei-suru*, in their ordinary usage, point to the function of communication or illumination, which are used for expressing one's thinking in a social context. Regarding the formulation of 'structures' (organisation of reasoning) (Miyakawa & Shinno, 2021), *beweisen* indicates students should demonstrate a proof by a deductive chain of reasons, while *begründen* often allows no specific form of reasoning. The usage of *beweisen* is almost identical to that of *shōmei-suru*. But what is asked by *setsumei-suru* in the proving context is different and varies before or after *shōmei* is introduced. Similarly, the meaning of *erklären* varies in the context of proving.

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FLUENCY AND REASONING PROFICIENCIES IN A HALVING TASK AMONG PRIMARY LEARNERS IN SOUTH AFRICA

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Poor mathematics performance in South Africa is coupled with a curriculum that stipulates working with tasks beyond what most children can actually do. This generated interest in looking closely at the kinds of Multiplicative Reasoning (MR) proficiencies manifested by primary school learners in South Africa. Fluency and reasoning proficiencies are crucial for overall math achievement. Fluency refers to choosing and carrying out procedures and drawing on known facts, whilst reasoning includes thinking about the relationships between numbers (Askew, 2012). The present study identifies fluency and reasoning proficiencies shown by Grade 4 and 6 learners in a halving task using Askew (2012) actions of proficiencies as a guiding framework.

A task-based interview assessment was administered to assess MR proficiencies with a sample of 18 learners (9 Grade 4 and 9 Grade 6). These learners were selected across the attainment range – three low, three middle and three high. Items tested were set below the curriculum expectation. The items include half of 202, half of 146 and half of 152 which were presented to learners in Grade 4 and Grade 6.

Across the attainment range in both grades, all the learners demonstrated fluencies with single digit halving when the digits were even. However, when the number included odd digits such as 146 and 152, these fluencies were absent and/or limited amongst all the low and some middle attaining learners. Low attaining learners worked by halving digits and were usually unable to work spontaneously with the underlying values i.e. "H of 146 = 123" and in H of 152, "H of 2 is 1, you can't halve 5 and H of 2 is 1", "H of 152 = 53 ½" and "H of 152 = 51". In contrast, middle and high attaining learners could partition numbers according to place value, halve these partitions and re-join the partial halves, indicating awareness of how to use place value decomposition relationships for halving. These proficiencies were more spontaneous among high attaining learners than they were with middle attaining learners who at times required prompting to either partition or re-join the numbers. In the Grade 6 sample, middle and high attaining learners demonstrated more spontaneous proficiencies than the Grade 4's in the same attainment range as would be expected given that they have had longer exposure to these concepts. While these proficiencies were more spontaneous in Grade 6, their responses indicted the extent of these proficiencies despite longer exposure to the content. Furthermore, by identifying these proficiencies, this study aims to contribute to closing the gap between learning and the mandated curriculum.

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CROSSLINGUISTIC TRANSFER OF MATHEMATICAL KNOWLEDGE AMONG CHINESE-FILIPINO STUDENTS

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The principle of linguistic relativity suggests that language influences thought. Jarvis and Pavlenko (2008) described it such that speakers of different languages who are looking at the same object are actually viewing the same thing, but the attention directed by the language(s) they speak allows them to see the same object differently. In relation, they also defined crosslinguistic influence or transfer as "the influence of a person's knowledge of one language on that person's knowledge or use of another language" (p. 1). While linguistic relativity and crosslinguistic transfer mostly dealt with grammar and semantics, Prediger et al. (2019) provided a demonstration of both principles vis-à-vis mathematical thinking through the translanguaging episodes observed among German-Turkish bilingual students.

Anchoring on the aforementioned ideas, this presentation reports on the crosslinguistic transfer of mathematical knowledge of ten Chinese-Filipino university freshmen who were part of a Bilingual Mathematics Program (BMP) during their K-10 schooling. That is, they were taught mathematics in English and then also in Chinese in separate classes. They were first asked to rate their perceived frequency of knowledge transfer between the two mathematics classes through a survey. They were then interviewed about their BMP experiences and also later observed while performing mathematical tasks, such as solving a word problem presented in one language but whose lesson was taught in the other. Results showed that most of the participants reported being able to reconcile the content taught in both languages. While the transfer from English to Chinese Mathematics was more prevalent, they found the BMP a beneficial experience overall. Moreover, most were also able to transfer their knowledge between the languages in the tasks. However, it was also found that the crosslinguistic transfer of mathematical knowledge may be dependent on the pedagogical choices of the teacher, among others. The findings suggest that learners are generally capable of utilizing mathematical knowledge as taught in two languages and these offer insight into the interdependence of languages with respect to mathematical thinking and performance.

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SUPPORTING STUDENTS' ENGAGEMENT WITH ELABORATED FEEDBACK BY MEANS OF INTERACTIVE REPORTS

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Elaborated feedback has the potential to enhance the quality of students' mathematical abilities. We refer to elaborated feedback as a dialogic process in which learners make sense of reported information and use it to enhance their mathematical skills (Carless, 2015). We address the challenge of engaging students with elaborated feedback by means of interactive report, i.e., reported information integrated within a technological environment. We studied the two most common representations of interactive reports: textual and measurements, and asked: Does students' engagement with elaborated feedback following interactions with each representation of interactive report help students improve the percentage of correct answers and diversify their examples? We used the Seeing the Entire Picture (STEP) platform that provides automatic elaborated feedback (Olsher et al., 2016), and designed activity in a dynamic environment. 29 students in grades 9-10 developed the activity in an environment with the interactive textual report (Group A), and 26 using an environment with the measurements (Group B). The activity contained two tasks. After the students submitted answers to the first task, they resaved the elaborated feedback automatically. They then proceeded to the second task. To compare the correctness and diversity of examples before and after the elaborated feedback, we conducted an independent-samples t-test. According to findings, following the elaborated feedback, both groups significantly improved the rates of correct answers and diversify their examples. This indicates that both types of interactive reports could be viewed important means for engaging students with elaborated feedback. However, the textual interactive report led students to significantly higher achievements in the first task than did the measurement report. Thus, the difference between the representations of interactive reports may lead to different learning processes that can be integrated into classroom situations.

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CONCEPTUAL TRANSITION OF A STUDENT WITH LEARNING DIFFICULTIES IN MATH: FROM COUNT-ALL TO COUNT-ON

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We studied how a grade-6 student (Mr. Happy, pseudonym he chose) with learning difficulties in math may shift from count-all to prenumerical count-on with *perceptual items*, then to prenumerical count-on with *figural items*, and then to numerical count-on. These shifts arose by working on tasks designed to create perturbations that lead to a reflection on his activity and its effects. Through such reflections, a student constructs a new scheme in two stages, *participatory* (prompted) and anticipatory (spontaneous; Tzur & Simon, 2004). We thus expand Steffe & Cobb's (1988) distinction of two count-on forms: (a) *simultaneous* use of figural units (prenumerical) and (b) *sequential* (numerical), double count of a second addend's items.

We conducted a constructivist teaching experiment as it affords insights into students' conceptualization, particularly units on which they operate. We focused on Mr. Happy because he spontaneously used count-all of perceptual items at the study start. We engaged him (and a researcher) in a playful task called *How Far from the Start* (HFFS), designed to help students transition from count-all to count-on. Each player takes a turn moving a number of spaces (floor or boardgame tiles), with the second starting from where the first stopped. The task is to find how far is the second player from the start.

In Episode 1 we saw Mr. Happy's shift from count-all to participatory count-on. Evidence included (a) his initial, spontaneous use of count-all of the tiles from 1 and then (b) sequential count-on once the researcher prompted him by blocking his view of those items (e.g., to add 8+7 he raised 5+2 fingers, then counted each of them from 9 to 15). In Episode 2, we saw his shift from participatory to anticipatory use of figural items while using sequential count-on (still prenumerical). Evidence included his spontaneous response to 7+4 steps, raising four fingers for the hidden tiles, then count-on from 8 to 11. In Episode 3, we saw his shift to participatory, numerical count-on. Evidence included his response to 7+8, uttering a number word coupled with raising one finger at a time and stopping when seeing 5+3 fingers raised on his hands.

Our study indicates that researchers/teachers can use theory-based designed tasks, like HFFS, to foster in students like Mr. Happy a transition (described here) similar to the transition promoted in other, first-grade students (Steffe & Cobb, 1988).

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PREDICTING STUDENTS' PREFERRED ASSESSMENT METHOD IN UNIVERSITY MATHEMATICS

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Fostering student agency in assessment is conceived to be beneficial for the development of self-regulation in learning (Adie et al., 2018). To enhance their agency, we let students choose between exam and self-assessment for the final assessment method in an undergraduate mathematics course. In this study, we aim to gain a better understanding of the reasons behind the assessment choice by analysing their self-regulation abilities and attitudes towards self-assessment.

Self-regulation refers to monitoring and directing one's own thoughts and behaviour in order to achieve desired outcomes. Self-regulation and self-assessment are intricately connected: self-assessment is a fundamental action in the process of self-regulated learning (Yan, 2020), and self-assessment interventions can enhance students' self-regulated learning (Panadero et al., 2017). On the other hand, students apply self-regulation when making agentic choices for their studies (Adie et al., 2018).

The participants were 333 students in an undergraduate mathematics course. We recorded students' choices of assessment method (exam or self-assessment) and their answers to a questionnaire concerning self-regulation and self-assessment attitudes in the beginning of the course. Logistic regression was used to model the assessment choice. The results suggest that perceived usefulness of self-assessment increased the likelihood of choosing it as a final assessment method, and strong self-regulation decreased the likelihood. These results will be discussed in the light of qualitative evidence, as well as the assessment culture in mathematics more generally.

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STUDENTS' FORMAL WRITTEN COMMUNICATION

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For centuries, students writing has been used to assess students' mathematical knowledge. While students' argumentation as well as their writing for other purposes than to communicate formal solutions has been the subject of several lines of different research (Morgan, 1998) we have found no previous design research focusing particularly on directing systematic teaching efforts to the development of students' formal mathematical writing competence. Such design-based teaching development is the purpose of our research.

Our approach hinges on separating the reasoning that makes up the solution to a mathematical problem from the formal written communication of that reasoning. This separation is upheld in the teaching design, where some lessons are solely devoted just to produce, discuss and improve formal written communications of previously established reasonings that solves some particular problem. The separation is also upheld in the framework for assessing formal mathematical writing that we develop within the project. Similar to how Stylianides (2007) deals with the concept of proof in grade three, our aim is to develop a framework for assessing formal written mathematical communication, that at the same time honor general principles for good mathematical communication and is communicationally relevant for any class and age group from 10-year-olds and up. In addition to being a guide for us (and later, others) when assessing students' formal written mathematical communication, our framework will also guide teachers in what should count as progress in the development of formal mathematical writing competence. A systematic literature review we conducted revealed that a framework of that type would be a novelty.

A main result from this early stage of our research project, is that when we gathered formal written communications from 77 Swedish students between 11 and 17 years old and analyzed them with a preliminary version of our framework, we found no improvement in the quality of the formal written communications over the six years of schooling. We take this as an indication of that no effective teaching directed towards improving students' skills in producing formal written communications is carried out.

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FROM ADDITIVE TO MULTIPLICATIVE THINKING: THE ROLE OF FEEDBACK FROM A COMPUTER APP

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Multiplicative thinking is an important aspect of mathematics education, such as with ratio and proportion. An educational challenge is to help students use additive and multiplicative thinking appropriately depending upon the inherent mathematical structure of the problem posed (Dooren et al., 2010). A particular computer App called *Stick and Split* (*SaS*) requires multiplicative thinking in tasks which require rods of a given length to be made by sticking together rods of equal length, and splitting rods into equal lengths. Computer games can be motivating, however any mathematics involved can be peripheral to the main gameplay. This is not the case with *SaS*. This research is a preparatory study which aims to examine whether the design of the *SaS* App can influence the nature of thinking which a young student might bring to the App's tasks.

We have used the theoretical framework of subordination (Hewitt, 1996) where the desired learning is not the main focus of attention for a student. Instead, it is subordinated to tasks which are understandable for the student irrespective of their current competency with the desired learning. This was chosen as it matched the design of the App, were making rods of a certain length is the explicit goal, whilst use of multiplication and division are implicit in its success. Video data were analysed in terms of actions taken in relation to the implicit feedback from the App (e.g. whether rods did join together or not). We analysed a case study of one Year 5 student (either 9 or 10 years of age) who expressed dislike of mathematics and required additional support in school. We found that she made repetitive use of additive thinking, such as trying to stick together rods of lengths 3, 2 and 3 to make a target rod of length 8 (where nothing happened due to the rods not being of equal length) for most of the 22 minutes of playing time. However, she gradually shifted and used multiplicative thinking consistently and fluently in successfully completing later, more demanding, tasks. This showed that carefully designed implicit, rather than explicit, feedback can shift thinking from an initially familiar additive thinking to multiplicative thinking.

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PERCEPTION OF FUTURE MATHEMATICS TEACHERS ON THE PROMOTION OF SELF-REGULATION OF LEARNING

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During their initial training, teachers must acquire important skills like the transversal ones, which they can later use and promote in their professional practice. One of these skills is the so called self-regulation, which is essential in the teaching and learning processes, since it allows controlling, organizing and adapting the training process under different contexts (Zimmerman, 2000).

This study describes the self-regulatory actions that future mathematics teachers (FMTs) claim to promote during their mathematics classes. The research follows a mixed methodology, analysing the perception of 100 FMTs during their practicum in the Master in Training of High School Teachers of a Spanish University, for the academic year 2020-21. The utilized questionnaire, "Promotion of self-regulation in mathematics class", has been designed, constructed and validated, and it consists of 23 items with a Likert scale from 0 to 5. It also contains two open-response questions, where FMTs have to justify why they promote some actions more than others. Such actions are previously classified according to the Didactic Suitability Criteria (Hidalgo-Moncada, et al., 2020), which are a tool that allows teachers to carry out a reflective practice through six facets: epistemic; cognitive; interactional; mediational; emotional and ecological. In this communication, the actions related to emotional and ecological suitability will be described, since they have yielded the most relevant results.

The results suggest that some FMTs always or usually promote the contextualization of mathematical activities, proposing intra and interdisciplinary connections, and implementing different forms of evaluation. It is also observed that some FMTs rarely or never implement self-regulatory actions related to knowing the interests of the students or promoting emotional, motivational or attitudinal self-assessment, which shows a need to reinforce these aspects in the initial training programs of teachers.

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PROPOTIONAL REASONING OF LOWER GRADE STUDENTS THROUGH LEARNING TRAJECTORY AND LESSON STUDY

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Most research on proportional reasoning in mathematics education focuses on students in secondary schools or upper grade students in primary schools. Alternatively, other studies that examined preschool children or lower grade students in primary schools pointed out the problem of underdeveloped knowledge in the multiplicative structure of numbers (e.g., Resnick & Singer, 1993). This study approaches the question of developing mathematics lessons to foster the proportional reasoning of lower grade students by adopting a design research methodology (Cobb et al., 2017). We propose a learning trajectory of proportional reasoning for this group of students and use it as a tool for examining the processes of learning. The researchers, including primary school teachers, conduct lesson studies and investigate lessons for supporting the learning of students based on the learning trajectory. Theoretically, we develop, test, and modify conjectures about learning trajectory and the strategies for supporting learning. The learning trajectory is based on two types of ratios, namely, scalar ratios (SRs) and functional rates (FRs; Vergnaud, 1994). Initially, we set five stages for SRs and three stages for FRs. In this presentation, we describe two lesson studies conducted in 2021 in four Grade-3 classrooms (8–9 years old) in Japan. One of these studies intended to support the SRs' processes of unitizing and norming in the context of comparing and measuring a quantity. The researcher and teachers collaborated to design a series of lessons for two teaching units in a classroom and pursued learning among several students. The objective of the other lesson is to elicit the processes of SRs and FRs by designing tasks that require coordination between two quantities, which is not typical in lower grades in Japan. The researcher and teachers designed six lessons across one year. We describe the results in terms of the capabilities of the students and examine implications for the learning trajectory and strategies for realizing progress.

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THE STORIES THEY TELL: EXPLORING STUDENTS' UNDERSTANDINGS OF DATA VISUALIZATIONS

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Data science is often described as thinking through and with data. Big datasets are becoming more accessible, and data-based arguments are increasingly used to understand phenomena and to persuade in the context of public policy debates. However, mathematics, and statistics in particular, is often taught as a set of definitions without a deep understanding of the application of the concepts (Bajak, 2014. Lee et al. (2021) describe a framework for data science education that argues for consideration of the personal, cultural, and socio-political layers of individuals' experiences.

Through the Data Visualization Project (DVP), we design and provide professional learning for teachers to support students' learning of data visualizations in order to analyze and summarize data addressing topics of personal/community interest. In this presentation, we focus on interviews conducted with students in the pilot phase of the project which investigated their understandings of data visualizations through a focus on storytelling. We draw attention to two research questions: 1) What are students' understanding of data visualizations? 2) What are students' relationships with data?

The project began in Fall of 2021, builds on a previous project conducted in the elementary grades, and will continue through 2024. In the pilot phase, the project team engaged 12-14-year-old students in activities that involved analyzing data and creating data visualizations. Project sessions included a subset of the authors and the 20 student participants. The sessions took place in the STEM (Science Technology, Engineering, and Mathematics) classroom of a school located in the Southeast United States. A future iteration of the project sessions will take place in an art class with student participants of a similar age. Preliminary findings from student post-interviews show that focusing on the stories that the data visualizations tell, both those created by the students themselves and those created by others, is an effective way to draw attention to the contextual nature of the data and to assess students' understanding of data analysis and visualization. Further, it is anticipated that the stories create opportunities to position students as authors of the data. Current findings will be shared.

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DEFENDING MINIMUM COMPETENCY STANDARDS FOR MATHEMATICS EDUCATION

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We have been involved in supporting intermediate pre-service teachers for over 15 years. We have based our research around representations and reasoning, a crucial aspect of specialised content knowledge (SCK) at this level (Mitchell et al., 2014). Researchers like Lubienski and Gutiérrez (2008) have pointed to concerns over inherent inequities in mathematics. We have focused on SCK development to support fearful pre-service teachers, and to effectively prepare them for high standards. In fact, our work in the education program has shown particular growth with a wide range of pre-service teachers, via this focus. In parallel, we have been encouraged by the strong alignment of the tenets of mathematics reform, which require such SCK, with the ways of learning and knowing mathematics that have been described in Canadian literature as being more culturally appropriate for Indigenous students (e.g., Lunney-Borden, 2018).

This presentation will look at the lessons we have learned and the specific political climate that we find ourselves in currently, as our Canadian province considers next steps for teachers. Recently, Ontario attempted to institute a Mathematics Proficiency Test as a licensing standard for all future teachers. A court decision came down that this requirement violated the Charter of Rights, since the Court found "racial disparities" (*OTCC v. The Queen*, 2021, p. 2) with the early test results. This "deleterious effect on diversity" (p. 2) has caused the requirement to be removed. Our position is that removing the requirement has grave potential to perpetuate inequities, and we argue for enhanced opportunities for growth and support, in working towards higher standards for all future teachers. In this presentation we share the data collected, the data used in the court case, and possible directions for the future.

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COMPARISONS OF BRAIN ACTIVITIES BETWEEN SOLVING FUNCTION TASKS AND MULTIPLICATION TASKS

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A growing number of researchers have used neuroscience as methods to examine student cognitive behaviour when solving mathematical problems. This study presents the examination of brain activities specific to problem-solving of function and multiplication tasks. Multiplication tasks refer to basic calculations (e.g., $3 \times 2 = 6$), whereas function tasks are those requiring to identify characteristics of graphs and their corresponding functions (e.g., quadratic function) (Waisman et al., 2014). E-prime software was used to perform the two kinds of tasks. Students' response accuracy (Acc) and reaction time for correct responses (RTc) for the tasks were adopted to determine the cognitive complexity of tasks.

We used Event-Related Potential (ERP) techniques to collect brain waves occurred during problem-solving processes. The analyses of brain waves focused on the components of P1, P2, P3, and N2. P1 can reveal the cognitive efforts related to information perception, where P2 refers to perception processing. P3 denotes the cognitive effort to synthesize stimuli and reason the outcomes, whereas N2 focuses on mismatch detector or reflect executive cognitive control functions. The wave peaks in terms of amplitudes and latency in accordance with each component were identified. The peak amplitude reveals cognitive effort made during problem solving processes. The latency refers to the speed of stimulus classification resulting from discrimination of one event from another. Shorter latencies indicate superior mental performance relative to longer latencies. 163 Taiwanese high school students participated in the study. Statistical analyses showed that students significantly performed quicker and more accurately on multiplication tasks than function tasks, indicating higher cognitive complexity of function tasks. Analyses on brain waves showed that function tasks caused significantly higher amplitudes and longer latency for P1 and P2 components. The analyses on P3 and N2 components revealed different results. Concerning P3, central area and the area between parietal and occipital of brain causes significant differences in amplitudes and latency between the types of tasks. However, no significant differences were found for frontal pole, antero-frontal, and temporal areas. Concerning N2, brain waves collected from electrodes of P08 and P8 also did not cause significant differences. We will discuss the analyses and research implications.

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INVESTIGATION OF DEVELOPMENT OF MATHEMATICS TEACHERS' NOTICING SKILLS: A TEACHER DEVELOPMENT EXPERIMENT BASED ON LEARNING TRAJECTORIES

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'Teacher noticing' which has placed student thinking at the center of the teaching process, is one of the knowledge, skills, attitudes and competencies that teachers are expected to have in order to a quality education. Teacher noticing includes the components to attending to students' strategies, interpret them and decide how to respond (Jacobs, et al. 2010). Research have revealed that teacher's content knowledge (CK), student knowledge (SK), beliefs and experience are effective on the noticing skill (NS). Video club, lesson study, video annotations and student written responses are used to develop NS.

In this research, it is aimed to improve the NS of a mathematics teacher by using learning trajectories (LT) for the 6th and 7th grade ratio and proportion topics. LT are described as a bridge between the 'curriculum' and "student's learning needs" as the developmental progress levels of students' thoughts. Because a LT addresses multiple curriculum goals based on the development of student thinking. So, it is assumed in the research that by using LT as a professional development tool, teachers' NS will be improved and the gap which mentioned can be reduced. This research is structured with a Teacher Development Experiment (Simon, 2000). The research process took 12 weeks. 2 hours of professional training, classroom observation and interviews were conducted each week. The analysis of the data continues. Expected results in the light of the findings obtained from the continuous analyzes of the research: it has been observed that the teacher's CK, SK and teaching knowledge on "ratio-proportion" subject limited to the curriculum goals is at an insufficient level. Teacher noticing and teacher knowledge are intertwined like this: "attending to students' strategies" based on CK; "interpret students' strategies" based on CK and SK; "decide how to respond" based of CK and teaching knowledge.

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RELATIONSHIPS BETWEEN OPPORTUNITY TO LEARN AND CLASS PRIOR ATTAINMENT IN MATHEMATICS

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Students within education systems can experience differentiated opportunities to learn (OTL) through content coverage, content exposure, content emphasis, and the quality of instruction. These different OTL may be consequences arising from variation in structures and policies within the system, or in cultural expectations around what students should or could learn, or in the decisions made by individual teachers. Students' OTL mathematics have been shown to be associated with their attainment (Klieme, 2013; Scheerens, 2017), and to mediate socioeconomic impact on attainment (Schmidt et al., 2015).

The analysis aims to examine if there are associations (Kendall's correlation coefficient) between the average prior attainment of a class and the different OTL that class has across seven educational contexts. It uses student-reported (n = 16,043) and teacher-reported (n = 594) measures of students' OTL and student pre-test scores from the OECD's TALIS Video Study. The analysis reveals that many teachers adapt what they teach depending upon the prior attainment of the class they are teaching, but the extent to which this happens and how this happens varies between cultural contexts. Specifically, content exposure and content emphasis vary more within some countries or jurisdictions than others. In most countries, the range of algebraic aspects of and emphasis on reasoning with quadratic equations did increase with higher attaining classes. In contrast, there was no evidence that on average there were any relationships between a class's prior attainment and the total time spent on each aspect of quadratic equations, or the proportion of time spent on each topic, except in England. These findings go someway to explaining the nature of class OTL gaps previously identified in English speaking contexts in earlier research (Schmidt et al., 2015).

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LOOKING INTO THE NON-COGNITIVE DIMENTION OF MATHEMATICS TEACHING

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Non-cognitive abilities are known to predict future success of students (Heckman & Kautz, 2012). In Japanese elementary schools, students' whole person development is typically assumed to be the primary goal of education (Lewis, 1995). The society embraces the holistic development of students as an important goal of education, and Japanese educators tend to consider education as a way to help students develop holistically as they make use of diverse situations of academic teaching. Then how do teachers in Japan actually promote students' non-cognitive abilities in their classes, and how do they master the expertise?

To investigate this issue, this study recruited 14 experienced elementary school teachers. The data collection involved video-taped observations of their math lessons and follow-up teacher interviews on their intentions of the key actions and interactions during the math lessons. In addition, students' questionnaire on non-cognitive learning developed based on the three key constructs (autonomy, relatedness and competence) of the self-determination theory was given to their students after the math lessons.

The study revealed that in the classes taught by the teachers, students reported a higher level of autonomy, relatedness and competence compared to students taught by novice teachers. The study also revealed that the teachers' key actions were targeted to nurture socio-emotional development of the students in the changing situations of their math teaching such as valuing others' perspectives in the process of problem solving, gaining confidence to speak up one's mathematical ideas in front of their classmates and learning to overcome challenges with others. Most of the teachers attributed this aspect of their expertise to local-level lesson study and mentorship.

Based on this finding, lesson study groups are now being organized across the world with the goal to help novice teachers master the expertise to elicit students' non-cognitive learning in their academic classes. The findings from project would inform how teachers in different cultural contexts could learn to promote students' non-cognitive abilities and what kind of localizations are necessary in each context.

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COGNITIVE PROCESSES IN PROBLEM SOLVING OF SYMMETRIC FIGURES

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This study was aimed to investigate cognitive processes in geometric problem solving among 52 first year students of mathematics teacher education program of Lampang Rajabhat University. To accomplish the study purpose, Duval (1998) was used as a theoretical framework which composed of three processes: visualization, construction, and reasoning. Research methodology employed participatory research as the researcher was a lecturer of Mathematical Problem-Solving course and used an Open Approach as a teaching approach (Inprasitha, 2011). Case-study problems were symmetric figures which adapted from Japanese textbook 'Gateway to the Future: Math 1 for Junior High School', these problems were taken from the Math Dojo which presents problems that involve students to use mathematics in a way of sparkling their further interests (Kerinkan, 2013).

Results of the study showed that 1) Reasoning-Visualization: after students saw figures (M- and N-shape), they tried to figure out these figures, then built a gestalt and configuration of the figures; 2) Visualization-Reasoning: not found because of the students actually think reasonably before visualization; 3) Reasoning-Construction: students argued about the figures composed of pairs of side which are the same length and area then they tried to make the figures on a paper by themselves; 4) Construction -Visualization: after the students made their own figures, they tried to differentiate sides and areas of the figures, then they found a symmetry line on the M-shape and symmetry point on the N-shape; 5) Reasoning (5A) students described by using their own language such as there is only one line that divide M-shape into two parts equally, (5B) students described by using propositions such as there are two corresponding points that connected by a line pass through a point for N-shape. Findings showed that students of the mathematics teacher education program could realize the students' difficulties of the problems will be used in schools.

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COVARIATION REASONING IN OPTIMIZATION TASKS

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This paper presents the results of a research study obtained from the implementation of a sequence of mathematical modelling activities focused on optimization. The experimentation was carried out with students recently graduated from high school in Mexico (17- to 18-year-olds). The resolution processes, as well as the diversity of models generated collaboratively, evidenced different forms of covariational reasoning used to determine the "most efficient" answers to the optimization problems posed.

It is possible to generate a variety of interesting contexts to involve students in the modelling of phenomena and situations that require finding an optimal value, which is done by recognizing patterns and covariation relationships between variables. This paper presents the forms of covariational reasoning manifested by a group of students, who were split up into collaborative teams, during the resolution process of modelling activities focused on optimization.

The tasks were designed from the perspective of models and modelling (Lesh & Doerr, 2003). The goal was to motivate students to repeatedly express, test, and refine their own ways of thinking through the creation of reusable conceptual tools. Students will be able to use their previous knowledge and develop new ideas/strategies by identifying optimal paths in a rectangular prism and designing cylindrical deposits with a minimum area while meeting certain restrictions. The forms of reasoning observed during implementation were analysed in light of Thompson and Carlson (2017) covariational reasoning framework. In this theory, covariational reasoning is the ability to visualize, coordinate and represent the changing nature of two variables which vary at the same time. The results obtained during this research show six different forms of covariational reasoning employed by the students, as well as how these forms of reasoning are mobilized according to the experienced scenarios.

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PRE-SERVICE TEACHERS' PLANNING OF CLASSROOM ACTIVITIES BASED ON THEIR REFLECTION ON STUDENTS' LEARNING NEEDS – A VIGNETTE-BASED STUDY

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Teachers' planning of classroom activities can occur on the base of the teacher's reflection on the students' learning needs and their prior knowledge. In this context, pre-service teachers' knowledge of a students' hypothetical learning trajectory (Ivars et al., 2020), related to a mathematical content area, can play a role, as in the analysis processes connected with the teacher's planning of classroom activities, both the mathematical content and characteristics of the learners have to be considered. The content structure, the learners' specific needs, and content-related goals have to be considered and balanced, supported by a criteria-based argumentation, which precedes the planning decision(s). Despite the high relevance, little is known empirically about teachers' analysis in this context and about the role the criteria-based argumentation plays for a pre-service teacher's decision-making related to the planning of classroom activities. Consequently, this study aims at exploring (1) how pre-service teachers analyse the needs of learners, (2) how they respond to these needs in their planning of classroom activities and (3) to what extent they support these planning decisions by criteria-based argumentation.

The sample of this study consists of 73 Spanish pre-service teachers (PTs) who have been asked to plan classroom activities based on a vignette showing a classroom situation that supports PTs' reflection on the learners' needs. In a parallel study, more than 22 German pre-service teachers have responded to the same planning and analysis tasks (this data set is currently being coded). The results from the first sample indicate that a successful reflection on the learners' needs did not warrant coherent planning of classroom activities and that criteria-based argumentation in this context was often a challenge to the pre-service teachers when planning the activities.

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DESIGNING FOR TEACHER REFLECTION AND ENGAGEMENT WITH RESEARCH ON CONNECTED MATHEMATICAL IDEAS

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A great deal of research has been done to characterise important connections and dependencies between mathematical ideas. These have been shown to play important roles in teaching and learning mathematics (Tall, 2013). However, there are challenges when it comes to incorporating this research into teacher learning. Considerably less research exists to inform strategies for helping teachers to engage with research implications for the nature and uses of these connections in their work.

In this design-based research project, we are exploring teacher responses to representations of mathematical connections and their research basis in order to develop and refine theoretically-grounded design principles for representing this information in a way that leads to positive outcomes when implemented with teachers. In Cycle 1, positive outcomes were initially defined as (1) stimulating teachers' reflection on their mathematical content knowledge, in particular with respect to connections, and (2) supporting critical engagement with research. We developed a framework for creating materials to aid teachers' engagement. The theoretical foundations for this framework include the DeFT framework for learning with multiple representations (Ainsworth, 2006) and the domain-specific theories of the structure of mathematics learning (Tall, 2013).

To complete Cycle 1, we implemented the initial version of the designed materials with 20 mathematics PGCE (education postgraduate) students as part of a three-day workshop. Afterwards, 20 students completed a qualitative questionnaire, and 11 participated in interviews. Results indicated that attitudes about various aspects of the materials were broadly positive and suggested several specific design updates and an additional design principle. The updated materials will be implemented in a digital format in Cycle 2. Overall, Cycle 1 provided qualitative evidence that teachers were being stimulated to reflect on their content knowledge and were engaging critically with research to the degree possible in the time frame of the workshop, and suggested ways that critical engagement with research could be better supported. Details and research in progress will be discussed further in the presentation.

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A MULTIMODAL PERSPECTIVE ON NUMBER SENSE IN DIGITAL LEARNING RESOURCES

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There is an increasing use of digital learning resources in mathematics education, which provides potential for multimodal approaches and new ways for students to meet and learn mathematics, but the outcome depends on the design and the implementation of the digital resources (Hoyles, 2018). This on-going study contributes by examining affordances of combinations of different modes in relation to number sense. We use Halliday's social semiotic theory (Halliday & Matthiessen, 2013) and McIntosh et al. (1992) number sense framework for analysing a specific app, *Vektor*.

We analysed the two exercises in the app that relate to aspects of number sense, *Numberpals* and *Numberline*. Each round in respective exercise was analysed with respect to Halliday's three metafunctions: *ideational*, *interpersonal*, and *textual*. For the ideational function, McIntosh et al.'s framework was used to characterise the mathematical content. As an example, figure 1 illustrates the second and third rounds in Numberpals. In these rounds, both coloured rectangles and numerals are used to present the mathematics, and the main aspect of number sense in focus is identified as *Multiple representations for numbers,* for example, two red + three green = one red + four green, and 2+3=1+4. In the first round there were no numerals in the bars, only rectangles, and in the later rounds there were only numerals in the bars.

Figure 1: Screen shot from the exercise Numberpals

Preliminary results, concerning the ideational metafunction, show that the mathematical object in focus may be perceived in different ways; either as just rectangles that should be combined (in a visual/geometrical sense), or as numbers of rectangles, or as numbers (and relations between numerals). This can impact students' development of number sense. Further analyses will include all metafunctions and a focus on progression in the exercises.

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THEY ARE DOING THE LESSON, BUT WHAT MATHEMATICS ARE THEY LEARNING?

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One way to attempt to understand students' learning of mathematics is to consider the dialectics in which they engage, as proposed by Brousseau (1997). Brousseau's theoretical framing suggests that students' engagement in dialectics of action, formulation and validation are all necessary for mathematical learning to occur. Additionally, it is important to consider whether, and how, students' activity moves between the pragmatic/empirical and mathematical/systematic fields (Noss et al., 1997) and, although there are occasions when all three of Brousseau's dialectics may be observable, if activity primarily remains, for example, in the pragmatic/empirical field, mathematical learning will be limited.

The Centres for Excellence in Maths research trial, in England, investigates the impact of adopting a specific teaching approach with students who have not achieved a 'pass' grade at in exams at age 16. Eighty teachers teach seven carefully designed lessons, which provide multiple opportunities for students, working in pairs, to engage in dialectics of action, formulation and validation and to move between the pragmatic/empirical and mathematical/systematic fields. The question at the heart of the sub-strand of the project that we report on here is 'What can we say about the mathematical learning of the students as they work on the tasks within the lessons?'. Answering this question is important not only for the implementation and process evaluation of the trial, but also to inform the design and teaching of future lessons.

For each of the lessons, two pairs of students, with different teachers, were closely observed. Their work was photographed and their discussions audio recorded. The analysis of their activity is ongoing, but preliminary results suggest that while students generally did engage in all of Brousseau's dialectics, dialectics of validation were more likely to occur in discussion with the teacher than with peers, and were frequently linked with movement between the pragmatic/empirical and mathematical/systematic fields. In the presentation, the work of two pairs of students is used to exemplify the approach and the findings; and design implications are discussed.

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A CASE STUDY OF TWO 1ST GRADE STUDENTS' STRATEGIC REASONING IN GEOMETRIC REPEATING PATTERN TASKS

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Patterns are sequences with replicable regularity varying along dimensions that provide opportunities to recognise, symbolise, compare and verbalise generalisations to predict how the pattern continues (Collins & Laski, 2015). Moreover, patterns may promote part-whole relational thinking, which is attention to relations between parts of patterns critical for number sense and mathematical proficiency (Hunting, 2003). Patterning strategies involve relational thinking in varying degree: *Duplicating* involves making an exact replica of a pattern and is considered less challenging, as a one-to-one matching strategy can be used. *Extending* involves continuing the repeating part of the pattern. *Transferring* involves creating the same pattern using superficially different materials, while *unit isolating* involves separating the repeating part from the pattern. The three latter task types are assumed to require a more cognitively complex and abstract strategy based on part-whole relational similarity (Collins & Laski, 2015).

The present case study of two 6-year-old students replicates and extends Collins and Laski (2015) by incorporating student-test administrator verbal interaction to investigate the role of language in patterning strategies. To that end, we applied a semi-structured task-based patterning interview addressing the following research questions:

What strategies do two 1st grade students use in visual geometric repeating pattern tasks of duplicate, extend, transfer, and unit isolate types? How do the students verbalise their thinking, and how does student-test administrator verbal interaction affect strategy use?

Preliminary analysis indicates that verbal interaction provided insights into student thinking not shown in their patterning strategies. The two students reasoned relationally but patterned using one-to-one strategies due to factors they verbally explained, such as shape or color preference giving a bias on what content regularity to rely the relational thinking and patterning strategy on. In some instances, verbal interaction helped the students overcome this bias changing their strategy use from a one-to-one to a relational strategy approach.

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HOW JAPANESE TEACHERS ELABORATE ON MATHEMATICAL TASKS THROUGH LESSON STUDY

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Japanese lesson study is widely used for teachers' professional development. However, what teachers discuss to improve mathematical tasks is an important aspect of lesson study (Watanabe et al., 2008), and there are few empirical studies focusing on it. This study explored what Japanese teachers consider and how to elaborate on mathematical tasks to make better lessons through a school-based lesson study.

In this study, the discussion for mathematical tasks consists of three stages from Fernandes and Yoshida (2004): lesson planning, pre-lesson discussion (reflection on research lesson1 and revised plan), and post-lesson discussion (reflection on research lesson2). Data was collected by observations and video recordings taken from October 4 to November 15, 2021, for the first and third grades' lesson study. Through thematic analysis (Braun & Clarke, 2006), codes were extracted to clarify the relationship between the elements that teachers consider to improve the mathematical tasks.

