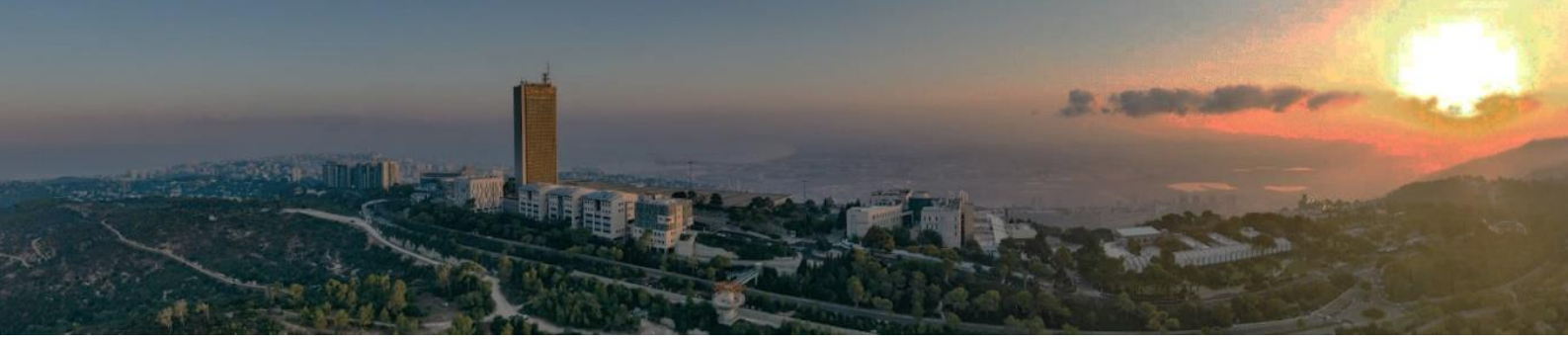

PROCEEDINGS

OF THE

46th CONFERENCE

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Haifa, Israel | July 16 – 21, 2023

PME-46

MATHEMATICS EDUCATION FOR GLOBAL SUSTAINABILITY

Volume 2

Editors: Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach

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Research Reports

A – G

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TABLE OF CONTENTS VOLUME 2

RESEARCH REPORTS (A-G)

VOLUME 2

RESEARCH REPORTS

A - G

PME-46

MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

CONNECTING REPRESENTATIONS FOR COMMUTATIVITY: STUDENTS' RICH DISCOVERIES IN A MULTI-REPRESENTATION TOOL WITH NON-EXPLICIT ARTICULATIONS

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Although digital dynamic multi-representation tools have been shown to provide potential for developing understanding, little is known about the exact conditions under which they should be explored. This paper reports from a design research study on fifth graders' exploration of a dynamic multi-representation tool on dot arrays for multiplication, seeking to deepen the understanding and justify the commutative property. The qualitative analysis of learning processes reveals that although various connections are discovered by different students, they articulate only some of them explicitly, so they are rarely combined to a justification. We conclude that for exploiting the potential of multi-representation tools in depth, more scaffolding for articulations is required, which can be realized by embedding the tool into a more comprehensive learning environment.

Digital technology can have different characteristics and follow different purposes in mathematics education, ranging from highly scaffolded tutorial systems to digital tools designed for very open explorations. Whereas early research overviews list these different technologies separately (Lagrange et al., 2003), they have become increasingly combined in digital learning environments with different features and degrees of openness (Hillmayr et al., 2021). For developing conceptual understanding of mathematical concepts and operations, the potential of *dynamic multi-representation tools* has often been identified qualitatively (Kaput 1986; Sacristan et al., 2010), while only moderate effect sizes were found in efficacy studies, with large variations across contexts and exact design features (Hillmayer et al., 2020). So, there is still a need to further disentangle the conditions under which *dynamic multi-representation tools* can productively enhance students' conceptual understanding (Lagrange et al., 2003; Drijvers et al., 2016). This research gap particularly exists for arithmetic content in early middle school which is less researched than functions and algebraic expressions and equations in later middle school (Drijvers et al., 2016). In our design research study, we focus on a dynamic multi-representation tool and investigate fifth graders' collaborative explorations of the commutative property, pursuing the following research question:

How do students work with a dynamically linked multi-representation tool to explore the commutative property and what connections of representations do they articulate?

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THEORETICAL BACKGROUND

Potentials of multi-representation tools for connecting multiple representations

Using multiple representations have been shown to bear potential for developing students' conceptual understanding: While the *treatments* within one representation often correspond to procedural rules (e.g., symbolic manipulations as following the commutative property), *conversions* between representations can enhance students' processes of constructing meanings (Kaput, 1986; Duval, 2006). For developing understanding for properties underlying the symbolic treatments, *representation-based justifications* (Schifter, 2009) convey meanings by conversions of treatments as will be exemplified.

Already Kaput (1986) promoted the potentials of dynamically linked multi-representation tools: "Information technology will have its greatest impact in … providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations" (p. 5). Since then, many digital tools were constructed for functions and algebra in which multiple representations are not only simultaneously depicted, but dynamically varied, and the effects of variation can be explored. Case studies have confirmed Kaput's (1986) early assumption that *dynamically linking multiple representations* can enhance students' understanding (Sacristan et al., 2010). However, quantitative efficacy studies found only small to moderate effect sizes, with large variations across exact design features (Hillmayer et al., 2020). This calls for further in-depth analysis of conditions under which multi-representation tools really enhance students' processes of meaning construction.

Empirical studies revealed a critical condition of success: Converting representations contribute best to constructing meanings *if* the connection between the representations is *explicitly articulated*, this involves addressing the relevant structure in all representations (Renkl et al., 2013). With respect to multi-representation tools, this raises the question how the automatic links really promote students' mental construction of connections and their explicit articulation of the relevant structures in view.

Topic in view: Commutative property and unit structures

Whereas most research on multi-representation tools was conducted for algebra and functions, our topic in view, multiplication and the commutative property, has attracted much less attention (Drijvers et al., 2016). Sinclair et al. (2020) constructed an interesting multi-representation tool with TouchTimes, but given the irritations they reported from teachers with its unfamiliar representation, we focus on *rectangular dot arrays.* Arrays have been identified as powerful graphical representations for conveying meanings for multiplication, e.g., visualizing 2 x 4 as 2 rows of 4 in an array (Wittmann, 1998; Schifter, 2009). Many students, however, convert only superficially between arrays and expressions by only hinting to length and width or the total numbers (see Figure 1, left). For learning to understand *multiplication as counting in composite*

units, students need to mentally impose multiplicative unit structures onto the dot array (Götze & Baiker, 2021; Askew, 2019). This can be supported by *multi-representation*

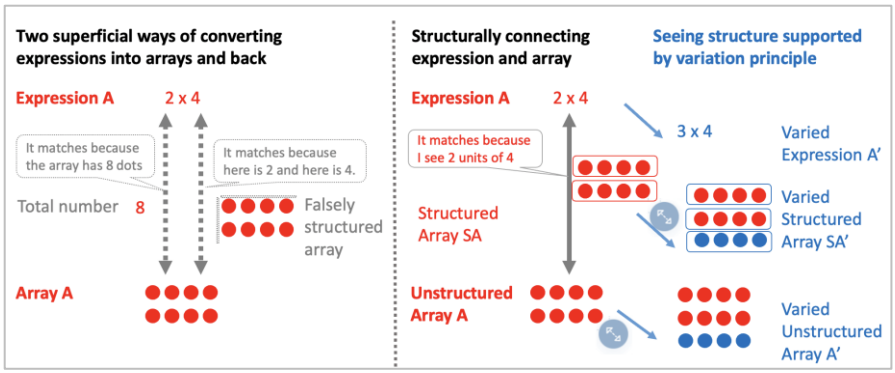

Figure 1. Meaning of multiplication in dot arrays requires structuring in composite units – Variation in multi-representation tools can support seeing structures

tools by dynamically linking symbolical expressions such as 2 x 4, 3 x 4, to arrays that grow in rows (depicted in blue in Figure 1), while composite unit structures must be explicitly articulated by teachers (Askew, 2019) and students (Götze & Baiker, 2021).

The *commutative property* allows for rule-based treatment of symbolic expressions (changing the order of factors from 2 x 4 to 4 x 2). For its *representation-based justification*, we convert one symbolic expression EA (2 x 4) into a structured array SA, conduct a treatment of representations (rotate the array) into SB and then convert it back to the second expression SB as in Figure 2 (Wittmann, 1998; Schifter, 2009). This justification fails short if the array is only converted without addressing the composite changing units (Strømskag & Valenta, 2017), and requires further prompts for generality. That is why we decided that our multi-representation tool should also engage students into examining variations (from SA to SA', from SB to SB'), while developing structural connections for a general justification of the commutative property.

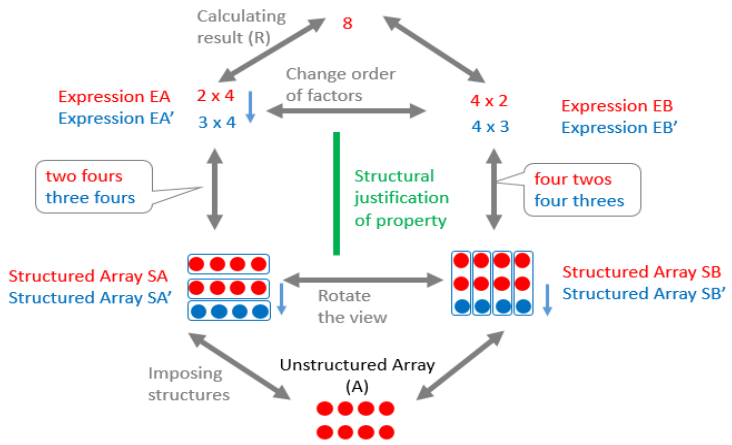

Figure 2: Representation-based structural justification of commutative property in dynamically linked multiple representation (adapted from Tondorf & Prediger, 2022)

Figure 2 shows the three-dimensional complexity of processes that students are invited to first *tacitless conduct and observe* and then *explicitly articulate* for explaining the meaning of multiplication and for structurally justifying the commutative property.

The figure includes vertical arrows for conversions from symbolic to graphical representations and horizontal arrows with treatments reflecting the commutative property, blue arrows signify dynamic changes conducted to focus students' attention on the structure of composite units and on generality. In total, the representation-based justification of the commutativity is a conversion of treatments. Its articulation requires to explicitly address the composite unit structure of the structured arrays and the combined verbalization of the horizontal and vertical arrows in Figure 2.

Design of the open exploration task with a multi-representation tool

In our digital learning environment *divomath*, students first have extended learning opportunities for connecting representations for multiplication with composite unit structures. Like in Figure 1, students study the systematic variation of arrays and articulate their effect on the expressions and vice versa. A later task in the learning environment is the open exploration task in Figure 3 in which two students are invited to systematically vary an array and see two symbolic expressions, EA 5×7 and EB 7×5, which they can understand as being connected to the array by structuring it in rows into the mentally structured figure SA or in columns in the mentally structured figure SB (see Figure 2). Both students can start structuring in rows, but as Student B looks from a 90-degree angle onto the iPad, this appears as columns for Student A and vice versa. Exchanging their ideas while systematically varying the array, students can discover rich relations, and explain different connections that can later be combined into the justification of the commutative property:

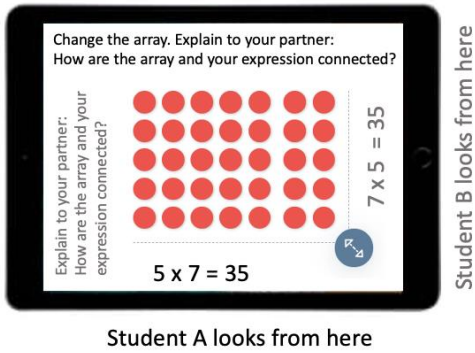

Figure 3: Exploring commutativity with a multi-representation tool from view A / B

- (already known) connection between their structured array and their expression looking at their rows, each (Student A: SA – EB, Student B: SB – EB, respectively)
- the effect of the systematic variations of their array on their expression (SA→SA' – EA→EA' or SB→SB' – EB→EB', respectively)
- justification of commutativity in static arrays (EA–SA–SB–EB or vice versa)
- consequences of varying an array in the context of commutativity (Figure 2).

As the exploration of digital tools alone rarely guarantees that students construct the targeted knowledge unless the processes are suitably scaffolded, either by the teacher or by tasks structuring the exploration (Drijvers et al., 2016; Hillmayr et al. 2020). It is critical for our design to study the students' processes and identify the potentials and limitations of this open exploration. This will later allow us to identify necessary task-

specific structural and language-related scaffolds for finding, articulating and combining the rich connections listed above and visualized in Figure 2.

METHODOLOGICAL FRAMEWORK

Methods of data collection. The case study documented here belongs to a larger design research project that develops and investigates a comprehensive learning environment on multiplication and division in Grade 5, with the dual aim of (a) developing and optimizing a digital learning environment, and (b) generating deep insights into conditions of productive learning with the tools (Gravemeijer & Cobb, 2006). In Cycle 1, design experiments were conducted with seven groups of 2-3 fifth graders (10/11 years old), in total about 300 minutes video or audio records and screen records per group. For pursuing the research question of this paper, we focus on their processes on the task in Figure 3 for which the video and audio material was partially transcribed.

Methods of data analysis. For the qualitative analysis of the transcripts and screen recordings, students' processes of interacting with the multi-representation tool and its automated links were coded with respect to the connections between representations that they addressed explicitly in their articulations (see Figure 2 that serves as deductive coding system, adapted from Tondorf & Prediger, 2022). We code EA – A when a match is simply stated and EA – SA – A when the unit structure is explicitly verbalized. The variation A→A' denotes verbalization on the array with focus on single dots, SA→SA' the structured arrays with articulating the unit structure in rows or columns. When the conversion of treatments is addressed, it is denoted EA-EB – SA-SB.

EMPIRICAL INSIGHTS

Episode 1: Aylin and Lisa explain composite units and swapped expressions

Aylin and Lisa work independently without a teacher. They first approach the task (from Figure 3) by checking that the given array and expressions fit (Turn 2) and then start changing the array (Turn 3).

02 Aylin [*looks at one dot in the array and the double expression 1 x 1 = 1*] Yes, that is correct. I think we need to go back. We have to do this. How are we doing this?

03 Lisa [*starts reading the text*] Change # [*Aylin is changing the array representation*] # Wait, wait, wait. Ey, the task has changed!

06 Aylin [*stops the array at four rows of five*] We need to explain why they are connected.

07 Lisa Okaaayyy. That is a swapped expression. Because here, um, it is five times four and over here, it is four times five. Swapped expressions.

08 Aylin We have to explain why it is connected. It is connected because there are four rows and there are five dots in each row. Thus, the expression matches because four rows of five dots each equals 20.

Lisa discovers that both expressions are "swapped", a standardized term for tasks with changed order (Turn 07: EA–EB). But they do not follow up this new observation, but go back to the tasks that they had practiced before, explaining the connection EA–A. They explain it with two arguments, first by explicitly articulating the composite units

in the conversion "four rows and there are five dots in each row" (Turn 08b: EA–SA–A), and second by the treatment and the total number as result "Thus, …, equals 20" (Turn 08c: EA-R-A). The swapped expression EB is not explicitly connected to the array A. After explaining the connection EA–SA–A, the students go on with the next task and don't focus on changes through variation or relations of commutativity.

Episode 2: Lena and Jenny qualitative consider changes without unit structures

Lena and Jenny start manipulating the array several times and observe the changes in the expressions. The design experiment leader acts as teacher and tries to elicit a description of the changes AB→AB' – EA→EA' and EB→EB'.

09 Teacher And how exactly do they change?
10 Lena If you pull the array, the expressions change.
11 Teacher But how do the expressions change?
12 Jenny When more dots are appearing.
13 Teacher And are the expressions completely different calculations?
14 Lena No, here is six times seven and there is seven times six.
15 Jenny So, it is swapped.
16 Lena So, swapped.

Their first descriptions of the changes are qualitative: The "expressions change" (Turn 10), so far without specifying how they change and how the changes are related. Jenny articulates on ordinal description, the number of dots increases, also without relating to the unit structures (Turn 12: A<A' – R<R'). Rather than eliciting an explicit articulation of more rows or more dots in a row, the teacher turns the focus to the two expressions (Turn 13). Again, the conversation stops here stating the swapped expressions (Turn 15/16: EA–EB), without exploring the intended richer connections.

Overview on articulated connections and variations in all seven groups

	Intended articulation of connections	Aylin & Lisa (independent)	Lena & Jenny (with teacher)	Group 3	Group 4	Group 5	Group 6	Group 7
Conversions within 1 view	EA–SA–A EB–SB–A	EA–SA–A EA–R–A		EA→EA'	EA–SA	EA–SA–R EB–SB–R	A EA–SA–R	
Systematic variation	SA→SA' – EA→EA' SB→SB' – EB→EB'		A<A' – R<R'	SA→SA' – EA→EA'			EA→EA'	
Addressing commutativity	EA – EB SA – SB	EA – EB "swapped"	EA – EB "swapped"	EA – EB	EA–R–EB		EA – EB	SA–R–SB
Justifying commutativity	EA-EB – SA-SB Generality for each ' EA'-EB' – SA'-SB'						SA→SA' – SB→SB' (turning the array)	EA-EB – SA-R-SB (equality of results)

Figure 4: Analytic summary with all connections articulated by seven groups

Unlike Aylin & Lina, the second pair Lena & Jenny uses the linked representations by varying the array and looks at the effects for the linked expressions. But again, they miss the rich opportunity to go deeper into the connections. Similar phenomena were found for other students, as the analytic summary in Figure 4 reveals. Most students focus on simple conversions and variations. They only address commutativity focusing

on the expressions and not on structured arrays. Group 6 and Group 7 try to justify commutativity but either they turn around the structured array instead of restructuring it or they include the result rather than the structures in their justification.

CONCLUSION AND OUTLOOK

The analysis of both episodes (and of the unprinted transcripts of five more groups) reveals that playing with a multi-representation tool with automated links of representations and systematic variation opportunities can indeed engage students in discovering various connections and variations (Kaput, 1986; Sacristan et al., 2010). This is visible in the analytic summary in Figure 4 in which for the first three steps of the learning pathway, each intended connection was discovered by at least one group of students. But by far, not all relevant connections were discovered and explicitly articulated by all students, like Aylin & Lisa in Episode 1 who did not really ask how the array changes with the expressions (systematic variation) and why the swapped expressions describe the same figure (justifying commutativity). Without the different connections being articulated, no complete justification can be combined out of them (Schifter, 2009).

Although students discover and articulate different connections which might invite a conversation about differences, the analytic summary in Figure 4 further reveals that only a few groups address conversions or treatments when describing the variation and rotation of arrays and expressions, as the combination of ideas was not sufficiently supported. The theory section highlights that the variation of treatments targeting the commutative property involves a number of treatments and conversions that need to be taken into account while exploring and understanding the whole concept. This corresponds to the often identified need that teachers facilitate students' rich processes in a more focused manner (Drijvers et al., 2016). Even in groups where teachers were present, however, their guidance was not always ideal to focus students' attention to a particular connection and to support them combining the ideas.

This case study supports the requirement not only to develop multi-representation tools, but to include scaffolds supporting students' focus of attention and students' articulations into the material itself. Rather than providing only tools for open exploration, a holistic learning environment is necessary to support teachers and students in successful and more focused learning processes. In our project *divomath*, the next design experiment cycle will experiment with a learning environment that includes focus questions and help for systematic explorations and explicit articulations, focusing the different connections to support students in exploring the conversion of treatments in commutative property. The support that should be included into the material must focus explicitly on the variation processes and its impact on the unit structures, with questions such as *"What happens to the array if student 1 adds a row?"* or *"How does each expression change if student 2 adds a row? And how do you see that in the array from your point of view?"*.

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REVEALING COGNITIVE PROCESSES WHEN COMPARING BOX PLOTS USING EYE-TRACKING DATA—A PILOT STUDY

Martin Abt, Frank Reinhold and Wim Van Dooren

University of Education Freiburg, Germany & KU Leuven, Belgium

Comparing data sets based on box plots is a challenging task. A typical error is due to a bias caused by thinking the box area is related proportionally to part of the sample, while it is inversely related to the density of the sample. How an area bias exactly affects different individuals, and what strategies individuals who are not affected by an area bias use, is not yet completely understood. We model different cognitive processes to make predictions for solution patterns in six item types, assign students to our a-priori defined patterns, and show that it is possible to validate our hypotheses for their underlying cognitive processes by analysing eye-movement gaze patterns.

INTRODUCTION

The box plot is a frequently used form of representation in descriptive statistics with a high content of information. Because of its compact representation of descriptive values, it is well suited for a comparison of several distributions (Kader & Perry, 1996). On the other hand, this compact representation makes it a complex and challenging subject to learn (Bakker et al., 2005; Edwards et al., 2017). One main reason for errors is the counterintuitive meaning of the box area (Lem et al., 2013): it is inversely related to distribution's density, and thus not proportionally related to the represented part of the sample (Bakker et al., 2005). Still, the latter is a way of interpreting diagrams that students usually establish in mathematics lessons over several years (Ben-Zvi & Garfield, 2004). One plausible explanation for students' struggles when handling box plots is the naïve concept 'the more area, the higher the proportion of the sample', derived from a strong curricular focus on other representations (i.e., bar and circle charts)—while the scientifically correct concept would be 'the more area, the lower the density of the data points. Based on these assumptions, students that have not (yet) acquired the correct conceptual knowledge will be systematically biased by the area of the box—leading to a characteristic error pattern in specifically designed items: we call items that would consistently be answered correctly with recourse to an area bias *area-congruent*; an area bias always leads to an incorrect answer in area-incongruent items (Fig. 1). This area bias was systematically investigated and reported by Lem et al. (2013). Besides this area bias (caused by the saliency of the box area), another specific error when comparing two data sets with box plots was recently discussed: If both medians are above or below the critical mark, a comparison of the medians is not meaningful with respect to the task. However, this does not prevent an inappropriate use of medians in these cases in terms of an overgeneralization of the median strategy (Abt et al., 2022). Considering this possible overgeneralization of the median in addition to the area bias, items can be area-incongruent and/or median-incongruent,

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leading to (at least) six different types of items that can be assumed to be solved systematically correct or incorrect (Fig. 1).

Stimulus: At two schools A and B, the same number of children answered the question about how many minutes they spend getting to school in the morning. At one of the two schools, there are more children with a trip to school of more than 10 minutes than at the other school. Decide at which one.

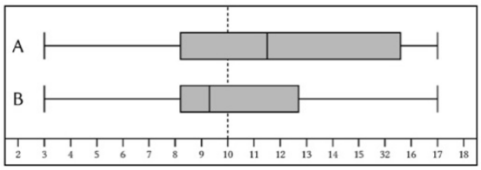
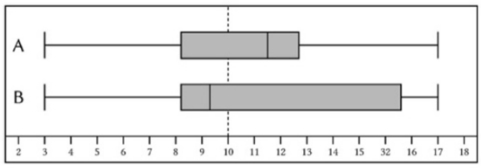
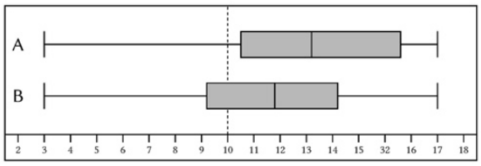
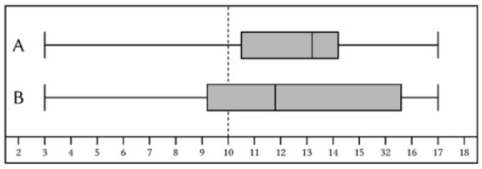
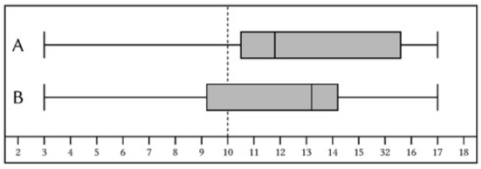
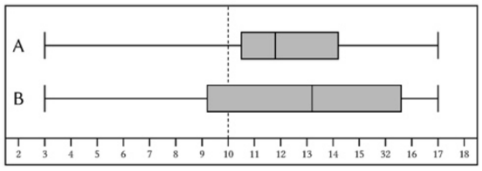
	area-congruent items	**area-incongruent items**
median items (i.e., the critical mark is between both medians—always resulting in median-congruency, because comparing the medians can be considered "correct" in these type of items)	 Applying an area bias leads to the same correct answer (A) as comparing the median (correct median-strategy).	 The correct answer after comparing the medians is A, but the use of an area bias leads to the incorrect answer (B).
median-congruent box items (i.e., one of the two boxes is completely above the critical mark, which means that these items can be solved by comparing the positions of the boxes)	 Applying an area bias leads to the same correct answer (A) as comparing the position of the boxes (correct box-strategy). Comparing of the medians leads also to the correct answer (A).	 The correct answer after comparing the position of the boxes is A, but the use of the area bias leads to the incorrect answer (B). Comparing of the medians leads also to the correct answer (A).
median-incongruent box items	 Applying an area bias leads to the same correct answer (A) as comparing the position of boxes (correct box-strategy). Comparing of the medians in contrast leads not to the incorrect answer (B).	 The correct answer after comparing the position of the boxes is A, but the use of the area bias and comparing of the medians both leads to the incorrect answer (B).

Figure 1. Congruency regarding area bias and median overgeneralization

This classification of six item types was used to distinguish cognitive profiles, which differ in whether both, none, or only one strategy (in-between profiles) has been acquired. Behaviourally, each of these profiles can then be assigned a specific pattern of accuracy in the given answers, which indicate how different item types are systematically answered correctly or incorrectly (Abt et al., 2022, Fig. 2). Consequently, samples should fall into different profiles, which was recently shown (Abt et al., 2022).

	cognition: applied strategies in specific item types		behaviour: expected solution rates in different item types					
			median items area bias		box items area bias			
					congruent median bias		incongruent Median bias	
profiles	median items	box items	cong.	incong.	cong.	incong.	cong.	incong.
area biased	area bias	area bias	1	0	1	1	0	0
guessing	guessing	guessing	0.5	0.5	0.5	0.5	0.5	0.5
no box-strategy 1	median strategy	median strategy	1	1	1	0	1	0
no box-strategy 2	“	area bias	1	1	1	1	0	0
no box-strategy 3	“	guessing	1	1	0.5	0.5	0.5	0.5
no median-strategy 1	area bias	box strategy	1	0	1	1	1	1
no median-strategy 2	guessing	“	0.5	0.5	1	1	1	1
proficient	median strategy	box strategy	1	1	1	1	1	1

Figure 2. Expected pattern in accuracy for the different profiles

The present study

Since now, the link between cognition and behaviour was solely made on the basis of solution rates; we aim at finding additional empirical evidence for the theoretically derived cognitive processes leading to different answer patterns. Therefore, we created an itemset according to the six item types (see Fig. 1). We firstly ask:

RQ1: Do the distinctions between box and median items, area-congruency and median-congruency largely explain the variance in item difficulty—validating the item generation process?

Regarding this research question our first hypotheses is:

Hyp. 1: The item type as well as item congruency has a significant effect on the items and explains variance to a large extent. Box items show higher solution rates than median items, congruent items show higher solution rates than incongruent items, median-incongruent items show higher solution rates than area-incongruent items.

Using an additional person-centered rather than a sample-based statistical approach we then assign each student to a profile based on the patterns in accuracy (see Fig. 2). In a third step, we are interested in whether eye-tracking data can be used to validate the assignment based on the patterns in accuracy. In particular, we are interested in which salient elements of the box plot representation are used during the problem-solving-process. Following the eye mind assumption we use eye-tracking data for this purpose (Strohmaier et al., 2020) to answer a second research question:

RQ2: Can we find evidence in the gaze data for the hypothesized profiles, in the sense that the gaze data show differences between groups of students—who were identified based on behavioural data—and that these differences indeed point in the expected directions (as reflected in the hypotheses)?

Our hypotheses regarding the eye-tracking data in this pilot study focus on students' gaze behaviour. We argue that:

Hyp. 2a: Median strategy is manifest in longer focuses on the median on the one hand and the critical mark on the other. An increased number of saccades between these foci indicates the comparison of these parameters.

Hyp 2b: Box strategy is manifest in longer focuses on the first quartile on the one hand and the critical mark on the other. An increased number of saccades between these foci indicates the comparison of these parameters.

Hyp. 2c: Area bias is manifest in longer focuses on area above the critical mark. This area is mainly captured by a comparison of the third quartiles, which shows up in fixations on the third quartiles and saccades between these fixation points.

METHOD

Sample

All participants (N = 27) were student teachers at the Freiburg University of Education who had taken a course in mathematics in the winter term 2022-23. Among the participants were 18 women and 9 men, aged 19 to 28 years. Students were between the 1st and 7th semester of study.

Instrument

For the study, 4 items were created for each of the 6 item types (cf., Fig. 1), i.e., a total of 24 items. When selecting the items, restrictions were applied in addition to those that are obligatory with respect to the task (the number of quartiles above the critical mark must not be identical in order to make a decision possible): For example, we focus on the area bias in this study, and the question of what influence different whisker lengths have on the decision process is not of interest in this study. Therefore, ranges e.g., were chosen identically in all items and all box plots.

Eye-tracking assessment

The Tobii Pro Spectrum eye-tracking device was used with Tobii Pro Lab software. First, all participants were shown an example box plot for a time of 10000 ms and then read the contextualization and the task. Then, the 24 items of the item set were answered, with each item visible for 5000 ms. Participants were informed beforehand of the time constraint. The answer was given during this time or immediately afterwards. No countdown timer was shown. The test leader noted down the participants' answers (A or B) on a sheet. For the first n = 13 participants, we presented the stimulus vertically as given in Figure 1. This did not result in reliable eye-tracking data, which could be traced back to an issue of a holistic perception of the iconic box

plot representations when below each other: The authors of the paper could reliably estimate the 'beginning and end' of both boxes while forcing themselves to focus only on the median of the first box plot in several items. Therefore, we altered the stimulus (diagonal design, cf., Fig. 3) for the remaining n = 14 participants.

Analysis of the data

Regarding our first research question, we used a generalized linear mixed model to explain the variance in item difficulty. Regarding the second research question, we assigned students in a first step to one of the profiles given in Figure 2 based on their solution pattern. This gives us a valid and reliable idea of their applied strategy in each of the six item categories. In this pilot study, we focus on a qualitative analysis and comparison of the gaze patterns as a "proof of concept" to derive further hypothesis for eye-tracking analyses. For this purpose, we selected one participant from each of the previously identified and investigated one item from each of the 6 item categories for each participant. For the same reason of showing a "proof of concept", we only used items where the participant's answer was in line with the previously assigned profile (e.g., a participant assigned to the profile 'area biased' answers 'area biased').

RESULTS

Descriptive results regarding overall item difficulty

We firstly estimated the effects of the item design (vertical vs. diagonal stimulus), *item type* (median vs. box item), *area-congruency*, and *median-congruency* on the estimated solution probability of our 24 box plot comparison items utilizing generalized linear mixed models; we allowed for random student and random item intercepts. In line with our hypothesis, a 'Model 1' *with* item characteristics fits the data significantly better than a 'Model 0' *without* item characteristics, $X2(3) = 32.3$, $p < .001$. It is noteworthy that we do not find a significant effect of the diagonal vs. the vertical presentation of the stimuli, yet this result needs to be interpreted with caution as it is a between-subject comparison. As expected, Box items were significantly easier to solve than Median items (Odds Ratio OR = 1.89*), and area-congruent items were significantly easier to solve than are-incongruent items (OR = 6.19*). Median-congruent items were easier to solve than median-incongruent items—in line with our hypothesis—but the effect did not yield significance in the present study (OR = 1.29). The proportion variance change in the random item intercept between the two models is 89.9%; the proportion variance change in the random student intercept is negligible (1.7%). This strongly supports our theoretically derived systematic variation of item characteristics (cf., Fig. 1).

Inducing applied strategies from gaze behaviour patterns

The central question for the current study was to what extent eye-tracking can validate the solution pattern previously assigned based on answer patterns. Our hypothesis was that students' solution pattern in the six item categories relates to their (item-specific) gaze behaviour. For this analysis, we used the results of n = 14 participants who were

presented the stimulus diagonally to avoid a holistic perception of the area and force saccades and specific fixation areas during the estimation of the area above the critical mark. This approach led to reliable eye-tracking data, which we used for the following analysis.

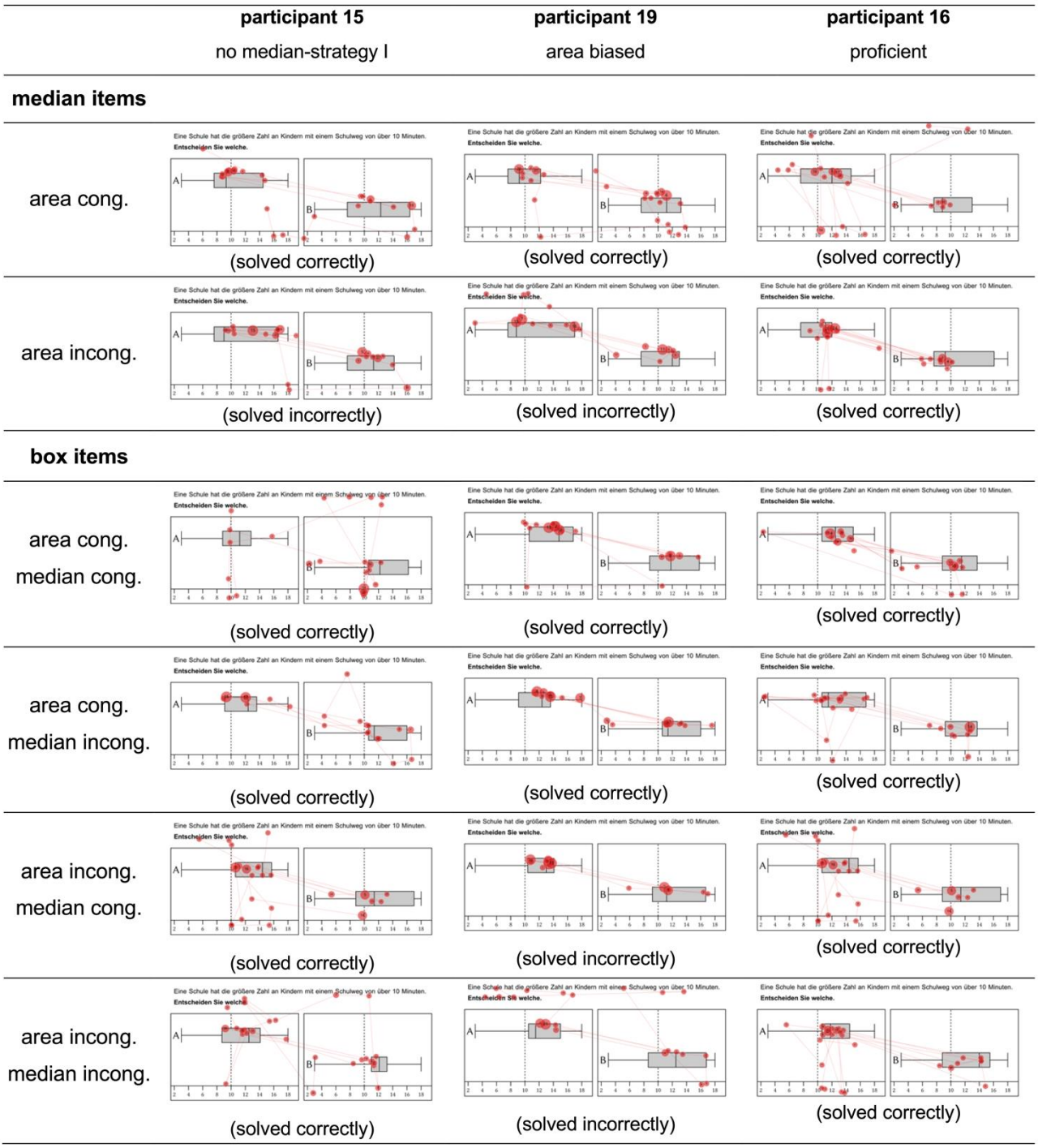


Figure 3. Gaze plots of three participants prior assigned to concept by their patterns in solution rates (download in higher resolution: https://bit.ly/3Qwvj4x)

In a first step we used the answer patterns to assign participants to one of the hypothesized profiles (see Fig. 2). In total we assigned $n = 2$ participants to the proficient, $n = 5$ to the area biased, and $n = 4$ to the no median-strategy 1 profile. The

remaining 3 participants could not be clearly assigned. For this proof of concept-approach, we did not evaluate all gaze plots qualitatively, but give an example for each identified profile and each of the six item types (Fig. 3) to show that gaze plots can be used for validation of the assigned profiles.

In Hyp. 2a, we assumed, that the use of the median strategy leads to longer fixations on the median and the critical mark with a high number of saccades between both. According to the assumed profiles we only expect participant 16 (proficient) to be able to use the median strategy and therefore only this participant should show the hypothesized eye movement in median items. We found that this is true for participant 16. Conversely and in line with the assigned profiles, this strategy is not apparent in box items, indicating the absence of a median bias. In contrast, participant 15 and participant 19 were assigned to profiles in which a non-existence of the median strategy is suspected. Their gaze behaviour is in line with this assumption, as can for example be seen in median items where both participants' eye movements allow the conclusion that they focus on the area of the boxes.

In Hyp. 2b we assumed that the box strategy is manifest in longer focuses on the first quartile on the one hand and the critical mark on the other, and in an increased number of saccades between these areas. According to the assumed profiles we only expect participants 15 and 16 (no median-strategy 1 & proficient) to be able to use the box strategy and therefore only this participant should show the hypothesized eye movement in box items. We found that this is true for both participants. In contrast, participant 19 here shows focuses on the third quartile as an indication of the presence of an area bias.

In Hyp. 2c we assumed that the area bias is manifest in longer focuses on the area above the critical mark. This area is mainly captured by a comparison of the third quartiles, which should show up in fixations on the third quartiles and saccades between these fixation points. According to the assumed profiles we expect the area bias in all items for participant 19 (area biased), only in median-items for participant 15 (area bias replaces the missing median strategy there) and in no items for participant 16 (proficient). We found that this is true for all three participants.

DISCUSSION AND FUTURE RESEARCH

Comparing data distributions via box plots is a challenging task. The difficulties arise from various difficulty generating factors, such as area-congruency and median-congruency. Our present study underlines that knowing about these factors explains most of the variance in item difficulty—and thus that these factors build a profound theoretical foundation for discussing box plot tasks both in research and in teaching scenarios.

These difficulty generating factors are closely related to cognitive profiles of students and can provide insights in the problem-solving process when comparing data sets using box plots. Considering the gaze behaviour of the exemplary selected study participants, our hypotheses could be confirmed to a large extent: the present pilot

study provides first indications that eye movements in comparing two datasets using box plots differ systematically and therefore can validate the assignment to the naïve area-based, the proficient, and the various other in-between profiles as proposed in Figure 2 based on the answer patterns. To use this promising approach systematically for qualitative analysis, a coding scheme is required and has to be developed. This scheme can subsequently be a first step to define characterizing areas of interest and typical patterns of transitions for both box and median strategy, so that a quantitative approach is possible, too.

It is noteworthy that we found that in a vertical alignment the area of both boxes could be perceived and compared without the need of additional eye movements (e.g., focusing on the third quartiles). Considering the eye mind assumption, this is a challenging result—not only for our study. This is where the interesting question arises whether a diagonal arrangement of the box plots only leads to changed eye movements or also influences the problem-solving process itself. On a more general level, we think that this result may also have relevance in a broader context of eye-tracking research in mathematics education—especially in studies where the perception of figures and shapes (and not symbolic representations of text or numbers) plays a crucial role.

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WHAT DIFFERENCE DOES TEACHER KNOWLEDGE MAKE? A FEASIBILITY STUDY ON USING ELEMENTS OF COMPREHENSION AS INDICATORS FOR SCHOOL-RELATED CONTENT KNOWLEDGE

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It is assumed that school-related content knowledge, as knowledge about connections between academic and school mathematics, is necessary for high-quality mathematics teaching at secondary level. Nevertheless, research on how teachers use this knowledge in action is lacking. In this study, we investigate the use of elements of comprehension (EoCs) as an indicator of school-related content knowledge in teaching situations. As part of a feasibility study, we analyzed video sequences of three pre-service mathematics teachers in short teaching simulations on the concept of limits of sequences. The occurrence of EoCs could be coded intersubjectively reliable, and the observations were in line with the participants' knowledge assessed prior to the study. We discuss the further potential of the approach for research on teacher knowledge.

MOTIVATION

The theoretical, deductive structure of university mathematics differs substantially from the descriptive and mainly inductive approaches in school. Prospective secondary-level mathematics teachers in many countries, including Germany, usually study academic mathematics at university. The connections to school mathematics are rarely addressed, which creates a gap between the two kinds of mathematics. This problem leads to a broad discussion on what kind of content knowledge mathematics teachers need. The *school-related content knowledge (SRCK)* by Dreher et al. (2018) is suggested as a domain of mathematics teachers' professional content knowledge (cf., *CK, PCK and PK* according to Shulman, 1986) referring to the necessary knowledge about connections between school and academic mathematics. This knowledge is seen as relevant to enable mathematics teachers to provide high-quality mathematics instruction. For this reason, there is an increasing number of approaches to promote SRCK in teacher training through integrating specific learning opportunities (e.g., Fukawa-Connelly et al., 2020). However, so far, there is a lack of empirical evidence on how (future) teachers may use their SRCK in teaching situations and how this may affect the student's cognitive activation as an aspect of high-quality teaching. Moreover, we also lack appropriate indicators to analyze the potential use of SRCK in teaching situations. Therefore, this study aims to identify and test possible indicators for an application of SRCK by (future) mathematics teachers in teaching situations and its potential for the cognitive activation of students.

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THEORETICAL FRAMEWORK

School-related content knowledge (SRCK) comprises the connection between school and academic mathematics by considering them in both directions, top-down and bottom-up (Dreher et al., 2018). The top-down relation of SRCK addresses how a mathematical idea can be reduced for the school context. For example, if we want students to discover that rational numbers are dense in $\mathbb{R}$, the academic approach (any real number is the limit of a sequence of rational numbers) is inappropriate. Teachers must hence reduce the idea and find an accessible yet conceptually honest (Bruner, 1999) approach, not referring to limits of sequences (e.g., find the smallest fraction greater than $\sqrt{2}$). In school mathematics, many underlying mathematical structures are only treated implicitly. Hence, teachers also need to know which mathematical definition, theorem or proof lies behind the school content, which is part of the bottom-up facet of SRCK. For example, when dealing with inverse functions, teachers should be aware that students may discover important mathematical ideas such as surjectivity or injectivity. These will be relevant later in the curriculum for investigations of the characteristics of functions (curve sketching, e.g., strictly de/increasing) and can be integrated well into the students' prior knowledge of the mathematical inverse. SRCK thus also includes a third facet, a meta-knowledge of the curriculum that comprises knowledge about the structure of school mathematics and its similarities and differences to the structure of the academic discipline (Dreher et al., 2018). If teachers introduce, for instance, the inverse function only as the reflection of the graph at the angle bisector or the concept of limits as “getting closer and closer” to a value, they not only miss an opportunity to prepare for future learning but, in the worst case, promote misconceptions. Several studies have shown the connection between teachers' knowledge and the potential to provide high-quality instruction, as well as the effect on student learning progress (e.g., COACTIV research program; Baumert & Kunter, 2013). Thus, SRCK, as part of teacher knowledge, is theoretically a prerequisite for high-quality mathematics teaching.

Although instructional quality is not operationalized consistently (for an overview, see Praetorius & Charalambous, 2018; Mu et al., 2022), many approaches are based on the three basic dimensions according to Klieme et al. (2009): clear-structured classroom management, supportive & student-oriented classroom climate, and cognitive activation of students. In the past, these basic dimensions were often considered interdisciplinary. However, a growing number of researchers are assuming a subject-specific perspective on instructional quality, which is especially relevant to the basic dimension of cognitive activation (Schlesinger et al., 2018). Since SRCK is also subject-specific and mainly addresses questions related to understanding mathematical concepts, this contribution focuses on the dimension of cognitive activation when investigating how SRCK may be used by teachers to provide high-quality instruction.

Drollinger-Vetter (2011) suggested to assess the potential for cognitive activation of instructional situations in mathematics by focusing on elements of comprehension (EoCs) and their use in instruction. EoCs are ‘*simple, interrelated sub-concepts/*

elements of a more complex concept that need to be understood in order to understand the whole, overarching concept' (Drollinger-Vetter, 2011, p. 198). In the associated Pythagoras study, for example, the Pythagorean theorem was organized into nine EoCs, such as "*the central figure is the triangle*" and "*it is about the lengths of the sides in the triangle*" (Klieme et al., 2009). EoCs, however, are not a fragmentation of the concept into digestible parts and are thus neither disjunctive nor stand-alone. Instead, they are interrelated and only become meaningful when combined. Consequently, the teaching objective is not merely listing the single elements but making meaningful connections based on the student's prior knowledge. Regarding their use in instruction, Drollinger-Vetter (2011) investigated (among other criteria) the occurrence, intensity, and quality of the connections by using lesson videos of 9th grade. To sum up, under this perspective, instruction is considered to have the potential for cognitive activation and so contributes to high-quality instruction when they succeed in covering EoCs.

RESEARCH QUESTIONS

As illustrated, it is assumed that SRCK contributes to high-quality instruction and supports deep understanding of students. More precisely, it is assumed that teachers with high SRCK are able to teach mathematics cognitively activating and promote students' mathematical learning, for instance, by building on prior knowledge or sustaining content-related discourse (Baumert & Kunter, 2013; Mu et al., 2022). To observe the use of SRCK in teaching situations concerning cognitive activation, we need suitable indicators that (a) represent established criteria for instructional quality, (b) are subject-specific, and (c) can capture aspects of school as well as academic mathematics. The latter condition rests on concepts of professional knowledge for teachers, involving both types of knowledge and, above all, the connections between them (SRCK). EoCs seem theoretically suitable, as they, firstly, have been used as subject- and concept-specific quality indicators for cognitive activation in mathematics education. Secondly, mathematical concepts may not always be treated identically in academic and school mathematics (see the density in $\mathbb{R}$), which can be captured by detailing EoCs that are part of the mathematical concepts under consideration.

So far, EoCs have only been used in real lessons with in-service teachers. Whether the approach can also be transferred to teaching simulations in teacher training is unclear. In this feasibility study, we want to investigate the theoretical suitability of EoCs for assessing cognitive activation based on teachers' SRCK in teaching situations. For this purpose, we would like to answer the following questions exemplarily for the concept of limits of sequences:

1. Is it possible to reliably identify EoCs in short teaching situations?
2. Are EoCs suited to surface differences regarding the SRCK of pre-service teachers in short teaching simulations?
3. How do differences in using EoCs in short teaching situations relate to the teachers' prior SRCK knowledge?

METHODS

In the first step, we determined EoCs for different mathematical topics through a theoretical analysis of the targeted mathematical concepts (this contribution exemplary presents the topic of the concept of limits). In a second step, we asked pre-service mathematics teachers to take part in short teaching simulations and analyzed their teaching with the EoCs. Additionally, the SRCK of the participants was assessed by a knowledge test and planning tasks administered prior to the teaching simulations.

Elements of comprehension for the concept of limits of sequences

For the development of EoCs, we investigated the question, *'which sub-elements of the concept of limits of sequences must be understood in order to understand the concept as a whole'* (Drollinger-Vetter, 2011, p.186) from a school as well as an academic perspective. For this purpose, we conducted an in-depth content analysis of the concept of limits of sequences based on the curricula, five school textbooks and four academic textbooks. Furthermore, findings from mathematics education research were considered (e.g., concept images, or typical misconceptions; Greefrath et al., 2016, Tall & Vinner, 1981). We synthesized the identified EoCs into a joint framework, and the resulting eight EoCs for the concept of limit (see Table 1). The table also indicates whether each EoC occurs in the academic concept, is used in school, or both.

Table 1: EoCs of the concept of limit of sequences

	academic mathematics	school mathematics
	It is about …	
1	… behavior of sequences in the infinite.	
2	. special sequences that converge to a value (convergent sequences), but other sequences do not tend towards a value (divergent sequences).	
3	The limit is represented as a dynamic process.	
	… the theorem on the boundedness of convergent sequences.	… bounded sequences.
	… the distance between the sequence elements and the limit becoming infinitely small from a certain sequence element onwards.	…the distance between the sequence members and the limit becomes smaller and smaller.
	… the existence and uniqueness of the limit of a sequence (object).	…the question of whether the limit is reached.
	… null sequences.	
	…the theorem of convergence of monotonous and bounded	… monotonous

sequences. sequences.

EoCs that were related but not similar in both contexts are placed next to each other. Note that for our intended use in coding teaching situations, the frequency and combinations of EoCs were not relevant.

Sample and data collection

To create comparable conditions, we asked pre-service secondary mathematics teachers to teach the concept of limits of sequences in a standardized laboratory setting. The teaching simulation took place with a group of four simulated 10th-grade students represented by other teacher students that acted according to defined profiles (e.g., regarding prior knowledge) like 10th-grade students (see Kron et al., 2022 for a similar approach) for about 20 minutes. The participants of the feasibility study were three volunteering pre-service mathematics teachers (three males, 23-26 years old) at the end of their studies for secondary school. For preparation, the participants had access to the background information (prior knowledge of simulated students, an overview of the teaching sequence, etc.) as well as school and academic textbooks. To gain insights into the knowledge (SRCK) of the participants, they were asked to complete a pre-structured planning document of the teaching simulation and SRCK test items (see Dreher et al., 2018 for examples). Among other things, we asked about the definition of the limit at academic and school levels, necessary prior knowledge of the students, and their concept images of limits (e.g., as an approximation/tends to (A), environment/ε-tube (E), and object (O)). The teaching simulation was videotaped.

Data analysis

The videos were analyzed with MaxQDA2022 by two independent raters for the occurrence and intensity (0 never, 1 short, 2 detailed) of the EoCs. In addition, the quality and linkage of the EoCs were assessed in terms of central sub-concepts that are particularly relevant for comprehension (see Table 1: EoCs 2, 4, 5, 6). For this purpose, a four-level coding scheme was developed for SRCK focusing on mathematical correctness (1- *incorrect mathematical content*, 2- *incomplete mathematical content*, 3- *correct and complete mathematical content but with formal deficiencies*, and 4- *both mathematical content and formally correct and complete*). To enable students to gain a deeper understanding, we preliminary decided to consider that level 3 should be aimed at from a normative perspective in this feasibility study. In doing so, we were aware that this decision, as well as the suggestion to distinguish EoCs regarding their relevance, need validation in further research (e.g., via a study with external experts).

RESULTS

The coding of the occurrence and quality of the EoCs could be applied intersubjectively reliable to all three videos (RQ1). The interrater reliabilities are substantial (Cohens $\kappa_{occurrence} = .783$, $\kappa_{intensity} = .715$ and $\kappa_{quality} = .750$). Table 2 shows which EoCs are used, the intensity (also distinguishing EoCs at academic and school levels) and the quality of the EoCs. We see differences between the participants in all three aspects (RQ2). In

this respect, the instrument seems suited to capture variance between the teaching sequences. There are differences in the approach to explaining the concept of limits from a school/academic perspective. While P1 and P3 primarily use EoCs at school level, P2 also uses academic approaches.

	EoCs used	intensity of EoCs (13 items, scale: 0-2)						quality of the EoCs (4 items, Scale 1-4)	
		total (13)		school (4)		academic (5)			
	detailed vs. (short)	mean	sd	mean	sd	mean	sd	mean	sd
P1	(1, 2), 4s, 5s, (6a+s), (8s)	.87	.72	1.50	.50	.33	.47	1.33	.47
P2	1, 2, 3, (4s), 5a+s, 6(a)+s, (7a), 8(a)+s	1.47	.72	1.75	.43	.83	.69	2.67	.47
P3	1, 2, (4s), 5s, 6(a)+s, 8s	1.07	.93	1.75	.43	.33	.47	2.00	.00

Table 2: Occurrence, intensity, and quality of EoCs

The results of the SRCK test (5 items, score 0-1), the participants' prior knowledge of the concept of limits, as well as an evaluation of the quality and implementation of the short teaching simulations, can be found in Table 3. We also find differences between the participants' prior knowledge and the quality and implementation of EoCs (RQ3).

planning & knowledge items							
SRCK	Prior Knowledge						description of the observed quality and implementation of the EoCs
mean	AD	P	A	E	O		
P1 .30	--	x	x	--	--		Cannot provide a suitable explanation (incorrect and incomplete). Limit concept only via approximation. EoCs only at school level.
P2 .50	x	--	x	--	x		A formally incorrect but mathematically mostly correct but incomplete explanation was provided. Limit concept via approximation, object, and environment (ε-tube). More use of EoCs at an academic level.
P3 .60	x	x	x	--	x		Cannot provide a suitable explanation (incorrect and incomplete). Limit concept via approximation and object. EoCs are mostly at school level.

Table 3: Prior knowledge vs. quality and implementation (AD-academic definition, P-prior knowledge, concept of A-approximation, E-environment, O-object)

P1, in contrast to P2/P3, was not able to provide a correct definition of the limit value. For all participants, the concept image of approximation (tends to) dominates, whereas P2 and P3 also referred to the environment concept (ε-tube). In line, P1 could not provide a suitable (correct) explanation for the students. Only P2 was, in part, successful in transferring the academic definition to a school level (correct but incomplete).

DISCUSSION

This contribution reported on a feasibility study to use EoCs for assessing the cognitive activation based on teachers' SRCK in short teaching situations led by (future) secondary teachers. Overall, the EoCs could be applied intersubjectively reliable to the videos and variations between the teaching situations were reflected by the coding. While only a few differences between the participants' use of EoCs are visible at school level, they are more evident in the case of EoCs at the academic level. Due to the small sample size, it is not possible to analyze whether the differences are significant. Still, triangulating the coding with the pre-service teachers' knowledge (SRCK) assessed prior to the teaching, the observed use of EoCs seems to be in line with the pre-service teachers' knowledge. This indicates the potential suitability of the approach using EoCs for further investigating how teachers use their professional knowledge in teaching. However, the SRCK of the pre-service teachers, as well as the EoC-based evaluations of the instructional quality, show low values hinting at a floor effect. Similar effects were found in other studies on the SRCK (Hoth et al., 2020).

Limitations and implications

Altogether, the results of our feasibility study support that EoCs may not only be suitable for studying the subject-specific quality of instruction in real lessons but also in teaching simulations. However, the feasibility study only provides a first impression based on three teaching situations of three students as a convenience sample. More investigations to validate the approach are required, such as using larger samples, more teaching situations per person, or an expert validation of the EoCs. It also needs to be clarified whether the procedure is transferable and can be used with other mathematical concepts. These open questions are currently being investigated in further studies. In addition, other aspects associated with cognitive activation seem relevant but not captured by EoCs (e.g., use of cognitively activating questions, student activation, and use of representations). So, further studies should also consider whether additional indicators are needed to not miss relevant features of high-quality mathematics instruction. In conclusion, the approach of capturing subject-specific aspects of high-quality teaching using EoCs is promising for future studies to investigate how (future) teachers use their SRCK in teaching situations and how this influences the effectiveness of learning opportunities.

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DECISIONS OF AN ADAPTIVE ENGINE FROM A DIDCATICAL PERSPECTIVE

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This study describes the considerations and decisions of the adaptive engine in adjusting a learning trajectory for each student for the "Fraction my way" for the 4th grade digital environment. We examined and formulated the decisions of the adaptive engine from a didactic point of view, translating algorithmic processes into mathematics teaching processes and learning processes - as much as possible. The research findings show a change in the sequence of tasks, skipping back and forth for practicing and reinforcement of the topics, treating identified misconceptions while keeping the student challenged and not bored or frustrated. Most, but not all, of the algorithmic decisions were described from a didactical point of view, enabling a critical perspective on the modifications made in student's learning trajectories.

LITERATURE REVIEW

The current settings of k-12 classrooms in terms of time limitations and student numbers, usually prevent teachers from providing student specific teaching and push teachers to teach according to the level of the average student. This strategy may leave struggling students behind, lead to a lack of interest for advanced students and improper time use (Vainas et al., 2019). The student centred approach, that sees the students as the center of the teaching and learning process and allowing each student to learn according to their own level, is difficult to implement under these conditions.

Adaptive learning systems provide a possible solution for personalized learning and have the potential to allow teachers to adapt their teaching to each student or group of students with similar needs, as they can interpret the data presented to the teachers through dashboards at any stage of their student's work. Utilization of technology can lead to an improvement in student achievement and can allow every student an equal opportunity to learn according to their appropriate level (Grant & Basye, 2014). One possible solution researchers proposed is the use of artificial intelligence technology that would offload some of the teacher's decision-making process to provide each student with support or guidance that is personally adapted based on their previous work (Pai et al, 2020). Artificial intelligence assesses situations and makes personalized decisions for students based on the algorithmic training (Akerkar, 2014), while creating a unique learning process that relies on the performance and characteristics of students to accommodate different learning goals (Yaghmaie and Bahreininejad, 2011).

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Adaptive learning gives learners control over the context, pace and scope of their learning experience (Martin & Whitmer, 2016). The adaptability of these systems stems from their ability to respond to each individual user according to the information that the system has processed and collected about him (Johanes, 2017). Teachers using an adaptive system for their students testified that when the system skipped tasks, the students interpreted it as positive reinforcement, which motivated them to learn, think and spend time before answering a task, so that their answer would be correct (Vainas et al., 2019).

During lesson planning, many teachers emphasize the interaction with the curriculum and the various learning materials, while only a few teachers plan their lessons and make decisions based on the students' thinking (Lloyd & Behm, 2005). During the lesson, the teacher responds and makes different decisions according to the conduct of the lesson. This type of decision-making contrasts on many occasions with long-term decision-making (or planning) by teachers performed after school hours, when they are not interacting with students. Another teacher's decision during the lesson is when choosing the next problem. This expertise requires not only knowledge of handling students' strategies and interpreting students' understandings, but also knowledge of student's mathematical development. This understanding can help identify the next step so that it will be adapted to the students by choosing a problem that will be accessible but also challenging for the students (Childress & Benson, 2014). In adaptive systems such as the "fractions my way" learning environment (Biton et al., 2022) that is studied in this report, the algorithm-based adaptive engine, rather than the teacher, determines the next task for each student. Yet the considerations of the engine are opaque for both teachers and students and are usually difficult to interpret and understand. This study investigates and describes the considerations of the adaptive engine in the learning environment "Fractions my way" for the 4th grade. The engine's decisions and the learning trajectories it created for students were coded for each of the mathematical sub-topics and were described from a pedagogical point of view.

RESEARCH QUESTIONS

How can considerations of the algorithm-based adaptive engine made while adapting student's personal learning trajectories be described from a didactic point of view? And what modifications are made in learning trajectories offered by the adaptive engine compared to a linear learning trajectory?

METHODOLOGY

Population

In this study we examined data representing 12 learning trajectories of 4th grade students in Israel. The learning trajectories included 30 hours of study of fractions in the 4th grade.

Research Tools

The “fractions my way” teaching and learning environment offers personal learning for the student based on an algorithmic model that builds each student's unique trajectory. The learning trajectory is constructed in real time and is constantly modified according to the student's personal achievements. These adjustments in the learning trajectory navigate the student to move back and forth in his personal learning space, so that he will be required to tackle tasks that the student is able to solve but will challenge him and he will have to invest time in thinking in order to succeed in solving these tasks.

Data Sources

The data consists several files generated from the digital platform: a file containing complete data on the linear learning trajectory, a file containing the twelve learning trajectories, a file containing the difficulty indices and a diagnosis index for all the tasks in the learning unit, and file that allows viewing various characteristics on all the trajectories passed in the system in 2021.

Data Analysis

In the first phase of the research, we used the file containing complete data on the linear learning trajectory in "Fractions My Way". This file included the sequence of tasks and topics; the mathematical skills for each of the tasks; The type of tasks - mandatory tasks, which the system is configured not to skip, additional practice tasks and enrichment tasks. Next, we created a focused file detailing the number of tasks in each subject and dividing them into different criteria. In the second stage, we used the file detailing the twelve learning trajectories representing students work within the environment, first, we coded three of the learning trajectories to examine the coding method, what can be learned from the data and validate the process. Next, we coded the rest of the trajectories. The coding was done relative to linear trajectory and described each learning trajectory individually. The description included the sequence of tasks completed, the answer for each task, correctness and characteristics of the task.

In the third stage, we coded the modifications made by the engine in the 12 learning trajectories, so that each of the skipped tasks was coded according to several categories: the chapter in which the task is located, the type of task (mandatory, enrichment or practice), mathematical skill for that task, rate of success in the sequence of tasks that preceded this jump, the type of skip (forward or backward), the size of the skip as well as the tasks to which the engine returned to after significant progress in the sub-topic. In the fourth and last step, we examined the coded data from a didactic point of view and proposed different interpretations for each of the engine actions that were coded. The process involved trying to find several interpretations in order to distil the most appropriate interpretation with supporting data. When an extreme skip was detected, the student's answers and the characteristics of the tasks before and after the skip were coded in order to locate the reason, leading to insights about the engine’s ability to

locate and identify misconceptions among the students and respond within the means available.

FINDINGS

In this section we present the findings that emerged from the data analysis as an answer to the research questions through case studies. For each case study, we will present the changes made in the learning trajectory by the adaptive engine compared to the linear trajectory, and present the interpretation of the considerations of the adaptive engine for modifying a personal learning trajectory from a didactic point of view.

In the first case, skipping tasks following success and significant jump back in the sequence of tasks were detected for tasks that the engine initially skipped. This case was identified in Track 4 on the topic "Fraction's Comparison", in fraction comparison tasks. The student experienced two skips in their trajectory. The first, forward skips following success: the first skip after 100% success, while the next jump after 58% success (in previous 20 tasks). At this point the student reached task 408, completed it successfully and continued to task 409 (the displayed task number is according to the cyclic position of the task in the full linear learning trajectory). Task 409 (figure 1) presented two fractions that are equal to one another, but divided into a different number of parts (3 and 4 respectively). The student chose the fraction that would be represented by the largest number as bigger. From this we learn that the student possibly needs reinforcement in the meaning of the fraction and that the skipping does not fit his performance.

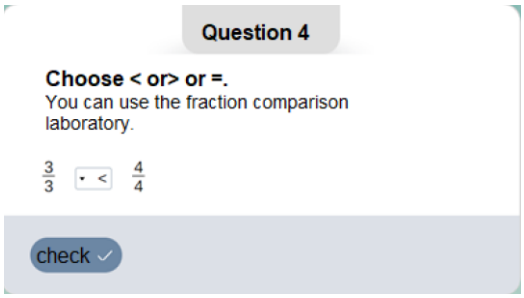

Figure 1: Fraction comparison task (409).

Examining the data from a didactic perspective suggests that to help the student, initially the engine skipped on tasks that he had a high likelihood to succeed in, based on his prior performance. Then, after making mistakes, the engine returned him to practice comparing fractions, task 207 (about two hundred tasks back) - comparing unit fractions, ninths versus tenths. Here too it can be seen that the student did not refer to the essence but chose the fraction in which the large number appeared, even though this number appeared in the denominator and represented a smaller part. The engine then returned the student to the task where the number of parts is the same - 3 parts in each of the fractions, and the denominators are different. That is, the student was shown the fractions three-fifths as opposed to three-sevenths. Here too the student repeated

the error. It is important to note that these tasks included a laboratory applets for the representation of the fractions in a visual manner (figure 2). If the student were to represent the fractions in the lab, he might have been able to visually identify the correct fraction and been able to better understand the meaning of this fraction. In this case study it can be seen that the system returns the student specifically to tasks that may help him overcome the difficulty and not to all the tasks he skipped, as there were other tasks related to the same subject but not suitable for this type of student error.

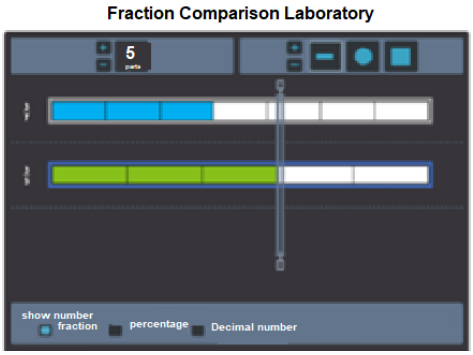

Figure 2: Task 207

In this case study, there are considerations regarding data that the engine developed and were not defined in advance. These skips were not only concerned with the type of task, but with the students' solution. significant jump across different sub-topics back in the learning trajectory, it was found that the engine was able to locate and identify misconceptions among students and respond to those perceptions with relevant tasks from the pool of tasks that the student did not complete.

In the second case study, the topic "Fractions Questionnaire - Recognition and Actions". The student made a series of mistakes. From tracing his mistakes, we noticed that when the student encounters addition and subtraction tasks with equal denominators, he performs the mathematical operation on both the numerator and the denominator, which shows that he treats the fraction as two different groups: the numerator represents one group and the denominator represents another group.

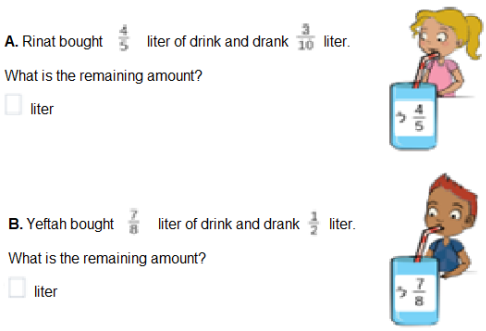

Figure 3: Change in the sequence of tasks.

In the task shown in figure 3, the student, after finding a common denominator, subtracted the numerator from the numerator and the denominator from the denominator. The student had 5 for the numerator and 0 in the denominator, and wrote "5 liters" as the answer. In the process of creating the common denominator, the notion that the numerator and the denominator receive the same operation did not interfere, since in expansion and contraction the same operation is performed on both the numerator and the denominator. The student may also apply this knowledge in addition and subtraction tasks in which he was challenged .

The student acted in the same way in section b, as well as in other tasks, in this case the engine returned the student to addition tasks with equal denominators. The student was asked to add different fractions with the same denominator (given in advance) that would lead to two wholes and then to different results, as shown in figure 4. The engine jumped back to practice and refresh the material he learned. These tasks are also supported with a fractions lab (interactive applet) with different shapes to represent the fractions and interact with them in the solution process.

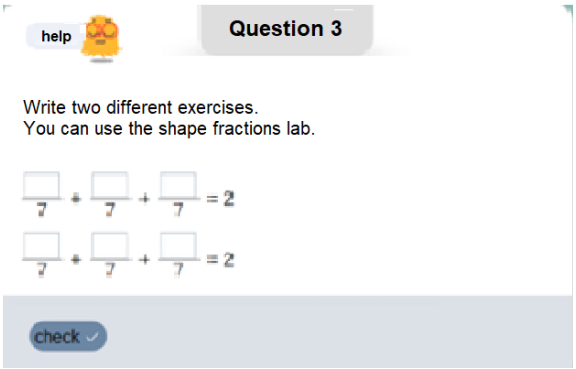

Figure 4: Task 544.

Examining the engine's considerations from a didactic point of view, additional trajectories were found in which the students made mistakes in this task and even in a similar sequence of tasks. An examination of the mistakes the students made revealed that the solution they submitted and hence the type of error, was different, leading the response and adjustment of the tasks to be different. These findings reinforce the fact that the jump back was not arbitrary but based on a misconception, a sequence of errors, and a type of answer specific to that student.

An additional phenomenon within these two case descriptions, as well as in various cases found in the learning trajectories, were two types of gradual skipping: skipping following correct answers and skipping following incorrect answers. Gradual skips are when the engine skips a sequence of tasks in a way that consists of a cluster of small skips and in which students are given individual tasks and some tasks are skipped. Examining the data from a didactic point of view indicates that the system keeps the student in the area where he is challenged, therefore when the student experiences a sequence of successes, it performs controlled jumps. The student does not get one skip over a large number of tasks at once, but gradually continues to practice and skip until the level where he is challenged enough to continue within the learning trajectory

without the need for skips. This also happens when the student experiences difficulties in order to keep the students at a level where they are challenged and not frustrated.

DISCUSSION

During the establishment of the learning environment and the data collection phase, thousands of users used "Fractions my way" in a full, linear trajectory to confirm the difficulty level. The algorithm-based engine accumulated large amounts of data helping to improve the decision-making of the algorithm based adaptive engine. This ongoing collection and analysis of the data, helps the engine to constructs connections and create new considerations that sometimes differ from the original rules coded into the algorithm. The adaptive learning engine updates and responds to students in real time by detecting correct or incorrect answers and responds accordingly. A student with high success rates skips easy tasks to stay challenged and not get bored, while a student with low success rates skips the enrichment episodes and gets a more gradual sequence of tasks, to prevent frustration. Even when the student has progressed in learning but it is evident that he needs additional reinforcements, the learning engine presents him with tasks he did not complete yet as part of a remedial practice treatment . It is difficult for a teacher in the classroom to make and carry out these decisions in real time for each and every student in the classroom, since the situation requires the teacher to constantly analyze and connect specific situations to what he knows about the mathematical development of children (Frank, Kazemi, & Battey, 2007; Heaton, 2000; Lampert, 2001). The adaptive learning engine identified recurring misconceptions among students and responded accordingly. In this way, the engine enables personal and precise, controlled and close attention to each of the students at any time on real-time performance as well as on past performance.

The continuous collection of data at each stage of learning allows the learning engine to create an infinite number of different learning trajectories. However, the adaptive learning engine has limited features on adapting a learning trajectory as well as limited information on the possible answers of the students. While many basic characteristics are coded for the adaptive engine such as the type of task , the success threshold of a student for this task , the type of answer, what is the correct answer and more, there are relevant characteristics that are not coded for it, for example, there are no recommendations for a possible response to a student who made a mistake in a certain task, or data about what is the sequence of tasks that establishes an understanding of a certain subject and to which it is recommended to return when necessary. Another disadvantage in adjusting the individual sequence of tasks for each student is the scope of the task pool in the adaptive environment. This database is limited so that the engine can only match a task from the existing activity database while maintaining a state in which the student cannot perform the same task twice, so that when the student studies and needs to additional material on a certain sub-topic, it is possible that the environment would not be able to give the student a task that can match the student's needs. In addition, misinterpretations of student difficulties might occur and lead to futile task sequences that are unrelated to the actual student needs.

It seems that the adaptive learning system provides a personal response to each student during learning, adjusting the sequence of tasks and keeping the student in a situation where he is challenged and leading the students to meaningful learning processes. But at the same time it is also limited in its abilities in various ways for example, affective considerations or other social aspects. From this study, it appears that a combination between an adaptive learning environment and a teacher that overlooks and intervenes at critical points could give a more comprehensive response to student's needs in their learning process.