From this analysis, the characteristics of discussions were identified: although all three stages discuss research lessons, teachers' perspectives were different at each stage. The stage of lesson planning focused on the concept of the task to connect textbooks with children's real lives, and the content of the unit. The pre-lesson discussion focused on mathematical content, and the post-lesson discussion focused on the children's reactions and mathematical content. In addition, it didn't matter whether the teacher taught as planned, because they continued to discuss the outcomes of the research lesson as well as its study themes and purposes, and also discussed plans to improve. The findings provided a landscape of how teachers can improve their tasks and lesson study works as part of their professional development.

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TYPES OF VISUAL REPRESENTATIONS OF FRACTIONS IN HUNGARIAN TEXTBOOKS FOR 5TH GRADERS

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Teaching fractions at the beginning of lower secondary or upper primary education is challenging. The way teachers address the topic of fractions can be largely influenced by the implicit or explicit expectations of students' performance manifested in textbooks (Charalambous et al., 2010). International textbook comparisons (e.g., Charalambous et al., 2010; Alajmi, 2011) revealed potential differences in the teaching and learning process which may be attributed to differences among textbooks. In Hungary, there is a system-level challenge in teaching fractions, i.e., after the first four years of school mathematics taught by a subject-generalist elementary teacher, from 5th grade on, subject-specialist mathematics teachers should build on students' already learned fraction representations, and further improve it while introducing the addition and subtraction of fractions.

The main aim of our research was to explore the types of visual representations of fractions in the textbooks of the two market-leading publishers. Visual representations were analysed both for individual fractions and for basic operations with fractions (addition, subtraction, and division and multiplication with natural numbers). Our main findings suggest that pie charts, rectangular and the number line are the most frequently used illustrations. As for the operations with fractions, there were relevant differences between the textbooks in that one of them abundantly used all these three main types, while the other relied more on just the rectangular model. Furthermore, there was a huge difference in whether the illustrations for the worked-out examples and the practice tasks were coherently provided. From a pedagogical point of view, we emphasize the importance of providing representations about multiple aspects of the multifaceted concept of fractions, and the feasibility of providing comparative and comparable representations of non-unitary fractions and on the addition of fractions.

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BIG IDEAS OF EQUIVALENCE AND PROPORTIONALITY IN A GRADE SIX MATHEMATICS LESSON

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Charles (2005) defined a "Big Idea as a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10). The revised mathematics syllabuses for primary schools in Singapore (MOE, 2019) reinforces that Big Ideas are central to mathematics as they connect ideas coherently from different strands and levels thereby facilitating a deeper and more robust understanding of individual topics in mathematics. These ideas include Equivalence and Proportionality.

Research has documented that teachers' lack of relevant content knowledge of Big Ideas in mathematics translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools and results in developing isolated compartments of mathematical knowledge in their students (Askew, 2013). A research study, Big Ideas in School Mathematics (BISM) is presently underway in Singapore and a part of it is situated in two primary schools where teachers are undergoing professional development (PD) led by mathematicians and mathematics educators. As part of the work in the schools, a group of grade-six teachers designed tasks involving equivalent fractions for the topic of percentage increase/decrease and enacted them in a 60-minute lesson it. The tasks were:

Task 1: $\frac{1}{2} = \frac{?}{?} = \frac{?}{?} = \frac{?}{?}$; Task 2: $\frac{50}{100} = \frac{60}{?} = \frac{80}{?} = \frac{100}{?}$

The lesson was video-recorded and a review of the video-record by the teachers and professors was carried out. A key finding was that teachers were cognisant of the big ideas of equivalence and proportionality and were deliberate in creating opportunities for pupils, where possible, during their lessons and that pupils were also articulating these ideas though in their own "words" such as "the same as" and "it grows in the same way."

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RECONCILING DIFFERENT DISCIPLINARY EPISTEMOLOGIES

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The epistemology in mathematics essentially differs from the other disciplines' epistemologies as it is based upon the notion of mathematical proof (Buldt et al., 2008). These differences may hinder students' ability to apply mathematics in real-life contexts. The current study is a part of a wider DBR project (Bakker, 2018) aiming to develop activities that acquaint students with different disciplinary epistemologies. Here we report on the enactment of an activity in which interpretations for a fictional archaeological inquiry, given by archaeologists and a mathematician, manifest their different epistemic perspectives.

The activity is divided threefold. In each stage, students are presented with some evidence and are requested to decide which of two artifacts is more valuable. First, the archeologists debate without quantities or mathematical terms. Secondly, they weigh the artifacts and find them unequal (though the difference is negligible). Third, the mathematician presents a geometry proof of areas' equality corresponding to the Four Lunas theorem (Nelsen, 2015, p. 44). We investigate students' argumentation to see the interactions between archeological and mathematical epistemologies.

Participants were 43 tenth Grade students working in 17 groups, 2-4 students each. Students' responses to an online questionnaire in the first and second stages of the activity (before the geometry proof was given) show that four groups used only archeological arguments relying on the archeologists' claims and authority. Five groups used mathematical epistemology manifested in measurements, quantities, calculations, or proving. The rest eight groups gradually transformed from an archeological to mathematical argumentation. Notably, all the mathematical-arguments groups argued that the weight difference was negligible (5/5), while the rest inclined to refer to it as significant (7/12). Several groups developed arguments from other disciplines (physics, chemistry, history, and economics). Our results suggest the potentiality of interdisciplinary activities to help students reconcile different epistemologies.

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AN ONLINE CAMP DURING COVID-19 FOR PROMOTING STUDENTS' MATHEMATICAL MODELLING AND MOTIVATION

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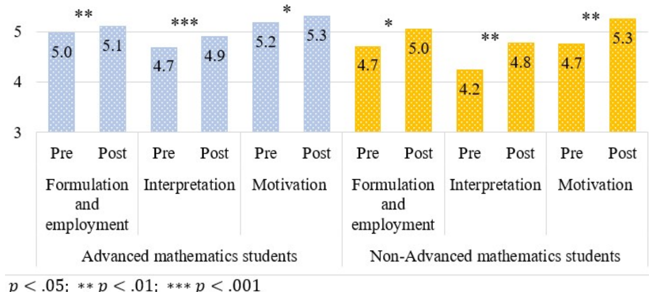
Mathematical Modelling (MM) is an important competency that students should acquire during their school studies. MM involves a cyclic process in which translation is made between the real-world into mathematics and back to the real-world. The modelling cycle involves three basic processes: *formulating* a mathematical model that responds to the real-world context, *employing* mathematical procedures, algorithms, and calculations for mathematically computing the model, and *interpreting* the results for verifying their adjustment to the real-world context (Blum et al., 2007).

In this study, secondary school students participated in a five-day online summer camp during COVID-19 where they experienced authentic MM tasks from engineering and technology contexts. We aim to explore development in students' MM competency and motivation to study mathematics during their participation in the summer camp.

Participants were 771 9^{th} grade Israeli students (44.5% girls), of whom majority, 90.6%, were intended to study mathematics at an advanced level at 10^{th} grade. Research pre-post questionnaires aimed to assess students' MM competency based on self-report regarding mathematics learning processes as well as students' motivation toward studying mathematics. Additionally, students were asked to write freely about what mathematics meant to them, also as indicator for their motivation.

Analyses revealed that this short-term intense involvement in MM learning increased student MM competency and motivation to study mathematics (Figure 1). Non-advanced math students benefited more from the camp compared to advanced math students. At the end of the camp, more students conceptualised mathematics applicability in their lives compared to the beginning of the camp where they mainly described it as an important profession. Our study suggests that linking mathematics to real life may help students understand the content learned in school, as well as motivating them to engage with it (Ferri, 2018).



$p < .05$; $** p < .01$; $*** p < .001$

Figure 1. The average score by mathematic study level on the pre and post attitude questionnaire separately

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EXPLORING A NEW ROUTE TO OLD ROOTS FOR INTERNATIONAL MATHEMATICS EDUCATION CONFERENCES IN THE POST-PANDEMIC ERA

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Since the first ICMI conference began in 1908, conferences have been playing an essential role in building the international mathematics education community. By networking and disseminating advancements in research and practices, the community established the discipline of mathematics education as a "learned society" (Leung, 2015). Despite the travel restrictions due to the global pandemic, international conferences have been serving mathematics education community by adopting virtual platforms to involve participants in our community.

In this study, we investigated how mathematics education researchers reflect on their experiences after attending face-to-face and virtual international conferences. We conducted three focus group interviews (Onweugbuzie et al., 2009) with international mathematics education researchers who have different backgrounds in terms of regions, genders, and academic careers. We grouped interviewees based on their roles in the conferences as attendees, presenters, and organizers. Each focus group participated in a two-hour semi-structured interview with three leading questions: (a) how did the participants perceive face-to-face and virtual conferences? (b) how did they view advantages and limitations of virtual conferences?, and (c) how did they envision mathematics education conferences in future? Focus groups shared advantages and limitations of virtual conferences compared to face-to-face conferences in terms of accessibility, presentation, network, and technology. For example, most of interviewees said virtual conferences allowed participants to attend conferences. This suggests that virtual conferences increased affordance, which could address the inequity. Many participants developed a new meaning of attending conferences after attending virtual conferences: (1) attending conferences without presentation, (2) attending conferences without traveling, (3) attending conferences in their own space such as home or workplace, (4) networking in virtual platforms. This study will shed light on seeking a new route in hybrid and virtual conferences to provide a broader community in mathematics education with meaningful conference experiences.

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UNPACKING DISAFFECTED STUDENTS' METAPHORS FOR MATHEMATICS LEARNING

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Students' earliest memories of mathematics in school often involve playing with numbers and identifying geometric forms. But as they progress through the grades, many face obstacles and become less motivated to learn math. This paper examines the metaphors for learning mathematics produced by disaffected students and considers whether these metaphors can serve as a resource for identifying ways to improve students' experiences and relationships with mathematics.

Scholars have used diverse approaches to explore students' conceptions and attitudes toward mathematics learning (Hannula, 2014). Our study used qualitative thematic analysis (Braun & Clarke, 2006) to study meaning patterns in interview responses to an open-ended prompt ('learning mathematics is like…') from over 100 Kindergarten to Grade 12 students, focusing on responses from those we identified as disaffected. Three themes were generated: learning math as path following; learning math as rule following; and learning math as emotional control. The findings indicate that some disaffected students interpret mathematics learning as a process that is externally constrained, a view also documented by Latterell and Wilson (2017), while others recognize their agency. These findings serve to enhance our understanding of students' experiences in classrooms, and point to possible pedagogical strategies to address students' perceived concerns, such as examining students' metaphors with them and exploring ways to address feelings of loss of control and emotional discomfort.

For researchers, we suggest that analysis of student metaphors for learning mathematics is an undervalued technique. In these times of restrictions on access to schools for (new) data collection, it is also one that does not necessarily need focused data collection and could potentially be applied to already-existing datasets if those materials contain free-flowing interviews on topics relating to student or teacher learning.

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FRUIT SALAD ALGEBRA – A COMPARISON OF EXPERTS' AND MATHEMATICS TEACHERS' NOTICING

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An adequate use of a representation has the potential of facilitating students' mathematical understanding, whereas its inappropriate use may lead to misconceptions. A well-known misconception in algebra learning is the *letter as object* misconception (Küchemann, 1981), where letters are conceptualized as objects rather than as unknowns or variables. Because often images of apples and bananas are used to represent variables *a* and *b*, it has also been called *fruit salad algebra*. Besides being aware of this misconception, teachers and experts in mathematics education need to use this knowledge to notice and make sense of classroom situations. One approach for assessing teachers' noticing is using representations of practice that include some breach of a norm. The identification and interpretation of such breaches are considered as an indication of teachers' noticing expertise. In this communication we examine and compare experts' and teachers' noticing regarding a classroom situation in which fruit salad algebra is used (breach of a norm). This study is part of a project that explores perspectives on instructional quality and teacher noticing (Dreher et al., 2021).

Data consisted of the responses of 12 German experts (professors in mathematics education) and 113 German secondary mathematics teachers to a vignette showing fruit salad algebra being used. The picture cards used in the vignette to illustrate the combination of like terms do not properly represent the nature of variables. Participants were asked to answer the following open-ended question 'Please evaluate the teacher's use of representations in this situation and give reasons for your answer'. The results of a qualitative analysis show that the majority of experts recognized the inappropriate use of objects to represent variables, but very few teachers did. Instead, many teachers appreciated that the representation relates to everyday life and that it is memorable.

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WHAT ERRORS DO PROSPECTIVE TEACHERS DETECT IN THEIR MATHEMATICS LESSONS?

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This study aims to characterise the errors detected by some prospective teachers when they reflected on their own practice during their educational internships, regarding two research questions: 1) Do prospective teachers identify errors when reflecting on their own practice? 2) What types of mathematical errors do they identify? In order to answer them, we analysed the reflection that prospective teachers made in their Master's Degree Final Projects (MFPs), in which they had to remember if they made errors and explain them. This reflection was analysed using the Didactic Suitability Criteria (DSC) proposed by the Onto-Semiotic Approach.

The DSC are a theoretical tool to assess teaching and learning processes (Breda et al., 2017), which is organised into six facets. Moreover, each of the DSC has components and indicators that allow them to be assessed in practice. We specifically focused on the 'Errors' component of the epistemic suitability criterion. In this study, we consider an error as a mathematical practice that, from the point of view of a mathematical institution, is not considered as valid; and an ambiguity (or inaccuracy) as an explanation by a teacher that may be partial and/or unclear, although not incorrect, that leads students to make errors or to confusion.

This study followed a qualitative research methodology from an interpretative paradigm, which mainly consists of a thematic analysis. The research context is a Master's Degree Program taught by the public universities of Catalonia (Spain). We considered 258 MFPs (from 2014-15 to 2019-20 academic years), in which the prospective teachers assessed the implementation of a lesson plan using the DSC.

From the errors identified, as the main result, we proposed a categorisation of six types of mathematical errors that the prospective teachers detected: error of definition, error of representation, error of resolution or procedure, error in posing a statement, error of proof or argumentation, and error of proposition or property. These results are susceptible to modifications, as we analyse the MFPs from other academic years.

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SOLVING ADDITIVE WORD PROBLEMS THROUGH CAUSAL DIAGRAMS IN ELEMENTARY EDUCATION

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Problem-solving refers to mathematical tasks that have the potential to provide intellectual challenges for enhancing students' mathematical understanding and development (NCTM, 2000). It also involves several cognitive operations such as reading and understanding the problem statement, detection of quantities and relationships between them, conversion of vernacular to mathematical language, execution of a designed plan and reflection on the solution obtained (Polya, 1945; Puig & Cerdán, 1988). In particular, solving addition and subtraction word problems requires mathematical skills and text comprehension techniques.

In this work we show a teaching model for primary or higher-level students, to improve the comprehension of an additive word problem (AWP) through reading mechanisms based on the division of the problem statement in different propositions using different colours and a graphic representation of causal diagrams (Puig & Cerdán, 1988).

The process is as follows: when reading a problem statement, the student must learn to detect the propositions or fragments of the text determined by an action verb. After this, a list of the quantities present in these fragments must be obtained, distinguishing whether they are data or unknown quantities. Finally, the student must try to obtain the relationship between the quantities identified through the action verbs mentioned. In this last phase, causal diagrams will be used to proceed to the numerical resolution.

The results of a preliminary exploratory study carried out with 16 students from 11 to 12 years old with three AWPs showed better results than those presented by Riley et al. (1983). In fact, the proposed method improves the probability of success in all cases and even the students who had not finished or correctly done the causal diagrams had improved their understanding of the problem statement.

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DEVELOPING FRACTIONAL REASONING THROUGH BODY PERCUSSION

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A great deal of literature exists on the connection between mathematics and music, and new ways in which teachers can harness the synergies continue to be discovered. This Oral Communication is based on aspects of the first author's broader doctoral study. The study is exploring the integration of mathematics and music to support the teaching and learning of fractions at primary school level. It is guided by the theoretical framing of realistic mathematics education (RME) (Van den Heuvel-Panhuizen, 2003), which encourages designing experientially real activities from which informal discussion can lead to formal, vertical mathematization. The first author conducted a preliminary piece of action research trialling one of her RME design ideas. The research question she asked here was: 'How might body percussion be used as an intervention strategy for teaching fractions?'.

This presentation will share some initial findings around the use of a specific body percussion activity designed to provide students with opportunities for deepening their conceptual understanding of fractions beyond the part-whole model. The activity trialled aimed at supporting Grade 6 students' understanding of the construct of fractions as ratio. It involved a clapping activity focusing on 'beats per bar' in music. This activity, based on the well-known clapping game, *Sevens!*, required students to clap a sequence of seven beats in different ways, getting faster and faster each round. Having taught the clapping activity to the Grade 6s, the first author then posed a set of problems which called on students' fractional reasoning. After completing the lesson, she critically reflected on her perceptions of the effectiveness of the strategy in her reflective research journal. She then transcribed the audio recording of the lesson and analysed the transcript data deductively using thematic analysis to identify patterns relating to the theoretical framing of RME. Data from her reflections and deductive analysis on this preliminary cycle of action research indicate that the body percussion activity certainly had the potential to serve as a realistic context for stimulating problem-solving requiring fractional reasoning around fraction as a ratio. Findings from the data will be shared in the presentation and will also be used to guide future iterations of using body percussion to support teaching fractions meaningfully.

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AN ANALYSIS OF TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE TOPIC OF FRACTION AFTER UTILIZING LESSON STUDY AND OPEN APPROACH

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Khon Kaen University

The development of Pedagogical Content Knowledge (PCK) is the result of teaching and teaching preparation (Shulman, 1986). Lesson study and Open Approach innovations can be utilized to recognize the central importance and difficulty of teaching by bringing the teaching process to frameworks and best practices in the classroom (Inprasitha, 2011).

This research was designed to analyse teachers' PCK in teaching the topic of multiplication of fractions after practicing the Open Approach incorporated in the Lesson Study process. Target groups consisted of two Lesson Study groups from two schools. Each LS group was comprised of 3-4 mathematics teachers at the project school, prospective teacher, and experts. Lesson study process in Thailand context was proposed by Inprasitha in 2004 consisting of three steps: collaborative design research lesson (Plan); collaboratively observe research lesson (Do); and collaboratively reflect on teaching practice (See) to change the paradigm of teachers' PCK in their teaching. The LS group started the 'Plan' step by meeting together to discuss and design a research lesson. This was followed by incorporating an Open Approach composed of four phases: posing open-ended problems; students' self-learning; whole-class discussion and comparison, and summarize by connecting students' mathematical ideas emerged in the classroom in the 'Do' step. Finally, the LS group participated in the 'See' step by reflecting on the teacher teaching and student learning activities. A qualitative research approach case study design was employed using research instruments: lesson plans, observation records, field notes, and interview protocol. Data was collected through the Lesson Study process and analysed using content analysis.

Results revealed that the target groups anticipate students' ideas thoroughly, but they still lack plans to organize with students' ideas. They plan to use appropriate questioning to whole-class discussion and trigger students to use their knowledge to reach the solutions and realized a new how-to for multiplying fractions by themselves.

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TRANSFORMING PRE-SERVICE TEACHERS' CONCEPTIONS OF DIVISION BY ZERO: A CASE OF SCAFFOLDING

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Research has highlighted that primary and secondary school teachers struggle with conceptually understanding division by zero (Quinn et al., 2008). Based on Quinn's (2008) study, we were motivated to examine whether first year mathematics pre-service teachers also share similar difficulties with understanding division by zero. Secondly, we were interested in exploring possible strategies that a lecturer can use to help deepen student teachers' conceptual understanding of division by zero.

In our qualitative study, a lecturer asked fifty-six first year mathematics pre-service teachers: "What is two divided by zero?" Forty-five students incorrectly stated that the answer was two. Based on the unexpected incorrect answer, the lecturer scaffolded the question and asked students to input x values ranging from 1000 to $\frac{1}{1000}$ into $y = \frac{2}{x}$ where $x > 0$. Subsequently, the lecturer made the students sketch the hyperbola and the students realised that as x becomes smaller, y becomes bigger. The students realised that as x approaches zero, the answer diverges meaning that division by zero is undefined.

Drawing on Vygotsky's socio-cultural theory, our study showed how the students' spontaneous concept was discounted and transformed into scientific concepts through the lecturer's scaffolding. Our study not only characterises pre-service teachers' initial difficulties with division by zero but provides an exemplary case of a lecturer scaffolding a difficult concept which may be of use to mathematics teachers and teacher educators.

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CALIBRATING THE MEASUREMENT OF INFORMAL MATHEMATICS IN PERUVIAN PRESCHOOL CHILDREN

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The development of mathematical competence begins from informal and intuitive learning in the family environment of children. Informal mathematics refers to mathematical notions and processes, generally learned in non-school contexts, which are developed from interactions with the physical and social environment, where scenarios such as games that generate meaningful learning in a more natural and spontaneous way are presented (Ginsburg & Baroody, 2007; Purpura et al., 2013).

One of the strongest predictors of future academic achievement is the early math skills with which children begin their school studies (Duncan et al., 2007). Because of this, it is essential to have proper tools for measuring the development of informal mathematical concepts and abilities at an early age in order to be able to intervene in a timelier, more effective way. The purpose of this research is to calibrate the items of informal mathematics from the Test of Early Mathematics Ability–3 (Ginsburg & Baroody, 2007) by applying the Rasch model. A total of 148 Peruvian preschool children (ages ranging from 5 to 6 years) participated in the study.

Our results show good psychometric properties of the informal mathematics dimension of the instrument, which indicates an adequate fit of the children sample and the items to the proposed model, as well as a tendency toward unbiased items and non-differential item functioning between girls and boys. Furthermore, we found that the items analysed exhibit a consistent internal structure at the theoretical level. We believe these results can contribute to new research on informal mathematics, and they provide elements for timely diagnoses and improved intervention proposals in early mathematics education, especially in the Peruvian context.

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TENTATIVENESS AS A STRATEGY AND AFFORDANCE IN PROBLEM SOLVING

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Spatial reasoning has been identified as integral to both general mathematical capability and the potential for individuals to flourish in life beyond formal mathematics education. Specifically, the ability to *visualize* shape and space is an aspect of spatial reasoning that is consistently associated with achievement in mathematics (Davis et al., 2015).

This research studies how spatial reasoning skills are enacted by secondary students as they work on other topics in mathematics, such as algebra. The study is guided by a theoretical frame of enactivism, in which cognition is viewed as a complex phenomenon emerging out of the interaction between an organism and the environment (Reid & Mgombelo, 2015).

Data was generated in two secondary classes with a total of 36 participants at a large, western Canadian high school. The classes covered topics in pre-calculus and calculus and were part of the school's International Baccalaureate program. All participants had high academic achievement in mathematics. Lessons were designed to elicit embodied and spatial approaches to problem solving (e.g., using origami to solve problems involving quadratic functions) and were video recorded.

Through an analysis of video and written work, I identify and describe a distinctive feature of classroom interaction, which I refer to as *tentativeness*. As a strategy employed by students, I show tentativeness to be an emergent, participatory quality of problem solving and posing. As a phenomenon to which students respond, I show it as an affordance that broadens the possibility for adaptive action in the classroom.

While tentativeness is often framed as a deficit, synonymous with indecision, I characterize it is an emergent process arising between the student, the problem, and the environment. Knowing more about how to structure it in the classroom and elicit it from students has important implications for students' mathematical understanding.

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CHARACTERISING PRE-SERVICE PRIMARY TEACHERS' DISCURSIVE ACTIVITY WHEN DEFINING

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An essential part of learning mathematics is learning definitions, but many studies report that students sometimes ignore the formal definitions or use them inadequately (Tabach & Nachlieli, 2015). However, there are fewer studies on how pre-service primary teachers (PPTs) propose definitions or choose one when several are available.

In this work, our aim was to study how PPTs do so through the lens of the commognitive framework (Sfard, 2021). This framework considers mathematics as a particular type of discourse and learning as a change in that discourse. It also distinguishes between object-level and meta-level learning. Meta-level learning is possible as the resolution of commognitive conflicts, which often occur when the participants' discourses are governed by different meta-rules. Therefore, our research questions were: (1) which meta-rules governed the discourse of PPTs when defining or choosing a definition? (2) did the existence of different meta-rules always lead to the existence of a commognitive conflict?

The participants were 45 PPTs organised into 12 groups of 3-4 students (G1 to G12). Each group had a worksheet with nine questions about defining geometric solids. They had to write down their answers and their conversations were recorded and transcribed. We later identified and categorised their meta-rules and the possible commognitive conflicts that appeared. This qualitative study shows that PPTs employed a variety of meta-rules. For instance, the PPTs of G7 used two different ones when defining the solids, which led to a commognitive conflict. On the other hand, the PPTs of G2 used two different meta-rules when choosing a definition, but a commognitive conflict could not be inferred because their discourses seemed commensurable. More research is needed to determine how to help PPTs to learn how to define and choose definitions.

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DIFFICULTIES WHEN SOLVING SHARING SITUATIONS WITH GROUPING ACTIONS

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While grouping actions can be used to solve sharing situations, Brown (1992) notes that grouping modelling actions do not coherently make sense of sharing situations per se. This means that for instruction that supports children's sense-making, translations from sharing situations to grouping actions must be chained appropriately in an instructional explanation pathway. In this presentation, we share episodes from the first author's doctoral study on division instruction where this logic was disrupted in two key ways (Mathews, 2021). These ways are exemplified through episodes drawn from one of the six teachers in the broader study. Semiotic analysis involving study of the sequence of production of mathematical signs was used. Ten of the 64 episodes taught by the focal teacher involved grouping actions to produce answers in sharing situations.

Illustrating the first type of disruption seen in these episodes, the teacher began by stating: "Share 24 among 5. You can count in five". She then wrote "24÷5=" on the board and proceeded to draw four groups of 5 dots and circling them and counted in fives to 20. She then drew a remainder group of 4 dots and inserted the answer 4 remainder 4. In this episode, while the symbolic division calculation can be solved with a sharing or grouping action, there is no highlighting or acknowledgement of the shift from a sharing situation (share 24 among 5) to a grouping-oriented diagram production (four groups of 5 dots with four dots remainder). From the child's perspective, making sense of the situation to produce the answer is disrupted here as the situation silently shifts. Illustrating the second type, the teacher described "equal sharing" between 3 people. She took 6 sweets and distributed 2 sweets at a time to each of three learners and wrote 6÷3=2. The answer is produced using prior knowledge of the quotient value, however, broader evidence in South Africa suggests that learners lack prior knowledge.

These two difficulties point to key ways in which sense-making is disrupted in the context of division teaching of sharing situations in South Africa. These disruptions shed light on low performance in division in the primary grades (Schollar, 2004).

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SCAFFOLDING MATHEMATICS LEARNING: A CASE OF GRADE 3 TEACHERS

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Research shows that students are more likely to succeed in their studies when learning is scaffolded (Walqui, 2006). Although vast research has been conducted on different scaffolding strategies to enhance mathematics learning, there has been paucity of research that is based on multiple means of representation (MMR) on scaffolding mathematics learning, and this study intends to fill that gap. The current study thus highlights how MMR was implemented to scaffold mathematics learning in primary mathematics.

The qualitative case study that is reported here involved four Grade 3 mathematics teachers who were purposefully selected based on their extensive experience of teaching mathematics in the foundation phase for more than 10 years. Data were generated through observations and focus group discussions; wherein critical emancipatory research (CER) was employed as a lens underpinning the study. CER promotes knowledge sharing and thus it afforded all the participants opportunities to share their experiences on how they implemented MMR to scaffold learning.

Findings from the study revealed that displaying content in different formats, reinforced the learners' understanding of mathematical concepts. Furthermore, the use of tables, wherein the key terms were outlined together with their meanings; application of key terms provided coupled with equivalent symbolic representations improved mathematical language and expression. Moreover, the use of pictures alongside the given scenarios assisted in clarifying content, and thus improved visual perception. The use of MMR was found not only useful in scaffolding mathematics learning and making learners independent but also empowered learners to be knowledgeable and resourceful.

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A TEACHERS' IMPROVED UNDERSTANDING OF GEOMETRY THROUGH A COURSE

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According to Levav-Waynberg and Leikin (2009), using multiple solution strategies to solve a geometry task improves understanding of geometry. They relate mathematical understanding to the connecting of different mathematical concepts, properties and representations by the use of multiple solutions in solving a geometric task. I used the framework developed by Levav-Waynberg and Leikin (2009) to guide my analysis of a teacher's (Ben) use of multiple solution strategies.

The focus of this paper is to show a teacher'simprovement in his geometric understanding through his use of multiple solution strategies as evidenced at the beginning, middle and end of a geometry course. In 2021 the Wits Maths Connect Secondary Project (WMCS) designed and implemented a basic geometry course in response to requests from secondary mathematics teachers who self-identified as having difficulty with the basics of Euclidean geometry (lines, angles and triangles). Tasks in the course were designed such that multiple solution strategies could be used and was one of the strategies that was emphasised in the course. At the beginning of the course Ben offered a single solution to a task that required different solution strategies. During a session, in the middle of the course, he offered two solutions to the task in Figure 1. Although the task could be solved in different ways using angles formed by parallel lines and transversals, Ben offered solutions that also made use of exterior angles of triangles. At the end of the course, he solved a task by using isosceles triangles and exterior angle of a triangle as well as parallel lines and transversals. While some teachers in the course could still not offer other solution strategies apart from those relating to parallel lines, Ben displayed a better understanding of basic geometry through his use of multiple solution strategies as suggested by Levav-Waynberg and Leikin (2009). Possible reasons for him being able to offer different solution strategies while other teachers could not can be further pursued by systematically mapping solutions of different teachers at different points during the course and through teacher interviews.

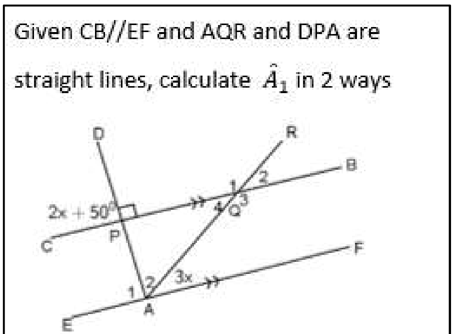

Figure 1: Task using alternate strategies

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VALIDATION OF AN INSTRUMENT TO MEASURE FUTURE TEACHERS' CONCEPTIONS OF MATHEMATICS TEACHING AND LEARNING

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Teachers' conceptions of mathematics teaching and learning have a significant impact on their knowledge and practices. Besides, given the increasing relevance of video analysis for teacher training, it becomes pertinent to determine whether pre- and in-service teachers' conceptions of mathematics teaching and learning bias classroom observations. Measuring teachers' conceptions still represents a challenge. Teachers find it difficult to describe their own conceptions, and inferences from external observations might result in glaring inconsistencies. Besides, existing instruments to measure future teachers' conceptions of mathematics teaching and learning have not been validated or have been used in empirical studies with small sample sizes.

Analysing teachers' conceptions of mathematics teaching and learning from their instructional tendencies is considered as a reasonable alternative. From this perspective, Climent (2005) developed the CEAM instrument (Spanish acronym of Conceptions about Teaching and Learning Mathematics) which consisted of a set of items aligned according to four instructional tendencies (traditional, technological, spontaneous, and investigative). This instrument has been recently adapted so it can be applied as a Likert-scale questionnaire and answered by individuals without teaching experience (Rodríguez-Muñiz et al., 2022).

Therefore, the original CEAM instrument was adapted to be applied to a large sample of future teachers and provided empirical evidence about their conceptions of mathematics teaching and learning. With another sample of undergraduate students from different bachelor's degrees: primary education, mathematics, and the education itinerary in psychology, a Confirmatory Factor Analysis validated the instrument and endorsed the assumption that future teachers' conceptions of mathematics teaching and learning can be described in terms of several combinations of instructional tendencies. This result also helped to thoroughly explore the multi-dimensionality of some of the items from the original instrument, and the complex nature of conceptions.

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IMPACT OF MATHEMATICS TEACHERS' REFLECTION ON THEIR PERFORMANCE AND BELIEFS

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Teacher observation during the teaching process allows for investigations that provide answers to important research questions, notably concerning the relationship between teachers' performance and aspects such as their attitude towards the discipline and their teaching or the work of their students. Different instruments for systematic observation of teaching practice appear in the literature, some specific to mathematics teaching (i.e. IQA rubric, by Boston & Candela, 2018), and others generic (such as the FtT, Danielson, 2013). The aim of this research is to study the impact of guided observation of a teacher's own (or another teacher's) teaching practice on teacher's beliefs towards mathematics and its teaching and on their performance in the classroom and to identify differences regarding the use of a specific or a generic instrument.

Considering that the objective concerns two variables taking two values each (instrument and origin of the observed video), we adopt a methodological design involving 4 pairs of teachers. In two pairs both teachers use either the specific IQA or the generic FtT instruments to analyse their performance or an external video. In the other two pairs, each member analyses with a rubric a video that can be their own or an external one. In this way we ensure comparisons between and within pairs. The study design consists of various phases. Participants complete a belief questionnaire (Coppola et al., 2012). This is followed by alternating recordings of each participant's teaching practice (3 times) with three guided observations using the rubric for each teacher. 10 months after the start, participants complete the belief questionnaire again.

The study started in October 2021 and we are currently collecting data. Our hypotheses are twofold: the guided observation of teaching practice will promote changes in this practice (thus there will be an evolution between the first and the third recording); and the guided reflection on self-observation through the IQA rubric will promote more evident changes in beliefs towards mathematics teaching.

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CONSENSUS ABOUT CONSENSUS? TEACHERS NEGOTIATING LESSON PLANS IN LESSON STUDY GROUPS

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Lesson Study (LS) has been adapted to many countries as a significant means to improve teaching and learning of mathematics (Lewis & Lee, 2017). We report on preliminary findings from an Israeli LS project, herein focusing on the notion of *consensus building* in collaborative lesson planning during LS cycles. Inspired by the Consensus Building Theory (Briggs et al., 2005), we view the strive for consensus in the context of our study as the degree to which teachers participating in LS are willing to commit to a possible future choice or action suggested as part of the lesson planning. The research question we posed was: which manifestations of reaching consensus can be found in Israeli LS groups, and how do these manifestations reveal teachers' perceptions about the aim of LS?

We collected data from 8 school-based LS groups across Israel, each including 8-12 secondary mathematics teachers. Two teachers from each group were randomly selected for individual semi-structured interviews. Data analysis employed Grounded Theory methods, with relevant key words identified in the interviews' transcripts (e.g., *convince*; *persuade*; *argue*). Utterances including these key words were classified into two emerging perspectives about consensus reaching during LS: (1) *Process-oriented consensus*. This perspective sees the process of striving for consensus as more important and significant for teachers than actually achieving agreement. Striving to reach consensus is viewed as an opportunity to discuss issues that usually do not arise in teachers' conversations. (2) *Product-oriented consensus*. Here, the focus of the group's activity is on reaching an agreed lesson plan that would address specific needs, with less interest in the process and little recognition of its added value for teachers. We propose that awareness to how the process-product perception affects the reaching of consensus in LS groups, especially outside Japan, may contribute to the ways LS communities are designed and led.

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THE DIALECTICAL PROCESS OF PROBABILISTIC THINKING

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We present a qualitative analysis of the probabilistic thinking process of six high school senior students in a Statistics course. The students were selected from a sample of size (n=270) as their answers were clear, detailed and representative of different thinking processes regarding predictions and justifications in three learning activities related to probability and their answers to five questionnaires.

The objective of the research was to find out what justifications are considered to predict or explain a result, or to make decisions when trying to win a game. The justifications were classified as intuitive (IJ), formal (FJ), or combined (CJ).

In the first activity, students threw a pair of dice 25 times, predicting the sum in each throw. Students answered a questionnaire (Q1A1) about what they thought would happen before the activity and answered another questionnaire (Q2A1) after the activity focused on their initial approaches to generalization. In Q1A1 only one student was classified as FJ, everyone else was classified as IJ and their predictions were very varied. On the other hand, in Q2A1 the student who was classified as FJ changed to CJ and everyone else maintained the IJ classification, although some of them observed that their previous predictions of Q1A1 had no sense. In the second activity, a simulator was used for throwing two dice 25, 100, and 1000 times. All students deduced that 7 was the most probable number based on frequency. All students' justifications in a questionnaire (Q2A2) were classified as FJ. The third activity was a game: a turtle race. A set of turtles were numbered from two to twelve. Two dice were thrown determining the movement of the turtle that coincide with the sum of the points of the dice. In the questionnaire (Q1A3) students had to choose one winning turtle justifying their choice. Most of them selected a number different than 7 but close. Four students' answers were classified as CJ and the rest of them as FJ. After the game in the questionnaire Q2A3 four of them were classified as FJ and the rest as IJ.

The results show a dialectical nature in the probabilistic thinking process, intuitive justifications were present again and again and live together with the formal ones. We concur with Batanero (2020) on the necessity to consider student's informal ideas stressing that probability should be viewed as a mathematical model.

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MEANING AND KNOWLEDGE COMPARTMENTALIZATION IN INTEGRAL CALCULUS

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We suggest a novel theoretical approach to knowledge compartmentalization (Vinner & Dreyfus, 1989), wherein compartmentalization is defined using the theoretical framework for meaning by Thompson et al. (2014). The main construct used from this theory is 'space of implications' – the actions or schemes that the current understanding brings to mind with little effort. We suggest that compartmentalization occurs when a learner has two different meanings for the same mathematical concept that have different spaces of implications – one does not bring the other to mind.

Oren is a 12th grade student taking advanced track mathematics. We interviewed him on a task relating to the expression $g(x) = \int_0^x f(t)dt$, given the graph of f – a step function with two positive values. Oren distinguished between what he called a 'regular integral' and an 'accumulating integral', the former referring to an integral with numerical bounds, and the latter referring to the accumulation function. While Oren connects the 'regular integral' to area, he connects the 'accumulating integral' to accumulating values, also stating that "it's hard to say that an accumulating integral represents area". Thus, the two terms have different spaces of implications for Oren, suggesting Oren's knowledge regarding integral is compartmentalized.

However, when Oren reasons about the accumulated values in a graphical context, he manages to create a connection between these values and area, albeit an unstable one:

Oren: Value 3 is ultimately, if we do the area of this rectangle here, 1 times 3, then we accumulate, eventually we accumulate this.

Hence, Oren briefly manages to overcome his compartmentalization. These and additional findings suggest different degrees of compartmentalization based on the existence of a cognitive conflict and the ability to resolve it.

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INVESTIGATING THE AFFORDANCES OF AI-POWERED MATHEMATICS LEARNING PLATFORMS

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PURPOSE AND RESEARCH QUESTION

Despite the ever-changing digital landscapes in education, especially with the leverage artificial intelligence (AI), schools and practitioners often lack clarity in terms of which pedagogical theories should guide them to evolve practices by embracing the affordances of such learning spaces (Hartwick, 2018). This study explored the affordances of AI-powered learning platforms in the context of K-12 mathematics learning and teaching. This study adapted Evans et al. (2017)'s view of affordance as "something that helps mediate behaviour towards an outcome" (p. 36) to mean that an affordance is a characteristic of the learning space that promotes or facilitates learning outcome. The research question of this study is: How do the affordances of AI-powered learning platforms influence students' usage of the platform for learning in general and mathematics learning in particular?

METHOD AND PRELIMINARY RESULTS

The main data instrument and data collection was done through the use of a five-point Likert scale structured questionnaire by 300 secondary students through the Google form. The exploratory factor analysis and logistic regressions were employed to analyse the data.

After conducting the exploratory factor analysis, four factors that influence students' usage of an AI-powered learning platform were identified: interface design, multimodal interaction, cognitive operation, and technical operation. Multimodal interaction was the most important factor affecting the usability of an AI-powered learning platform while technical operation was the least important factor. The findings of the study may suggest future studies examine design principles that provide users with multiple modes of interacting with a system.

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TEACHING A JOINTLY PLANNED LESSON ABOUT SUBTRACTION

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The implementation of teaching has been advocated to be an integral part of research, since it affects students' learning (Cai et al., 2017). If results from research are shared by teachers in instructional knowledge products like lesson plans, which has been advocated to promote student learning (Morris & Hiebert, 2011), the implementation in classrooms need to be studied, to see to what extent students are given similar opportunities to learn.

The aim of this study is to identify significant differences in teaching subtraction bridging through 10, to identify students' learning possibilities from lessons with the same lesson plan. This can shed light on issues in need to be addressed when teachers implement instructional products. The research question is: What is made possible to learn in the lessons and how can differences in what is afforded be explained? The study focuses on the teaching of one lesson held by four teachers in their classes. The lesson was planned by a group of teachers and researchers working collaboratively in a larger intervention project concerning addition and subtraction in grade 1. Variation theory (Marton, 2015) was used to analyse the video recorded teaching. The jointly planned lesson shows differences in aspects of the content brought out. These entail qualitatively different possibilities to learn subtraction bridging through 10 for the students. Teacher A drew students´ attention to *Subtraction tasks solved as dynamic operations* and used representations sequentially. The choice of strategy was based on student preferences rather than tasks. Teacher B drew students' attention to *Subtraction tasks as static part-whole relations*. Representations, used simultaneously, gave the task the meaning of a part-whole relation. While using parts in a part, i.e. 3 and 5 in the part 8, these were referred back to the task 13–8 = 5. The result shows that it might not suffice to co-plan lessons, have a script and a model lesson. Teachers might need to have a lived theory to meet student responses and direct their attention to relational and static aspects of subtraction instead of seeing subtraction as dynamic operations.

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SENSE MAKING AND REASONING IN ALGEBRA

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Researchers note that algebraic reasoning and sense making is essential for building conceptual knowledge in school mathematics. Consequently, pre-service teachers' own reasoning and sense making are useful in fostering and developing students' algebraic reasoning and sense making. We report here the features of pre-service teachers' reasoning and sense making in algebra, specifically in the process of analysing problem posing, with a focus on first-degree equations. The following research questions served as a guide in the analysis of data: What are the characteristics of the problem-posing tasks used for reasoning and sense making of first-degree equations? What are the characteristics of pre-service teachers' reasoning and sense making in problem-posing tasks?

This study is part of ongoing research carried out with pre-service teachers enrolled in the abovementioned mathematics teacher education course. All the pre-service teachers were older than 19, from diverse socio-economic backgrounds and attended class for all 10 weeks of the semester, including the seven weeks in which data was collected. Sixty-six pre-service primary teachers participated in an anonymous written exam and were informed about our research (characteristics, aim, confidentiality issues, etc.). The data analysis adopted a qualitative/interpretative approach, and the unit of analysis has three dimensions: reasoning, sense making and critical aspects. According to Olteanu (2020), reasoning and sense making are closely related to each other and to these dimensions in the manner of a 'rhizome' (Deleuze & Guattari, 1987).

Results revealed that the pre-service teachers create a rhizomatic reasoning and sense making that is characterized by lines of rupture. Those lines interconnect and arise from incorrect translations of rhizomatic problem-posing task (RPPT) into mathematical notations and the failure to discern the difference between variables and variables as unknown numbers; i.e., between algebraic expression and equation. The characteristics of reasoning in RPPT and of pre-service teachers are selecting, exploring, reconfiguring, encoding, abstracting, and connecting to highlight associations and relationships between different content, and the characteristics of sense making are recognition, relationships, profiling, comparing, laddering, and verifying.

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AN INVESTIGATION OF SOCIO-MATHEMATICAL NORMS BARRING THE TRANSITION FROM HIGH SCHOOL MATHEMATICS TO UNIVERSITY MATHEMATICS

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This study is a research carried out within the scope of project number 220K339 supported by The Scientific and Technological Research Council of Turkey. The aim of the study is to determine the socio-mathematical norms that prevent the transition from high school mathematics to university mathematics. The language used in the learning process, the discourses that take place in the classroom, some rules that are agreed upon during the acquisition of knowledge and social interaction constitute the social norms specific to that class (Lopez & Allal, 2007). Analyzing and revealing social norms is very important in terms of explaining the details of individuals' learning process. In the related literature, it is suggested that there are sociological, cultural and didactic changes as well as epistemological and cognitive difficulties in the transition process from high school mathematics to university mathematics (Yackel ve Cobb, 1996). This situation creates a need for research on the learning of the student community, which is in the process of transition from high school mathematics to university mathematics, in terms of factors that dominate classroom culture as well as cognitive aspects.

The participants of the research are secondary school mathematics teacher candidates who have just started university. Observations were made in the Calculus-I and Foundations of Mathematics courses taken by these teacher candidates during the 2021 fall semester. Interviews were held with both the instructors who taught these courses and the teacher candidates. The analysis of the data obtained from the observations and interviews was interpreted through the method of continuous comparative analysis. The norms determined as socio-mathematical norms that prevent the transition to university mathematics are grouped under main themes such as *definition*, *mathematical language*, *making connection, reasoning and proof.* The socio-mathematical norms under these themes, which have the potential to prevent the transition to university mathematics, will be explained with examples.

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PROSPECTIVE TEACHERS' SELF-ASSESSED CONFIDENCE IN THEIR OWN MATHEMATICAL KNOWLEDGE AND ON THEIR ABILITY TO EXPLAIN THIS KNOWLEDGE

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To teach mathematics, it is not only important to have mathematical knowledge but also to know the limitations of one's knowledge. That is to say, to be aware of things that one does not know. A possible manner to address this issue is to measure students' degree of confidence in their mathematical knowledge, as done by researchers such as Leclerq (2014).

In the context of an institutional diagnostic assessment on mathematical knowledge (Martínez et al., 2019) applied to prospective teacher students at the very beginning of their study plan, we included a set of questions asking students to indicate, on a scale of 0% to 100%: (a) how confident they were that their answer to a mathematical knowledge question was correct, and (b) how confident they were that they could explain to someone else why that answer was correct.

In this presentation, we will show preliminary results based on data from two teacher training programs of a Chilean university, Primary Education (PE, n=22) and Secondary Mathematics Education (SME, n=18). The results showed that SME students have a higher degree of both knowledge and confidence than those of PE students. However, in SME students this level of confidence remains very high even in the case of test items answered incorrectly, which suggests a lower level of awareness regarding the limitations of their own knowledge. We will discuss these and other elements that arise when considering confidence levels in diagnostic data analysis, including the value and importance of obtaining this information in evaluations in the context of initial teacher training.

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COMMUNICATIVE PROJECTS ABOUT MATHEMATICAL REASONING IN A NEW TEACHER'S CLASSROOM

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Mathematical reasoning (MR) is a concept in education research that comes in various shapes and is used with different purposes. In steering documents for education, MR is usually linked with students' mathematical competencies such as using mathematical concepts in different situations and engaging in mathematical conversations. However, creating and maintaining a discourse that promotes the development of such competencies, seems to be challenging for teachers (Højgaard & Sølberg, 2019). The research project presented here aims to further understand what communication about MR looks like in a new mathematics teacher's classroom. The dialogical approach to research on interactions foreground communication as relational (Linell, 2009). Then, to identify and analyse communicative projects, becomes a way to understand a locally produced discourse. A communicative project is understood as an organisation of communicative acts around a shared task. The research question is: What kind of communicative projects are established in the classroom when a new teacher and students are communicating MR?

The empirical material contains video- and audio-recorded observations from ten consecutive mathematics lessons with 16-year-old students, combined with a teacher interview after the observation period. The teacher has less than two years of teaching experience since graduating from teacher education. Jeannotte and Kieran's (2017) model of MR is used for identifying communicative projects around MR. The initial analysis is focusing on students' and teacher's communicative acts such as initiatives, responses, instructions, and questions that form tasks. The use of space and time and the functions of acts and activities are expected as important in determining sizes and organisation of communicative projects. Further, stories from the interview about planned versus enacted activities may influence the interpretation of observations. A more detailed description of the analysis will be presented at the conference.

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TEACHER TRAINING IN MATHEMATICS TEACHING USING DIGITAL TECHNOLOGY

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The literature has identified the importance of using technology for a more effective approach to teaching mathematics (e.g., Cai & Howson, 2013), which requires the implementation of Professional Development Programs (PDP) for in-service teachers.

We research the impact and sustainability of PDP (Zehetmeier & Krainer, 2011) on pedagogical practices with TPACK (Koehler et al., 2013) as a reference model. Methodology includes Parallel mixed methods design (Creswell, 2011), with qualitative and quantitative data analysis. The participants are 250 teachers who attended PDP, in eight courses, from 25 to 50 hours, in Mathematics Teaching using Milage Learn+ (www.milage.io) or GeoGebra (www.geogebra.org), in which the proposed activities were done in an educational context, so that they could be immediately applied in teaching practices, still during teacher training. Data was collected from the reports made by the teachers and from questionnaires applied to them at two moments: at the end of the PDP and one semester after it. Based on the data analysis, at the end of the PDP teachers showed high levels of satisfaction with the effectiveness of the programs. However, six months after the PDP, while the technology is used regularly by most of teachers, many still do not propose tasks for students to use it, namely specific practices developed in the PDP that mobilize the full potential of technology, which compromises its sustainability. Thus, there is a need for further research on how to promote the sustainability of PDP of this nature.

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EMERGENCE OF AN ANALYTICAL-STRUCTURAL WAY OF STUDENTS' THINKING IN LINEAR ALGEBRA

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This report concerns a part of my ongoing PhD project which investigates the difficulties encountered by students when learning fundamental linear algebra concepts. Sierpinska (2000) ascribes one reason for such difficulties to the fact that its understanding needs the concurrent use of three thinking modes: the *analytic-arithmetic (AA),* the *synthetic-geometric (SG)* and the *analytic-structural (AS)*. The literature about linear algebra instruction has shown that many students tend to use exclusively the first one and that this can disadvantage the development of conceptual knowledge. More recent researches have shown how the *SG* way of thinking can easily emerge, when the learning process is supported by the use of digital dynamic representations of concepts in geometric vector spaces. Conversely, there is less specific research about cases in which students show to be able to reason more structurally. What is clear is that the *AS* way of thinking is much more difficult for first-year university students to achieve, making its scaffolding one of the hardest challenges in linear algebra education. Nevertheless, one possible cause of the scarce evidence of such mode of thinking could be also the absence of recognizable representations that can be associated with it. In fact, instances of the *AA* and *SG* ways of thinking can be recognized in the use of algebraic inscriptions and geometric representations. Conversely, the third mode is not so easily detectable and our assumption is that, because of its structural nature, it could be analyzed studying the dynamic evolution of the different semiotic resources activated by students.

Hence, in my study a multimodal semiotic analysis of video-recorded interviews is conducted in order to detect instances of such an evolution of thought regarding the eigenvector and eigenvalue concepts. Semi-structured interviews were conducted with engineering students who had completed the linear algebra course approximately one year before. The aim of the interviews is to observe what representations students more naturally use for the purpose of reconceptualizing concepts that they had encountered and learned long time before and that perhaps might have forgotten by the time of the interview. In the presentation, one of these interviews will be introduced and analyzed. It will be showed how the dynamics of the representations and gestures produced by the interviewed student appear as indicators of the emergence of the *AS* way of thinking in her process of eigenvector's and eigenvalue's redefinition.

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HABERMAS' CONSTRUCT OF RATIONALITY TO BRING OUT MATHEMATICS AND PHYSICS DISCIPLINARY IDENTITIES

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The study is conceived within the European Erasmus+ project IDENTITIES, whose goal is to promote interdisciplinarity in prospective teachers' education. As a preliminary step in order to design teacher education activities, we investigate the disciplinary key aspects of mathematics and physics reasoning in disciplinary instructional materials (namely, physics textbooks) as well as interdisciplinary issues. As key aspects, we refer to the epistemic core of disciplines: aims and values, methods and methodological rules, practices, and scientific knowledge (Erduran & Dagher, 2014). To reflect on implicit reasoning structures, implicit goals, or implicit communicative strategies, we investigate the three dimensions of Habermas' rational behaviour: epistemic, teleological, and communicative (Morselli & Boero, 2009).

In this contribution we focus on the chapter about parabolic motion from an Italian high school physics textbook. The analysis shows a decreasing weight of the three dimensions of rationality: the communicative dimension (choices inherent to text presentation, for example use of bullet lists, highlighted or boxed words or sentences, repetition and use of terms) emerges more, followed by the epistemic one (explication of used hypotheses, laws and results), while the teleological one (goals, strategies, and decisions) remains more implicit. We also identify elements of the epistemic core of disciplines: physics aims, like modelling phenomena, and mathematical methods, like algebraic substitutions. Interdisciplinary issues emerge in the comparison between communicative strategies to introduce a concept (use of different examples vs definitions) and in the different structures of arguments (principles conciliating expected and deducted results vs deductive and algebraic reasoning from premises to general end). Further results and details will be provided in the oral communication. We are currently carrying out the second step of the study, i.e., involving prospective teachers in this kind of analysis of physics and mathematics textbooks.

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UNDERSTANDING PRACTICES FOSTERING ALGEBRAIC THINKING IN ELEMENTARY SCHOOL CHILDREN

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All over the world, researchers believe that algebraic thinking is a critical competence every citizen should develop and that the early development of algebraic thinking may contribute to the students' success in secondary mathematics. Researchers propose a great variety of tasks potentially contributing to the development of algebraic thinking in young students. Still, the algebraic thinking within the elementary school practices is a subject of theoretical discussions.

In our knowledge synthesis project, we analyzed more than 75 articles from scientific peer-reviewed journals to dress a picture of how researchers understand algebraic thinking and how they propose to develop it in elementary students. We collected task descriptions presented by authors as conductive to algebraic thinking. To understand better the features of tasks and their implementation contributing to the fostering algebraic thinking in students, we applied a theoretical framework combining the idea of semiotic representations (Duval, 2006) with the relational paradigm (Davydov, 2008). At early stages, the object of algebraic (theoretical) thinking is a quantitative relationship behind a situation or a task. According to Duval, the understanding of an object of study may accrue when the person transforms this understanding, expressed in one semiotic system, into another (different) semiotic system. Thus, to analyze our data we looked for the use of different semiotic tools to represent mathematical relationships at play within a task. Our analysis revealed that many tasks proposed as conductive to algebraic thinking present some specific characteristics. (a) The task starts with a problem formulated within a specific semiotic system and it requests students to find a solution within the same semiotic system. (b) During the task students need to analyze the initially given situation as a system of relationships and express (model) those relationships within a different semiotic system (or many systems). (c) Students should deduce some new information from the model and transfer the solution into the initial semiotic system. Our analysis suggests that the relational analysis, modeling and transformations from one semiotic system to another are key features of a task fostering the development of algebraic thinking in young children.

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PRIMARY TEACHERS' KNOWLEDGE DEVELOPMENT IN A LESSON STUDY

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Teachers' professional development concerns cognitive, affective, social, cultural, and organizational elements, in close relationship with the professional culture and institutional conditions. In this study we address teachers' mathematics knowledge for teaching (MKT) (Ball et al., 2008), especially in the domain of pedagogical content knowledge and aim to know how primary teachers develop it during a lesson study. To understand this process, we use the Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002). This model regards several domains which interact with enacting and reflecting processes, assuming that teachers' growth depends on the teachers' reflection on the domain of consequences, involving salient outcomes from activities carried out in the domain of practice.

The methodology is qualitative and interpretative. Data regards a lesson study with three primary school teachers in which the main facilitator was the first author. All the 12 lesson study sessions were audio recorded and individual interviews were made to the teachers. Data was analyzed through content analysis seeking to identify in the teachers' discourse illustrative aspects of the MKT mobilized or developed.

The results indicate that teachers developed their knowledge of curriculum, their knowledge of teaching, especially about tasks, about students' knowledge and their difficulties and, surprisingly, their specialized knowledge of mathematics about the measure meaning of fraction and its number line representation. This development arose from activities carried in the Domain of Practice, in many cases prompted by the External Domain, and were particularly important when these activities gave rise to unforeseen Consequences. The results also show how connections among these domains were established by enactment and reflection processes.

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IDENTIFYING MATHEMATICAL COMPETENCY DEMAND IN NUMERACY ITEMS FOR GRADE 5 AND GRADE 8

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Being numerate means being able to understand the world around us through a mathematical lens, evaluate mathematical information embedded in a context, and use mathematics as a tool to solve problems in the real world (Geiger et al., 2015). A broadly used framework for describing skills necessary to be numerate is the "mathematical competencies" framework developed by Niss and Højgaard (2019).

Numeracy was integrated into curriculums in Norway in the 2006 curriculum reform. To help monitor the numeracy proficiency of Norwegian students, a national testing system was implemented as well. The study presented aims to describe the extent the national testing system measures the mathematical competencies outlined by Niss and Højgaard. The presentation will focus on the mathematical competency demand of items used in the Norwegian National Numeracy Tests (NNNT) in grade 5 and grade 8.

The items analyzed in this study were the items used in the 2018 NNNT. The sample of items consists of 95 items, of which 45 were developed and administered to grade 5 and 50 items to grade 8. The competency demand of the items was classified according to a mathematical competency classification scheme developed by Turner, Blum, and Niss (2015) using three raters, two with prior experience using the scheme.

The present study will describe numeracy as measured by NNNT, showing the competency demand of grade 5 and grade 8 items respectively, as well as showing how the competency demand increases from grade 5 to grade 8. Furthermore, the methods applied in this study may strengthen comparisons of constructs across assessment systems. Additionally, the study will inform test development for numeracy in lower grades as well as provide evidence of the suitability of the classification scheme outside the context of PISA items.

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USING STUDENT-GENERATED EXAMPLES TO UNDERSTAND PRE-SERVICE TEACHERS' PREPAREDNESS TO TEACH MATHEMATICS

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Research has shown that when students are consistently required to generate their own mathematics examples as an integral part of their learning experience, they have a greater chance of undergoing cognitive shifts and developing broader example spaces than students who rely on external authority (e.g. teachers, textbooks) for examples (Essien, 2021; Watson & Mason, 2005). While there has been work done in the use of examples in mathematics teaching and learning, there is a dearth of research into how student-generated examples (SGE) can be used to promote teaching and learning in pre-service teacher education mathematics classrooms.

The present study reports on research that was conducted over a period of 4 years using SGE as a strategy to teach three mathematics topics (Sequence and Series; Statistics and Probability) to four cohorts of the Mathematics Methodology classes of 2015, 2016, 2017 and 2018. Pre-service teachers (PST) were given constraints within each of the above topics and asked to generate their own examples while being video recorded. The data was then transcribed and analysed with a focus on the nature and quality of examples generated by students and to examine PSTs' preparedness for teaching these topics using the Topic-Specific Teaching Knowledge Assessment Framework (T-STAF) (Ramabulana & Sedumedi, 2017). Preliminary findings from data analysis revealed that while the 2015, 2016 and 2017 cohorts of pre-service teachers constructed questions that were largely at the declarative knowledge level, the 2018 cohort, on the other hand, formulated high order questions, aligned to the curriculum, that did not only demonstrate sufficient declarative knowledge, but also deep conceptual knowledge necessary for teaching the three topics. T-STAF was used to interpret these findings.

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PRE-SERVICE TEACHERS' CREATIONS OF MATHEMATICS LEARNING VIDEOS

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There is an increasing number of students who look for or create videos with mathematical content and these cases illustrate how the classroom is transforming with the integration of the Internet (Oechsler & Borba, 2020). One of the aims of the project QLeV-Math (Quality Criteria for Learning Videos in Mathematics) is to design a catalogue for video creators who develop learning videos for mathematics (Ratnayake et al. 2020). The current version of the catalogue was tried out with groups of pre-service teachers in the universities of the authors.

We discuss the use of such a catalogue made by two groups of pre-service teachers (video-makers) as a fulfilment of an assessment of an undergraduate mathematics course at the University of Catania. They were free to choose the mathematical topic, a target audience from middle school to high school in the Italian education context and a software to edit the video. The constraints were: the length of the video should not exceed five minutes; the video had to include a real-world context (an application of the mathematics content) and the use of tools – either digital or manipulatives. The teacher of the course (one of the authors) put a great emphasis on the accuracy of the content, the used terminology and the picture and sound quality of the video. So, she introduced the creator's catalogue developed by QLeV-Math project to the video-makers before they develop the videos. Then the video-makers were guided to write a complete script for the video including all narrations and timing. The teacher checked the scripts and provided some feedback to the video-makers, sharing these with the whole class. The video-makers were then free to create a video either individually or in small groups of two or three. Finally, the video was accompanied by a teaching sheet in which the video-makers had to describe the structure of the video and explain the links between the mathematical content they have chosen and the Italian curriculum. The final product was evaluated using the same criteria in the catalogue that was discussed with the students at the beginning of the task. Two of the videos produced will be compared in our presentation.

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CHANGES IN ALMA'S ATTENTION TO CRITICAL EVENTS

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Teacher noticing – comprising of attending to, interpreting, and responding to students' mathematical thinking (Jacobs et al., 2010) – is one of the prominent frameworks used in pre-service mathematics teacher (PSTs) training programs. In many studies, the development of noticing skills is measured during the training period. To better understand how teachers learn to notice, this study explores changes in teachers' noticing skills from their training period to their time as novice teachers. Here, we focus on one teacher, Alma, and on the attention component, on which interpreting and responding rely. Alma participated in a large teacher training research program that used critical events to teach *teacher noticing*. We define *critical events* as moments in which the students thinking becomes apparent and can serve as an opportunity for the teacher to delve into the mathematics. We aim here to *characterize Alma's changes in attending to critical events by comparing events she identified during her training and those brought by her as a novice teacher.*

When collecting the data, we asked Alma to identify critical events from classroom observations during the program (2016-2017; 4 events) and in her first year of teaching (2017-2018; 3 events). Alma submitted the events' descriptions in reports according to a structured framework. Data analysis incorporated a three-axis model to characterize the events through different aspects (Rotem & Ayalon, under revision): participants (students, teacher, teacher and students), content (mathematics, pedagogy, both), and dimensions in learning and teaching (cognitive, affective and social). Whereas during her training period, Alma's critical events varied and characterized by multiple aspects, during her first year of teaching, Alma's events focused on specific characteristics, meaning students' mathematics, her teaching strategies, and social aspects. Her concentration on particular type of critical events provides us with insights into a teacher's process of becoming a teacher in real classrooms. In the presentation, we will elaborate more about how we gained these results using the model and discuss further implications.

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INVESTIGATING ARGUMENTATIVE PROCESS IN SOLVING PROBABILITY PROBLEMS

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The teaching and learning of argumentation play a crucial role among the learning objectives about mathematics defined by the Italian National Guidelines for primary and secondary school. We are hence interested in investigating the possibility of assessing the argumentation in math classrooms. Our explorative study focuses on argumentative processes developed during problem solving in probability context, which, in the experience of the research group, has been a rich context for developing argumentative activities. We follow the perspective of formative assessment (Black & Williams, 2007). This choice is led by the double aim of assessing and promoting the development of argumentative process and by the intention of taking into consideration the essential role of teacher, students, and peers. As pointed out by Stylianides (2007), the statements used in a mathematical argument should be accepted by, known to, or within the conceptual reach of the classroom community as well as recognized in today's standard mathematical culture. We consider argumentation from a social and interactionist perspective (Mercier & Sperber, 2017), according to which a good argumentation should display coherence relationships between the claim of the speaker and the beliefs held by the addressees.

Our study is in progress. The teaching experiment will involve a 11-grade class of 19 students, and it will take place during regular math class activities. The core of the activities will be group resolutions of some selected problems and the following whole-class discussion. The lessons will be video recorded, and the students' written productions will be collected. Written argumentations and interactive argumentation processes will be the focus of our qualitative analysis. In line with the formative assessment strategies, particular attention will be paid on feedback and students' reactions to them. We expect to find some evidence about argumentative processes aspects that can be emphasized and fostered during formative assessment activities. We will present the first results of our analysis by means of some examples.

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COURSE CONCEPTS BASED ON EDUCATIONAL VIGNETTES: THE CASE OF OPEN APPROACH AND CONCEPT CARTOONS

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Recently, various types of vignettes have been investigated as developmental or diagnostic tools in mathematics teacher education (Skilling & Stylianides, 2020). The vignettes include Concept Cartoons (Samková, 2020), individual pictures capturing a group of several children in a bubble dialogue. At our university, we created a six-week course based on Concept Cartoons, and implemented it into professional preparation of prospective primary school teachers. The aim of the course was to elicit discussion of prospective teachers about solving and assessing tasks that are open, i.e. tasks that have multiple interpretations of the assignment, multiple correct ways of solving, multiple correct results, or multiple interpretations of the results (Nohda, 2000). The course is based on ten pictures related to various primary school mathematics topics; each of them represents children solving an open task while some of the children propose correct solutions, others incorrect or unclear.

Twelve course participants got the pictures, with a set of indicative questions (Which children are right? Which are wrong? Why?) to answer in written form. The leader of the course analysed the answers, and then orchestrated subsequent discussions. The participants reflected the course in written form. Most of them valued the opportunity to observe and discuss different solutions, and also reported newly gained awareness of the variety of aspects that need to be considered when assessing pupils' performance. Within the coReflect@maths project, we prepare to share similar course concepts with other teacher educators.

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DRAWING INFERENCE FROM DATA WHEN COMPARING GROUPS: AN EYE-TRACKING STUDY

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When comparing groups and making an inference, several processes of information processing are involved, such as perception and attention as well as the interpretation of the data presented in the situation. These processes build the foundation for the construction of mental models of statistical situations (Eichler & Vogel, 2012). Preliminary findings from our still ongoing systematic literature review on data literacy in school education indicate that research to date has focused primarily on students' interpretation processes. These processes are typically analyzed based on students' given rationales after the statistical inference was drawn. It was found that students often struggle to consider variability in data and frequently fail to integrate local and global views of data (e.g., Ben-Zvi, 2004). While many studies focus on students' interpretative processes, little is known about the underlying perceptual and attentional processes as well students' internal processes of modeling statistical situations.

Using Eye-Tracking technology, this study aims to get insights into students' visual attention when comparing data distributions. A methodological triangulation with Eye-Tracking stimulated recall interviews provides further insight into internal processes related to mental modeling of statistical situations. In an Eye-Tracking stimulated recall interview, students are requested to describe their original thoughts as precisely as possible with the aid of gaze-overlaid videos that were recorded while they were working on the task. Data collection will take place in April 2022 and the sample will consist of primary and secondary school students. During data analysis students' performance levels of mental modeling will be differentiated based on an adapted version of Biggs and Collis' (1982) SOLO model. These levels differ by the number of features considered in the presented data (isolated or interrelated) and by the statistical nature of these features (center, spread, shape). A further research interest aims at an exploratory investigation of potential relationships between students' visual attention and their performance level of mental modeling.

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FOSTERING DATA LITERACY IN STEM SCHOOL EDUCATION: A SYSTEMATIC REVIEW

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Data and statistical literacy (DSL) are counted among the key competencies of the 21st century and are often set out as an expected outcome of school education (Gal, 2002). Accordingly, curricular adaptions of science, technology, engineering, and mathematics (STEM) education have been made in many countries with the aim of developing competencies that promote competent handling of data, statistics, and their forms of representation. Existing reviews identified promising methods for promoting DSL in higher education (Schüller & Busch, 2019). What has not been explicitly addressed so far is the question of what role school-based STEM education plays in building important foundational skills for DSL.

The main objectives of this systematic review are to compile, systematize, and interpret relevant findings from international research on the role of STEM subjects in fostering DSL. The research questions relate to definitions and conceptualizations of DSL, approaches to promoting it within STEM school education, and characteristics of students and teachers related to the development of student DSL. Following current PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analysis) guidelines, a broad systematic search of several literature databases was conducted that resulted in 16.865 records. The following title-abstract screening process was performed independently by two researchers with the support of the machine learning-based active learning software *ASReview*. A set of 598 full-text articles is currently being independently assessed for eligibility by two researchers. In each eligible full-text, findings relevant to the research questions are coded, extracted and synthesized using qualitative content analysis. Initial results indicate that the studies predominantly address the subject of mathematics and focus on student characteristics. Recent studies increasingly focus on the use of data and statistical concepts. Several effective approaches for fostering DSL can be identified.

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TEACHERS IDENTIFY STUDENTS' CHALLENGES WHILE ENGAGING IN MATHEMATICAL MODELING TASKS

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Applying mathematical modeling (MM) is a challenging process for both students and teachers due to its high cognitive demands. Effective modeling-based instruction adds an additional challenge for teachers who should acquire explicit MM teaching competencies. Among them is the ability to diagnose students' challenges while applying MM. The goal of this study is to explore whether and how do teachers diagnose their students' challenges while engaging in MM tasks.

The study was conducted in a designated professional development program, designed to prepare math teachers for modeling-based instruction, based on the model for competencies needed in teaching MM developed by Borromeo Ferri and Blum (2010), which presents four consecutive dimensions of MM competencies: *Theoretical*, *task*, *instruction,* and *diagnostic*. Participants were 24 low-secondary school math teachers who participated in the program. Six MM tasks were introduced and further implemented by the teachers in their classes. Research tools include teachers' reports regarding their ability to identify students' challenges while engaging in a MM task, and an observation on implemented lesson of *Image Resolution* task performed by Ronit (Pseudo), one of the participants, with a focus on her ability to anticipate students' challenges.

Findings indicate that teachers reported that 84.6% of students encountered challenges tied to the MM process. The most common challenge was understanding the real-world situation (45.5%), which was also reinforced in Ronit's lesson. Ronit explained how pixels are correlated with *image resolution* and anticipated that her students would have difficulty understanding the term pixels, by explaining this term several times and making sure it was understood before continuing with the lesson. This challenge was followed by various challenges, that correspond to modeling competencies, particularly applying mathematical routines (27.3%), mathematising (18.2%) and interpretation (9.1%). Theoretical contribution of this study lies in the integration of the model for competencies needed for MM instruction and the theoretical framework of MM. Methodological and practical contributions are also discussed.

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COMPARING THE INFLUENCE OF TASK CONTEXT ON THE PERFORMANCE AND MOTIVATION OF TALENTED AND ORDINARY STUDENTS

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Motivation, seen as the preference for doing or not an activity, is a key element to be considered when designing mathematical tasks (Rellesman & Schukajlow, 2018). In the case of mathematically talented students, challenging tasks encourage them to develop deep mathematical reasoning (Benedicto et al., 2018). Modelling tasks are complex and formulated in a real context. These tasks are not common in activities aimed at talented students (e.g., IMO problems). We present part of a research that aims to study the influence of the task context on student motivation and performance, comparing mathematically talented students with a group of standard students.

We adapted Rellesman and Schukajlow (2018), with tasks with common mathematical content and diverse levels of connection to reality: intramathematical, verbal and modelling problems. We have collected the productions of 22 students participating in mathematical talent programme EstalmatCV (13,8 years old) and 29 productions of ordinary students (13,6 years old). This is a mixed study in which, first, the performance of the participants is studied from a qualitative analysis of the resolutions and, second, data is collected on affective factors related to motivation: enjoyment, boredom, interest and value. Results confirm a significantly better performance by the talented students on all tasks. For these students, modelling problems are the most difficult, while for ordinary students intramathematical problems are equally difficult. Motivation is significantly higher in the talented group than in the ordinary group only for modelling and intramathematical problems. Differences between groups are not significant in the verbal problems. We conclude that the ordinary students feel more motivated towards verbal problems (the most accessible), whereas those talented show more motivation for modelling problems, which are the most challenging ones.

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ALIGNING CLASSROOM DISCUSSIONS WITH COMPETENCY GOALS AND PROBLEM-SOLVING ACTIVITIES

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A mathematically competent person has the knowledge, strategies, and skills necessary to handle mathematical situations encountered in everyday life and work (Niss & Højgaard, 2019). Developing students' mathematical competence is a central aim in many national curricula documents, such as those in the Nordic countries. Previous research indicates that students benefit from explicit teaching practices, such as setting clear goals and addressing key aspects of activities. However, research also indicates that even when teaching and learning activities are competency-oriented, mathematical competence is rarely the explicit focus of lessons. As such, this study aims at investigating the alignment between explicit competency goals and competency-oriented activities to explore the opportunities for teachers to facilitate students' development of mathematical competence.

Video material from a Swedish lower secondary mathematics classroom was analyzed to investigate the alignment between explicit learning goals and competency-oriented activities. Goal statements and activities were analyzed using the mathematical competency research framework (MCRF; Lithner et al., 2010), which comprises six competencies: problem solving, reasoning, applying procedures, representation, connection, and communication. The lesson centered on problem solving, comprising two problem-solving tasks and whole-class discussions. Analysis revealed that the explicit learning goals stated by the teacher targeted problem-solving and communication competencies. Furthermore, the problem-solving activities provided opportunities for students to engage with all six competencies described in the MCRF framework. However, during the whole-class discussion, the teacher focused on the outcome of the problem solving instead of discussing the communication and problem-solving processes the students had engaged in. Seemingly, the focus of the discussions, as steered by the teacher, did not align with the competence orientation of the learning goals and activities, thus clouding the purpose of the lesson rather than highlighting opportunities for competence development.

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STUDENT-TEACHERS' PERSPECTIVES AND PERFORMANCE ON TEACHING MATHEMATICS WITH PROBLEM POSING

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This study investigated student teachers' perspectives on teaching mathematics with problem posing and how they design and reflect on lesson plan that use lesson study team and problem-solving approach. The purposed group are 57 second year student-teachers in mathematics teacher education program, Lampang Rajabhat University. They are taking course on mathematics learning management in elementary level. They designed lesson plans by using lesson study and open approach (Inprasitha, 2015). After teaching experiments, they were asked to reflect on how to develop their lesson plan. The data were collected from the teaching mathematics with problem posing questionnaires, lesson plans, and reflecting notes based on Sawada (1997), Inprasitha (2015), and Cai and Hwang (2020).

Our findings demonstrate the student teachers showed evidence of positive results on their perspectives. More than 80% of the student teachers' perspectives on how to develop the problem encourage students to think from difference ideas; appropriate for the students; and lead students to mathematical thinking. And more than 75% of the student teachers' perspectives on how to develop the lesson plan for posing the problem. These results were supplemented with their lesson plan and reflections. Although the main constrain is time and teaching experiment factors, it was great challenge for the student-teachers to learn how to design, performance and develop the lesson plan on teaching the problem before posing problem in real classroom.

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INVESTIGATING MATHEMATICAL HABIITS OF MINF IN A FLIPPED CLASSROOM ENVIRONMENT

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A “Flipped Classroom” (FC) approach is based on “flipping” the traditional learning approach. The knowledge acquisition process which traditionally occurs in class, takes place outside of it via a variety of technological means (Lage et al., 2000). Owing to the significant amount of class time dedicated for exercising and problem solving, students can have more time mastering a huge variety of mathematical abilities (Bishop, 2013). Mathematical Habits of Mind (MHoM) are considered fundamental mathematical abilities that students need to acquire and improve. MHoM-based instruction enables students to engage in mathematical thinking processes that are similar to those used by mathematicians (Cuoco et al., 1996). Most research has looked into the application of MHoM in mathematics classrooms, yet scant research has been done on their application as part of an FC environment, particularly referring to the online component of the FC. The goal of the current study is to investigate the integration of MHoM in the online component of an FC environment.

The study focuses on an FC environment, which its online component is comprised of an advanced mathematics course for high school students and contains filmed lectures to numerous subjects. This study focuses on the MHoM of the instructor who delivers the content in the filmed lectures of the online environment, in addition to the those invited by the online exercises. Both quantitative and qualitative research tools are administered, including short filmed mathematics lectures, as well as interactive online exercises. The findings indicate that the instructional process in the examined lectures exemplify the characteristics of an MHoM-based instruction. In several filmed lectures for instance, it is clearly visible that the instructor is mainly focusing on implementing an MHoM that is applying mathematical reasoning. This research contributes theoretically to the limited literature about the interrelationship between MHoM and FC. The study also contributes to providing insights and probable solutions to challenging learning situations such as the current period of COVID-19 pandemic.