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IS BEAUTIFUL ALSO TRANSPARENT? STUDENTS LEARN FROM GRAPHS ABOUT WATER POLLUTION

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Mathematical graphs are among one the most used tools to communicate data regarding environmental issues such as pollution. Traditionally, research in Mathematics Education has focused mostly on the features of graphs that make them accessible and understandable, introducing the construct of transparency. Moreover, teachers in school tend to approach the teaching of graphs not per se, but strongly connected with the context they represent. In this paper, we explore the potential of the aesthetics of graphs in relation to their transparency by looking at how a sample of undergraduate students in Environmental Sciences read and appreciate different graphs representing water and air pollution in textile factories in Italy. Our study reveals the relationships between a graph's appearance and its transparency.

INTRODUCTION

Mathematical graphs are deemed to be an effective tool to convey information about, e.g., pollution issues, increasing temperatures, and climate change, by environmental scientists, especially when interacting with people outside the scientific community (Grainger, Mao & Buytaert, 2016). However, Ainley (2000) warns us about the fact that mathematical graphs are not always transparent for those who read them. This can have dramatic consequences when environmental issues need to be communicated through them: as Demeritt and Nobert (2014) observe, ineffectiveness of communication through graphs may prevent comprehension and, thus, create misunderstandings and inconsistencies or biased messages.

These considerations hold also for young students in mathematical classes, who learn both how to read graphs and how to create them (Roth & McGinn, 1996), usually strongly connected with the real context they refer to (Ivanjek, Susac, Planinic & Andrasevic, 2016). If the context is environmental education, and the aim is to convey information about the evolution of pollution, sea water level increases, and the like, it becomes even more crucial to understand graphs. Moreover, Thielsch, Scharfen, Masoudi and Reuter (2019) observe that a fundamental feature of graphs is their aesthetics: "users not only feel better if they use aesthetically pleasant interfaces, but, on average, they also perform a little better as well" (p.208). Thus, in the context of a mathematical lesson about the effects of textile production on air and water pollution, involving students in an Environmental Science undergraduate course, the research aims at answering the following research question: what the relationship is, if any, between the perception of graphs as aesthetically pleasing and the one as transparent? Implications for students' understanding is also focused on.

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THEORETICAL FRAMEWORK

Transparency, in a general sense, is the quality of being easy to perceive or detect. It is possible to apply this definition to a wide range of different contexts: someone is transparent when it is easy to understand his or her thoughts and intentions even if he or she doesn't say so directly, an administration is transparent when every bureaucratic piece of paperwork or decision is available to be read and understood by each citizen. There is an object, definite and circumscribed, and a story that *lies beyond* that; the term transparency denotes how easy the user can access such hidden significance. Lave and Wenger (1991) describe transparency as the combination of two characteristics: invisibility and visibility. This dual nature is well represented by the metaphor of a window: the window is highly visible in contrast to the wall that contains it, but the hidden meanings, represented by what lies behind the window, is clear if the glass is transparent enough (Ainley, 2000, p. 366).

It is possible to apply the definition of transparency even to the data that can be read from graphs; in this case the role of the window is played by the graph itself and the hidden significances are represented by data, upon which the graph has been created. The slope of the curve, the relationships between variables, the variation with time: these are some examples of the information 'lying beyond' a graph. For example, in their study, Berg and Smith (1994) asked a group of students to imagine walking across the room, and then re-presenting the imagined walk. Roth and Mcginn (1996) comment that in this case the relationship between the graph and the reality it shows is bidirectional "because it is assumed that a literate person can read the specifics of the walk from the graph or construct a graph after making (or imagining) a walk. However, there is evidence that this relationship must be constructed in the same way as the relationship between the word 'cat' and some furry creature that meows." (p.96). In fact, during mathematics classes, students learn this way of creating and interpreting graphs, but graphing, defined by Ainley (2000) as "drawing graphs, reading graphs, selecting and customising graphs for particular purposes, and interpreting and using graphs as tools" (p.1), becomes a cognitive ability unbounded from the context (Roth & McGinn, 1996). This is not always an easy task: Ivanjek *et al.* (2016) investigate first-year students' graph interpretation strategies and difficulties, finding that students' reasoning regarding the proposed problems involving graphs is "often very much bound by the context and conventions of the disciplines in which their knowledge was acquired" (p.11). In their research, the same graph is used in three different contexts and the students resort to different strategies, depending on the subject relating to the problem, even if the question is similar (Ivanjek *et al.*, 2016). On the one hand, it seems that graphs are strongly linked to both the context and the subject they emerge from, but at the same time their readability heavily depends on the cognitive ability of those who read and interpret them.

Mathematical objects are accessible through representations such as graphs, and not in a direct way: understanding a concept hidden inside a chart is a cognitive act which relates the signifier (information) to its signified (the graph) (Duval, 2006). Roth states

that: "graphs as objects do not exist as independent entities, but are a complex network that integrates entities and processes" (p. 305). Improving these cognitive processes could therefore be a way to increase graph transparency. Furthermore, research in eye tracking found a strong correlation between fixations and cognitive processing of information (Latour, 1962; Volkman, 1976); we think that increasing the time students spend looking at a graph may have a beneficial effect on its readability. We conjecture that the time spent in looking at a graph may depend on its aesthetic features and a meta-analysis by Thielsch *et al.* (2019) on general graphical choices, found that visual aesthetics of websites, software and other interfaces have a positive effect on user performance, improving attention and focus. Also, in an educational context, graphs can translate complex concepts in a succinct manner and help comprehensibility: "Design and aesthetics have a profound impact on how users perceive information, learn, judge credibility and usability, and ultimately assign value to a product." (David & Glore, 2010, p. 5).

In this paper, we question whether transparency of graphs, understood in Ainley's (2000) terms and having a dual nature depending on both the context and the student's ability, can be improved by a focus on the aesthetic features of graphs. We recall that our hypothesis is that a graph that is characterised with a better aesthetic appearance invites the observer to look at it and this give it more attention and what it contains. This may, in turn, make it possible to better understand what the graph wishes to express, thus improving transparency.

METHODOLOGY

The study took place during a lecture in a class attended by first year students of an undergraduate course in Environmental Sciences. Participants came from different educational backgrounds, resulting in a good representation of the sample for both gender and mathematical knowledge. The lecture was aimed at improving graph comprehension by inviting the students to interpret or draw graphs on air and water pollution of textile factories in the territory of Biella, Italy. Examples of graphs used are shown in Figure 1. During the lecture, which lasted three hours, a presentation was shown to the students, and it consisted of four different graphs for each chemical component analysed. The difference in style and method of construction between the four graphs presented was intended to emphasise the different aesthetical features of graphs representing the same data. In Figure 1, a 3D vertical bar chart, a radar chart and a line chart can be seen as examples.

Data collection consisted of a multiple-choice questionnaire made up of 15 questions related both to aesthetics and transparency and to the effective understanding of the graphs shown. For example, question 1 concerned four graphs (a vertical bar, a 3D vertical bar, an area and a line chart), for water toxicity of TEXMOL 400S, and the students were asked to indicate: (i) which graph they preferred (aesthetics) and (ii) which one better represented the data (transparency). After a second question about another pollutant, a question involving the comparison between the two was asked (understanding).

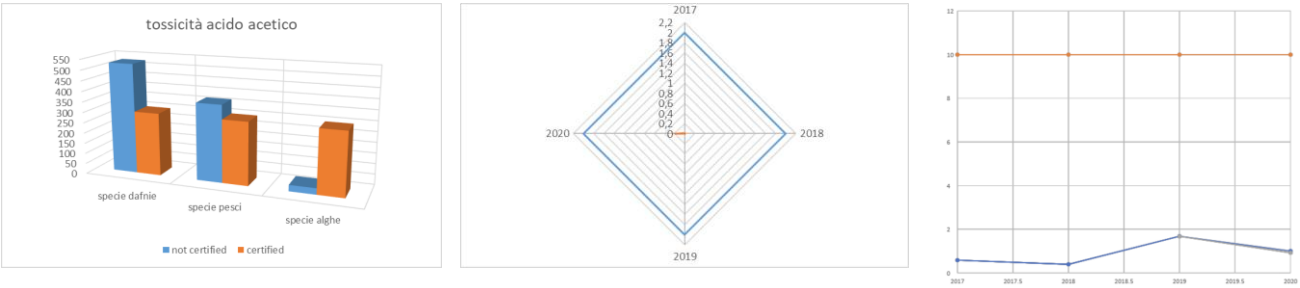

Figure 1 left: water toxicity of acetic acid;
middle: time series from 2017 to 2020 of chrome emissions of the fabric (orange line) compared to European standards (blue line);
right: time series for nitrogen emissions of the fabric (blue line) compared to European standards (orange one).

A first group of four questions concerned water toxicity (both fresh and saltwater) of certain chemical components used in wool finishing processes (see the graph to the left of Figure 1 as an example). A fifth, check-your-understanding question was also asked.

A second group of 9 questions concerned time series of the discharges into the water of two textile companies and their comparison with the European standards (i.e., maximum limits). Figure 1 middle and right show examples of graphs used. Also for this group of questions a check-your-understanding one was inserted. The goal of these two groups of questions was to explore students' appreciation and understanding of graphs representing pollution data.

Data analysis aims to reveal any possible relationship between aesthetics and transparency through the comparison of the answers given by the students. The analysis was conducted by calculating the frequency of responses related to aesthetics and transparency and pointing out any correspondence between the responses of each respondent. This part of the work tries to highlight if respondents aesthetically appreciate the same chart that they recognise as the most transparent.

The analysis of the two check-your-understanding questions aims at confirming whether the kind of graph that is perceived as the most beautiful/understandable really promotes students' comprehension by looking at the proportion of correct answers.

RESULTS

The first four questions (i.e., Q1-Q4 in Table 1), referring to the toxicity of the waste from the textile companies, reveal a clear aesthetic preference for 3D graphs. When the 3D graph is a bar chart, 22/35 (~63%) students chose the 3D charts in the first question, and 21/35 (60%) prefer it in the second one; speaking about the third question, where the one in 3D is the area chart, only 16/35 (~46%) like it the most among the others. When it appears among the four possible options, a 3D graph, regardless of being a bar or an area one, is aesthetically preferred, compared to the others. The check-your-understanding question (not in Table 1) for this group of questions shows the bar chart again and it is designed to check its effective understanding through comparing two toxic wastes. It proves good understanding of bar charts with 71% of correct answers.

Table 1: Frequencies of the answers given by the students in each question (Q).

		Bar charts			Area charts		Other charts		
Q		Simple vertical	Simple horizontal	3D vertical	3D	simple	Point charts	Radar chart	Lines charts
1	beautiful	2		**22**		2			9
	transparent	**13**		10		4			8
2	beautiful	7	2	**31**				5	
	transparent	**14**	**16**	5				0	
3	beautiful	6	1		**16**				12
	transparent	**25**	3		2				5
4	beautiful	9			**13**		3		10
	transparent	12			4		2		**17**
5	beautiful				9	3		10	**13**
	transparent				1	9		2	**23**
6	beautiful	**11**				6	7		**11**
	transparent	**17**				4	0		14
7	beautiful		**10**			7		**10**	8
	transparent		**16**			2		4	13
8	beautiful	11	3		**14**				7
	transparent	**17**	8		1				9
9	beautiful		8		**14**			6	7
	transparent		**16**		8			2	9
10	beautiful	**14**	7			8			6
	transparent	**17**	5			4			9
11	beautiful		5		**17**	4			9
	transparent		**13**		6	8			8
12	beautiful	**11**				10		8	6
	transparent	**20**				7		3	5
13	beautiful	8			**13**			8	6
	transparent	**15**			10			1	9

The mode for each row is in bold. The questions are numbered from 1 to 15 and each one is made of two sub-questions: one concerning the aesthetics and the other the transparency. The answers to the check-your-understanding questions are not reported.

No evidence of a relation between transparency and graph aesthetics has been found until this point; but some interesting results start to appear when students are asked to compare the amount of toxic waste of the companies year by year (to recall, the questions of the second type, from Q5 to Q13 in Table 1). In this series of questions, with reference to consistency (i.e., the percentage of students considering the prettiest graph as the most transparent) grows from values between 14-20% in the first three questions, to 34-48% in the last ones. When it appears among the four possible options, a 3D graph, regardless of being a bar or an area one, is aesthetically preferred to the others. The tendency to consider bar charts as the more transparent also remains unchanged, with a special mention to the line charts, which are deemed as the second more readable. When the 3-D one is an area chart, although defined as the most aesthetically pleasing, it is never considered as the most transparent; in fact, in the seven cases where 3D area graphs appear as an option (questions 3-4-5-8-9-11-13), they are considered as the least readable in four cases (questions 3-5-8-11). Looking at consistency, the highest levels of coherence between aesthetics and transparency are detected in the bar charts, then in line ones, area charts and, finally, in the radar charts, considered bad both for readability and aesthetics. This is further confirmed by the second check-your-understanding question, aimed at comparing the amount of waste dissolved in the water of the two companies through a radar chart: 31% of students answered correctly, a percentage that is much less than the previous 71% of correct answers given when a bar chart was shown.

DISCUSSION AND CONCLUSION

To understand the relations between graph aesthetics and transparency, if any, it is essential to understand which charts seem to be more student-friendly: considering the readability, it's clear that students prefer the bar charts as the easiest to read, with no big difference between vertical and horizontal ones; instead, speaking about beauty, the most appreciated aesthetic feature is their three-dimensionality. This characteristic, however, does not show any facilitation in reading what is hidden inside a graph, if not associated with transparency: in fact, 3D charts are always deemed as the most beautiful, the 2D analogous are chosen as the most understandable, when both 3D and 2D graphs of the same type are shown (see questions 1 and 2). High aesthetic appreciation of 3D area graphs, found to be opaque in Ainley's (2000) terms, demonstrates that beauty alone is not enough to improve the readability of a graph, in fact, it seems to hinder the accessibility of hidden information. The window (recalling the metaphor of transparency) turns out to be a coloured stained-glass window, through which it is very difficult to look.

When there isn't any 3D chart (questions 6, 7, 10 and 11), the most appreciated graphs are bar charts, also considered the most transparent, and in these cases consistency

reaches the highest level. Moreover, the check-your-understanding questions confirm that a big amount of students answered correctly when asked to compare two bar charts. Considering the sub-case of 2D-only graphs, we can argue that there is a relationship between beauty and transparency, but the direction of the relation is not clear: is the greater transparency of bar charts that make them appear more pleasant or is their aesthetics that increases the readability? A future, qualitative study can contribute answering these open questions.

Line charts come in second position for both consistency and transparency, but do not stand out for aesthetics. In this case their simplicity is able to make them appear very transparent, a feature which however is not enough for the students' appreciation, in our data. As per radar charts, questions 7, 9 and 12 reveal that they are more appreciated than considered transparent.

Generalising, we can say that the highest levels of consistency are achieved when there is a balance between transparency and aesthetics: the beauty of a graph alone does not guarantee that it will be transparent, just as a transparent chart does not ensure that it is also pleasant; it remains to see whether increasing the beauty of an already transparent graph might increase its understanding. This will be part of a forthcoming research.

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DYNAMIC INTERACTIVE MEDIATORS IN DISCOURSE ON INDETERMINATE QUANTITIES: A CASE STUDY

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This paper investigates, with a commognitive perspective, the role of dynamic interactive mediators (DIMs) in promoting students' discourse on indeterminate quantities. We analyze a case study of two high school students with a history of low achievement in mathematics, focusing on whether their discourse, developed in activities with DIMs, integrate the (meta)-arithmetical discourse. We show how words, narratives and visual mediators produced interacting with DIMs expand and compress the arithmetical discourse, shaping, in this way, the meta-arithmetical discourse.

CONCEPTUAL BACKGROUND

For more than three decades, research has shown how the teaching and learning of school algebra is a challenging issue, often source of difficulties for many students (e.g., Sfard & Linchevski, 1994; Kieran, 1992). More recently, a stream of studies focused on proposing characterizations of algebraic thinking, on studying the process involved in its formation, and on looking for forms of algebraic thinking in activities apparently distant from school algebra or that even precede it (e.g., Radford, 2014; Caspi & Sfard, 2012; Kaput et al., 2008). Radford (2014) characterizes thinking as algebraic when it deals with indeterminate quantities as if they were known. Kieran (2022) frames three dimensions – analytic, structural, functional - of early algebraic thinking, and defines as analytic the thinking dimension related to the dealing with unknows as if they were knowns. In this view, thinking algebraically is not necessarily related to the use of algebraic symbolism: unknows and variables can be represented with symbols, everyday language, gestures and different signs.

In this paper, we focus on the development of algebraic thinking, specifically regarding the use of indeterminate quantities, of low-achieving high school students. This study is part of a funded research project on the learning of high school students with a long history of failure, with support of digital environments.

We adopt the *commognitive framework* (Sfard, 2008) where doing mathematics is seen as to engage in an established discourse, learning mathematics consists in becoming able to participate in this discourse and to study students' mathematical learning means to analyze their mathematical discourse. The term *discourse* applies to a form of communication characterized by *specific words* (e.g. "equation", "variable", "function"), *visual mediators* (perceptually accessible objects pre-existing to the discourse or artefacts produced for communicative purposes), *narratives* (descriptions of objects, of relations between objects and of activities with them) and *routines* (repetitive patterns characteristic of the given discourse).

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Multilevel structure of algebraic thinking

In the framework of Commognition, mathematical objects are neither extra-discursive nor pre-existing entities. Rather, they constitute part of the discourse itself, they are discursive constructs (Sfard, 2008, p. 129). The process of object construction is called *objectification*. It may be achieved in three ways (*saming*, *encapsulating*, *reification*) that develop with the use of a noun which will be employed in depersonalized narratives. In these narratives, the human subject disappears, as if the referent of the noun exists independently of it. In this way, the discursive construct becomes an object of mathematical exploration and then new mathematical narratives can emerge. Therefore, mathematical discourse develops by addition of new discursive layers, any of which subsumes a previous discourse. In this perspective, elementary algebra can be described as "*metaarithmetic*, or more precisely, as the *unification of arithmetic with its own metadiscourse*" (Sfard, 2008, p. 120), that is the mathematical discourse on arithmetical relations and processes (Caspi & Sfard, 2012).

The development of discourse can be described in terms of alternating *expansions* and *compressions* (Sfard, 2008, pp. 119-123). The increase in the amount and complexity of routines and of new discourses leads to a discursive expansion, while the compression reduces the complexity of the discourse through the rise to the metalevel.

Dynamic interactive mediators

Following the distinction between static and dynamic mediators (Ng, 2016), some studies (Baccaglini-Frank, 2021; Antonini et al., 2020) have formulated the notion of *dynamic interactive mediators* (DIMs), mediators that are dynamic and that respond to a person's manipulations. Examples of DIMs are digital manipulable objects constructed within technological environments. In our study, we are interested in studying how the arithmetical and meta-arithmetical discourses are shaped by the discourses emerging from the interaction with DIMs consisting in digital representation of the balance model. The balance model is a common metaphor in teaching linear equations, for conceptualizing the equal sign and promoting strategies to deal with unknowns. Gains and pitfalls of this model are discussed in the literature (for a review, Otten et al., 2019).

The DIMs we have designed and that we analyze in this paper consist in different versions of two-pan balances. A first DIM consists of a two-pan balance with weights that are known and are represented as colored shapes with a number inside, while some weights are indeterminate and are represented by white shapes (Fig. 2,3). The user can insert one value for the white weights in an input field. By the key "Let's try!", one can interact with the balance which will move depending on whether the total weight on the right pan is less than, greater than or equal to the total weight on the left pan (Fig. 1b), where the white shapes are worth the value inserted in the input field. We call DIM_{TB} this DIM, where TB stands for "test-balance". It models linear dependency between quantities, as its position depends on the input.

To model an equation, we design a visual mediator of a two-pan balance with, on its pans, known weights, colored and with reference to their values, and unknown weights (the balance on the left in Fig. 1a, 4a). This is a fixed, not interactive, balance that we denote by VM_{FB}, where VM stands for "visual mediator" and FB for "fixed balance". Finally, we call $DIM_{(FB,TB)}$ the DIM embedding the DIM_{TB} and VM_{FB} (Fig. 1a), where the DIM_{TB} is a version of the VM_{FB} in which the unknown weights are white (Fig. 1a). In $DIM_{(FB,TB)}$, the fixed balance is unmovable since every weight has fixed value, even if unknown, while the test-balance can assume different positions according to the value given to the white weights. They correspond to two different ways of thinking about a relation between two algebraic expressions. For example, one can think about 3+2x=11 as an equality between two quantities where x is *one* unknown number (VM_{FB}), or as a relation that can be true or false depending on x (DIM_{TB}).

Research questions

This study is part of a wider research project investigating the impact of DIM-based teaching interventions with second year high school students with a history of low achievement in mathematics. Under the hypothesis that the discourse about DIM can foster students' participation in mathematical discourse (Baccaglini-Frank, 2021; Antonini et al., 2020), we are interested in investigating the role of the discourse emerging from the interaction with DIMs (hereon DIMs-discourse) in promoting students' mathematical discourse. To guide the study, we ask the following questions: *how do the $DIM_{(FB,TB)}$- and DIM_{TB}–discourses integrate the arithmetical and meta-arithmetical discourses of students with a history of low achievement? How can these DIMs-discourses shape the production of a discourse on indeterminate quantities?*

METHOD

Participants were 12 students of 10th grade from three Italian high schools, selected by their teacher for their history of severe and persistent difficulties in mathematics and on voluntary participation. The sequence consisted in 4 or 5 sessions of two hours each, which took place in an out-of-school center. Students worked in pairs under the guidance of an expert and were provided with touch-screen tablets for the activities. This paper focuses on a pair of students engaged in activities with DIM_{TB} and $DIM_{(FB,TB)}$ during the first two sessions. Data consist of audio-video recordings, students' written productions, and screen recordings of the tablets used for the activities. In tune with the Commognition, the analysis will focus on the use of words, narratives and visual mediators related to dealing with indeterminate quantities.

A CASE STUDY

We present three episodes from the case study of Andrea and Hugo, two students coming from a professional high school. During a preliminary individual interview, both students showed difficulties in dealing with indeterminate quantities. For example, in looking for a solution of $13 - a = 13 + 11$, Andrea said "I don't remember how to do it" and Hugo wrote "$a = 13 + 11 = 24$". His discourse appears purely ritualistic, focused on performing (meaningless) procedures.

Episode 1

Andrea and Hugo are asked to find out the weights of the triangle, knowing that the balance with the colored shapes (Fig. 1a) is balanced off. Hugo immediately answers:

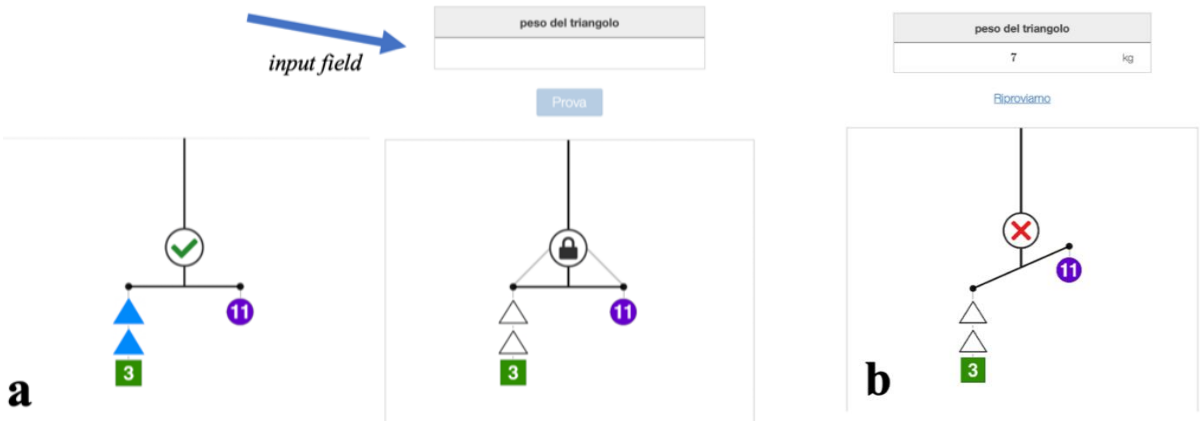

Figure 1: (a) $DIM_{(FB,TB)}$, with VM_{FB} (left) and DIM_{TB} (right);
(b) DIM_{TB} with number 7 in the input field.

1 Hugo: ... 4. Because here on one side is 3 […] here [*he points to one of the white triangles*] you put a weight is worth 4, this one is 4, 4+4, 8, +3, 11 [4+4=8 and 8+3=11]. If you put 5, the balance tends to dangle that way, […] To make it dangle on the right you have to put a number […] smaller than 4 […] if you put 3, here is 9 while here is 11.

2 Int: If we wanted to describe all this thing, in general terms, this balance, how could I say it? […] Let's make a summary.

3 Hugo: To keep the balance… balanced off, the number, here a triangle is worth 4, to make it dangle to the left you have to find a number greater than 4, while to make it dangle to the right you have to find a number smaller than 4.

The discourse on numbers (e.g., "4, 4+4, 8, +3, 11", "here is 9") is intertwined with the DIM_{TB}-discourse, especially with the narrations on the interaction (of the subject) with DIM_{TB} ("here you put a weight", "if you put 5", "you have to put a number", "if you put 3"). Students use the verb "to put" as a signifier of the action of inserting the number inside the white triangle (Hugo points to *one* white triangle but their discourse shows that they consider that the same number *is put* in every white triangle). However, the action of putting numbers into the shapes cannot actually happen. DIM_{TB} allows to put numbers only in the input field and, in fact, Hugo uses the verb "to put" as a metaphor. In [3] we can also observe a depersonalized narrative ("a triangle is worth 4") without a subject who "puts" numbers.

Episode 2

Observing the DIM_{TB} (Fig. 2), Andrea says that on the right there is "more space [where to] put […] the weights". He then justifies the choice of number 3 to balance off the balance summing the numbers in both pans ("I made 5 plus 5, 10, plus 3, 13, plus 3, 16 [5+5+3+3=16]) saying "I made also […] the spaces and it comes out 16, 3 plus 3". The calculation with specific numbers ["3 plus 3"] is an arithmetical discourse

while the sentence used to talk about this calculation ("I made also the spaces") can be used for whatever number ("the weights" in the "spaces") and therefore is a meta-arithmetical narrative. In summary, the use of "space" allows the construction of discourses about indeterminate quantities that can be considered as known quantities (even if Andrea produces a discourse on determinate quantities).

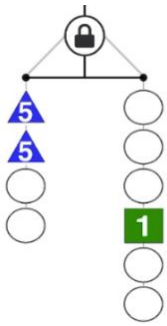

Figure 2: DIM_{TB} of the episode 2.

Episode 3

Andrea and Hugo are asked to identify the weights of the shapes that make the balance hang to the right, to the left or balanced off (Fig. 3). They find the couples of numbers (6,3) and (9,6) to balance off the balance. Then they summarize:

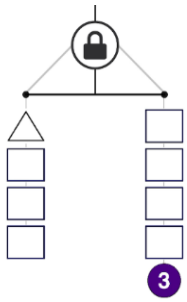

Figure 3: DIM_{TB} with two indeterminate weights and two input fields (episode 3).

4 Andrea: … to every weight [of the square] you add 3 [to get triangle's weight] […]

5 Hugo: It's enough that you take off 9 from the weight of the square with respect to the weight of the triangle […] it's enough that the weight of the square…

The problem is resumed during the next session:

6 Hugo: The triangle should be a square plus 3 […] because it is as if… you take off […] three squares you can tell that a triangle is a square plus 3 […] If you take off both sides three squares [*he makes a sketch*, Fig. 4b] here you are left with a triangle and you know that the triangle is a square plus 3.

The narrative in [4] is a meta-arithmetical summary of how the students have determined the numbers to balance off the DIM_{TB}. The object of the discourse is the human action ("you add 3") to add 3 to "every weight". In [5], Hugo still talks of a human action ("take off") but now the infinite actions, one for "every weight", are expressed as a single action on *one* weight ("the weight of the square"). The numbers 3 and 6 previously identified (as well as any other numbers) are here replaced by the single signifier "the weight", and then, this objectification can be considered as a

process of saming. In [6], the human subject disappears, the "weight of the square/triangle" collapses into "the square" and "the triangle", in the singular form, and the new narrative, completely depersonalized, is about their relation ("the triangle is a square plus 3"). In summary, the transition from the plural to singular form, the saming, and the depersonalization are the indicators of the process of objectification developing from infinite actions to one action, and from one action to the relations between objects. In table 1, from top to bottom, we can read this process.

Table 1: The development of the students' discourse on indeterminate quantities.

Narrative	Signifier (Math object)	Human action
"To every weight you add 3" [4]	Every weight (Every specific number)	Infinite actions, one for every weight of the square. Every action is on a specific number
"It's enough that you take off 9 from the weight of the square…" [5]	The weight of the square (An indeterminate quantity)	An action on one indeterminate quantity
"it's enough that the weight of the square…" [5]	The weight of the square (An indeterminate quantity)	No human subjects
"The triangle should be a square plus three" [6]	The square, the triangle (Two indeterminate quantities)	No human subjects

The process described before is also visible in the drawing (Fig. 4b) that the students use to endorse their narrative "the triangle is a square plus 3". This drawing is similar to the one made in a previous task (Fig. 4a) with the digital pen on VM_{FB} where the weight of the squares and triangles are 2 and 3 respectively and the circle has a (one), now unknow, *specific* weight. In the drawing produced in episode 3 (Fig. 4b), the signs are the same, but this time on white weights, where it is possible to "put" any number. Therefore, the process of saming described before, which allowed the compression of infinite discourses about numbers through the words "triangle" and "square", expanded the discourse too.

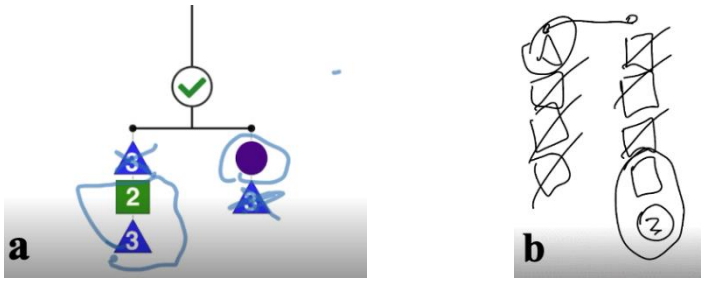

Figure 4: Hugo's drawings in solving tasks with (a) $DIM_{(FB,TB)}$; (b) DIM_{TB}.

DISCUSSION AND CONCLUSIONS

In response to the research questions, the conducted analysis sheds some light on how the DIMs-discourse ($DIM_{(FB,TB)}$ and DIM_{TB}) promoted the students' discourse on indeterminate quantities. The two students have generated words, narratives, and visual mediators linked to the DIMs and to their interaction with them. The continuous interaction with DIMs modifies the students' discourse. It intertwines with, and then expand, the arithmetical discourse, fostering the production of words, narratives and visual mediators that compress arithmetical discourses, shaping, in this way, the meta-arithmetical discourse. Several elements of the DIMs-discourse expanded. The use of the verb "to put" extends metaphorically to the assignment of numerical values to the "white shapes". The names of the shapes, firstly used for given or unknown but specific weights, extend to a variable value, and finally are used at singular as signifiers of objects that can be manipulated and studied (Table 1). The use of an encapsulated couple, where the objects were numbers (known or unknown), expands into a discourse on indeterminate quantities (Fig. 4a-b). The narrative "the triangle is a square plus 3" arises from a discourse compression accompanied by an objectification. This narrative, in which different numbers could substitute for "triangle" and "square", compresses (potentially infinite) narratives on numbers, reducing the complexity of the discourse. The words "triangle" and "square" allow to make a discourse on indeterminate quantities and to deal with them as they were known numbers. In this way the discourse moves on a meta-arithmetical (algebraic) level.

The case analysis shows how Andrea and Hugo, two high school students with school experience with algebra and a history of low achievement in mathematics, were able to participate to an algebraic discourse. Andrea and Hugo's discourse can be regarded still as *informal*. However, according to Kieran (2022), Radford (2014) and Caspi & Sfard (2012), thinking algebraically is not necessarily related to the use of algebraic symbolism and this can be considered a step towards the formal algebraic discourse. This is in line with Nachlieli & Tabach (2012, p. 24), who, in a study on functions, state that an informal mathematical discourse can be considered as a "solid foundation" for the students' "future discourse".

Furthermore, the analysis presented in this paper confirms previous studies involving low-achieving students that show, in different mathematical fields, how DIM can foster students' participation in mathematical discourse (e.g., Baccaglini-Frank, 2021).

Finally, this study has reported an example of development of informal algebraic discourse after the introduction of school algebra for low-achievers. Similar results are presented by Caspi & Sfard (2012) in a study about the emergence of informal algebraic discourses before the teaching of school algebra. These similarities underline the importance of investigating the relationship between informal and formal algebraic discourses during the long process of learning algebra.

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FROM INTERPRETATIVE KNOWLEDGE TO SEMIOTIC INTERPRETATIVE KNOWLEDGE IN PROSPECTIVE TEACHERS' FEEDBACK TO STUDENTS' SOLUTIONS

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This study broadens the notion of Interpretative Knowledge (IK) into Semiotic Interpretative Knowledge (SIK) by considering the role of semiotic systems in mathematical thinking and learning. We analyse the relationship between SIK and feedback structured in four categories according to the semiotic functions involved. Based on quantitative and qualitative data we scrutinize the SIK and feedback deployed by prospective primary school teachers. Although connected with teachers' mathematical knowledge, SIK is a specialized knowledge that requires specific training.

INTRODUCTION

Starting from research related to the conceptualization of mathematical knowledge for teaching (MKT) (Ball et al., 2008), Ribeiro and co-authors (2013) introduce the notion of interpretative knowledge (IK) as the part of the mathematical knowledge "that allows teachers to give sense to pupils' non-standard answers (i.e., adequate answers that differ from the ones teachers would give or expect) or to answers containing errors" (Ribeiro et al., 2016, p. 9). Being IK a kind of knowledge related to problem-solving strategies and errors, it is a piece of typically conceptual or strategic knowledge and does not consider explicitly the semiotic aspects related to signs and sign use in mathematical activity. As highlighted by Duval, conceptualization cannot be accomplished without an adequate competence of what Ernest (2006) calls patterns of sign use and production. As research shows (e.g., Duval, 2017; Ferretti et al., 2022), interpreting students behavior requires a strong semiotic competence.

In Asenova et al. (2023) the notion of semiotic interpretative knowledge (SIK) is introduced, broadening the seminal notion of IK proposed in the literature (Mellone et al., 2020; Ribeiro et al., 2013, 2016). This study undergoes a further step analysing prospective teachers' spontaneous use of SIK in interpreting students' responses and in providing feedback. In this sense, the approach on feedback based on the development of a suitable IK, proposed by Galleguillos & Ribeiro (2019), is extended by considerations related to SIK with the aim to show that SIK represents a theoretical tool able to further deepen the nature of teachers' feedback.

THEORETICAL FRAMEWORK

Ball and co-authors (2008) introduced the construct of mathematical content for teaching (MKT) as the mathematical knowledge needed by teachers to perform the usual tasks related to teaching mathematics. MKT is made of subject matter knowledge

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(SMK), related to the specificities of mathematics, and pedagogical content knowledge (PCK), related to the specificities of teaching and learning of mathematics. Two subcategories of SMK are common content knowledge (CCK) and of specialized content knowledge (SCK), while two subcategories of PCK are knowledge of content and students (KCS) and knowledge of content and teaching (KCT). While CCK is a MKT independent of the teaching-learning context, SCK is a SMK specific to it. Rooted in Ball and colleagues' notion of MKT, Ribeiro et al. (2013) introduced the construct of interpretative knowledge (IK) as a kind of SMK "in the intersection between the common content knowledge and the specialized content knowledge" (p. 4). Di Martino et al. (2019) derive the characterization of IK as belonging to SCK, but as strongly related to the CCK, from the conclusion that a strong CCK is necessary but not sufficient to develop a good level of IK, but at the same time, teachers with a strong CCK have difficulties in accepting unusual strategies that differ from their own (Asenova, 2022). Beside the conceptual, strategic and affective aspects (Di Martino et al., 2016) investigated in research on IK, the semiotic aspects of IK are still little explored. A strong semiotic competence is indispensable for a cognitively meaningful mathematical activity, but semiotic is not a MKT, as for instance geometry or algebra. It might be for this reason that IK does not consider the intrinsically semiotic nature of mathematical cognitive functioning. According to Duval (2017), in mathematics, ostensive references are impossible, as we cannot directly access mathematical objects through our senses. We can say that conceptualisation itself, in mathematics, is identified with this complex coordination of several semiotic systems (Duval, 2017; Ernest, 2006), rooted in semiotic transformations within the same semiotic system (treatments) and semiotic transformations between different semiotic systems (conversions). D'Amore (2003) identifies conceptualisation with the following semiotic functions, specific to mathematics: (1) choice of the distinctive features of a mathematical object; (2) treatment in the same semiotic system; (3) conversion between semiotic systems. The management of such semiotic complexity, within the structure of semiotic systems and the processing of semiotic functions, comes up against Duval's famous cognitive paradox (Duval, 2017): On the one hand we know abstract mathematical objects only through the semiotic activity mentioned above; on the other hand, such a semiotic activity requires the conceptual knowledge of the mathematical objects on the part of the student.

Taking into account the intrinsically semiotic nature of mathematical thinking, in Asenova et al. (2023) the theoretical construct of SIK is introduced as "the knowledge needed by teachers in order to interpret students' answers (be they standard or non-standard), as well as students' behavior, and to give an appropriate feedback to them, when conceptual knowledge is hindered, and thus remains hidden behind difficulties related to patterns of sign use and production, including individual creativity in sign use" (p. 11). SIK lies at the crossover of SMK and PCK because the control of semiotic functions is intertwined both with mathematical knowledge (noesis and semiosis are overlapped) and their implementation in the teaching-learning activity driven by the teacher (KCS, KCT). We show that besides a mere conceptual IK, in the context of

students with special educational needs, a strong SIK seems to be necessary to provide effective feedback.

Feedback is defined by Hattie and Timperly (2007) as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81). These authors distinguish, among others, between feedback about the task (FT) and feedback about the processing of the task (FP). FT and FP can be more or less elaborated and can go from simple correct/incorrect information to more constructive feedback related to additional information on content and strategies to be used. In teacher training, it is important to provide prospective teachers with the skills needed for a wide range of possible feedbacks and to foster the IK needed to give FT. Galegiullos and Ribeiro (2019) investigate prospective teachers' ability to use IK in giving FT: Teachers were asked to work in groups and first solve a task and then provide feedback to some solutions given by students to the same task. These authors classify the provided feedback into four categories: (a) Feedback on how to solve the problem; (b) Confusing feedback: When the feedback seems to be correct, but it can be confusing for the student; (c) Counterexample as feedback; (d) Superficial feedback: The content of such feedback was insufficient (too broad or inconsistent) to allow the solver to understand its meaning. In this paper we develop the kinds of feedback, introduced by Galegiullos and Ribeiro, consistently with the notion of SIK; FT is elaborated according to the implementation of the semiotic functions. We have outlined four main categories of feedback: type (i) - no mention of semiotic functions, which is framed by Galegiullos and Ribeiro's categories; type (ii) - use of semiotic representations confined to the recognition of the distinctive traits; type (iii) - use of distinctive traits and treatments; type (iv) - use of distinctive traits, treatments, and conversions. The semiotic categorization of feedback does not provide a level of effectiveness *per se* but filling the gap "between what is understood and what is aimed to be understood" (Hattie & Timperley, 2007, p. 82) is specific to the context. Feedback is intertwined with the nature of the task, the student's specificities, and the learning environment. In this paper, the accomplishment of SIK on the part of prospective teachers and its benefit for unfolding their feedback effectiveness is investigated, showing how it can be used to further deepen the understanding on the relations between prospective teachers' (S)IK and their ability to provide FT.

METHODOLOGY OF RESEARCH

In our investigation, we first focus on the classification and analysis of the answers given by 180 primary school prospective mathematics teachers (PPTs) attending an Italian University to a questionnaire related to the interpretation of the incorrect answers given by students to four math tasks and to the feedback they would give to the student. Then we focus on the way PPTs use SIK to support their answers to the questionnaire in a follow up interview. We use the PPT's answers to the questionnaire to classify the IK used to interpret the student's answer for themselves and the IK used to provide FT, adopting the following categories: (0) does not respond or the answer is

not classifiable; (1) Conceptual IK: The PPT does not mention representations but only concepts, strategies; (2) SIK-R: The PPT mention only semiotic representations without reference to semiotic transformations; (3) SIK-T: The PPT refers to treatments in the same semiotic system; (4) SIK-C: The PPT refers to conversions between different systems.

The 180 PPTs were in the first year of the 10-semester master's degree-course in primary education and 21 gave their permission to be interviewed after completing the questionnaire. In the following, we discuss and analyse the interview of a PPT.

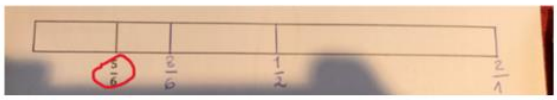
In elaborating the tasks we took our cue from the methodology used in Ribeiro et al. (2013) but instead of asking the PPTs to first solve the problem and then to give feedback to student's solutions, we presented immediately the student's solution and then asked the PPTs to first interpret the solution and then to provide feedback. We chose this approach because our focus was not on the development of the teachers' IK, but on the kind of IK used spontaneously by the PPTs in giving feedback.

Here we focus on the first two tasks of the questionnaire (Figure 1 and 2).

Figure 1: Task 1 (with the kind permission of prof. Cristina Sabena, inspired by prof. Elisabetta Robotti's research on teaching-learning of fractions)

Task 1

Starting from the representation of the number $\frac{5}{6}$, Gina have to represent the numbers $\frac{8}{6}$, $\frac{1}{2}$ and 2.



Question 1.1: What do you think has happened? **Question 1.2**: How would you intervene?

Figure 2: Task 2 (used by the authors in teacher training courses)

Task 2

Giovanni likes to invest his weekly tip in the stock exchange. He bought a share of Tetris, his favourite video game. In the table Giovanni represented the share's performance in the first three months of the investment he started in January.

Gennaio	7 €	
Febbraio	9,1 €	+30%
Marzo	7,7 €	+10%

Milena, after looking at Giovanni's table, claims that the share value of Tetris has decreased by 20 per cent from February to March.

Question 2.1: What do you think has happened in Milena's interpretation?

Question 2.2: How would you intervene?

Task 1 was chosen because it drives the use of semiotic functions (conversions involving symbolic language, natural language and figural representations). Task 2 was chosen because the implementation of the network of semiotic functions is not immediate and our conjecture is that the students would focus more on conceptual IK and treatments confined to algorithms (calculation and confront of percentages).

RESULTS AND DISCUSSION

Regarding Task 1, the quantitative data shows that most of the PPTs opt for IK both in the interpretation of the solution (52,2%) and in the feedback (34,4%). There was a

high percentage of invalid answers to Question 1.1 (22,2 %) and Question 1.2 (33,3%) that can be traced back to a lack of CCK and SCK that hinders IK and the ensuing feedback. A high percentage (22%) of the PPTs provide type (iv) feedback based on the implementation of the three semiotic functions even if only almost a third (7,8%) display SIK-C in the interpretation of the data. This case testifies that when PPTs possess suitable semiotic competences, which lie at the crossover of SCK and PCK, they set out SIK-C and feel the need to ground their feedback in the networking of different semiotic systems for higher effectiveness and clarity.

In Task 2, we notice a prevalence of invalid interpretations and feedback (Question 2.1: 40%, Question 2.2: 68,9%). There is a high percentage of PPTs who resort to IK in the interpretation (33%) but a lower percentage of PPTs who are able to give type (i) feedback (20,5%). A significative percentage (26,7%) of PPTs resorted to SIK-T for the interpretation but only 10,5% provided a type (iii) feedback. To make sense of this result we must consider that the task was challenging for the PPTs in that it was difficult for them to unravel the mathematical knowledge (percentages) in terms of CCK and SCK, and the ensuing KCS and KCT. Thus, on the one hand the IK was not backed by CCK and SCK to carry out an appropriate interpretation of the solution and provide effective feedback. On the other hand, it was difficult for the PPTs displaying semiotic activity at the crossover of SCK and PCK in the interpretation and feedback. Indeed, the SIK-T does not amount to a true semiotic interpretation but to meaningless calculations carried out in symbolic language. They testify the identification of the mathematical object with the semiotic representations accountable to the cognitive paradox.

Sara's Interview

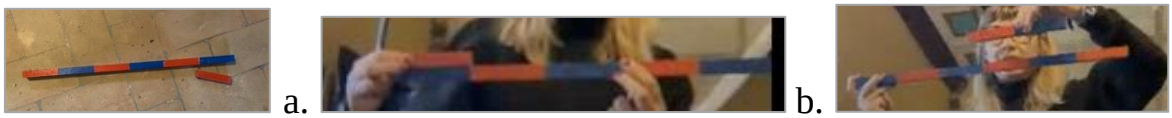
In order to operationalize our theoretical lens for interpretating tasks and providing feedback, we present an excerpt from the interview of Sara concerning Task 1. Sara interprets the solution with SIK-T explaining that the solution does not correctly consider the meaning of the denominators for the ordering of the fractions. She provides a type (iii) feedback based on the ordering of the fractions in the arithmetic symbolic system. The researcher asks Sara to explain her feedback and to make her feedback more effective. She spontaneously performs treatments and conversions that also involve Montessori materials she uses at school with her students. After transforming via treatment all the fractions to 6 as common denominator she grabs the Montessori-rod (Figure 3a).

Sara: If I want to represent $\frac{1}{2}$ which is halfway the length of the rod, three coloured rectangles on one side and three on the other. I consider three of the coloured rectangles [she scrolls the small red rectangle over the rod and counts 1, 2, 3 (Figure 3b)]. The same holds for 5/6 [she scrolls the red rectangle counting 1, 2, 3, 4, 5].

Researcher: How would you represent $\frac{8}{6}$ with Montessori-rods?

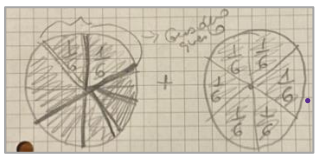
Sara: The only thing I can think of is to take … [she grabs a rod with 6 colored rectangles and a rod with 2 colored rectangles (Figure 3c)]. One represents the number 6 and the other represents the number 2, the 6-rod represents quantity 6 and the 2-rod represents quantity 2. I jump in quantity because I start with the number 1, this red one. When I count I do so 1, 2, 3 ... [scrolls the 1-rod over the 6-rod counting 1, 2, 3, 4, 5]. This is the useful step to take, this is the 6-rod but I consider 1, 2, 3, 4, 5 [referring to $\frac{5}{6}$]. If I consider this [the 2-rectangles colored rod] it means I consider 8 parts, it means I consider 8 units and I'm on the wrong path, I'm somewhere else like this. I no longer have the base and that's it, but I have the base plus two and it comes 8 as a whole. I mess up the student's understanding. So to do $\frac{8}{6}$ it's easier to use pie charts, where I consider all the quantities. The half divides me into two parts, because these wedges are made equal. I have the $\frac{1}{2}$ and $\frac{1}{2}$ and it gives me the whole, then I have the $\frac{2}{4}$, $\frac{3}{4}$ and so on. It's all divided in this fashion [drawing on a sheet of paper (Figure 4)] 1, 2, 3, 4, [counting the wedges on the pie chart] up to 6. So, I take another pie chart divided in six wedges and I consider this and this and this 8 times [writing $\frac{1}{6}$ on each of the 8 slices she is pointing to (Figure 4)].

Figure 3: The Montessori-rods used by Sara to represent the fractions



a. b. c.

Figure 4: Sara's drawing of the pie charts used to represent the fractions



The protocol shows Sara possesses SIK-T that allows her to interpret the solution and provide basic type (iii) feedback. When prompted by the researcher to explain the feedback she would share with the students she feels the need to include other semiotic systems via conversion transformations. Sara is aware that tapping into a network of semiotic transformations, which involve more semiotic systems, empowers the efficacy of the feedback she can provide. Nevertheless, Sara's interview highlights the lack of appropriate SIK that lies at the crossover of CCK, SCK and KCS and KCT. In fact, SIK as a specific MKT requires solid subject content knowledge (CCK and SCK) combined with PCK in order to implement patterns of sign use and production coherent with the mathematics at stake an effective to the student's learning. When Sara crosses the borders of treatment transformations in the arithmetic language to include Montessori- rods and pie charts, she loses control of the meaning of fractions and undermines the efficacy of her type (iv) feedback providing confusing information. Indeed, she mixes the meaning of the colored rectangles of the rods, without

recognizing the distinctive traits of the semiotic representation. On the one hand the rectangles represent the unitary fraction $\frac{1}{6}$, on the other hand they represent an integer quantity, the number of parts in which the whole is divided, 6 parts in the rod with six rectangles. When she wants to represent $\frac{8}{6}$, she is puzzled because "if I consider this (the 2-rectangles colored rod) it means I consider 8 parts, it means I consider 8 units and I'm on the wrong path". She means that she is not considering $\frac{1}{6}$ as unitary fraction but $\frac{1}{8}$. With the pie charts she recollects the correct meaning of the fraction because the pie has no fixed unit. She claims that: "The half divides me into two parts, because these graphs are made equal. I have the $\frac{1}{2}$ and $\frac{1}{2}$ and it gives me the whole, then I have the $\frac{2}{4}$, $\frac{3}{4}$ and so on. It's all divided in this fashion." So, each pie is the whole divided in 6 parts and each part is $\frac{1}{6}$. She then considers 8 parts that represent $\frac{8}{6}$. Sara has not yet reached an appropriate competence in semiotics that allows her to handle conversions in the context of fractions, thus developing an appropriate SIK.

CONCLUSIONS

The aim of the research was to investigate the spontaneous use of SIK by prospective primary school teachers, prior to specific training in mathematics specialized knowledge. We also analysed the impact of SIK on feedback categorized according to semiotic parameters. The quantitative data show that SIK does not belong to prospective teachers as a consequence of their subject matter knowledge. When prospective teachers recur to SIK they deploy the network of semiotic functions especially when providing feedback. Their need to provide effective information is characterized by a type (iv) feedback. Although SIK is not a spontaneous consequence of subject matter knowledge, we can infer that it is a necessary condition to trigger SIK because "there is no noesis without semiosis" (Duval, 2017, p. 23). Sara's protocol shows a spontaneous use of SIK-T both in the interpretation and the feedback but her need for further clarity and efficacy is backed by type (iv) feedback that requires the interplay of all the semiotic functions. Sara's type (iv) feedback clashes against the lack of semiotic competences that would allow her to position her SIK at the crossover of SMK and PCK. We can conclude that SIK is an important instrument in the hands of the teacher to interpret students' behavior and give effective feedback. The interiorization of SIK requires specific training in prospective teachers' professional development. Further research is required to design appropriate training programs that include SIK and validate its effectiveness in providing feedback able to improve students' mathematical learning.

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ALGEBRAIC DISCOURSE DEVELOPMENT IN A SPREADSHEET ENVIRONMENT AND DISCURSIVE-COMPUTER ROUTINES

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The current study aims at understanding the mechanism by which beginning algebra students learn in a spreadsheet environment. To this end we analysed the work of two seventh-grade students while working on purposefully designed activities in a spreadsheet environment. Adopting the commognitive theory, we identified the routines that the students enacted, and defined a new type of routine, relevant while working in a technological environment: a computer-discursive routine. Findings suggest that the bondedness between this routine and a discursive routine that is enacted at the same time is getting stronger, which is evidence of a de-ritualization process that characterizes learning.

INTRODUCTION AND LITERATURE REVIEW

The transition from arithmetic learned in primary school to algebra learned in middle school and high school is difficult for many students, especially for students who have not fully mastered elementary arithmetic (Rojano, 2002). Studies have shown that the use of spreadsheets facilitates the gradual transition from arithmetic to algebra and helps students to become independent (Tabach et al., 2013). Yet, a question arises - while learning in a spreadsheet environment, what is the mechanism that facilitates algebra learning? Our aim is to answer this question. To this end, we analysed the discourse of two seventh-grade students, Maya and Noa (pseudonyms), while working on purposefully designed activities in a spreadsheet environment. The analysis was conducted within the framework of the commognitive theory (Sfard, 2008), with a focus on routines.

The commognitive theory

The commognitive theory (Sfard, 2008) views mathematics as a discourse, and learning mathematics as a process by which the learner becomes part of the mathematic discourse community. Learning manifests itself in a change in one or more of the four features of the mathematic discourse: Words and the ways they are used, visual mediators, narratives, and routines. In this research, we focused on routines that the two seventh-grade students performed. Routine is a repetitive pattern of actions and is defined as a task and procedure pair (Lavie et al., 2019). The task is the task as the performer of the routine sees it in a given situation, and the procedure is the set of actions that the performer enacts to fulfill the task. Routines can be process-oriented or product-oriented. The purpose of a process-oriented routine is the procedure itself, how to perform the routine, whatever its outcome is. Process-oriented routines are

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performed according to an expert's expectations and for social purposes. The performer of the routine is not required to make independent decisions and she fully depends on the expert's approval. The purpose of a product-oriented routine is its outcome, and there can be many procedures to get to that outcome. The performer of the routine is required to make independent decisions while doing the routine. Process-oriented and product-oriented routines are on a continuum and most of them are not merely process-oriented or merely product-oriented. When a learner enters a new discourse, her routines tend to be process-oriented (ritual), and as the learning proceed and she becomes more independent participant in the discourse, her routines turn to be more product-oriented. This is a de-ritualization process.

One characteristic of a product-oriented routine is bondedness. A routine is bonded if one step of the procedure is connected to the next step. Product-oriented routines are inherently bonded because every step leads to the next step until the product is reached. In contrast, process-oriented routines can include steps that are not bonded to each other (Lavie et al, 2019).

A new type of routines: computer-discursive routines

Routines can also be classified according to what they achieve in the world, namely whether they are practical or discursive (Lavie et al, 2019). Practical routines are geared towards creation or changing concrete objects in the world. Discursive routines are aimed towards changing or manipulating discursive objects. In this work, we suggest a new type of routine, which is a hybrid of practical and discursive routines, and is relevant while working in a technological environment. We call this type a computer-discursive routine, and we demonstrate it in a spreadsheet environment. When writing an expression in a spreadsheet cell that makes use of a number in another cell (see Figure 1a) and pressing the "enter" key, a new number appears (see Figure 1b). When "dragging" this expression down a column, new numbers are created (see Figure 2).

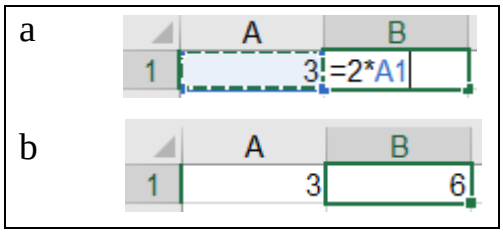


Figure 1a&b: writing an expression

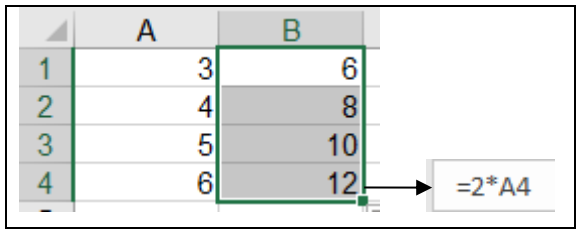


Figure 2: "dragging" cell B1 in a spreadsheet

A learner writes an expression in a cell and the computer "answers" her with a new number created in that cell. The learner drags the expression down a column and the computer "answers" her with new numbers in that column. In that sense, the learner operates on things in the world (pixels), but also "communicates" with the computer. Due to this hybridity of practical and discursive operations, and the uniqueness of the

"answering" feature, we claim such routines deserve a different name, which we term "computer-discursive routine". Notably, at the same time when working in a spreadsheet environment, a discursive routine may also be enacted: for example, by talking with the person you work with or making a mathematical calculation in one's head. We suggest to refer to computer-discursive routines and discursive routine, that occur sequentially and are performed to achieve a single task, as one routine, and call it a compounded-computer routine.

With the aid of these new conceptual tools, we can now re-formulate our research question: What are the changes in the compounded-computer routine along a serious of purposefully designed activities?

METHODOLOGY

Participants and tools

This study is part of a project that is aimed at understand in what ways can spreadsheet-based activities help students who did not fully master elementary arithmetic. Two seventh-grade mathematics students, Maya and Noa (pseudonyms), participated in this study. They were chosen according to their previous achievements and their teacher's recommendation. They met with the first author 10 times, once every two weeks in an out-of-school setting, to perform purposefully designed activities in a spreadsheet environment. Each meeting lasted about one hour. The activities were in accordance with the seventh-grade mathematics curriculum. The researcher did not intervene while the students worked unless they were stuck and did not know how to continue. Each meeting was recorded and transcribed verbatim.

Data analysis

The data was analysed according to the commognitive theory. We focused on the routines the students enacted while doing the activities, and looked for evidence of de-ritualization process. We looked at the first activity, to see the initial state of the routines, and continued to the second activity, where we found a change in the routines. This sub-set of transcripts were chosen in order to follow that change.

We recognized computer-discursive routines and discursive routines in the students' work, and classified them to process-oriented or product-oriented routines. We looked for changes in the relations between those routine, and in the way the students performed the same routine along the different activities.

FINDINGS

In this section, we present the learning process of Maya and Noa as it manifests itself in the compounded-computer routines they enacted.

Beginning of the learning process – activity 1

The first compounded-computer routine we present is taken from activity 1, where the students were asked to calculate the prices of bottles of water. The price of one bottle of water in the shopping mall was nine shekels and the question presented to the

students was: "There was a sale at the shopping mall – for every bottle of water you buy, you pay only two thirds of its price. Add a column with the new prices." Episode 1 presents the routine (The figures in episode 1 have been recreated from the original data for purpose of translation).

Episode 1 – calculating two thirds of the price of a bottle of water

854 Noa: two thirds of its price. In the mall it costs 9, two thirds,

855 Maya: it is 3, 6.

856 Noa: 6 because every third is divide by 3, so 6. So you pay 6.

…

863 Maya: How do we do two thirds *(In the spreadsheet environment)*?

864 Noa: What do we do now?

…

909 Noa: 9 *(from cell C2)*

910 Maya: and then divide

…

918 Maya: by 2…

919 Noa: *(clicks C2, writes '/2' and presses the "enter" key. When she sees the number 4.5 in cell F2, she happily exclaims:)* Boom boom boom! *(Then tries to drag the cell down the column)*

920 Maya: yes *(confirms the dragging)*

921 Noa: yes *(drags the expression down the column. When the students see the numbers created through the dragging, they both clap their hands.)*

	A	B	C	D	F
1	marketplace	street shop	shopping mall	number of bottles	sale at the mall
2	3	6	9	1	=C2/2
3	6	12	18	2	

	A	B	C	D	F
1	marketplace	street shop	shopping mall	number of bottles	sale at the mall
2	3	6	9	1	4.5
3	6	12	18	2	

	A	B	C	D	F
1	marketplace	street shop	shopping mall	number of bottles	sale at the mall
2	3	6	9	1	4.5
3	6	12	18	2	9
4	9	18	27	3	13.5
5	12	24	36	4	18
6	15	30	45	5	22.5
7	18	36	54	6	27
8	21	42	63	7	31.5
9	24	48	72	8	36

The first part of the compounded-computer routine in episode 1 is a discursive routine in which the task of the students was to calculate how much is two thirds of nine [854]. The procedure they performed was to calculate how much is one third and then two thirds. Maya said "3, 6" [855] and Noa explained the procedure [856]. The students authored the narrative that two thirds of nine is six. They referred to one third as a mathematical object and knew how to perform arithmetic operations with this object. We can thus say that the students performed a discursive routine that was product-oriented.

The second part is a computer-discursive routine, that had two sub-routines: (a) writing an expression in the first cell. The students' task in this routine was to write an expression for the computation of two thirds of a number in the first cell in the spreadsheet [863], and the procedure they enacted was to write the expression "C2/2" [909-918] and pressing the "enter" key [919]. The outcome of sub-routine (a) was the number 4.5 (in cell F2). This number is different from the outcome of the discursive routine (the number 6), but the students did not notice this difference and did not

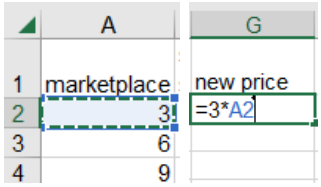
change the expression they wrote. Instead, they continued to sub-routine (b) - "dragging" the expression down the column F [921]. They were pleased to see the numbers they got and clapped their hands [921].

There was a disconnection between the computer-discursive routine and the discursive routine in episode 1, meaning the bondedness of the compounded-computer routine is weak, since the output of one step in the routine (the discursive routine) did not serve as the input of the next step (the computer-discursive routine). Thus, although both the discursive routine (multiplying 9*2/3) and the computer-discursive routine (enter formula and drag) were both product oriented, the compounded-computer routine was ritual, since there was no bonding between the sub-routines.

A change begins – activity 1

The next question posed to the students was to compute the prices of the bottles in the marketplace (each bottle costs three shekels), when a commission of three shekels was added to each purchase, regardless of how many bottles were bought. The compounded-computer routine is presented in episode 2.

Episode 2 – looking for an expression in the spreadsheet for the new price in the marketplace



1004 Maya: *(Maya writes "=3*" in cell G2 (the commission) and says to Noa:)* press the marketplace *(meaning the cell A2, where the original price of one bottle in the marketplace is written – three shekels)*

1005 Noa: why?

1006 Maya: press it.

1007 Noa: but why? *(And then presses cell A2 which creates the expression in the figure)*

The computer-discursive routine in episode 2 is combined from the same two sub-routines as found in episode 1. Whereas sub-routine (b) of dragging was practically the same, a change could be seen in sub-routine (a) – writing an expression in the first cell. The procedure Maya enacted was writing the expression "=3*A2", when the number in cell A2 is the original price in the marketplace. Maya wrote the same type of expression as they wrote before (a number multiplied by "the cell", which in this case is incorrect). We do not have evidence the students knew what number to expect after they would press the "enter" key, but Noa was not satisfied with the first part of the computer-discursive routine (entering the formula), as evidenced by her questioning of Maya's actions [1005, 1007]. This means that for her, it may be that this part is becoming product-oriented, whereas for Maya, it is still just "enter some kind of formula and drag". One possible explanation why this part became more product-oriented for Noa, is that Noa aims for the formula itself to make sense (for the "=3*A2", for example, to be linked to the story of the marketplace). This is the beginning of a change in the compounded-computer routine for Noa, where its bondedness is getting stronger. This is a start of de-ritualization process.

The change continues – activity 2

The next routine we present is taken from activity 2 that occurred two weeks after activity 1. In activity 2, the students were asked to write expressions for the weekly allowance of four children. One of them is Dina, whose allowance was given in a table format (see Figure 3). The routine is presented in episode 3.

Dina:

Number of weeks	1	2	3	4	5	6	7	8	9
The amount of money	7	14	21	28	35	42	49	56	63

Figure 3: Dina's allowance

Episode 3 – Dina's allowance

40 Maya: in the first week Dina had…

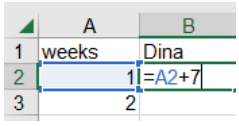
41 Noa: 7 shekels.

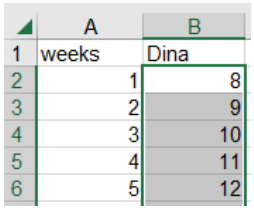
…

63 Noa: (*reads*) how much money did each child have after 3 weeks, ok, here 21 *(for Dina)*

For the discursive routine in Episode 3 the task was to determine the amount of money Dina gets after one week and after three weeks [40, 63]. The procedure was reading the numbers from the table [41, 63]. After that, the students were asked to create the table of allowances in the spreadsheet. The compounded-computer routine for creating the column for Dina's allowance is presented in episode 4.

Episode 4 – creating a column in the spreadsheet for Dina's allowance





219 Noa: in one week,

220 Maya: plus 7, its equal 8 *(writes =A2+7, and presses the "enter" key)* and then what do we do,

221 Noa: you should mark the 8 *(meaning B2, for dragging)*

222 Maya: Exactly *(after dragging the expression)*

223 Noa: no, but how does this help you? 8,9,10,11,12. It should be in jumps of 8.

224 Maya: of 7.

225 Noa: right, 7. and it's not in jumps of 7.

226 Maya: sure it is. In the second week? 2+7? 9. Third week, 3+7. Do you understand?

227 Noa: OK

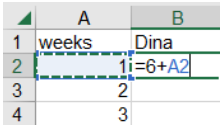
For sub-routine (a) of the compounded-computer routine, the task was to create an expression for the first week [219]. The procedure was to write the expression "=A2+7". The students pressed the "enter" key and saw the outcome of their expression – the number 8. They did not compare this number to the results of the discursive routine in episode 3, where they concluded Dina got 7 shekels in the first week, rather,

they enacted sub-routine (b) straight away. The task of sub-routine (b) was to create column of numbers for Dina's allowance and the procedure was dragging the expression to the whole column. Maya was pleased with the result [222], although the number eight for the first week and the number ten for the third week were not the number they decided on in the discursive routine in episode 3 (numbers 7 and 21, [41], [63]). For her, the computer-discursive routine was process-oriented, which meant the procedure was important and not its outcome. Here, too, like in episode 2, Noa was questioning the outcome [223]. For her, the computer-discursive routine was started to be linked to the discursive routine and the compounded-computer routine became more bonded. However, Maya convinced Noa that the numbers they got were fine. But, what Maya's argument related to was the calculations of the computer, meaning that its computation is according to the expression, and not that the expression is appropriate according to the discursive routine [226].

The students continued, but then Maya stopped and said that the first number in the column of Dina's allowance should be 7 and not 8. They tried to fix it. The compounded-computer routine for fixing Dina's expression is presented in episode 5.

Episode 5 – fixing the column in the spreadsheet for Dina's allowance

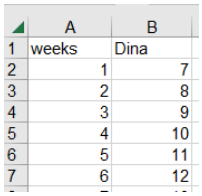
261 Maya: Dina gets every week 7 shekels, she doesn't get 8 shekels. OK, we do…OK *(writes the expression "=6+A2")*. Now what do we do *(drags the expression down the column).*



..

263 Noa: *(When she sees the numbers in the column)* no, what are you doing? It is exactly the same as before.

264 Maya: right. *(Deletes the numbers in the column, except the number 7 in the first cell)*



It can be seen in episode 5 that the compounded-computer routine has become bonded for Maya too. She said "Dina gets every week 7 shekels, she doesn't get 8 shekels", and she wrote an expression that gave the number 7 [261]. After they dragged the expression down the column, Noa said "no…it is exactly the same as before." [263], Maya agreed and deleted the numbers [264]. This time Maya compared the numbers in the column to the result of their discursive routine and she agreed that those are wrong numbers. The compounded-computer routine became bonded for her too.

DISCUSION

We showed a learning process of two seventh-grade students, Noa and Maya, who worked on purposefully designed activities in a spreadsheet environment, and focused on the routines enacted by the students. The contribution of the study lies firstly in the theoretical-methodological contribution. We defined a new type of routine, relevant while working in a technological environment: a computer-discursive routine, that is different from a discursive routine. We showed how computer-discursive routines can be linked to discursive routines and defined the linkage between these two types of routine as compounded-computer routine. This new conceptualization enabled us to

show, in micro-detail, a process of learning (or de-ritualization) that happens in a computer mediated environment.

The effectiveness of computer mediated learning environments, in general, and spreadsheet-based activities, in particular, for the learning of mathematics has been demonstrated in multiple studies (for example, Tabach et al., 2013). However, less is known about the processes of learning that occur in these environments. Commognition has been shown to be highly effective for describing processes of learning in non-computerized environments, yet its power has been harnessed less in computerized environments (as an exception, see Baccaglini-Frank, 2021). The conceptualization and analysis offered in this paper present a first step in this direction, that invites further development.

ACKNOWLEDGMENT

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UNDERGRADUATE STUDENTS' SECOND-ORDER COVARIATIONAL REASONING WHEN CONCEPTUALIZING PARABOLOIDS SUPPORTED BY DIGITAL TOOLS

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In this paper, we aim to investigate undergraduate students' second-order covariational reasoning. For this purpose, we designed a teaching sequence concerning the characterization of paraboloids and involving the use of two combined digital tools, GeoGebra AR and GeoGebra 3D, to help students develop covariational reasoning. The teaching sequence was experimented with 30 undergraduate students in mathematics. The research data were collected over two collective discussions and consisted of audio and video recordings. The data analysis was based on a descriptive coding of the emerging forms of covariational reasoning. Findings revealed that the adopted digital tools supported students' covariational reasoning when conceptualizing paraboloids.

INTRODUCTION

This paper focuses on the importance of covariational reasoning for Mathematics undergraduate students learning two-variable functions, three-dimensional surfaces, and related level curves while using digital tools such as GeoGebra 3D and GeoGebra AR.

The main motivation for this study was the difficulties students encountered in conceptualizing mathematical objects. One of the difficulties in learning mathematics arises due to the impossibility of conceptualization based on meanings referring to a concrete reality. On the one hand, every mathematical concept requires representations because there are no "objects" to exhibit, that is, conceptualization needs to go through representative registers; on the other hand, the management of representations is difficult because of the lack of concrete objects to relate the representations themselves, both in terms of their production and in terms of transformations. Some of these difficulties can also be found in many students attending university courses, where certain gaps and misconceptions remain (Eisenberg, 2002). For instance, university students often identify curves with functions of one real variable and surfaces with functions of two real variables. In particular, the difficulty of recognizing the graphical representation of a surface from its analytical representation persists, and students often do not recognize the graph of a surface from its level curves and vice versa (Tall, 1993).

Literature in Mathematics Education remarks the importance of a covariational approach for a deep understanding of functional thinking, including both one- and two-variable functions (Carlson et al., 2002; Thompson & Carlson, 2017). Covariational

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 67-74). PME 46.

reasoning is usually intended as "the cognitive activities involved in coordinating two varying quantities while attending to how they change in relation to each other" (Carlson et al., 2002, p. 354). Moreover, recent studies have highlighted how activities involving digital tools can help students to develop the multiple meanings of covariational reasoning. For example, Swidan et al. (2019) discussed the use of augmented reality (AR), Swidan et al. (2022) investigated the benefit of some digital GeoGebra applets, and Johnson et al. (2017) studied the use of some dynamic computer environments. In addition, recently, the construct of second-order covariation has been introduced by Arzarello (2019) and further explored by Bagossi (2022). Second-order covariation is intended as the ability to envision a family of functions and its characteristic parameters varying simultaneously.

In this study, the GeoGebra 3D experience is enriched with a new element with the AR functionality, that is the possibility of exploring mathematical objects generated by a computer placed in real-world environments. This should engage learners in exploring and interacting with mathematical objects resulting in a deeper understanding of the content. The great benefit of GeoGebra AR is that it solves the 3D navigation problem in a very intuitive way. In order to get users to explore 3D function graphs using GeoGebra AR, students can explore the surface from the front or any side; looking from above, it shows a cross-section of the graph walk around the object and explore it from different perspectives.