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DISRUPTING MATHEMATICS: SHIFTING PERSPECTIVES IN A SCHOOL COMMUNITY

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Responding to a call for the aesthetic and sensual in mathematics (de Freitas & Sinclair, 2013) and in-the-moment awareness of our virtual nature (Varela, 1999), my research includes facilitating experiential workshops for teachers. Recognising mathematics as imaginative, creative, inherently connective, and culturally and historically embedded, I use tasks which promote engagement, enjoyment, and deep understanding. What can mathematics be for the teachers, and subsequently for their students? How are issues of social justice enacted when implementing a more connected, less fragmented approach?

Working with seven non-specialist mathematics teachers, the study is longitudinal (eighteen months to date) and emergent, with a participatory element. Clips from the videoed workshops provided the stimuli for the teachers to reflect on their lived experiences. In response to teachers' emerging needs we have moved into the current participatory phase of "telling the story of mathematics in the school", for the benefit of teachers and the whole community. A teacher's request to experience how this approach to mathematics might manifest for his students has developed into more focused work in the classroom. On-going analysis, in line with my approach, is an emerging process imbricated in the unfolding of the project, supported by my meditative and contemplative practices (as suggested by Varela, 1999).

Requests by teachers for support to work with their students, in the way they experienced in the workshops, seems to indicate a transforming of relationships to mathematics in these teachers. One teacher commented on the mathematical thinking he is now witnessing in his students, of which he had not previously been aware; as well as his interest in the possibilities afforded for social interaction and development. Furthermore, as I work collaboratively with this teacher and move from leading the lessons, to co-teaching with him, to him leading the lessons, there are signs of his developing relationship to mathematics, and to how he is interacting with his students. By virtue of the emergent and participatory nature of the project, the presentation will include further research developments and reflections.

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TOWARD THE HISTORY OF MATHEMATICS EDUCATION FOR MARGINALIZED GROUPS

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The conversation about improvement of teaching processes to support diversity and inclusion in mathematics education continuously spark conversations and become topics of numerous publications. One thing in those debates is quite persistent — the underrepresentation and marginalization of students in mathematics education is centred around race and gender. Each generation might believe that the issues and complaints they have with their contemporary education are unprecedented (Ravitch, 2000). However, the relationship between past events and contemporary problems may be quite strong and intricate.

A historical review shows that the national debate over the purpose of education for black students unfolded at the turn of the 20th century when African Americans' efforts to establish a system of public education was influenced by their post-slavery experiences. Inclined by the efforts of freed citizens to create an education system that would end their academic, economic, and political oppression, high schools for Black students of the United States continued to offer college preparatory education until 1920; then, after 1920, secondary public education for Blacks started to shift to provide "industrial secondary education designed to train black children as a docile, industrial caste of unskilled and semiskilled urban workers" (Anderson, 1988, p. 199). In urban communities, the education of African American children revealed the large gap between beliefs and practice: "While publicists glorified the unifying influence of common learning under the common roof of the common schools, black Americans were rarely part of the design" (Tyack, 1974, p. 110).

As education professionals still debate on the inclusion of certain subjects or different requirements, the goal of this study is to outline and discuss historical events that might have played a role in the development of mathematics curricula for the historically marginalized groups.

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WHAT DO PRE-SERVICE TEACHERS CONSIDER IMPORTANT WHEN INTRODUCING THE CONCEPT OF FUNCTION?

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The contribution presents research focused on preservice mathematics teachers (PSTs) and their approach to introducing the concept of function in secondary schools. The analysis is based on the model of Mathematical Teachers' Specialized Knowledge – MTSK (Carrillo et al., 2018) with the focus on the subdomain Knowledge of Mathematics Teaching (KMT). We targeted on PSTs' awareness of the "didactical potential" of tasks, activities, strategies, and techniques for teaching the function concept and its four main aspects as described in Pittalis et al. (2020): function as an input-output assignment, function as a dynamic process of covariation, function as a correspondence relation, function as a mathematical object, and on their awareness of the "didactical potential" of different ways of representing a function and the need to intertwine these representations with each other. The research question is as follows: What do PSTs find important when considering suitable tasks for the first introduction of the function concept (aspects of the concept, its different representations)?

The research was conducted with 13 PSTs in their last year of study at our university. We divided them randomly into four groups. PSTs were given 29 tasks in order to cover different aspects of the function concept and its representations. 15 of them were chosen from Slovak textbooks, 10 were chosen and translated from journal articles and 4 were chosen from popular web pages intended for math teachers from the Czech Republic and Slovakia. During two 3-hour sessions, PSTs in each group solved collaboratively tasks and afterward picked and ordered several tasks which they found appropriate for their first lesson about functions in secondary school. Data were collected through lesson video-recording, a recorded semi-structured group interview, and written notes of each group. Analysis of the PSTs' tasks selection showed that for their lesson they preferred tasks in which the covariational aspect of the function concept dominates. As for diverse representations and connections between them, they paid attention to the transition from graph to word context and lacked attention to the transition from a graph to the table and from a table to the formula.

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ON-THE-FLY EVALUATION OF DIAGNOSTIC PROCESSES – EXPLORING THE POSSIBILITIES OF MACHINE LEARNING

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Diagnostic skills are considered essential for successful teaching, for example to accurately assess students' learning and adapt teaching and individual instruction accordingly. To support (pre-service) teachers in acquiring such skills, digital simulation-based learning environments are increasingly common. In their meta-analysis, Chernikova et al. (2020) showed that such environments can be used to support the acquisition of diagnostic skills and that different types of support are differentially effective, based on students' learning prerequisites. However, adaptively supporting learners based on their processes within these simulations appears important to increase learning effects. However, this requires that processes within such environments can be assessed and evaluated on-the-fly. First results suggest that Machine Learning (ML) algorithms are successful in evaluating such processes and in predicting individual learners' outcomes in such environments (Brandl et al., 2021).

The present study empirically compares three different strategies for the automatic evaluation of domain-specific and domain-general indicators of diagnostic processes within a video-based simulation (Codreanu et al., 2020) targeted at supporting pre-service teachers' diagnostic skills regarding mathematical argumentations skills: a key word approach and two machine learning approaches, for predicting. The comparison is based on data from more than 200 students (sampling ongoing).

Results especially underlined the effectiveness of the ML approaches, which yielded a high predictive accuracy and, in particular, also a high negative predictive value. The latter is especially important, as learners that struggle within the simulation can be identified reliably, which is essential for their support.

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PRE-SERVICE PRIMARY TEACHERS POSING PROBLEMS

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The activity of Problem-posing (PP) allows us to assess and understand the mathematical thinking and learning of students (Cai & Hwang, 2020). We developed a study of PP with pre-service primary teachers (PSTs) focused on PP as a cognitive activity (Liljedahl & Cai, 2021) due to its relevance for their professional development. The objective is to analyze PSTs' use of fractions to pose problems, a topic that investigation says that primary teachers still have difficulties (Yao et al., 2021).

A group of 18 PSTs were asked to pose three problems with two given fractions. We studied fraction understandings, mathematical structures and contextualization that PSTs most frequently use in their proposed problems. Results of the 52 problems proposed show that 28.8% of them had insufficient information, the most common fraction meanings used by PSTs were part-whole (30.8%) and operator (38.5%), and the mathematical structures additive (26.9%) and order (19.2%) were the most selected. On the other hand, contexts related to pizzas or pies (34.6%) and measurement situations (21.1%) are the most used, and all of them presented a continuous representation of fractions. The study reveals some notable absences: problems with multiplicative structure, use of the discrete representation of fractions and few problems using fractions with the meaning of division, ratio or decimal. The results suggest that PSTs construct fraction problems, but the variety of fraction structures and meanings are relatively limited. We continue further analysis of PP and the role of specific PP training.

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COLLABORATIVE LEARNING PRACTICES OF STUDENTS THROUGH SYNCHRONOUS ONLINE LEARNING ENVIRONMENT

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Recent reports revealed that while individual learning skills are respectively satisfied, collaborative learning skills are not effectively encouraged during pandemics (Lee, 2021). Several video conferencing applications developed artifacts such as breakout rooms to facilitate collaborative learning and peer-to-peer interaction. Online learning was reported more frequently on college and university levels, while documentation of online learning in elementary level was not adequately reported (Edwards, 2012).

This study investigates a collaborative learning environment and lower secondary level students' practices through online synchronous mathematics courses. This work adds value to the existing literature on the online collaborative learning environment for elementary mathematics courses with all benefits and limitations. This practitioner-led study seeks to address the research questions: What are the characteristics of a collaborative learning environment through online synchronous mathematics courses? What are the students' collaborative learning practices through online synchronous mathematics courses? Participants consisted of 15 seventh-grade students, and the data were gathered from online sessions, Students Attitudes Group Environment Questionnaire (SAGE), Collaborative Study Self-Evaluation Protocol. This study was conducted in 16 lesson-hour for eight weeks. The educational design research method was the guide to inform the teaching and learning process as well as data collection and data analysis processes. Four issues emerged during the study: the warming-up process of the students, students' collaborative actions under small group and whole-class discussions, teachers' and students' supportive practices for mathematical reasoning, and technical issues. The number of institutions that seem to offer online instruction continues to grow, as does the number of students desiring non-traditional delivery methods of teaching. Therefore, this study may contribute to the field to design effective instructional tools for collaborative learning.

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EXPLORING WOMEN OF COLOR'S VARIOUS EXPRESSIONS OF MATHEMATICAL IDENTITY

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One area in the growing call to create more equitable and just mathematics classrooms is supporting and strengthening students' mathematical identities (Voigt, 2020). However, limited scholarship in this area considers an intersectional lens when exploring mathematical identity at the undergraduate level (Leyva, 2021). Informed by Data Feminism, this research uses a transformative, sequential mixed methods design to ask: How do women of color in undergraduate Precalculus, Calculus I, and Calculus II group similarly and/or differently based on attributes of mathematical identity?

The data for this study include responses from women of color on a large-scale survey of students' undergraduate calculus experiences in the United States. We first outlined four domains of mathematical identity from 23 relevant survey questions using factor analysis. Based on literature, we labelled these domains class experience, inclusion, mathematics peer interaction, and self-efficacy. We then performed a cluster analysis over those domains, which suggests four distinct groups related to women of color's mathematical identities. Students in Cluster 1 perceive high levels of active learning in their class, frequently collaborate with peers, feel equally included in class, and experience no change in self-efficacy. Students in Cluster 2 perceive mainly lecture style teaching in their class, interact infrequently with peers, feel equally included in class, and experience no change in self-efficacy. Students in Cluster 3 perceive mainly lecture with some group work in class, interact fairly frequently with peers, feel less included in class, and experience a positive change in self-efficacy. Students in Cluster 4 perceive fairly high levels of active learning in their class, moderately interact with peers, feel equally included in class, and experience no change in self-efficacy.

We also used qualitative data through free response survey questions to better contextualize each cluster. Future interviews will explore the key ideas emerging from this analysis, including the role of peers and friendship, instructor care, class structure, and mathematical affect in supporting women of color's mathematical identities.

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THE ROLE OF THE DOTTED LINE: FROM 2-DIMENSIONAL TO 3-DIMENSIONAL GEOMETRY

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The relationship between the use of artifacts and mathematical knowledge has been studied. Theoretical perspectives such as the *semiotic potential of artifacts* are situated in this line. In this scenario, impact of dynamic geometry environments (DGE) in teaching and learning of mathematics has been recognized. However, the work in this way has been carried out mainly in 2D configurations, with little research in 3D. In this study we analyse the solutions to two problems by one student to find out the semiotic potential of some features of a 3D-DGE. The semiotic potential of an artifact refers to the double relationship that links it to personal and mathematical meanings when used to solve a problem (Bartolini-Bussi & Mariotti, 2008). In our study we used GeoGebra, with an emphasis on its 3D dragging function, and the relationship between the dotted line that evokes the projection of a point on the XY plane when it is dragged and the notion of perpendicularity in 3D.

We analyse the activity of a 11-year-old mathematically gifted student at secondary school when solving two problems of a sequence aimed to introduce objects and relationships in 3D geometry based on their 2D counterparts, where equidistance had a central role. The first author interviewed the student when he solved the problem to know the solution strategy elaborated by him, and the reasons why he considered that this solved the problem. We selected episodes where the dotted line had a prominent role in the student's solutions. An interesting result is that the student used the dotted line unexpectedly to justify the equidistance between objects, showing a connection between this relationship and orthogonality in 3D. Framed in the perspective of semiotic potential and considering our objective, the use to the dotted line suggests the potentialities of this characteristic of GeoGebra with respect to the mathematical meaning of orthogonality. These results contribute to the description of the semiotic potential of tools in 3D-DGE, specifically the unexpected potential of the dotted line, that deserves further investigation.

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THE EPIDEMIOLOGY OF MATHEMATICAL PERFORMANCE, ANXIETY, AND PSYCHOSOMATIC SYMPTOMS

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Learning math is a challenge for many students, but not all math performance weaknesses stem from difficulties in cognitive sub-skills (Szűcs & Mammarella, 2020). Many children's school performance is affected by performance anxiety, which can range from mild to intense stress, but does not always result in low performance (Devine et al., 2018). Somatic syndromes may occur during anxiety. These physiological symptoms are closely associated with emotional manifestations, they are the consequences of each other (Major & Ádám, 2013).

The aim of our research was to investigate the connections between mathematical anxiety, performance, and psychosomatic symptoms. We conducted surveys in four schools in Budapest, Hungary, and four schools in rural areas. We used the Mathematical Ability Test and the Mathematical Anxiety Test among 999 upper secondary students, followed by the PHQ (Physical Helath Questionnaire) in the second round. There is a significant ($r=-.928$ - $-.789$; $p< .001$) negative correlation between mathematical performance and anxiety for all factors and sub-factors. Psychosomatic symptoms occurred in 30% of students, showing a strong correlation with performance. Among those performing below average there is a stronger presence of psychosomatic syndromes. The results suggest that there is a clear significant relationship between mathematical anxiety, performance and the appearance of psychosomatic symptoms among the upper-class students studied. Besides the unique application of measures and methods from different fields of sciences, the educational conclusions are to be taken account in everyday pedagogical practice.

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MATHEMATICS-RELATED EMOTIONS BETWEEN CONTACT AND DISTANCE LEARNING ACROSS PERFORMANCE LEVELS

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During the COVID-19 pandemic, schools faced a new challenge when students could not attend school in person. Distance learning can impact students' emotions (Tannert & Gröschner, 2021), which are in turn related to their motivation and achievement (Pekrun et al., 2002). This large-scale quantitative study examines how distance learning affected the relationship between performance level and mathematics-related achievement emotions among Finnish high school students.

Previous studies have shown that adolescents with different performance levels exhibit different mathematics-related emotions (e.g., Holm et al., 2017). The data for this study were collected from 1367 high school students across Finland during spring term in 2021, using a modified Achievement Emotions Questionnaire instrument to assess mathematics-related emotions in contact and distance learning conditions. Students' mathematical performance levels were divided into 3 groups based on the self-reported grade point average for mathematics courses in high school. Mixed-design ANOVA was used to test mean differences and interactions between the high and low performance groups and learning conditions.

The results indicate that distance learning may have weakened the relationship between performance level and achievement emotions. For example, in distance learning conditions, although enjoyment was lower in both high and low performance groups than in contact learning, the difference between the groups was smaller than in contact learning, as enjoyment was reduced more in the high-performance group. On the other hand, anxiety was reduced more in the lower performance group, again reducing the difference between the two groups. Significant effects were also found concerning boredom and pride.

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APPROACHES TO STEM TASK DESIGN: A REVIEW STUDY

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The equitable representation of the four STEM disciplines is one of the issues in the current STEM practices. Particularly, mathematics is least treated as one core discipline in STEM education research (Martin-Paez et al., 2019). In addition, the current STEM systematic reviews have focused on instructional practices, perceptions, gender differences, and trends in publications, educational levels, disciplines or countries. However, few review studies have characterized STEM tasks. In this oral communication, we will present one part of the results derived from a review study – that is, approaches to STEM task design and specifically attending to tasks explicitly integrating mathematics.

A search of Scopus database using the search terms "STEM education" or "STEM learning" or "STEM teaching" AND “task*” or “activit*” AND "mathematic*” was conducted on January 2, 2022 and returned to 266 published articles. The inclusion criteria: the article must provide exemplary task, focus on students, and include the integration of mathematics with other disciplines. After screening all of them, 20 articles were related to our review study. The approaches to STEM tasks can be characterized according to aims of tasks (Laveault, 2014), viewing on mathematics (Bishop, 1988), and design methods (Watson & Ohtani, 2015).

The result shows six kinds of task design approaches: mathematical modelling, engineering design process, problem-based, digital-tool-based, social issues-based, and construction-and-practice-based. For each approach including the aim, view and method, critical principles for task design will be identified.

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A LONGITUDINAL STUDY ON THE RELATIONSHIP BETWEEN STUDENTS' MATHEMATICS LEARNING DISPOSITION AND ACHIEVEMENT: LATENT TRANSITION MODEL ANALYSIS

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Numerous studies have shown that students' math performance is related to learning strategies and goal orientation. However, few studies have investigated the association among these variables across students' grades. Due to the difficulty of longitudinal data collection, most studies were mainly cross-sectional with one-time data collection, but this might ignore the time factor and change in the learning process.

This study uses data from a county's 2017, 2018, and 2019 mathematics achievement tests in the Taiwan Education Long-Term Database. Exploring change over time of math achievement, goal orientation, and learning strategies for 1,500 students in grades five through seven. In this study, researchers collectively refer to goal orientations and learning strategies as learning dispositions and conduct a longitudinal study on the relationship between mathematics learning dispositions and achievement. In addition to discussing the number of latent categories for each grade level, this study used a Latent Transition Model Analysis (LTA) to assess the dynamic relationship between learning disposition and mathematics achievement, incorporating the time factor into the estimates (Collins & Lanza, 2009). The study results found that learning disposition can be divided into three potential categories in each grade. Category A is students with positive performance, mastery orientation, and elaborative learning strategies. Category B is students with positive performance, mastery orientation, and control learning strategies. Finally, category C is students giving up in learning or passive performance orientation and taking passive memorization learning strategies. The results of the LTA analysis found that gender was associated with learning disposition, and students with different dispositions had different math achievements. After equating the mathematics achievement scales of the three grades, it showed that the average performance of these three categories was the highest in the sixth grade and showed a downward trend in the seventh grade. It found that their learning disposition changed in lots of students in the cross-grade learning process. This study shows that the longitudinal analysis considering the influence of time is more precise and practical than the cross-sectional data analysis and proposes relevant suggestions for future study and instruction in mathematics education.

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UNCONVENTIONAL THINKING IN ONLINE LABORATORY SCHOOL: FRACTIONS

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For effective teaching, it is necessary to keep students' thinking central to the instruction. Jacobs et al. (2010) introduced professional awareness as: attending to children thinking (identifying), interpreting (making sense), and responding (deciding how to respond). Innovative teacher education programs prioritize preservice teachers (PST) acquisition of these skills as in our institution. This research investigated PSTs' noticing instances of students' unconventional fractional thinking and orchestrating online classroom discussions around those instances in our Online Laboratory School (OLS) setting (Tunç-Pekkan & Taylan, 2022). The OLS is founded during Covid-19 pandemic as a virtual school and has served to hundreds of low-income students since Spring 2020. OLS also aims to provide quality internship experience to mathematics PSTs by planning, teaching, and reflecting on the experience and providing research opportunities related to teacher education. For this project we primarily focus on the videos of teaching experiences of PSTs on the topic of fractions. Thirty percent of middle school curriculum in X which has centralized curriculum is on fractions and students experience most difficulty on this topic. It is also suggested that specifically fractions should be emphasized in teacher preparation programs (Lee & Lee, 2021). There were 10 PSTs who taught middle school mathematics in 8-week OLS and 24 fraction related lessons (video-recordings) were analyzed. Content analysis and grounded theory were utilized. Findings revealed that most common unconventional student thinking were on "operations" and "representation on the number line" categories. On most occasions, PSTs realized those instances in the teaching moment but some of their responding generated further misconceptions in students; when written lesson plans were analyzed, the problem statements were of poor construction. There is often a mismatch between PST intentions and student interpretations. A coordinated analysis of video-recordings and lesson plans also confirmed these patterns. We will discuss implications of findings for teacher education programs.

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INVESTIGATING THE INTERPLAY BETWEEN DIGITAL AUTOMATIC-ASSESSMENT AND SELF-ASSESSMENT

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It is well established that digital formative assessment can support student learning, for example by means of digital automatic assessment of students' work (Olsher et al. 2016). At the same time self-assessment is regarded as important to support students' meta-cognitive skills and to support students' ownership of their learning (Andrade 2019). Yet, little is known about *combining* automatic- and self-assessment. Starting from this research gap, the "Interplay between Self-assessment and Automatic Digital Assessment" (ISAA) project (Olsher & Thurm, 2021) scrutinizes how to combine automatic-assessment and self-assessment in the context of Example-Eliciting-Tasks using the digital formative assessment platform STEP (Olsher et al. 2016).

In the present study we investigate a newly designed STEP-self-assessment-module in which students first assess whether predefined mathematical characteristics are present in their generated examples. Subsequently students receive three types of reports: A) an overview of their self-assessment, B) an overview of the results of the automatic-assessment of the same characteristics in their generated examples, and C) an overview comparing their self-assessment with the automatic-assessment that highlights conflicts between the self- and automatic assessment. We conducted a qualitative video-case study in which we observed 16 students working on a EET on quadratic functions ("What is the relationship between the linear functions and the product function? Create three examples that are as different from each other as possible to represent your answer", see Olsher & Thurm 2021). In our presentation we focus on how students use and interpret the different reports (A, B, C) - in particular we expect that conflicts between students' self-assessment and the automatic-assessment encourage students to investigate differences and promote student learning processes.

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EXPLORING EIGHTH GRADERS' SPATIAL STRUCTURING OF UNIT CUBES: VISUALIZATION ABILITIES IN SKETCHUP CONTEXT

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This qualitative research aims to explore eighth graders' visualization abilities while spatial structuring of unit cubes to create specific buildings. Visualization abilities (which are figure-ground perception, perceptual constancy, positions in space, spatial relationships, and visual discrimination) regarding spatial structuring of unit cubes were elaborated by Gutiérrez et al. (2018), following the seminal work of Del Grande (1990). In this paper, we acknowledge the theoretical framework of "visualization abilities" (Gutiérrez et al., 2018) in a modelling software context, specifically in the SketchUp environment. To explore students' steps and discuss the function of the framework, we focus on two eighth graders' spatial structuring process and consider the following research question: How do eighth graders' spatial structuring in SketchUp context occur within the lens of visualization abilities?

Data comes from a series of task-based interviews with two eighth graders (one boy and one girl, both were fourteen years old) from a public school located in central Turkey. Two participating students were familiar with the software. The interviews included five different tasks that were related to constructing buildings with unit cubes. First, top, front and right views of the building on the paper were proposed and then asked to construct the given building. Later, it was asked to consider a zero-gravity environment (that SketchUp provides) and construct the same building with fewer cubes. Video-recorded interviews and screen recordings were triangulated within the lens of visualization abilities (Gutiérrez et al., 2018). Our tentative analysis indicates the central role of figure-ground perception and position in space in students' visualization abilities within SketchUp context. Furthermore, we observe and discuss the synergy and dialects among different visualization abilities.

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WORD PROBLEMS INVOLVING FRACTIONS: PERFORMANCE OF HIGH SCHOOL MEXICAN STUDENTS

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The study of temporal evolution of mathematics performance is important, since knowledge should be progressive and cumulative, so it should improve in higher levels (Schoenfeld, 1982). This fact, in Mexico, is not always observed in the mathematics results of national tests (PLANEA, 2017). Average score obtained by students, 17-year-old, differs by 8 points from that of 16-year-olds. Therefore, there is interest to investigate: What happens with problem solving involving fractions at this educational level? In this context, the investigation aims are: 1) to determine differences by age and type of school regarding problem solving success involving fractions, and 2) to analyze solving processes followed by problem type, age group and school.

The sample consists of 180 and 78 students aged 16 and 17 years old respectively, they studied in different type of institutions in Mexico City: general and technical high schools. A pencil-and-paper test with three word problems –P1, P2 and P3– was designed and applied; the common feature of those problems is a fractional distribution through "what is left".

Results indicate that younger students and those belonging to the technical high school obtained greater success solving the three problems posed. P3 was solved correctly by a higher percentage of students. It should be noted that, despite being introductory algebra courses (Filloy et al., 2001), students do not use algebraic methods. They employed arithmetic procedures to solve P1 and P2, and trial-and-error strategies to solve P3.

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SOME DIFFICULTIES WHEN ELEMENTARY PROSPECTIVE TEACHERS CLASIFY 3D FIGURES

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In recent times teacher training has been the focus of the research inside of the didactics of mathematics, but much of this research has centered on cognitive elements and ways to work with mathematical objects, and not on the processes to construct geometrical thinking. As proposed by Battista (2012) the early-years teacher has an important role in fostering children's mathematical knowledge, including their geometrical knowledge. Actually, teachers tend to neglect geometry and have difficulties with classification procedures, even with manipulatives (Hidayah et al., 2018). To identify the geometric knowledge and processes of prospective primary education teachers (PPT), a professional task is constructed and implemented with 134 PPTs in Spain. The aim of the task is to find the knowledge related to the classifications of 3D figures and the criteria that PPTs use to classify them in a constructing task using four half cubes. Using van Hiele's levels the PPTs' productions are organised and analysed. The results show the difficulties and errors of the PPTs when defining criteria for classifying corresponding solids and give later opportunities for reflection.

The most common difficulty of the PPTs at van Hiele's first level (93.5% of all PPTs) was the use of irrelevant attributes and/or properties of the solids. 80.6% had difficulties when using perceptual elements such as the shape of the faces of the solids to propose a definition or a classification. On the other hand, only 25.8% of the PPTs were placed in van Hiele level 2. In this case, a surprised difficulty found is the identification of many figures as prisms when the basis is not an expected convex polygon. Only 6.5% of the PPTs recognized enough elements of the solids to classify them, but this was mostly done using visual elements, such as number faces, vertices, and edges, without using common class' properties. Although a perceptual domain is identified in the characterization of 3D figures and a low level of structuring, the professional task offers nice reflections about the need for an adequate common mathematical knowledge. It was important for systematising for instance the process of enlarging well known 2D-classification criteria to 3D shapes (Patkin, 2015).

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PROSPECTIVE PRIMARY TEACHERS' MATHEMATICAL KNOWLEDGE IN A LESSON STUDY

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Difficulties in interpreting students' work and supporting students' learning, by prospective primary teachers, can be cause by misconceptions in mathematical knowledge (Tatto, 2018). However, provide many advanced mathematics' subjects in initial teachers' education courses does not seem to ensure, by itself, a sufficient development of mathematical knowledge. We argue that it is important prospective teachers apply mathematical topics in practical contexts, during their training.

In our lesson study, two prospective teachers worked together with three experienced teachers, planning, and conducting two lessons that aimed to solve classroom problems, focusing on student learning (Fujii, 2018). Initially, the future teachers revealed difficulties in terms of mathematical knowledge. Our communication focuses on this aspect and seeks to know the learning, in the field of mathematical knowledge, carried out by the two future teachers, when participating in the lesson study. We followed a qualitative approach and interpretive paradigm, within design-based research. Data were collected using participant observation, researchers' journal, document collection and semi-structured. We mobilized the Mathematics Teacher's Specialised Knowledge model to analyze data, more specifically, in the field of mathematical knowledge.

The results show that future teachers developed their mathematical knowledge, in the subdomain of knowledge about the topic, by practicing concepts, procedures and representation registers; knowledge about the structure of mathematics, when establishing connections based on simplification and increasing complexity and knowledge of mathematical practices, when applying their reasoning, justification and generalization strategies in solving tasks.

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PRESCHOOL CHILDREN'S NOTATIONS FOR DENOTING ORDINAL POSITION AND QUANTITY

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Written numerals are culturally influenced notational artefacts that can be examined as "referential-communicative tools" (Tolchinsky Landsmann & Karmiloff-Smith, 1992, p. 287). Children's own notations for representing mathematical concepts is a powerful lens for exploring the development of written symbolism. No previous research has compared children's notations for communicating ordinal position and quantity. We address this gap by aiming to answer the following research question: Do children's notations differ for the communication of ordinal position and quantity and if so, how?

A volunteer sample of 37 preschool children (mean age: 4.01 years) participated in individual interviews with two tasks. T01: Children were asked to produce notations showing the ordinal position of toy cars in a line. T02: Playing a 'Snap' game with the researcher, children were asked to note the quantity of cards won by each player. Notations were analysed using a framework informed by Lucangeli et al. (2012).

45.5% of all notations produced were *Idiosyncratic*, with 28.2% in T01 and 17.3% in T02. *Other symbolic forms* (crosses, dots) had the second highest percentage of notations, with an overall of 16.8%: 15.0% in T01 and 1.8% in T02. 29.6% of the overall notations for both tasks were *Arabic numerals*: 13.2% in T01 and 16.4% in T02. Overall, T02 (quantity) elicited more notations of conventional form in comparison to T01 (ordinal position). The results indicate cross-task notational variation for communicating ordinal position and quantity. Furthermore, when children were required to produce notations more than once within the same task, most children produced mixed types of notations, showing a range of within-task variation too. In the presentation, further results and notation categories will be discussed.

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A MODEL FOR THE PRIMARY SCHOOL STUDENTS' MATHEMATICAL MODELING COMPETENCY: A GROUNDED THEORY ANALYSIS

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Four important strands shaped the debate on mathematical modelling competency (Kaiser & Brand, 2015). The German modelling group proposed the concept of modelling competency based on different sub-competencies and found some different models of modelling competency, such as the five sub-competencies model. The Australian modelling group emphasized that modelling competency should include metacognition. Previous studies paid inadequate attention to primary school students, so this study constructed a model of the primary school students' mathematical modelling competency through grounded theory.

METHOD

This study carried out case studies and selected 6 fifth-grade students and 6 sixth-grade students from China as the objects of study. Thinking aloud was used to collect students' thinking process of solving mathematical modelling tasks which were translated and adapted from published modelling education studies and books.

RESULTS

This study obtained 370 codes and finally formed 7 categories and 2 main categories. Based on the coding results, a mathematical modelling competency model for primary school students was established:

Competency	Sub-Competency
Modelling Process Competency	Understanding Information; Making Models; Working Mathematically; Interpreting and Validating
Metacognitive Modelling Competencies	Orienting and Planning; Monitoring and Regulating; Evaluating and Improving

Table 1: The Model for the Primary Students' Mathematical Modelling Competency

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A FRAMEWORK FOR EVALUATING STUDENTS' SYSTEMS THINKING IN MATHEMATICS

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In the 21st century, people are living in an increasingly complicated world and thus should be prepared to think and act holistically and integratively rather than considering each component individually. In other words, individuals have to be equipped with systems thinking (ST). This is a 21st century skill that should be cultivated among students and has been included in the 2022 PISA math framework (OECD, 2020), indicating that mathematics is a subject that should help train students to be systems thinkers. However, ST connotation, structure, and evaluation when it is applied to mathematics have not been elaborated previously. Although PISA 2022 includes ST in its math framework, item developers have indicated that items in PISA 2022 are not specifically developed for evaluating this skill.

Studies exploring ST in mathematics are rare. Salado et al. (2018) studied four middle school students' ST skill in solving mathematical problems by investigating whether they can use relevant elements in the real-world context. However, the students only provided traditionally expected mathematical answers. In addition to research in mathematics, a systematic literature review of studies discussing ST connotations, structures, and evaluations in other fields, such as engineering, management, biology, and chemistry (e.g., Lavi et al., 2020), could provide insights regarding the picture of ST in mathematics. Based on such literature review of research from 2003 to 2020, this paper identifies the ST skill as a skill of identifying elements in a system and their interconnections and integrating them into a holistic structure, and a three-element operational framework is proposed for the evaluation of ST, outlined as follows. Test items in the math world and real world were also developed accordingly.

- Identify/construct relevant components: the key of evaluation is whether students consider multiple perspectives in their consideration.
- Recognise/construct relationships: the key of evaluation is whether students can consider diverse nonlinear relationships rather a simple linear one.
- Analyse/construct a system: the key of evaluation is whether students can systematically organise relevant components and relationships into a structure or analyse a structure in this manner.

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ANSWER PATTERNS OF JAPANESE SECONDARY SCHOOL STUDENTS IN TIMSS 2015 MATHEMATICS SURVEY

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Suzukawa et al. (2008) analysed the PISA 2003 mathematical literacy survey data, to reveal the Japanese students' answer patterns through comparison with data from 13 countries and areas. Their results indicated that Japanese students had peculiar answer patterns, while being especially good at solving questions in the 'educational' context of the PISA framework. In the secondary analysis of TIMSS mathematics survey data, Watanabe and Watanabe (2021) identified the answer patterns of Japanese fourth-grade students in TIMSS 2015, to show that Japanese primary school students had a peculiar answer pattern, and in particular, the items of calculations were found to be easier than for students from the 12 countries and areas. However, previous studies analysed the PISA and TIMSS data and examined the answer patterns of 15-year-old students completing compulsory education, and fourth-grade primary school education.

This study aimed to reveal the answer patterns of Japanese secondary school students. Since Watanabe and Watanabe (2021) analysed TIMSS 2015 data, this study followed their approach and targeted the TIMSS 2015 data. Some of the 13 countries and areas targeted by Watanabe and Watanabe (2021), did not participate in the TIMSS 2015 eighth-grade mathematical survey. Instead, 15 countries and areas, including Australia, Canada, Hong Kong, Hungary, Ireland, Italy, Japan, Korea, New Zealand, Russia, Singapore, Sweden, Taiwan, the United Kingdom, and the United States were targeted in this study.

The results show that Japanese secondary school students had a peculiar answer pattern in comparison to these 15 countries and areas, and in particular, the items of Number, especially the ones learnt at the primary school level, were found to be more difficult than they are for students from these other countries.

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CAN SCHOOL-RELATED MATHEMATICAL PROBLEMS AFFECT THE PERCEIVED DOUBLE DISCONTINUITY?

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There has been a discussion for more than 100 years, that (prospective) secondary mathematics teachers have difficulties to see connections between school mathematics and academic mathematics both at the beginning of their studies as well as at the entry to their career (*double discontinuity,* Klein, 1932). Hence, many universities started to support their prospective teachers in drawing these connections. A popular intervention are school-related mathematical problems (SRMPs) that address connections between school mathematics and academic mathematics (see *school-related content knowledge,* Dreher et al., 2018). Our research project aims to investigate whether SRMPs can lessen students' perceived double discontinuity. We assume that the use of SRMPs reduces the first discontinuity by fostering students' perception of connections between contents of academic mathematics and school mathematics. Furthermore, as most SRMPs address typical teacher tasks requiring academic knowledge, we assume that SRMPs also support students' perceived relevance of academic mathematics for teaching. In the long term, this might influence the second discontinuity.

We conducted our questionnaire-study at one German university with $N = 98$ first year students: an experimental group of 74 prospective teachers who received SRMPs and a control group of 24 students majoring in mathematics or related subjects who visited the same university courses, but did not receive SRMPs. Conducting a t-test and an ANOVA, we found that the perception of both groups did not significantly differ in the pre-test but that the perception in the post-test depended on an interaction of the factors group and time showing a more favorable trend for the experimental group. There were significant but small interaction effects on the perception of connections between academic and school mathematical contents ($F(1, 89) = 6.51$, $p = .01$, $\eta^2 = .03$) as well as on the perceived relevance of academic mathematics for the teaching profession ($F(1, 90) = 4.27$, $p = .04$, $\eta^2 = .02$). In sum, our results indicate that SRMPs might influence the perceived double discontinuity in a positive way. However, these first results need to be replicated in a larger sample.

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HOW IMPORTANT IS STRUCTURAL THINKING IN GRADE 2?

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Structural thinking is evident when learners calculate with an awareness of relationships between mathematical objects when solving problems. The literature in early mathematics education points to mathematical competencies being particularly powerful predictors of later mathematical learning. These competencies involve calculating with an awareness of relationships between mathematical objects, for example, numbers. Learners who use their knowledge of these relationships in solving problems demonstrate structural thinking. My study seeks to address the question: how does learners' understanding of structural thinking in Grade 2 (G2) determine later performance in Grade 6 (G6)? The same cohort of 64 participants in 8 government schools in a province in South Africa was followed in G2 and then in G6. I draw on the theory and Learning Framework in Number of Wright et al. (2006) and use its interviewed-based assessment (G2 LFIN) to evaluate the G2 participants' structural thinking. The G2 LFIN focuses on the sophistication of strategy used. A more sophisticated strategy entails a higher level of structural thinking. Data of four assessments were collected, a Grade 1 Annual National Assessment written test delivered at the beginning of G2 (G1 ANAdG2) and the G2 LFIN. The G6 assessments were the G6 Common assessment written test (G6 CA) and an interview-based diagnostic test developed within this study – the G6 Evidence of Structural thinking Assessment (G6 ESTA), focusing on evaluating and looking for evidence of structural thinking. I investigated how well the G2 assessments predicted the performance in G6.

I used SPSS to find Pearson's product-moment correlation coefficient to study the predictive power of the G2 assessments. The results of the correlations indicate that the G2 LFIN shows a much better correlation with the two G6 assessments (G6 CA $r = 0.738$, $n = 64$, $p = 0.01$; G6 ESTA $r = 0.750$, $n = 64$, $p = 0.01$) than the G1 ANAdG2 (G6 CA $r = 0.534$, $n = 64$, $p = 0.01$; G6 ESTA $r = 0.446$, $n = 64$, $p = 0.01$). The latter implicates that the G2 LFIN was a better predictor of the G6 assessments than the G1 ANAdG2. Since the G2 LFIN evaluates structural thinking and the G1 ANdG2 does not evaluate structural thinking to the same extent, the findings suggest that the assessment informed by structural thinking was a better predictor of performance in G6 and accentuate the importance of structural thinking in G2. Given that very young learners can appreciate mathematical structure, teachers and policymakers should ideally emphasize teaching numbers structurally in the early years by constantly making learners aware of the relationships between the numbers.

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THE INFLUENCE OF CONCEPT-BASED MATHEMATICAL INQUIRY CURRICULUM ANDINSTRUCTION ON 3RD GRADER'S MATHEMATICAL PROFICIENCIES

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The research is based on Concept-Based Curriculum and Instruction [CBCI], as well as mathematics inquiry teaching to design geometry and calculation curriculum, and applies it to a class of third grade to examine these students' performance of their mathematical proficiencies. According to the framework of CBCI (Erickson et al., 2017), which stresses applying learner's experiences and skills and emphasising the nature of mathematics, the researchers set the conceptual lens of curriculum, and learning goals for students, and design teaching activities by means of the 5E inquiry teaching model including stages of engaging, exploring, explaining, elaborating, and evaluating (Bybee et al., 2006). Besides, the study adopts the mathematical proficiency checklist to analyse students' performance of mathematical proficiencies (NRC, 2001).

The findings show that the students could understand the target mathematical concepts and give definitions of them, using different representations to solve problems, as well as interpreting the hidden concepts under the procedures of problem solving. For these third graders, even though they did not possess enough vocabulary to describe the generalisation of the concepts they found, they still managed to apply the generalisation to solving mathematical problems and developing new ideas. CBCI could not only promote the students to develop their mathematical conceptions and proficiencies, but also help the curriculum designers effectively connect their content knowledge of the relevant mathematical concepts and design interdisciplinary curriculum. Therefore, CBCI might be beneficial to mathematical teaching practices at primary level.

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MATHEMATICAL MODELING, EMOTIONAL ENGAGEMENT, AND PROMOTING PRODUCTIVE DISPOSITIONS

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Pre-service elementary teachers (PSTs) are a particularly vulnerable population in terms of mathematical identity and self-efficacy. Research has shown that they have higher levels of mathematics anxiety and poorer attitudes toward mathematics compared to other undergraduate students. They often enter college mathematics classrooms with feelings of uncertainty, irrational dread of the subject, and shame over their perceived lack of ability to do mathematics. Engaging PSTs in mathematical modeling is one possible avenue in promoting productive mathematical dispositions. By using mathematics to make sense of real-world problems, modeling provides powerful learning opportunities that allow PSTs to understand the role of mathematics in their world and its relationship to their lived experiences.

Through research funded by an NSF IUSE (Improving Undergraduate Stem Education) grant, we taught a semester-long modeling course for PSTs. To capture PSTs' perspectives during the modeling process, they kept and updated a journal across eight modeling tasks. We sourced prompts from Middleton, Jansen, and Goldin's (2017) attributes in understanding motivation and engagement including self-regulation, goals, interest, and utility. Through the lens of emotional engagement (DeBellis & Goldin, 2006), we examined emotions that PSTs report across different phases of the modeling cycle and how these emotions change across time with respect to the nature of the task and experience with modeling. Results indicate that modeling is initially very challenging and PSTs' report emotions like frustration or being overwhelmed tackling open-ended problems. As PSTs engage in multiple rounds of the modeling cycle, they are able to contextualize their emotions in relation to the process and build productive dispositions toward modeling and mathematics. Our results indicate that modeling expands PSTs' understanding of what it means to know and do mathematics. By expanding views of mathematics and reflecting on emotions, PSTs are empowered to engage in challenging work.

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SECOND LANGUAGE COMMUNICATION STRATEGIES IN LOUD READING OF MATHEMATICAL EXPRESSIONS

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Language-rich classroom activities are central for mathematics learning. When students daily discuss, explain, or argue about mathematics, at some point, transformation of written mathematical expressions into speech will occur. If a student is not familiar with the pronunciation of a symbolic expression, this might involve some problems. The situation is similar to a situation of second language (L2) communication; lack of vocabulary threatens to impair communication. For L2 users, a plethora of second language communication (SLC) strategies have been identified.

The purpose of the present study is to gain more knowledge about the situation when students struggle to transform symbols into speech, focusing on their strategy use. To explore whether oral SLC strategies are used in this type of situation, recordings of loud reading of short mathematical texts (3-8 lines) were analysed. Readers were 18 university students taking a preparatory course in mathematics. The texts included, e.g., integral expressions, differential equations, and double angle identities. The analysis was based on Dörnyei and Scott's taxonomy (1997), adjusted for analysis of sound recordings from a situation not including an active communication partner and normal L2-pronunciation problems.

The results showed that students do use SLC strategies when reading out mathematical symbols. Of 18 potential strategies, 15 were found at least once in the data. Strategies included different ways to *change the message of the text*, specific strategies for *change of vocabulary*, *vocabulary search*, and different types of *self-correction*. Strategies used to keep the communication channel open while struggling to read, were often combined with other types of strategies. The use of SLC strategies could indicate similarities in the obstacles encountered when transforming symbols into speech to obstacles in oral L2-communication. Whether this supports the idea of basing mathematics teaching on L2-learning methods, which has sometimes been suggested (Wakefield, 2000), is yet to be investigated.

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PRE-SERVICE TEACHERS' USE OF INFORMATION WHEN DIAGNOSING FIRST GRADERS' NUMBER SENSE

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Supporting first graders in developing their number sense from basic counting to more sophisticated understandings of numbers and operations is a challenging task (Anghileri, 2006). Teachers have to assess where the children are in their learning process, e.g., by analysing how they solve arithmetic tasks or how they explain their answers. Diagnosing first graders' number sense in such a way is an essential prerequisite for providing them with suitable follow-up learning opportunities. Studies in this context show that the quality of teachers' diagnostic judgements depends on whether valid information can be identified and used while invalid information can be ignored (Loibl et al., 2020). It is, however, still unclear to what extent pre-service teachers are able to use information when diagnosing first graders' number sense. Accordingly, our research questions are: (1) In what ways do pre-service teachers use information of different validity for diagnosing first graders' number sense? (2) To what extent is the quality of their diagnosis related to their use of information?

To address these research questions, we designed a vignette-based test instrument comprising of four authentic classroom situations based on text (student-teacher interaction) and images (photo of task and solution). The vignettes were developed based on literature and validated through n=14 experts (teacher educators in the field of arithmetic). According to our research aim, each vignette was varied in three experimental conditions with different information validity. The vignettes were embedded in an online questionnaire and answered by n=173 pre-service teachers at the end of a one-semester course on learning arithmetics in primary school. Two types of data were collected: written diagnostic judgements and process data about the use of information in form of mouse movements. A coding manual for analysing the participants' diagnostic judgements is currently developed. First findings will be presented and discussed at the conference. We expect that the findings indicate to what extent pre-service teachers succeed or struggle in identifying and using valid information when diagnosing first grader's number sense. The results of the study can help to understand how diagnostic competence can be fostered in university courses.

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HOW STUDENTS GET STIMULUS THAT SPARK THERE CREATIVITY AT MATH WORK？—AN EXAMPLE OF CONSTRUCTING QUADRILATERALS

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There is no doubt that creativity has been considered as a critical 21st century skill. Amabile (2016) referred creative skills to the ability of generating novel and useful ideas when individuals look at the same things and solve problems from different perspectives. This study aims to analyze how students get stimulus to generate novel and useful ideas in math work. A natural inquiry teaching experiment was conducted. Students were asked to draw as many quadrilaterals as possible with area 2 on grid papers. This work was designed by the researchers as an intermediary activity for teaching Pythagorean theorem that promoted students' creative skills. The sample included 51 eighth-graders.

The students drew an average of 8 quadrilaterals. The shapes they first drew were all normal or symmetric quadrilaterals, and then they started to draw shapes showing flexibility and originality dimensions in creativity. The results show: (1) The 21th century skills, "research & inquiry (RI)" and "systems thinking (ST)", posted in PISA2022 math framework are factors that spark creative thinking; (2) Both kinds of inquiries, no direction or with direction, have chances to stimulate creative works; (3) Computational thinking (CT) skills empower students to break down the task into manageable parts; abstract the feature associations of the same type of quadrilaterals; and generalize steps for creating various shapes which fulfilling the fluency dimension of creativity.

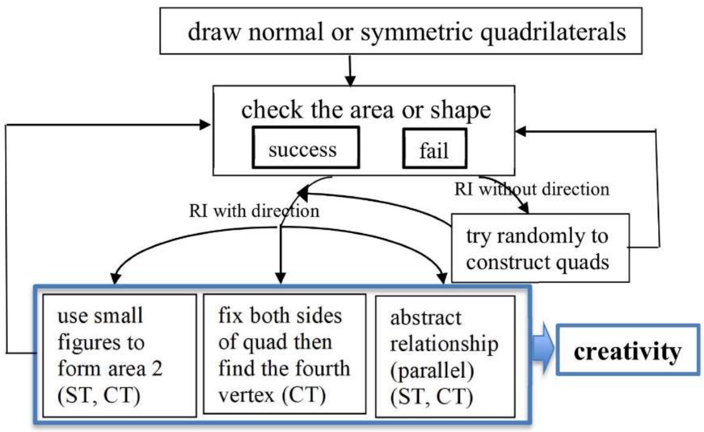

Figure 1. Scheme of the study

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COMPARATIVE ANALYSIS BETWEEN TWO VISUALIZATION OBJECTS IN DETERMINING THE SAMPLE SPACE OF AN EVENT: COINS AND TEKS

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Phillips et al. (2010) defined Visualization Objects as "physical objects that are viewed and interpreted by a person for the purpose of understanding something other than the object itself" (p. 26). However, the conventional way of teaching probability, often using coins, dice, and the standard deck of cards, has continually been a challenge for many Filipino students. Most of the time, they are unable to relate to these Visualization Objects causing them to hold numerous misconceptions. On the other hand, the context of "teks" which is one of the traditional Filipino games played by most students has been recognized as a context similar to tossing a coin. In fact, based on its physical structure, a teks card has also two faces similar to the heads and tails of the coin. The side with a picture is the front face, while the side without a picture is the back face. The front face is the winning side, and the back face is the losing side. Despite its unfamiliarity as a context inside the classroom, the researchers want to introduce the context teks to the students and compare their experiences to using coins in determining the sample space of an event. There were 62 Grade 11 students who were enrolled in a public school in Metro Manila, Philippines, who participated in the study. To capture the different aspects and richness of their experiences, phenomenography was used. All of them were asked to answer the Test on the Basic Probability Concepts (TBPC), which consists of coin and teks problem. Individual interviews were immediately conducted after the students answered the TBPC.

The issues encountered by the students in answering the coin problem were organized into four main categories: (1) failure to represent the number of coins in an experiment, (2) repetitive response, (3) lack of structural understanding of the general formula, and (4) difficulties in interpreting HHT and THH as two different outcomes. Generally, they tend to visualize the coin problem as something related to an "experiment or likelihood in mathematics" that requires a more sophisticated way (e.g. a diagram or a formula) in achieving the right answer. While on the other hand, they visualize the teks problem as something related to an "outcome," which is, for them, a non-mathematical term that describes the result or consequence of a certain event. This result unfolds qualitative aspects of an individual's construction of knowledge.

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MATHEMATICS TEACHERS' KNOWLEDGE OF STUDENTS' UNDERSTANDING OF ALGEBRAIC EXPRESSIONS AND EQUATIONS

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Middle school is a transition from concrete arithmetic operations to more abstract algebraic reasoning procedures. At this level of schooling, it is crucial to unpack teachers' thoughts about students' conceptions, difficulties, and errors to determine the gaps in teaching algebra, as well as to determine professional development needs. This study aims to contribute to the existing literature by focusing on teachers' knowledge of students' conceptions, difficulties, and errors in algebra (Asquith et al., 2007).

The purpose of this case study was to explore five middle school mathematics teachers' knowledge of students' understanding of algebraic expressions and equations. For this purpose, first, the researchers prepared an Algebra Diagnostic Test (ADT), then semi-structured interviews were done with mathematics teachers, where they were asked to predict and elaborate on their students' possible difficulties in ADT. Finally, ADT was administered to 267 eighth-grade students who were the students of those teachers interviewed. Students' responses in ADT were analyzed by coding their difficulties, conceptions, and strategies. Then, the findings of actual student responses were compared with teachers' predictions stated in the interviews.

Results indicated that teachers' predictions of students' understanding corresponded with students' actual responses in simple arithmetic and algebraic operations and doing simple transformations from arithmetic to algebra. However, teachers' predictions were not compatible with students' actual responses in more abstract and complex procedures, such as comparing two algebraic expressions and writing an algebraic expression of a word problem. Although teachers rarely identified some of the difficulties and errors about variables, they generally focused on using x to manipulate symbols rather than considering the abstract and complex meaning of x. The results will be provided with specific examples of students' responses and teachers' quotes.

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DOES INDIVIDUAL STUDENTS' VERBAL PARTICIPATION IN A MULTILINGUAL CLASSROOM CORRELATE WITH A MEASURE OF LEARNING?

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While learning gains can be achieved in mathematics classrooms where linguistically diverse groups of students engage in dialogic whole class discussions (Banes et al., 2020), quantifying the effect of individual students' participation in discussions on their learning remains elusive (O'Connor et al., 2017). Grounded in a sociocultural theoretical perspective, and building on the Academic Literacy in Mathematics framework (Moschkovich, 2015), we hypothesize that multilingual students' participation in mathematical discourse is central for learning. To examine this relationship, we investigated the question, what is the relationship between students' turns of talk in whole class discussions and gains on an assessment?

Data were collected in a classroom teaching experiment (TE) at a high school located near the US-Mexico border. About 30% of the students were classified as "English Learners" (ELs). Eighteen students took both the pre-assessment and the post-assessment (six ELs). Whole class interactions were transcribed and student turns of talk were tabulated. We correlated individual students' pre-post gain scores with their average proportion of student talk turns per day attended. The Pearson Correlation of these variables is 0.168 with a 95% Confidence interval of (-0.600, 0.340). We also tested the Spearman and Kendall non-parametric correlations. According our models, there were no significant correlations between the adjusted proportion of turns and the students' percent gain on the assessment.

This result suggests that individual turns in whole class discussions are not adequate for measuring the effects of dialogic whole class discussions. Future work may consider more sensitive measures of student participation such as nonverbal participation in whole class discussion and verbal contributions to group discussions.

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CHINESE TEACHERS' VIEWS ON THE DIFFICULTIES OF IMPLEMENTING PROBLEM-BASED LEARNING IN CHINESE MATHEMATICS CLASSROOMS

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Chinese students had high achievement in mathematics assessment, but low attitudes and confidence toward mathematics (Mullis et al., 2020). Moreover, Chinese students' collaborative problem solving were found to be below average in PISA 2015 (OECD, 2017). These issues might be attributed to Chinese dominant use of conventional teaching (CT), which is a teacher-centred pedagogy focused on information purveying and discipline that may inhibit students' learning interest and creative thinking skills (McCarthy & Anderson, 2000). Problem-based learning (PBL), a group-cooperative, student-centred approach whose effectiveness on students' mathematical interests and collaboration skills has been suggested by many researchers might be a potential solution for addressing this issue. Thus, drawing on a case study and focus groups with six teachers from three Chinese secondary schools, this study used thematic analysis to investigate Chinese teachers' perspectives regarding the challenges of implementing PBL in Chinese mathematics classrooms.

The findings revealed that China's centralised, high-stakes examinations limit teachers' pedagogical choices from frequently conducting PBL, a form of pedagogy that teachers perceived to be more time-consuming than CT. All participants disclosed that adopting PBL would increase teachers' stress levels because it sets higher standards for teachers in curriculum design and class management. Moreover, all participants had concerns about the adaptability of low achievers to PBL, and considered PBL to favour high achievers who possess more autonomy and time-management capabilities, and thus can explore material more deeply through PBL. This study also found that, for the sake of ranking and admissions rates, those top-tier schools might not be willing to take the risk of thoroughly changing their pedagogy to PBL, but rather consider PBL to be a backup plan instead of the best possible choice of pedagogy. This implies that PBL might have a larger market among those schools ranked in the lower tiers because such schools have less to lose compared to the top-tier schools.

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TEACHER LEARNING IN THE CONTEXT OF SECONDARY MATHEMATICS LEARNING LABS: A CASE STUDY

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Researchers, practitioners, and policy makers recognize that participation in classroom mathematics discussions can promote students' mathematical understanding. Studies on teacher practices that foster productive dialogue and learning have shown how teachers support participation in the classroom (Webb et al., 2017). However, little is known about how teachers learn and develop the practices to support student interaction in their classrooms and the role of collaborative discourse in the process.

This research is part of a larger study that examined teacher learning in a networked improvement community. The present study investigates how teachers learn to support students to engage around mathematics in the context of the collaborative learning lab structure. In this study we ask, what type of learning is occurring through teacher discourse in networked improvement communities and how do teachers make sense of and learn about asset-based practices in the classroom? Data consist of detailed observations from one full-day learning lab (two inquiry cycles) with six secondary mathematics teachers. Data analysis involved identification of individual and collaborative teacher moves to support students' mathematical thinking. Data were further coded to identify two types of teacher moves: moves to support students to explain their mathematical ideas and moves to support students to engage with others' mathematical ideas.

The analysis revealed that 1) teachers asked more follow-up questions when working collaboratively to elicit students' mathematical thinking; 2) teachers who worked together asked twice as many questions to support students to engage with others' ideas than a teacher facilitating momentarily on her own; and 3) teachers who made sense of student thinking during an intentional pause with their colleagues, called a teacher time out, initiated twice as many turn-and-talk moves as a teacher who initiated the moves alone. These findings suggest that learning labs create opportunities for teachers to learn to look for and notice students' mathematical ideas and assets, enact in-the-moment moves that are responsive to the details of students' thinking, co-construct moves to support student participation, and consider student thinking as a resource for teaching and learning. These findings suggest a need to examine the wider implications of teacher collaboration in classroom-embedded professional development.

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POSTERS

TEMPLATE AND REPRESENTATION: FRACTION IN INDONESIAN AND SWEDISH TEXTBOOKS

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If students are to develop understanding of fractions, they need opportunities to reason about fractions by use of representations (Rau & Matthews, 2017). However, textbook tasks seldom require students to argue and reason creatively (Jäder et al., 2020). In 12 countries' mathematics textbooks, most tasks could be solved using an available template as guidance. This type of task does not require students to reason creatively since if the template is available, students are likely to copy the template to solve the task (Lithner, 2008). Only a small proportion of the tasks, often at the end of a section, require students to construct the solution method by themselves. Following the perspective by Lithner (2008) and Jäder et al. (2020), an investigation of task relatedness to a template in textbooks tasks about fraction is needed.

We will address two research questions: 1) What is the proportion and sequencing of tasks categorized as High Relatedness (HR), Local Low Relatedness (LLR), and Global Low Relatedness (GLR) task? 2) What is the proportion and sequencing of tasks that require interpreting, creating and/or connecting representations? Three 7th grade textbooks are analysed: one Indonesian and two Swedish textbooks. The method follows Jäder et al. (2020) but additionally identifies required uses of representations.

Preliminary results show that HR task has higher proportion than LLR and GLR task for the Swedish Textbooks, while for the Indonesian Textbook GLR is higher. There are different typical sequencing of tasks, such as a task is placed directly after an example or template, separately after some examples, and example or template cannot be easily located. Indonesian textbook uses more text-based representations than the Swedish. In the future, it is necessary to study how actually teachers utilize the textbooks and students learn from the tasks.

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REVIEW OF FACTORS OF INFLUENCE FOR IMPLEMENTING LARGE SCALE DEVELOPMENT PROJECTS

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Mathematics education as a research field is approximately 50 years of age. Since its beginning, studies of practice contribute to the development of theoretical constructs, which then come to live their "own life" within the scientific domain and as a consequence often fail to inform practice. While expanding their explanatory power, these constructs grow in complexity. Since teachers may have trouble adopting complex theoretical constructs, this may unintentionally increase the distance between research and practice, which imposes a dilemma between the theoretical constructs for explaining elements of the teaching and learning of mathematics, and the implementation of these constructs into the actual teaching practice. The purpose of this project is to identify factors of influence for successful implementation through the perspective of implementation research (Century & Cassata, 2016).

To get an overview of what is already known about the factors of influence (Jankvist et al., 2021) for successful implementation, we conducted a systematic literature review. We searched for manuscripts with implement* in the title and/or abstract, in the top 20 ranked journals in the field of mathematics education. We found 1,093 peer-reviewed articles. After the full-text screening, 95 papers remained.

Results show that unless the innovation is adapted to the cultural context and presumably to the system of beliefs and manifested practices of the context of implementation, change is unlikely to occur. Given the results, we urge that it be needed studies where old lessons are used for an a priori approach specifically designed to shed light on the factors of influence for belief change and organizational support, as well as explicated ways of evaluating these mechanisms.

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MULTIDIMENSIONAL ASSESSMENT OF FLEXIBILITY – AN APPROACH

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Creativity is usually measured in terms of fluency, flexibility and originality. For the operationalization of these components various suggestions can be found in the literature which all have in common the calculation of a summarizing numerical value (Levenson et al., 2018). At least for flexibility which describes the variety of solutions and approaches, this is accompanied by greater difficulties because this diversity is not always easily assessable and these assessments are not always binary decisions. This challenge can arise especially in (geometric) invention problems. In order to describe flexibility in dealing with such problems, we propose a multidimensional approach and concretize this idea for the invention of figural patterns.

In a recent study with 24 third graders, each student was asked during a semi-standardized individual interview to use cubes for inventing as many different figural patterns consisting of four figures as possible. The first figure should contain one cube, the second figure five cubes (Assmus & Fritzlar, 2022). To describe the diversity of the invented patterns, different dimensions could be identified using qualitative content analysis: Types of mathematical relations (e.g., constant increase), shapes (e.g., cross, bar), building principles (e.g., building in multiple layers), number of extension directions (e.g., extension in four directions), focus of the student's oral descriptions (e.g., number). For each interview, the number of different states was determined for every dimension. Based on these five values, a differentiated *flexibility profile* was created for the interviewed student and visualized in a radar chart (Figure 1). This representation facilitated a simultaneous consideration of all dimensions. The size, the shape and the position of the spanned area made it possible to draw conclusions about *the extent and the way* in which a student had shown flexibility in inventing figural patterns.

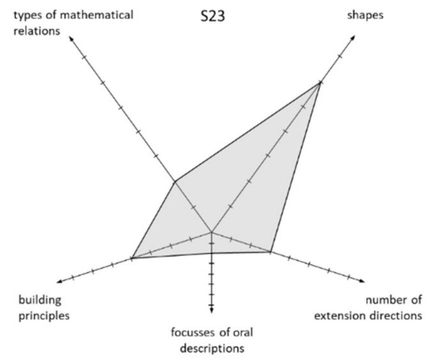

Figure 1: Radar chart

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USING LESSON ANALYSIS TO LEARN FROM TEACHING

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Considering the complexity of the teaching profession, it is important for the teachers to act as active learners in order to contribute to the students' learning. Lesson analysis as an approach to learn from teaching can be used in this sense (Hiebert et al., 2007; Santagata et al., 2007). It is important that the teacher educator puts different pedagogies into practice in the lessons in order to develop the lesson analysis skills of the pre-service teachers (PSTs). In this study, the lecturer leading the School Experience course redesigned the course assignments to include the lesson analysis tasks. PSTs analysed the lessons of the teachers they observed in real classroom environment. Based on observations and lesson analysis (that PSTs did), a class discussion on elementary teachers' mathematics teaching was held every week. It was intended that by utilizing this learning environment, pre-service teachers would be able to improve their lesson analysis skills as well as obtain knowledge about student-centered mathematics teaching. When the perspectives of PSTs on the significance of this learning environment to their professional development were examined, the importance of the class discussion emerged. The purpose of this research is to examine deeply how the class discussion takes place.

The sample of the research consisted of pre-service mathematics teachers (24 senior) studying at a state university in Turkey. The course lasted a total of ten weeks. The PSTs' end-of-term evaluation report, and the transcripts of the video recordings of the class discussions serve as data tools. The data collected was analyzed using qualitative data analysis methods. The strongest aspect of the class discussion setting is that it allows for discuss based on the teaching of four teachers. In this way, the PSTs access to information on the class levels they didn't observe as well as the teachers' approaches. PSTs exchanged ideas from peers and teachers about the situations they noticed about student understanding. PSTs were sometimes unable to interpret the students' learning difficulties. It is seen that PSTs give suggestions to improve teaching, however they are not always effective. Completing lesson analysis prior to the meeting, generated the ideas that will be presented in the discussion environment.

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ADAPTIVE TEACHING PRACTICES: AN EXPERIMENTAL STUDY ON PRE-SERVICE TEACHERS' NOTICING OF STUDENTS' THINKING

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Adaptive teaching includes teachers' adjustments to students' individual learning needs (student focus) and teachers' deliberate steering towards the content goals (goal focus) (Hardy et al., 2019). Although adaptive teaching has often been investigated, there is a lack of content-specific research capturing teachers' practices and responses with regard to the adaption to student and goal focus. Prediger et al. (2022) illustrated in a case study how adaptive teaching practices can differ in these two aspects (student and goal focus). However, there is a lack of experimental studies that examine to what extent teachers are able to choose such adaptive teaching practices.

We address the research question to what extent pre-service mathematics teachers notice students' thinking and choose adaptive teacher responses to incorrect student solutions. We were particularly interested in whether participants selected responses with student focus or goal focus, and the factors that guided their choices.

We developed ten vignettes, each of them presenting incorrect student solutions to fraction problems and three possible teacher responses varying in their student focus and goal focus. So far, we have collected data from $N = 22$ pre-service mathematics teachers who were previously taught about didactics of fractions. Data were collected in a computer-based experiment, where participants could choose between three possible responses to each vignette. Participants' verbal justifications of their choices were recorded afterwards (retrospective think-aloud).

Preliminary results show that the majority of participants selected the most adaptive response in only six of ten vignettes. Initial analyses of participants' verbal justifications suggest that their choices were strongly guided by motivational aspects of the possible responses, such as relevance to everyday life.

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DIFFICULTIES IN THE TRANSFER OF KNOWLEDGE BETWEEN MATHEMATICS AND PHYSICS IN THE RESOLUTION OF A TASK

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The relation between Mathematics and Physics cannot be dissolved (Abebe & Dirbeba, 2017). The results of different investigations (Planinic et al., 2012) with Secondary Education students (15-16 years) conclude that students do not realize that they are working with the same mathematical element when they solve tasks in a different context. The aim of this work is to identify the difficulties observed in the resolution of a task about the speed by high school students.

Knowledge transfer can be defined as the application of what you have learned in a situation to a different one (Rebello et al., 2017). The transfer of knowledge can only occur when a coherent and robust scheme has been built in the initial domain of learning. Using the theoretical framework proposed by Rebello et al. (2017), horizontal transfer occurs when students assign information read on a problem to an item of prior knowledge.

This research has been conducted with 104 students (17-18 years old) that have already studied the asked math and physics concepts. The students answered a task about instantaneous speed given the position function. Our methodological approach has been qualitative and inductive. Only 16 out of 104 students were able to solve the task (transfer of knowledge occurs). This shows the future difficulties the students will have to face in the study of STEM disciplines. The students have learnt the concept of derivative, but they fail to apply it to a statement out of the mathematic domain.

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FACTORS IN GEOMETRICAL TEACHING THAT SUPPORT CHILDREN'S UNDERSTANDING OF POLYGONS

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The children's understanding of polygon classification is linked to identifying common attributes in perceptually different figures (Battista, 2007), which involves recognizing parts of figures (dimensional deconstruction) and associating these to mathematical meanings (discursive apprehension) (Duval, 1995, 2017). The aim of this study was to identify factors in geometric teaching enacting dimensional deconstruction and discursive apprehensions, which support primary students' understanding of polygons.

We designed and implemented a geometric teaching experiment to support the children's understanding of polygons classifications in two 3rd-grade classrooms during 10 sessions. In each session, different problems were solved in whole-class, where the teacher questioned students to lead to more productive talk. The problems were focused on recognizing and justifying when a figure was an example of a class of polygon (and identifying when two different perceptually figures belong to the same polygon class); and construct polygons fulfilling some conditions. The teaching sessions were videotaped, transcribed, and analysed to identify instances in classroom lessons in which we could recognize different levels of sophistication in the children's understanding and the teacher's movements that seem to support it.

The findings reveal two interconnected factors featuring the teaching situations in which the children's understanding seems to be promoted: (1) tasks eliciting students' specific ways of thinking (features of children's geometrical thinking), and (2) the teacher movements that recognizing these features to encapsulate and des-encapsulate meanings of the geometric expressions (words) and strategies. These two factors provide information that allows helps to recognize the opportune moments for children's learning of polygon classification and the teacher's teaching decisions about which geometrical thinking is productive to pursue.

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REALISTIC NON-ROUTINE WORD PROBLEMS AND STUDENTS' FREELY CONSTRUCTED DRAWINGS

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Elementary students' difficulties when solving non-routine word problems are well documented (Verschaffel et al., 2000). To foster pupils' sense-making, they should be taught how to avoid the superficial strategy of meaningless execution of arithmetic operations with the data found in the text. The idea of providing clues and hints on how to make visual representations that help in problem solving has been addressed and proposed (Boonen et al., 2016). This study explores connections between students' freely constructed drawings and their solutions to non-routine realistic word problems. Seventy-four 4th grade students from two elementary schools participated in this study. The students solved realistic word problems and were invited to write down both their calculations and drawings for each task. Students' freely constructed drawings were categorized according to Berends and van Lieshout's (2009) terminology: bare, useless, helpful; while the numerical answers were scored as either realistic or wrong. E.g., "Students in a classroom form a single line in the P.E. class. Ash is the third from first; Onur is in the centre. Now that there are 10 students in between Ash and Onur, how many students are there in the line?" (Ulu, 2017, p. 561). Here the realistic answer is 27 which can hardly be obtained merely from the figures in the text, but a helpful drawing seems to be essential. Our results suggest that there are significant contingency coefficients between the type of drawing and obtaining a realistic solution for the task, ranging from .34 to .42. The possible causalities of these connections are analysed task by task since task characteristics influence the extent to which helpful drawings are accessible as freely constructed drawings. Furthermore, individual differences are to be considered when teachers plan to metacognitively scaffold students' solution steps.

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DEVELOPING RISK LITERACY AT SECONDARY LEVEL

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Risk-related questions and decisions under uncertainty determine everyday live. Thus, Gigerenzer & Meder (2014) call for fostering statistical thinking towards acquiring risk literacy in school. However, risk literacy has hardly found entrance into current curricula, particularly in Germany. The project *RisK-Design* addresses this gap by a design-based research approach (Bakker, 2019) of a lesson series to develop risk literacy in stochastics at Grades 9 and 10. Risk can be defined as both the probability and the expected value of an unwanted event, but risk is also relevant in decision-making related to statistical distributions. Risk literacy manifests in statistical reasoning within processes of risk decision-making and the manner statistical ideas as well as the own relation to risk are considered. Our research focuses on processes of decision-making providing a space for statistical reasoning. The questions are: *What are design principles for a lesson series that foster risk literacy? What are the facets of risk literacy? Which conditions promote the development of risk literacy in these settings?* We answer them by an iterative process of elaborating and exploring lesson designs in natural class settings including teachers as co-designers. We collected data in three iterative trials. Trials 1 and 2 took place in grade 10 of the same school allowing a first revision of the design by a delay of Trial 2. In Trial 3, the lesson designs were adapted to Grade 9 of another school culture. Data consist of video recordings of two focus groups and class discussions, students' worksheets and field notes in each class. Theoretically, inferentialism introduced by Brandom (2000) guides our design and our data analyses. An initial design principle to develop risk literacy requires the lessons to address multiple risk contexts (e.g., games, money, health) provided by various diagrams. Risk literate students handle risk situations based on statistical ideas; but they need a discursive classroom culture with space for open arguing. Teacher's confident handling of students' open-ended responses is a condition for practicing risk literacy since individual risk models are of huge importance. Results will provide a local theory related to a reference design of how to foster risk literacy.

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NOTICING FIRST GRADERS RELATIONAL THINKING: A STUDY WITH PRESERVICE TEACHERS

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Recognizing the importance of relational thinking in the context of Early Algebra (Steinweg et al., 2018) and that teaching must be responsive to children's mathematical thinking (Jacobs & Empson, 2021), this study intends to characterize preservice teacher's (PTs) noticing ability about first graders relational thinking, in two training tasks, in a teacher education course. Adopting a qualitative methodology, the research centers on a pair of PTs as participants. Data collection included audio and video recordings of the classes and the PTs' written productions on the tasks. A data analysis framework was developed by crossing the skills of describing and interpreting children's thinking with specific aspects of relational thinking. The results show that in the first task, while analysing children's written productions, the PTs noticed mainly formal aspects related to the representations children used. However, as PTs explored a classroom video, they deepened their analysis. In the second task, the PTs seem to be more conscious of the aspects that are essential to be noticed about relational thinking and focus their analysis on the way children explore the number sentences. The fact that the PTs had themselves the opportunity to solve number sentences similar to those solved by children and the collective discussions that took place in the course's classes seem to have contributed positively to their noticing skills, and thus can be considered in teacher education courses that intend to develop this ability.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR TEACHING VECTOR SUBSPACE CONCEPT

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Vector subspace concept is known for its high level of abstraction, and for this reason, many students have difficulty in its construction. Against this background, this exploratory research aims to evaluate a hypothetical learning trajectory (HLT) for teaching vector subspace concept. The vector subspace teaching process was started using the solution set of a system of linear equations (SLE) with infinite solutions. The research methodology was design research that consisting of three phases: preparation and design, teaching experiment, and retrospective analysis (Gravemeijer & Van Eerde, 2009). In the first research phase, we designed an HLT composed of four tasks on vector subspace concept in terms of the instructional design heuristic of emergent models (Gravemeijer, 1999) and the mechanism of reflection on the activity-effect relationship (Simon et al., 2004). In the second phase, corresponding to the teaching experiment, the designed HLT was used as a teaching tool in a Linear Algebra course composed of 30 first-year engineering students. In the third phase, we analyzed audio recordings and the written protocols of the tasks developed by the students to evaluate the designed HLT. The results show that the HLT on vector subspace concept helped students identify when a set was or was not a vector subspace. In particular, several students moved from their model of solution set of 2x2 SLE with infinitely many solutions that are vector subspaces to a model for identify which sets (other than vectors of $\mathbb{R}^n$) are vector subspaces.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR TRIGONOMETRIC EQUATIONS WITH INFINITE SOLUTIONS

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This research focusses on new forms of learning through the design and assessment of a hypothetical learning trajectory (HLT) on trigonometric equations with infinite solutions at university level. The research considers as the theoretical basis the instructional design heuristic of emergent models (Gravemeijer, 1999) and the conception of the HLT (Simon, 1995). Design-based research is used as a methodology. During the teaching experiment phase, the HLT was applied in a geometry course for the first year of an engineering. Audio recordings and the written protocols of the tasks were analysed.

The results provide evidence that the HLT contributed for students to learn how to solve trigonometric equations with infinite solutions. We observed that the students progressed from a known model-of mathematical activity (locating angles in the cartesian plane and the values of the trigonometric functions of known angles) to a more formal model-for reasoning (concept of trigonometric equations with infinite solutions). We recognize where the "model-of" and "model-for" are visible in the data, which is fundamental in Gravemeijer's work. However, the students showed drawbacks when interpreting the results in a real context.

This HLT may have some implications for the design of tasks in Geometry, as it provides a way to design a sequence of tasks for concepts that are difficult to learn due to their high level of abstraction. We hope to serve as a guide to design other HLTs that help students advance their learning in the area of geometry.

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INQUIRY-BASED TEACHING OF CONTEPT OF UNCERTAINTY IN COLEARING OF KINDERGARTEN TEACHER

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The purpose of this study was to explore the development of kindergarten teachers' knowledge and teaching practice of the concept of uncertainty and to develop a model for the inquiry-based teaching of the concept of uncertainty. In this case study, data on 20 teachers from a kindergarten, including video and audio recordings and interviews, were collected. These teachers learned together in a 3-month professional development community for 2 hours per week.

Teachers' professional development in co-learning

The teachers underwent 3 stages of professional development. Stage one, the teachers formed a co-learning community to learn theory of inquiry-based teaching and knowledge of the concept of uncertainty. Stage two, the teachers identified the differences between conventional teaching models and the inquiry-based teaching model. Stage three, a model for the inquiry-based teaching of the concept of uncertainty was developed, applied in class teaching, and modified accordingly.

Reflection on the practicing of inquiry-based teaching

Inquiry-based teaching was to help young children learn the concept of uncertainty and reasoning skills. However, the teachers did not fully understand the concept of uncertainty; they considered the concept too abstract to apply directly in teaching. Through their discussions in co-learning community, the teachers interpreted this concept as a variable; that is, the children had a various of views on the events they encounter in life. This increased the interactions between the teachers and the children, and enabling the teachers to develop the model for the inquiry-based teaching of the concept of uncertainty for their children.

Teaching effectiveness

Through co-learning, the teachers had their knowledge and teaching skills on the concept of uncertainty improved. The results revealed that after the teachers participated in the co-learning and action research, they expressed their intentions to conduct child-centred inquiry-based teaching. And built a teaching model on the concept of uncertainty was developed, comprising the steps of speculation, discussion, verification, and conclusion.