This contribution aims to refine the characterization of the second-order covariation construct by analyzing data from a teaching sequence involving undergraduate students engaged in purely mathematical activities about two-variable functions and three-dimensional surfaces. The data analysis revealed qualitative or quantitative forms of second-order covariational reasoning that the students developed throughout the different phases.

THEORETICAL FRAMEWORK

Learning two-variable functions with a covariational approach has been recognized as extremely important in the literature (Thompson & Carlson, 2017). To better frame the complex forms of covariational reasoning that can emerge when dealing with functions, we will adopt an enlarged theoretical framework about covariational reasoning, recently introduced, which considers covariation as a wider form of mathematical reasoning between mathematical objects rather than only variables.

First-order covariation, that is covariational reasoning as it is traditionally known in the Mathematics Education literature, is defined as the ability to envision how two quantities' values vary simultaneously. Six cognitive levels describing a person's capacity to reason covariationally have been identified (Thompson & Carlson, 2017). can reason covariationally in a "chunky and continuous" way when envisioning changes in the two quantities referring to intervals of a fixed size, or in a "smooth and continuous" way when students envision the two quantities as varying simultaneously through intervals in a smooth and continuous way.

Second-order covariational reasoning (COV 2) is the ability to envision a family of functions and its characteristic parameters varying simultaneously (Arzarello, 2019; Bagossi, 2022). Even if a rigorous cognitive characterization of COV 2 has not yet been initiated, some findings in this direction have already been presented. For example, data from learning experiments with high school students revealed that students could succeed in making explicit the direction of change of both the parameter and the related graph to condense the relationship between the quantities involved in a parametric equation (Bagossi, 2022; Swidan et al., 2022).

This study aims to highlight how students can be supported in developing second-order covariational reasoning using digital tools (GeoGebra AR and GeoGebra 3D) to conceptualize and characterize paraboloids. The research question we try to answer in the following is: Which characterizations of second-order covariational reasoning emerge in undergraduate students when conceptualizing paraboloids using digital tools?

METHODS

Context and participants

This work is part of ongoing research on the use of digital tools to improve mathematical conceptualization processes in students. The teaching sequence presented here was designed and implemented by exploiting the potential of joint use of GeoGebra AR and GeoGebra 3D, allowing students to visualize and manipulate 3D mathematical objects in the real world. Our teaching sequence was designed assuming that using these two digital tools could facilitate students in the characterization of several 3D surfaces through suitable manipulations of these mathematical objects involved and supporting the development of students' covariational reasoning. A class of 30 Italian undergraduate Mathematics students was involved in the teaching sequence on conceptualizing paraboloids based on the joint use of the two digital tools.

Teaching sequence

The teaching sequence consisted of three activities designed to foster students' development of covariational reasoning through GeoGebra AR supported by GeoGebra 3D in the processes of conceptualization of the mathematical object involved (Capone et al., to appear). Collective discussions led by the professor followed phases of small group work.

During the first activity, students were asked to vary the values of a parameter k via a slider and to observe and describe what these variations caused on the two worksheets in GeoGebra AR and GeoGebra 3D. The activity aimed to guide students in characterizing the curves in the 2D plane as the level curves generated by the intersection of the surface and the plane $z=k$ in the 3D plane. During the second activity, students were asked to find a relationship between the level curves and the surface of the first activity concerning their analytical representations. Specifically, students were asked to find the equation of the 3D surface by comparing the different

level curves corresponding to different values of parameter k (Fig. 1). The mathematical purpose of this activity was for the students to conceptualize that as parameter k varies, the level curves are circumferences with the center in the origin and radius, $\sqrt{k}$ and consequently, they have equation $x^2+y^2=k$. Moreover, to identify the equation of the 3D surface, students should observe that the dependence of the variable z on parameter k disappears.

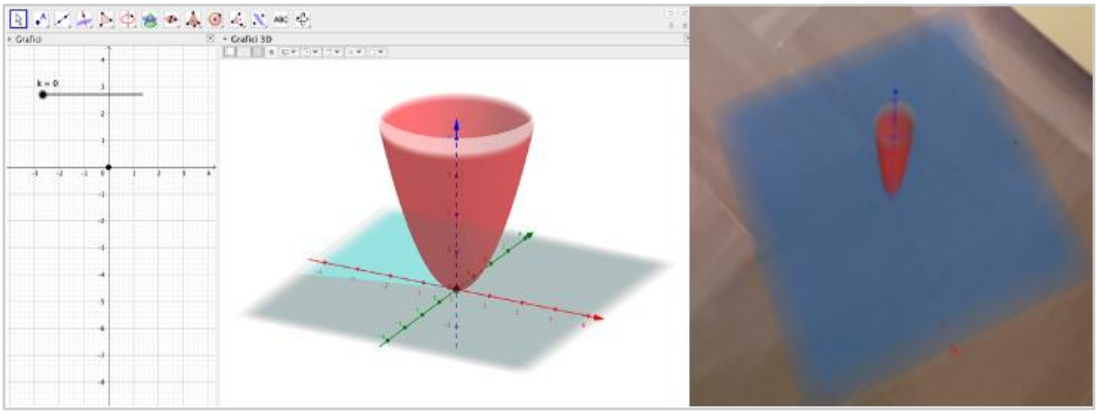

Figure 1: GeoGebra 3D and GeoGebra AR worksheets during the second activity.

Finally, in the third activity, students were provided with a new worksheet in GeoGebra AR and GeoGebra 3D. They were asked to consider the equation of a family of paraboloids in the form $z=ax^2+by^2$, to vary first separately and simultaneously the parameters a and b with the two associated sliders in both the worksheets and to observe and describe what these variations (Fig. 2). The mathematical purpose of this activity was to describe the characteristics of the different families of surfaces obtained from the variations of parameters a and b with the sliders, highlighting how both the surface obtained and their corresponding level curves generated by the intersection of the surface under consideration and the plane of equation $z=k$ varied when changing the value of the slider k.

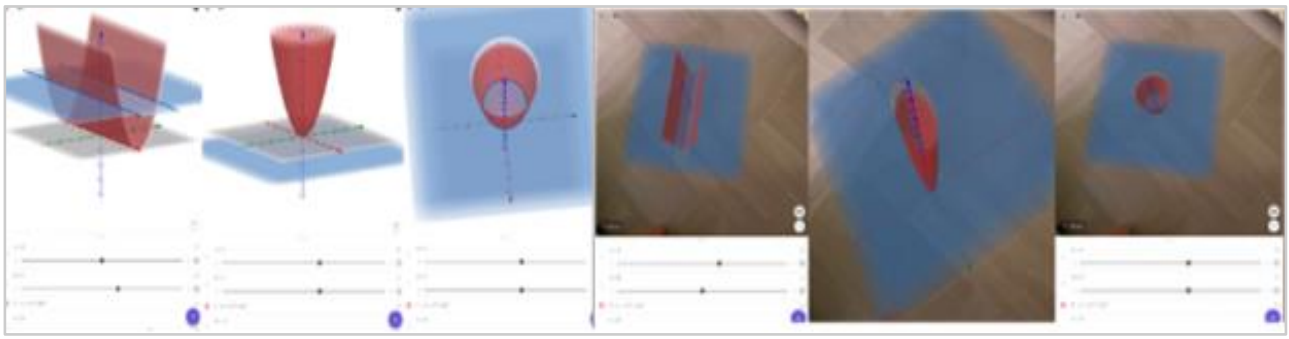

Figure 2: GeoGebra 3D and GeoGebra AR worksheets during the third activity.

Data collection and data analysis

All the activities, working group sessions and collective discussions were video-recorded, transcribed, and translated into English. The data were analyzed with the enlarged theoretical framework about second-order covariation. In this qualitative study, we adopt an interpretative approach based on a descriptive coding of second-

order covariational reasoning (Saldana, 2015). After having selected episodes revealing COV 2, students' reasoning was analyzed by describing the objects involved (variables, parameters, or functions) and their features (qualitative or quantitative forms of reasoning). Special attention was devoted to the chosen mathematical representations (graphical, symbolic, or verbal) and the features of GeoGebra AR and GeoGebra 3D that most supported students in developing their covariational reasoning.

RESULTS

In this section, we report the analysis of three episodes selected from collective discussions and revealing second-order covariational reasoning. Specifically, episode 1 is from the discussion conducted after the second activity, while episodes 2 and 3 are from the discussion held after the third activity. In the following, the professor leading the discussion is denoted with T, while the students are denoted as Si. The episodes are reported in chronological order.

Episode 1

This episode belongs to the discussion after the second activity. S1 is replying to the T's question about their answer to the activity "By comparing two curves obtained for specific values of slider *k*, can you get information about the surface in red? Can you describe its analytical representation with the available tools? Justify your answer". The worksheets students were provided with enabled them to observe that the variation of the slider *k* causes both the variation of level curves and the variation of the position of the plane *z*=*k* that cuts the 3D red surface (Fig. 1).

1 S1: We have observed that for $k = 1$ the equation of the circumference, that is obtained by intersecting the red surface with the blue plane $z = 1$ is $x^2 + y^2 = 1$; on the other hand, for $k = 2$ we get $x^2 + y^2 = 2$, and so on... So, we can generalize and say that we get $x^2 + y^2 = k$, depending on the *k* we chose… If this is the equation of the intersecting curve, then we can deduce that the surface in red has equation $x^2 + y^2 = z$, representing an elliptical paraboloid.

In this excerpt, S1 elaborates on the coordination of the numerical values of the slider k with the level curves, which are circumferences expressed in the analytical register. Even if it is not stated explicitly, it seems clear that in S1's understanding, the chosen values of *k* are increasing, and so are the level curves whose dependence on *k* is encapsulated in the radius. Hence, S1's reasoning was coded as a quantitative form of COV 2: the direction of change of the numerical values parameter and level curves is present even if in an implicit form.

Episode 2

In this episode, from the discussion after the third activity, students elaborate on the answer to the question in which they were asked to reflect on the level curves of each paraboloids' surface according to the values of parameter *a*.

2 S2: And therefore for $k = 0$. Then the level curves for elliptic paraboloids, that is when a is greater than 1, are precisely ellipses of the equations $ax^2 + y^2 = k$ *[T moves the slider* k *and makes the level curve appear from above]*. So, it is fixed b and, therefore, what would be the minor semi-axis, because a is greater than 1 and therefore exceeds b and, therefore, it is the major semi-axis that always increases; therefore, the level curves are ellipses. And the last case is when a is negative, and so for hyperbolic paraboloids, we have hyperbolas as level curves. If I'm not mistaken, we should also have asymptotes as level curves, I say. But I don't know if the boundary case has been reached. *[T moves the slider* k*]*.

In the first part of this episode, a covariation between parameter a and the level curves can be recognized. First, S2 recognizes the dependence of the level curves on parameter a and encapsulates this dependence in an analytical formula. Then, S2 specifies that the case analyzed is a greater than 1, which means it is the major semi-axis and always increases [2]. In this form of COV 2, the direction of change of parameter a is explicit ("always increases" [2]), and the covariation with the level curves is recognized, but how the level curves vary is not outlined. We coded this form of COV 2 as qualitative, absence of reference to the numerical values, and only the direction of change of the parameter is stated. S2 also claims "when a is negative, and so for hyperbolic paraboloids, we have hyperbolas as level curves" [2]: the student expresses the previously acquired knowledge by naming the resulting curve without mentioning how the curves change by varying the parameter. This sentence reveals the recognition of COV 2.

Episode 3

In this episode, from the discussion after the third activity, students keep elaborating on the paraboloid surface, varying the values of both parameters a and b.

3 T: So… [T shows the surface on the GeoGebra and starts according to S3's instructions to change the position of the values on the sliders].

4 S3: In the first case where $a = b = 0$, we have the plane $z = 0$. Instead, we get an elliptic paraboloid if we take a greater than 0 and b greater than 0.

[...]

5 T: If a is equal to b instead?

6 S3: Paraboloid, and that's it. We get the same thing with a and b both less than 0 and not equal.

7 S4: But with concavity downward.

8 S3: Instead, a different case is when we take a and b as discordant. We always get a hyperbolic paraboloid; however, if a is greater than 0 and b is less than 0, and vice versa, we obtain the y-axis or the x-axis.

9 S4: However, we observed that it is only a 90° rotation of the figure depending on the choice of a and b.

S3 identifies a second-order covariation between the parameters' values and the resulting surface [4-6]. S4 introduces a new feature of the paraboloid by referring to its concavity that becomes downward [7] when *a* and *b* are both negative. Eventually, S4 refers to a 90° rotation [9]: the shape of the surface is always the same, but its position in space changes depending on the parameters' values, and this tells how the parameters influence the family of surfaces. Hence, S3's understanding reveals a qualitative form of COV 2 in which a change in the values of the parameters determines a change in the concavity of the paraboloids. At the same time, S4 succeeds in recognizing an invariant relationship between the generated paraboloids consisting of a 90° rotation [9].

FINAL REMARKS

The preliminary results previously discussed suggest that the teaching sequence exploiting the two digital tools, GeoGebra 3D and GeoGebra AR, supported the emergence of forms of COV 2. Indeed, in the three analyzed episodes, we could detect these forms of second-order covariation: i) a recognition of COV 2 when students do not elaborate on how the curves are changing simultaneously with varying the parameter, but they recognize the simultaneous change (episode 2); ii) a qualitative form of COV 2, absence of reference to the numerical values, in which only the direction of change of the parameter is stated (episode 2); iii) a qualitative form of COV 2 in which a simultaneous change in the parameters determines a change in the shape of the paraboloids (a change of concavity or even an invariant relationship between the generated paraboloids, a 90° rotation - episode 3); iv) a quantitative form of COV 2 in which the direction of change of the numerical values of the parameter and level curves, expressed in an analytical representation, is present even if in an implicit form (episode 1).

We could not detect an evolution in the forms of second-order covariational reasoning: they seem to be intertwined throughout the two collective discussions. We also observed that often students focused on the elaboration of formal classification of the surfaces (elliptic paraboloid [4], hyperbolic paraboloid [8]) rather than envisioning its variation; the discrete values provided by the sliders associated with the parameters may also lead students to think in terms of discrete values [1] rather than as a quantity varying continuously. The teaching sequence's benefits and limitations for supporting covariational reasoning deserve a deeper analysis and reflection.

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MEANING-MAKING THROUGH QUESTIONING IN AN AUGMENTED REALITY ENVIRONMENT

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This study examines how using a specific augmented reality design may prompt students to ask questions, and how these questions may prompt the meaning-making of mathematical concepts. The phenomenological perspective, which considers meaning-making a process of disclosure, guided this study. Drawing on the case study methodology, we focus on a triad of 15-year-old students and analyze how the questions posed by the students help them to make meaning of the mathematics concepts embedded in a dynamic phenomenon. Three layers of disclosure were found, and the kinds of questions posed by the students were identified. The relationship between the questions posed and the layers of disclosure is also discussed.

INTRODUCTION

Posing questions by students is one of the key practices in the teaching and learning of mathematics, which may enrich the students' experience of mathematics. To emphasize the importance of the students' posing good mathematical questions, Mason argued that "it must be every competent teacher's dream that students will ask 'good' mathematical questions" (Mason, 2020, p. 710). Despite this recognized importance, research in mathematics education has focused mostly on the teacher's questioning as an instructional strategy but rarely on the students' questioning (Mason, 2020).

Understanding mathematical concepts and developing mathematical thinking require a significant investment of time and effort. Questions that students confront (and ask), the tools they use, and the social interactions between the students themselves and with the teacher, play a crucial role in developing mathematical thinking and understanding. Recently, Swidan et al. (2020) found that students who used a dynamic digital tool, whose display shows two graphs, one of a function and the other of its antiderivatives in two-linked Cartesian systems for learning calculus concepts in small groups, posed several types of questions. Answering those questions helped the students disclose the mathematical meanings of the graphs in the digital tool.

Following these insights, we conjecture that an augmented reality (AR) environment that collects real-time data of a dynamic phenomenon, while simultaneously augmenting the students' experience with mathematical representations of the dynamic phenomenon itself, can foster student engagement in the processes of questioning and answering sequentially. Furthermore, these questioning processes play an important role in disclosing the mathematical relationship embedded in the dynamic phenomenon. Specifically, in this study, we aim to examine how students make sense of mathematical concepts through questioning processes in an AR environment. In

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particular, we aim to examine how the use of AR technology may prompt students to ask questions, which, in turn, may prompt the meaning-making of mathematical objects.

THEORETICAL FRAMEWORK

Phenomenological perspective

To describe the meaning-making of mathematical objects, we refer in this study to the phenomenological perspective as elaborated by Rota (1991). Rota stated that there is "no such thing as true seeing," but "there is only seeing as" (1991, p. 239). According to Rota, this process is referred to as disclosure, a Husserlian concept whereby refers to the process by which people make sense of and interpret the world around them and various situations in the world. The disclosure process is far from natural. Students should be educated to make sense of what they disclose when they meet with mathematical objects (Radford, 2010). This is, of course, dependent on the students' background and age. Moreover, a given mathematical situation may evoke different contexts and lead to different sense-making. According to Rota, rather than being isolated, these different contexts are instead layered upon one another, and the layers can generate different meanings over time. To illustrate the idea of disclosure, let us imagine an increasing and decreasing continuous graph in a Cartesian plane. Young students may see the graph as the picture of a mountain; other students with a wider mathematical background may perceive the same graph as a function. Seeing the mathematical meanings (e.g., the graph of a mathematical function) that are in an object (e.g., the drawing) is a lengthy and delicate process.

Questioning process

Processes of questioning and answering sequentially are central to the teaching and learning of mathematics. Questions may direct students' attention to some mathematical features that characterize the phenomena that the students should disclose (Mason, 2008). In addition, having the students themselves engage in posing questions is generally considered "a useful process in their pursuit of learning, in that questioning is one of the most important ways students can support their own learning to become literate, well-educated people" (Boaler & Humphreys, 2005, p. 72).

In this study, we adapted Mason's classification of the questions that can be asked in mathematical classes (Mason, 2000): attention focus, testing, and enquiry. *Attention focus questions* aim at drawing the learner's attention to something that the asker is already aware of and would like the respondent to notice as well. *Testing questions* aim at emphasizing something that is already known to the respondent in order to ascertain something, or to establish control and dominance of the situation. *Enquiry questions* aim at getting the respondent to wonder what the possible answer (or answers) might be. There are no expectations for a certain output of the learner. The enquiry question may not necessarily require an answer.

METHOD

Augmented reality environment

The technological tool we used in this study is a specific design of AR that collects real-time data of a dynamic phenomenon during a physical experiment (Fig. 1a). In this experiment, a cube is moving along an inclined plane and real-time data of time and distance are collected by sensors and analyzed. The AR headset shows two mathematical representations, a table of values and a discrete distance-time graph, created simultaneously while performing the experiment. The students were able to observe both the real-world experiment and the mathematical representations of the dynamic object immediately in real-time (Fig. 1b).

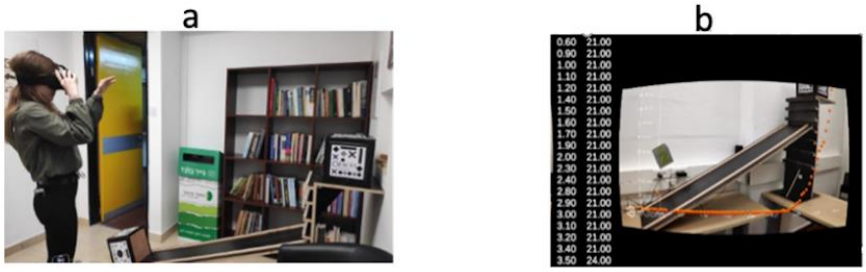

Figure 1: (a) Student uses the AR headset to collect real-time data;
(b) The mathematical representations of the dynamic object the student sees using the AR headset.

Participants and tasks

The study presented here is a small part of a larger project that aims to understand the integration of AR technology into mathematics classes. The present study reports on the case study about the interaction processes of three 15-year-old students from Israel, Shilat, Ori, and Shahar. At the time of the experiment, the students had already learned the concepts of linear function and quadratic function. The students worked together and spent 1.15 hours on the various tasks designed for the study (Fig. 2).

The experiment concerned the motion of a ball rolling along an inclined plane, the so-called Galileo experiment. Students were instructed through some guidelines provided on a worksheet (Fig. 2). First, students were asked to let a cube slide down along the plane and interpret what they could observe through the AR headset. They were asked to write some observations on their own and then to discuss those conjectures as a working group. In the second phase, students were requested to guess what would happen when changing the inclination of the plane. Hence, they were requested to conduct the experiment with a different inclination, to write some observations on their own, and then to discuss their findings together.

Galileo Experiment

Preparation
You will conduct an experiment with your AR-device.
The display will show many information.

Experiment
Conduct the experiment (the inclination is given) together.
Your task is to interpret what you see on the screen.

- Alone: Write down at least three observations you made.
- Together: Compare the observations. How can they be useful for you to understand the experiment?

Optional
What do you think will happen if you change the inclination of the plane?

Conduct the experiment together with a different inclination.
- Alone: Can you confirm your conjecture or do you need to modify it?
- Together: Discuss your findings.

Figure 2: Task worksheet given to the students translated into English.

Data collection and analysis

To collect the data, the learning experiment was recorded using two video cameras. One camera was located behind the students to capture their interactions. The second was set in an AR headset to record what the students see. To analyze the data, we repeatedly watched the video and then split it into episodes. Next, we analyzed each episode, distinguishing the kinds of questions asked by the students and the layers of meaning. Moreover, in order to focus on the role of the AR learning environment, we also analyzed the actions performed by the students distinguishing between actions with real objects (e.g., releasing the cube, changing the plane inclination) and with virtual objects (e.g., gestures reproducing the trend of the virtual graph). In the last round of analysis, we organized the data in a timeline as done in Swidan et al. (2020) to learn about the relationships between the kinds of questions and the disclosure processes. In the timeline, time is not distributed equally; each cell corresponds to a different interval of time functional for analysis purposes.

RESULTS

In this section, we present the results regarding the layers of disclosure and the type of questions. We found three layers of disclosure: *local layer*, *properties layer*, and *relationship layer*. In the local layer, the students disclosed a specific representation such as the graph that models the cube's movement or the real experiment itself. For example, they said "here is the graph!" or "the cube moved quickly." In the properties layer, the students disclosed the features and properties of a specific representation. For example, they said "first, the graph goes up, then it goes down" or "the numbers in the table are increasing" In the relationship layer, students disclosed relationships between and within the representations, for example, a relationship between the cube's movement and the graph, the relationship between cube speed and plane inclination,

or the relationship between the graph and the table of values. Regarding the questions asked by the students during the experiment, in total the students asked 19 questions: 10 were attention focus questions, seven were testing questions, and two were enquiry questions.

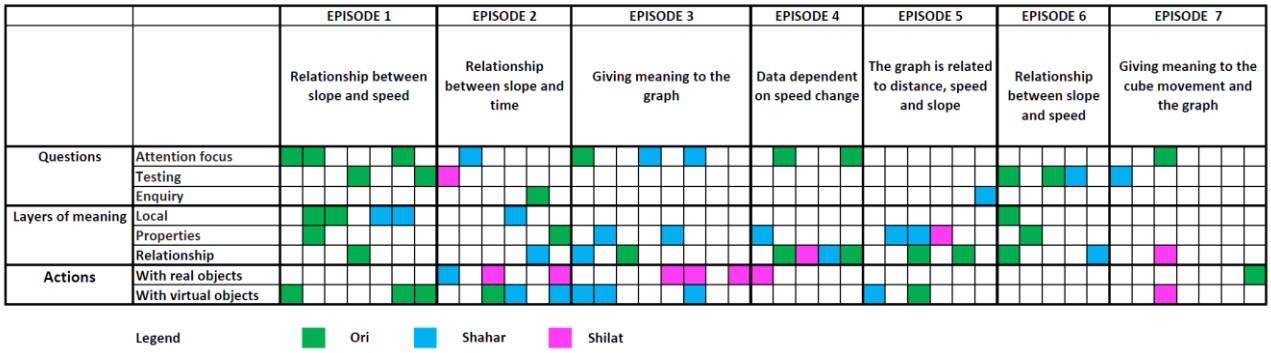

Figure 3: The timeline analysis of the learning experiment. Each color corresponds to a different student as described in the legend.

The timeline analysis (Fig. 3) shows the evolution and the relationships between the questions asked by the students, the layers of disclosure, and the actions performed by the students throughout the entire learning experiment. We observe that at the beginning of the experiment, the attention focus questions were dominant (the students asked nine attention focus questions versus three testing questions), whereas as the experiment progressed, the testing questions became dominant (the students asked four testing questions versus one focus attention question). It is worth mentioning that the students disclosed the properties of the graphs after they asked the testing and attention focus questions. In addition, we found that the virtual mathematical representations juxtaposing the real objects invited the students to interact with them through gestures. Following these gestural movements, the students disclosed the properties of the graph. This finding suggests that the mathematical representations juxtaposing the real objects served the students as embodied objects that could be touched and described, and hence helped the students to disclose the properties of the graphs.

In the two excerpts described below, we will illustrate the relationships between the questions asked, the actions taken, and the disclosures that were achieved. Specifically, we will focus on how the questions emerged and helped the students disclose mathematical relationships.

Excerpt 1

In this excerpt, the students explored the relationship between plane inclination and cube speed. In doing so, the students carried out several experiments each time with a different inclination. One of the students, Shilat, was in charge of releasing the cube from the top of the inclined plane, and the other two students, Shahar and Ori, traced the cube using the AR headset (Fig. 4a). The students referred to the graph and said:

1 Shahar: Wow, how different it is.

2 Ori: Yeah, (cube is) very fast (reproducing the trend of the graph with her finger as in Fig. 4b).

3 Shahar: It's like this... I don't even know what it looks like (gesture simulating the shape of the graph as shown in Fig. 4c).

4 Ori: Should we draw it? (She draws the graph).

5 Shahar: When the inclination is greater, the speed (of the cube) is faster.

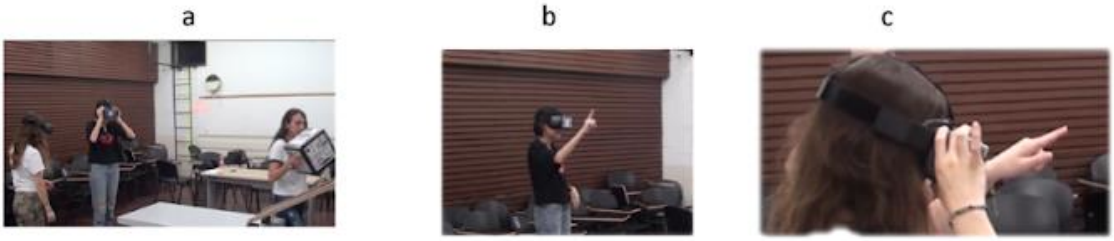

Figure 4: (a) Shilat released the cube from the top of the inclined plane; Shahar and Ori traced the cube using the AR headset; (b) Ori reproduced the trend of the virtual graph with her finger; (c) Shahar made the same gesture.

Shahar compared the graph she obtained upon releasing the cube with the graphs she had obtained in a previous experiment when the plane inclination was less steep [1]. Ori confirmed that the graphs were different and also referred to the cube speed [2]. Initially, Shahar's claim suggests that she had locally disclosed the graph [1], while Ori's statement [2] suggests that she had disclosed the properties of the graph while describing it through gestures (Fig. 4b). Shahar also disclosed the properties of the graph [3] while reproducing with her hands how the graph looked like (Fig. 4c). It seems that Shahar's difficulties in verbally describing the graph motivated Ori to ask a focus attention question [4], possibly because Shahar had disclosed aspects of the graph on which she wanted to focus her attention. Then, Shahar disclosed the relationship between two real elements, plane inclination and cube speed [5].

It seems that the disclosure process of the two objects in isolation – the virtual graphs and the cube speed – is fundamental for disclosing the relationship between virtual objects and real objects, as we will show in the next excerpt.

Excerpt 2

After the previous excerpt, the students realized that each of them had obtained a different graph representing the distance-time function of the moving cube – this is a feature of AR technology since each student captures the cube movement from a different position. Afterward, they drew the graphs they had obtained. The students looked through their classmate's headset (Fig. 5a). Ori disclosed that the graphs were different, while Shahar disclosed that although the graphs were different, they had some similarities. At this moment, Shahar suggested decreasing the plane inclination and examining how the graph would change consequently. Shahar began by asking an enquiry question that invited the students to conduct a new experiment.

6 Shahar: What would happen if you decreased the plane inclination? (Shahar changes the plane inclination).

7 Ori: Now it is clear.

8 Shilat: Mine is almost the same.

9 Shahar: Mine has a large amount of points on a straight line (gestures with her hand the many points as in Fig. 5b).

10 Shilat: Mine is also like this. Afterward, I have some points going up (gesture in Fig. 5c).

11 Shahar: Finally, we have a common conclusion.

12 Researcher: And it is?

13 Shilat: We all have a straight line and then some points that go up. We understand the straight line, it is when the cube slides down the inclined plane, and the points go up when the cube leaves the inclined plane.

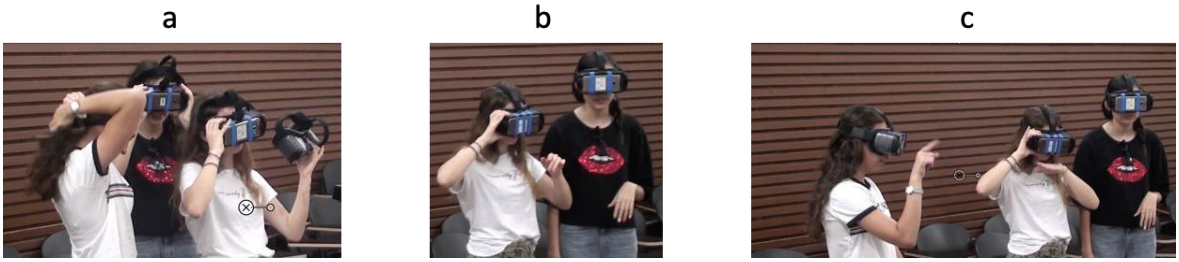

Figure 5: (a) The students looked through their classmate's headset; (b) Shahar gestured with her hand the large amount of points; (c) Shilat pointed to the points going up.

This excerpt starts out with an enquiry question posed by Shahar [6], possibly because she hypothesized that the slower the motion of the cube, the better the graph would have been displayed on the AR headset. Statements [7-8] suggest that Ori and Shilat locally disclosed the graph without describing its properties. Then Shahar disclosed the properties of the virtual graph that was present in front of her [9]. Her gestures suggest that she interacted with the virtual graph (Fig. 5b), and this interaction seems to help her in disclosing the graph properties. The same disclosure process happened with Shilat [10] who also used a gesture to disclose the properties of the graph. Shilat's response [12] to the researcher's question [11] suggests that she disclosed another layer of meaning which is the relationship between the virtual graph and the cube's movement, even though the conclusion she reached was not mathematically correct.

FINAL REMARKS

In this paper, we focused on the questions the students pose as they use AR technology. Our findings showed that the students mainly asked focus attention questions, which helped them to disclose layers of meaning that we called local, properties, and relationship. Following the testing questions, the students mainly disclosed the relationship between virtual objects (graph) and real objects (cube, inclined plane). Regarding the enquiry questions, we found only two of them posed by the students (Fig. 3). After the first question [6], the students disclosed the relationship between the

virtual objects and the physical objects; after the second enquiry question, the students disclosed the relationship between the graph and the table of values. We consider this layer of disclosure to be more sophisticated since the students tried to interpret the cube's motion by disclosing the relationship to the mathematical representation.

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ADAPTIVE STRATEGY USE IN PATTERN-RECOGNITION OF FIRST GRADERS WITH AND WITHOUT RISK OF DEVELOPING MATHEMATICAL DIFFICULTIES: AN EYE-TRACKING STUDY

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For early mathematics learning, adaptive strategy use by students at different levels of mathematical achievement is of central importance. For several mathematical tasks, there is evidence that high-achieving students are more able to use strategies adaptively, but not yet for pattern-recognition tasks. This paper presents results from an empirical study investigating whether students with and without risk of developing mathematical difficulties use pattern-recognition strategies adaptively for patterns with different units of repeat. Pattern-recognition strategies of 74 first-grade students were analyzed using eye-tracking. The results reveal that predominantly the first graders without risk of developing mathematical difficulties show an adaptive use of pattern-recognition strategies.

INTRODUCTION

The ability to use strategy adaptively, that is, consciously or unconsciously selecting the most appropriate strategy on a given task (Verschaffel et al. 2007), enables learners to solve mathematical tasks efficiently and correctly and is a key component of mathematical achievement (Heinze et al., 2009). Research on primary school students indicated that higher achieving students, in contrast to weaker students, are able to adapt their strategies, for example, for number line estimation (Van't Noordende et al., 2016) or arithmetic sums such as 8 + 7 (Torbeyns et al., 2005).

Another mathematical area that has become a focus of early mathematics learning research in the past decade is awareness of structures and patterns (Mulligan et al., 2020). This includes, for example, the ability to expand repeating patterns such as ● ● ● ● ● ● ● ●. While there is research on strategies first graders use in extending such repeating-pattern tasks (Baumanns et al., 2022; Lüken & Sauzet, 2021; Papic et al., 2011), there is lack of research on whether first graders show adaptive strategy use and whether students with and without risk of developing mathematical difficulties differ in adaptive strategy use in solving repeating-pattern tasks.

The aim of this study is to investigate whether first graders show adaptive strategy use and whether students with and without risk of developing mathematical difficulties differ in adaptive strategy use. To analyze the pattern-recognition strategies, we use eye-tracking which has been shown to be useful for investigating pattern-recognition strategies of children at early primary level (Baumanns et al., 2022).

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 83-90). PME 46.

THEORETICAL BACKGROUND

Risk of developing mathematical difficulties in early mathematics learning

Mathematical difficulties are manifested through difficulties students have with number sense, place value and base-10 number system, and arithmetic operations (Moser Opitz et al., 2016). As children transition from kindergarten to primary school, they may not yet exhibit mathematical difficulties but may be at risk of developing them. Standardized tests identifying students at risk of developing mathematical difficulties investigate mainly students' number sense. In recent years, there has been an increasing focus on students' awareness of patterns, in addition to their number sense, as factors associated with mathematical achievement and future success in mathematics (Mulligan et al., 2020). The ability to identify and understand patterns has been identified as a good predictor of future mathematical achievement (Rittle-Johnson et al., 2019).

Adaptive strategy use in early mathematics learning

Hereafter, the term *strategy* refers to general processes or approaches to solve a task. Verschaffel et al. (2007) define adaptive strategy use as the "conscious or unconscious selection and use of the most appropriate solution strategy on a given mathematical item or problem, for a given individual, in a given sociocultural context" (p. 19). This study focuses on first graders' adaptive strategy use in pattern-recognition tasks. To conceptualize adaptive strategy use more precisely, we draw on Lemaire and Siegler's (1995) model of strategy change consisting of four dimensions: (1) *Strategy repertoire* refers to the set of strategies a person uses to solve a task. (2) *Strategy distribution* refers to the frequency at which each strategy is used. (3) *Strategy effectiveness* refers to the speed and accuracy with which each strategy is used. (4) *Strategy selection* refers to the flexibility or adaptivity with which each strategy is chosen. In this study, adaptive strategy use is investigated considering mainly dimensions (2) and (4) by assessing flexible or adaptive strategy use through investigating the frequency at which pattern-recognition strategies are observed for different kinds of tasks (see below).

Several studies investigated primary school students' adaptive use of strategies for students at different levels of mathematical achievement. Van't Noordende et al. (2016) found that 9–11-year-old students with mathematical difficulties did not differ in their use of the strategies for number line estimation compared to students without difficulties. Studies in which students were asked to solve addition and subtraction tasks to 30 revealed that students with mathematical difficulties, compared to students without difficulties, use the same strategy for different addition and subtraction tasks (Torbeyns et al., 2005). In summary, there is evidence that students at risk of developing mathematical difficulties tend not to adapt their strategy to the respective task.

Repeating patterns and pattern-recognition strategies

In early mathematics learning, pattern skills are addressed, for example, through repeating patterns (Mulligan et al., 2020) which are sequences with a unit of repeat (e.g., AB) that continuously recurs (e.g., ABABAB). They differ along three dimensions:

(1) Representation: The patterns ABABAB (alphabetical), 1 2 1 2 1 2 (numerical), and ● ● ● ● ● ● ● ● ● (color) are isomorphic with a different representation.
(2) Length of unit of repeat: The unit of repeat of the pattern ABABAB has length two, while the unit of repeat of the pattern ABCABCABC has length three.
(3) Number of distinct elements: The unit of repeat of ABBABBABB has two distinct elements and the unit of repeat of ABCABCABC has three distinct elements.

Recognizing different lengths and number of distinct elements of the unit of repeat and recognizing isomorphic patterns are emphasized by NCTM Standards as the early introduction of algebra (NCTM, 2000).

Studies examining pattern-recognition strategies in preschoolers and first graders (Lüken & Sauzet, 2021) found that children use more sophisticated strategies with less difficult patterns (i.e., identifying and using the unit of repeat or focusing on the succession of elements on the unit of repeat). They also use more sophisticated strategies as they get older. Baumanns et al. (2022) identified four pattern-recognition strategies using eye-tracking and found that the use of these four strategies differs significantly between patterns with different units of repeat. However, there is no research on the adaptive strategy use for repeating-pattern tasks. In addition, because adaptive strategy use is a key component of mathematical achievement, the focus on first-grade students with and without risk of developing mathematical difficulties is of interest.

Based on the three dimensions of repeating patterns and the state of research regarding the adaptive use of pattern-recognition strategies, we pursue the following research question: *Do first-grade students show adaptive strategy use in solving repeating-pattern tasks? Do first graders with and without risk of developing mathematical difficulties differ in adaptive strategy use in solving repeating-pattern tasks?* Adaptive strategy use in repeating pattern tasks is investigated with respect to the following three dimensions: (1) representation of the unit of repeat, (2) length of the unit of repeat, and (3) number of distinct elements of the unit of repeat.

METHODS

Participants, procedure, and tasks

The present study had a sample of 224 first-grade students, comprising 102 students from three primary schools in Germany and 122 students from three primary schools in Cyprus. All students did the standardized mathematics test ZAREKI-K, which was used to assess students' mathematical performance level during the transition from

kindergarten to primary school (Aster et al., 2009). We used the adapted version by Walter (2020), which considers 6 instead of 18 subtests, reducing drastically the time required for the test while maintaining high accuracy for identifying students with and without risk of developing mathematical difficulties. The ZAREKI-K indicated that 37 students were at risk of developing mathematical difficulties (*RMD students*). Thus, the group of first graders who were not at risk of developing mathematical difficulties (¬*RMD students*) is disproportionately larger than the group of RMD students. For our analyses and investigating group differences, we included the 37 RMD students, i.e., the students with the lowest scores in ZAREKI-K, and the corresponding 37 ¬RMD students with the highest scores in ZAREKI-K, to avoid the analyses for the ¬RMD group to be overpowered in the statistical analyses.

In total, 12 repeating-pattern tasks were given to the students. Six were represented using numbers, and six using colors. One task each had the unit of repeat AB, ABC, AABB, AAB, AABC, and ABAC. Figure 1 shows all numerical and color patterns used in the study with their respective length of repeat and number of distinct elements. The students worked individually on the pattern-recognition tasks. Before the first numerical and the first color pattern task, the students worked on a sample task, to ensure that they understood the task correctly. The students answered by saying aloud the number or color they thought was behind the white blob. The students did not receive feedback and incorrect answers were not corrected.

	(1) Representation		(2) Length of unit of repeat	(3) Number of distinct elements
	Numerical patterns	Color patterns		
AB	4 1 4 1 4 1 4		2	2
ABC	1 5 3 1 5 3 1 5 3 1		3	3
AABB	3 3 2 2 3 3 2 2 3 3 2 2 3		4	2
AAB	7 7 1 7 7 1 7 7 1 7		3	2
AABC	1 1 7 5 1 1 7 5 1 1 7 5 1		4	3
ABAC	9 6 9 1 9 6 9 1 9 6 9 1 9		4	3

Figure 1: Numerical and color repeating-pattern tasks used in the study

Eye tracking

In the study, students' eye movements were recorded using the Tobii Pro X3-120, an infrared binocular eye tracker with a sampling rate of 120 Hz. The tasks were displayed on a 24" monitor and the students were positioned about 60–65 cm away from it. The eye-tracking data had an average accuracy of 0.98° (*SD* = 0.88°), which corresponds to an error of about 1 cm on the screen. Before the tasks were shown, a 5-point calibration and a 4-point validation were performed.

Qualitative analysis of eye-tracking data

Raw gaze-overlaid videos provided by Tobii Pro Lab software were used to analyze students' pattern-recognition strategies. For all gaze-overlaid videos, pattern-recognition strategies were coded by the first author using the coding manual illustrated

in Figure 2. More detailed descriptions of the gazes of each pattern-recognition strategy can be found in Baumanns et al. (2022). Twelve out of the 74 first-grade students (~16.2%) were analyzed independently by the penultimate author of this paper. The interrater agreement was calculated using Cohen's Kappa. With κ = 0.95 (95% CI [0.90, 0.99]), the interrater agreement is almost perfect (Landis & Koch, 1977).

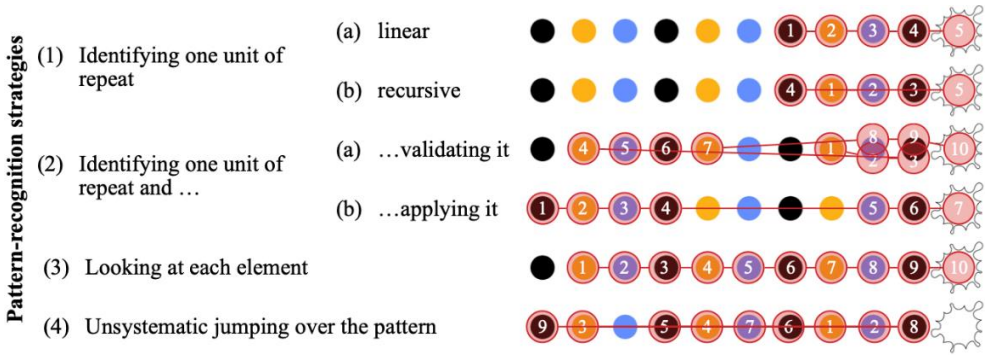

Figure 2: Pattern-recognition strategies coded in this study.

Quantitative statistical analysis

For statistical analyses, only correct answers were considered to make sure that in all cases, the students tried to solve the given tasks rather than just guessed. To determine adaptive strategy use for students with and without RMD for patterns with different units of repeat, twelve chi-square tests were conducted: First, chi-square tests were conducted for the six units of repeat (AB, ABC, AABB, AAB, AABC, ABAC; number and color patterns combined) for all students, then only RMD students, and finally only ¬RMD students. Afterwards, the same was done for the three dimensions of the unit of repeat ((1) representation, (2) length of unit of repeat, (3) number of distinct elements). Due to multiple testing of the present data, the alpha levels were adjusted using Bonferroni correction. Chi-square test is used to determine whether there is a significant difference between the observed and the expected pattern-recognition strategies used by the students. If the chi-square test indicates that there is a significant difference, it suggests that the students used the pattern-recognition strategies adaptively regarding strategy distribution (Lemaire & Siegler, 1995). To reveal separately for all chi-square tests which pattern-recognition strategies were observed significantly more often or less often than others, we analyzed adjusted standardized residuals (Sharpe, 2015). Adjusted standardized residuals were also adjusted using Bonferroni correction.

RESULTS

Figure 3 shows the distribution of pattern-recognition strategies for all tasks. Table 1 summarizes the results of the chi-square tests. Strategy (4), unsystematic jumping over the pattern, did not provide any correct answers and was excluded from the analyses.

All patterns: For the total sample, chi-square test indicates adaptive strategy use across all patterns with the same unit of repeat. If the ¬RMD and RMD groups are considered separately, the chi-square test reveals that only the ¬RMD students show adaptive strategy use across all patterns. Analyzing the adjusted standardized residuals reveals

that strategy (1), identifying one unit of repeat, was observed significantly more often for the unit of repeat AB and significantly less often for the unit of repeat ABAC than for the other patterns. Strategy (2), identifying one unit of repeat and validating/applying it, was observed significantly more often for the unit of repeat ABAC.

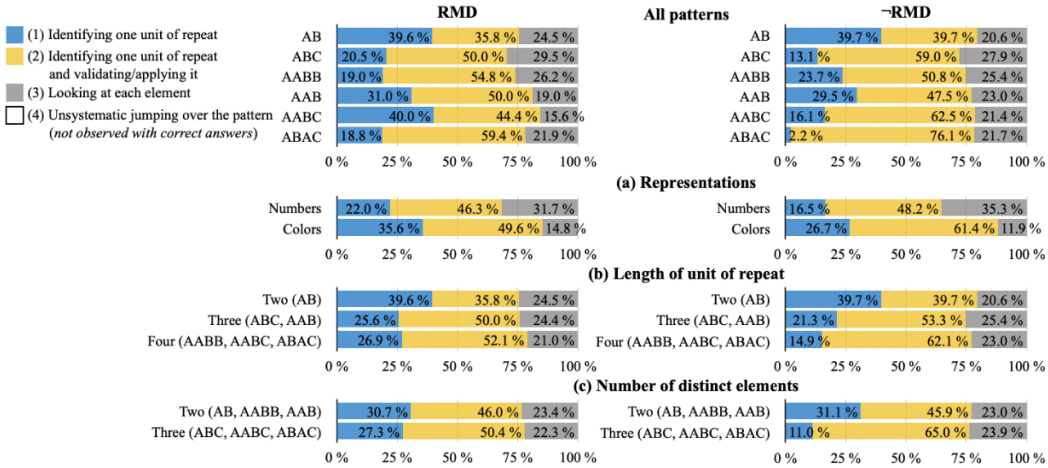

Figure 3: Distribution of the pattern-recognition strategies for all patterns and the three dimensions of the unit of repeat for RMD students and ¬RMD students

(a) Representations: The total sample shows adaptive strategy use for patterns with different representations (i.e., numbers and colors). Both ¬RMD and RMD students show adaptive strategy use. Analyzing the adjusted standardized residuals reveals that for both ¬RMD and RMD students, strategy (3), looking at each element, was observed significantly more often for number patterns than color patterns.

	Group	*N*	χ^2	*Df*	*p*	*V* (effect size)
All patterns	Total sample	614	35.11	10	< .01	0.17 (medium)
	RMD	261	13.22		= .21	
	¬RMD	353	31.50		< .01	0.21 (medium)
(a) Representation	Total sample	614	38.32	2	< .001	0.25 (medium)
	RMD	261	12.27		< .05	0.22 (medium)
	¬RMD	353	27.05		< .001	0.28 (medium)
(b) Length of the unit of repeat	Total sample	614	19.87	4	< .01	0.13 (small)
	RMD	261	5.03		= .28	
	¬RMD	353	17.05		< .05	0.16 (medium)
(c) Number of distinct elements	Total sample	614	15.00	2	< .01	0.16 (small)
	RMD	261	0.55		= .76	
	¬RMD	353	21.86		< .001	0.25 (medium)

Table 1: Results of chi-square tests for all patterns and three dimensions of unit of repeat for the total sample, only RMD, and only ¬RMD students (*N* is number of tasks)

(b) Length of the unit of repeat: The total sample shows adaptive strategy use for patterns with different length of the unit of repeat (i.e., length of two, three, or four). Chi-square test reveals that only the ¬RMD students show adaptive strategy use, but the RMD students do not. Analyzing the adjusted standardized residuals reveals that

for ¬RMD students, strategy (1), identifying one unit of repeat, was observed significantly more often for patterns with the length of two units of repeat and significantly less often for patterns with the length of four units of repeat.
(c) Number of distinct elements: The total sample shows adaptive strategy use for patterns with different numbers of distinct elements. Again, the chi-square test reveals that only the ¬RMD students show adaptive strategy use. Analyzing the adjusted standardized residuals reveal that for ¬RMD students, pattern-recognition strategy (1), identifying one unit of repeat, was observed significantly more often for patterns with a unit of repeat of two distinct elements and pattern-recognition strategy (2), identifying one unit of repeat and validating/applying it, was observed significantly more often for patterns with a unit of repeat of three distinct elements.

DISCUSSION AND CONCLUSION

The aim of this study was to investigate whether first graders show adaptive strategy use and whether students with and without risk of developing mathematical difficulties differ in adaptive strategy use in solving repeating-pattern tasks. Our results provide supporting evidence that ¬RMD students tend to show more adaptive strategy use in repeating pattern tasks compared to RMD students. This holds across the six patterns examined (AB, ABC, AABB, AAB, AABC, ABAC), as well as the three dimensions in which repeating patterns differ (representations, length of unit of repeat, and number of distinct elements). In general, ¬RMD students use a more efficient pattern-recognition strategy (i.e., (1) identifying a repeating unit) for simpler patterns (i.e., length of and number of distinct elements of unit of repeat is two) and more time-consuming strategies (i.e., (2) identifying one unit of repeat and validating/applying it and (3) looking at each element) for more challenging patterns (i.e., numbers as representations and number of distinct elements of unit of repeat is three). Students with RMD show adaptive strategy use in this study exclusively between different representations (numbers vs. colors). Thus, these results relate to other studies that also showed that for arithmetic tasks high-achieving first graders tend to show adaptive strategy use compared to weaker first graders (Torbeyns et al., 2005).
Regarding the dimensions of adaptive strategy use (Lemaire & Siegler, 1995), the present study is limited primarily to *strategy distribution*. Since strategy use and distribution depends on students' *strategy repertoire*, that is, the available strategies for solving pattern-recognition tasks, the strategy repertoire should also be investigated in future studies. For future research, it would further be useful to examine to what extent the analysis of adaptive strategy use in repeating patterns can be valuable to identify first-grade students at risk of developing mathematical difficulties. In addition, these findings can be used to develop materials that facilitate the development of adaptive strategy use in repeating patterns to help weaker students develop their mathematical competencies.

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PROSPECTIVE UNIVERSITY STUDENTS IN MATHEMATICS REFLECTING ON UNCERTAINTY: RESULTS AND COMPARISONS

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This paper investigates the written reflections of prospective undergraduate students in mathematics on a simple problem involving coin tosses. I analyze these in terms of recency and equiprobability effects understood non-normatively, observing in particular a major tendency of this type of students to equiprobability answers. The comparison of the present results with analogous results obtained in previous studies points to the fact that the influence of a university education in mathematics on the equiprobability effect is overall limited. It thus follows that the tendency to equiprobability answers of university students in mathematics is most likely acquired during previous compulsory education.

INTRODUCTION

The study of the recency and equiprobability effects constitutes a common ground of research for the psychology of mathematics and for mathematics education. These effects have been usually problematized when the respondents' answers to questionnaires or problems were deemed to be undesirable or wrong and thus often termed "biases" or "fallacies" (cf., e.g., Morsanyi et al., 2009; Chernoff & Sriraman, 2020; Batanero, 2020). For instance, in a seminal paper, Fischbein and Schnarch considered the following question.

> When tossing a coin, there are two possible outcomes: either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time? (Fischbein & Schnarch, 1997, p. 98)

These authors regarded the answer "equal to the chance of getting tails" to be correct. They further regarded the answer "smaller than the chance of getting tails" to be evidence of a negative recency bias, while they deemed the answer "greater than the chance of getting tails" to be evidence of a positive recency bias. As Beccuti and Robutti noticed, however,

> nothing in the statement of Fischbein and Schnarch's word problem as reported suggests that the hypothetical coin tossed by Ronni has to be considered a *fair* coin or that the way in which Ronni tosses the coin is not biased towards heads. As Gigerenzer (1991, 1996) has argued, probabilistic word-problems usually do not have only one correct answer over which there exists unquestioned consensus. It is true that often people's answers deviate problematically from the generally accepted norm. However, this discrepancy could be caused by the respondents' divergent interpretation of the situation presented to them (Chiesi & Primi, 2009, p. 152). (Beccuti & Robutti, 2022, pp. 1-2, emphasis in the original)

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 91-98). PME 46.

Indeed, among others, Sharma (2008) already evidenced the crucial role of the wording of these types of problems as well as the connected plausibility of students' realistic considerations when discussing them. In view of this, Beccuti and Robutti, building on Rubel (2007), evidenced the need to go beyond usual research methods based on multiple-choice questionnaires when investigating recency and equiprobability. They thus suggested a method to classify people's reflections which allows to understand possible divergent interpretations of such problems as well as to evidence the problematicity of answers which are usually deemed to be normatively correct. In order to do this, these authors pointed out the need to employ the following non-normative definitions of recency and equiprobability.

> The *positive recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is likely to happen again in the future. On the other hand, the *negative recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is less likely to happen again in the future. Moreover, the *equiprobability effect* is the tendency to judge a set of events as all equally likely to happen. (Beccuti & Robutti, 2022, p. 1, emphasis in the original)

Notice that these definitions deviate from the usual definitions of these effects given by researchers solely in the fact that they do not depend on whether the behaviors they describe are at odds with the usual normative interpretation of the involved problems: i.e., these definitions are non-normative. Hence, they are simply more general than the usual normative definitions, in the sense that the category of phenomena or behaviors they subsume includes as a subgroup the category of phenomena usually subsumed by normative definitions (cf., e.g., Chiesi & Primi, 2009; Gauvrit & Morsanyi, 2014; Morsanyi & Szucs, 2014).

These non-normative definitions were employed by Beccuti and Robutti to analyze the reflections of master's students in mathematics on a problem involving coin tosses analogous to the one employed Fischbein and Schnarch. In the present paper, I will thus employ the same methodology in order to analyze the reflections of a comparable population of prospective undergraduate students in mathematics on the same problem. I thus aim at investigating the following first research question with respect to such problem: *how do prospective undergraduate students in mathematics reason about uncertainty?* Moreover, a comparison of the results of the present study with those obtained by Beccuti and Robutti will also allow me to investigate the following related second research question: *what is the influence of a university education in mathematics on students' reasoning about uncertainty?*

SUMMARY OF PREVIOUS RESEARCH

Batanero and colleagues (1996) tested secondary school students with problems involving the throw of dice and spinners, finding that non-normative equiprobability answers are given less by younger students, while normative equiprobability answers are given more by older children. These authors argued that their results could be related to the participants' exposure to formal education. The aforementioned study of Fischbein and Schnarch found that the negative recency bias decreases with age (and

thus with degree of formal education), while overall the positive recency bias is almost negligible after a certain age. Furthermore, these authors observed an increase with age of the equiprobability bias, which they conjectured could be explained by increased exposure to formal education in probability. Rubel (2007) investigated secondary students' responses on problems involving coin tosses and further analysed the participants' justifications for their answers. She did not observe a correlation between age and the equiprobability bias or the recency biases. This fact, together with the fact that the students involved in her study were subjected only to a limited exposure to instruction in probability, appears to evidence that the prevalence of equiprobability answers is caused by formal education in probability rather than age.

Chiesi and Primi (2009) found (testing a problem involving drawing from a bag with replacement on primary school children and on undergraduate university students) that the positive recency bias decreases with age whereas they found no age-related differences for the negative recency bias. As to university students in particular, Chiesi and Primi observed a noteworthy manifestation of the negative recency bias as well as a less prominent manifestation of the equiprobability bias. A seminal cross-educational and cross-national study (concerning the answers of university students on various problems involving uncertainty) by Morsanyi and colleagues (2009) established a correlation between the equiprobability bias and formal education in probability and statistics. More recently, as mentioned, Beccuti and Robutti (2022) studied the reflections of master's students in mathematics on a problem involving coin tosses. By employing the non-normative definitions quoted in the previous section, these authors found manifestations of the positive recency effect while they observed an almost negligible negative recency effect. Most importantly, they observed a remarkable predominance of the equiprobability effect (which they further nuanced according to whether their participants problematized equiprobability or assumed it without questioning). These authors also argued that their participants' answers were connected to their formal education in probability and statistics.

The present study thus aims to extend the research on the equiprobability and recency effects on a type of students which appears to not have been studied before by previous literature: prospective undergraduate students in mathematics (first research question). Furthermore, the present study aims to put the results evidenced by Beccuti and Robutti under a new light and establish by comparison whether these were ascribable to their participants' university education in mathematics (second research question).

THE PRESENT STUDY

Participants

The participants of the present study are one cohort of 81 students (38 males and 43 females of median age 19) enrolled in the course "Introduction to mathematics" at the Department of Mathematics of the University of Turin, Italy (i.e., the same university and department where the aforementioned study of Beccuti and Robutti took place). The course is a preliminary non-compulsory course (which was lectured in presence

by the author of the present paper) taking place before the formal beginning of the undergraduate program in pure and applied mathematics. The course aims at reviewing basic mathematical concepts and techniques that are deemed by the faculty to be important to be mastered by the students before entering the program. Typically, the students had just completed their secondary education and were thus enrolled for the first time in a university degree.

Procedure

The following experiment was performed by testing the students with a procedure involving a short computer-based questionnaire administered in the first day of the course. The questionnaire was divided into two tasks which the students were instructed to address individually as part of a larger written assignment which included a selection of mathematical problems and tasks (aimed to evaluate the students' preliminary mathematical competences). As to the present questionnaire, each participant was presented with the following multiple-choice question.

> **Task 1.** Sara tosses for ten times a coin and for ten times she obtains heads. Sara then asks Luca to bet on the outcome of the next toss. Luca bets on the next toss resulting in heads again. Do you agree with Luca's choice?
> [Possible answers:] *Yes*; *No*; *In part*; *I am not sure*.

As soon as the students submitted their answers to the first task, the following related second task was immediately presented to them.

> **Task 2.** Explain your reasoning.

To complete this task, the students could type in the computer a text possibly containing mathematical symbols.

Explanation of the choices

Following Beccuti and Robutti (2022), I concentrate here on the students' written responses to Task 2, leaving an analysis of the responses to Task 1 (and of the relations between answers to Task 1 and 2) to a subsequent article. Except from minor details (e.g., the names of the characters of the fictional situation presented in Task 1), the wording of the tasks, the mode of administration as well as the procedure of analysis were chosen to be identical to that of Beccuti and Robutti (2022), in order to favor comparison in view of the second research question. Following these authors, the wordings of both tasks were selected in order to stimulate in the students the possibility of ample articulation of their reflections, in view of the first research question. In particular, as to Task 1, the range of possibilities of answers presented to the participants were selected with the aim to stimulate reflection over a decision problem rather than simply to confine the participants within the limited constraints of a yes/no or most-likely/least-likely answer (cf. Rubel, 2007). Similarly, the wording of Task 2 (and, crucially, the possibility to answer by submitting an open text) was chosen in order to give to the participants the amplest possibility to express their view over the fictional situation presented to them.

Method of analysis

The submitted answers to Task 2 were classified according to the deductive coding procedure (cf. Braun & Clarke, 2006) elaborated by Beccuti and Robutti (2022). Indeed, in view of the aforementioned definitions of recency and equiprobability, I classified each of the participants' submitted texts in relation to the conclusion of the argument that these texts presented as connected to the possibility of predicting the outcome of a hypothetical 11th coin toss in the fictional situation presented.

More specifically, a submitted text was classified as *Equiprobable tout court (Group A)* if it argued that the possible outcomes of a further hypothetical coin toss are equiprobable without any explicit doubt or reservation. Moreover, a text was classified as *Equiprobable with reservation (Group B)* if it argued that the outcomes are equiprobable but explicitly contained some form of doubt or reservation about this fact. Furthermore, a text was classified as *Heads is more likely (Group C)* or *Tails is more likely (Group D)* if it argued that outcome of an 11th coin toss is more probable or likely to result in heads or tails respectively. Finally, a text was classified as *Mixed (Group E)*, if the text was not unilaterally classifiable within any of the above groups (because it contained contrasting remarks without favoring any explicit conclusion).

RESULTS

Table 1 summarizes the participants' answers to Task 1 and Task 2.

	Group A	Group B	Group C	Group D	Group E	Empty	Total
Yes	2	1	5	0	2	2	12
No	12	13	0	4	0	0	29
In part	14	13	4	1	4	0	36
Not sure	0	3	1	0	0	0	4
Total	28	30	10	5	6	2	81

Table 1: Summary of the participants' answers to Task 1 and Task 2.

As said, I concentrate in the present paper on the students' answers to Task 2 (shown in the last row of Table 1). In the following subsections, I thus present a summary of each of the aforementioned groups of answers to Task 2 together with exemplifying extracts from the students' texts (translated as literally as possible from Italian).

Group A

Many of the students (28 participants) either simply affirm that heads or tails are equiprobable without doubt or reservation, or else they care to specify that the results of the previous tosses do not affect the outcome of the next toss.

> The coin landed 10 times on heads […] but this fact does not change the fact that, by performing a new toss, the coin may land on tails, since the probability is always 0.5.

Logically speaking, the independence of the events of heads and tails is assumed by these participants as the starting point of their reasoning. This is argued with reference to typical descriptions of sample spaces of idealized games involving coin tosses, or else it is simply stated as an unquestioned assumption.

Group B

A slightly more numerous group of students (30 participants) affirms the same conclusion of the previous group, but at the same time cares to specify that such conclusion depends on an unproven assumption: that the coin or the game is not biased or rigged.

> Getting tails or getting head should be the same […] if the coin is not biased.

Group C

Some of the students (10 participants) state that heads is more likely or probable in relation to the fact that there is evidence for deeming the coin to be loaded or the game to be unfair.

> Since the flip resulted ten times in heads, then probably Sara is cheating.

Group D

Few students (5 participants) argue that tails is more likely to be the outcome of a further hypothetical coin toss. These students argue by referring to mathematical principles (e.g., the law of large numbers) or else to arguments of a mathematical form (ultimately unsound).

> Tails is more probable, since the coin landed ten times on heads […] the law of large numbers make it so that at some point the coin has to land on tails.

Group E

A small number of students' answers (6 participants) were uncategorizable because they contained mixed conflicting statements and did not reach a conclusive decision on the fictional situation presented. 2 of these texts also contained reference to unsound mathematical arguments.

DISCUSSION

Overall, a comparatively small number of students manifests the positive recency effect (Group C), and an even smaller amount manifests negative recency (Group D). The largest portion of the participants manifests the equiprobability effect (Group A and Group B). With respect to Group A, in particular, unquestioned equiprobability answers appear to be stated in a similar fashion as those found in textbooks' formal presentations of idealized games involving idealized coins. Such answers can be deemed to be problematic and at odds with a full understanding of situations or decision problems in conditions of uncertainty (cf. Batanero, 2020, p. 685).

Beccuti and Robutti (2022), by using the same mode of administration and procedure of analysis of responses to Task 2, obtained comparable results from 84 master's

students in mathematics enrolled at the same university (all holding a bachelor's degree in mathematics comprising at least a course in probability and statistics). These are reported in terms of percentages in Table 2 together with the present results (already appearing in terms of absolute numbers in the last row of Table 1).

	Group A	Group B	Group C	Group D	Group E	Empty
prospective bachelor's students (present study)	34.57%	37.04%	12.35%	6.17%	7.41%	2.47%
master's students (Beccuti & Robutti, 2022)	40.48%	35.71%	13.10%	3.57%	5.95%	1.19%

Table 2: Comparison of the results of Task 2 in the present study with those obtained by Beccuti and Robutti (2022, p. 5).

As we can see from Table 2, older and more mathematically-educated students are overall slightly more subject to the equiprobability effect. In particular, answers of equiprobability without reservation are more prevalent in older students, while equiprobability answers without reservation are slightly less prevalent. We can further observe a very small relative difference in positive recency answers and a relatively consistent difference in negative recency answers (as well as in mixed or empty answers).

Globally, however, the present results are not substantially dissimilar from those found by Beccuti and Robutti, especially in terms of the overall prevalence of the equiprobability effect. This points to the fact that the tendency to equiprobability answers is likely not acquired during a university education in mathematics but possibly during previous compulsory education. Nevertheless, the equiprobability effect (in its unrestricted form) may possibly be slightly exacerbated by a university education in mathematics. Similarly, this type of education appears to reduce the negative recency effect, while positive recency appears to remain almost unaffected.

Further studies involving different types of students as well as involving the same type of students tested with different problems (or with the same problem presented in altered situations) may serve to further elucidate the relationship of recency and equiprobability with previous education and thus contribute both to mathematics education research as well as to research in the psychology of mathematics.

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PRESERVICE TEACHERS' ADAPTIVE TEACHING OF FRACTIONS: A VIGNETTE-BASED EXPERIMENTAL STUDY

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Adaptive teaching is necessary to support students individually. Adaptive teaching considers students' individual needs (student focus) and is directed towards a specific learning goal (goal focus). Research has not systematically explored the role of student focus and goal focus in (pre-service) teachers' adaptive teaching practices. We used text-based vignettes to investigate to what extent N = 48 pre-service mathematics teachers selected adaptive teacher responses (those with high student and goal focus) to incorrect student solutions on fraction problems. Participants chose the most adaptive response in only about half of the vignettes. There were large individual differences between participants. Our study contributes to a better understanding of pre-teachers' abilities and provides some guidance for teacher education.

THEORETICAL BACKGROUND

Good teaching considers the individual learning needs of the students and offers adaptive support (Hardy et al., 2019; Parsons et al., 2018). This adaptive support can be implemented at a macro level and at a micro level of teaching. Macro-adaptions tend to represent large-scale adjustments in instruction, informed by formal assessments. Micro-adaptive teaching, which is the focus of the present study, can be defined as a teacher response to stimuli supporting students' needs in a moment-to-moment teacher-student interaction, for example in a verbal teacher response to a student's solution of a mathematical task (Gallagher et al., 2020; Hardy et al., 2022).

Other than research on macro-adaptive teaching, most studies on micro-adaptive teaching have not considered the dimension of content goal. For example, the review of Gallagher and colleagues (2020) included 23 studies on micro-adaptive teaching in mathematics education, but only three of them briefly mentioned teachers' reflections on content goals. Prediger et al. (2022) addressed this research gap, illustrating in a case study with fractions how adaptive teaching can vary along the dimensions of student focus and goal focus. They conceptualized micro-adaptivity as teachers' strategies on a micro-level to achieve a high student focus and a high goal focus. Student focus includes teachers' adjustments to students' individual learning needs, and goal focus refers to teachers' deliberate steering towards the content goals (Prediger et al., 2022). For example, a teacher response with a high goal focus but a low student focus might be reflected in a teacher response presenting a correct solution or a new strategy to solve a certain task, without taking up the student's initial solution. A low student focus is problematic because the student may not be able to link the teachers' response to his or her own approach.

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In order to achieve micro-adaptive teaching with a high student and a high goal focus, mathematics teachers have to diagnose students' mathematical thinking and adapt their response to it. A common situation for mathematics teachers is, for example, to diagnose students' mathematical thinking in students' task solutions and to respond adaptively in order to support the students. Perceiving students' mathematical thinking and providing an adaptive response is particularly relevant when there are incorrect students' solutions that indicate misconceptions. Misconceptions are students' individual knowledge structures that are mathematically inaccurate and that could cause incorrect answers in certain mathematical tasks (e.g., Holmes et al., 2013). In the present study, we focus on misconceptions in the content-area of fractions, because proficiency with fractions is highly predictive of later mathematical achievement (e.g., Booth & Newton, 2012; Torbeyns et al., 2014) and students' typical misconceptions and faulty strategies are well documented (e.g., Clarke & Roche, 2009; Obersteiner et al., 2018; Siegler & Lortie-Forgues, 2017).

Empirical studies have shown that it is difficult, especially for pre-service mathematics teachers, to consider all information of students' solutions, and to distinguish between relevant and irrelevant information in order to make diagnoses and respond adaptively (Kellman & Massey, 2013; Levin et al., 2009). For example, Kuntze and Dreher (2015) used a vignette-based design and investigated pre-service mathematics teachers' ability to analyze teaching situations; they found that pre-service mathematics teachers often focus on information about students' learning motivation, ignoring other information relevant for understanding student thinking. Also using a vignette-based design, Wirth et al. (2022) confirmed these findings and found that pre-service mathematics teachers often used situational information for diagnostic decisions that are not relevant to the diagnostic goal. These findings suggest that strongly focusing on information that is less relevant for the diagnostic goal could distract people from student focus and goal focus, both important from the perspective of micro-adaptive teaching. There is yet a lack of empirical studies investigating to what extent mathematics teacher are able to consider a high student as well as a high goal focus in teaching situations requiring micro-adaptive interactions. Investigating this issue is relevant in pre-service mathematics teachers, as they may have difficulties in considering all relevant information simultaneously, and may need targeted support during their teacher education.

OBJECTIVE

The goal of the present study was to investigate quantitatively how strongly pre-service mathematics teachers consider a high student focus or a goal focus in micro-adaptive teaching. Specifically, the research question was to what extent pre-service mathematics teachers would choose adaptive teacher responses with a high student focus and a high goal focus, or to what extent they would prefer responses with less relevant aspects, such as purely motivational aspects. We used text-based vignettes in which we varied systematically the student focus and goal focus in teacher responses.