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STUDENTS' MATHEMATICAL SENSEMAKING IN CLASSROOMS LOCATED IN CONTEXTS OF DISADVANTAGE

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This study focused on the opportunities available for students to engage in mathematical sensemaking during their lessons. Sensemaking is a critical component of mathematics learning, as it allows students to connect new mathematical ideas, procedures, and practices to their existing knowledge (Battista, 2017). This qualitative study was located in a South African context of disadvantage with classrooms that usually have between fifty or sixty students, where traditional forms of instruction predominate, and instances of inadequate resources are common. I was interested in the opportunities available for students to develop an understanding through sensemaking of the mathematical content they were learning. I formulated the following research question: What opportunities do South African students in contexts of disadvantage have to engage in sensemaking during mathematics lessons?

Seven high teachers (four male and three females) and their students from different public high schools in Gauteng, South Africa, were conveniently selected to participate in the study. I audio recorded three lessons of 45 minutes for each teacher. The audio recordings were transcribed, and I analysed the transcripts thematically.

The major finding from the study was that the lessons observed were teacher-centered, which resulted in the learning context not affording the students opportunities for mathematical sensemaking. Teachers in the study showed and explained the mathematics without providing students opportunities to make mathematical connections. The lessons appeared to be well structured, with students being led through the content straightforwardly. Students responded to teachers' questions in a chorus, and there was no evidence of students making mathematical connections. The Initiation-Response-Evaluation sequence was prevalent in the observed lessons. The teachers initiated (I) with questions, to which the students responded (R), and the teachers evaluated (E). Teachers frequently asked questions that required students to regurgitate already known information. There were no instances where teachers asked students probing questions or needed them to clarify their understanding. This study is an important contribution to the mathematics education research literature because it provides insights into opportunities available for students in contexts of disadvantage to engage in mathematical sensemaking.

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POLYHEDRA CONCEPTUAL KNOWLEDGE WITH NEOTRIE VIRTUAL REALITY GEOMETRY SOFTWARE

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The educational research is increasing focus on the use of augmented and immersive virtual reality in the classroom. The possibilities of visualization and interaction with these technologies, as that offered by the immersive virtual dynamic geometry systems 3D "NeoTrie VR", are enabling new approaches to the teaching and learning of 3D geometry, although there is still a lack of empirical studies (Rodríguez et al., 2021).

We present a quasi-experimental investigation to detect if the use of NeoTrie in practical activities improves the conceptual understanding of polyhedra. We analyze seven specific items incorporated into the final assessment test of the subject "Geometry and Measure in Primary Education", of Primary Education Teacher degree at a Spanish university. The control group (71 students) received a traditional teaching with practical activities using pencil and paper and structured material. The experimental group (132 students) used NeoTrie and Wooclap activities in their practical activities.

The t-student contrast performed on the final test scores shows that there weren't significant differences between the two groups of students at the specified $p<.05$ level, $t(201)=-1.021$, $p=.309$, $d=-.30$, 95% CI [-.90, .28]. On the contrary, t-student contrast of set of seven items about polyhedra in the exam revealed significant differences ($t(201)=-3.91$, $p<.001$, $d=-.49$, 95% CI [-.74, -.24]). Students working with NeoTrie were more able to identify relevant properties of polyhedral (non-planar faces are not allowed) and irrelevant properties (prism bases can appear in any position and be any type of polygon, polyhedral angles can be concave or convex).

Accordingly, it seems that NeoTrie has improved the students' polyhedral conceptual knowledge. Nonetheless, the role of NeoTrie software for the development of spatial abilities needs further research, since spatial conceptual abilities are an important factor for geometry learning.

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PRACTICES OF NOVICE TEACHERS IN INTEGRATING DIGITAL TECHNOLOGY IN GEOMETRY TEACHING

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Drijvers et al. (2014) state that a challenge to integrate technology is to disseminate experiences in order to support mathematics teachers who start using them. In addition, Wickstrom and Langrall (2018) determine the uses that teachers make of the hypothetical learning trajectories (HLT), including, attending to the mathematical thinking of the student and designing tasks. In this exploratory research, we are interested in characterize the teaching practices of novice teachers in the integration of digital technology, when they use an HLT. For this, we apply an HLT that included dynamic geometry tasks and described its use, from the Instrumental Approach and the Technological Pedagogical Content Knowledge Model.

The participants of our study were two secundary mathematics teachers and their students. We collected data through semi-structured interviews and class recordings. Four metacategories emerged from the data analysis: teachers *guide* technical solutions and task management, teachers *report* on the development of tasks and the mathematical object, teachers *ask* for the expression of mathematical ideas and teachers *explain* about the concept mathematical. These actions were performed using the digital resource. Finally, we relate the metacategories with the elements of the HLT and the theoretical approaches.

The results indicate that the pedagogical technological knowledge of the content determines the characteristics of the teacher's practice in the development of the tasks. Likewise, the orchestrations called explanation and discussion of the screen, account the class management by novice teachers. In this sense, the design and use of the HLT contributes to the integration of digital technology by teachers because they took advantage of the learning tasks adapted to the applet and focused on the use of the resource to promote learning.

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COMPONENTS OF THE SPATIAL SENSE IN PISA ACTIVITIES FOR SECONDARY EDUCATION

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Research on geometry education has included spatial thinking and reasoning and visualization in geometry (Jones & Tzekaki, 2016). We present findings related to spatial thinking and visualization skills. Research in the field of spatial skills indicates a low development of the abilities related to spatial orientation, spatial transformations and relationships, as well as the understanding of spatial dimensions and positions.

We present findings that seek to determine the components of spatial sense (NCTM, 1989) that the written assessments proposed by PISA demand, in order to be solved by 15-year-old students in Uruguay. Among the activities aimed at the command of the mathematical culture released by PISA tests, in 2003 and 2012, when Uruguay participated, we analyzed those related to the space and shape content area. To examine the components of the spatial sense, each of the seven activities are solved in detail, and tables that deal with the use of geometrical concepts are designed: concepts of the figures, features of the shapes, geometrical relationships, location and movements, orientation; besides, the visualization skills required to solve them: motor-eye coordination, figure-context perception, conservation of the perception, perception of the position in the space, perception of the spatial relationships, visual discrimination and visual memory. Results emphasize the differentiation between spatial visualization and spatial orientation (Diezmann & Lowrie, 2009). We conclude that the visualization skills needed in the assignments are: figure-context perception, perception of spatial relationships, and conservation of the perception without the spatial orientation.

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ELICITING MATHEMATICAL KNOWLEDGE IN PRE-SERVICE PRIMARY SCHOOL TEACHERS: A CONCEPT CARTOON ON DIVISIBILITY

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Vignettes promote pre-service teachers' reflection and discussion of authentic classroom situations (Buchbinder & Kuntze, 2018; Fernández et al., 2018). Among vignettes, Concept Cartoons can be designed to elicit mathematical knowledge in pre-service primary school teachers (Samková, 2020). The aim of this study is to analyse what kind of knowledge pre-service teachers reveal when they participated in a Concept Cartoon on divisibility as an introductory task within a mathematics content course. Participants were 51 pre-service primary school teachers (PPTs). The Concept Cartoon on divisibility consists of a group of four student teachers, a divisibility activity, and four bubbles with different correct and incorrect statements. PPTs had to answer three questions (1) What thoughts could be behind the student teachers' thinking? (2) How could you help the other student teachers to correct their answers or to improve their argumentation? (3) Write your solution into the empty speech bubble. We performed an inductive analysis generating categories. The categories showed PPT's knowledge/lack of it regarding the key concepts implied in the activity, the type of arguments they provide to describe the thoughts behind the student teachers' thinking and different alternative ideas they proposed to improve their argumentation. Our results point out the potential of vignettes in teacher education programs.

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MATH COMPETENCE IN STUDENTS WITH AUSTISM

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Students with autism spectrum disorder (ASD) often show difficulties in learning mathematics. In order to further explore this issue, the formal and informal mathematical competence of 10 elementary students with ASD diagnosis is evaluated, observing age-appropriate informal mathematical knowledge in most participants but very varied levels of formal knowledge. The implications of the results for teaching children with these characteristics are discussed.

Given the positive effects of inclusive education, students with autism spectrum disorder (ASD) are increasingly incorporated into mainstream educational programs. Performance in the area of mathematics tends to be significantly lower in students with ASD compared to the general population. Given that early mathematical competence is a strong predictor of later academic performance, a better understanding of the factors that hinder ASD students' learning is necessary to improve their academic qualification and, consequently, their quality of life.

An exploratory and descriptive study has been carried out to deepen the informal and formal mathematical knowledge of 10 elementary students enrolled in 2nd to 4th grade, with a diagnosis of autism and without intellectual disability, by applying the TEMA-3 mathematical competence test (Ginsburg & Baroody, 2007). The data show age-appropriate informal mathematical knowledge in most students but widely varying levels of formal knowledge.

The results observed are in line with other works (Chen et al. 2019) that identify different mathematical competence profiles within students with ASD without intellectual disability. In particular, what was observed in the lower performance group could have implications for the construction of future mathematical learning of greater complexity. The work has implications for teaching students with these characteristics as they allow us to identify the categories of mathematical knowledge with which they show more deficiencies.

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ENGAGING STUDENTS TO LEARN CURRICULA CONTENTS WITH STEM EDUCATION

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Nowadays, school must provide knowledge through diversified pedagogies to prepare its students for an increasingly demanding society (Gadotti, 2020). In this regard, STEM education enables the development of cognitive and socio-affective skills, necessary to face the challenges of the labor market (Stracker et al., 2019).

This Poster presents STEM tasks implemented in two 6th grade classes of Mathematics and Natural Science, lectured by the same teacher (10 to 11 years old). In one of the classes (B class), it was used hands-on STEM tasks based on the school plant, which is a real scenario to solve real problems, as recommended in STEM education. In the second class (A class), mathematical contents were lectured the traditional way by reviewing contents related to areas and perimeters of the square, rectangle and triangle, and the area and perimeter of the circle. In class B, the same contents were worked with Google Earth to compute the areas and perimeters of the sports field (rectangle), the pavilion where they normally have classes (square), outdoor space (triangle) and the warehouse (circle). In addition, they were asked to build a model of their ideal school including outdoor plants and trees, among others. With a mixed methodology (Creswell, 2012), qualitative and quantitative, based on participant observation and questionnaires applied to students, before and after implementing the tasks, it was verified that class B revealed a greater enthusiasm and motivation in carrying out the tasks. It is concluded that the hands-on STEM approach was more effective to engage students to learn school curricula.

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FUNCTIONAL THINKING: CONCEPTIONS OF MATHEMATICS EDUCATORS – A FRAMEWORK FOR ANALYSIS

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Functional thinking as a "unifying principle" or "concentration principle" across K-12 curriculum can be described as a prototype of a fundamental idea and has caused many discussions about its importance for educational issues (Vohns, 2016). In the literature, there is a variety of definitions for functional thinking (Pittalis et al., 2020) what might cause differing understandings of this important concept. In order to inform the developmental work of the Erasmus+ project *FunThink – Enhancing functional thinking from primary to upper secondary school*, mathematics educators' conceptions of functional thinking are explored. Therefore, views on functional thinking from mathematics educators in European countries were gathered to answer the research question *which conceptions on functional thinking do mathematics educators in the five countries Cyprus, Germany, the Netherlands, Poland, and Slovakia hold?*

To answer this research question, a total of 34 semi-structured interviews were conducted with mathematics educators in all five participating countries. This poster displays the inductively and deductively determined categories of a framework as well as exemplary results of the qualitative content analysis of the interviewees' conceptions on functional thinking. The framework contains categories concerning the four aspects of functions, addressing functional thinking inside and outside of mathematics, patterning, the role of representations, mastery of semantic and syntactic elements, demarcation of this concept, and other interpretations. The determined categories represent the variety of definitions in literature and the diversity of conceptions of the interviewees. The presented framework will serve as a base for further analysis.

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LEARNING TO ANALYSE STUDENTS' PROBLEM-SOLVING STRATEGIES WITH CARTOON VIGNETTES

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Problem solving is a core mathematical activity and part of curricula and national standards worldwide. Studies show that the flexible use of strategies (e.g., adapting the strategy to a given task) leads to higher performance in problem solving but also that many children struggle with applying strategies in a flexible way (Elia et al., 2009). The question of how future teachers can be supported in their learning about different strategies and their use in the mathematics classroom is consequently of high relevance. In a university course (n=42 participants), we provided a unit focusing on the use of primary-school students' problem-solving strategies. Since cartoons have the potential to represent classroom practice in a systematic and theory-based way (Friesen & Kuntze, 2018), we designed two types of cartoon vignettes to support the participants' learning: short cartoons, each illustrating a problem-solving strategy (e.g., work backwards, draw a picture) and more complex cartoons providing the opportunity to analyse how students use different strategies or struggle while solving a non-routine problem. The evaluation of the course was based on the analysis of a complex cartoon (pre-post) and a questionnaire. Our findings show that the participants perceived the cartoons as valuable learning opportunities. Learning to analyse students' problem-solving strategies could be supported but also needed guidance and specific support.

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DIFFICULTIES IN THE RESOLUTION OF TASKS ON THE BASICS CALCULUS CONCEPTS

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The importance of Calculus and the difficulties in understanding its basic concepts have motivated the development of numerous investigations that address different aspects of this problem from various theoretical approaches. These studies indicate that the construction of the understanding of the concepts of Calculus is limited to the routine application of the algebraic rules in situations of artificial contexts or purely symbolic (Kouropatov & Dreyfus, 2014). Orton (1983) mentions that this practice causes a set of errors and difficulties observed when comparing students' performance when they solve routine problems versus those requiring a conceptual understanding.

In this study, we focus on the visible, namely, in analyzing the errors that students make to identify possible difficulties. For this, we consider the framework proposed by the APOS theory (Arnon et al., 2014), which has been operationalized through the use of mathematical elements and the logical connections that students establish when they solve tasks on the concepts of derivative and integral. The participants of this study were 40 first-year engineering students. The data were collected during the second semester of 2019 and correspond to the students' productions obtained from the application of a questionnaire, which was made up of 6 tasks that address different aspects related to the derivative and integral. We analyzed the answers regarding the mathematical elements and logical relationships connected by the students in their resolution process. From this, we established and interpreted the errors categories using the theory. The results show that there are difficulties associated with the construction of reversal processes that is manifested in a large number of errors associated with the use of logical equivalences between mathematical elements

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OBSERVATION PROTOCOL AS A PROFESSIONAL DEVELOPMENT TOOL FOR IN-SERVICE TEACHERS

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Teacher training and its influence on student practice and learning is a priority in Mathematics Education research (Bakker et al., 2021). Classroom observation of teaching processes can influence in the quality of instruction (Bostic et al., 2021). Several instruments have been developed for classroom observation in order to improve teaching practices (CLASS – Classroom Assessment Scoring System, MQI – Mathematical Quality of Instruction, Promate, among others). In this work, we study how some in-service primary and secondary school teachers understand and implement in their classess a training about non-routine mathematics problems, look for strengths and weakness of the training, by the analysis of their discussions.

We made an interpretive analysis of the dialogues developed by ten teachers throughout four video-recorded training sessions: three sessions before implementation in the classroom, and one session afterwards. Teachers were trained about a protocol for classroom observation, which is centered in math problem solving, focusing in four dimensions: mathematical strategies and representations, mathematical thinking, mathematical productions, and mathematical learning through problem solving. We analyse the discussions during the observation protocol develop and how teachers use it to reflect what happens in the classroom.

Results show that the observation protocol allowed the teachers to observe and reflect on aspects developed in the training, and some absences or weakness. On the other hand, some teachers said that they made some adjustments to their practice trying to follow the protocol used. Therefore, the observation protocol is a powerfull tool for reflecting and improving teaching practices.

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REPRESENTATION OF FRACTION DIVISION — EXPERIENCE IN A TRAINING WITH TEACHERS WITH FOCUS ON THE REFERENCE UNIT

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The number of publications on division of fractions has increased and, although studies on representation of division of fractions only address the different types and frequencies of models developed by teachers, they do not analyze whether the models focus on units of reference, and few studies have examined the teachers' understanding of units of reference and representations (Lee, 2017). Thus, our research is based on the following questions: Are teachers able to represent division operations in the context of fractions? Do teachers understand the units of reference to which numbers refer in their representations?

This work focuses on the knowledge revealed and mobilized by elementary school teachers (who work with students from 7 to 14 years old) through a task developed by the authors. The sources of information were produced in an online teacher training, lasting 6 hours, in 2 days, through questionnaires, observations during the online training, notes, productions of activities and recordings.

The teachers in this study had difficulties in performing the representations of fractional divisions, especially when the divisor was fractions. The investigation points out that this is since teachers do not have a broad understanding of the concept of division, partitive, and measure; as well as a weakness in flexibility with the reference unit, as this affected the ability to incorporate their division fraction length representations; in addition to the lack of experience with this type of task. Thus, it is necessary that teachers have experience with multiple representations to develop flexibility with the reference unit that allows the connection between symbolic notation and representation.

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MODELING PERFORMANCE IN VISUAL RATIO COMPARISON

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Research in numerical cognition investigates the mental mechanisms used by people to work with numbers and grasp mathematical concepts. A core concept in numerical cognition is the *approximate number system*, believed to support people's ability to construct mental representations of quantities and predict mathematics achievement (Halberda et al., 2008). Some authors claim that this ability extends beyond natural numbers to include ratios of numbers, and that the ability to visually discriminate ratios of quantities predicts performance with symbolic numbers (Matthews et al., 2016).

A widely debated issue regards the nature of these mental mechanisms. Many studies have shown that performance in comparing visually presented quantities is better modeled by using the ratio distance between numbers ($dist(m, n) = |\log(m/n)|$) instead of their linear distance ($dist(m, n) = |m - n|$). I will present data from young adults (N=24) performing a visual ratio comparison task. Participants viewed pairs of sets of dots for 1000 ms and decided which pair had a higher ratio of a given color (Fig. 1). Results showed that ratio distances provide a better description of participants' performance (Fig. 1), implying that this model is better not only for visual comparison of quantities but also of ratios of quantities. This indicates that the mental mechanisms underlying ratio comparison work similarly to those of number comparison. I will discuss the implications of these results for the research on intuitive work with ratios.

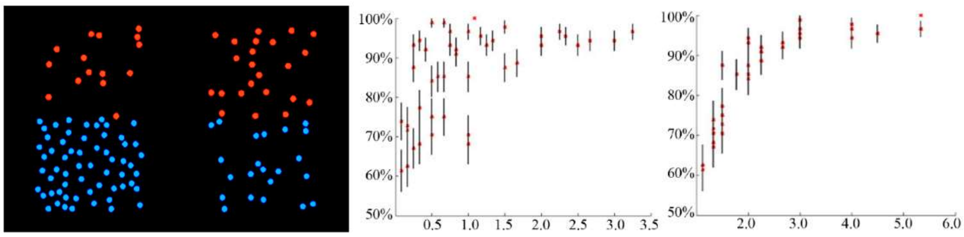

Figure 1: Example stimulus (left panel). Participants' accuracy depicted as a function of the linear distance between ratios (middle panel) or their ratio distance (right panel)

Acknowledgments

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ASSESSMENT OF THE SKILL “SELECTING DIGITAL TECHNOLOGY” OF MATHEMATICS PRE-SERVICE TEACHERS

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Since the role and the availability of digital technology (dT) in society is growing, educators need to increasingly more often decide when and what dT (Clark-Wilson et al., 2020) to use in their teaching. Given the importance of those decision-making skills (Tabach & Trgalová, 2020, p. 237), they need to be developed and the success of those development activities needs to be measured using reliable and objective assessment instruments and not by applying the frequently used self-assessment instruments. Based on an interview study comparing experts and novices (Gonscherowski & Rott, 2022), we have developed quantitative items to assess those decision-making skills. To evaluate the items, we conducted a pilot study with all bachelor degree mathematics pre-service teachers who participated in a fourth semester seminar introducing dT for the use in teaching. Participants (n = 26) were required to reason for or against the use of dT in context of learner age, learning content, and teaching phase. Because the decision whether to use or not use dT cannot be coded as “right” or “wrong”, following the approach of the interview study, the arguments are evaluated by level of argumentation: (i) no arguments (ii) overly general arguments (iii) substantiated arguments and if so suggested, the provided dT. Three raters independently coded the data; the Inter-rater Agreement varied from $\varkappa = .85$ to $\varkappa = .95$ and was in substantial acceptable range. The findings firstly confirm that different level of argumentation can be discriminated and secondly using pre-/post-measures a one-side Wilcoxon showing indications of skill increments which can be attributed to the seminar content ($z = -2.459, p = 0.0072$ with an effect size of $r = 0.482$ corresponding to a medium effect). Thirdly the knowledge of dT and their assignment to a teaching phase aligns with the results of the interview study. There are several limitations to this study, for one it is limited to pre-service teachers at one university in Germany and second the results are based on a small sample size. To ameliorate methodologically some of those limitations, we are planning a study with a larger sample size.

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GENERALIZATION TROUGH FUNCTIONAL TASKS BY A STUDENT WITH AUTISM

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Algebra is one of the subjects studied in secondary school with which many students struggle. To try to mitigate these difficulties, several studies suggest to start working with functional tasks from early ages, as they provide an appropriate context for developing generalization and algebraic thinking (Kieran et al., 2016). A task involving a function in the context of algebraic thinking usually show the first terms of an increasing sequence of natural numbers. The students evidence generalization and functional relationships when finding other terms of the sequence (near, far, or general terms).

Students with autism spectrum disorder (ASD) are increasingly incorporated into mainstream educational programs (Whitby, 2013) and often require additional support for mathematical learning. We present an exploratory study with a 9-year-old-student with ASD aimed at mobilizing generalization strategies through functional task involving near, far and general terms. Following previous adaptations of methodologies for ASD students, the instruction combined the use of manipulatives and self-instruction lists with visual support.

The results show that the student relied on the use of manipulatives and visual support for the near terms evidencing recurrence strategies, and managed to advance towards correspondence strategies for far terms, general term and inverse relation by relying on the use of tables and guided by the self-instruction list. Identifying aspects of instruction that may have facilitated the development of generalization strategies provides clues as to how to help develop functional thinking in students with these characteristics.

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A PHENOMENOGRAPHICAL STUDY OF CHILDRENS' SPATIAL THOUGHT WHILE USING MAPS IN REAL SPACES

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Using maps in real spaces is one way to challenge children to reason about conflicting frames of reference, that is, to reason about spatial relations and how they dynamically change (e.g. Liben, 2006). In addition, map-based orientation tasks such as locating on the map itself (self-location), determining the viewing direction (self-orientation), and navigation might be effective not only to foster children's map reading abilities but also to enhance their cognitive abilities being required for solving written spatial tasks (Heil, 2021). However, only little is known about how children use maps in real spaces and how they reason about the changing spatial relations when solving map-based orientation tasks. This research project aims to investigate qualitative variations in children's individual approaches on solving these tasks.

The sample comprised three children of grade 4 (aged 9 to 10) who completed a "treasure hunt" on the campus of the university: an experimental setting involving self-location, self-orientation, and wayfinding tasks. They were videotaped using three different camera angles. Once they completed a self-location and self-orientation task, they were asked to explain their approach. We then analyzed children's specific behaviour such as map aligning, looking around, or stopping by similar to the phenomenographical study of Gerber & Kwan (1994). Three different approaches were determined from these behavioural observations: (1) the uncoordinated and disoriented approach, (2) the careful and sequential approach, and (3) the visualized and coordinated approach. These three approaches also reflect their different abilities to maintain orientation, to constantly visualize spatial relations, and how they reason about transforming information from the map to real space and vice versa verbally. Further research, however, is required to investigate the underlying cognitive processes reliable for each of the three approaches. The poster also visualizes implications on how to enhance spatial thought in different educational settings.

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HOW PRIMARY SCHOOL CHILDREN UNDERSTAND THE SAMPLE SPACE LINKED TO DIFFERENT TYPES OF EVENTS

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Probability has been introduced recently in primary school level in Costa Rica (MEP, 2012), where no previous research until date related to children competence with probability has been carried out. In order to fill this gap, we summarise an exploratory study in which we analyse primary school students' understanding of various types of events. Understanding the sample space and the different types of events is a prerequisite for the child to be able to assign and compare probabilities, which requires thinking about the set of favourable and unfavourable cases as a whole set of possible cases (Nunes et al., 2014). However, little research has focused on children's construction of the sample space for a simple experiment. Thus, a sample of 55 6th degree students was given two tasks, in which they were asked to create a sample space such that, in a hypothetical game, the event *Mary wins the game* was certain, possible, equiprobable or impossible. In one task (urn context), the children were given a picture with an empty urn they could fill with black and white balls (Mary wins if she gets a black ball), and in the other task they were given a picture of a roulette with four equally sized sectors to number with 1 or 2 (Mary wins if she gets number 1).

We classified the children' drawings according to the type of event that could be obtained. There was a good intuition of the possible event in both urn and roulette context (92.8% and 96.4% of correct responses, respectively), and equiprobable event in both urn and roulette context (70.9% and 90.9% correct responses, respectively). However, only 34.5% (roulette context) and 29.1% (urn context) of children built a correct sample space for the certain event, which was interpreted generally as very likely. In the same way, the impossible event was only correctly understood by less than 25% children, being the most common interpretation as an unlikely event in both contexts. These results support other previous research that indicates the difficulty for students of the concepts of certain and impossible and point to the need to spend more time in the exploration of sample spaces and different types of events.

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ON THE CREATION OF RESEARCH-PRACTICE PARTNERSHIPS TO SUPPORT MATHEMATICS TEACHING AND LEARNING

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Given the depth and complexity of challenges in education broadly and mathematics education more specifically, scholars have recommended collaborative approaches known as research-practice partnerships (RPP) (Coburn & Penuel, 2016). RPPs create a new space for collective work that requires us to abandon the *research to practice continuum* metaphor and, instead, view research and instructional efforts as bidirectional, with researchers, K-12 educators, and community stakeholders capitalizing on their complementary areas of expertise learning from each other and forging new ways to improve mathematics teaching and learning. Importantly, RPPs use intentional strategies to foster productive relationships among researchers, K-12 educators, and community stakeholders so that research is relevant to all involved.

In this poster presentation, we draw on three projects to make sense of our learning in developing such partnerships. One project focuses on a mathematical professional development design and implementation involving elementary teachers in a rural school district ("Math Counts"). A second project is ongoing and involves community engagement supporting the integrated-mathematics learning of students and their families ("STEMhub"). A third project is ongoing and involves recruiting and supporting the preparation of secondary mathematics teachers for rural schools districts. All projects take place in the context of the Appalachian region of the United States. These projects draw on different research methodologies. However, they all leverage research-practice partnerships in different ways. For the purpose of this poster presentation, we make sense of our learning across these projects in terms of initiating, developing, and establishing research-practice partnerships. Preliminary analyses indicate that the themes of intentionally designing key artifacts and norms for communication, developing taken-as-shared ideas about the key constructs such as learning and mathematics, and positioning K-12 educators and community stakeholders as experts of their own experiences with mathematics and community challenges are significant aspects of establishing research-practice partnerships.

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USING VIDEO-BASED MODULES TO SUPPORT DEVELOPMENT OF MATHEMATICS FOR TEACHING

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Mathematics teachers must hold a flexible conceptual understanding of mathematics in order to support students effectively (Silverman & Thompson, 2008), and this understanding has been linked to student achievement (Baumert et al., 2010). In a teacher education program, there is limited time to develop these understandings when many do not enter a program already holding (e.g., Holm & Kajander, 2020). In order to support the development of mathematics for teaching, video-based modules were created in this research to allow for extra time in developing conceptual understandings by using an online platform. This poster considers two of the participants from the study: one who maintained a procedural view of mathematics throughout, and one who often switched to show a more conceptual view of mathematics. Although both showed gains in overall understanding of mathematics, only one showed gains in what would be considered conceptual understanding. In this poster, I present pre- and post-solution methods as exemplars from the two cases, as well as the structure of the online modules for discussion. Considering the two cases, beliefs may have played a role in the responses of the participants, since if one does not focus on supporting mathematical beliefs about the benefits of conceptual knowledge, the beliefs will "limit the value of learning the content" (Wilkins, 2008, p. 157). Although the online modules promoted different ways of solving mathematics problems, there was not an explicit portion that emphasized the importance of conceptual understanding. This poster opens a dialogue about making changes to the online modules to consider mathematics for teaching and not just general gains in knowledge of mathematical procedures.

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STUDYING CONCEPTIONS OF THE DERIVATIVE AT SCALE: A MACHINE LEARNING APPROACH

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In this poster presentation, we present a novel approach to study student conceptions of the derivative at scale using methods from machine learning (ML)—automated conversation disentanglement and natural language processing (NLP). Prior work on student conceptions have generally relied on studying small groups of students, and these ML methods provide a way to analyze mathematical conversations at scale, allow for patterns in conversations to emerge, and can help identify relationships between emerging conceptions.

In a seminal report, Zandieh (2000) developed a theoretical framework for analyzing students' understanding of the derivative concept. She used cognitive interviews with nine high school students to describe their understandings of the derivative and found the following conceptions of the derivative: taking derivatives symbolically, derivative as slope, derivative as velocity, derivative as rate or rate of change, derivative as a graph, and derivative as a formal definition or limit of difference quotient. Balacheff and Gaudin's (2002) conception model is used to create an operational (searchable) representation of each of the conceptions listed above. By using conversation data from an open-access, online mathematics tutoring platform we code for operators (what people appear to do in their work) and controls (what people take to be true when they do their work) in order to build a training set aimed at training a set of machine learning models to classify the rest of the conversations (~700,000 messages) by conception type, as well as find connections between different conceptions of the derivative. In our presentation, we (1) show how automated disentanglement models can be trained then deployed to disentangle chat logs into separate conversations and (2) provide details of this preliminary coding work to provide evidence how ML methods can be amenable for studying student conceptions.

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AWARENESS OF BUILDING UP NEGATIVE KNOWLEDGE – A VIGNETTE-BASED STUDY ON PRE-SERVICE TEACHERS' REACTIONS TO MISTAKES

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Mistakes in the mathematics classroom are opportunities for learning – in particular for building up negative knowledge (Oser & Spychiger, 2005) – which can (in short words) be described as knowledge regarding the boundaries of the "correct", examining what is incorrect and (possibly) why. Such negative knowledge is also helpful on a metacognitive level (Mevarech et al., 2018) for learners monitoring their mathematical thinking and activities. With negative knowledge, learners may be more successful in detecting corresponding mistakes in the future. When teachers react to students' mistakes, they should hence be aware of the learners' development of negative knowledge – beyond emphasising the corresponding "positive" knowledge, i.e. the knowledge needed to "do it correctly". However, relatively little is known about pre-service mathematics teachers' awareness (Kuntze & Friesen, 2018) of building up negative knowledge. Corresponding to this research need, this study explores whether pre-service mathematics teachers can draw on an awareness of building up negative knowledge when they are asked about reactions to a mistake shown in a vignette on a fraction calculation situation. Results from this study with 37 pre-service teachers indicate that although PTs could identify the mistake shown in the vignette, such awareness needs to be supported, with specific vignette-based learning opportunities.

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STRATEGY VARIATION IN COUNTING AND PATTERNING: PART-WHOLE REASONING

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Understanding pattern structures afford opportunities to develop generalisations and promote part-whole reasoning (Vanluydt et al., 2021), which is critical for understanding commutativity, number sense and algebraic development. Understanding pattern structure is mostly studied in visual patterns rather than numerical. Hunting (2003) propose that part-whole reasoning is vital for understanding numerical pattern structure in counting. As number sense is supposed to be an interplay between number and shape and by implication between numerical and visual geometric reasoning, we hypothesise that part-whole reasoning affects both counting and patterning. As such, categorising students' strategy use as one-to-one and relational part-whole strategies can be appropriate investigating students' counting and patterning proficiency.

Semi-structured task-based counting and patterning interviews were used to explore strategy use as part-whole reasoning in 75 grade 1 Norwegian students applying the following tasks: 1) counting and patterning repeating patterns (e.g., ABABAB, □ΔΔ□ΔΔ□ΔΔ), and 2) counting and patterning growing patterns (e.g., 1 3 5, ABAABAAAB). One-to-one correspondence and building-up approaches were categorized as one-to-one strategies, and unit or scalar approaches as relational part-whole strategies using NVivo. ANOVAs were conducted using SPSS.

The analysis revealed that the students could be divided into three subgroups based on their strategy use in patterning and counting 1) Students mastering only one-to-one strategies in counting and patterning, 2) Students mastering one-to-one strategies and showing inconsistent or emergent relational strategy use, and 3) Students mastering both one-to-one and relational strategies. The poster will present visual displays demonstrating strategy use employed by each of the three groups and the relations to students' overall number sense, nonverbal reasoning, and use of language during the task-based interviews. Our research indicates that one-to-one correspondence strategies and part-whole relational strategies can be a fruitful way to describe student strategy use on numerical patterns as well as for geometrical patterns.

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EXPLORING THAI MATHEMATICS TEACHERS' PERSPECTIVES ABOUT CLASSROOM NORMS

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The Teaching practices *elicit and use evidence of student thinking* and *facilitate meaningful mathematical discourse* (NCTM, 2014) can only be fully enacted if teachers recognize and value the potential of using student mathematical thinking to increase student learning. Enacting these practices in cultures that have traditionally used student thinking for the purposes of measuring content acquisition poses a challenge when the goal is to use student thinking to develop a mathematical concept. A first step in meeting this challenge is to learn more about the perspectives of teachers in such situations. This exploratory study focused on six Thai teachers' perspectives on classroom norms because norms have been shown to have a significant impact on what happens in classrooms (e.g., Yackel & Cobb, 1996). The six teachers were intentionally selected from government, demonstration, and private schools. We employed task-based individual interviews (developed from Van Zoest & Stockero, 2012) about teachers' perspectives while teaching ideal and realistic classes. Our analysis of these recorded interviews focused on identifying the presence or absence of general norms in mathematics classes that support productive *sociomathematical norms* (Yackel & Cobb, 1996). Based on our analysis, the findings indicate that these mathematics teachers' perspectives about teaching ideal classes seem to support more productive sociomathematical norms than their perspectives about teaching realistic classes. Specifically, it influenced their perspectives on students' characteristics, teachers' roles, and teaching approaches. We also found a relationship between the type of school the teachers taught and their perspectives on active learners, teachers' responses to students' contributions, and discussions (whole-class and small-group). This study informs educators who are working to support teachers to use student mathematical thinking to increase student learning. It does this by providing insight into how teachers' current perspectives can be built on to increase teacher learning about how to better use student thinking.

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COMMUNICATION AND COMMUNITY: ALGEBRA COLLABORATIONS ACROSS K-12 & UNIVERSITY

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Teacher collaboration supports sustained teacher learning, which inspires greater student outcomes and communities of professional learning among teachers (Ronfeldt et al., 2015). Using a communities of practice theoretical framework (Lave & Wenger, 1991), this research study explores the value of learning about algebra learning across K-12 and university algebra instructors in a professional learning community (PLC) that originated from an informal book study for math educators. The analysis shows development of the algebra instructors in pedagogical approaches to teaching algebra to engage students of different levels using meaningful content, emphasizing STEM activities on data and community engagement. The study informs an ongoing project that contributes to efforts to seek support of algebra learning for students who are historically lacking in mathematical development and are enrolled in remedial or developmental algebra classes. With the focus on engaging students with the value of learning math, the PLC emphasizes implementation of math tasks that engage students in communication with one another as well as application of the math to their lives beyond the classroom (Gutstein, 2012). The study asks the research question: What knowledge did K-12 and university algebra instructors gain about math learning from the PLC? Findings indicate that the instructors began to create lessons focused on noticing student understanding, adapting lessons to involve students in community engagement, and seeking community with instructors who teach at different levels to discuss curriculum, content, and pedagogy. The presentation will showcase current findings.

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MATHEMATICS LEARNING & TECHNOLOGY – A STRUCTURAL TOPIC MODELLING

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The role of ICT for mathematics learning has been discussed for years in mathematics education (e.g., Mariotti, 2012), and the pandemic situation has put even more emphasis on the topic. Recent research reviews often focus on a specific technology, a specific goal or process of mathematical learning. We aim to provide a broader picture of the different perspectives on hypothesized and reported effects of technology on mathematics learning by analysing theoretical and empirical research articles.

We searched Web of Science, ERIC, PsycINFO and MathEduc (last search 12/2021) for articles containing the terms "mathematics" and "learning", one technology-related term ("digital technolog*", "educational technolog*", "ict" or "multimedia"), and one term related to effects or processes of learning ("achievement", "effect*", "gain", "mathematic* learning", "outcome", "performance" or "process") in title, abstract or keywords. The search resulted in 652 admissible, peer-reviewed articles that focus on mathematics learning and technology.

Research field, mathematical content, technology, research methods, and grade level were coded for all articles (mean inter-rater reliability: κ = .75). The articles are analysed using topic modelling techniques (Roberts et al., 2019). Topic modelling is a probabilistic method to assign documents to clusters (topics) based on the co-occurrence of words. A first analysis based on 444 texts yielded 9 different topics. Some topics refer to specific types of technology (blended learning, dynamic visualizations, apps & games, feedback systems), some relate to specific theories (cognitive learning, activity theory) or specific variables (teacher competencies, gender, emotions). International comparison studies emerged as an own topic in the field. The topics in that initial search reflect established research traditions on ICT in mathematics learning. The poster will provide profiles of the topics, e.g., in terms of research methods and contents covered to stimulate a discussion on relations between the topics and on possible "blind spots" in the field.

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DEVELOPING KOREAN MATHEMATICS EDUCATION STANDARDS FOR FUTURE GENERATIONS

Oh Nam Kwon, Soomin Park, Jin Hee Park, and Kyungwon Lee

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We live in a time of accelerating changes such as the development of artificial intelligence based on big data, the entry of intelligent information society and hyper-connected era. The changes in mathematics education are needed for students to adapt to social and technological changes and have competencies that can lead the future society. Korean Ministry of Science and ICT (2018) suggested the development of Korean mathematics education standards for future generations. The purpose of the study is to develop the Korean Mathematics Education Standards (KMES) for future generations supported by Korean Ministry of Science and ICT.