METHODS

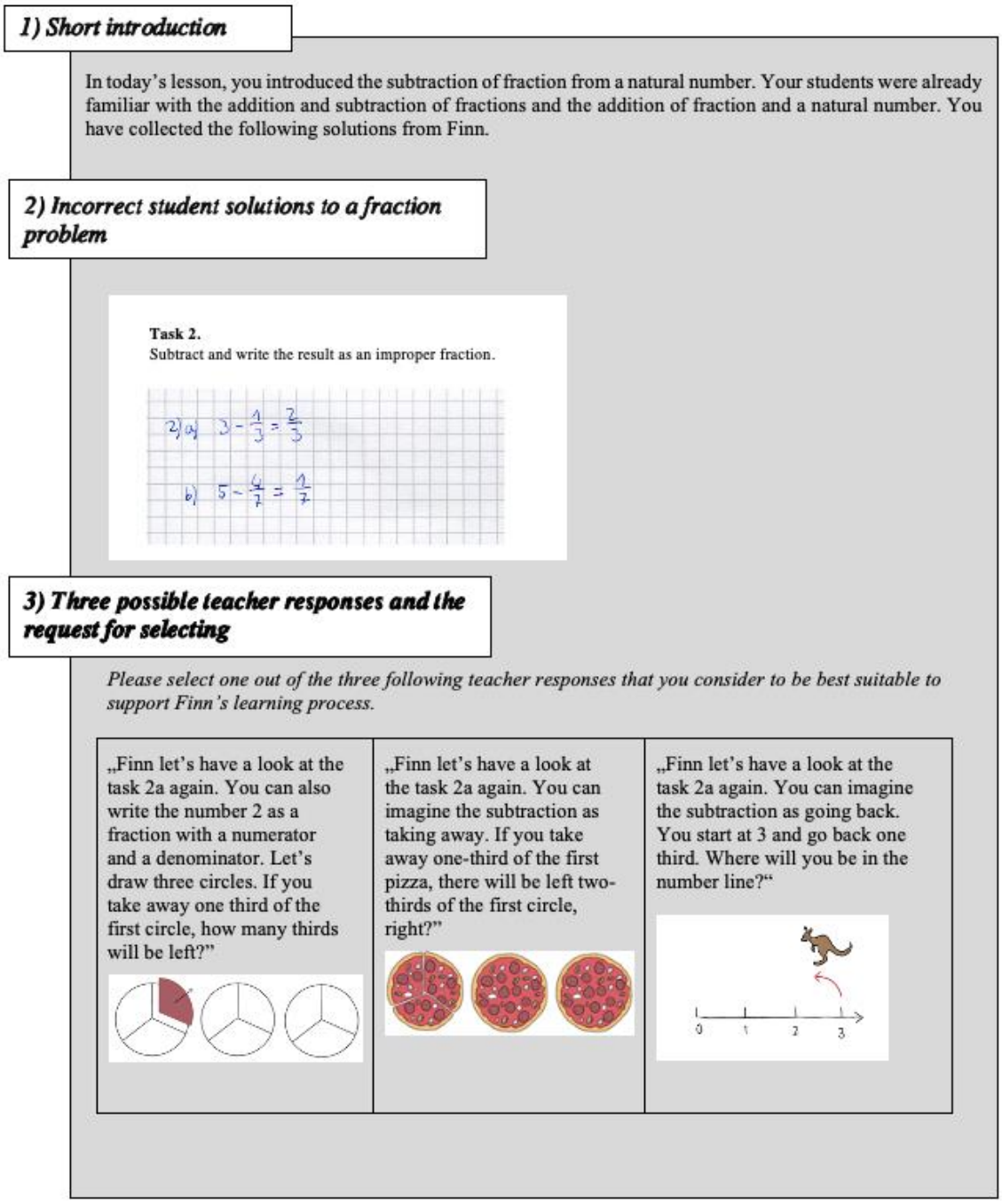
A total of *N* = 48 pre-service mathematics teachers (mean age = 24.77, *SD* = 1.82, 67 % female) participated in this study. The participants were recruited at a university in Germany. All participants had received instruction about teaching of fractions during their regular courses at university. In order to assess adaptive teaching in an authentic teaching situation, we used a vignette-based design. Such designs are suitable for our purpose because they allow representing authentically teaching situations without confronting pre-service teachers with the full complexity and immediacy of a real classroom situation. Furthermore, vignettes allow the development of teaching situations with regard to specific hypotheses that can be tested experimentally.

We developed ten text-picture vignettes that addressed the content domain of fractions. Each vignette includes three components (see Figure 1): 1) a short introduction to the lesson content and the fictitious students' prior knowledge of the specific content presented in the vignette; 2) incorrect student solutions to fraction problems (fraction addition, subtraction, or fraction comparison), and 3) three possible teacher responses consisting of a short text with a fictitious teachers' verbal explanation and a visualization.

The incorrect student solutions presented in the vignettes represented the most common misconceptions in the areas of fraction addition, fraction subtraction and fraction comparison, according to empirical studies (e.g., Clarke & Roche, 2009; Obersteiner et al., 2018; Siegler & Lortie-Forgues, 2017). For example, the following mistakes were presented in the vignettes: students add or subtract the numerators and denominators of two fractions as separate whole numbers; students fail to convert fractions to a common, equivalent denominator before adding or subtracting them and instead using the larger of the two denominators in the answer; students add or subtract a whole number to or from a fraction, ignoring the denominator of the whole number and only add or subtract the whole number to or from the numerator of the fraction (e.g., Clarke & Roche, 2009; Obersteiner et al., 2018; Siegler & Lortie-Forgues, 2017). In the tasks on comparing fractions, students consider two fractions to be of an equal size when both fractions are missing one piece to the whole; students compare the fractions' numerators and denominators separately, or students focus on the number of parts (numerator) without considering their size (denominator) (e.g., Clarke & Roche, 2009; Obersteiner et al., 2018; Siegler & Lortie-Forgues, 2017)

Participants were asked to select the one out of the three teacher responses that they considered to be most suitable. A closed format was chosen for the selection of the three teacher responses in order to be able to systematically vary the degree of student focus and goal focus in these responses. While some responses in the vignettes had high goal focus or high student focus, others strongly focused on situational information, which can be considered less relevant for students' learning. Situational information included, for example, motivational aspects (e.g., a kangaroo jumping along a number line to illustrate fraction addition, see Fig. 1). The vignettes were

provided on a computer using E-Prime software, and participants' responses were recorded by the computer.



1) Short introduction

In today's lesson, you introduced the subtraction of fraction from a natural number. Your students were already familiar with the addition and subtraction of fractions and the addition of fraction and a natural number. You have collected the following solutions from Finn.

2) Incorrect student solutions to a fraction problem

Task 2.
Subtract and write the result as an improper fraction.

2) a) $3 - \frac{1}{3} = \frac{2}{3}$

b) $5 - \frac{4}{7} = \frac{1}{7}$

3) Three possible teacher responses and the request for selecting

Please select one out of the three following teacher responses that you consider to be best suitable to support Finn's learning process.

„Finn let's have a look at the task 2a again. You can also write the number 2 as a fraction with a numerator and a denominator. Let's draw three circles. If you take away one third of the first circle, how many thirds will be left?"	„Finn let's have a look at the task 2a again. You can imagine the subtraction as taking away. If you take away one-third of the first pizza, there will be left two-thirds of the first circle, right?"	„Finn let's have a look at the task 2a again. You can imagine the subtraction as going back. You start at 3 and go back one third. Where will you be in the number line?"

Figure 1: Sample vignette.

Figure 1 illustrates a sample vignette. The first teacher response (left) is characterized by a high student focus and a high goal focus. It takes up Finn's misconception and supports the development of conceptual understanding trough the explanation and the visualization. In the second teacher response (middle in Fig. 1), there is a high student focus, because the response addresses Finn's misconception. However, there is a low goal focus because the response could encourage Finn's misconception; this is because the focus of the explanation is only on the first pizza and not on the whole visualization.

The third teacher response (right in Fig. 1) represents a low student focus because Finn's misconception is not addressed. The explanation and the visualization are not appropriate to support Finn's learning process, hence representing a low goal focus as well. Going back on the number line could enhance conceptual understanding of subtraction, which would be a high goal focus, but the specific representation does not fit well to the initial subtraction task, which includes an improper fraction. In both the second and third teacher response, the visualizations consider situational information that may attract attention and distract from the student and the goal focus.

RESULTS

We investigated quantitatively to what extent participants selected adaptive teacher responses with a high student focus and a high goal focus. These responses were always considered the best choice from a theoretical point of view. The results show that, overall, in 53.7 % cases (N = 480), participants selected the teacher responses with a high student focus and a high goal focus. The frequencies of selecting the teacher responses with a high student focus and a high goal focus differ strongly between the vignettes, ranging from 23.9 % to 78.3 % (see Table 1, column 2). In 46.3 % cases, the selected response was based on motivational aspects, with a low student focus and low goal focus.

Table 1: Frequencies of selecting the teacher responses with a high student focus and a high goal focus.

Vignette number	Frequencies of selecting the teacher response with a high student and goal focus
1	29.5 %
2	70.2 %
3	78.3 %
4	75.0 %
5	54.2 %
6	41.7 %
7	68.1 %
8	69.6 %
9	23.9 %
10	27.3 %

Analyzing the selection of the most adaptive teacher responses separately by participants, we found high individual differences. The average number of times one participant selected the most adaptive teacher response was M = 5.28 (SD = 1.83), ranging from one to nine selections out of ten vignettes.

DISCUSSION

The purpose of the present study was to investigate to what extent pre-service mathematics teachers are able to choose the most adaptive teacher responses to individual students' solutions of mathematical tasks. We considered teacher responses with a high student focus and a high goal focus as the most adaptive.

The results indicate that the pre-service mathematics teachers selected the adaptive response with a high student and a high goal focus in only half of the teaching situations provided in the vignettes. There was substantial variation of adaptive selections between vignettes as well as between participants. On the one hand, these results confirm the results of previous studies that found that pre-service teachers often focus on motivational aspects and do not always focus on aspects most relevant for their diagnosis (e.g., Kuntze & Dreher, 2015; Wirth et al., 2022). On the other hand, the results suggest that teaching situations vary in how readily pre-service mathematics teachers can assess them, and how straightforward it is for them to focus on the most relevant information. Furthermore, the variances between the participants indicate that aspects such as knowledge or experience could be influencing factors on micro-adaptive teaching.

To better understand the source of individual differences, we are currently analyzing participants' verbal justifications of their selection, which was recorded after they selected a teacher response (not reported here). We also assessed participants' pedagogical content knowledge (PCK) of fractions, so that we can analyze how participants' PCK is related to their selection of teacher responses. These analyses will be available by the time of the conference.

Overall, the results support the argument that pre-service teachers need support for acquiring micro-adaptive teaching skills during their teacher education program. Further research should more systematically develop effective support methods.

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THE ROLE OF IMPLICIT THEORETICAL ASSUMPTIONS IN EMPIRICAL RESEARCH

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There is much research on the role of theory in mathematics education research, at least from more overarching or theoretical perspectives. Micro analyses of the role of theory in particular research studies are rarer. We contribute by analysing one empirical study to allow for in-depth analyses and discussions around the role of theory in a specific case, concerning relationships between mathematics and reading. Our results show that studies that do not use an explicit theoretical model can still be strongly influenced by implicit theoretical assumptions. We conclude that it is important to identify existing theoretical assumptions in an empirical research study and try to convey them as clearly as possible, and we discuss specific issues concerning research on relationships between mathematics and reading.

INTRODUCTION

Theory is often considered to play an important, and sometimes crucial, role in mathematics education research. For example, it is sometimes described that theory should influence all parts of the research process. For example, this is done in the description of characteristics of a high quality JRME manuscript (NCTM, 2021), concerning influence of theory on "the study's design; its instrumentation, data collection, and data analysis; and the interpretation of its findings." However, there are also researchers that question whether theory should be so influential as described by NCTM. For example, Lester (2005) describes some shortcomings in relation to the use of theoretical frameworks and suggests that the use of *conceptual* frameworks is more suitable for mathematics education research. Furthermore, Niss (2019, p. 2) is critical towards an "ideal-typical research paper", which JRME (NCTM, 2021) can be said to describe, since this "represents a far too narrow and rigid understanding of mathematics education research". He describes different aspects of this "ideal-typical research paper", where theory is a key component.

We agree with critics concerning a type of over-reliance on theory in mathematics education research and a purpose with this paper is therefore to contribute to the discussion about the role of theory. We do this by analysing and discussing if and how certain aspects of theory have a role in certain parts of specific empirical research. We do not presume that theory is always needed in all parts of all types of empirical research, but we address this issue from an empirical standpoint, by examining the (potential) role of theory in specific research studies. We choose to focus on a specific research study, since much has been written about the role of theory in mathematics

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education research from more overarching or theoretical perspectives, as discussed more below, while less work has been done concerning more micro analyses of the role of theory in research studies.

THE CONCEPT OF THEORY AND ITS ROLE IN EMPIRICAL RESEARCH

The notions of "theory" and "theoretical" are used in different ways and can be considered vague and ill-defined (cf. Niss, 2019). In addition, it can often be unclear if and how theory actually has been used in a research study, "when some theoretical framework is being referred to in the beginning or at the end of the publication without having any presence in between" (Niss, 2007b, p. 1309). Therefore, we need to clarify both the meaning of "theory", or similar notions, such as "framework", and the potential roles of theory in empirical research. This work has been initiated by some researchers. Niss (2007a) has suggested a type of definition of "theory", as consisting of an organized network of concepts and claims, where the concepts are linked in a connected hierarchy. He also presents different roles theory can have in research, for example, to predict or explain phenomena, to organize observations and interpretations into a coherent whole, and to give a methodology for empirical studies. Radford (2008) also suggests a definition of "theory", which has much in common with the definition from Niss, but he also stresses that the use of theory does not only include explicitly formulated theoretical perspectives, but also "implicit views" (Radford, 2008, p. 320). Lester (2005) presents different types of research frameworks; theoretical, practical, and conceptual, and discusses their different roles in research, in relation to some general purposes of using a research framework; to give structure to a research study, to make sense of data, to come further than common sense, and in order not to be limited to finding answers to local problems.

These above perspectives on issues of theories show a complexity concerning relationships between theory and empirical research. There are different types of theories/frameworks that can function in different ways in relation to empirical research; there are different parts of theories, such as concepts, claims, and methodology, which can be more or less prominent (or explicit) in empirical research; and there are different parts of empirical research, such as purpose and research questions, and collection and interpretation of data, which can be affected by explicit or implicit theory in different ways. Therefore, when we want to discuss and analyse the role of theory in empirical research more specifically, we need to specify what type of theory and what parts of theory are addressed in relation to what parts of empirical research. In this paper, we focus on the implicit use of theoretical aspects in this situation.

PREVIOUS EMPIRICAL RESEARCH ON THE ROLE OF THEORY

Above, we discuss research that concerns the concept of theory and role of theory in empirical research on a general level, without addressing specific theories or specific empirical research. This type of research is important, but we also need more empirically based research on the role of theory in empirical research.

Most relevant for this paper is empirical research that analyses how theory is used (or not used) in specific empirical studies, which is the type of research presented in the present paper. Some studies highlight how different parts of empirical studies are dependent on which theory is chosen. For example, Gellert (2008) focuses on empirical data of students' collaborative problem solving where he shows how two different theoretical perspectives lead to different interpretations. A similar conclusion is drawn by Bergsten (2008) when he focuses on three empirical studies on limits of functions, in an analysis of how the use of different frameworks relate to the questions, methods, evidence, conclusions, and implications within these studies. Despite this type of conclusion, both authors address a potential of, but also a difficulty in, combining results from studies on the same topic that use different theories.

Other studies also highlight differences between theories in empirical research but at the same time see a potential of "translating" between these theories, which gives evidence that theories sometimes do not necessarily have a strong influence on (some parts of) empirical research. For example, Rodríguez et al. (2008) focus on empirical research on issues of metacognition in relation to problem solving. Their analyses show that it was not possible to do a "simple translation" of concepts concerning metacognition from one perspective to another. Instead, the problematic question that was the origin in one perspective could be "reformulable" in terms used in another perspective, which was also the case for some key aspects of metacognition (such as monitoring and self-regulation). Österholm (2011) comes to a similar conclusion when he compares two empirical studies about beliefs, where a main difference between these studies, concerning some specific aspects of theory, can be seen as a change of wording.

In summary, it is important to scrutinize the use of theories, including implicit assumptions regarding theoretical aspects, in empirical research. There is also a need for further studies of the relation between particular theoretical aspects and specific empirical studies, to understand how these can be related.

PURPOSE AND METHOD

The main purpose of this study is to deepen the scientific understanding of the role of theory in mathematics education research. We contribute to the line of research that analyse the role of theory in specific empirical studies, in particular when the theory is implicit. We analyse one empirical study (Caponera et al., 2016) that examines relationships between students' achievements in mathematics and reading, without explicitly relying on a theory or theoretical framework regarding the central concepts. We delimit our analyses to this study to allow for more in-depth analyses and discussions around the role of theory in a specific case, in particular, concerning if and how more implicit theoretical aspects can be of relevance in empirical research. The results can be added to previous similar type of research and allow for comparisons and cumulation of research results. However, as part of our analyses of this one study,

we also relate to other studies concerning the issues that come up from the analysis, concerning relations between achievements in mathematics and reading.

In line with the argumentation presented above, our analysis first focuses on identifying the implicit aspects of theory used in the article. These aspects concern the central *concepts* used and the *claims about relationships* between these concepts (cf. Niss, 2007a). Since the theoretic perspective is implicit, we will base our claims on how data is interpreted and how conclusions are drawn (cf. Radford, 2008). The implicit theory used in the study will be compared to other (implicit) theoretical perspectives used in research in the same area. We will also discuss the consequences of the chosen theoretical perspectives.

ANALYSIS OF THE EMPIRICAL STUDY

The article we primarily analyse is "The influence of reading literacy on mathematics and science achievement" by Caponera, Sestito, and Russo (2016), which has the aim "to evaluate the influence of students' reading literacy, measured by the PIRLS (Progress in International Reading Literacy Study) test, on their performance in the TIMSS (Trends in International Mathematics and Science Study) mathematics and science tests" (p. 197). Below we focus only on reading and mathematics, since mathematics and science are treated similarly. The article analyses correlations between students' achievements in reading and mathematics for 4,125 Italian students in Grade 4. The correlations between achievements were high and the authors conclude that the students' reading literacy influenced their mathematics achievement. Caponera et al. (2016) do not present a theoretical model or explicit definitions of the central notions of reading and mathematics and do not state explicit assumptions regarding the relationships between these notions. However, the study relies on implicit assumptions about the concepts and their relation, namely that reading and mathematics have nothing in common, as we specify in the following.

First, Caponera et al. (2016), as many others, interpret the correlation between mathematics achievement and reading literacy as a causal relation, since they state that "results confirmed the influence of reading literacy on mathematics achievement" (p. 197). Here, it is the word "influence" that signals causality. The authors do not (explicitly) consider that the influence could exist in the other direction, which is another possible conclusion. For example, such a conclusion has been drawn in another empirical study, where "mathematical performance predicted subsequent reading comprehension during the first year rather than vice versa" (Lerkkanen et al., 2005, p. 121). Furthermore, Caponera et al. do not consider that the correlation could be created by a common feature of these variables (e.g., when there is a third confounding variable). If a correlation implies that one variable influences the other, there is an underlying assumption that the variables have nothing in common, except what has been controlled for. In this case, it is assumed that *achievements in mathematics and reading have nothing in common*, except that both depend on students' socioeconomic status, which is controlled for in the study.

Second, the analyses by Caponera et al. (2016) show that good readers in general perform better than not so good readers on mathematics tasks. Based on this, the authors draw the conclusion that a "good reader had some advantages [...] independently on their mathematics ability" (Caponera et al., 2016, p. 202). The study does not control for mathematics ability in the analysis of the effect of reading ability, and therefore this conclusion is based on an implicit assumption that an effect of reading ability on mathematics performance cannot at the same time be an effect of mathematics ability. That is, any connection between reading ability and results on mathematics tasks is interpreted as saying something only about the influence of reading. The implicit assumption is that *reading ability and mathematics ability have nothing in common*, and therefore any connection to reading ability is interpreted as only an effect of reading ability.

Third, in the final statement of the article by Caponera et al. (2016), the authors make a connection between level of readability and validity of mathematics (and science) tests: "Our study seemed to indicate that the readability level of the mathematics and science test is a crucial aspect to consider to *correctly* assess mathematics and science achievement" (Caponera et al., 2016, p. 203, emphasis added). In the study, tasks with low and high reading demand are analysed, and a result is that "bad readers performed better on the mathematics low reading demanding scale than on the mathematics high reading demand scale" (Caponera et al., 2016, p. 201). Therefore, the authors' conclusion implies the implicit assumption that mathematics tasks with high reading demand do not "correctly" assess mathematics achievement. This is only reasonable if *reading and mathematics have nothing in common*, because then any effects of reading demands of tasks on students' performance on these tasks would be interpreted as a sign of lower validity for these tasks.

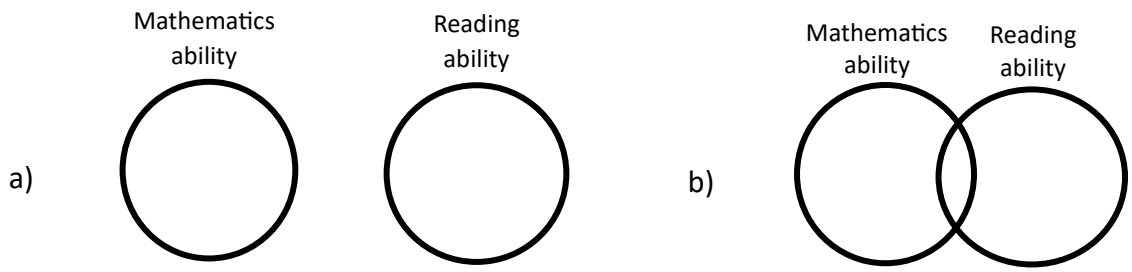

Figure 1: Two basic models of relationships between mathematics ability and reading ability.

We conclude that the study relies on an implicit theoretical model where mathematics and reading are separated. This includes a separation between mathematics ability and reading ability as well as between measures of achievement in mathematics and reading. We here suggest a simple theoretical model (see Figure 1a) that could be the current basis for conclusions by Caponera et al. (2016). Based on this model, any connection to issues of reading when focusing on mathematics tasks is unwanted, since reading ability has nothing in common with mathematics ability.

DIFFERENT THEORETICAL ASSUMPTIONS IN RELATED RESEARCH

Much research in mathematics education, as seen in different frameworks describing school mathematics, convey another theoretical model of the relation between mathematics and reading. For example, the framework of PISA (OECD, 2016) includes aspects of communication as important parts of mathematics. Mathematics performance is then not just *influenced* by reading ability but reading and interpreting mathematics text is one relevant and central *aspect or part of* mathematics ability. Such theoretical perspectives would better be illustrated using Figure 1b.

The second model is a slightly more complex model for the relationship between mathematics and reading ability. Here, the two circles of mathematics and reading ability are overlapping (see Figure 1b). The overlap symbolizes not merely the empirical results showing correlations between achievements in mathematics and reading, but also signifies that the two subjects have much in common *by definition*. This model is still quite simple but makes analyses a bit more complex. Any empirical connection between achievements in mathematics and reading (e.g., through correlation analyses) could be unwanted, if the result reflects an effect of the area in Figure 1b that lies outside mathematics but inside reading. At the same time, such a connection could also be highly relevant and nothing to avoid, if the result reflects an effect of the overlapping area in Figure 1b, which is part of both mathematics and reading.

For example, nominalizations are often described as making texts more difficult to read. Therefore, one would expect mathematics tasks with more nominalizations to have stronger connection to reading ability, so that students with lower reading ability would perform worse on such tasks compared to tasks without nominalizations. Caponera et al. (2016), as well as other studies, interpret this type of empirical result as a sign of lower validity for such mathematics tasks. However, nominalizations are not just surface features of a text that can be avoided without changing the meaning of the text, since “a nominalisation, by transforming a process into an object, opens up the possibility of a higher complexity of generalization” (Morgan, 2006, p. 233). For example, by transforming the *process* of adding into the *object* of addition, it becomes possible to talk about more advanced properties of addition, including that addition is commutative and that subtraction is the inverse operation to addition. Thus, tasks with more nominalizations could very well be more difficult to solve, since the language is used to describe more complex mathematics, in which case they also *should* be more difficult. Of course, there could as well be uses of nominalizations that are unnecessary and make the text more difficult to read without being part of mathematics.

DISCUSSION AND CONCLUSIONS

The present study aims to contribute to the scientific understanding of the role of theory in mathematics education research, in particular regarding the role of implicit theory in empirical studies. The article we have analyzed as a case (Caponera et al., 2016) has no explicit theory, but has implicit assumptions regarding the theoretical relation

between mathematics and reading, and these assumptions differ from what is assumed in other research in the same area. Our analyses show that studies that do not use an explicit theoretical model can still be strongly influenced by implicit theoretical assumptions. As mentioned in the background, we argue that it is not necessary to always use a theory in all parts of empirical research, which is also supported by previous empirical research (e.g., Rodríguez et al., 2008; Österholm, 2011). Therefore, it is important to identify which theoretical assumptions that are essential for the analyses in empirical studies and try to convey these assumptions as clearly as possible.

In particular, empirical studies on relationships between achievements in mathematics and reading should be explicit about assumptions regarding the relationship between these domains. Without relating to any theoretical model, it is difficult to compare and combine conclusions from different studies. For example, it is difficult to decide how to combine a conclusion that achievement in reading influences achievement in mathematics (from Caponera et al., 2016) with a conclusion that achievement in mathematics influences achievement in reading (from Lerkkanen et al., 2005).

Since connections between mathematics and reading *can be* relevant and wanted but *can also be* irrelevant and unwanted, we cannot rely on too simplistic models for analyses of these issues. It can make us draw unfounded conclusions and lead us to practical recommendations that are not helpful. For example, let us say that we have a study showing a correlation between the number of nominalizations in mathematics tasks and task difficulty, and the study is based on a model that separates reading from mathematics (Figure 1a). The authors of this study might then recommend teachers and other task creators to avoid nominalizations, perhaps primarily for students with lower reading ability. That recommendation could lead to fewer opportunities for these students to become familiar with objectifications in mathematics, which would be negative for their learning of mathematics. Therefore, we suggest that a theoretical model takes the overlap between reading and mathematics into account (Figure 1b), that is, assumes that some part of reading ability is also a part of mathematics ability, by definition.

Furthermore, studies only focusing on associations between the existence of certain linguistic features of mathematics tasks and students' results on these tasks are not relevant since these studies are not informative. It is not possible to draw any meaningful conclusions based only on such an association, since it is not possible to know if the association is relevant or irrelevant, as described above. We encourage literature reviews of empirical studies to examine what types of conclusions and recommendations that have been made that are not valid when placed within a more relevant model.

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TO JOIN SEEING AND DOING: CREATING A FORMULA WITH A VIRTUAL AND A PHYSICAL 3D-PUZZLE

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We focus on the role of artefacts when students are expected to set up a geometric formula. For that, we employ the lately created concept of 'views on formula' in a case study design situated in a social constructivist approach, and explore how a pair of students in upper secondary creates a formula for a pyramid using two artefacts, a virtual and a physical 3D-puzzle, which share the same structure of six pyramids building a cube. Our aim is to characterize the contributions of the two artefacts on the students' way of setting-up the formula. Main results are that the transparency of the virtual puzzle shows that the six pyramids fit together while the physical artefact is not transparent but allows embodied arguing for the formula. Joining viewing and doing through both artefacts fosters an in-depth understanding of creating the formula.

RATIONALE, A BRIEF REVIEW AND THE RESEARCH QUESTION

In many math classes all over the world students struggle when using algebraic expressions (Arzarello et al. 2001; Kieran, 2020; McNeil et al., 2006). This holds specifically for formulas, an area that is widely under researched. One exception is a recent study from Schou and Bikner-Ahsbahs (2022) about geometric formulas. In line with this study, we consider a formula to be an algebraic representation of an equality relating measures of a solid. Students' difficulty with formulas partly depends on how they view them. E.g., the view *Shape* foregrounds strings of signs and therefore takes the two versions of a volume formula of a cylinder tube $V = \pi \cdot h\,(r_1^2 - r_2^2)$ and $V = \pi r_1^2\, h - \pi r_2^2 h$ as distinct. In the second version, it is much easier *to make sense of its parts*, i.e., by the view *Reading*. One relevant view on formula is *Blueprint*. This view makes the students see a formula as a blueprint for a building plan. It develops when students create formulas. In the case of the volume formula of the cylinder tube the first step is the concrete experience of taking a small cylinder out of a big one and then stepwise translating this action into a symbolic expression.

We have identified Blueprint in situations where students were expected to set up a formula from a single artefact, a prototype solid. However, recent research has stressed the relevance of using complementary artefacts for learning. Soury-Lavergne (2021) elaborates the concept of *duo of artefacts* as a system of artefacts designed to foster joined *instrumental genesis* (Trouche, 2020). Mariotti and Montone (2020) point to the potential of *synergy* of a duo of artefacts highlighting the synthesis of their semiotic

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 115-122). PME 46.

potentials. Maffia and Maracci (2019) have coined the notion of *semiotic inference* for mutual effects transcending cross-references when using various artefacts. Connecting several artefacts for learning has also raised questions of benefits and pitfalls in the transitions between artefacts, e.g., Rich (2022) investigates how teachers orchestrate such transitions and Panorkou et al. (2022) investigate the "messiness" of transitions taking students' perspectives seriously. From all this research, we infer that a collection of complementary artefacts that share a mathematical structure may foster an in-depth creation of a formula but it is not exactly clear how. By focusing on the views students have on formula, in particular the view Blueprint, the complexity of setting-up a formula becomes evident. In a new research path, we unpack this complexity for such a pair of artefacts by asking: *How does the duo of artefacts support the students' setting-up of a formula by activating their views, particularly Blueprint?*

THEORETICAL FRAMEWORK: THEORIZING VIEWS ON FORMULA

We adopt a socio-constructivist approach (Krummheurer, 2007) building on an object-related adaptation of symbolic interactionism. According to Blumer (1969), students make sense of objects (e.g., formulas) based on the meanings these objects have to them. They gain these meanings through social interaction as result of mutual interpretation of the sayings and the doings with their peers related to these objects. In these processes, students modify and adapt their interpretations taking into account their peers' expressed expectations and construal, specifically of the object. When faced with a task addressing a formula in social interactions, students activate various views (Schou & Bikner-Ahsbahs, 2022) as part of a conceptual frame. A conceptual frame (Arzarello et al, 2001) is an "organised set of knowledge and possible behaviours" (p. 68), which shape their interpretations and expectations in social interaction when they handle formulas related to a piece of knowledge. Within this frame, students may handle a formula showing a "locally coherent pattern" (Scherr & Hammer, 2009, p. 151) of repeated mathematical behaviour, where "coherent" means that "the pattern holds together for some length of time and 'local' in that the coherence may be particular to the moment or context" (p. 151). Views on formula are observable by such patterns. As Schou and Bikner-Ahsbahs (2022) have empirically identified and characterized eight 'views on formula', we can now use these views as scientific tools for investigating students' creation of formulas with pairs of artefacts.

METHODOLOGY: TASKS, TOOLS, AND METHODS

Figure 1: 3D-puzzle folding a cube with six pyramids (using www.GeoGebra.org)

To answer the research question, we use data from an upper secondary classroom study. The data stem from the implementation of a task to set up a formula for the volume of a pyramid based on a transparent digital 3D-puzzle, in which the students

can drag sliders to build a cube of six pyramids by folding it (Figure 1). As a complementary physical artefact, the students were provided with an opaque 3D-print of a cube that can also be composed of six pyramids, but by hand (Figure 2). When building a cube, the underlying structure of the two artefacts is the same, but the affordances when arguing are very different. Although they fulfil the conditions for a "duo of artefacts" (Soury-Lavergne, 2021), we abstain in this paper from investigating the students' instrumental geneses in favour of foregrounding our concept of object-related social interaction that will allow us to take the students' views on formula seriously.

The students were asked to follow a task sequence consisting of recalling the formula of the pyramid volume $V = \frac{1}{3} B \cdot h$, considering the measure of the volume of a pyramid as $\frac{1}{3}$ of the volume of a box in which it precisely fits, interpreting the volume formula by the use of the 3D-print (Figure 2), and interpreting the volume formula of the pyramid by the use of the virtual 3D-puzzle (Figure 1). The task sequence was implemented in ordinary teaching of an upper secondary classroom in Denmark where the work of one focus group was video recorded, and all the material all students produced in class was collected for comparison. We transcribed the video recordings verbatim and translated them into English. Next, we have used the concept of 'views on formula' to the guide data analysis, thereby reconstructing the views in play during the social interaction of the students mediated by the tools. Based on our theoretical approach, we conducted an interpretative turn-by-turn analyses (see Krummheuer, 2013) of the students' sayings and their doings. This allowed us to identify how the students interpreted the way their peers worked with the artefacts related to the goal of setting-up a formula for one pyramid, and how they proceeded. We traced the reconstructed views and their relations in the course of solving the task, focusing particularly on transition phases between artefacts in the duo.

SOME DATA, DATA ANALYSES AND A SUMMARY OF THE RESULTS

We evidence our results by analysing three episodes extracted from a much longer transcript. The sequence of individual expressions is kept, [….] marks lines left out.

Blueprinting sub-artefacts

In the first episode, we focus on the students' exploring of the physical 3D-puzzle and how the puzzle can be composed into a box.

120	S:	S: but it is double as tall, remember that, as the pyramid. There are two pyramids on top of each other [gesturing thumb and index finger for the two pyramids]
122	K:	yes, and so it will be also be one six when you calculate it, yes

The students are mainly concerned with how to compose the 3D-print. Going back to the six pyramids, due to the previous task they see a box–not yet a cube–and two pyramids on top of each other (#120), a justification of the fact that the box "is double as tall" as one pyramid. They add "one sixth" (#122). In the minutes they take the parts

together to form the expression $\frac{1}{6}s^2 \cdot 2h$, which is typical for Blueprint, where this would usually be followed by '=V'. We do not observe one complete blueprint but various blueprints of sub-artefacts that appeared by comparing pyramids with the box: s^2 for the base, $2h$ for "double as tall as the box", $\frac{1}{6}$ for considering one out of six pyramids. The terms are put together to get the "formula" $\frac{1}{6}s^2 \cdot 2h$ (Figure 3).

Preparing the complete blueprint by *seeing* how the cube is folded virtually

In the second episode, the students turn to the virtual 3D-puzzle. K tries out the sliders.

185 S: What are you saying? It's just, you just fold, you fold up the top to 90 degrees, and then you fold up all the others to 90 degrees. And then it becomes a nice box.

207 S: right, ... What on earth are you doing [K works with the animation]

208 K: I pull the figure here [twists and turns it on the screen]. Look, it is a cross!

209 S: it is actually a nice cross. But it is difficult not to make a cross, if it has to fit … yeah, you could NOT make a cross but it would be confusing.

The students are engaged in how the box–not yet the cube–is built by folding. Their focus is the cross, but not translating their actions into symbolic expressions.

Blueprinting by *doing* and *arguing* how the pyramids fit to build the cube

After having explored the virtual 3D-puzzle, the students return to the 3D-print in the third episode, and this time put up as a cross like in the virtual case.

218 K: say, if we now for instance say [inaudible, takes the model apart and places it as a cross in the same way as in the animation when unfolded] Then this side, that is the height [slides her index finger back and forth along the bottom edge of one pyramid] (Figure 2)

By aligning the 3D-print as was done in the virtual case, K merges the two artefacts conceptually and immediately begins to interact bodily with the 3D-print to argue.

223 K: and these there are two of (the horizontal distance from vertex to the edge of the base) [shows the distance with two index fingers] because this is [erects her right pyramid so it forms one side in the cube] two times the height [points along the same (now) vertical edge of the bottom of the pyramid starting at the table and ending at the upper end and back again]

229 K: and there we also have [shows the same distance by pointing with her index finger at the middle of the edge of the pyramid placed on the grey basis, … She erects some of the other pyramids and almost have a cube] (Figure 2)

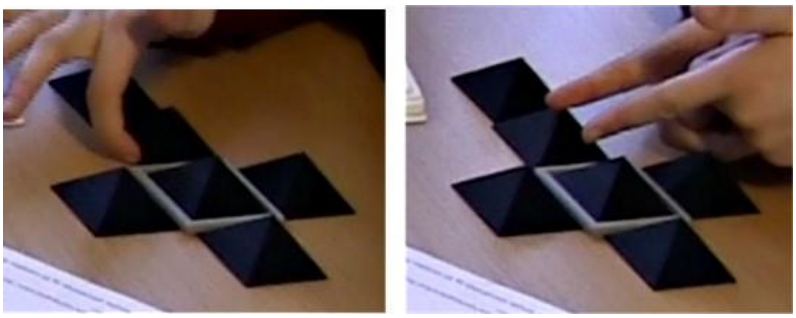

Figure 2: Physical 3D-puzzle (left: showing the height of the box as the length of the bottom edge of a pyramid, right: showing half the side length of the box)

233 S: so, actually it is a squared, squared square [they look at each other for a long time] It is a square (≈ cube). That is what I am trying to say.

235 K: It is a squared, squared square and we only use one sixth of the square (≈ cube) when we want a pyramid [places the pyramids in the cross shape again.] I don't know how to explain

238 S: I don't know how to explain it differently

239 K: Precisely. [Takes the computer and looks at the animation] we can see that there are six of them, and then we can fold them together. … but here we have a lid, right? [places the last pyramid vertically … to form the "lid"].

244 S: You must do it synchronous [takes the computer, while she folds the lid to make a cube on the screen, K does the same thing with the material cube]

247 S and K: wooouw [they laugh]

248 S: that was a bit crazy [laughs]

249 K: …. [Now seriously, looks at the screen that shows the folded cube] This side, right, it is two height, [let her finger slide up one of the vertical bases of the assembled model, still looking at the screen].

253 S: What?

254 K: this side [let her finger slide up and down] it is two height.

255 S: That means that we have side times side times side because it is a squared quadrangle

257 K: [looks puzzled] yes … okay, so it is side times side times side and that is just s to the power of three, (pause) no?

262 S: yes

263 K: s to the power of three divided by one third

264 S: yes … because s to the power of three just means s times s times s

266 K: so it is just s to the power of three divided by one sixth [S writes] why do I say divided by one sixth? I should say divided by six or multiplied with one sixth, yes

For the first time K expresses *verbally* that the box must be a cube (a squared, squared square) (#233/235). The students *infer* that only 'one sixth' is needed for a pyramid (#235). They *align* both artefacts by *synchronous* folding (#239/244). Then the students produce a *hybrid translation* 'side·side·side' (#255/257). They read it as a "squared quadrangle", used as a justification. The *hybrid* translation "*s* to the power of 3" (#257) is followed by "divided by one third" (#257). Next, the students *symbolize* $s \cdot s \cdot s$ (#264) verbally and change this into 's to the power of 3 divided by 6' (after one correction (#266)). In the minutes, we find the "formula" $s^3 \cdot \frac{1}{6}$ as the result of blueprinting.

Expressed by five stepwise translations, the students blueprint (Figure 3). At the same time, they add two kinds of actions: (1) *argumentations* (using finger sliding to

measure the sides of the pyramids; comparing the heights of the pyramids with the height of the box by a distance gestures; repeating this for the other pyramids, Figure 2) and (2) *coordinating* both artefacts (erecting side pyramids after comparing heights; repeating this for other pyramids; folding both artefacts synchronously).

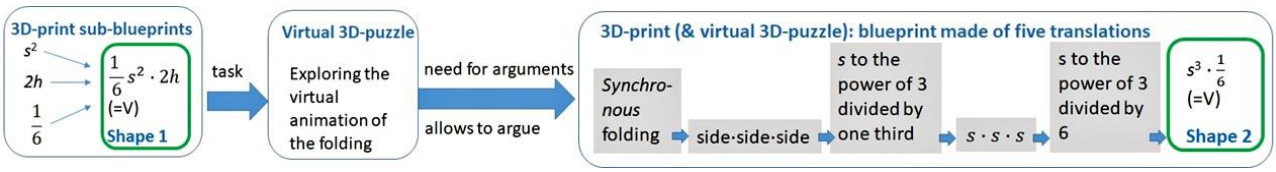

Figure 3: The three episodes with transitions, blueprints and shapes

Figure 3 summarizes our results. First, the students achieve a formula by putting blueprints of sub-artefacts together. There is no transition within the 3D-print. The transition to the virtual artefact is initiated by the task. The virtual animation is transparent, so that the students could see through the cube that the six pyramids seem to fit together, but with the artefact they could not check it. Thus, they changed into the physical world when building a cross and thereby coordinating the two artefacts and folding them synchronously. The transition seems to be initiated by a need for arguments (Kidron et.al., 2011). The haptic nature of the 3D-print allowed to fulfil this need, resulting in transitions through a *pattern of five translations*, i.e., *Blueprinting*.

DISCUSSION, CONCLUSION AND A TAKE AWAY

When the students use the two artefacts, blueprinting is not a straightforward process of a sequence of translations as in other cases, even though the artefacts share a common structure. The digital artefact shows transparently how the cube can be folded, making the students look through it from all sides, thus *seeing* that the six pyramids fit perfectly together. The students were much engaged in the folding procedure and its strictness; but as there was no embodied interaction possible with the virtual artefact, the students could not test conjectures. They had to take what they *see*.

This is different with the 3D-print, where we immediately observe the students begin to measure and compare lengths by measuring-gestures. These were used to argue for why the box is double as tall as the pyramid–considered from different sides. Thus, they could infer that the box was in fact a cube. The students had used the 3D-print already earlier, before they explored the virtual puzzle, but with trial and error and based on what they could *see*. That the box is a cube, was not taken into account. Making a cross came from *seeing* the virtual folding of the cube: it provided a systematic procedure, an offer to overcome trial and error. As *the fit cannot be seen* in the 3D-print, a need for arguments emerged, which itself necessitated to measure and compare. In contrast to the virtual artefact, the concrete 3D-print could fulfil the need for arguments. An argumentation about how the pyramids fit was the only way to be sure that the pyramids fit–symmetrically. In addition, the 3D-print allowed to coordinate the folding of both artefacts synchronously, substantiating that the artefacts share the same structure. Remarkably, S reflected in the end: “we have realized that the formula for the pyramid *also* [referring to $\frac{1}{3}B \cdot h(=\frac{1}{6}s^2 \cdot 2h)$] can be explained

as one sixth of s to the power of three [$\frac{1}{6}s^3$] as it is a square [meaning a cube]". She then convincingly cited the reason, that both artefacts follow "the same principle". Thus, they *read* the same principle in the two different formula *shape*s.

Neither of the two artefacts alone led to the complete blueprinting, both were needed. The strength of the virtual artefact to *see* through the cube however lacked certainty initiating a change from folding one to synchronous folding of both artefacts. The strengths of the 3D-print allowed the students to expand their ways of acting including argumentation and coordination. Data analyses have shown that *seeing* the common structure of the two artefacts was not enough for creating the formula, the students needed to *do* the aligning of folding both artefacts to make the same structure procedurally *visible* and hence, justify what they *saw*. *Seeing and doing were jointly intertwined* in the blueprinting of the third episode.

Therefore, *joining 'seeing and doing' initiated by composing 'virtual and physical' artefacts* seems to be a promising principle for task design to foster an in-depth conceptual understanding of a geometric formula that–in our empirical case of setting up a formula–enabled the students *to see, in the course of blueprinting, two shapes of the formula expressing the same "principle"*. Further research is needed to underpin the relevance of this design principle and thereby the relevance of *joining seeing and doing by design*.

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THE RECONSTRUCTION OF MATHEMATICAL INTERPRETATIONS – ACTIONS OF PRIMARY SCHOOL CHILDREN ON DIGITAL AND ANALOGUE MATERIAL

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This paper presents the results of the qualitative study MatheMat. It aims to analyse primary school children's actions on comparable digital and analogue materials to reconstruct their mathematical interpretation from a semiotic perspective on learning mathematics. For this purpose, a semiotic adapted qualitative analysis is applied to analyse the actions of two third graders in a geometrical learning situation to reconstruct their interpretation of the diagram realised with various materials. The comparison of the results shows that sometimes learners make the same interpretations of the digital and analogue material arrangements despite different actions because they recognise and interpret the same mathematical relationships as relevant to their actions.

MATHEMATICS LEARNING WITH DIGITAL AND ANALOGUE MATERIAL

Material is of great importance for mathematical learning processes and is often investigated in mathematics education research. However, the focus is often on the digital or analogue material itself rather than what learners do with it. For example, Larkin et al. (2019) use the Artifact Centric Activity Theory (ACAT) for evaluating digital materials to help teachers navigate the wide range of digital materials and understand their potential for mathematical learning. Such an instrumental approach, "[...] which is often used for research on technology learning settings, fails to attain insight into the epistemic process in all its aspects." (Behrens & Bikner-Ahsbahs, 2017, p. 2721). Therefore, the paper focuses on the results of the qualitative study *MatheMat – Mathematical Learning with Materials*, which aims to investigate learners' actions and their usage of the material. For the empirical investigation of learners' interpretations as they act on various materials a semiotic perspective on mathematical learning according to C. S. Peirce (1931-35) is adopted, which defines the actions on *diagrams* as the core of doing math (Dörfler, 2006). Specifically, this paper considers two cases in which the actions of two third-grade learners (9-year-olds) on comparable digital and analogue diagrams to solve a geometrical problem are analysed in order to reconstruct their mathematical interpretations. These interpretations are then compared to identify whether the learners make the same diagram interpretations in similarly designed digital and analogue learning situations and, thus, gain the same mathematical insights. The results of the comparison can be used to exploit the possibilities of the different materials for practice in mathematics teaching.

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THEORETISCAL FRAMEWORK

Semiotic Perspective on Learning Mathematics

Diagrams can be considered as relationally connected signs, namely, complex signs whose main function is to represent relationships (e.g. Wille, 2020). By this definition, a diagram is manifold: it can be a table, a function graph, a geometrical drawing, an arrangement of materials (digital or analogue), or an argument, as long as it is about the representation of the relationships. However, diagrams do not have a fixed reference that determines their meaning or significance (Dörfler, 2006). Instead, the activities on the complex signs ground them and make them meaningful (Roth & Bowen, 2001). According to Peirce (NEM IV), a diagram has certain features that would always belong to it, even if its non-essential features could be changed.

To illustrate this, Dörfler (2016) makes an everyday example: the same card can have different meanings due to the different activities in various card games. Thus, the meaning of a card is inseparable from its use and the rules and relationships that are recognised and established in the actions. If these are changed, the meaning of the cards also changes as they are among the essential features of the diagram. The cards' appearance, however, can be changed and is one of the non-essential features. In this sense, recognising and observing the relationships between the signs are constitutive of the activities on the diagram and require interpretations by the actor.

MATHEMATICAL INTERPRETATION FROM A SEMIOTIC POINT OF VIEW

In order to be able to examine learners' interpretation of complex signs more precisely, it is useful to consider the Peircean definition of a sign in more detail. Here, a sign is something that stands for someone in some respect or quality and consists of a triadic structure that includes the representamen, the object, and the interpretant of the sign (CP 2.228). By using the word representamen, Peirce means the outwardly perceptible sign that stands for something meant; he calls it its object. The perceptible sign triggers in the mind of the sign reader an interpretation called the interpretant. This interpretant can be an "equivalent sign, or perhaps a more developed sign" (CP 2.228) of the perceptible sign. Looking at communication (with oneself and others), the sign reader's interpretant can be expressed in a reaction to the sign, a new representamen, and thus there is a continuous translation into new signs (Maffia & Maracci, 2019). The new representamen produced by the sign reader in communication can be, for instance, an action, a gesture, or a phonetic utterance.

Peirce distinguishes between three different types of interpretants: an emotional, an energetic, and a logical interpretant. He describes the *emotional interpretant* as the first effect evoked by a sign and describes this effect as a feeling (CP 5.475). It arises in the sign-reading person but does not have to be expressed as a perceptible sign. The *energetic interpretant*, on the other hand, can be seen as a spontaneous action that the sign-reader performs as an effect on the sign, which involves an effort on the part of the sign-reading person; this effort can be physical as well as mental (CP 5.475). However, the energetic interpretant is not an action that the sign-reading person has

already repeated many times and that has become a habit with a specific goal. When it becomes habitual it is called the *logical interpretant* (CP 5.486).

The description of the different interpretants is important for the analysis of the data described in this paper, as they are used to reconstruct the mathematical interpretations that the learners make during their actions on the digital and analogue material. Based on the logical interpretant, an interpretant based on research is formulated, which can be described as the habitual reaction of experts in a community to a complex sign. The research-based interpretant describes relationships between the signs and the resulting rule-based actions that are necessary to establish these relationships. As Peirce highlights, the action must be described with specifications of the motive (CP 5.491). Therefore, in the description of the research-based interpretant, only the relationships and manipulations that are important regarding the task are dealt with. This research-based interpretant is compared with the learners' energetic interpretant. Through this comparison, it is possible to reconstruct which relationships between the signs the learners may have recognised and used to express their interpretant. In this way, it can be determined whether the learners working with digital material may recognise and focus on different relationships than those working with analogue material, even though they are working on comparable diagrams.

RESEARCH FOCUS

From the theoretical considerations, the objective of the *MatheMat* study is to reconstruct the learners' mathematical interpretation by analysing the actions on comparable diagrams realised with digitally and analogue represented signs. By comparing the reconstructed interpretations, it will be investigated whether the learners make the same interpretations even though they work with different materials. Geometrical and statistical learning situations with digital and analogue material were examined, which were developed especially for this study. The learning situations were designed so that, based on the same mathematical tasks and the same mathematics education considerations, the same mathematical relationships between the different materialised signs could be recognised. Following Dörfler (2016), diagrams that have the same mathematical structure and relationships are expected to enable the same mathematical engagement with them, so that the different materials should not interfere with this. This allows for comparison of material as the learners engage mathematically with the same diagrams of a different materiality.

This paper focuses on a geometrical example, where two third-grade learners investigate the relationship between the area and perimeter of similar squares digitally using GeoGebra (Hohenwarter, 2001) and analogue using an adaptation of the OrbiMath material (Huber, 1972) (see Figure 1). For the analysis, one part of Nils's and Marleen's work on the geometrical problem is considered, in which they create squares of different sizes with the material provided. These squares are the basis for completing various sub-tasks, which they have to solve.

Considering the geometrical example, the following research question is addressed: Which mathematical interpretations of Nils and Marleen can be reconstructed from the actions on the digital or analogue geometrical diagrams, and which possible differences can be described between the reconstructed interpretations of the two learners?

METHOD AND DESIGN

Method of Data Generation

For data collection, material-based interviews were conducted in the summer of 2019 at two German primary schools with 16 learners at the end of grade 4 (10-11 year-olds) and with 16 learners at the beginning of grade 3 (8-9 year-olds). Each learner worked in a pair on two learning situations, once with digital and once with analogue material. In addition, each learner worked on a geometrical and statistical task. Working on the learning situations the learners themselves could choose which sub-tasks they worked on in which order (Billion, 2021). In this way, the learners could decide at which mathematical level they wanted to work and could put difficult sub-tasks aside first and work on them later after having dealt with other sub-tasks.

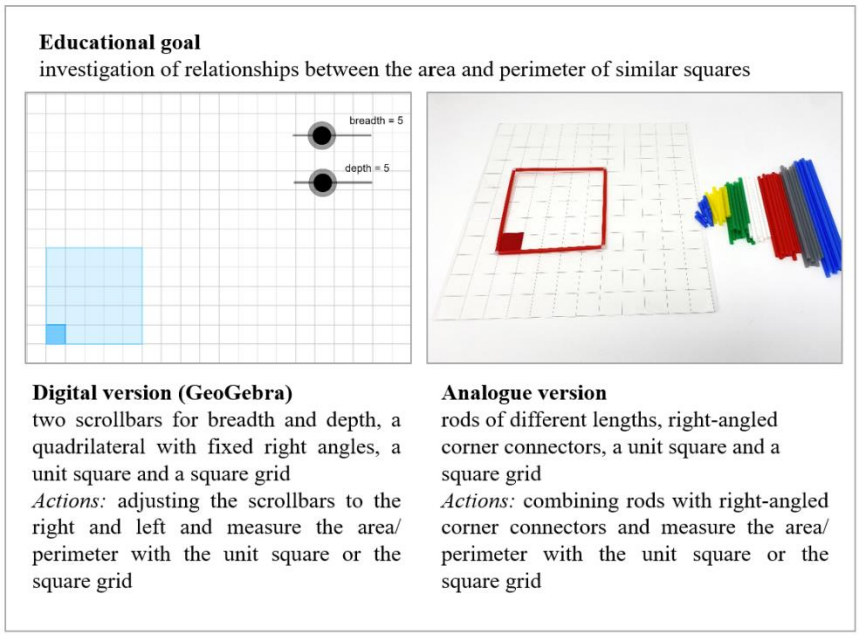


Figure 1: Geometrical learning situation

METHOD OF DATA PREPARATION AND DATA ANALYSIS

The learners' processing was recorded with two cameras. One camera focused on the actions and gestures made on the digital and analogue material, the other recorded the whole scene. In the learning situations with the digital material, the manipulations on the screen were also recorded with a screencast. In the videos, passages are sought in which the learners work on the same sub-task and these are transcribed for analysis.

For the reconstruction of the learners' mathematical interpretations, a semiotic adaptation of the qualitative context analysis according to Mayring (2014) and Vogel (2017) is provided. As already described, the learners' energetic interpretant is compared with the research-based interpretant in order to reconstruct which relationships between the signs the learners have recognised, interpreted, and used. In the first step, an energetic interpretant (i.e. a spontaneous action) of the learner is selected from the transcribed passage. Then, in the second step, the research-based interpretant is developed for this selected spontaneous action and is compared with it.

This is where the first reconstruction of the learner's mathematical interpretation takes place, which he or she does at this point for acting on the diagram. In the third step, the narrow context analysis (Mayring, 2014), all the same, and similar energetic interpretants of the learner that can be found in the transcript are compared with the research-based interpretant formulated in the second step. Depending on whether the learner's energetic interpretants are actions on other diagrams, the research-based interpretant needs to be adjusted for the comparison of interpretants. The reconstruction of the learner's mathematical interpretation at further passages in the transcript can confirm, extend or discard the one already reconstructed. In the fourth step, the broad context analysis, all the same, and similar energetic interpretants to the first energetic interpretant from the entire videotaped processing are added. These are in turn compared with the (possibly adapted) research-based interpretant for further reconstruction. In this way, the reconstructed mathematical interpretation can be described across the advancing sign process. In the final step, the reconstructed mathematical interpretations of the learner are presented in summary.

Given the passages selected for analysis from Nils's and Marleen's treatment of the geometrical problem, the focus of the research-based interpretant is on the relationships necessary for the construction of a square. The two important relationships to consider in the actions on the digital and the analogue material are listed in Fig. 2.

Digital	**Analogue**
Relationship between the lengths of all sides: To establish an equal relationship between the side lengths, the learner has to set the two scrollbars (depth and breadth) to the same position. GeoGebra automatically changes the side length of one dimension.	*Relationship between the lengths of all sides:* To establish an equal relationship between the side lengths in actions, the learner needs to select four rods of the same length.
Relationship between the connections of two adjacent sides: The relationship does not need to be established in the learner's actions as the right angles of the quadrilateral are fixed and the square grid also represents the relationship.	*Relationship between the connections of two adjacent sides:* The relationship does not need to be established in the learner's actions as the right-angled corner connectors and the square grid already represent the relationship.

Figure 2: Research-based interpretant

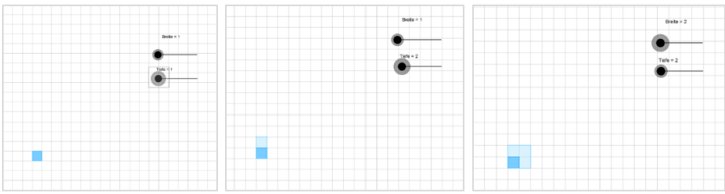
RESULTS

Analysis of the Actions on Digital Material

The focus is on the analysis excerpt where Nils sets the scrollbar *depth* and *breadth* to length 2. Therefore, he guides his finger towards the screen (see Fig. 3, Panel A), which shows a square with side lengths 1 (see Fig. 4, Panel A). Then he performs drag movements over the scrollbar *depth* (see Fig. 3, Panel B). Ultimately, the slider of this scrollbar changes to position 2, resulting automatically in a rectangle with a breadth of 1 and a depth of 2 (see Fig. 4, Panel B). Meanwhile, he utters "two\". Subsequently, Nils touches and makes drag movements over the scrollbar *breadth* (see Fig. 3, Panels C-D). Nils's actions move the slider to position 2, creating a square with side length 2 (see Fig. 4, Panel C). Nils then releases his finger from the screen (see Fig. 3, Panel E).

(A) (B) (C) (D) (E)

Figure 3: Nils's actions on the digital material



(A) (B) (C)

Figure 4: Manipulations in GeoGebra triggered by Nils's actions

By comparing the research-based interpretant (see Fig. 2) with Nils's actions, it can be reconstructed that he wants to establish an equal relationship between the lengths of the sides. Initially, Nils makes drag movements over the scrollbar *depth* to set the scrollbar *depth* to length 2. His phonetic utterance confirms this intention. His further actions suggest that he wants to make a square, as he also sets the scrollbar *breadth* to length 2. Since Nils withdraws his hand from the screen after he has set the second scrollbar, he is most likely finished with his action. He probably recognises that the relationships between the lengths of the sides are established, but it remains open whether he recognises the relationships between the connections of the sides, since these are already present in the material. Overall, Nils can interpret the parts of the material arrangement as a diagram, since he establishes new relationships in his actions and uses the relationships already implemented in the material to construct a square.

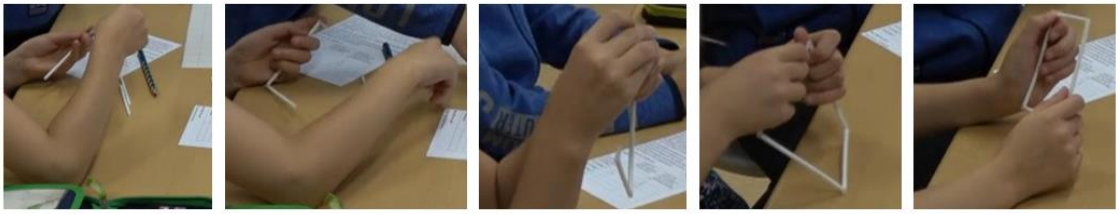
ANALYSIS OF THE ACTIONS ON ANALOGUE MATERIAL

The focus is on the analysis excerpt in which Marleen selects four rods of length 4 and joins them together to form a square. Initially, Marleen chooses three rods of length 4 and places them in the workspace in front of her (see Fig. 5, Panels A-B). She then selects another rod of length 4 and lays it alongside the others (see Fig. 5, Panel C).

(A) (B) (C)

Figure 5: Marleen selects four rods of length 4

Subsequently, she joins them together with the right-angled corner connectors to form a square (see Fig. 6, Panels A-E). Meanwhile, Marleen talks about her classmates. The spoken language does not refer to the mathematical content and will be neglected.



(A) (B) (C) (D) (E)

Figure 6: Marleen joins the four rods together to form a square

Comparing the research-based (see Fig. 2) and Marleen's energetic interpretant, it can be reconstructed that Marleen probably realises that she needs four rods of equal length to form a square. She uses the right-angled corner connections to put the selected four rods together. In her actions, she only establishes the relationship between the lengths of the rods; for the relationship between the connections of two sides, she uses the relationship already present in the material. The analysis of her actions does not reveal whether she is interpreting the relationship that is already visible in the material. She establishes the relationship between the lengths exclusively through her actions and does not refer to it linguistically. Overall, it can be reconstructed that Marleen interprets the material arrangement as a diagram by recognising relationships to create a square.

COMPARISON OF THE ANALYSES OF THE ACTIONS ON THE VARIOUS MATERIAL

Comparing the extracts from the analysis results reveals that both learners interpret the material arrangement as a diagram. It can be assumed that Nils and Marleen interpret the relationship between the side lengths of the square in the same way and perform actions that correspond to the relationship. However, it is noticeable that they perform different actions to do so. Furthermore, it becomes clear that they both use the relationship between the connections of two sides that is present in the material arrangement, but it cannot be reconstructed whether they explicitly interpret it.

DISCUSSION AND OUTLOOK

Concerning the research question, the same mathematical interpretations can be reconstructed for Nils and Marleen, since they establish the same relationship despite different actions on different materials. Thus, the haptic of the action does not influence the mathematical interpretation, so the learners are likely to gain the same mathematical insights when working with digital and analogue material. Not only the appearance of the signs is insignificant to the meaning of a diagram (e.g. Dörfler, 2016), but also the appearance of the actions when the same relationship is established in these. Further results of the *MatheMat* study show that there are passages in the data where the digital material functions as a tool (by abbreviating actions and relationships

to be established) and, thus, different mathematical interpretations can be reconstructed based on the actions on the digital and analogue materials (Billion, 2022).

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SECONDARY SCHOOL STUDENTS INTERPRETING AND COMPARING DOTPLOTS: AN EYE-TRACKING STUDY

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Dotplots can increase students' reasoning about variability and distribution in statistics education but literature shows mixed results. To better understand students' strategies when interpreting non-stacked dotplots, we examine how and how well upper secondary school students estimate and compare means of dotplots. We used two item types: single dotplots requiring estimation of the mean and double ones requiring comparison of means. Gaze data of students solving six items were triangulated with data from stimulated recall. Most students correctly estimated means from single dotplots; results for comparison were mixed. A possible implication is that single, non-stacked dotplots can be seen as a step towards teaching students to interpret univariate graphs but further research is needed for comparing graphs.

THEORETICAL AND EMPIRICAL BACKGROUND

The ability to interpret graphs is an important educational goal. For instance, graphs can reveal patterns in data that may not be noticed when looking purely at computational measures (such as means or correlations). In this paper, we will focus on graphs that are used to represent the distribution of a single variable. The distribution of a variable is one of the key concepts of statistics, and a prerequisite for understanding more complex distributions. Research has started to investigate what role various graphical representations (histograms, boxplots, and dotplots) have on the reasoning about the distribution of a variable (e.g., Lem et al., 2013a). More specifically, it has revealed a range of strategies and possible misinterpretations of each graphical representation.

In recent years, such strategies and misinterpretations are being investigated by means of eye-tracking data, that can yield a unique insight in strategies students use when interpreting the graphs and drawing conclusions. A recent review (Boels et al., 2019a) revealed a range of difficulties when interpreting histograms, and eye-tracking data have shown that students tend to interpret them as if these were case-value plots (Boels et al., 2022). Also for boxplots, various misinterpretations have been documented (Lem et al., 2013b), and currently, attempts are made to reveal these by eye-tracking data. The current paper focusses on strategies used on the third graph type, i.e., dotplots.

According to a local instruction theory on developing students' statistical literacy (Bakker, 2004), dotplots can increase students' understanding of variability in data (delMas & Liu, 2005), support students' reasoning about distribution (Bakker & Gravemeijer, 2004; Garfield & Ben-Zvi, 2008) and scaffold students' interpretation of

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 131-138). PME 46.

histograms (Lyford & Boels, 2022). However, the literature on students' dotplot interpretations showed mixed results. For example, Lem et al. (2013a) demonstrated that first-year university students tended to employ a local view on the distribution of a variable when interpreting dotplots, thereby focusing on individual observations rather than the distribution as a whole, more often than with the other representations. Moreover, they found that students had more difficulties comparing means, medians, and variation of data presented in dotplots when distributions were asymmetric compared to symmetric distributions. In addition, students used heights of dotplots to compare skewness of distributions, similar to what they applied to histograms. Lyford (2017) showed that in several cases students interpreted dotplots better than histograms. For example, although various students used stack heights in stacked dotplots, they did so less often than in histograms. However, when students compared 'bumpy' and 'spaced uniform' graphs, students answered correctly significantly more often for histograms than for dotplots. Therefore, also for dotplots, we want to achieve a better understanding of students' strategies when interpreting them, thereby relying in part on eye-tracking data. The current study addresses the research question: *how and how well do upper secondary school students estimate and compare arithmetic means of dotplots?*

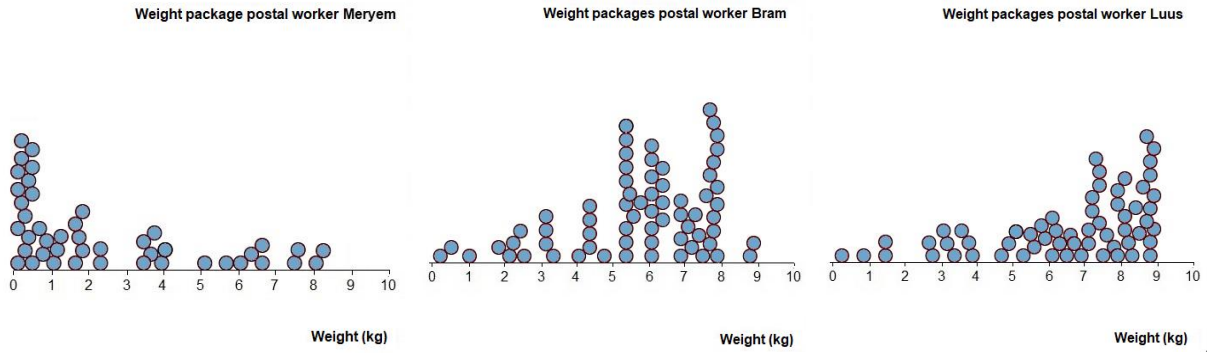

Figure 1: Item13, 15 and 18 of the original data collection for which students were asked to estimate the arithmetic mean from each dotplot. For item 13, for example, the actual mean is 2.7 (Table 1) and the range for correct answers was [1.6 – 3.8].

METHOD

We present answers and gaze data of five Grades 10–11 secondary school students. The students followed a pre-university track. They solved a total of six dotplot items. We designed two item types: open ended questions requiring estimation of the mean (Figure 1) and multiple choice items requiring comparison of means (e.g. Item 17, Figure 2). Note that our students had never seen a dotplot before in their education, but are familiar with case-value plots (where the height of each bar is the measured value) and histograms (where the position of each bar indicates the range of measured values). For each item type we designed three items. Gaze data were triangulated with verbal data from stimulated recall (cued retrospective thinking aloud) for which students' own eye movements were used as a cue (Van Gog et al., 2005). Data triangulation is needed because there is no straightforward relation between students' solution strategies and

gaze patterns (Schindler & Lilienthal, 2019). The data presented in this article stem from a larger data collection with 50 upper secondary students solving 25 items with various statistical graphs (e.g., histograms, case-value plots). In line with recommendations of Orquin and Holmqvist (2017), stimuli differed systematically on relevant features (e.g., positions of dots) but were kept similar for irrelevant features (e.g., color of dots, weight scales).

A Tobii Pro X2-60 eye-tracker with a 60 Hz sampling rate was used, mounted on a HP ProBook 6360b laptop with a 13-inch display (refresh rate: 59 Hz). The Tobii Pro Studio 3.4.5 software (n.d.) recorded in real time where people were looking on the screen using harmless infrared light to detect the gaze. A chin rest was used for better gaze data quality. Mean accuracy was acceptable (1.16°) with highest accuracy on the for this research most relevant graph area (0.27°; considered good); average precision (0.58°; RMS-S2S; Holmqvist et al., 2022) is considered good (see Boels et al., 2022 for more details).

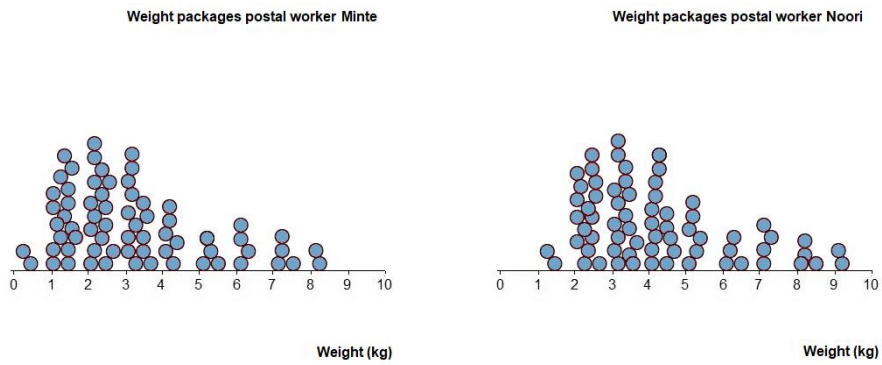

Figure 2: Example of a double dotplot item. Students were asked to compare arithmetic means, with three answer options: higher mean on the left, higher mean on the right, or approximately the same means. Here, the higher mean is on the right.

MAIN RESULTS

Regarding how well students interpret dotplots: four of the five students correctly estimated the mean from all single dotplot items (Table 1). One student (L03) overestimated the mean for the first dotplot item (Figure 1). However, for comparing means, results were more mixed, and only one of the students consistently gave a correct answer (Table 2).

For length reasons, the elaboration on how students interpreted the dotplots is restricted to the *single* dotplot items. Interpretations of double dotplot items will be presented during the PME 46 conference. We found four different strategies for single dotplots. The most common strategy is a strategy that we previously called a *histogram (interpretation) strategy* (Boels et al., 2019b): Students estimate the mean by finding the ‘balance’ point of the graph, or a ‘clump’ of dots. When students apply this strategy, a vertical scanpath pattern is visible in their gaze data.

Table 1: Characteristics of students and students' estimations of means from *single* dotplots items. Answer ranges were set to actual means [m= …] +/- 1.1. Correct answers in bold. Item numbers refer to their placement in the original item sequence.

Student	Age	Grade	Sex	Answers		
				Item13 [m=2.7]	Item15 [m=5.7]	Item18 [m=6.4]
L01	16	11	M	**3**	**5**	**7**
L02	18	11	M	**2**	**5**	**6.5**
L03	16	10	F	*4*	**6**	**7**
L04	17	11	F	**2**	**6**	**6.5**
L05	15	10	F	$\mathbf{2\frac{1}{3}}$	**6**	**5.5**

Table 2: Students' answers for comparing means from double dotplots items. Correct answers in bold.

Student	Answers		
	Item14	Item16	Item17
L01	Frans	**Same**	**Noori**
L02	**Sam**	**Same**	**Noori**
L03	Frans	Mustafa	**Noori**
L04	**Sam**	Mustafa	**Noori**
L05	**Sam**	Ilse	Same

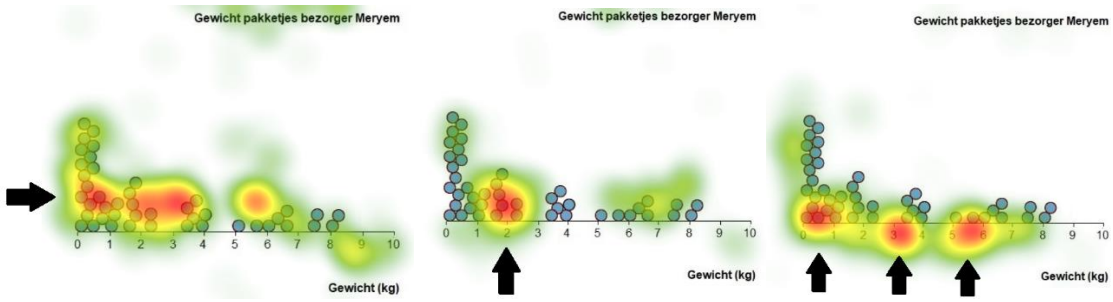

Figure 3. Heatmaps of Item13. Left: case-value plot strategy (L01). Middle: histogram strategy (L02). Right: computational strategy (L05). The colours indicate where students' gaze was less (green), medium (yellow) and most (red).

In a *computational strategy*, students add the measured values (positions of dots). An indication for this strategy is long fixations on each stack or number along the axis. Surprisingly, as shown in Figure 3, student L01 seemed to have used a strategy that incorrectly used the *heights* of the dotplots, instead of their horizontal positions. In such strategy, the dots are equally spread out along the horizontal axis. The height of the resulting stack is then estimated. We previously (Boels et al., 2019b) called this a *case-value plot (interpretation) strategy.* The difference between this strategy (Figure 3, left) and a histogram strategy (Figure 3, middle) is clearly visible in the heatmaps by the difference in horizontal spread-outness of gazes. The verbal data of student L01 do not substantiate this claim, but we think that is most likely due to this student switching to a correct strategy for later items and only reporting the latter for all items.

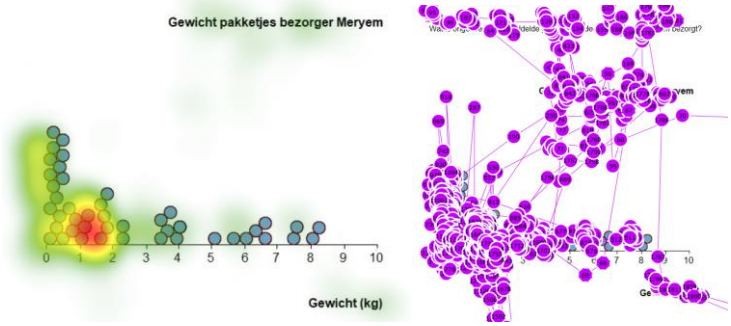

Figure 4. Gaze pattern of student L04 for Item13: heatmap and gazeplot .

Table 3: Students' strategies for single dotplot items. Correct strategies in bold.

Student	Strategy		
	Item13	Item15	Item18
L01	Case-value plot strategy	**Histogram strategy**	**Histogram strategy**
L02	**Histogram strategy**	**Histogram strategy**	**Histogram strategy**
L03	Unclear strategy	**Histogram strategy**	**Histogram strategy**
L04	**Histogram strategy**	**Histogram strategy**	**Histogram strategy**
L05	**Computational strategy**	**Histogram strategy**	**Histogram strategy**

Eye-tracking data showed that initially students did not know quite how to approach the first dotplot item. This is also visible, for example, in Table 3 where for Item13 four different strategies were found for these five students, compared to one strategy for Item15 and Item17. In addition, from the *video* of the eye movements we inferred that some students switched strategies. For example, the video of L04 for Item13 showed at the start long fixations around the numbers 0, 1 and 2 and the corresponding stacks of dots, and a longer fixation on the top half of the highest stack. Such long fixations might indicate thinking, which is necessary for a computational strategy. However, for a full computational strategy we would expect long fixations around all numbers and stacks along the horizontal scale (Figure 3, right). Instead, there are much

fewer and much shorter fixations on the dots at the higher numbers. These shorter fixations seem to indicate that the shape and location are looked at and that ultimately no computations were performed. As these shorter fixations occurred toward the end of the trial, shortly before the answer 2 was given, it appeared that this student switched strategy. The verbal data confirm the computational start and strategy switch:

L04: And then I saw that a lot of them had a weight of between zero and one and because of that I could work out that [this] was a pretty low mean. And then I did an approximate estimate.

Researcher1: Yes, okay. And I had the idea that you were also going to count here [started with counting] is that possible?

L04: No[t] with this one […] With this one I first thought I'll count. So I had already started counting but then I thought that's too much counting work and then I just started making an estimate because then I saw that, I guess so much was [in the left part] relative to the right.

The computational strategy that student L04 used at the start cannot be clearly inferred from the heatmap and gazeplot (Figure 4), although the heatmap shows that this student focused on the stacks with lower numbers. However, the video of the gazes *does* show a gaze pattern—at the start—that belonged to a computational strategy.

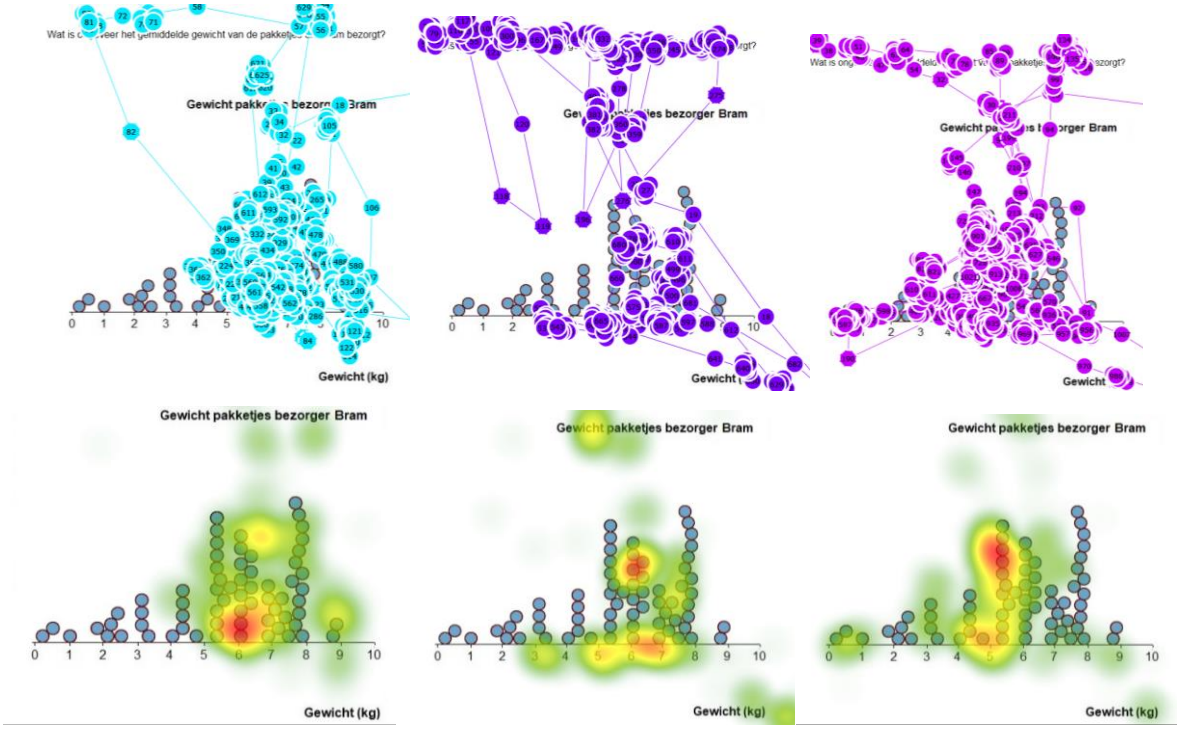

Figure 5. Gazeplots (top) and heatmaps (bottom) of correct strategies for estimating the mean, applied to Item15 by student L02 (left), L03 (middle), and L04 (right).

Both from the videos of the gaze data and Table 3 it became clear that students settled their strategy for single dotplot items (Figure 5) after Item13.

CONCLUSIONS AND DISCUSSION

From our study it appears that students are quite capable of estimating means from single dotplots, although they never learned about dotplots in school. For comparing means of dotplots, students answers suggest mixed results. Of course, we need to consider that this paper has the limitation that we involved a small number of students, and that graphs were presented in a fixed order due to technical restrictions.

Contributing to the local theory of interpreting statistical graphs, our study suggests that single non-stacked dotplots are well understood by upper secondary school students who never encountered these graphs in their curriculum. A possible implication is that single non-stacked dotplots can be seen as a step towards teaching students to interpret univariate graphs (e.g., histograms, boxplots, stem-and-leaf plots). However, for comparing distributions, students' variation in answers are in line with the mixed results Lyford (2017) found for undergraduate students. Therefore, further research is needed to investigate when and how students correctly *compare* dotplots.

This study is the first to reveal by means of eye-tracking the kind of strategies that students employ when interpreting dotplots. This is the major methodological advantage of eye-tracking data in this context: It reveals more details about students' thinking processes compared to concurrent thinking aloud (Van Gog et al., 2005). Concurrent thinking aloud may affect the actual thinking process and thereby not provide valid measures. In that sense, eye-tracking may even be seen as a—for research purposes—less obtrusive investigation method. The eye-tracking data may also reveal the entire range of strategies employed by students (both correct and incorrect strategies), and even show that students switch from one strategy to another while solving a specific problem. Such data are not only useful for research; they may also be relevant for educational practice. For instance, teachers can use a selected number of gaze patterns to draw students' attention to correct and incorrect interpretations of dotplots.

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TEACHING EDUCATION FOR SUSTAINABLE DEVELOPMENT-CHALLENGES AND SUCCESSES OF PRE-SERVICE MATHEMATICS TEACHERS

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Integrating Education for Sustainable Development (ESD) into the classroom is a declared goal worldwide. Teacher professional research is a desideratum in this regard. The intervention study presented here, which has been running for one and a half years, aims to alleviate this desideratum. Several cohorts of pre-service math teachers (PSMTs) took a semester-long an ESD-modelling seminar developed by the authors. Results of two cohorts are reported. The results, including questionnaires before and after the seminar, show the enormous challenges for teachers, because knowledge about ESD is initially or almost non-existent. After the seminar, knowledge and competence development became evident. Teaching ESD through modelling problems, as a practical part of the seminar, is a success factor.

INTRODUCTION AND RESEARCH QUESTIONS

With the 2030 Agenda for Sustainable Development (United Nations, 2015) as a global framework, 193 countries have also agreed to the conception of sustainability strategies as a basis for local, national as well as international learning and design processes in terms of sustainability. The 17 Sustainable Goals (SDGs) form the basis and orientation of sustainable development with an equal weighting and equal ranking of ecological, economic and social impacts as well as compliance with intra- and intergenerational justice of existing and future development processes. Subgoal 4.7 shows international agreement that education should be cross-cutting in the process of "Transforming our World" (United Nations, 2015). In order for Education for Sustainable Development (ESD) to be taught in the future and necessarily continuously as well as integratively and interdisciplinarily from mathematics lessons, well-trained teachers on the one hand and suitable tasks formats on the other hand are needed in the areas mentioned. The knowledge of mathematics teachers necessary to fulfil SDG 4.7 cannot be assumed, but must be taught as part of mathematics teacher education. In Germany, there is a lack of such mathematics education training courses in which the necessary specialised knowledge in the field of sustainable development and explicitly the SDGs is taught and in which the focus is also put on a mathematical or interdisciplinary interrelationship. However, teacher professional research in the field of ESD and mathematics education, especially in the field of secondary school teachers, is also very rare internationally (Firth & Winter, 2007). Based on this, the motivation of the two authors arose to reorient a mathematical modelling seminar to an ESD-modelling seminar using the interdisciplinary nature of mathematical modelling already known from integrative STEM learning (Borromeo Ferri & Mousoulides, 2017; English,

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2009). This ESD-modelling seminar for pre-service secondary mathematics teachers (PSMTs) at the University of Kassel (Germany) has already been conducted for one and a half years as an intervention study investigating different research foci. In this paper, we will report on two cohorts of PSMTs who provided a self-assessment of knowledge about the SDGs and ESD in a pre- and post-questionnaire. In addition, PSMTs' self-assessments were also asked about how they rate ESD/SDGs as opportunities to connect with mathematics and embed them in the mathematics classroom. Since there are no intervention studies of PSMTs to date that examine the above-mentioned aspects over a semester, we will focus on the following questions from our research agenda in this contribution:

1. How highly do the PSMTs rate their knowledge of SDGs and ESD before and after the seminar?
2. How highly do the PSMTs rate mathematical modelling for understanding the topics of environment, globalisation, democracy, everyday life phaenomenon and sustainability before and after the seminar?

This study draws attention to both the challenges and the successes of teaching and learning ESD (Vare & Scott, 2007). We will present also a case study of one such success here.

THEORETICAL BACKGROUND

According to de Haan (2002), teaching topics must fulfil certain ESD criteria in order to be able to teach sustainable thinking and action. Consequently, to this, an ESD topic must be a central, local and/or global topic for sustainable development processes, with longer-term significance, an interdisciplinary knowledge claim and action potential. ESD thus aims at developing visions and providing creative solutions, under the requirement of a participatory, inter- and transdisciplinary teaching and school culture. The core themes of ESD include development and environmental issues as well as the complex economic, environmental and social causes and solutions to these problems. Thus, the focus of ESD goes beyond isolated subject knowledge to thinking in contexts and in age natives, to systemic and transformative thinking. The mathematical ways of thinking and working contribute to such a knowledge of world systems, an understanding of nature, values, democracy and the One World, and provide data for critical reflection and risk assessment, as well as confronting the learner with conflicts of interest and target dilemmas in a data-based manner.

Teacher professional research on ESD in mathematics education

Although ESD is of high educational policy importance and knowledge about the SDGs and ESD forms the basis for practical teaching in schools, it is not yet naturally anchored in teacher education and training worldwide (Dahl, 2019). Furthermore, research desiderata exist on the necessary knowledge and competence areas for successful teaching of ESD through mathematical content, especially with a focus on the mathematics education perspective and on PSMTs. Even though there are some

studies that report on successful longer ESD training programmes for practising teachers, e.g. investigating teachers' self-efficacy regarding ESD during the implementation phase in the classroom (Boeve-de Pauw et al., 2022), there are hardly any evidence-based best-practice examples for mathematics teacher education. In the Helliwell and Ng (2022) study, although PSMTs are introduced by the researchers to the critical study of teaching and learning mathematics for a sustainable future, it was not clear in this study what specific background the PSMTs received on SDG/ESD and how these were concretely, integratively linked to mathematical problems. There is also a lack of intervention studies that illustrate knowledge and competence development of PSMTs for teaching ESD, including the relevant research instruments. Vásquez et al. (2020), in their qualitative study of ESD-beliefs of early childhood and primary PSMTs, found that of the 136 respondents, 78% answered "no" when asked if they felt well prepared to teach ESD. In this paper, we therefore focus on the domain of knowledge (Alexander, 1992) regarding SDG/ESD in the context of teaching mathematics of secondary school PSMTs, among others.

Teaching and learning ESD 'concrete' – through mathematical modelling

For teachers to teach the most integrated and interdisciplinary learning approach (Hobbs et al., 2017) of ESD, we believe that ESD should be taught and learned through real-world problems and task formats that incorporate also age-related mathematical content. Appropriate interventions by the teachers can thus succeed in encouraging the learners to critically reflect on real contexts by applying mathematical ways of working and thinking. Mathematical modelling, briefly described as the translation of a real problem into mathematics and the translation of the result back into reality (Pollak, 2007), offers an excellent opportunity to experience the sustainability goals in an interdisciplinary and integrative way. This is ultimately also reflected in the successes of the PSMTs in our study, who recognised on the possibilities of ESD, especially through the development of an ESD-modelling problem and its teaching at school. Fundamental to this was, among other things, the examination of the content and methodology of the Sustainable Development Goals Report 2020 and 2021 (United Nations, 2021) as part of the seminar. In addition to an intensive insight into the contents of the SDGs, the teachers also gained an idea of the importance and possibilities of mathematical modelling in the context of sustainable development.

METHODOLOGY OF THE STUDY

The intervention study on PSMTs' knowledge and competences for teaching and learning ESD through modelling activities is based on a mixed-methods design (Buchholtz, 2019). Qualitatively, the written reports of the PSMTs to be submitted after the seminar are a survey instrument, which are not the focus of this paper. Here we report on parts of the questionnaire developed by the authors, focusing on selected scales and show results from two cohorts. We report on two seminars with a total of 26 participants. The PSMTs, aged 19-21, were all in their second year of study. The modelling seminar by Borromeo Ferri (2018), which was developed on the four

dimensions of theory, task, instructional and diagnostic dimension, has been extended by SDG/ESD content with an integrative approach. Within the 14-hour ESD-modelling seminar divided into four temporal blocks, the PSMTs develop ESD-modelling problems with the help of the two seminar leaders, the authors of the paper, among others. For this purpose, the PSMTs design the associated lesson planning and held the lesson in the school between the 3rd or 4th session or afterwards. Without being able to go into detail about the seminar here, the required self-organised and interdisciplinary way of working of the PSMTs in the ESD learning arrangements should be emphasised at this point. The aim was the individual and cooperative acquisition of knowledge on and the examination of concrete case studies on non-sustainable development processes:

- To Know, describe and be able to relate the 17 SDGs as the core of the 2030 Agenda and the sustainability strategies.
- To Know the genesis, content and meaning of ESD.
- To Know the possibilities of mathematical ways of thinking and working for the screening of (non)sustainable development processes.
- Be able to describe the importance of mathematical modelling in the context of ESD.

In our opinion, this knowledge base enables a meaningful and comprehensible link to concrete contents and the development of ESD-modelling problems.

DESIGN OF THE STUDY AND QUESTIONNAIRE

In order to investigate the knowledge and the competence development of the PSMTs, a questionnaire was developed and used at the beginning and end of the seminar. The questionnaire on "Knowledge of Mathematical Modelling in the Context of Education for Sustainable Development (ESD)" consists of three parts, A through C. We will only present parts of A here. Before part A, the following questions were asked as initial questions (EF): (1) "I have not yet dealt with the topic of Education for Sustainable Development (ESD) at all. "(2) "I have already dealt with the topic of Education for Sustainable Development (ESD)."

In Part A, a 4-point Likert scale (from (1) 'strongly disagree' to (4) 'strongly agree') is used in a first part with the following items to ask for self-assessment regarding the knowledge domain (KD) SDG/ESD: “I can explain the following terms: - the 17 Sustainable Development Goals, - SDGs, - sustainable development, - the 2030 Agenda, - education for sustainable development.” The PSMTs' subjective views, if mathematical modelling is helpful for understanding local and global development processes in terms of the 2030 Agenda are asked in another part of Part A (SMM): “In my opinion, mathematical modelling can contribute to the understanding of the following topics: - Environment, - Globalisation, - Democracy, - everyday phenomena, - Sustainability.” The terms used here are representative of the criteria and dimensions of education for sustainable development.

RESULTS OF THE STUDY

Results are presented here, focusing on mentioned parts (EF), (KD) and (SMM) of the questionnaire. For this purpose, the data of both cohorts were combined and descriptively analysed in order to clarify differences before and after the intervention, which includes not only the seminar but also PSMTs' task development and their teaching. Rank sum tests were used for the mean comparisons, the medians, standard deviation and statistical outliers can also be seen within the below shown boxplots created for the scales KD and SMM. A psychometric analysis of the data is still being carried out as part of the further data analysis of the study.

Regarding (EF), only 2 of the 26 PSMTs had stated that they had already dealt with the ESD topic before the seminar. The lack of knowledge becomes even clearer through the PSMTs' further self-assessment with the Knowledge Domain (KD) scale at the beginning of the seminar, which leads to the answer to the first research question. Before the seminar, the mean score was 1.535 and afterwards 3.432 (the numbers 1-4 in Fig. 1 and 2 on the y-axis is the 4-point Likert scale). Thus, the mean value has increased by 1.89 points, which corresponds to a knowledge growth of about 45%.

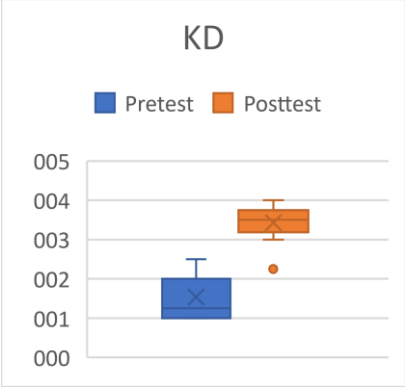

Fig. 1: ESD/SDG Knowledge Domain

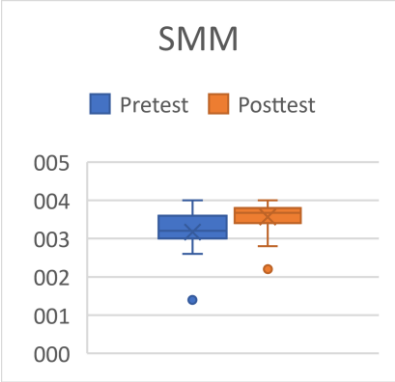

Fig. 2: Subjective view: Modelling and ESD

The background to mathematical modelling and the development of an ESD-modelling problem during the seminar and finally teaching the problem also changed the subjective view of the PSMTs before and after the seminar (Fig. 2), which leads to the answer of the second research question. The mean differences of 3.1 to 3.5 and thus with a growth of nearly 10% show that the PSMTs experienced modelling as an activity that enables them and the learners to capture the dimensions of sustainability in a task-based manner. The results therefore make clear that knowledge of SDG/ESD cannot be expected. It is a great challenge for PSMTs and certainly for practising teachers to teach ESD in an integrative and interdisciplinary way without specific training. Knowledge and skills increase over time and in particular the development and teaching of the ESD-modelling problem with a subsequent reflection is a factor for success. As an example, we present a PSMT (Tessa) to show that challenges at least over a semester lead to success. Tessa specifically used mathematical modelling to enter into a current social discussion in a fact-based way. In the following, it will be made clear how Tessa succeeds with her ESD-modelling problem not only in

considering the actuality and the closeness of learners to everyday life, but also in covering a large part of the ESD criteria of a topic at the same time.

Exemplary case study of success in learning and teaching ESD with modelling

Tessa developed the following ESD-modelling problem to raise awareness of energy consumption quantities and to thematise renewable energies, in line with the 2022 FIFA World Cup in Qatar:

> "Model how much energy is consumed for the air conditioning of the Al Bayt soccer stadium in Qatar during a 90-minute World Cup match. Consider under which conditions Qatar can still meet the claim of a climate-neutral World Cup. "

At the time the ESD-modelling problem was developed, neither data on the energy consumption of the air-conditioning systems nor on the exact dimensions of the soccer stadiums in Qatar were available on the internet. But Tessa wanted to design a problem specifically for this, "because on the one hand the World Cup in Qatar was already very controversial (especially human rights in this country) and [...] this energy consumption aspect (especially in times of an energy crisis) also provides/can provide an enormous amount of need for discussion". So, Tessa modelled the size of the Al Bayt football stadium using Google Maps and the energy consumption using the energy consumption data of German soccer stadiums. In the end, it is only standard calculations that lead to the conclusion that this amount of energy can supply about 140 German households for a year. "The mathematical result can then be related back to the ESD topic and provides a good basis for raising awareness of ESD-related problems, discussions about them and possible further modelling addressing the achievement of the Sustainable Development Goals (SDGs)". The implementation of the lesson confirmed Tessa's intention, as the learners discussed the result, questioned it due to the higher outdoor temperature in Qatar and also related it to their everyday life (energy crisis). As a suggestion for a climate-neutral World Cup, reference was made to the use of photovoltaic systems, which is probably particularly suitable in Qatar.

Tessa was one of the few PSMTs who already had a basic knowledge of the SDGs and the guiding idea of sustainable development before the seminar. Nevertheless, she describes the seminar as instructive and profitable, because it changed her view of her future role:

> "Mathematics teachers can also make an important contribution to promoting education for sustainable development. Before dealing with ESD in mathematics lessons, I tended to place this educational task in other subjects. [...] I consider the raising of awareness of the current way of life of people on the environment, society and the economy to be essential, which should also be the goal of mathematics teaching, as mathematical modelling is particularly suitable for illustrating, raising awareness of the dimensions and modelling a more sustainable way of life."

In addition, Tessa mentioned the particularly current application reference as further important aspects to promote the motivation and cognitive activation of the learners,

as well as the hope that such problems will create an impulse to live more sustainably in a private context as well.

SUMMARY AND OUTLOOK

In this paper, challenges and successes of PSMTs in teaching and learning ESD were illustrated through a one-and-a-half-year study so far. We argued that the SDG/ESD knowledge domain is central to promote an integrative teaching-learning approach for teacher education, because only then teachers concretely can link SDG and ESD goals to content or tasks, in our study mathematical modelling problems. PSMT's self-assessed increase in knowledge of ESD/SDG over a semester could be shown statistically on the one hand and, as a consequence, also qualitatively on the basis of a case study. In Tessa's case, it was clear that she was able to transfer her knowledge into action, engaging learners in critical thinking discussions and decision dilemmas to think about sustainability issues.

Limitations are given in the study with regard to the small sample. However, this is also due to the fact that, in contrast to other studies, we are not recording the status quo of the knowledge and competences of PSMTs, but are investigating the process over a semester. The many contents in the seminar regarding modelling, SDG/ESD, task development and teaching are central aspects that require time and thus allow the results to be interpreted accordingly. Learning and teaching ESD does not happen overnight, so we see our study as a start for ESD modules in teacher professionalism in mathematics education, especially in the context of mathematical modelling. In a further study, learners in secondary school are currently being surveyed before and after the PSMTs lesson on their beliefs and knowledge regarding ESD/SDG and the interdisciplinarity of mathematics. In this way we are creating a bridge and a transfer of knowledge on ESD from the PSMTs to the learners in school.

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CULTURAL ASPECTS IN THE CONCEPTUALIZATION OF ACTIVE, BODILY EXPERIENCE MATHEMATICS LEARNING ACTIVITIES.

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An essential first step to investigating the implementation of a mathematics education research finding is to characterize within the research field the innovation being studied. The conceptualization and characteristics of the object under investigation may include aspects related to the mathematical culture of the explored specific context. Therefore, analysing the research perspectives of different cultural contexts may be relevant to bring out these features that would otherwise remain hidden. To this end, researchers in mathematics education can be pivotal enablers. This report presents cultural aspects in conceptualizing active, bodily experience mathematics learning activities that emerge from interviews with a selected group of Italian and Australian researchers.

INTRODUCTION

The relevance of body and movement in the mathematics teaching-learning process finds general agreement, although research perspectives that consider this aspect have different roots. Over the years, many research findings, both experimental and theoretical, have emphasized, on the one hand, the importance of actively engaging students in experiential activities and, on the other hand, the role played by perception and movement in mathematics teaching-learning processes. According to the *embodied cognition theories* (Lakoff & Núñez, 2000; Varela et al., 1991), some relevant examples are the *enactivist pedagogy* (Abrahamson et al., 2022), the *inclusive materialism* (de Freitas & Sinclair, 2014), and the *multimodal approaches* (Radford et al., 2017). However, we do not possess specific information on how and to what extent this idea influences school practice, i.e. the nature and scope of educational proposals in schools that are aligned with what is indicated by research in this regard. To investigate primary and secondary school teachers' prospects in integrating activities consistent with the enactive-embodied perspective in their teaching practice, the first step is to characterize these activities searching for common ground within the different research perspectives. Setting aside the theoretical differences, we identified an operational construct that could be clear and easily accessible to teachers, to be the object of our research on implementation. Thus, the terminology active, bodily experience mathematics learning activities, abbreviated hereafter in ABM activities, refers to activities designed according to the perspective of enactive-embodied learning or, more generally, to activities in which students are actively engaged in exploring mathematical concepts using manipulatives, tools (virtual or physical), or whole-body movements. This construct encapsulates two main components: the students' active

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 147-154). PME 46.

engagement in mathematical exploration and their perceptual motor involvement. Looking at the implementation, we inscribe our research in the methodological framework of the Implementation Research (IR) in Mathematics Education (ME) (Jankvist et al., 2017). According to the model emergent from the work of Century and Cassata (2016), the first essential step is to clarify the components and to shape the attributes, possible declinations, and adaptations in different contexts of the operational construct under investigation. In other words, the elements that can characterize ABM activities and their implementation. To this end, in addition to reviewing the direction of research findings and official guidelines at national and international level, we conducted an exploratory study with researchers in the field of mathematics education. Indeed, they hold a privileged position to pursue such a goal because, from a research perspective, they are in continuous dialogue with school contexts and prospective teachers, representing a *trait d'union* between research and school. Thus, they play a significant role in recognizing the core elements and expected outcomes of the ABM activities, and to classify determinant factors in and for their implementation. Moreover, their participation allowed a contextual characterization, drawing interpretive lines on possible differences in implementation related to the specific structural characteristics of the school system and the culture, both mathematical and educational, in which students and teachers are immersed. As Clarke (2017) pointed out, the educational system is, at the same time, produced and constituted by the culture of the context, which is composed of an amalgam of innovations and educational traditions. Therefore, how the possibility to innovate and implement is profoundly limited within these cultural boundaries. According to Huang and colleagues (2020), the comparison of contexts with dissimilar teaching cultures, such as Italy and Australia, can lead to a better understanding of the activities being studied and their current implementation, revealing the presence of both features of contextual specificity, which may find reason in the specific cultures, and shared characteristics, which cross cultural boundaries. In particular, it may allow us to consider possible differences in the characterization of ABM activities depending on the context's culture. Thus, in this contribution we focus on the following research question: considering the two groups of selected Italian and Australian researchers, are there any cultural differences in their conceptualization and characterization of ABM activities?

METHODOLOGY

Participants

The involved experts in mathematics education were selected based on experience alongside teachers, for their research expertise in implementing innovation at school, and for research interests, which were akin to our research topic. The selection process consists in contacting experts who possess the aforementioned characteristics; the researchers in the sample are the ones that, after an email invitation asking them to contribute to the research, joined the project voluntarily. The six experts in Australia are academics, belonging to MERGA (Mathematics Education Research Group of Australasia) and three of them are also former secondary school teachers. Australian

experts' research interests range from initial and professional teacher education to inquiry-based learning; from the use of technology in mathematics education to mathematical Literacy and Numeracy; and from implementing teaching innovations in elementary schools to reforms in curriculum and assessment. They all have experience in teacher education and in implementing research findings in schools. In Italy, we have selected and interviewed 9 researchers in mathematics education. They are 7 accomplished academics and two teacher-researchers, who have a wide range of different research interests: mathematics difficulties and the use of representations, teachers' beliefs and problem-solving, teacher education, semiotic mediation, cultural transposition, multimodal approaches and gestures, Montessori method education. All of them have experience in teachers' professional development courses and empirical research in classrooms, and they are familiar with the topic. Seven of them are members of the National association of research in mathematics education (AIRDM).

Interview protocol

To collect the academic experts' opinions, we conducted individual semi-structured interviews via Zoom, approximately one hour long, in the period between May 2021 and December 2021. The interview prompts were designed to assess the researchers' views on key aspects of implementing ABM activities at school, especially in relation to teaching practices. The first goal of the interviews was to gather the researchers' opinions on the terminology and prototypical examples that might be commonly known and recognized by teachers, at different school levels, to identify ABM activities. Furthermore, the interviews helped shape the characterization of ABM activities and their implementation from the researchers' perspective. The prompts for the interviews are listed in the box below (Tab.1).

Table 1. Prompts for mathematics education researchers' interviews.

I	*Whether and why is it important to implement ABM activities at school?*
II	*What are the beliefs that should guide teachers in proposing them?*
III	*Which levels of awareness and knowledge should accompany teaching when implementing ABM activities?*
IV	*Which characteristics concerning the implementation of ABM activities at school determine their teaching effectiveness?*
V	*What are the main limitations of the use of ABM activities in daily teaching practice?*
	What are factors that could hinder/favour their implementation at school?

DATA ANALYSIS

The narrative material, manually transcribed in Jeffersonian simplified style, was analysed via MAXQDA. Interviews were analysed according to the Qualitative Content Analysis method, using an inductive category formation procedure (Mayring, 2015). The so-called open coding in Grounded Theory was used in the first instance to

refine the results with focused, axial coding (Cohen et al., 2017). Concept maps were used as a tool to represent the conceptual framework of the academic experts' perspective that emerged from the data gathered on each theme. The main themes addressed by the interviewees in response to the protocol prompts were coded and grouped into categories. The system of codes and sub-codes generated represent the core of the analysis and the interpretation of the results. It was conducted by making use of concept maps for each macro-category, in which the nodes consist of the codes' labels. Each code represents a natural unit of meaning, that is, a relevant theme that emerged in the narratives, and the codes are organized according to the categories and subcategories. In the procedure, we adopted a hermeneutic approach to refine the system of emerging categories and codes, based on criteria of interpretive clarity and informational accuracy by rereading and cross-analysing the narrative materials in several cycles. Proceeding hermeneutically could ensure the stability of the analysis, however, it does not guarantee that the codes assigned to the text units will be reassigned in the same way by another independent coder. In order to get a measure of the trustworthiness, we made use of investigator triangulation (Denzin, 2009). In the research presented here, a reliability process was planned, as outlined by Syed and Nelson (2015). We involved two external researchers in a refinement phase, for partial analyses of significant coding patterns, and two coders whose task was to conduct a final validation of inter-rater agreement, analysing the 20 percent of the total narrative material. The percentile agreement index shows that the analysis is sufficiently reliable ($\mathrm{i}_{\mathrm{agreements(Cod.1,Cod.2)}} = 83\% > 80\%$). In addition, the accuracy indexes of both external coders are quite good, respectively $\mathrm{i}_{\mathrm{accuracy\ (Cod.1)}} = 85\%$ and $\mathrm{i}_{\mathrm{accuracy\ (Cod.2)}} = 96\%$. We also attempted to limit the risk of the researcher's autonomy in creating the coding system, by inductively eliciting analysis from text units and reverting to coding narrative texts based on code and category systems, questioning fidelity and interpretive clarity reshaping coding systems until no more inconsistencies or ambiguities were found. Finally, to further ensure reliability, we returned the entire interview transcript to each interviewee.

FINDINGS AND DISCUSSION

The analysis of the interviews revealed the presence of common elements and characteristics that distinguish the opinions of the two groups of researchers, Italians and Australians. We report below an overview of some indications provided by the researchers to observe differences that emerged concerning the two investigated contexts. The examples of data and findings reported here come from a reading of the materials organized into two macro-narratives, expressing the totality of the Italian contributions, on the one hand, and the Australian ones, on the other. A more extensive discussion of the results of the explorative study may be found in (Boscolo, 2022a). We will report the experts' contributions by indicating the name, in the Italian case, or a pseudonym (e.g., Au_Expert 1), in the Australian case, followed by the citation paragraph number. This difference in the treatment of direct quotations is a consequence of the different guidelines of the research ethics committees that

evaluated the project in the two countries. Further, the Italian contributions are translated by the author.

From the analysis of the researchers' interviews, a significant cultural difference was particularly evident. For Australian academics, ABM activities represent an instrument for investigating and interpreting the world, e.g., "to visualise, [..] envision mathematics in the world" (Au_Expert 2, p. 38-42), and they tend to consider these as a way of bringing mathematics closer to students, as shown in the following paragraph:

> I think math could be taught in a very abstract way and if - particularly for younger children- if you want them to engage and enjoy maths I think it's gonna be practical and real, and using manipulatives just helps them to see this being something real. (Au_Expert 1, p.28)

Otherwise, the Italian ones mainly related them with the possibility, for a greater number of students, to access a deeper and more relational understanding of mathematics (Skemp, 1976), through a meaningful construction of knowledge that also considers its history and evolution. For instance, the Italian expert M. Mellone, addressing the first prompt, stressed to what extent ABM activities promote meaningful learning of mathematics, giving students the possibility to be actively engaged in the construction of mathematical meaning:

> [In these activities there is] the possibility of more meaningful learning, where students are actually active protagonists in the construction of their knowledge […] allowing for a multifaceted approach to mathematical meanings. (p.30)

Furthermore, the expert F. Arzarello emphasized that in such activities clearly emerge "what role the body plays in the solution [of mathematics task] and thus the multimodality with which we relate to mathematics, which is fundamental"(p.21). The multimodality, as an essential aspect of ABM activities particularly aiming at allowing broader access to mathematics, is also emphasized by the Italian expert A. Baccaglini-Frank. In the following, she focuses on the beliefs that a teacher should possess when implementing ABM activities:

> [she has to believe that their integration is relevant for] opening up the [teaching-learning] proposal on multiple channels and having the belief that this actually facilitates more students to follow the teacher in the construction of knowledge, which is crucial. (p.32)

The different characterization, evident throughout the interviews, clearly emerges when analysing the examples of the ABM activities proposed by the experts, both in terms of the content areas concerned and the typologies of materials and tools involved. The Italian researchers showed a greater interest in more traditional mathematical disciplines (e.g., activities from the geometrical tradition) with an emphasis mostly on the conceptual and theoretical construction of knowledge. For instance, as emphasized by M.G. Bartolini Bussi, among ABM activities, the integration of activities related to the history of mathematics is considered especially significant:

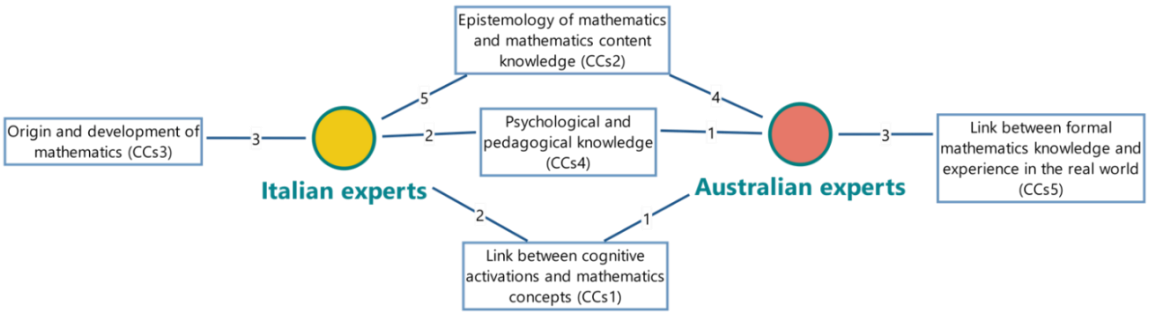
> [It is relevant to include] examples that relate to the history of mathematics, because the mathematics that we know today has been mainly developed from these examples. And thus, by the way, not always consciously. (p.30)

On the other hand, Australian academics cited many examples of mathematical modeling and real-world problems or activities related to the area of probability and statistics, which are completely absent in the Italian context. In addition, Australian academics quite commonly referred to examples in interdisciplinary areas, unlike Italian researchers. In the Italian context, ABM activities are instead much more often conceptualized as ends for the discipline itself. Evidence of this is the many references to the history and development of mathematical ideas that emerged repeatedly from their narratives, involving references to examples with classical tools that have characterized the evolution of mathematics (such as the abacus, the ruler and compass, or mathematical machines). Finally, Australian academics gave much less space to examples that recalled the use of a specific material designed for instructional purposes, for the conceptual learning of mathematics, preferring materials related to everyday life and contexts. Beyond the examples, this characteristic emerges cross-categorially in the researchers' contributions. For instance, as illustrated in the conceptual map below (Fig. 1), showing the indications regarding the knowledge a teacher should possess to implement ABM activities, although most of the indications are in common, the Italian researchers stressed the importance of knowing the history and development of mathematical ideas: e.g.,

> Teachers need to know the epistemology, the philosophy, and, nonetheless, the history of mathematics: how humans first came to certain concepts can be a fairly natural way to present them to children. Thus, it is necessary to know mathematics and, furthermore, some ancient mathematics. (B. Scoppola, p. 63).

Meanwhile, Australian academics highlight the need for specific knowledge to link formal mathematics to the experience of reality, as emphasized in the followings:

> It requires more experience in the teacher to be able to envision the mathematics in the world […] They have to see the mathematical ideas that are at play. And I think for most teachers- both primary and secondary, they don't have that experience. So they don't yet know how to make the links. They might know the mathematics but they haven't linked it. (Au_Expert 2, p.42)



The number on each link indicates the occurrences within the contributions of each group of researchers (Italian or Australian) of each unit of meaning (indicated in the corresponding box).

Figure 1. Concept Map: Necessary teachers' knowledge for implementing ABM activities (Realized with XMap, MAXQDA Analytics Pro 2022).

CONCLUDING REMARKS

In this report, the opinions of experienced mathematics education researchers regarding ABM activities are presented, comparing respondents according to the context to which they refer. It allowed us to identify contextual and cultural factors possibly determining the characterization of this proposal and its implementation in Italian and Australian schools. These activities are conceptualized as promoting access to a deep understanding of mathematical concepts by the interviewed Italian researchers. This aspect emerges in the arguments answering the specific interview prompts, as well as in the proposed examples. Indeed, they leaned toward activities that refer to strictly disciplinary content and classical and historical examples. Moreover, in these examples, didactical materials specifically designed for mathematics or tools used in mathematical practice are expected to be used to access conceptual aspects of mathematics, rather than looking at applications or envisioning mathematics in the world. In contrast, for Australian experts, the proposal for this activity is more explicitly aimed at linking mathematical knowledge to the real world, favouring modeling activities, representing phenomena, and dealing with real-world problems. Answering the interview prompts, they emphasized the relevance of linking mathematical contents and the real world. Furthermore, they proposed examples that refer to content areas belonging to applied mathematics and to exploratory tasks concerning the reality in which the students are immersed. From the perspective of IR in ME, if, as an initial step, it is essential to identify at the research level the characteristics of the innovation under study, search for the cultural aspects of this conceptualization seems also pivotal for investigating the implementation taking into account the culture of the context (e.g., the country). To this end, involving researchers in mathematics education which refer to different countries, with diverse mathematical and educational cultures, could be a helpful instrument to become conscious of these cultural features. In our case, the comparison between the conceptualizations of the two groups of researchers, Italian and Australian, emerging from interviews, allowed us to outline the research directions for the exploratory study conducted with teachers in Italy and Australia (Boscolo, 2022b). Comparing the opinions of diverse groups of experts, or experts from other different countries, the conceptual framework that emerges may highlight still further differences. Furthermore, it seems essential to further analyse if emerged features in the conceptualization could be traced in the curricular trends and mathematics education cultural traditions of the two contexts.

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STUDENTS' MATHEMATICAL WELLBEING DURING A CULTURALLY SUSTAINING MATHEMATICS PEDAGOGY PROFESSIONAL DEVELOPMENT INITIATIVE

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Developing equitable outcomes for all students in mathematics has been an ongoing focus in both research and policy. In New Zealand, a professional learning and development (PLD) initiative, Developing Mathematical Inquiry Communities, supports teachers to use culturally-sustaining mathematics pedagogy with marginalised students. The study presented in this paper investigates student mathematical wellbeing (MWB) in schools undertaking this PLD. We focus on students' MWB across the number of years of school participation in the PLD and following the PLD completion in the fourth year and beyond. Findings showed higher student MWB in schools that had completed the PLD. Implications for PLD programmes in mathematics education are discussed.

INTRODUCTION

Both within New Zealand and internationally, an ongoing challenge has been persistent inequity in a range of outcomes related to mathematics education for marginalised groups of students. New Zealand schools, similar to many other Western countries, have a changing student demographic including a large number of Indigenous Māori students and increasing numbers of students of Pacific descent. People of Pacific descent are a diverse heterogeneous group of people with heritage to the Pacific nations such as Samoa, Tonga, the Cook Islands, Niue, Tokelau, and Fiji. Research studies illustrate teachers' deficit perceptions in relation to marginalised students' capability in mathematics and others show deficit beliefs are highly resistant to change (e.g., Louie, 2017; Turner et al., 2015). Both deficit teacher perceptions and the ongoing documented challenge of achieving equitable outcomes for Māori and Pacific students (Allen & Trinick, 2021; Hunter & Hunter, 2018) have the potential to influence students' MWB.

When students' values are fulfilled in the mathematics classroom they feel good, are more engaged, and become more resilient to challenge and adversity – that is, they have high MWB. Conversely, because wellbeing is value dependent, cultural mismatches in the mathematics classroom (i.e., teachers or pedagogical values incongruent with students' cultural values) can contribute to anxieties, disengagement, or poor achievement (Stephens et al., 2012; Hill et al., 2021) – all indicators of ill-being. In New Zealand, Māori and Pacific students experience a cultural mismatch between the values of a Eurocentric educational system and their own cultural values

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 155-162). PME 46.

and often believe mathematical success is possible only when they suspend their own cultural identity and values (Hunter & Hunter, 2018). Milne (2013) describes this as 'white space' - or when marginalised students assume the mindset of the dominant culture. This cultural identity flipping reinforces deficit beliefs; degrades student autonomy, and agency; impacts on students' motivation to learn; and ultimately undermines their wellbeing (Ryan & Deci, 2000). We argue shifting marginalised students' mindsets so they can begin to embody and celebrate their cultural values in the mathematics classroom requires a sustained, dedicated, and long term approach.

Culturally responsive/sustaining mathematics pedagogy (CSMP) is a means of responding to cultural mismatches and addressing challenges in relation to equitable outcomes for marginalised students (Gay, 2010; Paris, 2012). In New Zealand, a three-year professional learning and developmental initiative called *Developing Mathematical Inquiry and Communities* or DMIC (see following section) focuses on CSMP and aims to counteract deficit theorising, support teacher practices for inclusivity and enactment of lessons of higher intellectual quality; and support student MWB through cultural value alignment. This study reports on students' responses to a survey measuring their MWB. Specifically, we investigate the difference in MWB for students attending schools engaged in the first, second or third year of DMIC PLD or those students' attending schools which have completed the PLD and are working to sustain the practices that align with the PLD.

DMIC PROFESSIONAL LEARNING AND DEVELOPMENT INITIATIVE

The DMIC model is a research-based professional development and pedagogical change initiative. A key component is the construction of collaborative learning communities across schools, with groups of teachers, and in individual schools and classrooms. The PLD is set within the central tenets of a culturally sustaining model (Paris, 2012) as well as supporting teachers to create, plan and enact ambitious mathematics pedagogy to raise mathematics achievement (Kazemi et al., 2009). New Zealand schools serving marginalized Pacific communities are prioritised for inclusion in the initiative, which is funded by the New Zealand Ministry of Education. Many of these schools also have significant numbers of students of Indigenous Māori heritage.

The DMIC PLD takes a whole-school approach predominantly working with teachers from primary and middle schools (Year One to Year Eight) with some involvement in lower secondary schools (Year Nine and Year Ten). Over the three-year period it involves two complementary forms of PLD, out-of-class PLD sessions and in-class mentoring. In the third year, a lesson-study process (Hunter & Back, 2011) is introduced to develop sustainability. The out-of-class PLD offers opportunities for exploration, discussion, and reflection on pedagogical practices aligned with CSMP and ambitious pedagogy. This includes identifying and building on student funds of knowledge and cultural values, using challenging tasks, implementing mathematical practices with students, and noticing and responding to students' mathematical thinking. During the in-class mentoring, the mentor and teacher work together to co-

construct mathematics lessons and critically reflect on and shift pedagogical practices. This includes explicitly noticing student strengths and creating a classroom environment for respectful social interactions. A key focus in working with teachers is to support them to engage students in a range of mathematical practices such as explanation, justification, and generalisation while also considering the cultural values and beliefs of the students (Hill et al., 2019; Hunter & Hunter, 2018). Following the third year of the professional development, schools move to independently sustaining the pedagogical practices with support when necessary in the form of out-of-class PLD sessions and in-class mentoring only for new teachers at the school. Aligning the teachers' pedagogical values with students' values underpins DMIC with the aim of promoting positive learning outcomes like enjoyment, interest achievement, and engagement – all components of MWB.

THEORETICAL FRAMEWORK & BACKGROUND

Here we define MWB as feeling good and functioning well (Huppert & So, 2013) accompanied by a positive state of functioning from students' experiences in the classroom aligning with their personal, or cultural values (Tiberius, 2018). We explore MWB according to seven dimensions previously identified in the literature (see review by Hill et al., 2022): accomplishments, cognitions, engagement, meaning, perseverance, positive emotions, and relationships. In this study we also include cultural identity as an eighth dimension because of links to Pacific and Māori wellbeing (Matika et al., 2021). Pacific and Māori people embrace collectivist cultural values like respect, relationships, family, belonging, and inclusion. Studies indicate cultural identity, language, and the ability to live in accordance with values, are closely tied with positive wellbeing for Pacific and Māori people (Matika et al., 2021). Applied to the classroom, Pacific and Māori students who feel a sense of cultural pride, cultural belongingness, connectedness, and have their cultural norms/values embraced (e.g., community, love, reciprocity) would likely experience higher wellbeing. These eight dimensions are interconnected rather than being mutually exclusive, e.g., feeling accomplished is often accompanied by positive feelings.

METHODS

In this study, students self-reported their MWB using an 11-point likert scale with 21 questions covering eight MWB dimensions (Hill et al., 2022). This survey was developed from existing wellbeing surveys (e.g., the PERMA wellbeing profiler with 23 survey questions, Butler & Kern, 2016) however, here we adapted the survey questions to reflect mathematics education. For example, survey questions included: *In my maths class I have friends that support me when I need it* (relationship dimension); *Maths is an important part of my culture* (cultural identity dimension); or *When I am doing maths I feel happy* (positive emotion dimension). Cronbach's alpha showed an acceptable internal consistency across each of the eight wellbeing dimensions ($0.68 < \alpha > 0.93$).

Participants included 4218 students (50% male) across New Zealand, including students from Years 3 to 10 (aged 7 to 15). Students were culturally diverse, self-identifying as Pakeha/European (*n* = 2502), Māori (1014), Pacific (384), Asian (291), or unspecified (27). Students attended schools that were either in their first year (*students n* = 1101; *schools n* = 13), second year (1752; 27), third year (1174; 12), or post PLD (191; 4). Students completed the survey in 2021 at the start of Term 1, thus Year 1 were schools right before they had started the PLD (i.e., a baseline group), Year 2 was after a full year of DMIC (and into the second year) and similarly with Year 3. Post PLD were students from schools who had finished the three-year DMIC PLD and were working on sustaining the pedagogical practices aligned with the PLD.

Survey data were imported into SPSS 28, with univariate ANOVA (using post-hoc Tukey tests) assessing statistically significant main effects in students' ratings of MWB across each year schools had participated in the DMIC initiative. Thus, our Year 1 group (shown in Figure 1) included students from multiple Year levels and schools and likewise with the second, third, and post DMIC PLD groupings.

RESULTS

The mean ratings for overall MWB (dotted line) and for each of the eight dimensions (solid colored lines) are displayed in Figure 1.

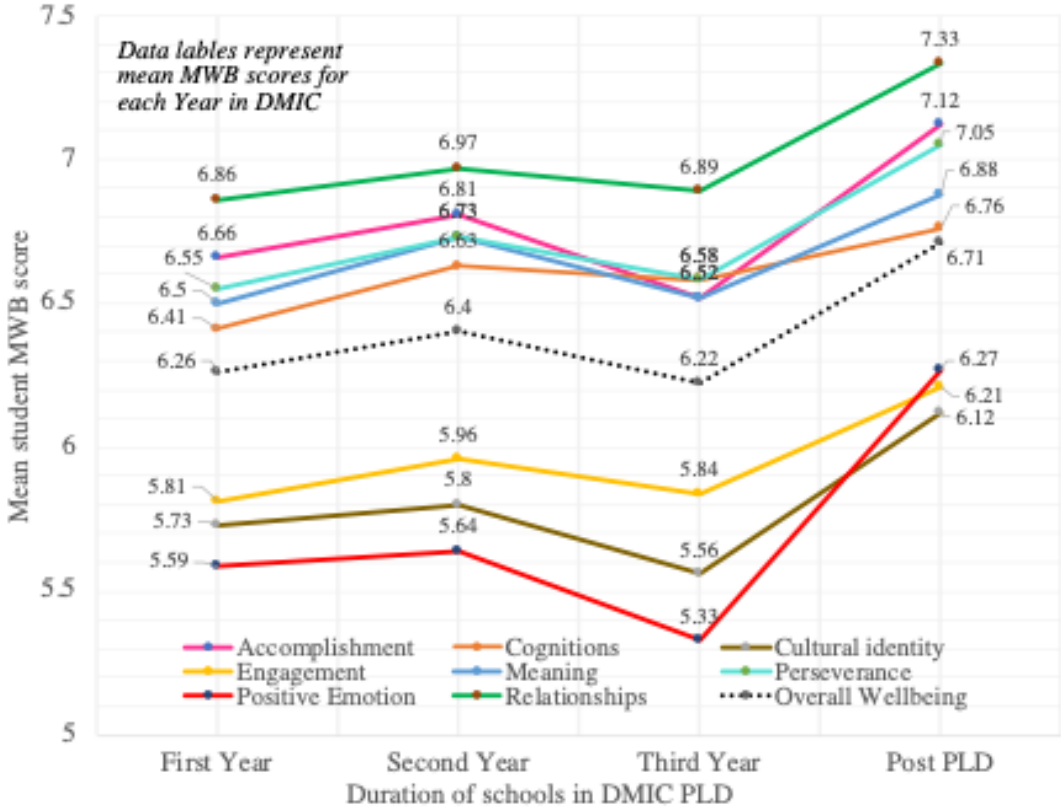

Figure 1: Mean MWB scores across length of time in DMIC PLD.

As shown in Figure 1 students across all years in the DMIC initiative tended to rate relationships (green line) the highest and positive emotions (red line) the lowest, except for the post DMIC period where positive emotions rapidly improved relative to the

other MWB dimensions in the same period. Statistically significant main effects were found for overall student MWB $F(3, 4227) = 5.5$, $p < .001$, attributed to students having significantly higher overall MWB post DMIC (see Figure 1 for all mean values) compared to both the first and third, but not the second years of DMIC. Concerning the eight MWB dimensions, main effects were found for all dimensions except engagement (accomplishment $F(3, 227) = 5$, $p = .017$; cognitions $F(3, 4227) = 2.61$, $p = .05$; culture $F(3, 4203) = 4.12$, $p = .006$; meaning $F(3, 4227) = 3.4$, $p = .017$; perseverance $F(3, 4226) = 3.88$, $p = .009$; positive emotions $F(3, 4226) = 7.39$, $p < .001$; and relationships $F(3, 4227) = 3.36$, $p = .018$). These effects were predominantly because students rated relationships, positive emotions, accomplishment, and perseverance significantly higher in the post DMIC period compared to first and third years of DMIC; positive emotions were rated higher in the post DMIC period compared to all other years; meaning was rated significantly higher only in the second compared to third year; cultural identity higher only in the post DMIC period compared to the third year. Taken together the DMIC PLD programme appears to enhance students MWB broadly and across multiple MWB dimensions from the post PLD period.

DISCUSSION AND CONCLUSION

Our findings demonstrate a positive association between schools participating in the DMIC PLD with higher student MWB, with improvements in students' MWB more likely when they attend schools who have completed the 3-year PLD initiative and are sustaining the pedagogical practices (i.e., the post PLD period). This is an important finding because it demonstrates improvements to student MWB can be sustained even after the PLD has concluded. Key aspects of the DMIC PLD include a focus on CSMP and the introduction of ambitious pedagogy and mathematical practices in ways that align with marginalised students' cultural values. For example, to align with values of respect and collaboration teachers introduce mathematical argumentation as "friendly arguing" and teach students how to disagree with a mathematical idea in a polite way while positioning this both as a respectful action (i.e., it is respectful to show that you have thought deeply about a peer's idea) and an action to collaboratively support others (i.e., through sharing and constructing mathematical ideas). Introducing these high leverage practices in ways that draw on value alignment/fulfillment may in part explain why students' MWB continued to improve post PLD. Additionally, we conjecture that the schools which were working on sustaining the pedagogy post the PLD may have implemented the pedagogical practices with high fidelity (e.g., aligning pedagogical values with students values) over an extended period of time.

Our findings suggest improving students' MWB may not be a quick process, thus a long-term approach to mathematics PLD, to ensure changes in teacher practices and beliefs is required. Earlier research (e.g., Horn, 2010) highlighted the extended time and incremental process for teachers to enact new pedagogies. As teachers see the results of their actions in classrooms impacting student outcomes, the evidence promotes a shift in deficit beliefs and affirms the pedagogical actions, leading to more sustained classroom practices. However, because student outcomes are often delayed

following PLD enactment, as acknowledged in a review of educational PLD (Kennedy, 2016), teachers may doubt the efficacy of their actions resulting in a premature reversion to initial practices. This phenomenon may be contributing to the results shown here in our study where MWB across a range of dimensions decreased slightly (though not significantly) between the second and third year of the DMIC PLD. Alternatively, we conjecture that this may be related to the timeframe in which these schools began the DMIC PLD in 2020. The PLD delivery in 2020 was significantly disrupted due to COVID-19 lockdowns and school closures in the first part of the school year and middle of the year. We will continue to examine this as part of the longitudinal data collection in future to see whether the dip in MWB is consistent in the third year of the PLD.

Our earlier study (Hill et al., 2022) demonstrated a decline in MWB as students progressed through school. Follow-up studies will also investigate how the year level of students interacts with the length of time in the PLD in relation to student MWB. The ultimate effects of PLD on student MWB and other outcomes are likely not completely visible until towards the end of the period of professional learning. Our results support an argument that mathematics PLD evaluation should take a long-term view (Kennedy, 2016) in studying both teacher practices and student outcomes beyond the end of the PLD and as schools themselves sustain the changes in practice.

Notably, our findings point to the positive emotion MWB dimension showing the most change from beginning to post DMIC PLD period (a 0.7 point increase), followed by relationships, and perseverance (both 0.5 point increases). Across many countries, a significant challenge of mathematics education is the negative feelings, dislike, and anxieties students hold towards mathematics, and once developed, these negative feelings often carry through well into adulthood (Grootenboer & Marshman, 2015). Key components of the DMIC PLD focus on elements which potentially improve positive emotions towards mathematics by shifting student perceptions of what it means to do mathematics. For example, a previous qualitative study (Hunter et al., 2019) focused on the changes in Pacific and Māori student perceptions of mathematics after the first year of the DMIC PLD, responses shifted from the majority of students describing learning mathematics as number and calculations at the beginning of the PLD, to students predominantly describing learning mathematics as participatory practices and problem solving.

The DMIC PLD advocates teacher pedagogical practices that align with key collectivist Pacific values such as relationships and collaboration. The strong improvement in students' ratings of the relationship and cultural identity dimensions suggests the PLD is supporting teachers to use practices that fulfill students' cultural values by promoting closer relationships and allowing students to embrace their cultural identity thus counteracting the 'white space' mindset (Milne, 2013). Similarly, the increase in students' ratings of perseverance is encouraging because students who persevere are more likely to recognise their strengths and attempt to persist for a greater

length of time with challenging mathematics tasks. This in turn raises the potential for student achievement and higher accomplishment MWB.

We acknowledge the need for further studies that also draw upon qualitative data to examine and identify both teacher and students' perspectives in relation to the reasons for the shifts in MWB. Additionally, further research should focus on the relationship between teacher actions in the mathematics classroom and student MWB. It would also be beneficial to interrogate longitudinal matched data on student MWB from the same schools to provide a more robust view of the changes in student MWB over time point data. These areas will be the foci of future studies which are currently in progress.

To conclude, we have shown the potential of developing more positive student MWB through the use of PLD which addresses both CSMP and ambitious pedagogy. We note the importance of recognising and building upon cultural values to ensure values alignment and fulfillment. We also illustrate the importance of both a long-term approach to mathematics education PLD and longitudinal data collection related to student outcomes following PLD.

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THE SUPPORTING EFFECT OF DIFFERENT VISUALIZATIONS FOR JUDGING COVARIATION AS PART OF BAYESIAN REASONING

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Bayesian reasoning is a way of thinking in situations of uncertainty that is crucial for experts such as physicians or lawyers but also for lay people. Bayesian reasoning could be understood to calculate specific conditional probabilities, but also to be able to estimate the result of a change of independent variables on dependent variables, i. e. to judge covariation in the context of Bayesian reasoning. Research yields that (expert and lay) people struggle with calculation in a Bayesian situation, but also yields strategies to improve Bayesian reasoning, e.g. by using natural frequencies and visualisations. In this paper we refer to the effect of different visualisations on people's performance regarding covariation. We found a supremacy of visualisations such as a 2x2-table and a unit square over a well-known visualisation such as a tree diagram.

THEORETICAL BACKGROUND

Bayesian reasoning is understood as "the process of combining conditional probability information and base rate information to update a posterior probability" (Reani et al., 2018, p. 63). For example, a person receives a positive medical test result concerning a specific disease and uses this information to update the knowledge about his or her condition of health. Before getting this information about the test result (I), the person's hypothetical knowledge about his or her condition of health concerning the specific disease (H) may (without additional information) be represented by the base rate (sometimes also referred to as prevalence) for this specific disease in the population ($P(H)$). With this base rate and additional knowledge about characteristics of the medical test, a Bayesian situation, i.e. a situation in which applying Bayes' formula is appropriate (Büchter, Eichler, et al., 2022), is given (Fig. 1).

The probability that a person in a certain population has the specific disease (H) is 10% (base rate, $P(H)$).

If a person has the disease, he or she will have a positive result in the medical diagnostic test (I) with a probability of 80% (sensitivity of the test, $P(I|H)$).

If a person does not have the disease, he or she will nonetheless receive a positive test result in the medical diagnostic test with a probability of 10% (false-positive rate of the test, $P(I|\overline{H})$).

Figure 1: A Bayesian situation concerning a fictitious disease

By processing the data with Bayes' formula the so-called positive predictive value (PPV) as the probability of having the disease given a positive test result can be

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel., & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 163-170). PME 46.

calculated: $P(H|I) = \frac{P(I|H)\cdot P(H)}{P(I|H)\cdot P(H)+P(I|\overline{H})\cdot P(\overline{H})} = \frac{0.8\cdot 0.1}{0.8\cdot 0.1+0.1\cdot 0.9} \approx 0.47$. The PPV is the update of the base rate and, thus, an improved information of a person's condition of health. Similarly, the so-called negative predictive value (NPV) is the probability of not having the disease given a negative test result, $P(\overline{H}|\overline{I}) = \frac{P(\overline{I}|\overline{H})\cdot P(\overline{H})}{P(\overline{I}|\overline{H})\cdot P(\overline{H})+P(\overline{I}|H)\cdot P(H)} = \frac{0.9\cdot 0.9}{0.9\cdot 0.9+0.1\cdot 0.2} \approx 0.98$. Bayesian reasoning is important among disciplines, such as medicine (Ashby, 2006) or law (Lindsey et al., 2003), but also for lay-people within the general society when judging risks (Spiegelhalter et al., 2011).

However, for the aspect of calculation referring to the PPV a meta-analysis of McDowell and Jacobs (2017) showed that only about 5% of people are able to calculate a PPV in a Bayesian situation as given in Fig. 1 without further support. This is particularly concerning, as the performance is even similarly low among experts, who require Bayesian reasoning professionally, such as medical practitioners (Hoffrage & Gigerenzer, 1998) and legal experts (Lindsey et al., 2003). A bias which is often mentioned as the cause for the low performance of calculation is the so-called base rate neglect (Kahneman & Tversky, 1982), by which people tend to overlook the influence of the base rate. However, research in psychology and mathematics education consistently showed that it is possible to improve people's Bayesian reasoning by using natural frequencies instead of probabilities as the format of data in a Bayesian situation (McDowell & Jacobs, 2017). The medical situation with the information in the format of probabilities which is given above is given with natural frequencies in Fig. 2.

100 out of 1000 persons in a certain population have a specific disease (H).

80 out of 100 persons having the disease will have a positive result in a medical diagnostic test (I).

90 out of 900 persons not having the disease will have a positive test result in the medical diagnostic test nonetheless (I).

Figure 2: Bayesian situation of a fictitious disease with natural frequencies

In this case the PPV, that is $P(H|I) = \frac{8}{8+9} = 0.47$, can be computed by dividing natural numbers. Following the meta-analysis of McDowell and Jacobs, using natural frequencies increase the proportion of correct calculation of the PPV from about 5% to about 25%.

Visualisation of the data in a Bayesian situation is found to be a second strategy for improving the performance of calculation as part of Bayesian reasoning (e.g. Binder et al., 2015; Böcherer-Linder & Eichler, 2019; Brase, 2009), particularly if they are combined with the natural frequency strategy (Binder et al., 2015). With appropriate visualisations, the proportion of calculating the correct PPV in Bayesian situations increases from about 25% (with natural frequencies and no visualization) to about 60-70% with double-trees, unit squares or 2x2-tables (as displayed in Fig. 3) combined with natural frequencies (Böcherer-Linder & Eichler, 2019). A well-known tree

diagram was found to be also supportive, but significantly less supportive compared to the three visualizations mentioned before (e.g. Böcherer-Linder & Eichler, 2019).

Tree diagram Double-tree Unit square 2×2 table

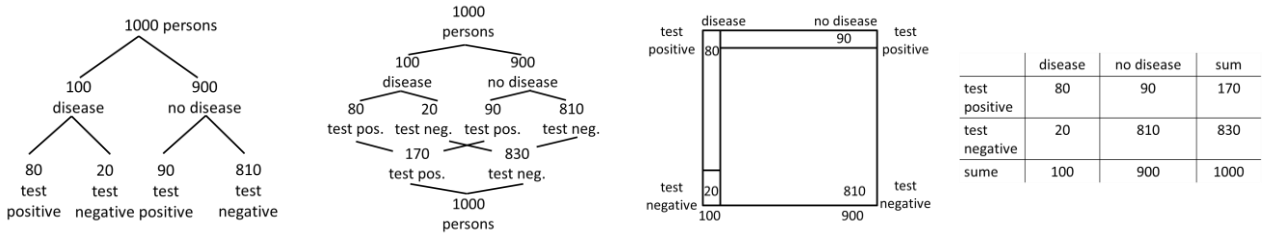

Figure 3: Visualization for supporting Bayesian reasoning

However, an in-depth understanding of probability and Bayesian situations goes beyond only calculating a conditional probability such as the PPV in a Bayesian situation, but also entails to be able to evaluate the "influence of variation of input parameters on the result" (Borovcnik, 2012, p. 21; cf. also Kazak & Pratt, 2021). For example, a variation of the base rate in a Bayesian situation involves a variation of the PPV as well. By regarding Bayes' formula as a function, we rely on Thompson and Carlson (2017) to understand a function (in our case Bayes' formula) covariationally when variations of two quantities (in our case probabilities or proportions) are conceived simultaneously. We further regard covariation as an additional aspect of Bayesian reasoning which builds on calculation (Büchter, Eichler, et al., 2022). Covariation has hardly been investigated in the field of Bayesian reasoning so far. Yet, Büchter, Steib, et al. (2022) argue that double-trees and unit squares can both be used to support covariation in Bayesian situations. This may be based on their structural characteristics on the one hand but on the other hand also on their superiority over the simple tree diagram regarding calculation. We are not aware of any empirical studies comparing the effects of different visualization for covariation apart from the comparison of unit square and simple tree diagram in Böcherer-Linder et al. (2017). Additionally, it has been shown that the parameter of the Bayesian situation which is varied affects covariation (Steib et al., under review). However, we are not aware of studies comparing the judgements on different dependent variables when one independent variable is varied. For this reason, our study refers to people's performance of covariation, that is to judge the influence of varied parameters in a Bayesian situation on an independent variable such as the PPV, as part of Bayesian reasoning supported by four different visualisations combined with natural frequencies (Fig. 3). In this regard, we investigate two questions.

First, we investigate, which visualisation is appropriate to support judgements about covariation in a Bayesian situation. For this, our hypothesis (H1a) is that a simple tree diagram should be least supportive and the double-tree and unit square should be better as they both outperform the simple tree diagram regarding calculation (Böcherer-Linder & Eichler, 2019) and also regarding covariation at least for the unit square (Böcherer-Linder et al., 2017). Additionally, the area-proportionality could be

particularly supportive for covariation (compare Büchter, Steib, et al., 2022). For this reason, a superiority of the unit square over the 2x2-table may be expected. However, since Böcherer-Linder and Eichler (2019) found a significant superiority of the 2×2 table over the unit square for calculation, we explore differences between a unit square and a 2x2-table (H1b) for covariation without a directed hypothesis.

Secondly, in Böcherer-Linder et al. (2017) people's performance of covariation varied descriptively regarding the affected variables when the base rate was varied. The descriptive results yielded the best performance for estimating changes of the PPV. However, the differences were not explored statistically. Therefore, our second research question investigates, if the performance of covariation varies when different dependent variables of a function, that is the PPV or other probabilities of a Bayesian situation are analysed. We hypothesize (H2) that the performance should be higher for estimating changes of the PPV than for changes of other variables (i.e. sensitivity or NPV) when varying the base rate.

STUDY DESIGN AND METHOD

The sample consists of 221 undergraduates at the University of Kassel (Germany) who were enrolled in a course of mathematics education for primary schools. This course does neither include the four visualizations (Fig. 3) nor Bayes' rule in the curriculum.

The students worked on two Bayesian contexts with two tasks each. One of these contexts is the fictitious medical context given in Fig. 2. In the first tasks the students were asked to calculate the PPV in the Bayesian situation. In the second task which is the main focus in this paper, the students answered the three items about covariation given in Fig. 4. Each Bayesian context was visualized with one of the visualizations and the same visualization was used in both contexts (cf. Böcherer-Linder et al., 2019).

In another population the proportion of people who have the disease among the 1000 people is higher. Indicate the correct proposition:

Item 1: The proportion of the negatively tested people among those people who have the disease

☐is higher ☐ is lower ☐ remains the same

Item 2: The proportion of people having the disease among the positively tested people

☐ is higher ☐ is lower ☐ remains the same

Item 3: The proportion of people not having the disease among the negatively tested people

☐ is higher ☐ is lower ☐ remains the same

Figure 4: Second task on covariation with three items in one of the Bayesian contexts

Item 1 refers to a change of the sensitivity, item 2 refers to a change of the PPV and item 3 refers to a change of the NPV each based on the change of the base rate. For

item 1, the third option is the correct answer (remains the same), for item 2, the first option is the correct answer (is higher) and for item 3, the second option is the correct answer (is lower).

To introduce the visualizations for students, we gave a brief description of the visualizations on the front pages of the questionnaires (cf. Böcherer-Linder & Eichler, 2019). In addition to the introduction, we had no further intervention.

For analysing the data, we estimated a generalized linear mixed regression model with a maximum-likelihood method (Hilbert et al., 2019). To test hypotheses H1a and H1b about the impact of the visualization and also H2 about the impact of the items, we defined the visualizations and the items as fixed factors and the participants and the context as random factors. We used the free software package R for the analysis.

RESULTS

On a descriptive level, our results are shown in Fig. 5, where the proportions of correct and incorrect answers are given for each item and each visualization.

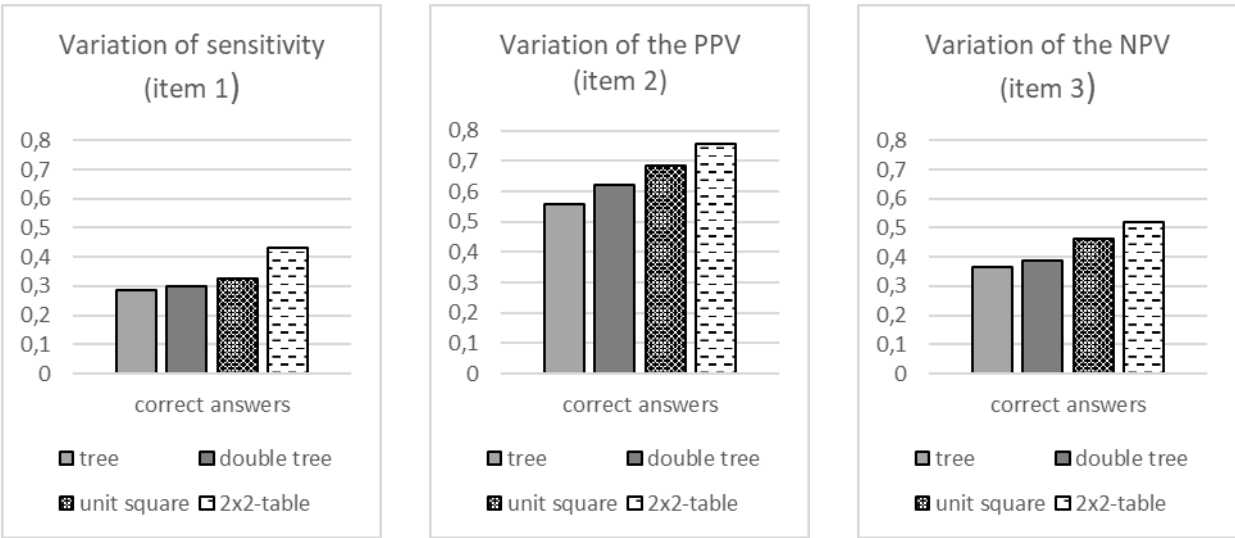

Figure 5: proportions of (in)correct solutions for each item and each visualization

A linear mixed model compares different subgroups of the whole sample to a reference group. As our hypotheses H1a and H1b both address comparisons with the unit square and H2 states that performance should be best for Item 2 (about the PPV), we set the reference group of the linear mixed regression model to the given answers with a unit square about Item 2 (the PPV). Then, all other visualizations (double-tree, tree diagram and 2×2 table) as well as the other items (Item 1 about the sensitivity and Item 3 about the NPV) are implemented as fixed effects. The interactions of the effects are also investigated. Results for the main effects are shown in Tab. 1.