KMES were developed through literature review, the expert advisories, the focus group interviews and a survey from June to November in 2021. The expert advisories were conducted three times with 9 experts in science and technology, social sciences, mathematics, and mathematics education, as well as those in the media field. The focus group interviews were conducted four times with 10 mathematics and mathematics education experts. The learner profile for the future society and mathematical competencies developed were validated through a survey conducted on 295 mathematics curriculum stakeholders.

The learner profile for the future society pursued by mathematics education was established as creative global citizens with mathematical competencies. The concept of mathematical competencies in KMES is the ability to understand mathematical knowledge necessary for the future society, to form it through a mathematical process, and to recognize and practice mathematical values. Mathematical competencies include three dimensions of knowledge, process, and action. Mathematical knowledge includes numbers and quantification, changes and relationships, shapes and spaces, data analysis and predictions, mathematics and science/technology, and mathematics and society/culture. Mathematical process includes problem solving, reasoning, connection, communication, creativity, and computational/algorithmic thinking. Mathematical action includes agency, flow, resilience, collaboration and openness, enjoying the mathematical culture, and global citizenship. The results of this study can be utilized in the development of mathematics education standards and mathematics curricula in many countries.

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A DIGITAL ADAPTIVE LEARNING SYSTEM FOR DIAGNOSTICS AND SUPPORT OF BASIC ARITHMETIC COMPETENCIES

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Basic arithmetic competencies are a core content of primary mathematics education. However, some students leave primary school without acquiring sufficient basic arithmetic competencies, which then often cascades to greater difficulties in the first years of secondary school (Ehlert et al., 2013). This problem also emerges since, in practice, teachers may lack resources to individually diagnose student difficulties and to provide individual support. The project aims at developing a digital adaptive learning system for diagnosis and support of basic arithmetic competencies, to facilitate individualized diagnostics and support of students with difficulties.

Eye tracking, the recording of students' eye movements, has been proven to provide valuable insights into students' approaches on arithmetic tasks—especially for students with mathematical difficulties. Therefore, the digital adaptive learning system developed in the project uses eye-tracking data to diagnose and support students in basic arithmetic competencies.

The poster shows the results of a first study with 24 fifth-graders in a comprehensive school, in which we piloted the digital adaptive learning system. For this purpose, students worked on different kinds of arithmetic tasks on a computer screen, while their gazes were tracked. The tasks involved, for example, quantity recognition in structured representations, number line estimation, as well as addition, subtraction, and multiplication in different representations. In the poster presentation, we will present the digital adaptive learning system as well as the findings on the diagnosis of students' approaches on and difficulties with the basic arithmetic tasks.

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A STUDY ON THE RELATIONSHIP BETWEEN ICT USAGE AND MATHEMATICAL LITERACY AMONG STUDENTS IN TAIWAN AND MACAU

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The rise of information technology allows students to study anytime, anywhere. Integrating information technology into teaching is a topic that has been discussed for decades. Both Taiwan and Macau have participated in the OECD's three-year PISA programme over the past few years. While Taiwan's math performance remains one of the top performers, scores have been on a downward trend since 2015; moreover, Taiwan's math performance was initially better than Macau's, but Macau began to surpass Taiwan in 2015. The changing trend in performance between the two places may be due to the change of PISA assessment to computer-based assessment in 2015. In addition to students' subject knowledge and skills, the computer-based assessment focuses on the usage of ICT to solve problems related to mathematical literacy (Lin et al., 2021).

The purpose of this study was to investigate the relationship between the usage of ICT in learning and mathematical literacy among Taiwanese and Macau students using a hierarchical linear model (HLM) by analyzing data from PISA 2018. We also included students' gender and economic, social, and cultural status index (ESCS) in the HLM analysis. The results of the analysis show that Macau's ESCS was worse than Taiwan's but had a higher average level of math literacy and ICT use. After controlling for the effects of gender, student ESCS, and school ESCS, in Taiwan, students' usage of ICT for homework activities outside of school was positively associated with students' mathematical literacy; in contrast, ICT usage in school learning was negatively associated. The Macau analysis showed no significant correlation between the usage of ICT in learning and students' mathematical literacy. This may be due to the relatively extensive usage of ICT in Macau, which makes ICT usage not significantly correlated with students' mathematical literacy performance. Based on the findings, this study recommends that school authorities promote greater usage of ICT by teachers and students in work activities both in and out of school. At the same time, the government should properly supervise the usage of ICT by students and avoid under-use or over-use.

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SECONDARY SCHOOL STUDENTS' STRATEGIES IN SOLVING ARRANGEMENT PROBLEMS

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Despite the relevance of combinatorics in the school curriculum and in discrete mathematics, not much attention has been given to the teaching of this topic in Italy. To fill this gap, our aim was exploring Italian students' combinatorial capacity and solving strategies, as well as its changes with instruction. In this poster we analyse performances and strategies in two arrangements problems. We based our work on Fischbein and Gazit (1988), who analysed the combinatorial capacity of children since 10th year of age, and on the set of combinatorial solving strategies described in Godino et al. (2005). Our sample was made of 115 secondary school Italian students (grade 10, 11 and 12), 51 of which had received instruction on combinatorics and 64 with no instruction. The students were given two open-ended arrangements problems of distribution and selection type each one, and an analysis of the content of their written responses was performed in order to codify the correctness and strategies used.

Both problems were difficult (4.7% and 7.8% correct solutions in the no instruction group in the distribution and selection problem, respectively), although there was an improvement in the instruction group (45.1% and 47.1%). As regards the solving strategies, the students with no instruction mostly solved the problems with either systematic or a-systematic enumeration or the product rule. Students in the instruction group used a wider set of strategies, including systematic or a-systematic enumeration, a formula, product rule or sub-problem decomposition, with a few students using sum or quotient rule or tree diagram. Nevertheless, students of both groups tended to commit the same kind of mistakes, such as producing incomplete lists of configurations, not considering the order of elements, or incorrectly using additive rule instead of the product rule. In summary, this exploratory research provides new insights on the procedures that Italian students activated and their errors, an information that would allow a teacher to improve his/her teaching of combinatorics, by correctly reinforcing the students' spontaneous strategies.

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A PROPOSAL FOR THE MATHEMATICAL-DIDACTIC TRAINING OF TEACHERS ABOUT LOGICAL KNOWLEDGE

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In the 1960s, as a result of the work of Piaget and Szeminska in the 1940s on evolutionary psychology, there was an increasing interest in the idea that classification and seriation were required for the acquisition of numerical knowledge. There is a strong agreement about the incorporation of such logical knowledge in the early childhood education curriculum, evolving from seriation and classification to patterns and algebraic thinking. In the Spanish context, these notions appeared in the Education Law in 1970 in the early childhood curriculum and were consolidated as part of the mathematical-didactic training of early childhood teachers with a strong Piagetian approach.

However, some authors (e.g., Sarama & Clements, 2009) consider that activities involving logical knowledge are of interest themselves since this knowledge is central to addressing most of the problems one person can tackle. In our research, we focus on the following research problem about teacher education: What knowledge do preservice teachers need to teach logical knowledge in early childhood education? How might teacher education be organised to work with teachers to analyse, design, and manage the teaching of logical knowledge in a functional and articulated way?

To address this question within the Anthropological Theory of the Didactic (Chevallard, 2015), we design a *study and research paths for teacher education* (SRT-TE) about logical knowledge. With this proposal, we expect that the preservice teachers question how to interpret what logical knowledge is in early childhood education, and how to analyse its teaching and learning in these initial school stages. Results from two recent implementations in two Spanish universities show how the questions addressed along this SRP-TE help teachers and educators to progressively build the necessary praxeological equipment to analyse and design activities involving logical knowledge.

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EXPLORING TEACHER PROFESSIONAL DEVELOPMENT IN PROMOTING SECONDARY SCHOOL STUDENTS' SELF-REGULATION OF MATHEMATICS LEARNING

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Taiwan implemented a new mathematics curriculum in 2019. One of the curricular goals aims to develop students' self-regulation of mathematics learning (SRML) to lay the groundwork for life-long learning and career readiness. Teachers play a critical role in achieving the goal. However, we know little about how secondary school mathematics teachers interpret SRML and how to facilitate their practice of guiding students' SRML. Thus, this ongoing study constructed a teacher professional community and applied Lesson Study to promote teachers' proficiency in developing students' SRML.

The researchers adapted the conceptual framework of classroom observation proposed by OECD (2020) toward SRML to promote teachers' profession. Four mathematics teachers serving in a secondary school participated in the study. The teachers and researchers met once or twice a month to plan lessons and discuss teaching materials based on the adapted OECD framework. When the lesson plan and teaching materials were completed, one teacher conducted the teaching materials in a lesson, and other participants observed and recorded what the students performed and responded to the instruction. In addition, the adapted framework contributed to the construction and application of the observational protocol.

After the lesson, we referred to the observational records to reflect on and revise the lesson plan and teaching materials. Subsequently, another teacher used the revised lesson plan and teaching materials in his classroom, which initiated a new cycle of Lesson Study.

Data collections included teacher interviews that were conducted at the beginning of this study and teacher talks that occurred in community meetings. Data analysis revealed that the teachers' views of SRML were varied. Although teachers could conduct a formative assessment to monitor students' progress, they rarely guided students to self-assess their learning and provided self-regulated feedback. We expect to report the transformation of teachers' views on and instruction supporting students' SRML.

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EXPLANATIONS OF CONSTITUTIVE ELEMENTS OF ALGEBRAIC EXPRESSIONS: AN INTRODUCTORY LESSON

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The explanation of the basic elements that construct the algebraic expression is important to understand in order to manipulate and operate on the algebraic expression successfully. However, it is not clear what the constitutive criteria of explanation are because literature in the field conflates and uses the words explanation and explaining interchangeably as though referring to the same thing. The conceptual work of the study distinguished explanation, as a noun, from explaining, a continuous verb. To pursue this, I recruited Ruben's (2012) interpretation of Aristotle's four causes as a framework and criteria for explanation. For this presentation, I focus on the teachers' explanation of constituent elements of algebraic expressions in their introductory lessons.

I obtained the data for the study of explanations from five teachers who successfully participated in a professional development (PD) project. From the PD, I collected document data in the form of PowerPoint presentations and all handouts concerned with sessions focused on the idea of explanation. I also conducted a follow up audio-recorded interview with the PD presenter. Three consecutive lessons from each teacher were video-recorded and transcribed verbatim after participating in the PD. Follow up audio-recorded interviews with the teachers were undertaken. All this data was analysed with the aim of answering the broader research question: what is constituted as explanation and acts of explaining algebra in the PD and in teachers' practices? The ideas shared here are explanations offered for introductory lessons to algebraic expressions in grade 9.

The results of the qualitative analysis illustrate a dominance of talk in the explanation that privileges, a meaningless string of symbols and steps of the procedure. The results for acts of explaining display that regulation and revoicing were the most privileged acts of explaining.

The results highlight the importance of distinguishing criteria for explanation from criteria for acts of explaining. I argued that access to and awareness of these criteria have the potential to assist teachers to determine the quality of the explanation in the planning phase of the lesson.

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WRITTEN RESOLUTION OF A MATHEMATICAL PROBLEM BY 11TH GRADE STUDENTS

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Problem solving and written communication are strongly connected, since the resolution of a problem presupposes the use of written communication to record the reasoning, either to communicate with another person or to review the resolution in the future. Bearing in mind this relation, and also considering the relevance of both in learning mathematics, our research question is: how students communicate their resolutions of a mathematical problem in writing? To answer this, we made a qualitative research with an interpretative paradigm. The participants were 29 students of 11th grade, divided into six working groups, who voluntarily signed up for a problem-solving project developed online and in an extracurricular format. In this poster, we show the analysis carried out on the written resolutions of each group to one of the problems proposed in that project. With this study, we try to contribute with categories that help to analyze written communication, and to improve and highlight the importance of that topic in solving math problems, and in the teaching in general.

The analysis' categories we used in this investigation was inspired by the ones written by Santos and Semana (2015), considering four points: correctness, completeness (divided into justification level, justification type, and final answer), representations, and organization. In the analysis of the results, only one group presented an incorrect answer, being also the only one with an absent final answer and null justification level. Still regarding the justification level, two of the correct resolution presented a low level, one a medium level, and the remaining two a high level. The justification type and the representations may be related to these results, since the resolutions with a null and a low justification level used exclusively rules and only symbolic representation. The medium and high-level ones, on the other hand, conjugated symbolic representation with verbal language, and resorted to more than one justification type – which could mean that the use of more than one type of representations can lead to better written communication. As for organization, the high-level resolutions were the only ones with an organized resolution.

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MISSING DATA'S IMPACT ON PROBLEM-SOLVING MEASURES

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PROBLEM AND PURPOSE

During assessment completion, students elect to answer items or not. A student may leave an item unanswered due to lack of content mastery, time, interest, or other reasons. Missing data presents a challenge to data analysis due to a significant number of factors contributing to the meaning of a student's choice and the nature of the assessment (Cheema, 2014). Ways in which missing data have been addressed during analysis are manifold (Cheema, 2014), and a one-size-fits-all approach to solving the issue is not possible. While DeMars (2002) reported the Rasch model was quite "robust" in the estimation of parameters where missing data were observed, issues raised during analysis surrounding the extent of missing data make the solution to the problem vexing. Most missing data studies reflect multiple-choice assessments (Rose et al., 2014). There is a gap in understanding missing data's impact in higher-level thinking constructed response assessments like problem-solving measures. Such tests require students to use different skills of producing rather than selecting answers and thereby require more time in item completion and fewer items tested. Thus, the impact of any single missing data point may be critical. The current study explored whether differential outcomes exist when missing data are analyzed as "incorrect" versus "missing" within the *Problem-Solving Measure* for Grade 5 (*PSM5* for age 10-11-year-olds). Research questions guiding this study were: (1) Was there a significant relationship between student *PSM5* measures from each scoring method? (2) Was there a significant difference in student *PSM5* measures depending on the scoring method?

BRIEF FINDINGS AND CONCLUSIONS

Student test scores from both scoring methods shared a strong and statistically significant relationship (r=.983, p<.001). This result suggests the Rasch model was a robust predictor of rank regardless of scoring method with low item tests (N=18). A paired samples t-test revealed a significant difference in student measures when unanswered items were considered incorrect versus missing; $t(366)=1.65$, p<.001. Student measures were slightly lower when missing data were considered incorrect.

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BENCHMARKING FOR A PROBLEM-SOLVING MEASURE

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PROBLEM AND PURPOSE

Student evaluation of learning progress is a common international practice (OECD, 2013). Educational content standards have been developed in many countries to blueprint classroom instruction (OECD, 2013) with aligned assessments for valid measures of student learning (Moore et al., 2017). Content standards and associated assessment construction have rightfully received much attention. However, the establishment of student performance (or proficiency) level benchmarks on these assessments has been analyzed less. To be meaningful, proficiency level benchmarks (e.g., below, proficient, above) should be developed following a criterion-referenced process, comparing students to content standards. However, it is more traditional for student performance benchmarks to be generated through norm-referenced processes that compare a student's performance to that of other students (Oescher et al., 2014).

This study examined the establishment of criterion-referenced multi-level performance benchmarks for the *Problem-Solving Measure* in grade 6 (*PSM6*; 11-12-year-olds). Objective Standard Setting (OSS), a modern Rasch-based method for benchmarking performance levels, was used because of its criterion-referenced outcomes. OSS has never been used in mathematics education with student assessments associated with this age level. Thus, the research question was: Is OSS an effective model for establishing multiple criterion-based performance benchmarks on the *PSM6*?

ABBREVIATED RESULTS AND CONCLUSIONS

Seven expert judges were engaged in an OSS exercise. Judges were trained and then rated *PSM6* items as either Essential, Advanced, or Non-Essential. Two student benchmarks, *Proficient* (2.70 logits) and *Advanced* (3.06 logits) *Problem-Solvers*, were established. Findings suggest OSS was a productive process for developing meaningful, content-related benchmarks for assessing student growth on the *PSM6* in conjunction with grade-level academic content standards.

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ERROR-BASED TEACHING TO IMPROVE PRIMARY SCHOOL STUDENTS' UNDERSTANDING OF DECIMAL NUMBERS

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Misconceptions about decimals usually persist over students' learning process, and, in addition, these errors use to be systematic and common (Durking, 2012). Erroneous examples (EE) during the teaching of decimals can be helpful for students to remedy their misconceptions and improve comprehension, as students have to find, solve and explain the errors encountered (Siegler, 2002).

A pretest–post-test experimental study is designed to examine the effectiveness of EE in the learning of decimals compared to correct examples (CE). On-site pre- and post-tests will be sat to assess students' knowledge before and after the intervention, which consists of 15-minute sessions that participants will complete at home through an online platform. Participants will be randomly assigned either to the experimental (EG) or control group (CG). In the sessions, students will visualize correct examples (in the CG) or erroneous examples (in the EG) before solving decimal tasks. In the pre- and post-tests, we will use an instrument to measure students' understanding of decimal numbers based on a validated test (Durking, 2012), which addresses three types of misconceptions (whole number, role of zero and fraction) and consists of four different kinds of tasks (e.g., comparison of two decimals). A preliminary pilot study with sixth-grade students was conducted to identify the most prevalent misconceptions, on which to design intervention sessions. Although participants had already completed the unit devoted to decimals in their schools, some tasks continued to be challenging for several students. In particular, tasks involving the identification and placement of numbers on the number line and the comparison of decimals were more prone to the commission of errors.

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THE CONSTRUCTION OF A DIAGNOSTIC TOOL FOR "ADDITION" AND "SUBTRACTION": AN EXAMPLE OF RESEARCH SUPPORTING PRACTICE

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"Addition" and "subtraction" are key concepts in Mathematics. Children begin to study them from as early as pre-school and work with them throughout their primary school years. It is, therefore, imperative for teaching practitioners to be able to diagnose the quality of their pupils' understanding and plan their instruction accordingly (Carpenter et al., 1988).

The study reported here aspired to fill a significant gap in the related literature with the construction of a comprehensive diagnostic instrument for addition and subtraction. The basis for its design was the classic categorization of additive word problems by Carpenter et al. (1983) according to their semantic structure ("change", "combine", "compare" and "equalize" problems with each category including "join" or "separate" types). The result was an 18-item instrument which represented all the possible cases of semantic categorization. For example, the problem "Peter had 6 apples. Anna gave him 9 more apples. How many apples does Peter have now?" was a "change-join-final quantity unknown" case.

The instrument was administered to a convenience sample of 121 (7- and 8-year-old) pupils and 4 of them were interviewed in depth about their scripts. The test data and, more importantly, the interview data indicate the diagnostic strength of the proposed tool: all the conceptions and misconceptions mentioned in the literature were revealed through the problem solutions and the children's explanations for them. As a result, the use of such an instrument can effectively aid and enrich the teachers' diagnostic competence, empowering them in terms of their teaching practice.

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A FRAMEWORK OF DESCRIPTORS TO CHARACTERIZE FLEXIBILITY AS A TRAIT OF MATHEMATICAL GIFTEDNESS

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Mathematical creativity is an element of mathematical giftedness (MG), consisting of fluency, flexibility, and originality (Leikin & Lev, 2013). Problem solving is an accurate way to identify MG students (Kattou et al., 2013). We used the context of mathematical olympiads to observe MG students' flexibility when solving problems. Our research objectives are i) to present a theoretical framework characterizing flexibility by a set of descriptors of MG students' problem-solving processes and ii) to particularize it to some types of problems used to identify MG students.

Researchers have related mathematical flexibility during problem solving to: using several representations; adapting to changes in the demand of the problem; changing the line of thought if an obstacle arises or the solution is too long or complex; producing significantly different solutions (Leikin & Lev, 2013). We argue that flexibility is also related to changing to a new and more efficient way of solution. As for the first objective, we define mathematical flexibility as the ability to change the way of solving a problem when its conditions change or when an obstacle or a more interesting new idea arise. Operatively, through this definition, mathematical flexibility can be evidenced in MG students' solutions by the following descriptors:

DF1. Moving from one system of representation to another more efficient or combining several systems of representation.

DF2. Beginning a new way of solution because the student: blocked and did not know how to continue the solution; noted that the current solution does not lead to the answer, or it is too long or complex; had a new idea suggesting another more efficient solution procedure; adapted to changes between parts of the statement.

DF3. Producing significantly different multiple solutions of the same problem, based on: different systems of representation; different solution strategies; performing the steps of the solution in different orders.

As for the second objective, we shall present in the poster examples of the problems we used in the olympiad and the particularizations of some descriptors to those problems.

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PLANES OF SYMMETRY IN REGULAR POLYHEDRONS WITH VIRTUAL REALITY AND MANIPULATIVE MATERIALS

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Visualization difficulties are a source of errors in spatial geometry, where conceptual images are usually very poor, in spite of their importance (Gutiérrez & Jaime, 2015). Physical manipulation of objects and the recognition of properties in geometric solids requires specific spatial skills (e.g., Ramírez et al., 2018), which are not spontaneous in many students, but can be trained. Virtual reality has the potential to enrich these conceptual images and help students to overcome errors and limitations (Rodríguez et al., 2019). This poster presents the analysis of secondary students´ errors and limitations when locating the planes of symmetry in regular polyhedrons, comparing the work with manipulative materials and with virtual reality.

The sample comprises 30 students of third year of secondary education (age 14-15). In a one-hour session, the class was divided into two groups: Group 1 of 10 students working with virtual reality, and Group 2, with 20 students carrying out the activity with manipulative materials. The students who used manipulative material only managed to point out correctly the planes of symmetry of the tetrahedron and the hexahedron, while all the students who worked with the immersive virtual reality NeoTrie VR software completed correctly the planes of symmetry of the five regular polyhedrons. Most frequent errors made during the activity were: 95% of Group 2 students did not correctly indicate the midpoint necessary to do some planes of symmetry, while all the Group 1 student did so, by using the tool provided by the virtual reality interface. Similarly, 80% of the Group 2 students failed to place the diagonal planes in the hexahedron properly, compared to 10% of those who used virtual reality. Finally, none of Group 1 students made mistakes in checking planes of symmetry, while the Group 2 students were unable to do this check. All this will be detailed in the presentation.

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USING IMMERSIVE VIRTUAL REALITY WITH NEOTRIE TO PROMOTE STUDENTS' CONCEPTUALIZATION OF QUADRILATERALS

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Current trends in geometry teaching and learning are emphasizing spatial reasoning, the geometry of transformations and the active construction of meaning, including the composition-decomposition of figures, spatial orientation, and the mental comparison and manipulation of two- and three-dimensional figures (Sinclair & Bruce, 2015). In this regard, the recent development of three-dimensional dynamic geometry immersive virtual reality software (3D DGS IVR) opens new interest focus on how to enhance these aspects, although more empirical studies are still lacking (Rodríguez et al. 2021).

This research aims to examine the contextualization of quadrilaterals by primary school children using a 3D DGS IVR called Neotrie. We are carrying out a teaching experiment with 17 students (2 fourth graders, 7 fifth graders and 8 sixth graders). Five lessons have been designed to study quadrilaterals from projections of a variety of prisms, which children need to choose and manipulate in order to identify and analyse the different types of quadrilaterals. In one hour and a half lessons, the students work with Neotrie in small groups. All sessions are being videotaped. To assess the students' quadrilaterals conceptualization, we use a pre-post-test and a qualitative observation.

The early results show that the students' motivation has been increased, as well as their development of figural, instrumental and discursive genesis (Kuzniak et al., 2016). Consequently, it appears that 3D virtual immersion has a positive impact on the development of a complete and rich geometrical work on the conceptualization of quadrilaterals. In any case, lesson design and instruction need further research. Further results will be discussed in detail in the presentation.

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ASSESSING STUDENTS' PROOF SKILLS: SUPPORTING PRESERVICE TEACHERS WITH SCAFFOLDING

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INTRODUCTION

Introduction to proof is a key learning goal in secondary mathematics education. To optimally support students, their teachers need to be able to assess their proof skills adequately. However, this has been shown to be challenging for preservice teachers. To support the acquisition of assessment skills, simulations including cognitive (e.g., conceptual prompts, CP) and motivational scaffolding (e.g., utility value intervention, UVI) have proven effective in other domains. In this regard, research also pointed out that learners' success expectancy moderates their effectiveness. In the present study, we examine if these findings can be transferred to scaffolding of preservice teachers' assessment skills regarding students' proof skills.

METHOD

Assessment skills regarding proof skills of $N = 108$ preservice teachers were measured using a video-based simulation (Codreanu et al., 2020). A week after a pre-test, they were randomly assigned to a 2x2 intervention design with a UVI and CP condition.

Only the CP condition improved significantly from pre- to main test. Descriptively, CP and UVI conditions were on par in the main test. The UVI+CP condition increased least. We conclude that CP and a UVI effectively support preservice teachers' assessment skills regarding proof skills; for the effectiveness of their combination, more time for reflection might be needed. Regarding the role of success expectancy, students with one standard deviation above (below) the mean success expectancy benefitted most from a UVI (below: CP), which is in line with prior research.

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LESSON STUDY: A RESEARCH SYNTHESIS

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Lesson study (LS) plays a promising role in improving delivery of a lesson, promoting student-student discussion, and changing the attitudes of teachers in reflecting on their lessons by implementing new strategies (Huang et al., 2020). Despite such a role, LS does not seem to gain sufficient attention by teachers in the US. This research synthesis aims to critically analyze LS research outcomes and practices. Research studies published these past 20 years in the U.S were systematically content-analyzed to conceptually synthesize LS research focusing on preservice and in-service teachers in mathematics education. This conceptual synthesis is best presented in visual form as a research poster presentation to examine the extent to which LS has received the attention that is comparable to its merits to student learning and teachers' professional development.

Findings from this review reveal that LS is not part of the US classroom teachers' culture. Research on LS is scanty and even when there was research on LS, its impact in changing teacher dominance in the classroom is minimal. In-service teachers show interest in LS and seem to see its impact on student learning and their views of lesson preparation during the research debriefing and discussions. However, these views were short-lived, and teachers switched back to their authoritarian classroom culture. Research involving preservice teachers centers on modified versions of LS and emphasizes lesson analysis and evidence of students' learning, mathematics talk, and examination of teaching practice. Through LS, preservice teachers are provided opportunities to be critical thinkers, make connections between professional practice and research, and engage in reflective approach to teaching (Phelps-Gregory & Spitzer, 2021). Mathematics educators need to invest on teacher development for in-service teachers and training of preservice teachers. Desired change cannot be achieved by just designing attractive curricula but rather it requires an investment on teachers' professional development as well.

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ADULTS' EYE MOVEMENTS WHEN COMPARING DISCRETIZED OR CONTINUOUS FRACTION VISUALIZATIONS

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Visual representations of fractions, such as tape diagrams, are thought to be helpful for comparing fraction magnitudes quickly. Textbooks often use discrete or discretized fractions visualizations that represent fraction numerators and denominators with individual units or segments. However, continuous representations (without any segments) could be more intuitive to process mentally (Boyer & Levine, 2015) and to reduce natural number bias (Ni & Zhou, 2005). This assumption is based on results from studies of people's accuracy rates and response times in comparison tasks with discretized or continuous visualizations, but we do not well understand the cognitive processes leading to these results.

We used eye tracking to assess the process of comparing tape diagrams in discretized or continuous formats. We were particularly interested in differences between these formats regarding participants' eye fixations and saccades (transitions between fixations). The sample consisted of 34 university students (21 female, 19.0 years), who solved 44 comparison items as quickly and accurately as possible. The two tape diagrams in each trial were represented on a computer screen one above the other, either in a discretized or continuous format.

On average, participants compared tape diagrams in the continuous format more accurately (85% vs. 80%,) and faster (3729 ms vs. 6128 ms). They also used fewer fixations (7.4 vs. 11.9) and fewer saccades (9.7 vs 14.5) per item in the continuous than in the discretized format. There was also a reduced natural number bias in the continuous compared to the discretized format.

This study suggests that adults use continuous tape diagrams more efficiently than discretized ones to compare tape diagrams of fractions. Continuous fraction visualizations could be helpful in fraction instruction to support learners' intuitive focussing on fraction magnitudes and to reduce natural number bias.

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IMPACT OF ERRONEOUS EXAMPLES ON FOURTH-GRADE STUDENTS' WORD-PROBLEM-SOLVING PROFICIENCY USING AN INTELLIGENT TUTORING SYSTEM

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Erroneous examples (EE) –resolutions that intentionally include one or more errors– are used as a pedagogical technique as they encourage critical thinking about mathematical concepts, provide new opportunities for problem-solving and motivate reflection and inquiry (Borasi, 1994). In addition, EE can be effective in combination with intelligent tutoring systems (ITS), environments where students are provided with automatic feedback and scaffolding on the resolution process (Booth et al., 2013). Based on this, we present an ongoing study with a hybrid nature, as it will combine instructional videos in HINTS –an ITS that allows students to individually solve word problems– and the viewing of embedded instructional videos. This study aims to analyse if (i) instruction on the arithmetic solving of word problems based on EE is more effective compared to correct examples (CE), and if (ii) the use of an ITS with integrated explanation videos is effective in improving students' arithmetic word problem-solving proficiency. The sample will consist of fourth graders, randomly divided into two groups, each working with CE or EE. The intervention will last 6 sessions including pre- and post-tests, whose problems were selected based on previous research on the problems' attributes associated with students' difficulties. In the experimental phase, participants will complete a blended instruction based on watching three-minute videos and problem solving. The videos are based on EE and CE in the experimental and the control group, respectively. A preliminary paper-and-pencil study (part of this project) with 97 pupils yielded positive results for the EE group. The current study goes a step further by integrating the teaching sequence into an ITS, combining EE's educational potential and artificial intelligence's tutoring capabilities.

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THE DISCOURSE OF MATHEMATICAL LANGUAGE AND NATURAL LANGUAGE IN FIFTH-GRADE MATHEMATICS CLASSROOMS

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There is a bridge between mathematical language and natural language. Students needs identify textual components and carry out logical operations mathematically, therefore, natural language can help students understand the linguistic and mathematical deeper structure (e.g., Ilany & Margolin, 2010). Researches that provide evidence related to how discourses and voices weave representations of mathematical language and natural language are still quite few, but well needed within the rising field of mathematical research.

This study aims to investigate the use of mathematical language and natural language for fifth graders and the linguistic interaction between teachers and students. 52 fifth graders from 2 classes and their mathematics teachers participated in this study. Video recording was used to collect data. We focus on 480 minutes of the video illustrating specific critical events. Transcriptions of the video conversation were analysed by analytic induction method. Data were categorised into 4 stages of Mayer's model of problem solving, which are problem translation (PT), problem integration (PI); solution planning and monitoring (SPM); and solution execution (SE), and then cases are inspected to locate common factors and provisional explanations.

The findings of this study showed as follows. Teachers' conversation in mathematical language often went back and forth between the stage of PI and SPM. Students tended to use both mathematical and natural language, when explaining the mathematical definition or relationship, e.g. perimeter of a square/total length of a square. Some 5th graders ignored the necessity of using mathematics vocabulary during describing situation of problem, instead, they used natural language. The scaffolding that teachers ask questions with mathematics vocabulary affects how often students use mathematics language to answer teachers, explain problems or communicate with classmates. Students use natural language more in group discussion than in full class discussion. In addition, students tended to ignore classmates' errors in mathematical vocabulary or confusion in mathematics semantics when they discuss in small groups. More details will be presented in the conference.

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EXPLORING MIGRANT STUDENTS' MATHEMATICAL EXPERIENCES AND STORYLINES

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The global mobility phenomenon is creating a multilingual and multicultural society in the world, also in Norway, turning mathematics classrooms into multicultural mathematics classrooms. Mathematics learning and language learning go hand-in-hand in introductory classes for newly arrived migrant students. Students struggle to connect their previous mathematical knowledge into the present learning situation when time is spent on translating mathematics texts from Norwegian to their respective mother tongue. When mathematics teachers listen to students' perspectives about their mathematical experiences (e.g., from their previous learning context or from outside school), they are more able to plan and develop math lessons that support students' learning. This research aims to further understand newly arrived migrated students' mathematical experiences to identify possible storylines that are at play. Storylines are "the ongoing repertoires that are already shared, or they can be invented as participants interact" (Herbel-Eisenmann et al., 2015, p.188). The research questions are: How do migrant students describe their mathematical experiences? What storylines do students take in use when they describe their experiences?

This research uses positioning theory (e.g. Herbel-Eisenmann et al., 2015) to support further understanding of migrant students' descriptions of mathematical experiences during mathematics lessons. I draw on a participatory research approach (e.g., Groundwater-Smith et al., 2014). The focus group is newly arrived migrant students from the introductory program in a lower secondary school in Norway. The data is collected through class observations, and through semi-structured interviews. The data analysis is in the initial stage. A careful analysis of the communication acts, and the positions/positioning that occur during the communication acts will help to identify the potential storylines that migrant students take in to describe their mathematical experiences. Based on the initial analysis I identify that there are multiple storylines at play when migrant students describe their mathematical experiences. A more detailed description of the initial findings will be presented at the conference.

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SPECIAL EDUCATION TEACHERS' BELIEFS ABOUT THE USE OF MATERIALS IN FIRST LESSONS OF ARITHMETIC

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In the qualitative study special education teachers’ beliefs about the use of concrete and tangible materials in arithmetic lessons are reconstructed. The interest in researching beliefs is mainly based on the fact that there is a high relevance of material-supported work for children with special needs (Ratz, 2012) to support the learning process in arithmetic lessons with regard to the development of mental ideas about numbers, operations and arithmetic strategies. Furthermore, the interest based on the assumption that the intensive and long-term use of materials is influenced by the beliefs of their teachers. Many studies show that teachers' beliefs influence the design of lessons which also influences the quality of learning at school (Bräunling, 2017).

The data of the study were collected by eight interviews whose design corresponds to the narrative-based and guideline-based interview. For the data analysis, beliefs were reconstructed as a construct of cognitive, affective and conative components. The cognitive component of beliefs refers to the individual knowledge about the object to which one holds a belief, the affective component refers to an existing emotional relationship to the belief object and the conative component refers to the action-guiding character of the belief. In order to proceed in this way, the evidence-based classification of texts according to Schütze (1987) was be used.

The interviews with eight teachers were the basis for reconstructing 65 beliefs about the use of materials in first lessons of arithmetic. The beliefs determined three central conclusions: Firstly, many beliefs suggest that materials are used as a general tool for arithmetic learning and/or as a means of calculating and solving problems. Secondly, a significance of motivation and emotions in the use of materials can be concluded. Thirdly, a low mathematics-didactic content can be assumed in the beliefs of special education teachers. This does not suggest a solid didactic knowledge, which can be attributed to the fact that special needs education does not necessarily include mathematics didactics training.

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PROSPECTIVE ELEMENTARY TEACHERS' STRATEGIES WHEN SOLVING MISSING NUMBER SENTENCES

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Research has shown the importance of prospective elementary teachers (PTs) developing relational thinking to be prepared to promote this type of students' thinking in their future practices (Magiera et al., 2011). Taking this into consideration, in this study we seek to identify the strategies used by PTs when solving missing number sentences aimed at mobilizing relational thinking (Molina et al., 2009). The participants were 94 (35 male and 59 female) Spanish PTs who attended the 1st year of a degree in elementary education teaching at a public university. In the analysis we considered a relational thinking framework concerning: (1) relational strategies, when PTs attended to the numbers' characteristics and used arithmetic properties (e.g. properties of operations, compensation, and composition or decomposition); and (2) non-relational strategies, when PTs used arithmetic or algebraic procedures not based on the previous properties and numbers' characteristics.

The results show that the PTs used preferentially non-relational strategies, most of them based on arithmetic calculations (in about 50% of the answers) and others consisting on solving algebraic equations (13%). In turn, only 22% of PTs' answers evidenced relational strategies. In the remaining cases, the PTs used incorrect procedures or computations (9%) or did not justify their answer (5%).

Although no conclusive claims can be made about these PTs' relational thinking, it is evident that they mainly adopt mechanical procedures that lead to correct answers but are less efficient than relational strategies. Making a diagnosis of these situations may constitute a point of reflection on the PTs' relational thinking to design interventions in teacher education courses that support them to further develop this type of thinking, and to understand its importance when teaching mathematics to their future students.

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TEACHING INTEGRAL CALCULUS USING RECOGNITION AND HEURISTIC SEARCH

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Successfully solving integrals involves recognition of which techniques are useful and, when recognition is incomplete, heuristics on how to effectively explore and reduce the remaining problem space (Gobet, 2015). Based on work by Kop et al. (2015) we developed an expertise framework that distinguishes multiple levels of recognition and corresponding heuristics. This framework provides direction on what students need to learn and how this can be taught.

The research question was: how can a series of lessons based on the framework contribute to the development of recognition and heuristic search in calculating integrals for students in upper secondary education?

In the series of lessons, the techniques of integration were taught using a whole task approach with reflective questions. The framework was extended step-by-step, incorporating newly learned techniques and heuristics.

Data included written exams, thinking-aloud protocols when calculating integrals, and card-sorting tasks in which students, at the end of each lesson, had to categorize functions according to the different techniques of integration.

The results of the card-sorting tasks showed that students were able to make increasingly precise distinctions between the categories. These results were validated by the thinking aloud protocols, which showed that students used similar approaches. These thinking aloud protocols also showed that students expanded their repertoire of heuristics during the series of lessons. The written exams showed that most students could perform the techniques properly after the lesson series.

Step-by-step explicit teaching of recognition and heuristics seems to be a promising direction to support students' learning of integral calculus.

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DEVELOPING A MATHEMATICS CLASSROOM TASKS SURVEY

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PROBLEM AND PURPOSE

Early mathematics instruction is critical to building a foundation that allows students to be successful in later grades (Jordan et al., 2010). However, many preschool (age 4-5) teachers do not teach mathematics in their classrooms because they do not feel confident (Copley, 2004). Understanding preschool teachers' instructional practices in mathematics can inform professional development opportunities. Thus, the purpose of this study is to describe initial steps in developing and qualitative findings from validating a survey used to assess preschool teachers' mathematical practices used.