Table 1: Estimation of the main effects in a generalized linear mixed model

Fixed effects	Estimate	Std. error	z-value	P(>|z|)
Tree diagram	-0.61	0.32	-1.90	0.057
Double tree	-0.31	0.33	-0.95	0.341
2x2-table	0.35	0.35	1.00	0.316
Item 1 (Sensitivity)	-1.71	0.31	-5.45	<0.001
Item 3 (NPV)	-1.05	0.30	-3.46	<0.001

All interactions show no effect. The probability for the comparison of a unit square and a tree diagram regarding performance of covariation in Item 2 is slightly above .05. Thus, the unit square tends to outperform the tree diagram referring to covariation. With the interactions not being significant this can also be assumed for the comparison of unit square and tree diagram for Items 1 and 3. The comparison between a double-tree and a unit square does not show a significant difference for Item 2, but the negative estimate for the regression coefficient indicates a descriptively lower performance of the double-tree compared to the unit square. Therefore, we ran a post-hoc Chi-squared test for the comparison of double tree and tree diagram, which did not yield a significant result. For this reason, hypothesis H1a is partially confirmed regarding the unit square, but cannot be confirmed regarding the double-tree. Regarding H1b, we did not find a significant difference between a unit square and a 2x2-table although the results yield a descriptive advantage for the 2x2-table. Yet, a post-hoc Chi-squared test is significant, suggesting a superiority of the 2×2 table over the tree diagram.

Finally, people working with the unit square struggled significantly more, if the influence of the base rate on the sensitivity or the NPV is regarded compared to the influence of the base rate on the PPV, which is provided by the significant estimates for these fixed effects. Further, the non-significant interactions suggest that this different performance between the items is similar among all other visualizations. It is noteworthy, that people were able to solve item 2 significantly better than by guessing (by guessing a proportion of 1/3 is expected), whereas there is no significant difference between the observed performance for item 1 and guessing.

DISCUSSION

First of all, hypothesis 1a about the unit square was partially confirmed. The constraint refers to the p-value concerning the difference between a tree diagram and a unit square that is slightly above .05. However, the results imply in addition to the study of Böcherer-Linder and Eichler (2017) a supremacy a 2x2-table over tree diagrams. This result could be interpreted as a supremacy of a style of visualisation, i.e. the nested style (2x2-table, unit square; Khan et al., 2015), over an alternative style of visualisation, i.e. the branch style (tree diagram; Khan et al., 2015). However, a double-

tree diagram seems to have characteristics, e.g. the amount of numerical information, which may balance the differences between a nested style and a branch style, as the descriptive advantage of the double-tree over the simple tree diagram may suggest even though this difference is not significant.

We did not find a supremacy of the unit square over a 2x2-table, although a plausible hypothesis is that the area-proportionality of the unit square may have an effect on people's covariational reasoning. By contrast, on a descriptive level, a 2x2-table supported people's covariational reasoning more effectively than a unit square. We assume two reasons for this phenomenon. First, a 2x2-table is a common visualisation in school and university (e.g., Veaux et al., 2012; cf. Büchter, Eichler et al., 2022). Thus, the familiarity with a visualisation may impact people's ability to reason on the basis of a visualisation. Further, it may be based on the difference of performance regarding calculation with a 2x2-table, as people's ability to calculate a PPV was found to be higher with a 2×2 table compared to people who used a unit square (Böcherer-Linder & Eichler, 2019). Also, the ability to calculate a PPV in a Bayesian situation could be a predictor for people's ability to judge covariation, particularly concerning the PPV. However, both suggestions must be further investigated.

Finally, people's performance of covariation seems to be strongly impacted by the specific variable that should be investigated with variations of the base rate in a Bayesian situation. Thus, not only the particular supportive strategy but also the situational characteristics of the specific tasks in a Bayesian situation strongly impact people' performance of covariation (also compare Johnson & Tubau, 2015 for a similar observation regarding calculation). For this reason, covariation in Bayesian situation could be a field for further research.

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PRE-SERVICE TEACHERS' CURRICULAR NOTICING WHEN PROVIDING FEEDBACK ON PEERS' LESSON PLANS

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This paper reports an investigation into what 13 pre-service teachers (PTs) enrolled in a secondary mathematics methods unit in one university noticed when providing written feedback on peers' lesson plans. Drawing on the first two elements of the curricular noticing framework (Dietiker et al., 2018), we identify aspects of the lesson plans the PTs comment upon in their feedback (attending) and how they make sense of those aspects (interpreting). Results highlight three main themes of PT noticing from the lesson plans: the pedagogical approach, the nature of tasks, and the learning intentions. We discuss how the context of the methods unit and the design of the activity impacted the PTs' feedback.

INTRODUCTION

Recent years have seen a growing interest in research about mathematics teacher noticing. In part, this is because the ability to notice salient features of a lesson and make instructional decisions based on what is noticed is regarded as an indicator of quality teaching (Bastian et al., 2022). The focus on teacher noticing also recognises the complexity of teachers' work, particularly deciding if, when, and how to attend, interpret and respond to students' mathematical thinking amid the multi-dimensional interactions that occur in classroom settings (Sherin & Star, 2011). There is also general agreement among researchers that teachers can be taught to improve their noticing ability (van Es & Sherin, 2008). Studies have investigated the role of, for example, learning tasks (Ivars et al., 2019), interviews (Lesseig et al., 2016), video (van Es & Sherin, 2008), and student work samples (Simpson & Haltiwanger, 2017) in developing teachers' noticing skills.

Jacobs and Spangler (2017) differentiate three approaches to conceptualising noticing: a sole focus on how teachers attend to incidents or ideas (e.g., Star et al., 2011); others combine teachers' attending to how they also interpret what they notice (e.g., van Es, 2011), though most researchers also include teachers' responses through their decisions or reasoning about what is noticed and interpreted (e.g., Kaiser et al., 2015; Lee, 2021). So, although teacher noticing has been described in a variety of diverse ways, the conceptions typically include "attention to and interpretation of students' thinking, resulting pedagogical decisions, and the relation between those interpretations and broader principles of teaching and learning" (Amador et al., 2021, p. 1).

Conceptualisations of noticing have also moved beyond classroom interactions to include other teaching-related contexts (Dindyal et al., 2021) such as planning (Lee & Choy, 2017) and reflecting on a lesson (Choy et al., 2017). Consequently, there has also been an expansion of the actions and materials which are the focus of research

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studies of teacher noticing. These include, for example, professional noticing of children's mathematical thinking (Jacobs et al., 2010), curricular noticing, which includes how teachers interpret lesson objectives (Dietiker et al., 2018), and productive noticing when teachers design tasks as part of lesson planning (Choy, 2016).

PT NOTICING

It is generally recognised that experienced teachers and novice PTs notice aspects of classroom instruction differently (Bragelman et al., 2021) since PTs often fail to discern the most relevant and significant aspects of a learning situation being distracted by superficial details (van Driel et al., 2021). For example, Lee and Choy (2017) report that during the planning stage of a Lesson Study, PTs often focus on aspects of classroom management and organisation rather than examining the mathematical content or the pedagogical approaches of the lesson.

Even when PTs attend to meaningful incidents, they can struggle to interpret them, so teacher educators must help PTs learn to notice (Earnest & Amador, 2019) through authentic activities that reflect teachers' work (Dindyal et al., 2021). Lee (2021) cautions that using videos of PTs' or other teachers' lessons, or of task-based student interviews, might be too complex for PTs and make it more difficult for them to selectively attend to noteworthy elements. Instead, Lee recommends reducing background distractions to foster PTs' ability to notice from videos.

One way to achieve a more controlled situation is through what Grossman et al. (2009) describe as 'decompositions of practice'. These break down complex practices so that PTs can more effectively recognise and enact elements of practice. An example of a decomposition of practice for PTs which Grossman and colleagues identify is focusing on the elements of lesson planning. Lesson planning is what Gueudet and Trouche (2009) refer to as an authentic task of 'teachers' documentation work' which they describe as "looking for resources, selecting/designing mathematical tasks, planning their succession and the associated time management, etc." (p. 201).

CURRICULAR NOTICING FRAMEWORK

It is important for teachers to capitalise on the affordances of recent curricular reforms by applying the principles of teacher noticing. Dietiker et al. (2018) refer to this work as curricular noticing which they describe in terms of three "strategic and purposeful professional practices that must be learned and developed (p. 524). Curricular *attending* concerns teachers' skills in reading information contained in curriculum materials to inform teaching, where curriculum materials can take a variety of forms such as lesson activities, mathematical content, and teaching advice. For the purposes of our research, we contend that a prepared lesson plan can also be regarded as a curriculum document. This is also consistent with Earnest and Amador (2019) who describe curriculum materials as physical or digital resources for the purposes of guiding instruction.

Curricular *interpreting* is a sense-making activity whereby teachers connect the curriculum information to their own content and pedagogical knowledge for teaching. Teachers' interpreting skill draws on their prior experience, their knowledge of their students, and how they understand the rationale for the design of the materials. Curricular *responding* relates to teachers' decision making that occurs in response to their interpretation of the curricular materials. It includes not only how teachers decide to respond, but also how they enact that in the classroom. In doing so, teachers take account of the affordances of the curriculum and consider any alignment between the curriculum and the kinds of learning experiences they want for their students.

This paper reports an investigation into what pre-service teachers (PTs) notice when providing written feedback on peers' lesson plans. Drawing on the first two elements of the curricular noticing framework (Dietiker et al., 2018), we identify aspects of the lesson plans the PTs comment upon in their feedback (attending) and how they make sense of those aspects (interpreting).

METHOD

Participants and context

The participants for this research were 13 PTs in a two-year, full-time equivalent Master of Teaching (MTeach) degree for secondary teachers at a large, metropolitan university in Sydney, Australia. Entrants to the MTeach hold a bachelor's degree with relevant subject content knowledge for their nominated teaching areas. The participants were all intending to teach secondary mathematics and were undertaking the third and final mathematics methods unit in the first semester of their last year of study. All three mathematics methods units were taught by the first author and adopted a reform-oriented approach based on constructivist learning principles.

Data collection

The final assignment for the mathematics methods unit, called a Joint Lesson Plan (JLP), was a group task based on a modified Lesson Study. PTs worked in three groups of three and one group of four to collaboratively plan a lesson on a given topic from the Australian secondary curriculum. The topics allocated were Networks (Shortest Paths) (JLP1), Networks (Critical Path Analysis) (JLP2), Bivariate Data Analysis (JLP3), and Introduction to Vectors (JLP4). A 500-word 'background section' was also included to provide a description of the target class (real or hypothetical), information about the students' prior knowledge for the topic, the key concepts, skills and attitudes to be developed in the lesson, a rationale for the pedagogical approach taken and how this met the learning needs of the target class, and likely student misconceptions and how the lesson addressed these. The completed JLPs (background sections, lesson plans and accompanying materials such as worksheets) were uploaded to the university learning management system where three or four PT peers were randomly assigned as reactors. Their task was to individually critique another group's JLP and write a 500-word response for them to provide useful and constructive feedback on their JLP along

with suggestions for improvement. The reactor feedback commentaries from the 13 PTs are the data for the present study.

Data analysis

All commentaries were uploaded into NVivo for analysis and we applied an emerging coding scheme to the data. We independently read the 13 reactor commentaries, noting aspects the PTs attended to and how they made sense of them. We then met to discuss the unit of analysis and our coding notes. After reaching consensus, we coded one commentary together and revised our coding scheme. We applied constant comparative analysis, using consensus coding to code all data and kept refining our views until we agreed on the code. For example, one author coded a statement as 'attention to coherence' when the other coded it as 'alignment'. After clarification, we agreed on 'alignment' to signify how the activities addressed the learning intentions set up at the beginning of the lesson plans or how the intentions addressed curriculum outcomes. After emergent coding was done, we applied axial coding to group the codes into themes. For example, the codes related to the approaches used in the lesson plans and the specific ways PTs introduced and developed specific understanding were grouped into 'pedagogy' as they refer to how to teach mathematics. Based on the frequency of the codes and themes, we report here on the items that describe what PTs noticed when reading their peers' lesson plans and how they interpreted them. Given the limited space of this paper, we present the three main themes that emerged from our analysis.

RESULTS

The first main theme that emerged from the data analysis was pedagogy (39 instances). This theme relates to the ways the PTs focused on how the teaching approaches (e.g., constructivist, inquiry-based) (10 instances) were adopted in the lesson plan and the specific approach of teaching those topics, including students' misconceptions and difficulties when learning the topics (12 instances), the choice of representation models and sequencing of the tasks in the lesson (10 instances), and specialised content knowledge (5 instances). For example, JLP3 Andy wrote,

> It [the justification] could be stated why a constructivist/social constructivist approach suits the topic ... because it can engage higher-order thinking which may result in better conceptual understanding.

Andy focused on the approaches that can help develop higher-order thinking without saying specifically how the approaches were used in the lesson. In contrast, JLP1 Eva reacted,

> The abstract model used … is much larger than previously seen networks, which could be too large a leap. Could it be just as effective with more vertices or more edges, still illustrating the point desired that an algorithm is useful? There would also be less set up in the concrete model required by students, saving time. There would be opportunity to increase the size of the network diagram in the following lesson, also on shortest paths.

Eva commented on how the approach to helping students develop an understanding of shortest paths with the specific selection of the number of edges or vertices that will not create "too large a leap" for students.

PTs comments on the nature of tasks used in the lesson plan were prominent (36 instances). PTs focused on how engaging the tasks were (16 instances) and if they included real-life or practical contexts (11 instances). For example, JLP1 Gina wrote,

> Your lesson on Shortest Paths is … engaging, practical, hands-on, and relevant to real life. I particularly admire your use of public transportation, as this is accessible to students of all socio-economic status and relevant to their age demographic. This real-life application will drive the mathematical literacy of the students.

The PTs focused on the use of contexts that are relevant to students, which might keep them engaged and help develop their mathematical literacy. In some instances, the PTs commented on purpose and utility of tasks, which help students see why they need to learn the topic (5 instances). For instance, JLP2 Susan wrote,

> The warm-up activity can be more related to the topic introduced. For example, it can be modified to help students recall certain prior knowledge needed for the new topic. One way is to present a question on finding the shortest path, which they have learned recently. It can also be related to the purpose of critical path analysis.

Although acknowledging the excellent ideas of the warm-up activity, JLP2 Susan suggested that the task should be designed more on purpose to lead to the learning goals of critical path analysis by bringing students' attention to their prior knowledge that links to the current topic. Some others focused on details and thoughts about how to implement the tasks in classrooms.

PTs noticed the learning intentions of the lesson when reading their peers' lesson plans (24 instances). This theme relates to how PTs commented on the alignment of the learning intentions with the tasks as well as how they address curriculum outcomes. JLP1 Gina commented,

> I also think it's important to consider how the learning intentions are used. Whilst they are valuable for both students and teachers to direct the focus of the lesson, this is only if they are addressed later. In the mathematics lessons I have observed and taught, it is best if the learning intentions are used like a checklist which the teacher refers to at the end of the lesson, so the students can see their success (visible learning).

Here Gina focused on whether the learning intentions are used for both teachers and students to direct the lesson flow. In other instances, the PTs noticed the nature of the learning intentions, especially higher-order thinking. For example, JLP4 Roger commented on the critical and analytical skills addressed in the lesson.

> Working in groups can help motivate and encourage students to learn the content more effectively together and be on the same page when later doing individual questions. Group work can also help develop critical and analytical thinking skills whilst helping struggling students reduce the strain and stress when tackling difficult mathematical problem.

DISCUSSION

This study examined what PTs notice from their peers' lesson plans. The results highlight three main themes they attended to and made sense of: the pedagogical approach, the nature of tasks, and the learning intentions. Interestingly, pedagogical approaches used in lesson plans received more attention from PTs than issues about classroom management and organisation reported in previous studies (Lee & Choy, 2017). The difference might be explained by the inclusion of the background section in the JLPs, where the groups provided information about their target class and justified their teaching approaches. This writing might have stimulated PTs' attention to pedagogical aspects of the lesson.

PTs' noticed and commented on the nature of tasks, especially on their practicality, real-life contexts, and whether the tasks were engaging. This result can be partly explained by the assigned topics which were chosen for their potential link to real-life contexts (e.g., networks, statistics). Also, a major focus of the mathematics curriculum units was to make mathematics more engaging for school students. Research has also highlighted the crucial role of tasks in student learning (e.g., Stein & Lane, 1996).

PTs' feedback also focused on the learning intentions, especially the comprehension of the intentions, how they were aligned with the mandated curriculum, and their coherence with the learning tasks and activities used in the lesson. This focus of noticing is encouraging as is suggests PTs were aware of explicitly identifying learning intentions for students and using them to guide lesson planning. This reinforces the research interest in learning intentions as a focus (cf. Spitzer & Phelps-Gregory, 2017) of teacher noticing.

This study contributes to research on PT noticing by shifting the object of noticing from more commonly used sources such as videos and student work samples to lesson plans. We think that lesson plans provide an authentic decomposition of practice (Grossman et al., 2009) for PTs while avoiding the diversions that often beguile PTs in classroom settings. Lesson plans can also be viewed multiple times but without the distractions that videos can sometimes hold for PTs (Lee, 2021). Peer lesson plans might also allow PTs more scope for noticing and critique since the work of their peers is more accessible and open to suggestion than that derived from experienced colleagues or in published curriculum materials.

CONCLUSION

The results of our study suggest that lesson plans, particularly created by peers, can be a fruitful source of curriculum noticing for PTs through *attending* and *interpreting* (Dietiker et al., 2018); *responding*, though not reported here, is also evidenced in PTs' suggested improvements for the lesson. In the JLP assignment, PTs were co-creators of lesson plans and reactors to their peers' plans, and we intend to investigate if adopting these dual roles might impact what PTs choose to focus on in their feedback. We surmise that their concurrent experience of planning might encourage them to be more thoughtful in their reactor comments to provide feedback commensurate with

what they would want themselves. Future research could also compare how PTs notice peers' lesson plans with and without doing their own lesson planning.

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PROSPECTIVE MATHEMATICS TEACHERS' LEARNING THROUGH GENERATIVE METAPHORS

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Generative metaphors are useful to support meaningful changes in teachers' thinking and practice. This paper reports on a qualitative study that explored whether prospective secondary mathematics teachers held pre-existing metaphors or could create metaphors for mathematics that are generative or potentially generative to support their learning of inquiry-based mathematics pedagogy. Findings indicated that while they did not hold pre-existing metaphors for mathematics, they were able to create descriptive metaphors with potential to be generative if they are helped to identify and explore meaningful attributes of the metaphor domains in teacher education.

This paper builds on the idea that metaphors can play an important role in teachers' development or growth and in framing their practice (Chapman, 2017; Tobin, 1990). Tobin explained that metaphors can help teachers make sense of their beliefs and encourage reflection which can lead to improvements in their practice. He suggested that "significant changes in classroom practice are possible if teachers are assisted to understand their teaching roles in terms of new metaphors" (p. 123). Generative metaphors in particular are considered useful to support meaningful shifts in teachers' thinking depending on the metaphor. Schön (1979) explained that we hold *certain pervasive, tacit generative metaphors and we ought to become critically aware of them. Thus, if* prospective mathematics teachers [PMTs] *hold metaphors on entering teacher education, understanding them from a generative perspective could provide a basis to support their learning through them. This paper reports on a study that explored the initial metaphors secondary school PMTs held or created for mathematics and the potential of the metaphors to be generative of a perspective of mathematics that aligns with an inquiry-based perspective of teaching.* It considers the nature of the metaphors' target domains, relationship between mathematics and the target domains, the generative perspective of the metaphors, and implications for teacher education.

USE OF METAPHORS IN RESEARCH AND TEACHER EDUCATION

Metaphors have been used in a variety of ways in studies of teachers and teacher education. Saban (2006) identified ten of these ways consisting of metaphor as: a blueprint of professional thinking; an archetype of professional identity; a pedagogical device; a medium of reflection; a tool for evaluation; a research tool; a curriculum theory; a mental model; an instrument of discovery; and a springboard for change. These ways are related to the perspective that metaphors provide useful windows into teachers' professional thinking and cognition. Teachers' knowledge, when expressed

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metaphorically, could communicate meaning that is difficult to access through literal language (Carter, 1990). Research on prospective teachers have indicated that the use of metaphors offers opportunities for them to articulate, explore, examine, gain insight into, and understand their education-related pre-conceptions, beliefs, experiences, and emerging professional identities (e.g., Buchanan, 2015; Casebeer, 2015; Eren & Tekinarslan, 2013; Lynch & Fisher-Ari, 2017). Metaphor creation and analysis could help them to frame and deeply understand their own ideas about teaching and learning (Massengill & Mahlios, 2008; Saban, 2010).

The few studies specific to PMTs have used metaphors as a tool for examining preservice elementary teachers' beliefs about mathematics teaching and learning (Reeder et al., 2009); in mathematics teacher preparation (Noyes, 2006); to investigate PMTs' thinking about mathematics (Erdogan et al., 2014; Güler et al., 2012) and to address mathematical identity (Latterell & Wilson, 2017). So studies on PMTs have not addressed the generative aspects of metaphors, which is the focus in this paper.

GENERATIVE METAPHORS

Generative metaphors (Schön, 1979/1993) or structural metaphors (Lakoff & Johnson, 2003) facilitate a process by which we gain new perspectives on the world: a process that involves generating or structuring one domain in terms of another. According to Schön, a generative metaphor is characterized by the mapping of frames or perspectives from one domain to another and allows for frame re-structuring when conflict exists. It uses experiences directly appropriate for one domain as a lens for seeing another, that is, seeing domain A as domain B where A and B had previously seemed to be different things, which requires a restructuring of perception to see A as B. Thus, a generative metaphor generates perceptions or explanations of new features of a domain or give rise to a new view of it. But Schön explains that not all metaphors are generative. Some simply capitalize on existing ways of seeing things. In this paper these will be considered as descriptive metaphors.

Given the nature of generative metaphors, they have the potential to help in changing beliefs about mathematics to impact practice positively. There is a lack of information regarding PMTs' ability to construct generative metaphors, to which this study contributes. The study focused on creating generative metaphors for mathematics since it is well established that there is a direct relationship between beliefs about mathematics and teaching mathematics and these beliefs could be difficult to change. Generative metaphors have the potential to help PMTs to make changes to their beliefs about mathematics in a way that could support inquiry-based teaching depending on the attributes of the metaphors. For example, by relating mathematics (the base domain) to attributes of a target domain, they could explore the meaning of mathematics in new ways. This would require them to choose a target and attributes of the target domain, which must include at least one attribute that is different for them regarding the relationship to mathematics and prompts exploration of attributes of mathematics that could result in a shift in perspective about mathematics.

RESEARCH METHODS

This initial stage of a project that investigates supporting PMTs' learning through metaphors addressed the research question: Do PMTs hold pre-existing metaphors for mathematics or can they create initial metaphors for mathematics that are generative or have the potential to be generative in relation to framing inquiry-based practice?

Participants were 65 secondary school PMTs enrolled in a post-degree Bachelor of Education program. Most of them had a degree in mathematics, while the others had an engineering or physics degree with the minimum requirement of mathematics courses for the program. They were in semester 2 of the 4-semester two-year program.

Data collection was conducted at the beginning of the first class of the PMTs' first mathematics education course to obtain a baseline of what they were able to do. Participants were first asked to think about how they viewed mathematics as a discipline and to respond to: (1) *What is mathematics to you?* (2) *What view of mathematics do you want your (future) students to have at the end of your (secondary school) mathematics courses?* They were then asked to respond to: *Mathematics is like _[something]_ because* _____. They were told that the *something* could be anything that made sense to them and the *because* should explain what about it and mathematics were considered and how they were related. They were also asked to indicate whether they already had a metaphor and to briefly describe their experience in creating one including whether it allowed them to think differently about mathematics. In this paper, the focus is only on the metaphors and experience in creating them.

Data analysis included: (1) Identifying and categorizing the targets (i.e., what math is compared to) of the metaphors, which produced seven categories that showed the diversity of the metaphors and the PMTs' thinking. (2) Identifying attributes of the base domain (mathematics) and the target domains for each metaphor. This involved examining the explanations or descriptions of the domains (i.e., their "because …") for attributes of the domains. For example, attributes for the target domain of "skyscraper" were "solid foundation" and "different levels" and for the mathematics domain "everything builds on the previous concept." The attributes were highlighted and compared for patterns across the metaphors. This resulted in the metaphors being grouped into four categories based on whether the attributes focused mostly on: the mathematics domain, the target domain, mapping of both domains, or solving problems. Each group was then interpreted from a generative perspective, e.g., whether the attributes for each domain were explicitly mapped to each other, the attributes for mathematics aligned with inquiry-based teaching/learning, and the targets had potential to generate an inquiry-based perspective of mathematics, e.g., mathematics as a way of thinking (mathematical thinking, inquiry, problem solving), as conceptual structure (e.g., patterns and relationships), and as authentic real-world applications. (3) Examining the PMTs' written responses of their experience creating the metaphors to determine pre-existing metaphors, shifts they perceived in their thinking about mathematics that related to generativity of the metaphors, and learning from the task.

FINDINGS

From the PMTs' perspective, they did not have a pre-existing metaphor for mathematics, which suggested that they did not hold their conception of mathematics as a metaphor. This was the first time they were asked to think of one, which many found to be challenging because it was also the first time they had to explicitly consider what is mathematics. Creating the metaphor allowed them to think about what they knew about mathematics based on how they experienced it, particularly at the school level, and then to try and relate it to something (the target). This process did not generate a shift in their perspective of mathematics but awareness of some aspects of their view of it that stood out for them. Thus, their metaphors were mostly descriptive than generative and suggested what they were able to do in creating them and their thinking about mathematics prior to taking any mathematics education courses. As presented next, there were limitations in their identification of attributes and interpretations of the domains of the metaphors, which affected their meaningfulness to generate a perspective of mathematics that aligns with inquiry-based mathematics pedagogy.

Target domains

While the PMTs did not have pre-existing metaphors for mathematics, all, but one, of them were able to create a metaphor with base domain (mathematics) and a target domain. Based on their explanations of the domains, the targets were grouped in seven categories: object (largest group), place, the arts, activity, person, language, and miscellaneous. The following are examples of the targets for each category.

- Object: wrench, Swiss army knife, house, tree, heart, book, road map, car
- Place: foreign country, river, mountain, staircase
- The Arts: art, work of art, music
- Activity: a puzzle, dancing, cooking, baking, riding a bike, hockey
- Person (real/imagined): hockey dad, ghost, bogeyman
- Language: language, spoken language, alphabet
- Miscellaneous: journey, epiphany, new day, never ending

These targets show the diversity and personal orientation of the PMTs' thinking. Many of these targets could provide a basis to create generative metaphors for viewing mathematics by identifying attributes of them that could be used to explore the nature of mathematics. However, the PMTs' use of these targets in their metaphors lacked meaningful attributes as reflected in their explanations of the two domains of the metaphor and relationships between them.

Attributes and Relationships of Domains

The PMTs' explanations of the meaning of the metaphors (i.e., their "because") were effective in conveying their pre-existing ways of thinking about mathematics, but they did not consider or address attributes of the targets in a way for the metaphors to be generative. The following four categories of examples of the targets and explanations of the domains highlight some of the limitations in relation to generativity for math.

Category 1 – focus on mathematics domain, e.g., mathematics is related to:

- *Journey* because as a teacher you can use mathematics to explore history, to look into the future or to examine the present.
- *Art* because how [math] flows when it works. An example would be proving a theorem. Every line flows beautifully and every term is connected.
- *Dancing* because you can really get into [math] with your mind, body, and spirit and let it lead you to new conclusions.
- *Music* because portions of math are indeed very strict and logical and some areas are more "open to interpretation" and different approaches.
- *Spoken language* because [math] is an abstract language that is used to define and express ideas about reality.
- *A book* because [math] contains knowledge that had been built up over centuries by hundreds of thousands of individuals. You can use it when needed, but rely on it too much and you may miss other valuable perspectives.

This category consists of metaphors with explanations that focused more on describing the mathematics domain than the target domain. Attributes of the target domain are not explicitly or meaningfully considered, which is important for generativity. The metaphors also seemed unnecessary since the description of mathematics could stand by itself without the target. The metaphors have potential to be generative with clearer and deeper exploring of attributes of the target domains to prompt further exploration of their understanding of the mathematics domain regarding the usefulness of mathematics and doing mathematics that seem to be the underlying focus of their thinking.

Category 2 – focus on target domain, e.g., mathematics is related to:

- *A tree* because it grows from common roots but has many branches.
- *Wrench* because it is a tool which holds all other bits tightly in place.
- *Work of Art* because there is beauty that exists, but it can take a lot of understanding and contemplation to see it.
- *Swiss army knife* because it is really useful once you know how to use the different parts.
- *Hockey dad* because he never fails to answer a question, organize anything that's out of order and explain the number of wins and losses.

This category consists of metaphors with explanations that focused more on the target domain than the mathematics domain. They explicitly addressed at least one attribute of the target domain (e.g., for tree, roots and branches; for wrench, tool and holds in place; for art, beauty). However, the implied mapping with mathematics is ambiguous regarding whether it has identical attributes since no explicit or appropriate meaning for mathematics is considered. More clarity of the mapping of the mathematics domain to the target domain attributes is important for generativity. Thus, these metaphors have potential to be generative with deeper explorations of the attributes of the target domains and particularly the mathematics domain regarding the nature of mathematics, its utility, and doing mathematics that seem to be the underlying focus of their thinking.

Category 3 - focus on mathematics and target domains, e.g., mathematics is related to

1. *Music* because music notes have sound and symbol. When combined they form rhythm, structure and a type of completeness. Similar to math, if a phrase or calculation is missing, the piece or theory may fail as a coherent composition or provable theory.

- *Language* because it defines an alphabet, namely, numbers, rules of grammar, for example, algebra is used to relay or express information. The information is based in fact, that is, were the information not based in fact, the information could not be used in building larger concepts without greatly putting the integrity of that information at risk.
- *A skyscraper* because we start with a solid foundation and build different levels from there as in math. Each level in math represents a new concept or idea. Everything builds on the previous concept. Everything is interrelated and depends on the previous level.

This category consists of metaphors with explanations that explicitly address both math and target domains. They explicitly state attributes of the target domain (e.g., for music, sound, symbol, rhythm; for language, alphabet, grammar; for skyscraper, foundation and levels). There is explicit mapping to mathematics through specific ways of viewing it that demonstrated limitations or misconceptions in their understanding of it and lack of generativity. But the metaphors have the potential to be generative depending on the choice of attributes, e.g., structure is an attribute for all three of the examples that could generate exploration of mathematical structure in considering nature of mathematics.

Category 4 – focus on solving problems, e.g., mathematics is related to:

- *Road map* because there are so many different directions one can go when trying to solve a problem. There are many different paths to get to a location, some being more direct than others. Sometimes new roads and paths can be accidentally created or discovered.

  Alphabet because you need to be able to have basic skills before you can form a sentence or solve a problem. You need to be able to hold a pen, write letters, compose words and so forth until you can finally write a sentence or story. In math you need to be able to read a problem, understand what the numbers and symbols represent and the steps involved before you can solve it.

  Cooking/baking because you must learn the concepts, and then you can repeat the process by applying them to other questions (like recipes). Although you may follow the steps or directions, the answer (final product) is not always what you expect or right.

  Puzzle because you have to piece together the right information, tools, and knowledge to solve a problem
- *Staircase* because it is filled with steps to get to your answer (or in the case of stairs, a door of your destination).

This category consists of metaphors with explanations that directly or indirectly

address attributes of the target domains that are related to solving problems mostly as an algorithmic activity in mathematics. Except for road map that suggests alternative approaches to a solution, the other targets are tailored to an algorithmic view of problem solving and are unlikely to be generative regarding a view of problem solving or mathematics from an inquiry-based or mathematical thinking perspective.

CONCLUSIONS

The findings suggest that PMTs may not hold pre-existing metaphors for mathematics or may not able to access them because, as Schön (1979/93) suggested, they could be tacit and thus not easily accessible. The participants were able to create descriptive metaphors (ones that capitalize on existing ways of seeing things) that provided a snapshot of their thinking about mathematics. The metaphors presented a view of mathematics that did not align with an inquiry-based perspective of learning/teaching it.

While creating the metaphors allowed the PMTs to think about something that stood out for them in how they viewed mathematics, it did not allow for the metaphors to be generative mainly because of lack of depth in considering attributes or meanings of attributes of the domains. As Ashton (1994) stated, an "essential feature of metaphor is that it demands the interpreter becomes actively involved in searching for meaning. This is done by seeking for [attributes] that the two parts of the metaphor have in common in order to share insight" (p. 358). The PMTs did not demonstrate this level of engagement in searching for meaning and seeking attributes to share insights. They chose attributes of the target that allowed them to express or map a traditional classroom view of mathematics they already held. This suggests the need for intervention to support their creation and use of generative metaphors. Many of their metaphors have the potential to be generative if they are helped to consider attributes and meanings of them that will require them to search for new meanings of mathematics.

The study suggests that mathematics teacher education could benefit from engaging PMTs in creating metaphors to support reflection on and shifting perspective of mathematics and mathematics pedagogy. For example, their initial metaphors could be used to (i) articulate and reflect on their initial thinking of mathematics; (ii) form a base line for their learning that they could return to during a course to critique and revise in terms of the domain attributes as they learn more about the nature of mathematics; and (iii) be the basis of intentional intervention to make them generative by exploring other attributes of the target domain and the mathematics domain as in my ongoing project.

The paper provides examples of metaphors that other PMTs could critique and revise to support their learning. It also offers categories of the metaphors that could form the basis of a framework for use in working with PMTs or in designing future research. Future research could explore PMTs' end-of-course or program metaphors and the impact on their actual future teaching. It could investigate practicing teachers' metaphors to identify those that generate inquiry-based teaching that could be used as examples in PMTs' education/learning and practicing teachers' development.

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UNDERGRADUATE STUDENTS' UNDERSTANDING OF THE CONCEPT OF DERIVATIVES IN MULTIVARIABLE CALCULUS

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In this study, we interviewed 8 undergraduate students about their understanding of the concepts of derivatives in multivariable calculus. Students used single-variable functions, multivariable functions, and multivariable vector-valued functions as process-object layers and showed geometric, symbolic, verbal, and linear approximate representations. Also, students related, formed, and extended their understanding within a representation by generalizing it along the process-object layer. We observed that the clearer the generalizing actions were, the more students showed a structural understanding in the context of the concept of derivatives in multivariable calculus. These results contribute to college calculus education.

INTRODUCTION

The importance of the concept of derivatives is not only limited to single-variable calculus but also applies to multivariable calculus. From pure mathematics to natural and social science, the concept of derivatives for multivariable calculus is essential when dealing with continuum and rate of change with plural variables. Therefore, we can say that examining the understanding of the concepts for multivariable derivatives of undergraduate students who need to study mathematics in their major, becomes a major goal of college mathematics education. The concept of derivatives in multivariable functions learned in college calculus courses, is a generalized extension of that in single-variable functions, but students struggle with learning such contents (Trigueros, & Martinez-Planell, 2010).

In this paper, we explore the process of how undergraduate students represent and construct the meaning of the concepts of derivatives in multivariable calculus. And based on the research about the understanding of the concept of derivatives in single-variable calculus, we intend to analyze that in multivariable calculus in respect of the representational contexts and generalizing actions seen by students. As a result, this paper tries to offer a theoretical framework for how undergraduate students understand the concept of derivatives in multivariable calculus.

THEORETICAL BACKGROUND

Understanding of the concept of derivative in single-variable function

The concept of derivatives requires a comprehensive understanding as they are interrelated with the instantaneous rate of change, tangents, slopes, and limit concepts, while students' understanding of the concept of derivatives is mainly concentrated on a calculation of differential coefficients and slopes (Orton, 1983; Sahin, Yenmez, & Erbas, 2015).

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Zandieh (2000) identified that students' understanding of the concept of derivatives consists of a 'ratio-limit-function' process-object layer, which is both an operational process and a structural object at the same time. And based on the interview responses from students taking AP Calculus BC, Zandieh (2000) classified the following representations or contexts about the understanding of the concept of derivatives; Graphical, Verbal, Physical, and Symbolic. Table 1 below is the two-dimensional framework corresponding to process-object layers and representational contexts.

Process-Object Layer	Graphical	Verbal	Physical	Symbolic
	Slope	Rate	Velocity	Difference Quotient
Ratio				
Limit				
Function				

Table 1: Zandieh's framework for the understanding of derivatives

Many studies have used Zandieh's framework as a fundamental theoretical background for integrated analysis of students' understanding of derivatives, and modified in their way maintaining the two core dimensions; representation and process-object structural understanding (Jones & Watson, 2018; Roundy, Dray, Mangue, Wagner, Webber, 2015). The concept of 'pseudo structural' understanding is about when students understand the concepts only as an object and do not recognize that as a structural process (Sfard, 1991). Zandieh argued that the concept of derivatives should be understood structurally in each representation, not pseudo-structurally, and the process-object structural understanding that occurs in one representation should be linked to another.

Generalization

Generalization is one of the most important components of mathematical thinking (Lannin, 2005), and generalizing action is a core of mathematical activity and a major means to construct new concepts in mathematics (Ellis, 2011). Ellis, Lockwood, Tillema, and Moore (2022) developed a comprehensive framework of generalizing activity called the *Relating-Forming-Extending (RFE) framework* by researching which aspects of mathematical activity become productive in more diverse and challenging areas of mathematical generalization. *Relating* is the identification of similarities across situations, problems, or strategies that the learner perceives as distinct contexts. *Forming* is the development of an initial, sometimes tentative generalization, and *extending* is the use of that generalization, sometimes to a broader domain (Ellis et al., 2022, p. 362). The authors said that the RFE framework has been successfully applied widely in middle school through college in the domains of algebra, advanced algebra, trigonometry/pre-calculus, and combinatorics, especially when the concepts extend their range.

The concepts learned in single-variable functions extend to those in multivariable functions and again, to those in multivariable vector-valued functions, which students have difficulty understanding it. Kabael (2011) and Reed (2018) analyzed the generalizing process of students when extending the function space in multivariable calculus and real analysis, respectively. And it is found that students generalized the concept of the average rate of change, domains-ranges, and the graphs of function for multivariable functions from those for single-variable functions by relating objects expanding and modifying previous understanding (Dorko, 2015; Dorko, Lockwood, 2016; Dorko, Weber, 2014). In addition, researchers claimed that generalization is a useful framework for exploring students' understanding of the concepts in multivariable calculus.

METHODOLOGY

Participants and data collection

We selected voluntary 8 undergraduate students for a semi-structural interview for about an hour and a half. All students were enrolled in first-year multivariable calculus at a university in Seoul, South Korea. They learned about the concepts of derivatives in multivariable functions and multivariable vector-valued functions through the course. Students were legitimately compensated for their participation. To discuss multivariable calculus, the participants need to understand well of high school-level calculus. All of the participants in the study understood school mathematics very well enough. They could explain what derivatives mean and how to calculate differential coefficients in single-variable functions with their own mathematical representations, and could smoothly solve various problems by combining structural understanding.

In the semi-structured face-to-face interview, a series of mathematical problems were given that asked to calculate and explain the meaning of derivatives in several single-variable functions, multivariable functions, and multivariable vector-valued functions in order. Then participants were asked to solve them over time and explain them to the researcher along with the description. In this task-based semi-structured interview, the responses of the students were suitable for identifying similarities from specific cases, forming general rules, or expanding reasoning from previous knowledge. We recorded the video with the consent of the participants. Through the recorded videos, it was able to identify the nonverbal responses like gestures and dynamic meanings in their responses.

Analysis

We observed how the structural understanding shown in the interview is based on the process-concept layer of Zandieh (2000)'s framework from the students' responses. In addition, the contexts of expressions and representations shown by students while explaining the tasks were classified. In other words, from the students' responses, the comprehensive understanding of the concept of derivatives in multivariable calculus was analyzed concerning the process-object layer and representations. In this process, we focus on the reasoning in which students' understanding of mathematical concepts

such as linear approximations, tangent lines, tangent planes, slopes, gradient vectors, the instantaneous rate of change, Jacobian matrices, and the context in which the answers are connected. We divided the students' responses by representations and process-object layer into 3 levels: structural understanding, pseudo-structural understanding, and lack of understanding.

And we applied the RFE framework of Ellis et al. (2022) to identify the generalizing actions of students during the interview. Following the actor-oriented data collection methodology (Ellis, 2007), we paid attention to the similarity and commonality of the representation in the responses, regardless of the mathematical or logical accuracy. We observed the contexts in which the process-object layer of the generalizing actions shown by students was made and apply it to Zandieh (2000)'s framework. Through this, the relationship between structural understanding and mathematical generalizations for each representation was derived.

RESULT

From the interviews, we observed that students recognize single-variable functions, multivariable functions, and multivariable vector-valued functions as both structural objects and operative processes concurrently in multivariable calculus. For example, the concept of derivatives in multivariable vector-valued functions does not only mean the acquisition of the consequential object. It is also obtained through the process of associating it with the derivatives of the functions through partial differentiation of each component of the multivariable function. Thus according to Sfard (1991), the understanding of the concept of derivatives in multivariable calculus is of a *'single-variable functions - multivariable functions - multivariable vector-valued functions'* process-object layered structure.

We classified students' responses by substituting them into newly obtained process-object layers. As a result, students showed the following four representations or contexts for the concept of derivatives in multivariable calculus; (1) *Geometric* representations: contexts related to graphic or geometric objects of the functions like tangent lines, tangent planes, or linear mappings by matrices. (2) *Symbolic* representations: contexts of algebraic expressions and computations like differential coefficients, gradient vectors, or Jacobian matrices. (3) *Verbal* representations: contexts of the instantaneous rate of change for each component or infinitesimal rate of change of volume, and (4) *Linear approximate* representations: contexts that combine the above contexts into the concept of local linearity and linear approximation functions.

Graphic, symbolic, and verbal representations in previous studies were also identified in the context of the responses, but due to the limitations of calculus textbooks for freshmen, physical representations were not come out from students' answers. Meanwhile, the term 'geometric' was used instead of 'graphic' because multivariable vector-valued functions cannot be expressed in a general image of graphs. And since linear approximations require the algebraic calculation of the geometric objects and

have to explain the numerical degree of the instantaneous rate of change verbally, the linear approximate representation was classified into a separate category rather than falling into any existing ones.

	Representations / Contexts			
Process-Object Layer	Geometric	Symbolic	Verbal	Linear Approximate
Single-variable functions	tangent line of the graph of the functions, slope of the tangent line	existence and value of derivative	instantaneous rate of change	local linearity, well approximated by a linear function
	↓	↓	↓	↓
Generalizing actions	RCB, FAF, EC	RCB, FAO, EC, ET	RCB, RT, FAO, FAF, EC	RCB, FAO, EC
	↓	↓	↓	↓
Multivariable functions	tangent plane of the graph of the functions, slope of the tangent plane	existence and value of gradient vector	instantaneous rate of change by components, including directions	local linearity, well approximated by a linear function
		↓	↓	↓
Generalizing actions		RCB, FAO, ERP	RCB, RRE, RT	RCB, RRE, RT
		↓	↓	↓
Multivariable vector-valued functions	linear mapping by the Jacobian matrix	existence and value of the Jacobian matrix	infinitesimal rate of change of volume, the absolute value of Jacobian	local linearity, well approximated by a linear function

Table 2: a framework for the understanding of derivatives in multivariable calculus and generalizing actions used by the students

Students related, formed, and extended their understanding within a representation by generalizing it along the process-object layer. In Ellis et al. (2022)'s RFE framework, the generalizations students shown in the process of solving and answering interview problems of the concept of derivatives were the following; *Connecting back (RCB)*: forming a connection between a current and previous problem or situation, *Recursive*

embedding (RRE): embedding a previous situation into a new one as a key component of the new task, *Transfer (RT)*: influence of a prior context or task is evident in a student's current operating, *Associating operative objects (FAO)*: associating objects by isolating a similar property or structure, *Associating figurative objects (FAF)*: associating objects by isolating similarity in form, *Continuing (EC)*: continuing an existing pattern or regularity to a new case, instance, situation, or scenario beyond the one in which the generalization was developed, *Transforming (ET)*: extending a generalization by changing the generalization to be extended, and *Removing particulars (ERP)*: extending a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally.

Table 2 below is the framework outline and rationale of representations along with the process-object layer of students' understanding of the concept of derivatives in multivariable calculus. The framework also includes the generalizing actions used in each context and the process-object layer that students used during the interviewFor instance, the following Figure 1 and extract are part of one of the student’s responses about the concept of derivatives in multivariable functions.

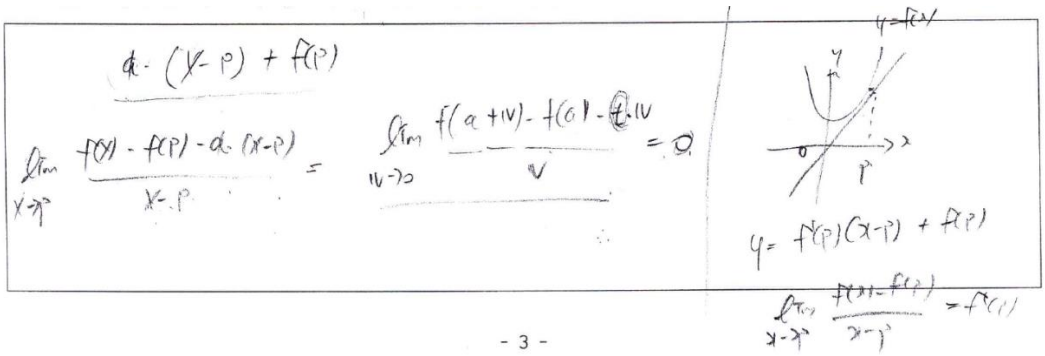

Figure 1: Student’s written answer about the derivatives in multivariable function

Student 5: There was an expression I learned last semester, but I couldn't remember exactly. So I induced it from the equations of the tangent line I know (Write down the right part of Figure 1). Then you can write down a similar expression, and if you make it easier to understand, this line comes out (Write down the left part of Figure 1). It's still zero even if you divide it by limit zero.

Researcher: Is there any word to describe this?

Student 5: Approximation.

In this case, answering the concept of derivatives in multivariable functions, the student related it to what he understood in single-variable functions and connect it back with the tangent situation (RCB). Then he formed expressions of similar structures, which are operative objects that can be associated (FAO). Finally, he extended the domain and continued the previous concepts into a new one (EC). Through these generalizing actions, this student showed a structural understanding of the concept of derivatives in the multivariable function process-object layer in the linear approximate representation.

DISCUSSION

In this study, the representations and process-target layers of Zandieh (2000)'s framework were newly classified, modified, and extended so that to figure out how undergraduate students understand the concept of derivatives in multivariable calculus. Undergraduate students understood the concept of derivatives in the representational context of geometric, symbolic, verbal, and linear approximate along the process-object layers from the single-variable function to the multivariable function, and from the multivariable function to the multivariable vector-valued function. In Ellis (2007)'s actor-oriented perspective, students' mathematical thinking and actions while they showed during the interview could be identified as generalizing actions from Ellis et al. (2022)'s RFE framework.

When students strongly used generalizing actions as behavior and description, we observed that students take the conceptual objects from the previous layer as an operative process so that they could reach a structural understanding of the derivatives. If generalizing actions occurred weakly or did not occur, they could only reach a pseudo-structural understanding or failed to understand it. Based on the research results, we combined the process of mathematical generalization into a framework for the understanding of the concept of derivatives extended to multivariable calculus. This study emphasized the importance of the representational contexts in the research of concepts of derivatives in multivariable calculus. Adding to that, the conclusion about the importance of structural understanding through process-object layers with each representation in a single-variable function, claimed by Zandieh (2000) and the subsequent researchers, is also presented in a further area. In addition, it was observed that generalizing action influenced the structural understanding through process-object layers. The plural process-object layers were related, formed, and extended so that the flexible structural thinking of conceptual objects and operative process aspects of derivatives occurs at the same time.

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SHARED EXPECTATIONS? AN EXPLORATION OF THE EXPECTATIONS BETWEEN PRIMARY MATHEMATICS LEADERS AND TEACHERS

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Primary mathematics leaders are middle leaders working between both school leadership and classroom teachers. They influence teacher development and classroom practice with strategic direction, developing shared goals with teachers. There can be challenges to their role due to the absence of clear guidelines, and the contextual needs within schools. This paper reports on one aspect of a research project examining how primary mathematics leadership is conceptualised and experienced. It discusses interview findings regarding the expectations primary mathematics leaders have of teachers, and teachers' understandings of those expectations.

INTRODUCTION

Primary Mathematics leaders have a formal and significant responsibility for improving student learning, working between, and with, both school leaders and teachers, positioning them as middle leaders (Copping, 2022). A positive relationship has been found between the work that middle (curriculum) leaders carry out and student achievement where curriculum leaders augment school leadership through regular, direct contact with teachers and students (Leithwood, 2016). In Australian primary schools, where teachers are generalists, and middle leaders often also have a teaching aspect to their role, their time both in their own classroom and in the classrooms of other teachers can have a significant impact on classroom practices and educational outcomes (Grootenboer et al., 2015). As such, middle leaders are key to the successful implementation of improved practices in mathematics because they are critical in connecting the vision of the school to the enacted curriculum at the classroom level (Jorgensen, 2016). This reflects the transferral in role from manager/coordinator to leader, shifting the role to a more strategic leadership focus instead of administrative (De Nobile, 2017). Lipscombe et al. (2020) believe the role of middle leaders across Australia lacks clarity. There are no clear policies, hours, expectations, titles, training, or support. While contexts influence the role of the primary mathematics leader in each school, there are challenges as a result of this lack of clarity.

LITERATURE REVIEW

Explorations of the different aspects of the role of middle leaders identify common practices of successful leaders. This includes: a focus on student learning; having and building a clear vision or strategic direction; developing a culture of shared responsibility and trust; fostering teacher learning; and having high expectations of themselves and others (Gurr & Drysdale, 2013; De Nobile, 2017). Dinham (2007) concurred that it was important to foster a constructive and committed team, who would work collaboratively, with a common purpose. Successful curriculum leaders in

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 195-202). PME 46.

schools with exceptional student outcomes sought the input of team members to develop common and shared goals. The team members had ownership of these goals as they had contributed to the development of them. Although there were particular expectations, the team members still felt they had input and control over the shared goals and purpose and had a high level of perceived self-efficacy.

Successful middle leaders have a clear vision and set high expectations for themselves, for teachers and for the students (Gurr & Drysdale, 2013). To achieve their vision, they set clear objectives, provide guidance, and are consistent in their expectations and actions. They model their expectations and create a sense of the importance and worth of their subject area for others (Dinham, 2007). This is supported by Balka et al. who say that mathematics leaders work with teachers to develop a vision for what mathematics will look like in their school. They have a common purpose of what effective mathematics teaching looks like in their school and know what is needed to achieve this vision. The work with the teachers and are committed to the shared goals (2010). And more importantly for this vision for improving mathematics, that it is explicit and understood by all (Bezzina, 2007). The ability to influence and impact student learning and teacher development through those shared goals can be challenging for primary mathematics leaders (Millett & Johnson, 2004; Sexton & Downton, 2014).

Leading is not an isolated practice, done by one person. Improved student learning is a shared responsibility of staff and students within schools (Grootenboer et al., 2015). Therefore, it is important that all involved have a shared understanding of the school's goals and their role in supporting the enactment of that vision. Expectations from school leadership of middle leaders was studied in NZ secondary schools (Bassett, 2016). Expectations for the middle leaders' role were found to focus on three areas. Firstly, student learning through curriculum leadership, such as developing the curriculum, planning learning programs, meeting government curriculum requirements, assessment, assessment moderation, and improving student learning outcomes. The second expectation was Teacher development, consisting of professional development of staff, mentoring, coaching, improving teacher practice, offering guidance for example in report writing, and understanding the pastoral care needs of the staff. The final expectation focused on administration, tracking and recording assessment and student achievement, managing budgets and resourcing, and compliance. The study also examined an interesting tension that middle leaders were simultaneously responsible for developing teachers professionally, but also supervising and monitoring their performance. This could undermine the relational trust they were building with teachers (Bassett, 2016). Yet, as the literature review reveals, the expectations primary mathematics leaders have of teachers, and whether these are shared by teachers, has not been previously explored.

METHODOLOGY

This study uses a representative case study model (Yin, 2017). Research questions for the study are:

- How do mathematics education stakeholders conceptualise primary mathematics leadership?
- What formal and informal structures are available in primary mathematics leadership and what are the affordances and challenges associated with them?
- What are the experiences of the roles and responsibilities that primary mathematics leaders are expected to fulfil?

Individuals from a range of contexts were interviewed to provide insight into how primary mathematics leadership is conceptualised and experienced. Semi-structured interviews were conducted with a targeted, stratified sample representing a range of demographics in eight primary schools within Victoria. Within each school the primary mathematics leader (PM leader) was interviewed, and where possible, a member of school leadership, and two teachers (with different levels of teaching experience) were also interviewed. The schools included State, Catholic and Independent sectors in metropolitan, regional and rural settings. The schools were diverse in size; Index of Community Socio-Educational Advantage (ICSEA), an Australian national ranking of educational advantage; and levels of English as an Additional Language (EAL). This is summarised in Table 1 below for the schools with quotes included in this report, with the school names being pseudonyms.

Table 1: School demographics

School	Wattle Tree Primary School	Banksia Gardens Primary School	Sundew Patch Primary School	Acacia Grove School
Type	F-6 Government	F-6 Catholic	F-6 Government	F-12 Independent
Location	Metropolitan	Metropolitan	Regional	Metropolitan
Student enrolment	Medium 200-400	Large 500-1000	Large 500-1000	Very Large >1000
EAL	>90%	20-30%	<10%	30-40%
ICSEA	Well below average	Well above average	Above average	Well above average
Time allocation for roles	0.6 PM leader 0.4 classroom teacher	1.0 PM leader No classroom allocation	0.5 PM leader 0.5 classroom teacher	0.2 PM leader 0.8 classroom teacher

Only one aspect of the interviews will be focused on in this paper, that is, from the primary mathematics leader and teacher interviewees from four schools. Responses from school leaders have not been included. Primary Mathematics leaders were asked,

"What do you expect from the teachers at your school?" and the teachers were asked, "What does the primary Mathematics Leader expect of you as a mathematics teacher?" Qualitative analysis of the interview data was undertaken using an inductive approach to determine emerging themes (Thomas, 2006). NVivo software was used to organise the data and support analysis. Themes and categories were created, and then summarised, refined and modified based on subsequent analyses (Thomas, 2006). Pseudonyms have been used for all participants in this report.

RESULTS AND DISCUSSION

Examination of the data revealed that most schools had shared expectations for mathematics across several areas. Each school had shared expectations which were clear, explicit and understood across the school (Bezzina, 2007). These were largely related to student learning of mathematics, such as the instructional model, planning, assessment, or catering for diverse learning needs (Bassett, 2016). Also evident were expectations around teacher development of content knowledge, or pedagogical content knowledge. A shared expectation associated with student learning is included from Wattle Tree. Two primary mathematics leaders and two teachers were all consistent in the expectations of the use of a particular instructional model for the teaching of mathematics.

> Zara [F-2 PM leader]: I think that the main goal is really just that, um staff understand how to, you know, they understand the basics in relation to teaching maths, they understand, I mean the, the PD [professional development] that Mei and I have done this term has been all around, um, the instructional model and what the different parts of the instruction model mean.
>
> Mei [3-6 PM leader]: Um, first of all, we have our numeracy instructional model let's say, that's the structure that our school is doing, and I would expect every single teacher to, to follow that instructional model.
>
> Justin [early career teacher]: Well they would expect the consistency across, across the classrooms. And knowledge is the focus
>
> Anna [experienced teacher]: Um, there is an instructional model in place, in terms of how the mathematical, mathematics is expected to run

Zara explained there had been specific training of teachers in the model and Mei reinforced the expectation for every teacher to be implementing the model, which Anna explicitly referenced. While Justin did not mention the model, he discussed consistency across classrooms. This demonstrates in this school a very clear model for mathematics teaching, that has been well communicated (Balka et al., 2010). The primary mathematics leaders have focused on student learning with a consistent instructional model (Gurr & Drysdale, 2013). The time allocation for the leadership roles for both Zara and Mei has provided opportunity to embed the instructional model and impact mathematics teaching across the school. As classroom teachers, they are also implementing the same instructional model and leading practice through their own teaching (Grootenboer et al., 2015)

Banksia Gardens also had shared expectations for student learning as well as another on high expectations, which no other school shared.

> Michelle [PM leader]: Um, I have high expectations, probably too high, and I've got to bite my lip quite a few times… I'm trying to lift the cognitive load of the actual lesson, so it caters to more kids and those rich discussions happen.

> Genevieve [early career teacher]: Um, she's got quite high expectations, I think. Um, which is good. Uh, she, um, is expecting, I guess, …that we're like, you know, using, um like language to deepen the children's understanding…. Because sometimes I know she'll give us tasks, we're like, Oh, my goodness! Like, are they going to be able to do this? But they can. So, I guess her having those high expectations of us then transfers, to us, having high expectations of the children.

> Carmen [experienced teacher]: …and not sort of like pigeonhole kids, but to really open it up and see how far they can go with maths and with their thinking, and really challenge them, but also to support them.

Although Michelle is concerned that she is setting her expectations too high, the impact of those expectations on teachers and students, can be seen in the responses from the teachers, particularly the early career teacher. Genevieve explains how the high expectations of her have extended to the students in her classroom. The example set by Michelle, creates a sense of the elevated expectation for student learning (Gurr & Drysdale, 2013). Students are required to be able to think deeply about mathematics and all staff are expected to support this initiative. The primary mathematics leader has high expectations of not just the teachers, but the students too. Michelle's role as a full-time leader has allowed her to influence the teacher's perceptions and effect a change in teaching approach within classrooms (Millett & Johnson, 2004).

While there were similarities across all schools, there were some interesting differences. At two schools there were no common themes in which all interviewees agreed. At each of these schools there were shared expectations between the leader and the experienced teacher, but the graduate teacher (i.e. at the beginning stage of their teaching career) did not share these understandings. These shared expectations were focused on student learning and teacher development, similar to other schools. For example, at Sundew Patch coaching for teacher development was an expectation which both the leader and experienced teacher shared.

> Michael [PM leader]: My next priority is around coaching… Um, so, I'd come in and do a bit of an observation. I don't like to sit down the back, I'm not the kind of coach who would sit down the back with a pen and paper, and kind of take notes. It was more like, oh I'll come in, and …we'd work together on something. You know, it'd be a bit of responsive teaching.

> Laura [experienced teacher]: Um, the expectation that I'm still learning all the time, and also that, um, he always talks about when he comes in, in that coaching role, he never wants to be like up the front, us watching, and him, or the opposite, him watching and writing about it. He always wants it to be a bit

> of a joint effort… So, I guess it's an expectation is that we are learning together.

While Michael and Laura both see the coaching as a partnership and have a shared understanding of the expectation that they need to work together (Bassett, 2016), this was not clear to the graduate teacher in the school. Although she mentioned coaching, it was not in the context of a partnership, and she found it difficult to articulate the expectations that Michael had of her. Instead, Rachel appeared to have personal goals that she was working towards. Not having shared goals is a recognised challenge for mathematics leaders (Millett & Johnson, 2004).

> Rachel [graduate teacher]: I suppose… well I don't know. I suppose from our working together, so the sort of coaching piece, um, the expectations, was around the goals we set, …and I guess the expectation, the expectation would be to have fun, because that's what I suppose my biggest goal was. I wanted to loosen up and have fun. So I suppose that it was that expectation that I was able to have fun. Um… To be honest, I don't… I could put words out there, but I don't know I haven't asked. I assume… Um… But no, I… [shrugs shoulders]

Michael was new to the role of primary mathematics leader this year. His time allocation for the leadership role was 0.5. He identified time as a challenge, particularly with his additional classroom teaching role. Providing the support needed to graduate or new teachers within a school can be challenging due to their lack of knowledge and practices for teaching mathematics (Sexton & Downton, 2014). Rachel's perspective as a graduate teacher is focused on her personal goals around improving classroom practice. Michael has supported her to create a personal goal which builds on the individual needs of that teacher for the mathematics classroom (Koellner et al., 2011).

Amongst some schools, while there were shared expectations, there were also other instances where the primary mathematics leader had an expectation that was not shared by the teachers, and similarly teachers identified expectations that the leader had not. For example at Acacia Grove, one expectation that the primary mathematics leader had of the teachers went beyond student learning, encouraging an understanding of the value of the subject area.

> Christine [PM leader]: I guess that they're positive about maths, and that they provide their kids with a love for maths to the best of their ability.

This expectation was not noted by either of the teachers at the school. In this case, this was not a shared expectation that could be articulated by the teachers (Bezzina, 2007). While at this school there was some shared understanding of the instructional model and expectations around planning, there was less correlation between responses. Again, Christine was new to the role and had limited time to work with a large staff across multiple campuses. The mathematics leadership role was nominally 0.1, but this was taken as six release sessions throughout the week, rather than as one day. Christine also had a full-time classroom teaching role. Time and the working environment provided a challenge in ensuring the goal was communicated effectively and clearly (Lamb et

al., 2015). This can be seen through Ethan's response to his understanding of the expectations.

> Ethan [graduate teacher]: um I would say that we are, that we are teaching the, the concepts, teaching each of the concepts to our students each week. On top of that, to ensure that we are… improving the students' automaticity and, and fluency of maths, maths number facts to regularly refresh their, or regularly, refresh their memories of previously taught concepts.

While this is a localised response it reflected that Ethan's focus was the teaching of mathematics in his classroom. He was supported in his access to planning documents and resources provided by Christine. The planning outlined the concepts for each week and the documentation for this was consistent throughout the school. The implementation of a whole-school approach had been coordinated by Christine to support all staff in planning and teaching (Bassett, 2016).

CONCLUSION AND IMPLICATIONS

Primary mathematics leaders in Australia have some shared expectations with teachers to support the strategic directions within their respective schools. Although there were areas which demonstrated correlation between expectations, these were primarily related to student learning and teacher development. Most primary mathematics leaders demonstrated they had explicitly shared expectations for student learning, particularly for classroom instruction. Primary mathematics leaders also had expectations beyond this, which were less likely to be understood and shared by teachers. This was more evident for those with a lower time allocation for their leadership role. Graduate teachers frequently revealed a more localised focus, sometimes not understanding the shared expectations of more experienced teachers. There were factors that influenced or challenged the success of the effective communication and understanding of primary mathematics leaders' expectations. These included time allocation for the role and teachers' experience. Implications indicate that time allocation impacts the ability of primary mathematics leaders to fulfill their role and there is a need to ensure that there is adequate time for the size and context of the school. It is also important for primary mathematics leaders to be aware of the needs of graduate and early career teachers, to ensure that expectations are explicitly communicated, and where needed, extra support and guidance is provided.

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MULTIDIRECTIONAL SHIFTS IN ELEMENTARY TEACHERS' MATH TEACHER IDENTITY: UNDERSTANDING THE ROLE OF INSTRUCTIONAL COACHING

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Teacher identity significantly shapes how teachers enact their role in the classroom. An underdeveloped math teacher identity can manifest in actions detrimental to student outcomes, thus exploration of teacher identity within math education and to support its development is essential. We explored five Ghanian teachers' math teacher identity development across participation in year-long instructional coaching. Findings support the notion that teachers' identity development trajectories are unique to the teacher and that teachers' general professional identity is distinct from their math teacher identity. Additionally, the factors that underlay their perceptions of themselves as math teachers align with Carlone and Johnson's (2007) tri-partite conception of identity involving competence, performance and recognition.

INTRODUCTION

Teacher identity can be defined as "teachers' understandings of themselves as teachers, shaped through ongoing processes of interpretation and re-interpretation of personal and professional experiences embedded in multilayered social [historical] contexts" (Author, 2020, p. 208). Teacher identity is dynamic, develops over time, and continuously evolves (Watson, 2006). It is thought to influence how teachers define their purpose and commitment to the profession; thus, identity shapes how teachers enact their role in the classroom (Day et al., 2006). Given the connection between a teacher's identity and their instructional practices, exploration of teachers' math identity has gained traction within the last two decades (Brown & McNamara, 2011; Kaasila, 2007; Lutovac & Kaasila, 2011).

Elementary teachers often struggle to identify as math teachers due to how they are educated as teachers (Author, 2018). They are considered generalists; thus, teacher education programs focus on developing skills across four core disciplines. An underdeveloped math teacher identity can manifest in actions that are detrimental to student outcomes (Author, 2020; 2022). Hence, identifying effective approaches to supporting teachers in developing robust professional identities is essential. Instructional coaching has emerged as an effective means of improving teachers' practices (Kraft et al., 2018), but less is known about how coaching influences identity development. This study addresses this gap. Situated within the context of elementary classrooms in Ghana, we focused on answering the following question, *in what ways do professional identities of Ghanaian elementary teachers of math shift through participation in individualized instructional coaching?*

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MATH TEACHER IDENTITY

Conceptualizations of teachers' professional identity development in the field of education vary (Beijaard et al, 2002). However, a common thread across these framings is the understanding, or conception, of self or self-image in the development of identity (e.g., Kelchtermans, 2005). Teachers' identity development is situated within a context (e.g., the Ghanaian education system, a specific classroom, or a subject area), and to make sense of the development of this identity, one must explore both the perceptions a teacher has of themselves and the ways in which their teaching context influences these perceptions. Professional identity captures an individual's attributes, beliefs, values, experience, knowledge, and skills required in a specific professional role (Park et al., 2018). Individuals use identity to "justify, explain and make sense of themselves in relation to others" (McKeon & Harrison, 2010, p. 27). They have to negotiate this conception of themselves with the values and standards of the broader professional community and their local context (Pillen et al., 2013).

In considering how math teacher identity develops, we draw on the framework proposed by Carlone and Johnson (2007). They proposed a model which conceptualizes science identity as a process of interaction between the interrelated dimensions of competence (i.e., one's knowledge and understanding of content), performance (i.e., one's ability to communicate and use tools in the ways accepted in the discipline) and recognition (i.e., acknowledgement as a legitimate participant in the field by self and others). The study examined different ways participants' identities developed over time and suggested that there is potential for supporting shifts in identity through interventions targeting an individual's experiences of competence, performance and/or recognition. This model was developed to define and explore science identity, but acts as a useful framework to understand math teachers' identity as these components capture one narrating their identity through foregrounding credibility (Authors, 2018).

With respect to elementary teachers, the degree to which their conceptions of their own identities align with being a math teacher is influenced by a variety of contextual factors. The institutional discourses around math (i.e., when a system emphasizes student achievement or a society values formal education in higher math) as well as experiences teachers have had in math throughout their lives contribute to how a teacher's self-image as a math teacher is formed (Neimayer-Depiper, 2013). Willis et al. (2021) explored math teacher identity and found that sense of belonging to a mathematical community, self-efficacy in teaching math, and enthusiasm for teaching math were positively correlated with a strong self-image as a teacher of math. Further, Yeigh et al. (2022) found that having a strong professional identity was fundamental to teachers' well-being. In sum, research on math teacher identity suggests that teachers' self-efficacy and enthusiasm related to math teaching are central aspects of their professional identities as math teachers, and that developing a robust identity bodes well for student learning and professional well-being. The question then

becomes, how can we support teachers in developing robust professional identities as math teachers?

Teacher identity development research focuses less on in-service teachers, and domain-specific identity development. In addition, given its effectiveness as professional development approach, the field needs to better understand the role instructional coaching plays in teacher's identity development (see Kraft et al., 2018). Thus, we focus on investigating the ways in which the math teacher identity of Ghanaian elementary teachers change through participation in a model of instructional coaching - Holistic Individualized Coaching (HIC).

METHODOLOGY

This study was conducted within a larger project focused on supporting Ghanaian elementary teachers in advancing their math knowledge and instructional practices as called for in a nationwide math education reform initiative.

Participants

Five elementary teachers working across two different schools within the same district participated in this study. All teachers had taught for at least three years, identified as teachers, and all (except Fabian) had a longer-term vision of being a teacher. Table 1 shows teachers' demographic characteristics.

Table 1. Participants' Demographics

Teacher	Gender	Grade Level
David	Male	6th
Rainy	Female	4th
Rachel	Female	2nd
Coby	Male	6th
Fabian	Male	6th

Participants were involved in a professional development program over the course of a year that incorporated an instructional coaching model - Holistic Individualized Coaching (HIC) developed by the first author.

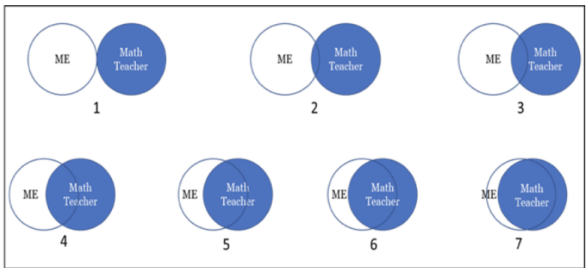
Holistic Individualized Coaching (HIC)

This coaching model considers teachers to be individuals and professional learners. It focuses on their holistic development by attending to their emotions, beliefs, identity, knowledge and current instructional practices (Author, 2019). In other words, if centres the psycho-social-emotional aspects of teaching as they strongly influence teachers' instructional decision-making and wellbeing. HIC involves six steps. *First*, the coach develops a general teacher profile from information about teachers' emotions, beliefs, identity, mathematical knowledge for teaching and instructional practices. These interviews also serve to initiate the development of trust and rapport. *Second*, each teacher has a conversation with the coach (pre-coaching conversations) before teaching the lesson. These conversations allow the coach to gain an insight into the teacher's

history with, feelings about, and understanding of the content. *Third*, the coach uses the information from the conversations to determine the nature of pre-lesson support. *Fourth*, the coach collaboratively works with the teacher on the lesson design. *Fifth*, teachers teach the lesson; the coach does not co-teach but remains present as a means of support and guidance. This allowed teachers to maintain ownership of their instructional decisions, which validates their professional identity. All the lessons taught were videotaped. As the *last step*, the teacher and coach each watch the video, select clips which provided insight into student thinking and instruction (post-coaching conversations), then discuss the clips together. Teachers completes 4-5 rounds of HIC.

Data Source and Analysis

To determine teachers' math identity, we adapted McDonald et al. (2019)'s single item measure to capture a "snapshot" of the teacher's identity at a particular moment (Fig. 1). Teachers were asked to respond to the prompt (Fig. 1) at the start and end of the coaching experience [1 = no overlap (do not identify as a math teacher) and 7 = near complete overlap (closely identify as a math teacher)]. Teachers were also asked to justify the reasoning of their choice using an open-ended prompt.



PROMPT: "Select the picture that best describes the current overlap of the image you have of yourself and your image of what a math teacher is. Explain why this picture best described the image of themselves as a math teacher"

Figure 1: Single Item Measure (adapted from McDonald et al., 2019)

We entered teachers' image selections in a table, then notated the change. Then we read through all the reasons teachers provided and summarized the reasons using phrases capturing the essential meanings (Fig. 2). This process revealed three major themes that helped us understand the reasons underlying the participants' conceptions of themselves as math teachers and shifts in these conceptions: pedagogical content knowledge, instructional practices, and validation by others. In what follows, we describe these themes in further detail.

FINDINGS

There were no consistent patterns in the nature or degree of teachers' identity change. Specifically, there was no shift for one teacher, three more closely identified as math teachers, and one identified less as a math teacher. Teachers' identities related to math were closely connected to their feelings of efficacy and competence in relation to knowledge of math, and their performance as math teachers.

Pedagogical Content Knowledge

David identified fully as a math teacher prior to and after he participated in the HIC cycles. His identity as a math teacher was grounded in his strong knowledge of math, and flexibility with the content such that he was able to utilize multiple representations and explain concepts in multiple ways. He stated, "*The picture I have painted about myself as a math teacher is so because of my understanding of the concepts and the way I utilize multiple ways to represent and explain and apply and use math skills and concepts".*

Teacher	Pre-Selection	Reason	Post-Selection	Reason	Direction of Shift
David	7	Strong pedagogical content knowledge	7	Advanced his pedagogical content knowledge	None
Rainy	4	Fear of teaching some math topics	6	Increased confidence in teaching; supporting students in learning and communicating mathematically	Up
Rachel	5	Find some math topics complex, challenging to understand	6	Increased interest in subject; Increased understanding of math topics	Up
Cobe	7	Influence on students' success; being told he is a good math teacher	6	Needs to improve strategies for facilitating learning	Down
Fabian	3	Teaching strategies need improvement	5	Increased confidence that his instruction is support learning; being told by an expert that his instruction is good	Up

Figure 2. Teachers' Math Identity Shifts

In his post-responses, he referred to his strong math knowledge in more nuanced ways. He also alluded to the ways in which his accumulated experiences in teaching reified his identity as a math teacher.

> I always say, over the years' experience, I built on my experience as a math teacher. And these experience actually had made me to grow…I am a maths teacher because I know my content. I know what to do to achieve results. I know what to do to make my students or learners achieve results. I know what to do to make my lesson fun, my lesson interesting. I know how to solve problems, mathematical problems for my students

At the end of the year, David seemed to be expanding on the content knowledge to include the skills he used to get students to achieve, he knows how to make math engaging, and he knows how to solve the problems he gives to his students.

Rachel also referred to her math knowledge as an influential factor in math her teacher identity. In contrast to David, whose identity appeared to be stable over the year, Rachel initially selected an image indicating medium overlap. She described that "*This is because there are some topics in Maths that I understand with ease after solving few questions but with the others, I really find it difficult when it becomes complicated*".

Due to this content-based struggle with some math topics, she did not fully identify as a math teacher. However, we noticed shifts after working with the coach over the course of a year. She reported an increased interest in math as she identified areas of growth in her teaching and worked on those with the coach's assistance. Her interactions with the coach helped her begin to accept herself as a math teacher.

> The [coach's] lessons and interview has also brought about some improvement and love for maths because there are some mistakes I do and since she [the coach] started to pinpoint them I'm improving gradually in this subject but not fully.

Instructional Practices

Some teachers emphasized their instructional practices as the core reference for thinking about how they conceptualized their math teacher identity. Cobe, Fabian, and Rainy pointed out the ways their confidence in the effectiveness of their math teaching was connected to their identity. Rainy more strongly identified as a math teacher at the end of the year, selecting image 6, and stated,

> … now I have so much confidence when it comes to maths. I come up with my own ideas when it comes to a particular topic for the kids to get prepared, to make them think where they talk more, the teacher talks less, the kids are able to answer more questions.

In the post interview, she seemed to have greater confidence in her ability to draw on her own ideas to prepare students to learn and in in orchestrating discussions which encouraged students to talk more in comparison to her explaining the ideas.

In his post-response, Cobe expressed confidence in his understanding of math concepts and how his students learned but indicated that he needed to "*improve on my facilitation".* He was the only teacher who identified less as a math teacher at the end of year. It appeared that he developed greater awareness of the role high-quality teaching plays in being a math teacher, resulting in a greater perceived gap between his current self-image and what he saw as an ideal math teacher.

Initially Fabian identified minimally as a math teacher. He described a lack of confidence in his ability to teach math due to the minimal teacher training he had. Having not *"majored in math at the university" he believed "I still lack some major techniques needed to teach math."* In his pre-response, he selected image 3 indicating the low level identification as a math teacher.

Validation by Others

Fabian and Rachel described the ways working with the coach and being identified as a math teacher by her supported them in strengthening their professional self-images. Fabian mentioned that although students recognized him as a math teacher, it was not until his skills were recognized by the coach (an expert), that he was able to embrace his identity as a math teacher. He explained

> And then the comments I get about math is just from some students I've taught, and they still remember how you used to teach them math. So I think maybe this remark is just coming from the children. But they are not well-versed in that area. Then you came in and

> then you study what I'm doing. We discuss and then the comments you pass, "Okay, well whatever the children were saying, now you are saying that from the professional now. So okay, that means that whatever they were saying was right." So it shifted my self-confidence from that level to the level that I chose right now, that's number five…

Rachel also identified her interactions with the coach as integral to expanding both her love and interest in math. Thus, indicating the role that individualized support from an expert can play to helping teachers negotiate shifts in their identities.

DISCUSSION

The government reform mandate (broader societal context), and participation in the HIC cycles (local context), encouraged teachers to reconceptualize their role in the classroom and how they engaged with students. We observed that at the start of the year the participants' images of themselves as math teachers varied, and over the course of the year the direction and degree of their identity shifts were distinct. These findings align with current literature showing that teacher identity evolves over time based on teachers' experiences within multilayered contexts (Author, 2020; Hong, 2010), and their identity development trajectories are unique based on interpretations of their professional experiences (Author, 2020; Day et al., 2006).

The participants' reasons for the conceptions of themselves as teachers were captured by three themes: pedagogical content knowledge, instructional practices, and validation by others. These themes are synonymous with Carlone and Johnson's (2007) interrelated dimensions of identity: competence, performance and recognition. In this regard, the participants' identities were grounded in their feelings of competence about their math knowledge relevant to teaching (i.e. David, Rachel), how effective their instructional practices were in supporting student learning (i.e.,Cobe, Fabian, Rainy), and the extent to which they were acknowledged as a math teacher by an expert (i.e., Fabian, Rachel). Thus, math teacher identity development is deeply content-based, distinct from a generalized teacher identity, and is strongly integrated with two components of teacher efficacy – personal efficacy and knowledge efficacy (c.f., Roberts & Henson, 2000). We also observed that in the cases where participants more strongly identified as math teachers, they identified experiences in the HIC cycles as contributing to this shift. Further, the cases where there was no shift, or a downward shift, there was greater awareness of their own math-related competence and performance as reflected in their nuanced descriptions. We note the potential of instructional coaching that is individualized and designed to be responsive to teachers' developing identities and other psycho-social-emotional needs.

Our findings contribute to the identity literature in the following ways. They showed that in-service teachers' math teacher identities are distinct from their general teacher identity. Also, teachers' math identity trajectories vary and can make positive shifts when engaged in interventions targeting their experiences of competence, performance and/or recognition. We encourage the development of interventions with in-service teachers that directly attend their identities and that acknowledge the nuanced approach needed to attend to content-specific identities.

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PROSPECTIVE TEACHERS' DEVELOPMENT OF GOAL STATEMENTS AND ALIGNMENT TO A TECHNOLOGY-INFUSED LESSON

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This study investigated the nature of prospective teachers' (PTs) goal statements and the nature of alignment between goal statements and lessons involving technology by a group of 13 PTs across three time points. Only two PTs were able to construct a goal addressing medium or high level conceptual knowledge on the initial lesson, but by the final lesson this had increased to seven PTs. There was a gradual improvement in alignments between lesson and goal statement with one during the initial lesson, five during the field experience, and six during the final lesson. Despite the positive results some PTs struggled in constructing medium or high level conceptual goal statements and aligning goal statements with lessons. The implications of these results for the design of technology methods classes are discussed.

BACKGROUND

Tyler (1950) first emphasized the importance of setting objectives (hereafter referred to as goals) in developing curriculum and in doing so brought educators' attentions to this foundational idea. Hattie's (2009) synthesis of meta-analyses shows clear evidence linking the setting of challenging goals to increased student achievement. Consequently, professional mathematics teaching organizations in the U.S. (e.g., National Council of Teachers of Mathematics [NCTM], 2014) and elsewhere (https://www.aitsl.edu.au/standards) have highlighted the importance of setting goals in developing school mathematics lessons. Indeed, NCTM lists *establishing mathematics goals to focus learning* as the first of its mathematics teaching practices. Additionally, conceptual knowledge is one of the three foundations upon which the de facto U.S. national standards rests. We also know that conceptual knowledge is an important foundation upon which procedural knowledge and procedural fluency is built (Fuson, Kalchman, & Bransford, 2005). Moreover, as Nilsson (2020) has pointed out both procedural knowledge and conceptual knowledge can be learned at different levels of complexity. Thus, due to the importance of conceptual knowledge and the importance of setting goals this study focuses on prospective teachers' (PTs) development of conceptually oriented goal statements or COGs.

Morris, Hiebert, and Spitzer (2009) have argued that instead of producing accomplished teachers at the end of their teacher preparation programs educational institutions should instead provide teachers with the skills they need to learn from teaching. Hiebert, Morris, and Glass (2003) suggest that these skills consist of identifying student learning goals, examining the alignment between instruction and goals, and examining evidence of student learning to revise subsequent instruction.

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Research by Morris and colleagues suggests that prospective elementary teachers (PSETs) struggle to unpack a concept into its underlying sub-concepts. Drake, Land, and Tyminski (2014) argue that educative curriculum materials (textbook materials designed to help teachers learn important content knowledge and teaching skills) can be leveraged to help teachers develop the skills identified by Hiebert and colleagues. Indeed, early empirical work by Land and Drake (2014) suggests that PSETs can identify, unpack, and extend the learning goals appearing in educative curriculum materials. While no research currently exists regarding the presence of educative curriculum materials in U.S. public schools, the most recent research at the primary and secondary levels suggests that conventional curriculum materials continue to dominate the U.S. textbook market (Banilower et al., 2018). This research study differs from previous work as it focuses on the development of goal statements by PSETs and prospective secondary mathematics teachers (PSTs) in lesson plans where mathematical action technology has been infused into more conventional U.S. beginning algebra mathematics textbooks. Throughout the remainder of the paper, I will use the acronym PTs to refer to both PSETs and PSTs. Additionally, I use lesson plan and lesson interchangeably throughout the paper. This study was designed to answer two research questions.

1) How does the nature of goal statements created by PSETs and PSTs change across three time points during a technology methods course?
2) What is the nature of the alignment between a goal statement and a technologically infused mathematics lesson created at three different time points?

FRAMEWORK

For ease of communication, I use the terminology *mathematical idea* to encompass elements that students are expected to learn in school mathematics and consist of mathematical definitions, proofs, procedural knowledge, conceptual knowledge, etc. Following Anderson et al. (2001) I define procedural knowledge and conceptual knowledge in the following ways. Procedural knowledge consists of knowledge of procedures, methods, or ways of calculating. Conceptual knowledge consists of knowledge of classifications, structures, and principles. Knowledge type goal statements were broken down into procedural knowledge oriented goal statements (POGs) and conceptual knowledge oriented goal statements (COGs). There were two levels associated with procedural knowledge (low and high) and three levels for conceptual knowledge (low, medium, and high). I used the following acronyms to refer to goal statement knowledge type and level: POG-L (low procedural knowledge); POG-H (high procedural knowledge); COG-L (low conceptual knowledge); COG-M (medium conceptual knowledge); and COG-H (high conceptual knowledge). The initial lesson and final lesson (discussed in more detail in the methods section below) involved transformations of linear and absolute value functions, respectively. Field experience lessons involved different types of content depending on the grade where PTs were placed for their field experience.

Following Nilsson (2020), I categorized different levels of conceptual knowledge based upon an increasing presence of justifications. Conceptual-low knowledge entails making a connection across two representations. For instance, students who can identify the change to the equation representation of the function $f(x)$ when it is changed by k as a vertical translation of the graphical representation of the function would be considered as exhibiting a low level of conceptual knowledge. Here the student is making a connection between the equation representation and the graphical representation of the function. Conceptual-medium knowledge involves knowing that the reason why adding k to a function causes the function to translate vertically is that it is affecting the output values of the function. Conceptual-high involves knowing why as defined under conceptual-medium, but also knowing that the reason the y-values are affected is because we can write the equation as $f(x) + k$ and because the output values are represented by $f(x)$, the addition of k affects the y-coordinates while leaving the x-coordinates untouched. Like previous research I have conducted in this area (Davis & Witt, 2022), I considered procedural-low knowledge to consist of graphing a transformed function point by point using a table. Procedural-high knowledge consisted of graphing a transformed function as one object moving by the amount represented by the parameter. It was possible for PT goal statements to address different types (procedural and conceptual) and levels of knowledge and in these cases both codes were applied. Each goal statement was coded for the highest level of procedural knowledge or conceptual knowledge appearing.

METHODS

A total of 13 PTs (seven seeking to become elementary school teachers or PSETs and six seeking to become secondary mathematics teachers or PSTs) participated in the study during the spring 2022 semester with the site being a teaching mathematics with technology course in a large university in the midwestern portion of the U.S. PTs were asked to create a lesson plan involving a goal statement focused on COG-H involving linear function transformations during the first week of instruction (*initial lesson*). At various time points during the semester, PTs created a lesson plan with a COG-H for a lesson involving mathematical action technology that they taught in a nearby primary or secondary school. The first of these lessons occurred during the third week of instruction and the last occurred during the 12th week of instruction (*field experience lesson*). At the end of the semester, PTs crafted a lesson with a COG-H involving transformations of absolute value functions (*final lesson*). If I was not able to determine the knowledge type from the goal statement it was given the code *unable to code*. A lesson plan was defined as the materials needed by a teacher to implement the lesson (e.g., goal statement) as well as the activity for students to complete. All final lessons used Desmos Activity Builder for the student activity. All PTs created a Word document that contained other components of their lesson such as the goal statement.

The technology methods course consisted of four components intended to improve PTs' construction of COG-M and COG-H. First, to construct a COG-H that described a rich conceptual knowledge of a mathematical idea, PTs had to first develop that

knowledge. PTs did this by working individually and with their peers during the rehearsal of field experience lessons in our technology methods course that involved rich conceptual knowledge of a variety of ideas. Second, PTs engaged in a **D-F-R** cycle multiple times individually and when working in small cooperative groups. These cycles consisted of the **D**evelopment of a COG-H, receiving **F**eedback on that COG from their peers or the instructor of the class, and **R**evising that COG using peer and instructor feedback. Third, my earlier research (Davis & Witt, 2022) suggests that PTs who engage in a field experience involving content associated with the initial lesson and final lesson were more likely to create a final lesson focusing on high level conceptual knowledge. Thus, during the 10th week of class all PTs worked with their peers in small collaborative groups to develop lessons involving why the parameter *c* causes a vertical translation of a quadratic function and to construct a COG-H involving understanding why the parameter *c* has this effect. The fourth component of the course involved the experience of rehearsing and teaching a lesson focusing on high level conceptual understanding and reflecting on that lesson. The COG for Donald's field experience lesson was created by the instructor and therefore led to a rating of *not applicable* so that it would not be considered in analyses.

Analyses of the alignment of the student activity (hereafter referred to as activity) and goal statement led to the creation of seven codes: *alignment*; $mismatch_A$; $mismatch_G$; $mismatch_B$; $mismatch_T$; *content;* and *indeterminate*. Alignment occurred if there was an exact match between the activity and the goal statement regardless of its knowledge type and level focus. A mismatch_A code was assigned if the activity provided students with opportunities to learn all the mathematical ideas appearing in the goal statement as well as others that were not listed in that goal statement. That is, the goal statement was underspecified. A mismatch_G code was given if all the mathematical ideas appearing in the activity were contained within the goal statement and the goal statement contained other mathematical ideas that did not appear in the activity. A mismatch_B code indicated that there were mathematical ideas common to both the activity and goal statement, but there were learning opportunities in the activity that did not appear in the goal statement and the goal statement contained mathematical ideas that did not appear in the activity. This was considered a bi-directional mismatch. A mismatch_T code was assigned if there were no common mathematical ideas between the activity and goal statement. A content code was assigned if there were any mathematical inaccuracies in the goal statement or the activity. Mathematics educators agree that goals should have a high degree of specificity (Hiebert et al., 2007; Stein & Meikle, 2017). An indeterminate code was assigned if a lack of specificity in the goal statement did not permit alignment to be measured. This lack of specificity might have occurred due to the use of one or more words that were not defined, not elaborated upon in the goal statement, or that I did not understand as a result of reading the goal statement. There were two cases (Todd and Donald) where the lack of specificity did not permit the determination of a knowledge type and level for the goal statement. The frequency of each of these codes was noted across the three lesson plans for each PT, broken down by group (PSTs and PSETs), and were examined for patterns across and

within the three lessons. In the case of Donald, as the instructor created his COG for the field experience, the alignment was given the code *not applicable*.

RESULTS

The goal statements by knowledge type and level for PSTs and PSETs are seen in Table 1. Despite the school mathematics background of many PTs in the U.S. focusing on procedural knowledge due to their school mathematics experiences (National Research Council, 2001), only four PTs initial lessons contained POGs. On the final lesson only one PSET (Kevin) created a goal statement that involved procedural knowledge. This certainly may have been due to the requirement that PTs create an initial lesson involving the use of mathematical action technology. Many PTs offloaded the graphing of the transformed function to the technology as an amplification of student skills and as a result this opened space for a focus on the effects of a change of the equation on the graphical representation an example of low conceptual knowledge.

Table 1 shows that one PST (Julie) and one PSET (Anne) made a transition from crafting a goal statement targeting procedural knowledge to targeting conceptual knowledge which I considered a positive effect of the technology methods course. An additional important effect of the course appeared to be the transition from the construction of lower level COGs to higher level COGs or staying at a medium/high conceptual level. This movement occurred for a total of seven out of 13 participants.

The alignment between the lesson and the goal statement for the three lessons are shown in Table 1 in the second/third row for each PT. There is a gradual increase in alignment from the initial lessons with only one alignment, to the field experience with five aligned lessons, and the final lesson with a total of six alignments. These numbers show that while PTs made progress in developing lessons that aligned with their goal statement, most of the PTs still struggled with this skill at the end of the course. There were also twice as many alignments (8 vs. 4) overall among the PSETs than among the PSTs. The next highest frequency code after alignment was mismatch_A. This code only occurred once in the initial lesson, five times for the field experience lesson, and three times during the final lesson. PTs struggled with specificity in their goal statements for eight total lessons as seen by the appearance of the indeterminate code. Most indeterminate codes occurred during the initial lesson with one occurring during the field experience and two occurring the final lesson. Another frequently occurring code regarding alignment was content. Specifically, more PSETs than PSTs struggled with mathematical accuracy, but this was primarily during the construction of the initial lesson plan. There were no content issues in the field experience lesson and only two during the final lesson – one by Matt (PST) and the other by Cathy (PSET).

Table 1: Goal Statement Categorization and Lesson-Goal Alignment

PT	Initial	Field Experience	Final
		PSTs	
Jerry	POG-L, COG-L	COG-L	COG-M
	Indeterminate	Mismatch_{A}	Mismatch_{A}
Julie	POG-L	COG-L, POG-L	COG-L
	Mismatch_{T}	Mismatch_{A}	Indeterminate
Matt	COG-L	POG-L, COG-L	COG-H
	Alignment	Mismatch_{A}	Content
Paula	COG-M	COG-M	COG-H
	Mismatch_{G}	Alignment	Alignment
Joseph	COG-L	COG-H	COG-M
	Content Mismatch_{B}	Mismatch_{A}	Alignment
Todd	Unable to code	COG-L	COG-M
	Mismatch_{A}	Mismatch_{G}	Mismatch_{B}
	Indeterminate		
		PSETs	
Donald	Unable to code	Not applicable	COG-L
	Indeterminate	Not applicable	Mismatch_{A}
Kevin	POG-L	COG-L	POG-L
	Content	Alignment	Alignment
Rebecca	COG-M	COG-M	COG-H
	Content	Alignment	Alignment
	Indeterminate		
Maria	COG-L	COG-L	COG-L
	Mismatch_{B}	Alignment	Alignment
Anne	POG-L	POG-L	COG-L
	Content	Indeterminate	Indeterminate
Cathy	COG-L	COG-L	COG-L
	Content	Alignment	Content Mismatch_{A}
Phillip	COG-L	COG-L	COG-M
	Mismatch_{T}	Mismatch_{A}	Alignment

DISCUSSION

The work that PTs experienced during the technology methods class around crafting COG-H paid off in several different ways. First, there was an increase in the number of goal statements that were characterized as high level conceptual knowledge from the initial lesson to the final lesson. Second, the lack of specificity in goal statements that others have noted among teachers (Stein & Meikle, 2017) decreased from the

initial lesson to the final lesson. While the PTs increased in their ability to construct higher level COGs, their ability to construct an activity that aligned with their goal statements also increased. If one adopts a preservice teacher instructional model described by Hiebert et al. (2007) this is an important finding as it suggests that these future teachers are better positioned to learn from their teaching. Despite these positive results, six PTs still struggled to construct a medium or high level COG on the final lesson. Most of these individuals, however, were PSETs suggesting that while the focus of the content was at the upper middle school level (ages 13-14), this content may have been too advanced for these individuals. Consequently, one implication of this study is to use content for the initial and final lesson that is more aligned with lower-level middle school content for this group of participants.

To evaluate the impact of a lesson on student understanding it is not only important to have a clearly specified and articulated goal statement, but also align that goal statement to a lesson plan (Hiebert et al., 2007). The participants clearly made progress in this area as there was only one alignment on the initial lesson while the final lesson had a total of six alignments. Especially important was the fact that of these six alignments, four involved either a COG-M or COG-H. The final lesson also illustrated some of the problems that PTs experienced in alignment. Namely, issues with goal specificity as seen in the appearance of indeterminate codes. Also, several PTs created a student activity that involved more mathematical content than the goal statement associated with it as seen in the $\text{mismatch}_{\text{A}}$ code. This was especially problematic during the field experience lesson with this code occurring five times out of 13 lessons. This finding implies that PTs need specific assistance in the construction of a goal statement that aligns with an activity instead of working solely on the construction of an activity, which is typically what happens during this component of the class. These findings have led to modifications of the teaching with technology course for the current semester; the effects of which I intend to investigate in the future.

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THE *DIALOGUE* BETWEEN MATHEMATICS EDUCATION AND ANTHROPOLOGY: THE CASE OF TERTIARY TRANSITION

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The discussion on the identity of the research in mathematics education has often underlined the distinction between mathematics education and many other fields of research. On the other hand, this discussion also highlighted how mathematics education research has traditionally drawn and still draws on other disciplines. Several studies in mathematics education referred to anthropological approaches and constructs over the last decades. This contribution is the result of a dialogue between a mathematics educator and an anthropologist: analyzing the use of the rite of passage model for describing the tertiary transition in mathematics, possible developments of this research are discussed, as well as the potential of a continuous dialogue between specialists in the two fields of research.

INTRODUCTION

In the last three decades, many scholars debated the nature of the research in mathematics education (Sierpinska et al., 1993; Schoenfeld, 2000; Presmeg, 2008). The distinction from other disciplines such as mathematics, psychology, sociology, anthropology, philosophy, epistemology, neuroscience, semiotics, etc. but, at the same time, the relationship with these disciplines were the main issues in this discussion. On the one hand, mathematics educators wanted to clarify what research in mathematics education and its results are, their specificities. On the other hand, the need for drawing upon the established knowledge bases and methodologies of other consolidated fields of research for understanding the complex process of teaching and learning mathematics has always been evident:

> The central focus is, of course, the teaching and learning of mathematics, and thus the nature of mathematical activity and thinking are a crucial focus for study in the field. The nature of mathematical activity and thinking have to be studied using those fields, psychology, sociology, anthropology, philosophy, and so on. (Dreyfuss, 2014, p. 65).

In line with Romberg's work (1992), we can recognize four phases of the research process (see Fig. 1).

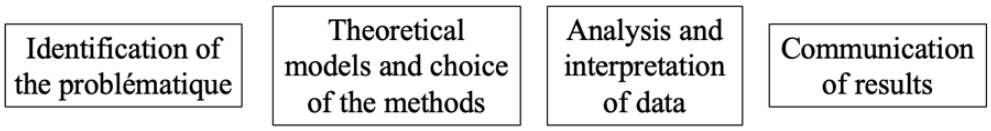

Fig. 1: the four phases of the research process

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If the identification of the research problem and the development of the research (and, in particular, the interpretation of data) appear to be characteristic of research in mathematics education, in some sense defining it (Sierpisnka & al., 1993), the second phase is where interaction with other disciplines is most evident:

> why do we need psychology, sociology, and so on – even linguistics? I am convinced that the lenses of research methods used in these fields of the humanities are essential tools for mathematics education researchers to have at their disposal. The reason is simply because mathematics education involves *people*, with all their complexities. (Presmeg, 2014, p. 47).

Within the lenses mentioned by Presmeg, the anthropological point of view had and still has special relevance for the interest in developing a cultural perspective on mathematics education (Bishop, 1988), and for the recognition of the role of socio-cultural factors in the teaching-learning process (Presmeg, 2009).

In this frame, Chevallard (1992) developed the Anthropological Theory of Didactic (ATD) assuming that praxeologies underlying all human activities are strongly affected by cultural and environmental factors, and Cobb (1989) underlined the significance of the anthropological perspectives in mathematics education research. The main purpose of this anthropological perspective is the identification and analysis of regularities in the interaction between teachers and students, seen as "members of a classroom community with its own microculture" (Cobb, 1989, p. 33). This purpose requires appropriate research methods and, in this perspective, Eisenhart (1988) proposed the ethnographic approach in mathematics education, a significant approach for the typical anthropological interest of understanding the lifestyle of exotic groups from the native's point of view. As Presmeg underlined:

> the ethnographic methodology of anthropological research is peculiarly facilitative of the kinds of interpreted knowledge that are valuable to mathematics education researchers and practitioners. After all, each mathematics classroom may be considered to have its own culture. (Presmeg, 2009, p. 134).

The union of mathematics education and anthropology within phase 2 in Romberg's model (Fig. 1) is not limited to the choice of methods: some theoretical models originally developed in anthropology were adapted to phenomena of interest to mathematics education.

On the other hand, as Connors warned:

> such "borrowings" are often not successful; a researcher in one field is not always aware of the issues surrounding, or the current status of, a particular paradigm in another. Every discipline is dynamic (…) Unless the "borrower" is aware of this disciplinary debate, the result can be the application of an outmoded idea to a new field, where it may very well be accepted, and perpetuated, by naive readers. (Connors, 1990, p. 462).

For this reason, we strongly believe in the need of a retrospective comparison between mathematics educators and the specialists of the field where the used theoretical models or methods were originally developed (anthropology in our case).

In this paper, we will develop this retrospective analysis focusing on tertiary transition in mathematics and in particular, critically analysing the use of the theoretical model of the rite of passage to describe the phenomenon.

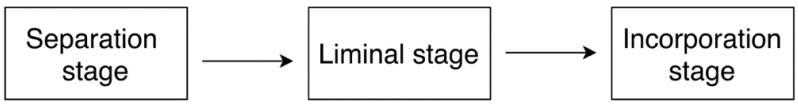
THE TERTIARY TRANSITION IN MATHEMATICS AS A RITE OF PASSAGE

Tinto described the generic transition from school to university as a modern rite of passage, making explicit reference to the rite of passage model (see Fig. 2) introduced by the anthropologist van Gennep (1909):

> the problem of becoming a new member of a community that concerned van Gennep is conceptually similar to that of becoming a student in college, it follows that we may also conceive of the process of institutional persistence as consisting of three major stages or passages – separation, transition, and incorporation – through which students typically must pass in order to complete their degree programs. (Tinto, 1988, p. 442).

Clark and Lovric (2008), underlying how the existing body of research on the tertiary transition in mathematics was substantially characterized by the absence of a theoretical model, adapted the model of the rite of passage to fill this gap. They described the tertiary transition in mathematics as a modern-day rite of passage for students composed of three stages: *separation* (from secondary school), *liminal* (from secondary school to university) and *incorporation* (into university).

Fig. 2: The three stages of the rites of passage



How showed by a recent literature review (Di Martino et al., 2022a), the introduction and use of the rite of passage model has inspired much subsequent research on tertiary transition in mathematics, including those conducted by the first author of this report (Di Martino & Gregorio, 2019; Di Martino et al., 2022b).

The adoption of the rite of passage model permitted researchers to highlight some crucial issues in the research on students' difficulties in tertiary transition, until then characterized by a purely cognitive approach.

First, the rite of passage is always related to specific cultural routines, therefore sociocultural aspects cannot be neglected in the study of tertiary transition in mathematics. This means we should be extremely cautious about generalizing results across different university contexts and we need to develop more cross-cultural studies.

Second, the transition shock is inevitable: no passage and no incorporation in the new community are possible without putting into crisis the individual's routines and identity. These kinds of crisis are related to the emergence of strong affective reactions: in particular, feelings of inadequacy often emerge. Therefore, the rite of passage model also highlights the role of the affective component in the tertiary transition in mathematics and the consequent need of considering affective aspects for understanding students' difficulties in this transition.

The adoption of the rite of passage model has therefore led to an emphasis on the need to adopt a more holistic approach in tertiary transition research. Moreover, the rite of passage model points out that the actions for smoothing tertiary transition in mathematics cannot prevent students' crisis if we want the transition to be successful, their main goal should be to prepare students for the crisis and for dealing with it.

On the other hand, are we sure that the rite of passage model – originated in another field to describe different phenomena from a different era – has been used appropriately?

According to Connor (1990), the answer to this question cannot be given from an inner perspective (the perspective of mathematics education), but we need the answer from the perspective of an anthropologist.

THE ANTHROPOLOGICAL POINT OF VIEW: THE *PARS DESTRUENS*

Time, culture, and context are obviously crucial aspects of anthropological research and its interpretation of ethnographic data. van Gennep develops the theory of rites of passage at the beginning of the last century and he refers to *simple* societies, characterized by predictable life trajectories of its members, where deviations from the shared idea of normality are practically absent. In these realities, the transient moments can be ceremonially accompanied by the community and the ritual language is highly formalized and repetitive.

Cultural frames and practices for transitional phases are present also in the so-called *complex* societies of our era: anthropology recognizes that discontinuities and transitions are universal conditions of the individual life trajectories (Benedict, 1938), what changes is the meaning given to these transitions and the way they are overcome through rituals. The definitions and meanings attributed to the ritual category have changed over time: in this discussion we are adopting a generic definition of ritual as a shared cultural practice that symbolically accompanies moments in the calendar of people's and groups' lives.

In this frame, two main issues appear to be the particularly critical in the use of the rite of passage model for tertiary transition in mathematics.

First, the cultural worlds considered by van Gennep for developing his model are historically and culturally very far from the tertiary transition contexts studied in mathematics education and, from an anthropological point of view, this cultural-historical distance does not appear to be sufficiently problematized in the use of the rite-of-passage model for tertiary transition made in mathematics education.

A second and more critical issue, the model is abstractly applied to a social unit considered indistinguishable (in some sense unique) and internally homogeneous (whose members have similar characteristics, living the discontinuities in similar ways): the mathematics class of the first year of undergraduate education. Both the assumptions are strongly questionable from an anthropological point of view. Members of any university cohort show differences relative to gender, generation, social class,

political horizon, background, schooling, socialization, etc. In her very famous book Coming of age in Samoa, Margareth Mead (1928) focused on the perception of adolescence in Samoan and U.S. girls, showing how a critical transitional phase in a culture (the adolescence in U.S.) cannot be perceived as critical in a different cultural context. This shows how the cultural contexts – in our case, the different universities in different countries – play a crucial role in identifying, defining and living the transitions. Therefore, it is important to identify the stages of growth in the different considered contexts for recognizing their cyclicities and, consequently, the stages of transition and status change. Cyclicity is a needed precondition of any ritual form, enabling individuals and groups to predict transitions.

The anthropological literature shows how various ritual forms exist in contemporary schooling systems (Segalen, 1998). Not all these ritual forms follow the logic of van Gennep's tripartite sequences, however, all of them are recognizable as ritual forms, even by external observers.

In this frame, the main questions we need to reply are: are we sure that the entrance to the university – in particular, in a Mathematics degree – is externally recognized as a moment of passage? If the answer to that first question is yes, what is the status transition cyclically produced by the ritual?

The answers to these questions are not obvious. It would be interesting to experiment new interdisciplinary research paths in order to find solid answers. This reflection is the right introduction for the pars construens.

THE ANTHROPOLOGICAL POINT OF VIEW: THE *PARS CONSTRUENS*

The critical study of the social and cultural phenomena through the connection between the emic perspectives (from inside the studied group) and the external perspectives is characteristic of anthropology. In our view, this characteristic represents the added value that anthropology can offer to the research in mathematics education for realizing the *social turn* evoked by Lerman (2000). This approach can be particularly significant for the understanding of the tertiary transition in mathematics, offering the main argument for the pars construens of our reflection.

As we said, the rite of passage cannot be considered an abstract model applicable to any transition: in particular, the tertiary transition in mathematics appears to be neither socially recognized as a passage nor ritually characterized.

Surely, it represents a critical phase of strong discontinuity for freshmen (Di Martino & Gregorio, 2019; Di Martino, Gregorio & Iannone, 2022b), but we need to know if and how much this is specific of the transition in mathematics. Comparisons with other kinds of tertiary transition would allow the recognition and distinction between the identifying elements of the transition in mathematics and those common to the different university transitions.

The identified elements (difficulties, trials, and coping strategies experienced in the first year) of the tertiary transition in mathematics could be interpreted as a form of

initiation into a specific community of initiands, representing the liminal stage within the van Gennep's sequential logics in a perspective that recognizes the cyclic nature of the rite of passage from freshmen to graduate. In this frame, the crisis – what Turner (1982) calls the social drama – would extend throughout the college period: the university enrolment would represent the separation stage and the graduation would represent the incorporation stage with the social recognition of the new status.

In this way, this phenomenon is recognizable as a rite of passage even outside the considered community (the undergraduate students in mathematics) and this external recognition is crucial to investigate any subcultural specificities of the transition in mathematics. Such perspective would allow for an in-depth study considering several crucial elements: the factors affecting the choice of the degree in mathematics, the social representations of mathematics, the students' expectations and experiences noted year by year. It would also allow for the collection and interpretation of quantitative and qualitative data on the mathematics cohort concerning the internal differences of gender, social membership, background, etc.

CONCLUSIONS

Our reflections bring to conclusions on two different levels.

Concerning the specific focus on the tertiary transition in mathematics, the discussion in the pars costruens paragraph offers ideas for developing a new research program, starting with the critical issues reported in the pars destruens paragraph: what characterizes a rite of passage? Which elements to consider for identifying the first-year mathematics class and its possible subculture from an internal and an external perspective? What are the elements connecting this subculture to other subcultures?

The interdisciplinary dialogue highlighted the limitations of the transposition of the rite of passage model and is therefore essential to take a step forward. However, the "naïve" use of the rite of passage in the research on the tertiary transition in mathematics was not at all useless: on the one hand, the result obtained are however significant having a crucial role in the development of research toward a more holistic approach, on the other hand, the critical analysis of these naïve use of the model suggests new and interesting directions of research.

In line with Connor's conclusion: "as mathematics educators become more familiar with the field of anthropology, their borrowing will become more sophisticated" (Connors, 1990, p. 467), this research program should necessarily be characterized by strong interdisciplinarity between mathematics education and anthropology, not limited to the phase 2 of Romberg's model (see Fig.1) but including the identification of the problématique (phase 1) and the interpretation and discussion of the results (phase 3).

At a more general level, we want to emphasize the significance of developing a retrospective dialogue when we conduct research based on constructs borrowed from other disciplines.

This interdisciplinary dialogue involves some evident difficulties related to the differences in language, communication styles, jargon, and modes of argumentation between our field of research and a different field such as anthropology. However, once this diversity is mutually understood and accepted, it is possible to create the basis for designing original and significant research without abdicating our respective disciplinary specificities.

Research in mathematics education and research in anthropology are particularly well suited for this kind of dialogue and contamination: they have proven it in the past and they will be able to continue to do so, with greater awareness, in the future.

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EXAMINING THE ROLE OF FACILITATORS IN THE CONTEXT OF PLANNING AN INQUIRY-BASED MATHEMATICS LESSON

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This paper presents categories and codes of an analytical framework that combines both Knowledge Quartet and Mentoring Strategies to understand the role of facilitators in collaborative learning community through lesson study. The data is from a Norwegian lesson study involved three facilitators and six elementary teachers working together on an inquiry-based lesson. The new categories reported in this paper contributes to build a robust analytical tool to explore and discuss the role of facilitators in a future large-scale international study including several countries and all three phases of lesson study.

INTRODUCTION

The role of facilitators in collaborative work with teachers has been identified as central to the professional development of mathematics teachers (e.g. Borko & Potari, 2021). A difficult task is how to conceptualize the contribution of facilitators in such collaborative settings. For instance, Shulman's pedagogical content knowledge (PCK) (Shulman, 1986) could be used to understand the difference of knowing something for oneself and being able to help others to know it. Yet, it is difficult to use PCK to interpret the differences we (Skott & Ding, 2022) found in how facilitators talked with teachers and what content-related aspects they talked about in mathematics lesson study (LS) in Denmark and China. This is not only because of the fact that LS is new in Denmark, but also because of cultural, social and power-related aspects of the different roles of facilitators in the two cultural settings.

Based on this previous study, we will in the present study further develop categories and codes to describe, understand and interpret the tacit and abstract nature of how facilitators interacted with teachers and what content-related aspects they talked about in a mathematics LS in Norway that targeted inquiry-based learning (IBL) in elementary school. Our goal is to build a robust analytical tool to explore and discuss the role of facilitators in a large-scale international study including several countries and all three phases of LS (i.e. planning a research lesson, conducting the research lesson and observing student activities, reflecting on the lesson based on observations). Our research question in this paper is: how do facilitators interact with a group of elementary school teachers to learn to plan an inquiry-based mathematics lesson?

THE ROLE OF FACILITATORS IN TEACHERS PROFESSIONAL DEVELOPMENT

There has been a growing recognition of the significant role of facilitators in supporting teachers' professional development through collaborative work related to classroom

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practices in general and in LS in particular. Nevertheless, the research into the nature of the facilitators' interactions with teachers in LS is scare (Skott, 2022). Thus, there is a need to develop a new analytical tool to describe, understand and interpret the complicated role of facilitators in teachers' professional development. Gu and Gu (2016) developed an analytical tool based on empirical data to conceptualize how teaching research specialists (facilitators) mentor teachers during post-lesson LS-debriefs based on more than 100 hours of videos of 50 facilitators in China. They developed a two-dimensional framework for analysing the mentoring activity: the first dimension is of the mentoring strategies, which encompass the dynamic between the facilitators and the teachers; the second dimension is the knowledge that mentors pay attention to (i.e. mathematical, pedagogical and practical knowledge). Regarding the mentoring strategies, Gu and Gu found that "the conversations between [the facilitators] and teachers were … monologues rather than dialogic in nature", with the facilitators paying most attention to "what they know and what they anticipated, rather than … what teachers were concerned about in their teaching" (p. 451). Regarding the knowledge, the facilitators focused on practical knowledge, helping teachers to analyse concrete cases that embraced mathematical and pedagogical ideas. Ding and Jones (2018) propose that there is a *dual nature* of Chinese facilitators' expertise in Chinese LS. The first nature is scaffolding the teachers to learn concurrently the act of the multiple theoretical ideas (i.e. teaching with variation) through the LS. The second nature is scaffolding the teachers to learn to reflect on their own beliefs about the subject, pedagogical thinking and action, and to develop their identity as mathematics teachers.

Recent studies have showed that the different qualities of facilitators' mentoring may promote alternative aspects of teachers professional development in special contexts. In a previous comparative study (Skott & Ding, 2022), we found big differences in the ways that the facilitators in Denmark and China interacted with the teachers. While the facilitators in the Danish LS rarely talked more than 2 minutes and interacted in a dialogic-relational mode with teachers during LS meetings (about 60 min), the Chinese facilitator dominated the conversation in a seemingly authoritative way in LS meetings (45 min). An important result of this study is that "cultural, social, and power related issues at the interactional level as well as at a broader level have high influences on their [the facilitators] engagement" (p. 8). An analytical tool should to some extent be able to take into account such issues. Comparing two mentoring strategies in a two-year large-scale study in Germany, Richter et al. (2013) conclude that constructivist-oriented mentoring (i.e.; when mentors initiate inquiry stances towards teaching) supports new mathematics teachers more appropriately than transmissive-oriented mentoring (i.e.; when mentors convey their own teaching ideas and often focus on technical skills). The authors further suggest that the quality of mentoring rather than its frequency and close guidance explains successful career paths. Their conclusions are based on teachers' self-reported experiences of (among other things) self-efficacy and teaching enthusiasm. Skott (2022) investigates what happens in lesson study beyond its initial adaptation in countries outside East Asia with no use of external

support (facilitators). Based on a comprehensive Danish case, the study suggests that external support (or other external knowledge sources) are crucial at the mature stages of adaptation too. Skott (2022) highlights that compared with LS in Japan and China, two activities are new for the external support in the Danish context: 1) prevent teachers' use of artefacts and actions from transforming into rote procedures. 2) identify problems to be analysed during reflection and challenge teachers' views.

THEORETICAL APPROACH

Given the research question of this paper, two analytical tools are referred to in the initial data analysis: the Knowledge Quartet (KQ) (Rowland, 2013) and the Mentoring Strategies (MS) (Gu & Gu, 2016). The KQ is an empirically based conceptual framework developed particularly for the analysis of the relation of mathematics content with teaching in classrooms. We refer to the KQ rather than other frameworks, because it enables us to largely focus on the classification of situations of the collaborative learning and working between facilitators and teachers in which the subject matter is related to teaching. The KQ includes four components: (1) *Foundation*, about knowledge 'possessed', meaning the teacher's theoretical background and beliefs in terms of what they learned at school and teacher education etc. The other three components are about knowledge-in-action, as they refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching" (p. 200). (2) *Transformation*, about the capacity to transform the content knowledge one possesses into forms that are pedagogically powerful. (3) *Connection*, about the coherence and mathematical connections in mathematics pedagogy. (4) *Contingency*, about responses to classroom events that were not anticipated.

Here, we wish to make clear two points of the KQ that enable us to develop further reflection on the categories for accounting for the ways facilitators interact with teachers in our study. First, the strength of the KQ is to enable researchers to focus on the relations between content-aspects and teaching situations. Second, we are aware of the differences between *transformation* and the capacity of a facilitator to enable other teachers not only to learn about, but also to agree on how to make *transformation* in their classes. We are thus open to the data to enrich the description of this category in our analysis. Nevertheless, the social and affective aspects are not less important in teachers' learning and working in collaboration (Skott & Ding, 2022). This is what the second analytical tool of mentoring strategies enables us to examine, namely the complex relations between more cognitive aspects and affective and social aspects of the ways facilitators contribute to the social interactions in our study.

Gu and Gu (2016) identify four types of mentoring strategies in their study of Chinese facilitators: (1) *General comments*, what teachers should know and do in classroom teaching in general; (2) *Comments on anticipated problems*, that teachers may encounter and advice on how to deal with them; (3) *Responses to teachers' questions*, raised and related to the issues in the class taught; (4) *Dialogues with teachers*, discussing problems that occurred in class.

Based on the Danish data, Skott and Ding (2022) showed that Gu and Gu's four types proved to be insufficient to capture all the strategies used by the facilitators, thus adding three new types: (1) *Encouraging comments*, such as emotional recognition of teachers' ideas and suggestions; (2) *Challenging comments*, such as disagreeing with teachers' proposals and understandings; (3) *Building on or reformulating teachers' ideas* that are expressed in the conversation. The three new codes are also examined in our analysis of the Norwegian lesson study.

THE INQUIRY-BASED MATHEMATICS LESSON STUDY IN NORWAY

The context of the Norwegian lesson study

The project involved two schools in the outskirts of a city in central Norway. The goal was to enhance the use of inquiry-based pedagogies in mathematics and science teaching by engaging teachers in lesson study. Explicitly, the goals were being defined in the project as focusing on inquiring activities (i.e. open tasks with multiple solutions or solution strategies) to develop such as students' engagement, critical thinking, asking questions; cooperation and communication, etc.

In the IBL LS meeting discussed in this paper, three teacher educators and researchers (called facilitators in the rest of the paper) participated. One of the facilitators DS (all anonymous) was a mathematics teacher educator while DR and DJ were natural science teacher educators. Six teachers participated: 1) TM2 (the school principal), 2) TR (1st grade classroom teacher), 3) TI (1st grade classroom teacher), 4) TM1 (special education teacher with specialization in mathematics and Norwegian), 5) TT (2nd grade classroom teacher), 6) TS (2nd grade classroom teacher). TM2 and TM1 do not do classroom teaching. The school had previously participated, with the facilitators, in two projects focused on IBL, but of the teachers only TM2 had been actively involved in those previous projects.

Given the research question of this paper, we concentrate on the planning meeting of the first lesson. The teachers had talked briefly about the meeting in advance, and agreed that the first lesson is on mathematics, and partly spurred by TS, that the mathematical goal should be related to functions (one task of her mooc education was on the learning of functions). In the initial analysis, we focus on the first meeting (2 hours). Using Microsoft Word, the words in the transcript (in Norwegian) were counted as follows: teachers uttered 14843 words and facilitators uttered 3843 words.

Data analysis strategy and procedure

In this section, we describe how we have developed our strategies for the data analysis and interpretation according to the two analytical tools presented above: KQ (Rowland, 2013) and MS (Gu & Gu, 2016; Skott & Ding, 2022). All three authors analyzed the transcriptions individually (see Table 1 for an example of coding data in the analysis). We then validated our interpretations and use of the codes in digital meetings. In the same way we analyzed and discussed our use of MS codes (see Table 2).

Table 1: The use of KQ categories and codes in the data analysis

Categories	Codes of KQ (Rowland, 2013)	Example of data
Foundation (F)	F1. awareness of purpose; F3. overt subject knowledge; F4. theoretical underpinning of pedagogy; F5. use of terminology.	DS: mathematics is about recognising patterns and do generalisations and such, and you can say that with functions it is also about finding expressions or formulas for it

Table 2: The use of MS categories and codes in the data analysis

Categories	Codes of MS (Gu & Gu, 2016)	Example of data
Dialogues with teachers	Gu 4. Facilitators and teachers dynamically and dialogically discuss and share their own opinions	TR: Maybe the task could be to find different ways to continue the pattern. DR: Or if you first start the way you were thinking, how do you think the next one will look like, and then you can say that there are at least three ways to do it, can you find them. TM2: More advanced, more creative. TR: I think that was very smart.

In this paper, we examine the facilitators' interactions with teachers to gain new insight into the differences between the components of the KQ and the capacity of a facilitator in collaborative working with teachers in the LS. That is, it is necessary to address analytical vs. holistic ways of thinking (Ding & Jones, 2021) to develop a relational understanding of the key codes. For instance, two questions were largely discussed in the planning meeting regarding the IBL LS: (1) Is the mathematics goal clear? (2) Does the plan offer students' enough space to inquire, explore, discuss and cooperate? In the next section, we describe our identification of one new strategy of MS and two new codes of one new category of KQ.

FINDINGS

The first episode (see Table 3) illustrates the identification of the new strategy and one new code of the new category of KQ. The analysis of DS' utterances, in the first twenty minutes, showed how DS played an important role in facilitating LS, namely seeking and actively listening to feedback from teachers towards promoting a collaborative learning and working culture of being open, honest and constructive (new category of MS). In so doing, the facilitators played a significant role towards developing a shared mathematical knowledge foundation for planning and teaching the IBL lesson.

Table 3. Data analysis in the first episode.

Facilitating LS, and seeking and actively listening to feedback from teachers towards promoting a learning process of open, honest, constructive.	DS: … and we can see the importance of planning, but reflection after the first lesson is also very important, so we don't have a perfect lesson planned even if we are several people now spending hours at planning. […] It is also important to know what we are looking at in the first lesson so the discussion afterwards becomes productive …
Probing and understanding what knowledge and thoughts teachers actually have about functions (F1, F3, F6).	DS: Maybe you could say more about that task and what kind of literature that is relevant.
After having sought and actively listened to feedback from teachers, DS responded to teachers' expression of lack in F3 and F4. And he also tried to extend teachers' mathematical knowledge about functions in general to establish a shared mathematical knowledge foundation for planning and teaching at grade 1.	DS: It is about discerning patterns and relations. E.g. that a triangle has three edges and that the number of edges and vertices are related, and how it is in a quadrilateral: a quadrilateral has four edges and four vertices. And this are words and concepts and figures that are also relevant at grade 1. TS: Absolutely. DS: And I am thinking, at grade 1 we are looking for simple connections, and it is important to discuss connections already at grade 1, since it lays the foundation for further development, as mathematics is about finding patterns and make generalisations.

In the second episode, the analysis of DS' utterances in the main part of the meeting (16:39-1:00:33) could be largely coded as *Building on or reformulating teachers' ideas* towards establishing shared goals of the LS (e.g., generalization). Here, DS tried to support teachers' knowledge-in-action (i.e. C20 (contingency of KQ) in Table 4) and enhancing their awareness of the relationship between teachers' role in teaching and students' role in learning mathematics (students' situation, individual differences) (see Table 4).

Trying to focus on the theme-the learning goal, not time (at this moment). (Facilitating LS)	16:39 - 28:38 DS: I believe it is important that we don't spend much time discussing how long things take but rather discuss the theme for the lesson.
Focus on learning goal (F1, F3).	TI: What we are supposed to do.
Gu4, pointing to generalisation (F1, F3).	DS: Yes, and what the learning goal is. What are the students supposed to learn from the lesson.
Teachers openly expressed their feelings to the facilitator's shared mathematical knowledge for	DS: Yes, then you have a simple start and several possible paths ahead. How do you think figure number 20 will look like? Then you are approaching generalization, … TS: Yes, so that is interesting, what will number 20 be. And then

establishing the shared learning goal.	building on that … TR: That would be fun. TI: Yes, really fun.
After having sought and actively listened to feedback from teachers, the facilitator, for instance, responded to teacher insight (C20), to highlight the role of teacher and teaching for helping pupils to move on in learning mathematics. (*Building on or reformulating teachers' ideas*).	40:04 - 1:00:33 TT: But do you understand the difference. … if you think as a teacher, the process is important, that they sit there reflecting, right, that they are reflecting together and are able to launch lots of ideas. That is the important part. Whether I get to know all those reflections is maybe not so important. … DS: The first thing we have to do is to discuss how do you think the next figure will look like, and if they think it is only one more, it is clear what number 13 will be, but if they take one, two, one, two, there needs to be some kind of reasoning, so there can be different solutions.

Table 4. Data analysis in the second episode.

DISCUSSION

In this paper, we have identified a new category of mentoring strategy (MS) and two new codes of a new category of KQ of the communication and collaboration in the lesson study community, to contribute to the existing study of the role of facilitators (i.e. Skott & Ding, 2022; Gu & Gu, 2016; Richter et al., 2013). The new category of mentoring strategy is: Seek and actively listen to feedback from teachers towards promoting a collaborative learning and working culture of being open, honest and constructive. The new category of KQ is an effort to address the relational understanding of the components of KQ (Rowland, 2013) in our collaborative learning and working LS community in the context of real-world work. The first code of this new category is: Trying to establish a shared mathematical knowledge foundation for planning and teaching the IBL lesson (tackling the conflict between the facilitators' mathematical knowledge foundation and the teachers' lack of mathematical knowledge). The second code is: Towards establishing shared goals of teaching mathematics (e.g., generalization) (a need to construct together about a foundation of knowledge-in-action of both the facilitators and teachers).

As the analysis shows, the number of teachers' words (14843) in the planning meeting was almost four times than of the three facilitators (3843). Also, the strategy *Building on or reformulating teachers' ideas* was common in both the Norwegian LS as well as in the Danish LS. Apparently, teachers are provided much space in communications with facilitators in the context of Norwegian and Danish LS. In Chinese LS (i.e. Gu & Gu, 2016; Ding & Jones, 2018; Skott & Ding, 2022), the facilitator appeared to give more respect and space to openly talk in the post-lesson debrief meetings. Further study

is needed to understand in depth of the different role played by facilitators regarding building up an open, honest and constructive learning and working community at real-world work. Given the different respects and space given to LS participants to talk in the different cultural settings of LS (e.g., China, Denmark, Norway in our study), it is necessary to reconstruct the oversimplified dichotomy between a constructivist-oriented vs. a transmission-oriented learning (Richter et al., 2013) in the context of teachers' professional collaborative learning and working. Noticeably, Richter et al. (2013) highlighted the importance to reduce attrition that particularly happened to beginning teachers in the context of Germany school teacher professional field by support them to adjust to their new work environment, reduce stress levels and enhance job satisfaction. In China (and likely in other east Asian countries), attrition is not largely reported as a major problem for teacher professional development. Thus, as pointed in Skott and Ding (2022), future study needs to take into account the cultural social and power related issues particularly regarding the value-oriented pedagogy in teacher professional development, in order to make high influences on the participants [i.e. facilitators and teachers] engagement in collaborative learning and working. Our future research also needs to deal with further extending the tool by using it in the other phases of LS that have not yet been investigated in our study, and by using it with more LS cases. We also aim to test it against data from possibly new countries.

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APPLYING A CONSTRUCTIVIST PROGRESSION TO CHINESE STUDENTS: DO EARLY ERRORS INDICATE LATER REASONING?

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Using a multiplicative reasoning progression rooted in a constructivist units-and-operations model, we addressed a twofold problem: To what extent is the model applicable to Chinese primary students' reasoning – using the second, Same-Unit Coordination (SUC) and third, Unit Differentiation and Selection (UDS) schemes in the progression – and how may their errors on SUC tasks be related to strategies they used to solve more advanced, UDS tasks? We found that ~50% of the Chinese students (grades 3-6, n=545) made errors in solving SUC tasks, of which nearly half were totaling errors – and that those errors correlated significantly with the less advanced strategy (total-first) in solving UDS tasks. We discuss implications of these findings in terms of lessened opportunities for students to learn ways of reasoning multiplicatively.

INTRODUCTION AND CONCEPTUAL FRAMEWORK

A problem of interest to the PME community that we examined in this study is to what extent models about mathematics learning, developed through research programs in one culture, may provide an applicable tool to explain, and study, related phenomena in other cultures. Here, we focus on the extent to which a western-born, constructivist model of conceptual progressions in students' multiplicative reasoning, in terms of units and operations a learner brings forth and uses to solve tasks (Steffe, 1992), can help address questions about aspects of Chinese primary students' strategies for solving such tasks. Specifically, we examine how errors they make in solving tasks that require less advanced conceptualization may link with their solutions to more advanced tasks.

Studying the applicability of such a model seems strategic, as researchers agree that multiplicative reasoning is a conceptual leap from additive reasoning (Kamii & Clark, 1996; Lamon, 2007). When reasoning additively, the units operated on and the resulting units are all of one type (e.g., 5 slices + 6 slices = 11 slices). When reasoning multiplicatively, one operates on two types of units and the unit resulting from coordinating those are of a different type (e.g., 5 pizzas x 6 slices/pizza = 30 slices).

Tzur et al. (2013) elaborated on prior constructivist works to distinguish a 6-scheme progression. The second scheme, termed *same-unit coordination* (SUC), entails being cognizant of equal-size sets of composite units as being composed of 1s while operating additively on the number of composite units in each set. For example, a task may involve 9 pagodas with 3 floors each ($9P_3$) and 4 pagodas with 3 floors each ($4P_3$). Correct *SUC-sum* reasoning yields 13 *pagodas* ($13P_3$) and *SUC-difference* yields 5 pagodas ($5P_3$). A common error type we found among Chinese primary students (Ding

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 235-242). PME 46.

et al., 2022), termed *totaling error*, leads them to find the total of 1s (here, floors) in each set (27 and 12), then add or subtract those totals accordingly (e.g., 39 or 15, respectively). Figure 1 illustrates Chinese students' totaling error in finding the sum.

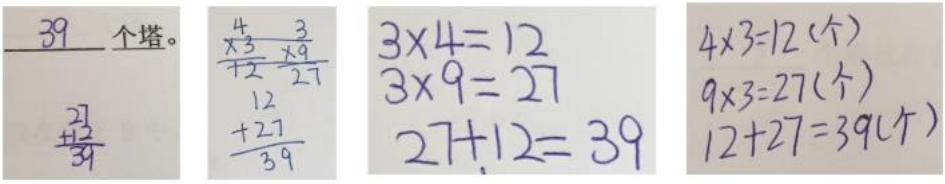


Fig 1. A sample of Chinese students' totaling error on SUC-sum tasks.

The third scheme in the progression, termed *unit differentiation and selection* (UDS), involves finding the difference (or sum) of 1s in two given sets of composite units. Consider this task: "Store A has 198 packs of Coca-Cola; Store B has 201 packs; each of those packs has 6 cans. How many more Coca-Cola cans does Store B have than Store A?" Researchers (Tzur et al., 2013; Wei, 2022) identified two strategies students may use to solve such tasks, a total-first and a difference-first. A *total-first strategy* involves finding how many 1s are in each set and then subtracting (e.g., 198x6=1188 and 201x6=1206; 1206-1188=18). A difference-first strategy involves finding the *difference in composite units* and then multiplying only that difference by the unit rate (e.g., 201-198=3; 3x6=18). The difference-first strategy is more advanced as it involves selecting the sets of composite units to operate on first (additively, as in SUC) and only then on the 1s in the resulting difference, whereas in total-first strategy all units are first changed to 1s (multiplication) and then subtracted as 1s. As our example shows, the difference-first strategy can improve efficiency and reduce computational errors. Critically, coordinating total- and difference-first is a necessary conceptual foundation for mindfully using the distributive property of multiplication over addition (e.g., [201-198]x6 = 201x6 – 198x6; see McClintock et al., 2011). As totaling errors on SUC tasks involve a total-first strategy for UDS tasks, our research questions were:

1. Using SUC and UDS as a lens, what solution strategies, and errors, can be identified in Chinese primary school students' solutions to SUC and UDS tasks?
2. Are Chinese students' totaling errors related to their UDS solving strategies?

METHODOLOGY

We present the sample, instrument, and data collection and analysis used for this study, which was part of a larger, Chinese-USA research project, *Primary School Students' Performance on Multiplicative Reasoning Assessment*.

Sample

Following local procedures for human subjects, all students (N = 545) in third (n=124), fourth (n=127), fifth (n=155), and sixth (n=139) grades at one school in a mid-size city in northeast China participated in this study (~50-50 gender split). These students took a new 31-item assessment developed and validated in the USA and China. For this

study, we extracted data from their solutions to six SUC tasks (3 SUC-diff and 3 SUC-sum) and to 4 UDS tasks.

Instrument

Due to space limitations, in Table 1 we present only key information of the 10 tasks used for this study. In the student's written assessment, each item appeared in full language on a separate page, with space and request to show their work. Two cycles of feedback from experts in multiplicative reasoning, followed by three cycles of back-translation process (English to Chinese to English) ratified construct validity. Then, we also used Rasch analysis. The Cronbach's alpha (0.93), the eigenvalue of variance explained by the measure (21.0) accounted for 40.4% of the observed variance (expected=40.1%), all eigenvalues of unexplained contrasts being less than 1.8, and satisfied criteria for reliability and unidimensionality (Brentari & Golia, 2007) of this assessment of multiplicative reasoning.

Task	Composites / 1s	Set A	Set B	Answer
2.1 SUC-diff	Shelves / Boxes	23 / 10	23-9 / 10	16 shelves
2.2 SUC-diff	Boxes / Cookies	17 in all / 5	17-11 / 5	6 boxes
2.3a SUC-sum	Bags / Candies	4 / 3	9 / 3	13 bags
2.3b SUC-diff	Bags / Candies	4 / 3	9 / 3	5 bags
2.4 SUC-sum	Rows / Chairs	18 / 7	23 / 7	41 rows
2.5 SUC-sum	Boxes / Tomatoes	15 / 6	7 / 6	22 boxes
3.1 UDS	Teams / Players	7 / 5	10 / 5	15 players
3.2 UDS	Pizzas / Slices	6 / 4	15-6 / 4	12 slices
3.3 UDS	Packs / Cans	198 / 6	201 / 6	18 cans
3.4 UDS	Bunches / Balloons	16 / 7	19 / 7	21 balloons

Table 1: Key information of SUC and UDS tasks used in the assessment.

Data Collection and Analysis

Graduate research assistants (GRA) administered the assessment to all classes during a regular math lesson, guiding all students to show their work. For the error type on SUC tasks (none, totaling, other), and UDS strategy (difference-first, total-first strategy, other), a GRA entered codes after Kappa statistic showed they reached 0.90 interrater reliability. We used a t-test and ANOVA to analyse mean differences and Cramer's V to test the association between totaling error and UDS strategies.

RESULTS

Table 2 shows students' performance on SUC and UDS tasks. Students in grade 6 scored highest on both SUC and UDS tasks (4.41 and 2.87, respectively); students in grades 3 (2.44) and 4 (2.78) scored lowest on SUC tasks; students in grade 3 scored lowest on UDS tasks (1.69). By showing the gradual growth in reasoning about and solving SUC and UDS tasks, these data lend support to our claim that the conceptual progression in multiplicative reasoning, at least for these two schemes, may be

applicable for students in cultures (e.g., China) other than in western cultures where it was initially studied and established.

Grades	3	4	5	6	All	F	Post Hoc
N	124	127	155	139	545		
SUC (max=6)	2.44 (1.98)	2.78 (1.98)	3.64 (1.96)	4.41 (1.64)	3.36 (2.03)	29.00***	6>5>4,3
SUC-sum (max=3)	0.98 (1.12)	1.17 (1.13)	1.64 (1.17)	2.13 (1.02)	1.50 (1.20)	35.45***	6>5>4,3
SUC-diff (max=3)	1.47 (1.05)	1.61 (1.07)	2.00 (0.99)	2.28 (0.83)	1.86 (1.03)	18.68***	6>5>4,3
UDS (max=4)	1.69 (1.22)	2.48 (1.12)	2.64 (0.99)	2.87 (0.95)	2.44 (1.15)	30.16***	6>4 5,4>3

*** Between-grade ANOVA (F-value) is significant at $p < .001$; a Bonferroni post-hoc analysis shows all grades' contribution to this difference.

Table 2: Students' mean performance scores (with SD) on SUC and UDS tasks.

To further examine these data, we first focus on students' scores on SUC and UDS tasks and then on how their totaling errors correlate with scores on UDS tasks. We note that, as hypothesized, overall students' scores on SUC and UDS tasks had statistically significant correlation ($r = 0.55$, $p < .001$).

Students' Performance on SUC Tasks

Students performed better on SUC-diff tasks than on SUC-sum tasks (mean = 1.86 vs. 1.50; $t = 9.07$, $p < .01$). Figure 2 shows percentages of their correct responses, totaling errors, and other errors on each SUC task. Overall, students made errors on 35% of SUC tasks, of which 51% are totaling errors. The percentage of totaling errors on SUC-sum tasks (65%) is higher than on SUC-diff tasks (37%), indicating a possible impact of the Chinese instructional focus on mastery of multiplication facts.

We divided all students' responses to the six SUC tasks into four categories: correct/no-error (n = 146, 27%), totaling error (3 or more, n=66, 12%), partial totaling error (1-2 errors, n=212, 39%), and other (n = 117, 22%). ANOVA of these categories shows statistically significant differences in students' scores on SUC tasks ($F = 92.0$, $p < .01$). Bonferroni post hoc analysis shows that the gap between students in the "no error" category (M=5.11, Std.=1.64) and students in the "totaling error" category (M=1.35, Std.=1.02) contributed the most to this F-value, while the "partial totaling error" and "other error" categories were somewhere in between (M=2.94, Std.=1.69; M=3.14, Std.=1.88).

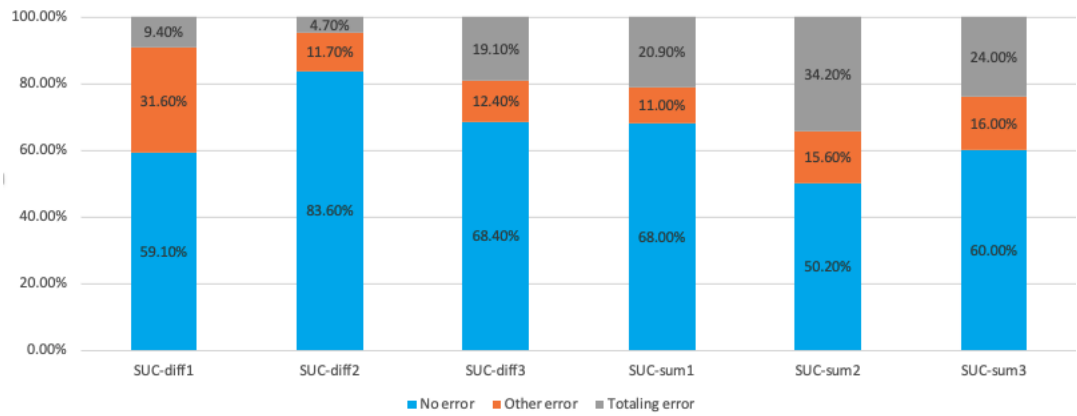
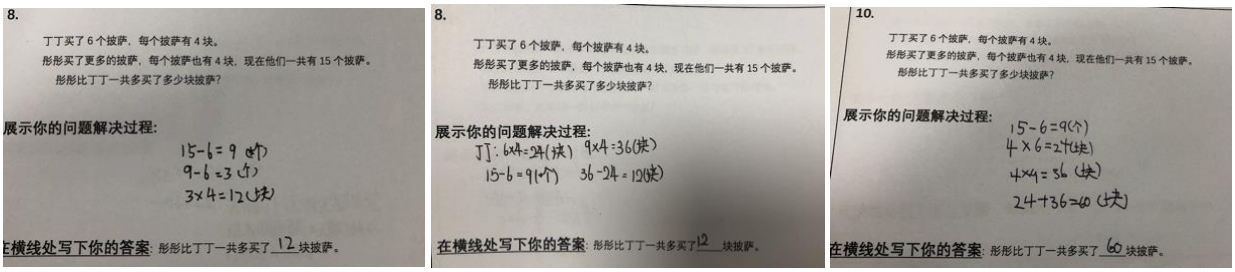


Figure 2. Students' responses to SUC tasks (percentages).

Students' Performance on UDS Tasks

Figure 3 illustrates three common strategies used in solving the UDS-3.2 task: (3a) first finding the difference in composite units between the two compilations, or (3b) first finding the total of 1s in each compilation, or (3c) finding the total of 1s in an UDS task, which requires the difference (3c). Figure 4 summarizes these statistically.



a. Difference-First b. Total-First c. Other

Figure 3. Examples of student responses to UDS tasks.

The data in Figure 4 further indicate that most participating students are accustomed to using the total-first strategy in UDS tasks (UDS-3.1, UDS-3.3 and UDS-3.4), while more students (47.4%) used the difference-first strategy for UDS-3.2. A plausible reason for this could be the complicated information given in UDS-3.2, as it includes the total number of pizzas for two children, the number of one child's pizzas, and the number of slices in each pizza. These givens may be taken by more students as a basis for first operating on the composite units. Also, the numbers in UDS-3.3 are larger than in the other tasks (201 packs and 198 packs, 6 cans in each pack). A student's choice to use a difference-first strategy makes it quite easy whereas the total-first calculation can be time-consuming and error-prone. Yet, most participating students (62.9%) used the total-first strategy.

To quantify levels of UDS, we focused on their responses to the two items that were most likely to be solved using a difference-first strategy. Our choice to focus on just two items intended to be strict with our results because students' choice to use total-first on the other two items could be attributed more to the task characteristics. We remind the reader that to establish a UDS scheme it would be necessary to construct

and coordinate the two solution strategies, with difference-first being more advanced conceptually (Wei, 2022). We thus categorized responses as "no UDS" if students

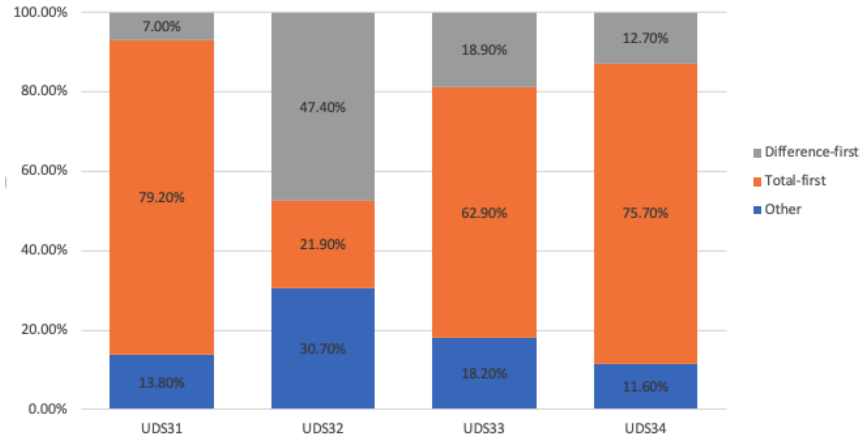


Figure 4. Students' strategies in solving each UDS task (percentages).

used neither difference-first nor total-first strategy, "Partial (diff+)" if they used difference-first for one of the two likely items, "Partial (total+)" if they used totaling for one or both items (but not difference-first), and "Diff-first" if they used difference-first for both items. Table 3 shows that ~13% had Diff-first, 41% had partial (diff+), ~35% had partial (total+), and ~12% had no UDS. Considering all results above, we next analyzed correlations between them.