The Standards for Educational and Psychological Testing (AERA et al., 2014) discuss the need to collect, evaluate, and document multiple forms of validity evidence for an instrument to be deemed suitable for specified intents. Five types of validity evidence should be collected: content, response process, consequential, internal structure, and relationship to other variables. This study focuses on qualitative validity evidence collected, analysed, and used in a rigorous and iterative design-based-research process.

ABBREVIATED RESULTS AND CONCLUSIONS

A frequency of use survey was developed with 12 preschool mathematics tasks. Content validity evidence (items align with construct) was evaluated by an expert panel (n=7) and deemed acceptable upon minor revision. Response process (teacher survey responses align with researcher intent) was assessed using preschool teacher cognitive interviews (n=8). Feedback suggested four wording revisions and one item addition. Consequential validity evidence (negative impact from completing survey) was investigated at the end of preschool teacher cognitive interviews. No teachers reported a negative effect. This study demonstrates the importance of incorporating feedback from various stakeholders when developing educational instruments. A revised survey will be quantitatively piloted with an estimated 100-150 preschool teachers to study internal structure and relationship to other variables validity evidence next.

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A CAPTIVE BODY IN MATHEMATICS CLASSROOM NORMS?

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Whatever the mathematics classroom culture is, the classroom norms could specify the movement of the students and in many cases there is little body movements, usually the studens are sitting on their chair with a desk in front, this may happens due to "much attention to the body in mathematics education remains mostly captivated by the representational politics through a discursive mediation of classroom norms" (Chronaki, 2019, p. 321). The classroom norms can be observed in the "particular culture [which] is created at every instant in each classroom as a result of the participants' actions, including interactions with tasks" (Lozano, 2017, p. 897). The term culture is related to cultural behaviour which is "communicative interactions that give a certain continuity to the history of a group, beyond the particular history of the participating individuals" (Maturana & Varela, 1992, p. 201). Examples of norms that can be categorized explicitly are, "saying hello when the teacher arrives at the classroom" or "raising a hand to express an idea in the classroom setting".

The poster presents a reflection about body movements of teacher and students in the learning of mathematics. I explore a range of examples of being outside (as a natural setting to promote body movements) and inside the classroom. I am bringing more questions such as: How we can promote a natural awareness of body movements related to mathematics wherever the class takes place? Being inside or outside the classroom, are the students or teacher adapting to their environment noting their own body movements to understand mathematics? How can standard mathematical classroom culture and norms be challenged to promote body movement in mathematics learning? Even though mathematical situations that involves being outside the classroom, are students captive under their classroom norms and culture? In other words, they are still doing the same "solving mathematics task" in another place, Does it matter?

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LEARNING GEOMETRY THROUGH M-LEARNING

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Mathematics education is key to the development of responsible future citizens, who must be able to understand and analyse their environment. To that end, using ICT we have proved a noticeable improvement in students´ attitudes towards mathematics, making them demonstrate a greater critical spirit, perseverance, precision and creativity, as well as an increase in their flexibility of thought (García & Romero, 2009). We introduced m-learning methodology based on the use of mobile phones in classrooms, which takes advantage of their interest in technology and is closer to the students' reality. This methodology gives them access to information without any spatial limitation, thus offering learning opportunities from beyond the classroom (Castro et al., 2016). Focusing on learning geometry, we used GeoGebra 3D, a virtual reality mobile app, which encourages them to create, manipulate and interact with geometric 3D models.

We carried out an action-research in a second-grade high school, with 28 students. We detected important deficiencies after an initial observation of students´ three-dimensional geometric knowledge. We developed a didactic mathematic proposal using m-learning, and we observed what was happened in the classrooms. The goals were improved student spatial vision and an improvement in the learning of the principal geometric models. We got evidence, which proved the benefit of using m-learning in geometry learning. We did interviews with students, and we concluded that 100% of them think ICT, as mobile applications, are beneficial. They feel that they have improved their learning capabilities and what they have learned is useful.

From the teacher´s point of view, thanks to GeoGebra 3D, students have had the opportunity to contextualize the mathematical content and visualize it, and to develop abstract thinking later. Further to this, the group have improved their final qualifications. We will continue with a next cycle of research-action, improving the educational proposal, the following academic course.

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SIMULTANEOUSLY DEVELOPING CK & PCK — DESIGN RESEARCH OF COURSES FOR PRE-SERVICE TEACHERS

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New university courses that combine subject content and subject didactics at the curricular level have been designed, evaluated and further developed in interactive cycles using design research (Prediger et al., 2015). The courses aim to simultaneously develop pre-service teachers' content knowledge (CK) and pedagogical content knowledge (PCK). On the basis of theories on teacher professional knowledge, in particular the concept of a school-related content knowledge (SRCK; Dreher et al., 2018) specific to the teaching profession, content-related design principles have been developed and used as reference point for both conceptualization and research. The design principles and their background theories are visualized on the poster.

The poster also presents the methods and results of two design research cycles within the course 'Arithmetic and its Didactics', using the example of the density of rational numbers. On the basis of a math problem for Grade 6 students ("Is there a number between two given fractions, and if so, how many?") test items were developed that address different domains of SRCK and PCK. As part of the final exam, a written test was administered to all pre-service teachers (N=123 in 2019 and N=112 in 2021), followed by problem-centered group interviews (n=9). Research questions were: To what extent have the pre-service teachers acquired the desired knowledge, and what characterizes their understanding of rational numbers? What conclusions can be drawn regarding the effectiveness of the course? What are the implications for course design?

Qualitative content analysis of the written test results and interview transcripts gives detailed insight into pre-service teachers' knowledge building with regard to different domains of professional knowledge. Results show that the pre-service teachers have acquired well developed CK on the school math level and PCK, but much less developed SRCK in the sense of connecting the school math problem with academic mathematical knowledge and fundamental ideas of mathematics. In order to increase the effectiveness of the course, one of the design principles, namely "make knowledge explicit and reflect it on a meta-level", should be strengthened in the re-design.

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INTERPRETATIVE TASKS FOR TEACHER EDUCATION TO ACCESS AND DEVELOP TEACHERS' INTERPRETATIVE KNOWLEDGE

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The design and implementation of tasks for students is a crucial part of teachers' practice (Mason & Johnston-Wilder, 2006). In that sense, with the aim of disrupting the vicious vision of learners as passive listeners and instead giving them an active role in the learning process, getting experience in task design and implementation must also be a core element in teacher education.

Considering the specificities of teachers' knowledge and the need to take into consideration what the students know and how they know it, the conceptualization of the Interpretative Knowledge—IK (Jakobsen et al., 2014) has been developed. With IK we refer to a deep and wide mathematical knowledge that enables teachers to look at mathematical problems from different points of view and to give meaning to students' productions (Jakobsen et al., 2014). Assuming the key role of teachers' IK for developing students' mathematical understanding, and knowing that IK does not spontaneously develop over time in teachers practice, we have developed a special kind of tasks named Interpretative Tasks, to stimulate the development of teachers IK. These tasks are situated in different practice contexts and grounded in mathematical critical situations in students learning and in teachers' practices.

We will discuss the nature, focus and design of interpretative tasks for teacher education and present an example in the context of area measurement. Different elements considered in the task design and how they are used, both from the research point of view and for teacher education for accessing and developing teachers' Interpretative Knowledge, will be discussed.

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THE CONTRIBUTION OF INTELLIGENCE, COGNITIVE REFLECTION AND BELIEFS TO MATHEMATICS ACHIEVEMENT IN SECONDARY SCHOOL

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Many factors contribute in an interrelated way to predict individual and developmental differences in mathematics learning and achievement. Findings indicate that it rests on underlying domain-specific abilities, and on domain-general cognitive and non-cognitive factors. Reasoning processes and cognitive reflection abilities are specially involved. In addition, student´s beliefs about learning and self-efficacy in math have a meaningful impact (Gómez-Chacón et al, 2014). However, it is unclear their relative involvement in math achievement in secondary. Dual-process theorist (Evans, 2007) argued a dual system of reasoning processes: Type 1 (T1: intuitive & unconscious, not linked to Gf) and Type 2 (T2: analytical & conscious, linked to Gf), both involved in math reasoning and achievement. Preference for T1 or T2 thinking may be belief biased. This study examined the relationships and contribution of fluid intelligence (Gf), cognitive reflection processes and beliefs in predicting math achievement in a sample of 250 students attending 2nd (N=121) and 4th (N=106) grades of secondary school. Participants were assessed on Gf (Ravens Progressive Matrices test), cognitive reflection in problem solving (Cognitive Reflection Test-CRT: Frederick, 2005) and beliefs (Mathematics-Related Beliefs Questionnaire-Creemat: Gómez-Chacón et al., 2014). As for math achievement, student grades were collected. Results showed significant differences between groups on CRT correct and intuitive responses, and a pattern of low to moderate correlations between math, Gf, CRT and Creemat (*rs* ranged from .21 to .43) in 2nd, as well as moderate associations between math and Gf and Creemat (r = .49 and r = .58, respectively) in 4th graders (p < .01, all cases). Multiple regression analyses indicated that Gf and Creemat explained about 30% and 53% of the variance in mathematics in 2nd and 4th graders, respectively, being the significant variables Gf (β_{2nd} = .26; β_{4th}= .3) and Creemat (β_{2nd} = .36; β_{4th} =.56). Findings confirm the role of Gf, whereas they note a greater influence of the belief system on math achievement in adolescence.

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TRANSITIONAL OBJECTS AND EARLY ALGEBRAIC THINKING

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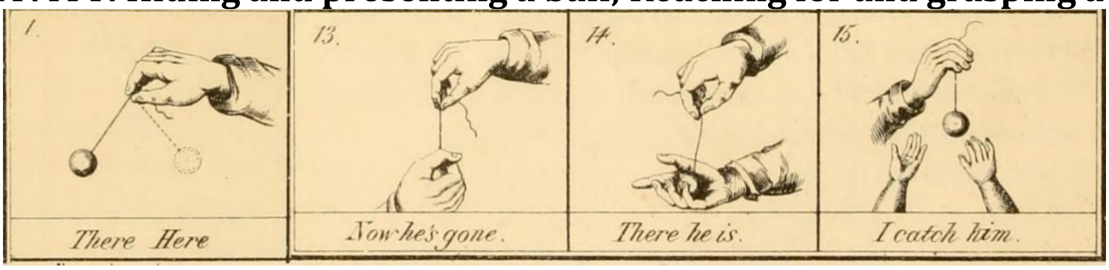
According to Winnicott (1953), when a caregiver is absent, young children, to cope with separation anxiety, may use 'transitional' objects, such as thumb-sucking, or a soft toy, to stand in temporarily for the caregiver - an early form of symbolisation. Friedrich Froebel, the inventor of Kindergarten, similarly related babies' play with objects such as balls and cubes to pre-verbal enactment of relationships with caregivers (Froebel, 1895). This work-in-progress study aims to explore theoretical links between psychological projection of relationships with caregivers onto objects in Kindergarten play, and proto-algebraic thinking, where an object may stand for various things, which underpins e.g. Davydovian and Gattegnan primary mathematics curricula. I reviewed Froebelian activities with balls and cubes and attempted to interpret them through first a psychological and then a mathematical lens, mapping concepts broadly gathered from developmental psychology and the mathematics curriculum in England.

ACTIVITY: Hiding and presenting a ball; Reaching for and grasping a ball



What the little one has up to this time directly felt so often by the touch of the mother's breast – union and separation – it now perceives outwardly in an object which can be grasped and clasped (Froebel, 1895, p.36)

PSYCHOLOGICAL:	**MATHEMATICAL**:
Attachment; Separation anxiety; Containment; Splitting and integrating; Shared gaze; Paranoid-schizoid and depressive positions; Object permanence.	Problem and solution; Doing and undoing, inverse relations; Sphere, line, circle; One; Finding the unknown, algebra.

Figure 1: A Froebelian activity mapped to psychological and mathematical concepts

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DESIGNING FOR AWARENESS: EARLY YEARS EDUCATORS COMING TO TERMS WITH THEIR MATHEMATICAL POWER

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There is a dichotomy between young children's abilities in mathematics and their early childhood educators' (ECE) experiences with mathematics. Children in the early years begin school with both a lot of mathematical knowledge and with much potential for mathematical growth (Clements & Sarama, 2007). At the same time, their ECE has often had negative experiences with mathematics and view themselves as non-mathematical. Yet, like their students, each ECE has much potential for mathematical growth and hidden mathematical knowledge that needs to be (re)discovered. We posed the question: How do we educate ECEs for awareness of their own mathematical potential and power? We designed a mathematics course for ECE teachers to become aware of their own mathematical potential, knowledge, and power. Our study is situated in mathematics task design research (Cevikbas & Kaiser, 2021) where we study the output of the task, the learning interactions and experiences, to understand the design effects of the task elements (Bikner-Ahsbahs & Jansen, 2013). We used the framework of Mason's (2002) educating awareness to frame the course design. Our findings highlight how specific aspects of the task design, namely opportunities for being mathematical, explicitly introducing the language of awareness, and creating moments of cognitive conflict, uncovered mathematical knowledge and supported ECEs' awareness development of their own mathematical power.

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REQUESTS FOR REASONING IN GEOMETRICAL TEXTBOOK TASKS FOR PRIMARY-LEVEL STUDENTS

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Mathematical reasoning may lead to a deeper individual understanding. Although geometric learning at the beginning of school is still essentially visual and holistic, geometry becomes a popular content area for learning mathematical reasoning, argumentation and proof in secondary school (Battista, 2009).

Our textbook analyses from 2010 and 2016 (Ruwisch, 2017) showed that even over time, only less than 10% of the textbook problems in grades 3 and 4 ask for reasoning. Roughly two types of requests could be distinguished: explicit and more implicit requests. In nearly all textbooks, more problems asked implicitly for reasoning than explicitly. On average, implicit requests were twice as frequent as explicit ones, although the amount of explicit requests was slightly higher in fourth grade.

RESEARCH QUESTIONS

1. How often are requests for reasoning in geometrical tasks in comparison to other contents in the textbooks?
2. How can explicit and implicit reasoning prompts in geometrical tasks be linguistically differentiated in detail?

PROMPTS FOR REASONING IN GEOMETRICAL TEXTBOOK TASKS

The textbook series differ both in terms of the total number of (geometrical) tasks and the number of prompts for reasoning (in geometrical tasks). Whereas 8.5% of all textbook tasks require reasoning, a different picture occur, when focussing on the geometrical tasks only: On average, 14% of the geometrical tasks ask for reasoning, but only about 4% do it explicitly. Four explicit (reasoning, arguing, explaining, and proving) and five implicit (assuming, detecting, deciding, checking, and judging) reasoning competencies could be identified as task requirements. They will be presented in detail on the poster.

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COMPUTER PROGRAMMING IN THE PROFESSIONAL DEVELOPMENT OF FUTURE MATHEMATICS TEACHERS

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We present part of a 5-year ongoing research (see Mgombelo et al., 2022) where we examine how university students use computer programming as a computational thinking instrument for mathematics inquiry, using a mixed methodology and an iterative design approach. We focus here on some students who are pre-service mathematics teachers and present an analysis on how they use computer programming both for their own learning and for educational resources they create for others. The research takes place at Brock University, where mathematics students as well as future mathematics teachers have the option to take a sequence of three one-semester courses called Mathematics Integrated with Computers and Applications (MICA). In the progression of these courses, students engage in 14 programming-based mathematics investigation projects, including three projects where pre-service teachers may choose to create learning objects (LOs) to teach a mathematics concept. In our larger research, we have been using the instrumental approach to analyse how MICA participants develop programming from an artefact into an instrument (Gueudet & Trouche, 2009) that they can integrate into their practice as mathematicians and teachers; and have analysed their instrumented schemes. Here, using as framework the documentational approach (Gueudet & Trouche, 2009), we focus on how future teachers use programming for their learning and in teaching resources they develop. Data collected include each participant's LO, their associated report, their final projects which are teaching lessons, and semi structured interviews with them. In Mgombelo et al. (2022), we inferred some instrumental schemes that future teachers develop during the development process of an LO. Here, we identify elements of the documentational genesis of future teachers, in how they appropriate programming as a resource, first when they create LO's, and later in their design work for the final teaching lessons.

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SERVICE-LEARNING AND GAMES AS A STRATEGY FOR THE TRANSFER OF MATHEMATICAL KNOWLEDGE

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4 researcher-professors and 168 students 2nd year of Primary Education Degree at the University of Malaga, subject Didactics of Arithmetic, carried out a Service-Learning experience. The objective was to design and develop games with manipulative materials, which were taken to two Public Primary Education centers. We wanted to show how Service-Learning promoted greater practical training and formation of values in future teachers who will have a key role in society (Rodríguez-Gallego, 2014), and to assess the groups' opinion. According to Alsina (2011), the game is a symbolic content activity, through which students carry out a process of adaptation to reality and solve problems that they could not do with real content in the real world. Games and the handling of materials promote mathematical skills development.

The university students organized themselves into 32 groups of 4 to 6 people. They went to the schools with mathematical games adapted to the different primary levels. They took some traditional materials, such as Cuisenaire rods, vertical abacuses, tables of 100 number and Multibase Arithmetic Blocks. Furthermore, the university students made some of the materials by hand, and some of them were also customized with different themes. In addition, they adapted traditional games such as Monopoly (called "Mathpoly"), Bingo, dominoes, roulette and cards with numerical operations, among many others. These university students showed the pupils from primary schools how to play and played with them. Finally, they left the different elaborated games and materials at schools for them to continue using. The four researcher-professors analysed the students´ participation in the Forum after the experience, as well as some of the interviews, in which we can see that this practice was received well and did facilitate a better approach to the reality. They also felt that their workshops were useful, connecting theoretical content with practice. The students at school also improve their learning while they enjoyed themselves. Finally, this experience can be considered a good methodological strategy that favours curricular, personal and social development from university students and students in school.

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EMPIRICAL ANALYSIS OF COMPLEXITIES IN A HLT ON REASONING WITH LATIN SQUARES

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Completing Latin squares, in addition to being a fun pastime, can provide significant learning opportunities to simultaneously work on logical and spatial reasoning processes with Primary Education students (Křížek & Solcova, 2021). It is known that the problem of deciding whether there is a solution to the task of completing a Latin square or not is NP-complete and, consequently, empirical ways must be explored to analyse its complexity. These analyses turn out to be of interest from the field of psychology, since they use the same ordinal scale to link the ability of individuals and the complexity of the task itself (Birney, 2006). Questions other than complexity are the objective, the sequence and the hypotheses about the process that make up a hypothetical learning trajectory (Clemens & Sarama, 2012). The analysis of the exhibited complexity can be used to redefine and improve. In this research, a sequence of activities has been developed in a virtual environment with the aim of developing logical and spatial reasoning processes, based on a series of hypotheses regarding the potential difficulty that each of the variables involved would entail (dimension of the square, number of constraints, nature of the task required…). Thus, a sample of 50406 students from 76 countries and between 6 and 12 years old, has been studied. Its activity has been monitored from 2019 to 2022, analysing the distributions of efficiency and response times by activity. Likewise, the average time it takes for a student to go from one activity to the next has been statistically estimated. In view of the data, it can be concluded that, for the perceptual description task, no differences related to the dimension of the square are observed. Regarding the completion task, a marked increase in complexity is observed when moving from dimension 2 to 3, which does not happen again in the successive dimensional increases. Finally, it has also been noticed that a lower number of imposed restrictions explains a decrease in the complexity of the task.

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PRE-SERVICE TEACHERS' TASK SELECTIONS FOR ASSESSING STUDENTS' MISCONCEPTIONS

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Diagnostic competencies are an important facet of mathematics teachers' professional competence (Leuders et al., 2022). In assessment situations, teachers need to select and assign appropriate tasks that have the potential to reveal an individual student's misconception in a certain topic. Previous research suggests that teachers do not always select the most appropriate tasks to support student *learning*, but we do not well understand teachers' task selection processes in *assessment* situations (Kron et al., 2021). Specifically, there is little empirical evidence for the assumption that selecting tasks with high diagnostic potential leads to more correct diagnoses.

We asked pre-service teachers to identify individual students' misconceptions in the domain of numbers. We were interested in the extent to which the *quantity* and the *diagnostic quality* of selected tasks were related to participants' accuracy in identifying students' misconceptions. The sample comprised 461 pre-service primary school teachers (403 female). We used a digital simulation with six virtual students, each of whom had one specific misconception that was revealed in their task solutions (e.g., errors in tasks requiring understanding of the place-value system). Participants had to identify the virtual student's underlying misconception by selecting tasks from a given portfolio and reviewing the student's answers. We analyzed the number of selected tasks (*quantity*) and their diagnostic potential for revealing the respective student's underlying misconception (*diagnostic quality*).

General linear mixed models showed that across all assessments, selecting one additional task increased the odds of accurately identifying the misconception by 10%, whereas selecting one additional task with high diagnostic potential increased the odds by 101%. Thus, in task-based assessments, the quality of the selected tasks is indeed more important than the total number of tasks used. Further analysis is needed to better understand how pre-service teachers identify relevant task features.

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PERSONALIZATION OF MATHEMATICS TEACHING BASED ON STUDENT'S MATHEMATICAL-COMMUNICATION-FORMS

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a way that encourages excellence, love of learning, and reaching high achievements by tailoring teaching and learning to each student's characteristics. In this poster, we present a case study (Stake, 1995) of applying personalized instruction in-a Teach-In program that has two components: Implementing of learning style (Pashler et al., 2008) related to individual's differences concerning the mode of study that is most effective for them; Using artificial intelligence (AI) system, which enables navigation of the personal learning path (PLP) with human decision-making based on access and analyses of extensive data about student learning (Wolf, 2010) such as preliminary scores and target scores (TSs). Five mathematical communication forms (MCFs) were defined: Verbal, Concrete, Formulaic, Visual and Contextual. The diagnosis of a student's MCFs relies on an admission questionnaire; a personal interview upon admission to the program; the information obtained from the school system. The teaching and learning consist of two components: (i) Teach-In: a tutor-student interactive method of personalized teaching and learning. (ii) I-Teach: an artificial-intelligence-supported computer system. The study aimed to address the question: How and why do the TSs and PLPs match students' MCFs? Participants were four ninth-graders with different MCFs who came from the same class in the same school and received personal instruction from the same tutor. The results indicate the importance of personalized learning paths assignment to students according to their strengths in mathematics communication. This agrees with Pashler et al. (2008), who emphasized the importance of creating an environment that matches students' learning styles. The results highlight the potential of personalized teaching supported by I-Teach which enables navigating each student through their PLP (Wolf, 2010).

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RELATION BETWEEN PRE-SERVICE TEACHERS' PERFORMANCE AND THEIR SELF-ASSESSMENTS WHEN USING COMPUTER ALGEBRA SYSTEMS: A PILOT STUDY

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Self-assessments form the de-facto standard for measuring pre-service teachers' digital competence, for instance, according to the TPACK framework. Such self-assessments are prone to biases (e.g., social desirability, Dunning-Kruger effects) (Kan et al., 2018). So, it is suggested to shift towards performance assessments, requiring teachers to work on standardized problems close to practice (Tabach, 2021). The latter is still rarely used as objectively rating responses is costly. Hence, it is important to investigate innovative test designs that combine the benefits of both assessment approaches.

Thus, we examined, whether pre-service teachers correctly self-assessed their performance regarding the use of digital tools immediately after working on a performance task. If so, efficient post-performance self-assessments may be used instead of a costly objectively rated performance indicator. We developed a test of teacher competence regarding the use of digital tools, combining performance tasks in an open and self-assessments in a closed format. Pre-service teachers had to do a CAS tutorial from a student's perspective and to self-assess their competence level concerning the use of tutorials in teaching (4 steps, e.g., 4="*I can develop such tutorials myself."*). Depending on the self-assessment, we presented a performance task (e.g., 4: create a tutorial from scratch), before they self-assessed their performance again.

A pilot study with $N = 29$ pre-service teachers showed a correlation between the objectively rated performance indicators and post-performance self-assessments (Spearman $\rho = .32$, $p < .05$). In general, pre-service teachers with lower performance evaluated their competence to be lower. Further, we observed tendencies to overestimate the competence. The low-medium correlation indicates that self-assessments cannot be considered as a reliable indicator. So, we suggest focusing future development efforts on performance indicators independent of self-assessments.

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ATTENDING TO RELEVANCE IN MATHEMATICS TEACHING

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The overarching goal of mathematics education is for students to develop mathematical competence necessary for everyday life, work, and to partake in a democratic society (Niss, 2007). However, previous research indicate that teachers seldom attended to the relevance of goals, except when referring to exams and tests (Selling et al., in review). Furthermore, research (e.g., Sealey & Noyes, 2010) indicates that mathematics education should be relevant and help students understand mathematics role in society, emphasizing that such aspects should be attended to during lessons. Thus, focusing on clarity and coherence, rather than relying upon students making inferences about the lesson goals and intra and extra-mathematical connections. The research question for the present study is as such: *how does teachers in the Nordic countries attend to relevance in the classroom, in terms of learning goals and activities?*

The study is part of a larger study, with data from the LISA Nordic study (Selling, et al., in review). The data comprises video recording from lower secondary mathematics classrooms in Finland, Norway, and Sweden. These countries are selected as overarching relevance goals can be found in their national curricula.

A two-camera setup is utilized, with audio collected simultaneously from the teacher and the classroom as a whole. Content analysis is applied to investigate teachers' goals and activities focusing on relevance. Analyses are conducted on a sentence-level, based on utterances, written communication if available, as well as classroom discussions. Preliminary results indicate that relevance is seldom attended to as an explicit goal, except when it is a result-oriented goal. While some lessons connect mathematical topics to everyday life, such as shopping or understanding economical aspects, there are few instances of lessons that target more overarching questions of society and democracy.

The poster will present an overview of teachers' goals centered on relevance and real-life connections in classroom samples from Finland, Norway, and Sweden.

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TIMESCALE ANALYSIS OF TEACHERS' TALK: A THEORETICAL FRAMEWORK

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Mathematics teachers' conversations about teaching are filled with talk about time. Teachers may express the dilemma of covering many precise procedures and concepts in a short span of time. This task-based cycle of mathematics classroom organization sits within wider, regimented temporal cycles—days, learning modules, courses, grade levels—typical of educational institutionalization. Framing mathematics as a tool of contemporary social critique or as a timeless, abstract symbolic system brings forth contested attitudes towards time that lie at the heart of the discipline. Mathematics teachers' experiences with temporality are a central yet poorly understood dimension of their work (Staats & Laster, 2019).

The construct of timescales in discourse—with the more general construct of sociolinguistic scales—can elucidate teachers' experiences of temporality (Barwell, 2020; Lemke, 2000). Timescales are features of discourse that invoke socially-recognized but differentially-valued forms of action over time (Barwell, 2020; Blommaert et al, 2015). A timescale could emerge in teachers' talk through direct reference to temporal dilemmas of teaching, or as a performance of action, a style or manner of speaking that is conditioned by temporality, for example, if a teacher speaks quickly and leaves out information in order to complete a mathematical task in the remaining class time. We argue that this seeming contrast of "reference to" and "performance of" timescales is a false distinction, which we exemplify with excerpts of secondary mathematics teachers' professional development discussions. This poster reviews and responds to theoretical and ontological debates in timescale scholarship to propose a framework to further timescale research in mathematics education.

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ANSWER-CHECKING STRATEGIES REVEAL GROWTH IN COGNITIVE POWERS: NUMERICAL REASONING OF A STUDENT WITH LEARNING DIFFICULTIES IN MATH

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This is a case study of a 6th grade boy identified as having learning difficulties in mathematics. He was a participant in a larger research and professional development project focused on teaching math conceptually (grades K-5). We used constructivist teaching experiments (Cobb & Steffe, 1983) to make inferences about and build models of how the learners constructed conceptual structures and operations.

A major challenge for students experiencing difficulties in learning mathematics is to overcome a long-learned coping mechanism of looking to others as authorities for "right or wrong answers" (Brousseau & Warfield, 1999). Therefore, we examined what strategies the student used to check his answers. This led to the research question: how do the student's answer-checking strategies relate to his problem-solving strategies? The video data was collected across four teaching episodes during which we used an activity designed to support an advance from count-all to count-on.

Episode	Problem-solving strategy	Answer-checking strategy
1	Count All	No strategy
2	Prenumerical Count-On	Count All
3	Numerical Count-On	Prenumerical Count-On

Table 1: Correspondence of problem-solving to answer-checking strategies

As seen in Table 1, this boy's answer-checking strategies remained one conceptual step behind the ones he used for problem-solving. His answer-checking strategies were evidence of what he could achieve independently, whereas his problem-solving strategies were those he was still developing. Therefore, answer-checking strategies can serve as a formative assessment for what strategies children use independently.

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THE EFFECT OF ASKING FOR INTERMEDIATE QUANTITIES WHEN SOLVING MULTI-STEP WORD PROBLEMS

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In this study, we use HINTS, an intelligent tutoring system for the arithmetic and algebraic solving of word problems, that stands out for its ability to provide tailored assistance and its high level of granularity (Arnau et al., 2014). We explored to what extent a problem statement that includes a larger number of questions related to unknown intermediate quantities can be useful to develop students' ability to solve word problems. This quasi-experimental mixed study lasted eight once-per-week sessions and involved fifth-grade students divided in two groups. There were 27 students in the intervention group (IG) and 28 in the control group (CG). First, students of both groups completed a pre-test to evaluate their initial level in the resolution of word problems. Then, students used HINTS to autonomously solve word problems. However, while the statements of the problems presented to the CG only asked for the final unknown quantity, statements in IG included extra questions related to intermediate unknown quantities. Finally, both groups took a post-test with equivalent problems to the pre-test ones. In addition, a case study was conducted with five of the CG students who had had the most difficulties during the post-test. Regarding the quantitative results, the comparison of the pre- and post-tests reveals a significant improvement in both groups' ability to solve word problems arithmetically. In this sense, the use of HINTS could increase the competence of the participants to carry out one-step analytical processes supported using conceptual schemes. However, the non-existence of differences between groups would lead us to conclude the inclusion of intermediate unknowns in the statement has not resulted in a higher level of students' ability. The case study shows that, after failing to solve problems in their short version with HINTS, some students quickly solved them with the intermediate questions.

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TECHNOLOGY USE: ANALYSIS OF LESSON PLANS ON FRACTIONS IN AN ONLINE LABORATORY SCHOOL

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This study aims to investigate how pre-service teachers (PSTs) use of technology when preparing fraction lesson plans in an Online Laboratory School. The school is established during Covid-19 pandemic under the roof of a university. The school served to hundreds of public-school students all around Turkey since Spring 2020. It also aims to provide quality internship experience to mathematics PSTs by planning, teaching, and reflecting on the experience. For this project we focus only the planning experiences of PSTs on the topic of fractions. Fractions are 30 %of middle school curriculum in Turkey which is centralized and students having most difficulty. There were 17 PSTs participated and planned lessons as a group. PSTs prepared 12 written lesson plans on fractions (ranging from 5-13 pages) and taught 4th, 5th, and 6th grade mathematics during 2020-2021 academic year. Content analysis of the written lesson plans were conducted. We first analyzed whether technological tools were used, if yes then which technological tools were used, and for what purposes.

Results show that the web 2.0 applications were present almost in all plans, sometimes more than one applications (Nearpod, Padlet, GeoGebra, etc.) were used in a lesson. More specifically, Math Learning Center was used in 7 out of 12 lesson plans, and Nearpod was used in 6 out of 12, Powtoon and Wordwall was used in 3 out of 12 lesson plans: Animaker, Math Playground and GeoGebra were used twice, Mentimeter Learninggaps, Conceptua Math, Plotagon, Phet Colorado Edu. were used once in 12 plans. One of the most used applications were the Math Learning Center; it was used to model fractions and to provide solutions of fraction problems through the model and this tool was teacher directed. The other highly utilized tool was Nearpod Applications, and they were used to provide students with an individual study environment, and to measure what students learned at the end of the lesson.

We can conclude at least one technological tool was used in all the online lesson plans. Some of the technological tools were used only by the teacher and all students observed. In some cases, students were given the opportunity to study online individually. We are also implementing TPACK framework to the same data to study the details of the PSTs' technological pedagogical knowledge which we might share results in PME also.

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TEACHING MULTIPLICATIVE WORD-PROBLEM SOLVING TO A STUDENT WITH AUTISM SPECTRUM DISORDER

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Recently, there has been a growing interest in teaching mathematical problem solving to students with learning difficulties. Several studies focus on students with autism spectrum disorder (ASD), who often exhibit deficits in key cognitive skills required for problem solving, such as working memory and executive functions. While most studies focus on simple arithmetic word problems involving addition and subtraction, little research has been conducted on teaching students with ASD to solve problems involving multiplication and division (Polo-Blanco et al., 2022).

In this work, we tested the conceptual model-based problem-solving (COMPS) methodology for its effectiveness in teaching a student with ASD and intellectual disability to solve multiplicative word problems. COMPS methodology employs equation-like, conceptual model diagrams that emphasize algebraic expressions of relations. In particular, we conducted a single-case, multiple-baseline across behaviors design. The intervention was carried out to help a 14-year old ASD-diagnosed student improve his ability to solve one-step equal groups, multiplicative comparison and Cartesian product problems, each of which was addressed as a separate behavior.

The results of the study show that, after a low initial baseline, performance both in identifying as well as in performing the correct operations rose quickly to 100% for the three problem types, and remained over 75% during follow-up and generalization stages. During problem-solving training, the student also showed higher levels of concentration and interest. Finally, an experiment was performed in which the acquired skills were generalized to real-life situations, in particular cooking. In conclusion, the findings of this study are promising in terms of the content that can be addressed with appropriate methodologies in students with these characteristics.

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THE TEACHING PRACTICE OF BUILDING ON MOSTS

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To better understand and improve teachers' ability to engage in complex practices, Grossman and her colleagues (2009) suggested that practices be decomposed into their "constituent parts" (p. 2069) for the purpose of helping teachers develop these practices. We share a decomposition of *Building on MOSTs* (Mathematical Opportunities in Student Thinking; Leatham et al., 2015)—a teaching practice that takes full advantage of high-leverage student mathematical contributions made during whole-class interaction. We theorized that Building is comprised of four elements: (a) *Establish* the student mathematics of the MOST as the object to be discussed; (b) *Grapple Toss* that object in a way that positions the class to make sense of it; (c) *Conduct* a whole-class discussion that supports the students in making sense of the student mathematics of the MOST; and (d) *Make Explicit* the important mathematical idea from the discussion. We initially proposed this conceptualization of Building to a group of 12 teacher-researchers who then worked with us to refine that conceptualization by creating instantiations of Building in their grades 6-12 classrooms (ages 11-18). As we studied these instantiations, we continually shared with them our developing understanding of Building and they created new (and improved) instantiations. We will present the aspects of the elements of Building that emerged from this work, and also provide evidence for the value of this practice in improving in-the-moment use of high-leverage student mathematical thinking during instruction.

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ALGEBRAIC RATE WORD PROBLEMS: HIGH SCHOOL STUDENTS PERFORMANCES WHEN INTEGRATE THE SPREADSHEET

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Problem solving in the mathematics curriculum has been proposed for more than four decades (Gallagher, 1997). However, according to Sanz and Gómez (2018), the problems have been used as practice to apply or evaluate the knowledge acquired, and not necessarily to reflect on the resolution process. Furthermore, what about the processes followed when solving problems in a scenario that integrates digital technologies? In this sense, the research aim is to identify the epistemic value of the processes of solving verbal algebraic rate problems that high school students follow when integrating the electronic spreadsheet. The theoretical framework is the Instrumental Approach (Haspekian, 2005), so it was interesting to know the epistemic value of resolution processes (techniques) to solve problems (tasks).

The study was carried out with 18 students from an Industrial Technological and Services High School Center in Mexico. It consisted of three phases: 1) Design and selection of activities and problems. 2) Implementation, it was divided into two parts, first activities were carried out so that the students became familiar with the electronic spreadsheet, and later they solved the problems in pairs. 3) Collection and analysis of the data obtained. The sessions were videotaped, and the files of the students' procedures were compiled.

The results indicate that the use of the electronic spreadsheet led to the development of algebraic reasoning associated with generalization and the expression of generalities in students, using increasingly formal symbolic languages. Pattern recognition was observed where one or two variables appeared and these were related to the data through formulas (epistemic value of the techniques).

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DEVELOPING A CODING SCHEME FOR EXAMINING PRE-SERVICE PRIMARY TEACHERS' SPECIALIZED KNOWLEDGE ON FRACTIONS DIVISION

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Having a profound knowledge of fractions division (FD) is still a challenge for pre-service primary teachers (PTs). The current work aims to develop a coding scheme to examine PTs' specialized fractions division knowledge (SFDK). This coding scheme will be pivotal in unraveling the PTs' initial knowledge, such as their use of representations or errors when solving FD problems before designing instructional activities to develop the knowledge.

Two stages were carried out to develop the coding: (1) development and (2) testing, assessment, and refinement. The coding scheme was developed referring to the SFDK model (Wahyu, 2021) that represents connected and flexible knowledge of FD. Connected SFDK is the PTs' ability to move forth and back when using multiple representations in solving FD problems. Flexible SFDK is PTs' ability to differentiate each FD, which guides her/his movement across the representations.

PTs were given five tasks before being introduced to fractions division. For instance, *write a different word problem for 4÷2/3; 2/3÷4; 3/4÷1/2; and 1 2/3÷1/4!* (Task 4). There were 57 PTs' answers to the tasks and 10 answers were purposively selected for the coding to establish reliability. After five iterations of coding, discussion, and refinement, the coding scheme resulted in 7 categories and 36 codes. For example, category 6 (*symbolic-verbal*) has five codes (e.g., *correct answers-partitive*) to examine task 4. The percent agreement of three coders is over 0.7 and the AC_1 coefficient (Gwet, 2014) is greater than 0.7 for each category, both of which show a very good agreement.

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