UDS scheme	Diff-first	Partial (diff+)	Partial (total+)	No UDS
N	70	221	188	175
Percentage	12.9%	41%	34.6	11.9%

Table 3: Students' UDS strategy-levels (percentages).

Correlating students' UDS strategy-levels with their error types on SUC

ANOVA of students' error types in SUC shows statistically significant differences among the four categories ($F = 6.31$, $p < .01$; see Table 4). We thus analysed association between students' error-type on SUC items and their level of performance on the two UDS tasks as explained above. Figure 5 shows that 24% of students in the SUC "no-error" category indicated having the UDS scheme (Diff-first) as opposed to only 9.0% of students in the partial totaling error category and 9.1% of students in totaling error category. A Cramer's V test shows that this difference is statistically significant (Cramer's $V = 0.144$, $p < .001$).

Taken together, the results presented in this section support our hypothesis that participating Chinese student responses to SUC items are correlated with their UDS strategy-levels. This correlation, considered through a units-and-operations model of progression in multiplicative reasoning, suggests that participating Chinese students' solutions to SUC and UDS tasks may be explained by this model. Importantly, our

results shed light on the needed attention to the kind of errors students (here, Chinese) may make in solving SUC and UDS tasks – particularly totaling errors.

Table 4: ANOVA of different error groups performance on UDC tasks.

	Group	N	Mean	Std.	F	p
UDS (max=4)	No error	146	2.70	1.06	6.31	0.00
	Partial totaling error	212	2.50	1.11		
	Totaling error	66	2.35	1.03		
	Other error	117	2.10	1.30		

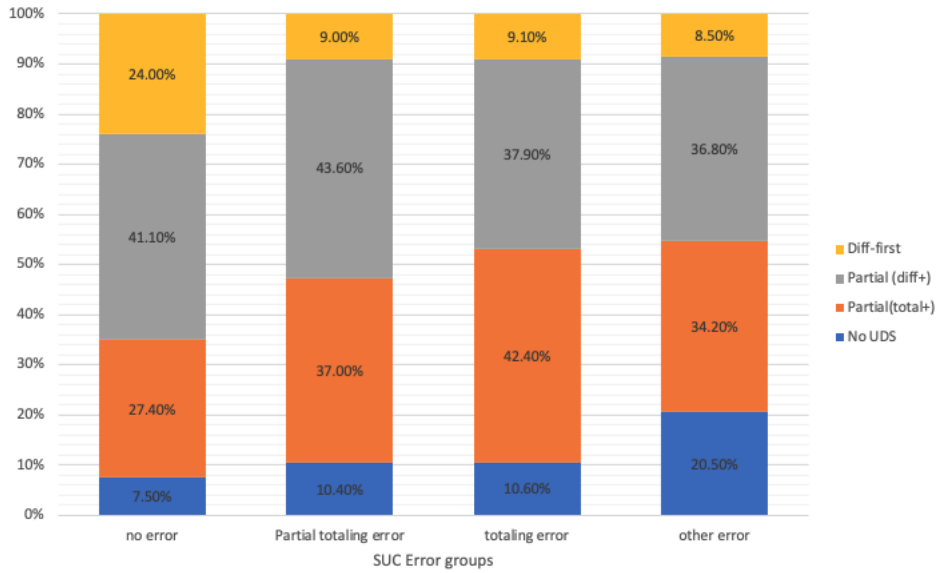


Figure 5. Students' UDS levels per SUC error-type categories.

DISCUSSION

Keeping in mind that a theory is needed to explain correlation among variables, our findings support three claims. Our first claim is that, seen through the model of conceptual progression from SUC to UDS, the correlation between totaling errors on SUC tasks and UDS strategy-levels indicate the former may serve as an indicator of the latter. Our second claim is that primary school Chinese students who made a totaling error (SUC) are more likely to use a total-first strategy to solve UDS tasks. Whereas a total-first strategy clearly enables to solve any UDS task, it may constrain the student's development of the more conceptually advanced, difference-first strategy. Taken together, these two claims imply that students who are making a totaling error are less likely to construct a coordinated UDS scheme (i.e., total- and difference-first), which is the conceptual basis for the distributive property of multiplication over addition (McClintock et al., 2011). They also imply to pay close attention to students' reasoning particularly when instructional practices (e.g., in China) seem to stress speedy and accurate use of multiplication facts, to which totaling errors seem related.

Our third claim is that this study lends support to a line of research looking to examine possible cross-culture models of conceptual development. Specifically, the western-born, conceptual progression in multiplicative reasoning that is rooted in a constructivist model of units-and-operations (Steffe, 1992) seemed applicable to students in a rather different culture (China). Our study of linkages among students' solutions to SUC and UDS tasks adds to a prior study of the totaling error in relation to the first scheme of multiplicative reasoning (Ding et al., 2022). Between that study and the present one, we begin to see how the first three (of six) schemes in the conceptual progression model may also help explain Chinese students' development. Thus, in a future study, we will focus on their development of the fourth scheme in the progression, which underlies mindful operations in a place value, base ten system.

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PRESCHOOL CHILDREN'S REPRESENTATION OF DIVISION WORD PROBLEMS THROUGH DRAWINGS

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Research has shown that preschool children can make sense of multiplication and division situations. However, researchers suggest that sharing is the most intuitive way for young children to solve division problems using manipulatives. Starting from the hypothesis that drawing might be another means by which young children can represent division situations, we conducted a task-based interview with a small group of Australian and Italian 6-year-olds about measurement and partitive division situations. Results indicate that the children could interpret both types of division and their drawings and gestures captured the strategies used. We contend that children's drawings capture their thinking more effectively than their use of manipulatives.

INTRODUCTION

Research has highlighted that children can represent multiplicative situations prior to commencing formal schooling (Bakker et al., 2014; Vanluydt et al., 2022). For instance, Bakker and colleagues (2014) found that first graders (6-7 year olds) could solve multiplicative word problems even when still unaware of how it is formally represented. Others reported that kindergarten children could detect a multiplicative relation (the ratio) in proportionality problems (Vanluydt et al., 2022) and after specific instruction could solve multiplication and division problem by modelling the situation with tally marks, counting, or recall of facts (Carpenter et al., 1993). Although the children in Carpenter et al.'s study were permitted to use counters or pencil and paper to help them solve the problems, many used counters only while modelling division problems.

We wondered if these findings indicate that preschool children can model division problems only with manipulative materials, or if drawings may be another means of representation for division situations. The study presented in this report explored this idea, as several authors consider drawing as a powerful tool for problem solving (e.g., Soundy & Drucker, 2009).

THEORETICAL FRAMEWORK

Young children's drawing in mathematics

Carruthers and Worthington (2006) investigated the development of young children's (3 to 8 year-olds) mathematical graphics. They defined graphics as the full range of marks children make when exploring mathematical ideas. These included dynamic, pictographic, iconic, written, and symbolic marks. They claimed that exploring mathematics through their own intuitive marks helps young children to make sense of standard symbols and bridges the gap between informal mathematics and the abstract

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel., & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 243-250). PME 46.

mathematics of school. Also, children's drawings, and narrative about them, are a "window into the mind of child" (Woleck, 2001, p. 215) allowing teachers and researchers insight into children's mathematical thinking.

Young children's drawings generally serve two purposes in the mathematics classroom. Drawings may support the process of mathematical work and/or represent the product of mathematical work (e.g. Smith, 2003). Smith (2003) described these purposes *as drawing as problem solving* and *drawing of problem-solving*. Similarly, MacDonald (2013) argued that drawings are "not just a procedure by which children record their knowledge about a concept; it is also a process through which understandings can be constructed, re-considered and applied in new ways" (p. 72). Furthermore, Woleck (2001) identified that first graders used drawings in mathematical problem solving "as if they were manipulatives" (p. 216) to carry out the steps of organising and counting, that supported their problem-solving efforts. She also reported that children might use drawings as a prewriting tool to communicate their mathematical thinking to others.

In the context of multiplication and division, Mulligan (2002) found that children's drawings and their explanations of the drawings could be used to identify how they notice multiplicative structures. She reported that the images drawn by low attaining children in the primary years tended to lack structure and were poorly organised. This was attributed to an underlying lack of awareness of the equal groups structure and a reliance on using counting by ones when solving problems.

Division of integers

While it is acknowledged that division is more than just sharing (Squire & Bryant, 2002), the physical act of sharing a quantity equally is division, in that to share a quantity successfully one divides a dividend into equal quotients. Previous research has found that young children (4 to 5-year-olds) can share out quantities using one-to-one correspondence, and model division problems using concrete materials long before any formal introduction to division (Carpenter et al. 1993; Frydman & Bryant, 1988). Furthermore, these initial strategies tend to reflect the action described in the problem (Marton, 1996). These findings led many to suggest that sharing is the schema for action from which an understanding of division develops (e.g. Correa et al., 1998; Squire & Bryant, 2002). Frydman and Bryant (1988) found that although most 4-year-olds in their study could share items equally between two groups, only half of the children (10 out of 24) were able to infer the number of items in each set. This suggests that these children have an understanding of the numerical significance of sharing. It also suggests the developmental nature of this concept.

Division word problems can be interpreted and represented in two different ways, namely division by the multiplier (partition division) and division by the multiplicand (quotitive or measurement division) (e.g., Correa et al., 1998; Greer, 1992; Verschaffel et al., 2007). According to Greer (1992, p. 276):

> Dividing the total by the number of groups to find the number in each group is called partitive division, which corresponds to the familiar practice of equal sharing […]. Dividing the total by the number in each group to find the number of groups is called quotitive division (sometimes termed measurement division, reflecting its conceptual links with the operation of measurement).

The difference between quotition and partition problems relates to the textual structure of the problem (Nesher, 1988). For example, the expression 12 ÷ 4 could be interpreted as a partitive problem, such as: Twelve lollies are shared equally among 4 children. How many did they each receive? In solving the problem the action is one of sharing or distributing the twelve lollies equally between the four children. Interpreted as a quotitive problem, using the same context: There are 12 lollies and each child receives 4. How many children will receive lollies? While the quotient is the same for each, the model is quite different, so is the action. Rather than an action of sharing it is a grouping or count of the twelve lollies into groups of fours.

In summary, critical ideas for students to construct are: that collections/objects can be divided into equal groups; division involves part-whole relations that include three elements: the size of the whole, number of parts, and size of the parts; there is a relationship between three values represented by the dividend, divisor, and quotient; and that division is the inverse of multiplication, in which case multiplication can be used to solve division problems.

Our focus in this report is on four division problems (two measurement and two partitive) (see Table 1), which are part of an interview in our larger study.

Measurement division	Partitive division
P1. Tad fished 12 tadpoles. He put 4 tadpoles in each jar. How many jars did Tad put tadpoles in?	P3. Mr. Gomez had 12 cookies. He put the cookies into 3 boxes so there was the same number of cookies in each box. How many cookies did Mr. Gomez put in each box?
P2. John had 6 crayons. He put 2 crayons in each box. How many boxes did John need?	P4. You have 12 marbles and you give the same amount of marbles to three friends. How many marbles do they each get?

Table 1: Division problems used in the interviews.

In the context of using manipulatives, Carpenter et al. (1993) noted that young children enact different solution strategies when solving a division problem. When solving measurement division of the type M:N, some children counted M items and then arranged them in sets of N items each (*measuring* M in terms of Ns). Others made sets of N items until they reached a total of M items and then counted the number of sets. Some students counted by Ns (*double count* according to Kouba, 1989). In the context of partitive division A:B, children arranged A in B sets with the same number of counters in each set (*grouping*). Some children distributed the counters one by one into

the B sets (*sharing*). Others put a number of items in each B set then adjusted the number in each (*trial-and-error* strategy) or used *known or derived facts*.

METHODS

Since we were interested in understanding how children represent division situations using drawings, we needed suitable methods. In particular, we wanted to identify the elements that allow understanding if the situation is modelled as expected, that means if a measurement or partitive division is modelled as such. We needed to distinguish between drawing-as-problem-solving and drawing-of-problem-solving (Smith, 2003). We expected some of the strategies described by Carpenter et al. (1993) in the context of manipulatives to appear in the context of drawings as well.

Research shows how children can produce drawings together with other means of communication like talk, writing, movement, and sound (Soundy & Drucker, 2009). With the aim of taking into consideration all the possible means of communication involved in the production of the drawings – including gestures, spoken words, and the use of manipulatives – we decided to adopt a multimodal semiotic approach by referring to the construct of a *semiotic bundle* as presented by Arzarello et al. (2009). A semiotic bundle is:

> a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. (Arzarello et al., 2009, p. 100)

In particular, the *synchronic* and *diachronic* analysis of the semiotic bundle may give hints about how the children represent the proposed situations, what are the relations between the different representations and how such representations (do or do not) help the child during the process. The diachronic analysis focuses on the evolution of signs over time, and the transformation of their relationships; the synchronic analysis, instead, focuses on the relations among the signs used in a certain moment. In particular, we will speak of *genetic conversion* (Arzarello, 2006) when the conversion rules between semiotic sets have a genetic nature, namely, one semiotic set is generated by another one, so enlarging the bundle. Our research questions can be then rephrased: Do the same strategies observed for manipulatives appear in the case of graphics? How do the different components of the semiotic bundle correspond with preschool children's mathematical graphics of the division situations?

We conducted our research in countries with different languages (Australia and Italy, so English and Italian languages). In particular, these two languages differ in the way in which multiplication is verbally represented. In English (as in many other languages) the word 'times' is used to read multiplications (like expressions such as 3×4. This is not the case in Italian, where the symbol × is read 'per' is unrelated to the word 'volte' ('times' in Italian) but refers only to the name of the symbol itself. In the Australian context, children start preschool (kindergarten) when they are 4 year-olds; from there

they transition to their first year of primary school at the age of 5. In Italy children move directly from kindergarten to primary at the age 6.

Prior to commencing the main study, the interview tasks and recording sheet were trialled, following which refinements were made to the language and the number range. We interviewed Australian children during their Foundation year, while Italian children were interviewed at the very beginning of first grade. Our sample consisted of 19 children with an average age of 6 (6 males and 4 females were Australian; 5 females and 4 males were Italian). The researcher interviewed the children individually using an interview script to ensure consistency. The task could be repeated as many times as needed. Each interview took for approximately 15 minutes. Each interview was video recorded with parent consent. Video recordings were transcribed verbatim, and the transcript was enriched with images of gestures and drawings to describe the semiotic bundle (Radford & Sabena, 2015).

RESULTS

The interview data were categorized in terms of strategies used for measurement or partitive division situations. The video analysis took into account the different modalities of representations, not only the drawing, but also the relations between the different signs both synchronically and diachronically, exemplified below. Table 2 includes the strategies we observed in the children's drawings, which correspond to those observed by Carpenter et al. (1993) when children used manipulatives (see Theoretical Framework section).

Table 2: Occurrences of each strategy for addressing the proposed division situations

	Measurement division		Partitive division	
	P1 (tadpoles)	P2 (crayons)	P3 (cookies)	P4 (marbles)
measuring	✓	✓		
double count	✓			
grouping			✓	
sharing			✓	
trial-and-error			✓	✓
known/derived facts		✓		✓

The diachronic analysis of the process of drawing (including speech and gestures) allowed us to distinguish between the different categories. For instance, Figure 1 shows the final graphics from two different children. The graphics depict P1 (the tadpole's problem) and in both cases it is possible to see a human character and three jars, each containing four tadpoles. However, the processes behind these two drawings are completely different and, we contend, highlight two different strategies of interpretation of this measurement division situation. The Italian child started by drawing one jar with four tadpoles inside (Figure 1a). In contrast, the Australian child drew a long fishing pole and the twelve tadpoles; the jars (represented by C-like lines)

were added as last element (Figure 1b). He is measuring the number of tadpoles (12) in terms of the number of tadpoles-per-jar (4) We noticed that this second procedure follows the same order of appearance of the information in the verbal presentation of the situation: the character first, then the number of tadpoles, and finally the number of tadpoles per jar. The first procedure did not present, at least initially, the information about the total number of fished tadpoles while the sequence of drawing appears as a genetic conversion of the sentence 'He put 4 tadpoles in each jar'. The character was added as last element.

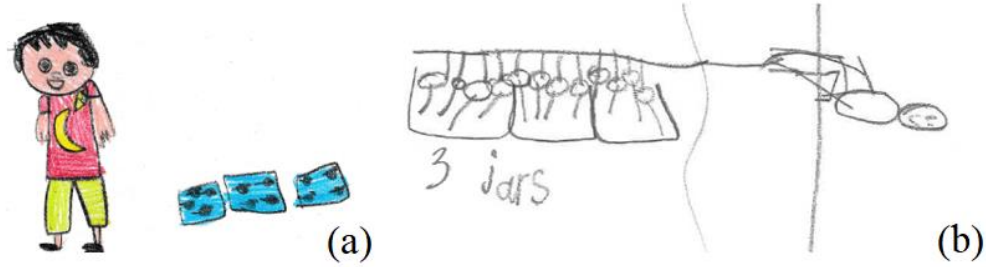

Figure 1: Two children's mathematical graphics of the tadpoles' problem (P1).

Indeed, after drawing the first jar, the child whose graphic is depicted in Figure 1a, drew a second jar with four tadpoles inside (Figure 2a). Then, she counted the drawn tadpoles pointing to them one by one (Figure 2b) After realizing there were eight tadpoles, she stated that another jar was needed and drew the last jar (Figure 2c). We can see that she is using a double count: she is keeping the count of the number of tadpoles-per-jar in each jar and, at the same time, checking that the total amount of tadpoles reaches the expected quantity of twelve.

Figure 2: Video screenshots of the process of generation of the graphic in Figure 1a.

DISCUSSION AND CONCLUSION

Our results show that when young children represent division situations through drawings, they adopt solution strategies similar to those who used manipulatives in earlier studies (Carpenter et al., 1993). A key finding from this study is that drawing may be an efficient tool not just in the sense of *drawing of problem-solving* but *as problem solving* (Smith, 2003) as well. Through the diachronic analysis of the semiotic bundle, we were able to exemplify how different processes involved in the creation of mathematical graphics may correspond to different strategies. Also, that some of these strategies only appear as genetic conversion of the words used to present the situation into inscriptions.

Our results suggest that drawing can be an efficient tool for problem solving in preschool as much as the use of manipulatives. We contend that our analysis allows us

to argue that drawings may help researchers (and possibly teachers) capture children's thinking more effectively than manipulatives. Indeed, the diachronic analysis of the relationships between the different inscriptions clearly shows the model of division adopted by the child while allowing the researchers to distinguish the presence of those that we have called the critical ideas related to division (see Theoretical Framework section). Observing the sequence of each child's drawings and gestures they used enabled us to notice whether the child had a sense of the equal group structure and or relied on counting by ones to solve the problems, as observed by Mulligan (2002). Also, we observed grouping strategies for both partitive, and measurement division situations. This result contradicts the commonly held view that sharing is the schema for action from which an understanding of division develops (e.g., Correa et al., 1998; Frydman & Bryant, 1988; Squire & Bryant, 2002).

It is possible that through the production of mathematical graphics that children are able to adopt such strategies: while drawing they produce a permanent record of their previous thinking which may allow them to reconsider it or reason about it – by themselves or with the scaffolding of an adult. Also, symbols like connection-lines and arrows may help in re-tracing the enacted processes of moving, grouping, or sharing. Such strategies are not possible with manipulatives, as only the final product of the process remains visible.

We acknowledge further research is needed to provide large-scale evidence for these speculations. In particular, the study presented in this research report was explorative in its nature and involved a very small sample. The fact that the sample was constituted of children with different schooling experiences and different languages supports the possibility of generalization of the obtained results. In the future, we plan to repeat the same study on a larger scale to support our conjecture.

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MATHEMATICAL KNOWLEDGE FOR TEACHING FOR COLLEGE ALGEBRA AT COMMUNITY COLLEGES

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In this report, we present an initial analysis from a study that is developing an instrument to assess community college instructors' mathematical knowledge for teaching college algebra. This instrument contains multiple-choice items that are organized around two types of tasks of teaching, understanding student work and choosing problems, and covers topics in linear, rational, and exponential functions. The instrument is being administered to instructors teaching at a nationally representative and stratified sample of community colleges in the United States. Using the current partial sample of 289 instructor responses from 184 institutions, we describe item characteristics and preliminary patterns in the responses.

Substantial work has been performed to understand the nature and composition of mathematical knowledge for teaching. Foundational work by several scholars (Ball et al., 2008; Rowland et al., 2005) has indicated that there is a strong relationship between teachers' knowledge about mathematics and its teaching, and the quality of their work in the classroom. Furthermore, the quality of instruction has been shown to have a positive impact on student learning. Ball and colleagues (2008) hypothesized five components building upon Shulman's (1986) distinction between content and pedagogical knowledge but failed to empirically show their hypothesized multidimensionality of the construct with their instrument (Hill, 2010). On the other hand, Ko and Herbst (2020), theorizing about knowledge needed to perform two specific tasks of teaching instead, identified a two-dimensional structure in the context of high-school geometry teaching, one dimension related to the knowledge needed to choose givens for a problem and the other dimension related to the knowledge needed to understand students' work. We build upon this work, hypothesizing a two-dimension construct for the knowledge based on two different tasks of teaching college algebra, one related to the knowledge needed to understand students' work, and the other related to the knowledge needed to choose problems for teaching a mathematical concept, differently from Hill and colleague's work, and from Ko and Herbst, we focus on topics usually taught in a college algebra course—linear functions, rational functions, and exponential functions.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 251-258). PME 46.

THEORETICAL FRAMING

We grounded item design in the theoretical work on mathematical knowledge for teaching and in the context of community colleges.[1] We adopted the approach of studying *tasks of teaching* related to the *instructional situation* of presenting concepts to students via examples. *Tasks of teaching* refers to tasks that arise when teaching, for example "planning for … lessons, evaluating students' work, writing and grading assessments, explaining the classwork to parents, making and managing homework, attending to concerns for equity, and dealing with the building principal who has strong views about the math curriculum" (Ball et al., 2008, p. 395). In an instructional situation, instructors need to manage the interactions between students and content; they have the responsibility (and obligation) to offer students work that will be directly related to learning a piece of mathematics and of analysing the mathematics evident in students' utterances or in their written work produced while learning mathematics. Community college instructors typically use examples to anchor the presentation of the material and solve them collaboratively with students (Mesa & Herbst, 2011). This is an ideal instructional situation that showcases the two types of hypothesized knowledge as teachers need to both be able to choose problems that exemplify specific key ideas of the content and be able to understand students' work (approaches or mistakes) when solving those problems. We posit that the knowledge needed in each case is of different nature.

College algebra at community colleges in the United States encompasses many algebra topics with great variation across institutions and textbooks used to teach the topics. To make the instrument manageable, we narrowed the content to address linear, rational, and exponential functions, as related concepts (e.g., covariation, function transformation, algebra of functions, rate of change, behaviour of a function over a whole domain, etc.) are essential for building a conceptual understanding of ideas that are needed for courses within a calculus sequence—a sequence required for many students interested in pursuing a degree in science, technology, engineering, or mathematics (VMQI-AI@CC 2.0 Team, 2021; Watkins et al., 2016). The Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA) instrument that we developed includes items within the two organizers, tasks of teaching and topics, resulting in a proposed knowledge structure with six dimensions (Figure 1).

[1] Community colleges are tertiary-type B institutions that offer courses in the first two years of post-secondary education (OECD, 2017).

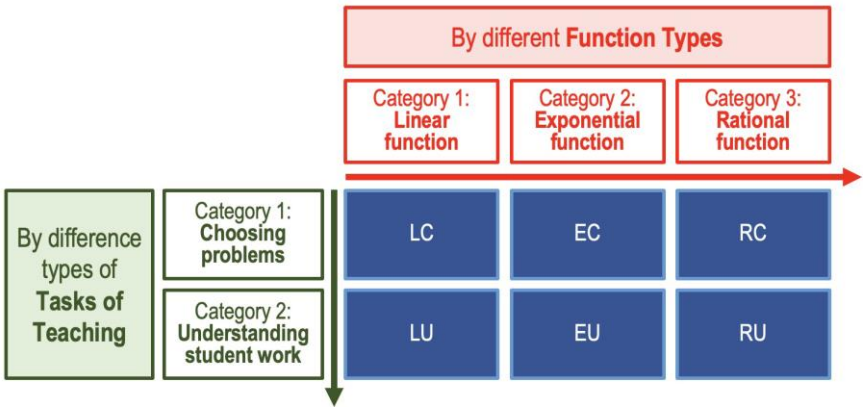

Figure 1: Blueprint for the development of the MKT-CCA instrument.

In earlier work, using responses from an instrument developed to assess mathematical knowledge for teaching high-school algebra (Phelps et al., 2014), we identified a positive connection between the types of courses that community college instructors had taught and their scores in the instrument. Specifically, those instructors who had taught advanced algebra courses performed better than community college instructors who have not (Ko et al., 2021). As we are only about halfway through our data collection process (we are targeting a sample of 600 respondents), we decided to examine participants' performance at an item and an average of the item-level rather than at a construct level that may require testing dimensionality with a larger sample size. However, and for the same reason, we investigated participants selecting the options that we deemed correct with respect to their teaching experience specifically, years of full-time equivalent mathematics teaching experience and number of times they taught various types of courses. We answer the following question: What is the relationship between instructors' teaching experience (reports of years of full-time equivalent mathematics teaching and courses taught) and the proportion of correct item responses?

METHODS

Participants

We used random and census approaches to recruit participants. We first used the National Center for Education Statistics to select a stratified random sample of 799 public colleges in the United States that primarily grant associate's degrees (out of 1374 institutions). The sample was representative of two-year institutions in terms of region in the country, size of the student body, setting (e.g., city, rural, suburban), and student racial diversity. We then recruited instructor participants by emailing full- and part-time instructors in our sample who teach at those institutions. We also sent email invitations to instructors who are members of the American Mathematical Association of Two-Year Colleges and to all full- and part-time faculty at minority-serving two-year institutions. This report is based on the responses from 289 community college instructors, about halfway from our target of 600.

Instruments

We use two instruments: an MKT-CCA instrument assessing the knowledge and a Background Characteristics survey. The MKT-CCA instrument consists of balanced sets of items assessing the knowledge used in choosing a problem and understanding student work across three types of functions (see Figure 2 for an example of an item on exponential functions and understanding student work).[2] The MKT-CCA instrument has 27 multiple-choice items and seven testlets (an item with a common stem and four options each of which can be either correct or incorrect). The Background Characteristics instrument contains 22 questions, including four questions inquiring about participants' gender, age, race, and ethnic backgrounds and four about their teaching experience: the total number years of full-time-equivalent teaching experience in mathematics (FTE_MT), and the number of times they have taught (a) mathematics courses before college algebra (B_CA), (b) college algebra (CA), and (c) courses that follow college algebra (A_CA) (the options *Never, Less than 5 times, More than 5 times and less than 10 times, More than 10 times* apply to these four questions). The instruments are distributed via Qualtrics. The MKT-CCA items are presented in the same order to all the test-takers, with five items per page, and they are not timed. Respondents are asked not to use any additional resources to answer the items and encouraged to complete all in a single session.

Analysis

To answer our research question, this proposal used item difficulty, each participant's average proportion of correct answers, and their teaching background variables. Item difficulty is calculated as the percentage of respondents who chose the option deemed as correct; thus, the higher the value in item difficulty, the easier the question. According to Lord (1952), for a four-response multiple-choice item, the ideal difficulty would be 74, or 74% of respondents choosing the correct response. We used four variables related to teaching background: the reported FTE_MT (continuous) and three categorical variables regarding the number of times they taught college algebra (CA), and courses below (B_CA) or above it (A_CA). Using SPSS (IBM, 2020), we tested the correlation between each participant's proportion of correct answers and their FTE_MT and performed chi-square tests to identify an association between whether participants choose a correct or an incorrect answer per item and their CA, B_CA, and A_CA. In addition to check how teaching experience was related to responses to individual items we used a non-parametric test, the Mann-Whitney U, to test the difference in responses of choosing a correct or an incorrect answer per item with teaching experience. The responses from 289 participants who responded to all assessment items as of 1 January 2023, constitutes our analytical sample. They represent 184 different institutions, 44% of which are in the West, 30% in the South, 19% in the Midwest, and 7% in the Northeast of the United States. About 51% of the

[2] We are sharing items that are not part of the instrument as the data collection is ongoing. The presentation of the item is modified to fit the template requirements.

institutions are in urban cities, 27% in the suburbs, 12% in cities without suburbs, and 9% in rural areas. Fifty-two percent of the institutions have between 5,000-19,999 students, 26% have 20,000 or more, and 22% have less than 5,000 students. About 45% of the participants identified as female and 51% identified as male; about 76% identified as Caucasian, 10% as Asian, and 7% as Black or African American. On average, the participants reported having 16 years of full-time mathematics teaching experience (SD = 9 years); regarding the number of times that participants reported teaching (a) courses before college algebra, (b) college algebra, (c) courses that follow college algebra, and (d) courses not related to a college algebra sequence a relatively high percentage indicated reporting the courses 10 times or more (59%, 64%, 53%, and 43% respectively); while a small percentage of teachers reported never teaching courses before college algebra, college algebra, or courses not related to a college algebra sequence, almost one fifth of the participants (18%) indicated not having taught courses that follow college algebra.

Students have been working with exponential models. The following question was assigned as homework.

In certain medical diagnostic imaging scans, a radioisotope is used to diagnose medical conditions. The amount of the radioisotope in the body decays exponentially so that 90% of the amount injected remains one hour after injection. Complete the table showing the amount of radioisotope remaining in the body every hour after 10 millicuries are injected.

Hours:	0	1	2	3	…	*n*
Amount:	10					

Ashwin's work is shown below.

Hours:	0	1	2	3	…	*n*
Amount:	10	9	8	7	…	0

Which of the following (four options, A-D) is the most likely incorrect interpretation Ashwin made about the model?

A. Ashwin thinks that $n = 10$.

B. Ashwin thinks that the amount of radioisotope decreases by 1 millicurie per hour after injection.

C. Ashwin seems to know that the amount of radioisotope in the body decreases by 10% but thinks that only applies to the initial value.

D. Ashwin thinks that at the start of each hour, the amount of radioisotope in the body is 90% of the amount that was in the body the previous hour.

Figure 2: Sample item of understanding student work on exponential functions

Limitations

The results of our preliminary analysis are impacted by two limitations. First, to ensure that the instrument reaches faculty in various parts of the country and that they answer it at a time when it is convenient for them, we administered the instrument online and did not proctor it. While this strategy increases response rates, a negative impact is that while we request respondents not refer to outside resources, we do not know that this is the case. Second, to reduce the number of pages a respondent must go through, and the potential fatigue created by answering a multiple-page instrument, we presented

five items per page. This way of presenting items makes it impossible to determine how much time on average each item takes, which is a consideration when assessing item quality. At the same time, we can reach more faculty across the nation and from different institutions, which enriches our data set.

FINDINGS

Because we do not yet have a sizable sample of respondents to perform an analysis of the dimensionality of the measure, and for space reasons, we report only on the percentage of correct answers of the 27 multiple-choice items (most respondents selected the expected options in the seven testlets). The average percentage of correct responses for the multiple-choice items varied from 20% to 79%, with an average of 57.23% correct (and a standard deviation of 13.86%). The item that had only 20% of respondents selecting the correct answer is an exponential function item on understanding student work (see Figure 2 for a comparable item). Figure 3 shows the distribution of the average of correct responses per item across the 289 respondents. The distribution suggests that the respondents are choosing the answers that we have identified as correct within a reasonable range.

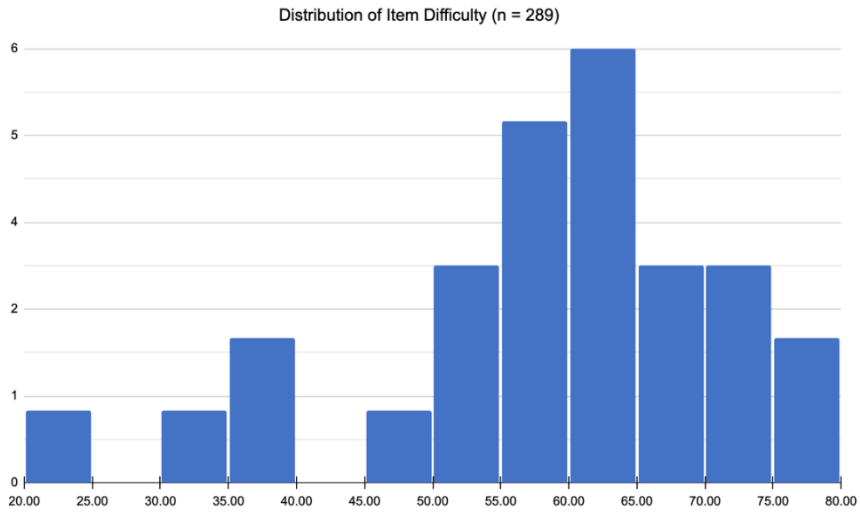

Figure 3: Distribution of item difficulty
(percentage of participants who answer an item correctly)

No items are extremely difficult or easy, and on average, the item difficulty is closer to 60%. The distribution of the percentage of correct responses per item is skewed towards the right, suggesting that the items are within the ideal difficulty of 74% (Lord, 1952). We found a positive and significant association between participants' average proportion of correct answers and their number of years of teaching experience (r = 0.164, p < .027). Mann-Whitney U tests for six individual items (3-rational, 2-linear, and 1-exponential) suggest that instructors with more full-time mathematics teaching experience more frequently chose the correct answers on these items than instructors with less teaching experience (U ranges from 6575.5 to 11947.5, 0.001 < p < 0.05; and an average effect size of 0.132). Chi-squared tests indicate that in 12 items (6-rational,

4-linear, and 2 exponential), instructors who had taught college courses more than 10 times tended to select the correct option more frequently than instructors who had taught college courses less than 10 times. This was true for instructors who indicated teaching college algebra or college-level courses that build on college algebra skills more than 10 times. We were unable to identify a pattern in these items that related to tasks of teaching, function type, or other aspects of the items (e.g., graphs, tables, context information).

DISCUSSION AND NEXT STEPS

The positive correlation between years of full-time equivalent mathematics teaching and choosing a correct answer on items designed to assess mathematical knowledge for teaching college algebra suggests that the items might be related to some construct of teacher knowledge. This finding is promising, and if the analysis with the full sample confirms such a pattern, our work would be aligned with current literature. Equally promising is that for a sizable number of items, more instructors who have taught college algebra, or courses below or above it more than 10 times, selected the assigned correct answer more frequently than instructors who taught less. If confirmed with the larger sample, we would be corroborating prior work that indicates that teachers who have specific teaching experience in related content perform better in instruments that assess mathematical knowledge for teaching algebra. As we progress in our data collection, we are working at making sure that the sample of institutions reflect the stratification scheme by region, location, and size (we have larger proportions than national estimates of institutions in the West, institutions that are medium size, and institutions in urban areas). Additionally, we will be recruiting a sample of students who are competent in the mathematics content assessed by the items but who have no college teaching experience, to ascertain that the instrument indeed captures knowledge needed for teaching rather than mathematical content knowledge.

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Erratum: Data contains 17 unverified cases. Corrections are available upon request.

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THE EFFECTS OF DIFFERENT TEACHING APPROACHES ON ENGINEERING STUDENTS' MODELLING COMPETENCY

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This paper reports on results from the CoSTAMM project. A modelling unit was implemented in 2022 at a public university in South Africa in a first-year engineering mathematics class, with 112 students following an independency-oriented teaching approach and 89 students following a teacher-directive approach. The same unit, with the same two teaching designs, had been implemented in 2019, with 150 first-year engineering students. The unit consists of a diagnostic test, a pre-test, five lessons with ten tasks, and a post-test. Linear mixed regression models were used to evaluate and compare the results. In both years the teaching designs yielded interesting effects on the development of students' modelling competency.

THE COSTAMM PROJECT

Various empirical studies have shown that solving modelling problems is cognitively challenging for students at all levels (for an overview see Niss & Blum, 2020, chapter 6). Hence, an important goal of research is to explore which teaching methods are effective for teaching mathematical modelling, where "effective" is measured by how far the goals for teaching modelling are reached. One essential goal is to advance students' modelling competency (Niss & Hojgaard, 2019), that is their ability to solve real-world problems by means of mathematics. What is particularly needed in research are *comparative studies* into the effects of different teaching methods on students' acquisition of modelling competency (Cevikbas et al., 2022; Schukajlow et al., 2018).

One such project where teaching methods for modelling were compared is the German *DISUM project* (which was the basis for our CoSTAMM project on which we report in this paper; for the conception and the results of DISUM see Blum, 2011; Blum & Schukajlow, 2018; Schukajlow et al., 2012). The global research question in DISUM was: How can students' mathematical modelling competency be advanced effectively in everyday teaching practice? The focus was on the lower secondary level, and *quality teaching* (König et al., 2021; Kunter et al., 2013; Schoenfeld, 2014; Schlesinger et al., 2018) was the conceptual frame. In DISUM, five dimensions of quality teaching were distinguished: effective classroom management, student orientation, cognitive activation, meta-cognitive activation, and demanding orchestration of topics (see Blum, 2015, for details).

In the DISUM main study, three teaching designs for a ten-lesson mathematical modelling unit were compared concerning their effects on students' achievement and attitudes: an independence-oriented design called "operative-strategic" teaching, a teacher-directed design called "directive" teaching, and a blend of both designs called "method-integrative" teaching, with a classical pre-post-test research design.

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The guiding principles of the *independence-oriented design* were: a permanent balance between students' independent work and teacher's guidance, with adaptive teacher interventions, encouraging individual solutions; a systematic change between students' independent work in groups and whole-class activities; and a "Solution Plan" (see Schukajlow et al., 2015), that is a four-step modelling cycle, as the basis of the teacher's diagnoses and interventions (not in students' hands).

The guiding principles of the *teacher-directive design* were: The development of common solution patterns guided by the teacher; and a systematic change between whole-class teaching, oriented towards "the average student", and students' individual work on exercises.

The third design resulted from a comparison of the effects of the first and the second design, which led to the so-called *method-integrative design* which means the independence-oriented design with some directive elements (in particular, the teacher as a "model" who shows once in the beginning how to solve modelling tasks along the Solution Plan), and with the Solution Plan as a meta-cognitive tool in students' hands.

In addition, all three designs were oriented towards some basic criteria of quality teaching (especially effective classroom management).

The essential results of these comparisons were: There was significant progress in mathematics for all three designs, but significant progress in modelling only for the two independence-oriented designs, and here the progress for the method-integrative design was substantially higher than for the operative-strategic design.

An interesting question was: Will similar effects be visible also in other environments? In October 2018 the idea arose to form a South African/German research team (consisting of the three authors), to conceive and carry out research studies similar to DISUM at the tertiary level, and to compare these studies with analogous studies at the secondary level. This was the beginning of the *CoSTAMM project* ("**Co**mparative **S**tudies into **T**eaching **A**pproaches for **M**athematical **M**odelling"). Our studies were so far (for details see the following sections):

- In 2019 a comparison of the method-integrative and the teacher-directive design for first year engineering students from an extended curriculum programme in South Africa (see Durandt et al., 2022a).
- Parallel studies for South African engineering students, conceived for 2020 and 2021, both cancelled due to the Covid-19 measures in South Africa at that time.
- In 2022 a parallel study with the same two teaching designs for South African engineering students from a normal curriculum programme (the results will be presented in this paper), a parallel study for German grade 9 students (the results will be presented at ICTMA-21), and a method-integrative study for South African analytical chemistry students (the results have been presented at ICTMA-20).

DESIGN OF THE COSTAMM STUDIES (2019 & 2022)

The CoSTAMM studies at the tertiary level mentioned in section 1 follow a classical design of an entrance test, a pre-test, a treatment, and a post-test.

- The *entrance test* (90 minutes) focusses on six content areas (algebra, geometry, functions, trigonometry, calculus, and modelling) and is used as a diagnosis of basic mathematical competencies which are taught at school. Details regarding this test are documented in Durandt et al. (2021a).
- The *treatment* consists of a modelling unit with ten tasks, organised in five lessons (45 minutes each) and embedded in the topic area of functions. This unit is presented in detail in Durandt et al. (2022b).
- The *pre- and post-test* (45 minutes each) are aligned with each other and with the treatment, and consist of three sections each: Section A with open modelling tasks, Section B with mathematical tasks, and Section C with multiple choice modelling tasks (taken from Haines et al., 2001). Both tests are administered in two versions with parallel items, following a rotation design, randomly and equally distributed to groups, which allow also for comparing pre- and post-test results. Further details on these achievement tests (e.g. with regard to evaluation objectivity, internal consistency of the scales, etc.) are documented in Durandt et al. (2022a) and Table 1.

The study in 2019 was conducted from February to April at the University of Johannesburg. The participants were randomly divided in three equal groups (approx. 50 per group) according to the university timetable. One group was exposed to the *method-integrative* (MI) teaching design, while the two other groups were exposed to the *teacher-directive* (TD) design. Both the MI and one TD group (TD1) had the same lecturer. The main research question was: How do the modelling competency and the mathematical competency of students develop through the modelling unit, depending on the teaching designs? The study also included an attitudinal component, but we do not deal with that component in this paper.

To examine the effects of the teaching intervention (development of students' achievement), *linear mixed regression models* with dummy-coded predictors were estimated for the three test sections and for the overall test score as dependent variables using the statistical software R (R Core Team, 2022) and additional packages such as "lme4" (Bates et al., 2015). In these models the longitudinal data structure nested by participants was considered, the different teaching designs (groups) were directly compared (for details see Hilbert et al., 2019), and the total score of the entrance test was included in the model as a (global) covariate. The results from the 2019 study were presented at ICTMA-19 (see Durandt et al., 2021b). The essential result was that all groups had a significant learning progress, so both the MI and the TD teaching design had effects, and the MI group had the biggest competency growth, particularly for mathematical modelling.

To compare the results with another sample, the study was repeated with the same unit and the same test instruments at the same university from February to April 2022 with 201 first-year engineering students from the mainstream programme studying towards a qualification in either mechanical or industrial engineering. Students were divided in two unequal groups based on their official university tutorial timetable; 112 students followed the method-integrative approach (group MI) and 89 students followed the teacher-directive approach (group TD). The modelling unit was again implemented during the mathematics tutorial classes. The question was whether the TD and MI approaches will show similar effects as in 2019. Both groups in 2022 had the same lecturer as the MI and TD1 groups in 2019. The 2022 study did not include an attitudinal component.

RESULTS OF THE ENGINEERING STUDY IN 2022 (AGAINST THE BACKGROUND OF 2019)

The same test instruments were used in 2022 as in 2019, so the results of the two studies are directly comparable. The internal consistencies of the scales were estimated using the reliability indicator McDonald's Omega (McDonald, 1999) and are similar to those in 2019 (Durandt et al., 2022a). They are sufficient overall for the diagnostic test, the pre-test and the post-test, but low for the individual sections, as expected, given the small number and the different contextual focus of the items (Table 1). The respective sum values can nevertheless be interpreted as performance indicators (Bühner, 2011). In the scale of test section C (multiple-choice modelling tasks), one item each had a negative part-whole corrected item-total correlation in the pre-test and in the post-test (in both versions), which is why both items (like 2019) were removed from the scale in both the pre- and the post-test. The two test versions of Section C differ significantly in the pre-test ($t(199) = 3.74$, $p < .01$), but not in the post-test ($t(199) = 0.07$, $p = .94$). The equivalence of the items of Haines et al. (2001) must thus be questioned (as already in 2019), but any differences average out in further analyses due to the (almost) balanced, randomised distribution of the versions in the pre-test and post-test.

As shown in Table 1, the MI group performed significantly better on average in the diagnostic test than the TD group. This was tested using a *t*-test for independent samples ($t(199) = -3.79$, $p < .01$). Therefore, performance on the diagnostic test is included as a covariate in all further analyses (as in 2019) to control for differences between samples. Initial descriptive comparisons between the results of the groups TD and MI in the pre- and post-test reveal overall and in two of the three test sections (B and C) tendencies towards higher mean values for the group TD in the pre-test. In addition, increases in achievement are visible for both groups TD and MI in all sections and overall between pre-test and post-test. The mean achievement of group MI is slightly higher than that of group TD in all sections in the post-test (see Table 1).

Linear mixed regression models with dummy-coded predictors were estimated to check whether the achievement gains of group MI were significantly bigger than those of group TD. The achievement growth of group TD was modelled as reference and the

achievement change of group MI was compared to it. The difference in the achievement development of both groups is represented by the interaction effect.

Table 1: Internal consistencies (McDonald's Omega) per test (section), means and standard deviations per group and test (section).

Test (section)	**Number of items**	**Internal consistency** *Omega (ω)*	**Teacher directive (*N* = 89)** *M (SD)*	**Method integrative (*N* = 112)** *M (SD)*
Diagnostic test	31	.76	12.24 (6.09)	15.42 (5.77)
Pre-test (total)	11	.50	8.55 (2.81)	8.21 (2.46)
A: Modelling tasks	2	.16	0.57 (1.12)	0.62 (0.95)
B: Mathematical tasks	6	.56	4.37 (1.63)	4.26 (1.47)
C: Multiple-choice tasks	3	.22	3.61 (1.40)	3.34 (1.67)
Post-test (total)	11	.56	9.49 (3.31)	10.24 (3.32)
A: Modelling tasks	2	.20	1.20 (1.49)	1.48 (1.77)
B: Mathematical tasks	6	.57	4.91 (1.54)	5.27 (1.60)
C: Multiple-choice tasks	3	.44	3.38 (1.75)	3.49 (1.54)

Note. ω = McDonald's Omega; *M* = mean; *SD* = standard deviation. The evaluation objectivity of the tests (interrater reliability) was already checked in a previous study (Durandt et al., 2022a) and is satisfactory (Cohen's $\kappa \geq .72$).

As Table 2a illustrates, group TD shows significant achievement gains in test sections A (open modelling tasks) and B (mathematical tasks) between pre- and post-test. The achievement development of group MI does not deviate significantly from this, but at least tends to be somewhat higher than that of group TD ($b_{\text{Group x Time, A}} = .17$; $b_{\text{Group x Time, B}} = .29$).

Table 2a: Linear mixed regression model for test section A (open modelling tasks) and test section B (mathematical tasks)

	Model A					**Model B**				
N | *Obs.* | *ICC*	201		402	22.3%		201		402	18.1%	
Fixed effects	***b***	***SE***	***df***	***T***	***p***	***b***	***SE***	***df***	***t***	***p***
Intercept	**-0.28**	**0.10**	**360.64**	**-2.69**	**0.01**	-0.15	0.10	374.53	-1.47	0.14
Diagnostic test score	0.04	0.06	199.85	0.71	0.48	**0.21**	**0.05**	**196.35**	**3.89**	**<0.01**
Group	0.01	0.14	361.37	0.09	0.93	-0.18	0.14	374.59	-1.29	0.20
Time	**0.44**	**0.12**	**202.30**	**3.69**	**<0.01**	**0.33**	**0.12**	**198.97**	**2.68**	**0.01**
Group x Time	0.17	0.16	202.30	1.04	0.30	0.29	0.17	198.97	1.74	0.08
marg. R^2 | cond. R^2	.08		.37			.11		.31		

Note: *N* = sample size; Obs. = number of observations; ICC = intraclass correlation; *b* = (standardized) regression coefficient; *SE* = standard error; *df* = degrees of freedom; *t* = *t*-value; *p* = probability of committing a Type I error; R^2: coefficient of determination.

In section C, the achievement development of both groups is not significant (but comparatively positive for group MI: $b_{\text{Group x Time, C}} = .24$; Table 2). As in 2019, this

result is at least partly due to the shortening of the scale and should therefore be interpreted with caution. In Model D for the overall test result, not only does the TD group show a significant development in achievement, but that of the MI group is significantly higher ($b_{\text{Group x Time, D}}$ = .35; Table 2).

Table 2b: Linear mixed regression model for test section C (multiple choice modelling tasks) and the complete test

	Model C					**Model D**				
N | *Obs.* | *ICC*	201		402	31.5%		201		402	30.3%	
Fixed effects	***b***	***SE***	***df***	***t***	***p***	***b***	***SE***	***df***	***t***	***p***
Intercept	0.10	0.11	358.66	0.95	0.34	-0.15	0.10	343.77	-1.49	0.14
Diagnostic test score	0.00	0.06	197.84	0.03	0.97	**0.13**	**0.06**	**197.07**	**2.21**	**0.03**
Group	-0.17	0.15	359.48	-1.18	0.24	-0.17	0.14	345.54	-1.25	0.21
Time	-0.14	0.12	200.28	-1.13	0.26	**0.31**	**0.11**	**199.38**	**2.71**	**0.01**
Group x Time	0.24	0.17	200.28	1.42	0.16	**0.35**	**0.15**	**199.38**	**2.32**	**0.02**
marg. R^2 | cond. R^2	<.01			.32		.09			.44	

Note: *N* = sample size; Obs. = number of observations; ICC = intraclass correlation; *b* = (standardized) regression coefficient; *SE* = standard error; *df* = degrees of freedom; *t* = *t*-value; *p* = probability of committing a Type I error; R^2: coefficient of determination.

The results of the 2022 replication study show that the MI group overall had the biggest competency growth, similar to 2019. In sections A and B, however, the differences are smaller than in 2019, but still tend to be in favour of the MI group. This is possibly due to the already higher entrance performance of both groups in 2022 (both in the entrance test and in the pre-test; cf. Durandt et al., 2022a), which reduces the absolute effects regarding the overall achievement increase (TD: $d_{2019,\text{ total (TD1)}}$ = 1.09, $d_{2022,\text{ total}}$ = 0.31; MI: $d_{2019,\text{ total =}}$ 1.37, $d_{2022,\text{ total}}$ = 0.70; effect size *d* according to Cohen, 1992). As in 2019, the multiple-choice modelling tasks did not prove useful for measuring achievement progress.

DISCUSSION AND CONCLUSION

The CoSTAMM studies are a continuation of the DISUM study with a focus on the principles of quality teaching. These studies followed a similar design to DISUM at a different educational level with a different mathematical content. A modelling unit consisting of five lessons with ten tasks, a diagnostic test, a pre-test, and a post-test were implemented, and students were exposed to either an independency-oriented or a teacher-directed teaching approach.

Results from the engineering study in 2019 show that all groups had *significant learning progress*, so both the method-integrative and the directive teaching had effects. The method-integrative group had the biggest achievement growth, particularly for mathematical modelling.

The replication study from 2022 showed very similar results. Again, both teaching approaches led to significant competency growth, with advantages for the MI group for modelling and overall.

In conclusion, results from the CoSTAMM studies may encourage first-year lecturers to consider innovative teaching approaches in their classrooms, linked with the principles of quality teaching. Further ideas are to improve the modelling unit (e.g. including more examples in more lessons) and to test the study on larger samples in other context also including again an attitudinal component similar to 2019. To include the lecturers' professional competencies as a variable in future studies may also be a promising idea (Casinillo, 2023; König et al., 2021; Kunter et al., 2013).

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LOST AND FOUND IN TRANSITIONING BETWEEN MULTIPLE COMPUTERIZED VISUALISATIONS DURING STATISTICAL MODELING

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Transnumeration, the ability to reason with and transition between multiple representational forms, has unique pedagogical affordances. Technological advancements can provide novices with more extensive transnumeration experiences. However, much is yet to be discovered about the software's mediational role, especially when representing stochastic behaviour. We present a case study of two prospective primary school teachers' participation in a learning sequence inspired by the practice of statistical modeling that calls for representing sources for systematic and stochastic variation. The affordances and hindrances of their use of TinkerPlots software are discussed, explaining how, although initially lost in transnumeration, they were ultimately found.

BACKGROUND

Transitioning between different representations and learning mathematics

Mathematical objects are abstract, intangible constructions. Any form of mathematical engagement therefore necessitates the use of representations for these abstract notions (Duval, 2006). While representations are instrumental, a single representation can only symbolize certain aspects of the mathematical object, and often conceals others. Developing learners' reasoning with mathematical objects thus requires to mediate multiple representations, to support their gradual construction of the meaning of each, as well as establishing relations between them (Duval, 2006). Nurturing learners' 'representational fluency', the ability to transition between different representations, can support their transferring knowledge from one representation to another as well as develop new insight through this engagement (Zbiek et al., 2007).

In the context of learning statistics, Wild and Pfannkuch (1999) introduced the term 'transnumeration', defined as changing representations to engender understanding, as one of the types of thinking that is fundamental to experts' statistical thinking. Transnumerations accompany each step of statistical data investigations, from identifying variables to measure in the data collection stage, using graphical representations to explore the data, applying different statistical models in the analysis of the data, to communicating the resulting findings via different representations (Wild & Pfannkuch, 1999). Because transnumeration is omnipresent in most statistical practices, many of the currently explored pedagogies in statistics education engage learners with various forms of it, often facilitated by educational technologies.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 267-274). PME 46.

Statistics educational technologies for transumeration

While supporting learners to manually invent their own representations has been proven beneficial for deepening conceptual understanding (diSessa et al., 1991), technological advances allow them to generate representations more quickly and automatically. Thus, learners can more feasibly explore and transition between these different forms, and develop an appreciation of the different purposes they are well-suited for (Biehler et al., 2013). TinkerPlots (Konold & Miller, 2015) is a widely used dynamic statistics education software that embodies these affordances (Biehler et al., 2013).

Despite the pedagogical potential of engaging novices with technologically-mediated transnumeration, dealing with different, simultaneously presented representations as part of a statistical task can be challenging. For instance, young learners sometimes prioritize one representation without considering others (Schnell & Prediger, 2014), and require support to consider additional representations to nurture more mature perspectives. Instructors likewise might overly rely on very few representations, and fail to make explicit the relations between different representational forms (e.g., Lee at al., 2016). Additional challenges are implied, such as associating different purposes for a computerized representation (Dvir & Ben-Zvi, 2022). Furthermore, the underlying stochastic nature of the process of creating the data that is represented is often not reflected in a statistic data depiction, even if explored through various representational forms (Dvir & Ben-Zvi, 2021). Therefore, in light of the proliferation of new technological tools, it is necessary to further elucidate the pedagogical affordances and hindrances of computerized representations of stochastic notions (Pfannkuch, 2018). To attend to this lacuna, we focus on the practice of statistical modeling to engage learners with representing stochastic aspects of the data generation process (Dvir & Ben-Zvi, 2021).

Informal statistical modeling and transnumeration

A model is a representation that offers a simplified depiction of a phenomenon with a descriptive, explanatory, or predictive purpose (Hesse, 1962). Offering only a partial depiction, a model can be found to be ill-suited for the purpose it was intended to fulfill. Therefore, the modeling process – the process of constructing and using a model – consists of constant evaluation of the model, in light of its intended purpose, and transforming the representation it offers to better attend to its goal (Dvir & Ben-Zvi, 2021).

In the context of statistics, transnumeration is particularly abundant in novices' engagement with activities based on statistical modeling, as their modeling process often includes two separate, yet not independent, sub-modeling processes: (1) data modeling, modeling what can be seen in the data (e.g., drawing an ellipse circumventing a cloud of cases that seems to be gradually ascending); and (2) conjecture modeling, creating and refining a representation for an abstract generalized conjecture (e.g., representing a linear positive relation through an ascending line) (Dvir

& Ben-Zvi, 2021). Expert statisticians have a ready-made toolkit of models and representational forms, and would thus likely focus on evaluating the fit of one or more of them to the data. Novices, however, need to concurrently develop such generalizable tools to represent their initial and changing conjectures. These may vary greatly from the formal models that experts might consider suitable representations. The result is that novices often hold two concurrent, often rivaling, data and conjecture models. Comparing the two rivaling models to notice discrepancies between them is a central action that promotes shifts in either model or both. The Reasoning with Informal Statistical Models (RISM) framework summarizes these key entities and offers a means to describe and analyze novices' informal statistical modeling process (Dvir & Ben-Zvi, 2021). The dual modeling process offers plentiful opportunities for novices to create, refine and transform various representations. Thus it is rich with transnumerational actions.

Furthermore, the informal statistical models that the students create (as opposed to mathematical models, English & Watters, 2004) are specifically intended to represent sources of systematic variation, the signal in the data (e.g., a line to describe the ascending tendency of a cloud of cases), but also its nonsystematic sources, the noise (e.g., the extent to which the data vary from the line, as a result of natural variation or sampling variation, Dvir & Ben-Zvi, 2021). Thus, much of the novices' transnumerational actions focus on representing stochastic behavior. Therefore, we chose to engage novice pre-service teachers (PSTs) in informal statistical modeling activities, and examined: *(1) What forms of representations or models did the students create and reason with; (2) How did the students transition from one representational form to another; and (3) How did the students' use of TinkerPlots facilitate these transitions?*

DESIGN OF THE STUDY AND METHODOLOGICAL CONSIDERATIONS

With the purpose of providing a detailed in-depth account for a phenomenon that has not yet been deeply explored, we adopted a qualitative approach and provide a case study of a pair of PSTs for primary school, Cora and Tim. The pair participated in an online semester-long seminar on informal statistical modeling pedagogies taught by the authors. We focus on one activity sequence that was particularly rich with transnumerational actions, and the pair was chosen as their written documentation was extensive, and concentrated primarily on representing stochastic aspects.

Activity sequence

We focus on an activity that the students engaged with, in pairs, over the span of sessions 2 to 5, primarily as part of their homework assignments. Overall, the task followed the Integrated Modeling Approach (Manor & Ben-Zvi, 2017), starting with the PSTs' exploration of a sample size of 60 of real-world data collected by primary students. The PSTs formulated research questions and conjectures, organized and analysed their data in multiple ways with TinkerPlots, made informal inferences about a larger population, and articulated uncertainty regarding the representativeness of a

sample size 60. To explore these concerns, the PSTs were asked to conduct a probability-modeling activity that began with their constructing a dynamic computerized model based on their conjectured population with the Sampler option in TinkerPlots. The Sampler enables creating and running probability simulations and drawing simulated samples from a conjectured population. After designing the Sampler model, the PSTs were asked to generate and compare multiple random simulated samples from it. Lastly, to further explore sampling variation, the PSTs chose a statistic (e.g., the mean or relative percentages of some of the cases) they wished to track over 100 simulated samples, and, using TinkerPlots, created a sampling distribution for it.

Data collection and analysis

The data corpus consisted of recordings of the ten four-hour sessions of the seminar, along with all the electronic artefacts that the students created throughout their participation, including written documentation of assignments given during and after the course. Cora and Tim's written assignment on the presented learning activity included their responses to 24 questions. The unit of analysis was the PSTs' response to a single question. These were sequentially microgenetically analysed (Siegler, 2006) to identify any representation implied in each statement, along with visualizations that the PSTs provided. If the representation was purposeful (e.g., the learners allocated a specific meaning to some aspect of a graph) it was labelled as a model. Using the RISM framework, each model was either classified as a data or conjecture model, and other elements of the framework were interpreted when possible. Any changes made to one or both of the models elicited a new RISM snapshot and additional interpretations for the other RISM elements. The result was a series of seven snapshots that together describe the students' modeling process. Reviewing these snapshots, the role of the students' use of TinkerPlots for any representational shift was identified.

FINDINGS

Cora and Tim chose to examine "Is there a correlation between the flexibility of children and their time spent exercising?" and offered an initial conjecture: "We assume that the more time a child spends exercising and training their body, their flexibility will increase". Throughout the activities, the PSTs made use of two main representational forms: (1) The grid: each attribute as the x-axis or y-axis, divided into bins (Fig. 1); and (2) Columns: showing the distribution of a single attribute, divided into bins (Fig. 2). The students began with expressing a clear preference to the grid representational form (Fig. 1), as it was more "detailed". Despite this, their initial description of their expected data, if their conjecture were true, indicated imagined columns (e.g., the left column would have many cases on the top and few or non in its bottom). The grid display, however, sufficiently convinced them to abandon this initial (column-based) conjecture. It also allowed the pair to begin to account for the non-systematic variation they observed (referred to as "no connection") and gradually refine a model for it that was more row-based (e.g., "There are many children who exercise more than two hours per day but they are not necessarily more flexible

compared to others who do not exercise this much”). Furthermore, while their initial column-based depictions were fully deterministic, the row-based accounts, mediated by the grid representation, were more stochastic (e.g., "not necessarily").

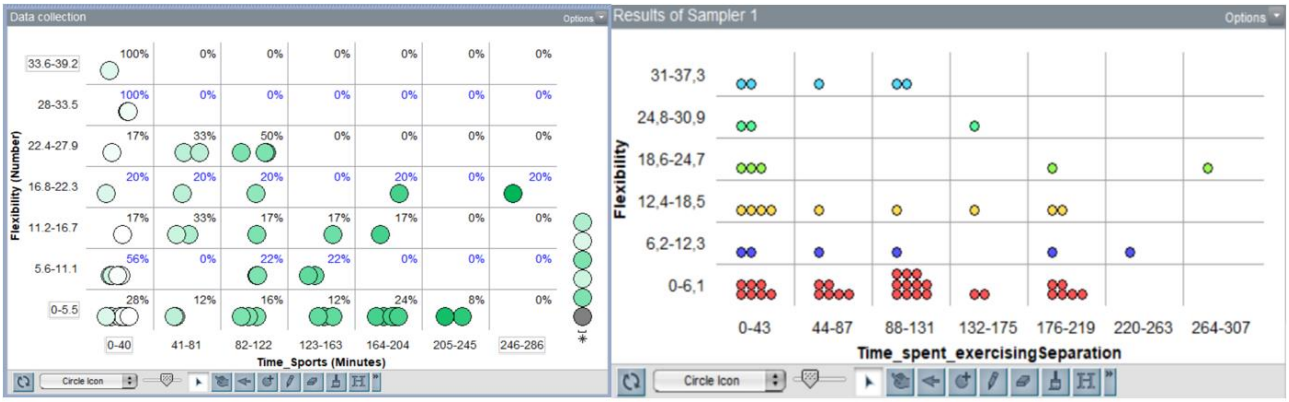

Figure 1: Grid representation of the real-world sample (left) and of the first simulated sample (right); Y-axis is for flexibility, x-axis is for time for sport

To initiate their exploration of sampling variation, the PSTs were asked to design a Sampler model that offered visualization tools that differed from those utilized earlier to form the grid representation. As a result, they returned to their earlier column-based representational form, and designed the Sampler model based on it (Fig. 2 middle).

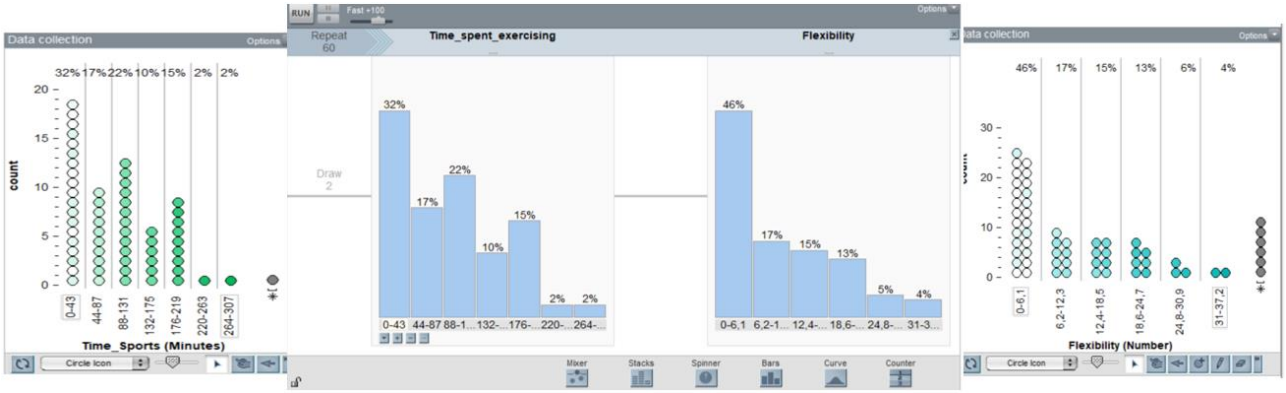

Figure 2: The column-based representation of the first Sampler (middle) based on same representation of the real data for flexibility (right) and time for sports (left)

The first simulated sample generated from the Sampler model was again displayed in their preferred representational grid form (Fig. 1 right). This lead the PSTs to emergently compare the two similar representations: the simulated and the real-world sample (Fig. 1). By doing so, the PSTs were conducting a non-endorsed comparison of samples generated by two different populations (one real, one imagined). The learning activity intended to focus the PSTs on the comparison between the simulated sample and the population it was generated from (i.e., the Sampler model). To do so, the learners had to compare the two different representational forms, the column-based Sampler (Fig. 2), and the grid (viewed in a more row-based form, Fig. 1). However, the students compared between two incomparable aspects of the two forms: the percentages of entire columns in the Sampler model, with percentages of only single

bins in the grid. Furthermore, the students reverted back to expressing much more naïve deterministic expectations (e.g., “the simulated sample does not reflect the original percentage”).

Generating a second simulated sample, the PSTs again chose to organize it in their preferred grid representational form (as in Fig. 1), and used it to compare the new sample to the first simulated sample. At this stage, the PSTs engaged in an endorsed comparison, but also in an endorsed manner (through comparable aspects). The result was a more stochastically mature conclusion (e.g., “Even though there are some differences between our 2 representations, we think that the overall trend is similar”).

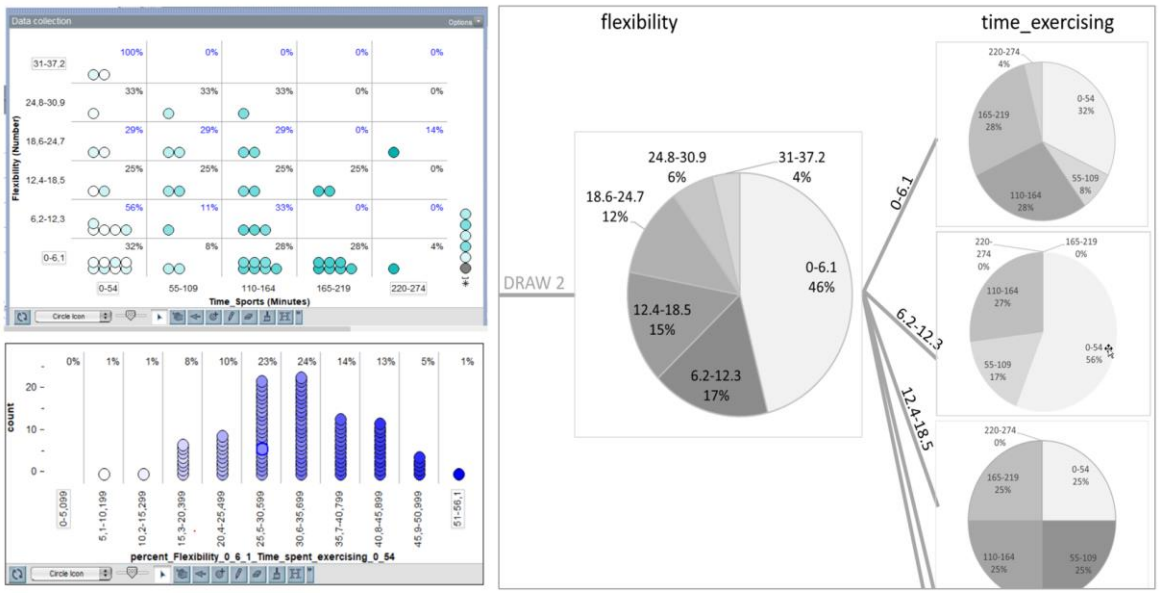

Figure 3: The grid-inspired Sampler design (right, recreation) based on real-world data (upper left); Column-based sampling distribution of the percentages in the bottom left bin in the grid, across 100 samples (bottom left)

Before continuing the activity, inspired by another pair that introduced their work in class, the PSTs chose to redesign their Sampler model. While earlier they relied on column-based data representations, they seemed to have found a way to utilize the Sampler tools to reflect the grid representational form of the real-world data (Fig. 3, right). This allowed them to become fully engage in endorsed comparisons, focusing on the bottom left bin (Fig. 3, upper left; cases with little exercise time and high flexibility), which they had already addressed in their first analysis of the real sample as not fitting to their expected data. Furthermore, they chose its percentages as the statistic they tracked across additional 100 simulated samples. To represent the sampling distribution they created, being a single attribute, they returned to the column-based form (Fig. 3 bottom left). At this stage, however, they accompanied its use with much more stochastically mature accounts of sampling variation: “In the original sample population the percentage of flexible students who don’t exercise is 32%. The average in our imagined population is between 31,5% and 33%...We are definitely more confident that a small sample can represent the population, however,

we would not say that we are 100% certain as there are still some variances within our sampler history".

DISCUSSION

This study set out to explore what forms of representations of stochastic behaviour PSTs create, how they transition from one representational form to another, and the role that the TinkerPlots software plays in these transitions. Despite the multiple options TinkerPlots affords, our pair primarily used two representational forms, and showed a clear preference to one. This is in-line with prior depictions of both students' and teachers' tendency to prioritize one or few representations (Schnell & Prediger, 2014; Lee et al., 2016). While both forms were used to represent a shared phenomenon which the PSTs referred to as "no connection", each form seemed to mediate different views of the stochastic behaviour the PSTs were modeling: The column-based representational form seemed to initially highlight (in the eyes of the PSTs) the more systematic variation they initially expected, and was accompanied with relatively deterministic expectations. The grid representational form seemed to provide the PSTs with a more "detailed" view that allowed them to better account for the non-systemic variation. After developing a more mature appreciation for these stochastic sources, the PSTs ultimately returned to the initial column-based representational form, which mediated even more mature accounts of sampling variation. This illustrates how providing learners with the freedom to design their own representations and explore their preferred visualization can be beneficial in supporting their sense-making of the representation itself, and their utilization of it to gain broader insight on the represented idea (Zbiek et al., 2007; diSessa et al., 1991).

The role of the technological tool, along with the design of each activity, was particularly consequential. First, it afforded the learners' gradual construction of a complex representation of stochastic behaviour that was understandable and accessible to them, and allowed them to explore sampling variation more deeply. Second, the PSTs reverted to the column-based form only when challenged with the different representational tools offered by the Sampler. In this regard the tools served as hindrances, and mediated more naïve, deterministic views. Furthermore, the students were somewhat lost in this transnumeration, resulting in their assigning similar meanings to aspects that were incomparable between the two representational forms. However, when the students discovered a way to re-design the Sampler model to become aligned with the grid form, they became fully emerged in endorsed comparisons and, when returning to the column-based sampling distributions, ultimately expressed relatively mature views of sampling variation and representativeness (as in Dvir & Ben-Zvi, 2021). In this regard, despite initially being lost in transnumeration, they ultimately were found.

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THE BODY PROBABLY UNDERSTANDS

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Learning probability can pose difficulties for students at all levels. Based on studies indicating that conducting probability experiments with concrete means can encourage students to develop an understanding of probability we tested the feasibility of learning abstract concepts in complex mathematics by applying embodied learning through movement-based games. Following a careful analysis of probabilistic concepts, a stratified movement class was structured according to the graded construction of the concepts. These were studied as metaphorical images for concrete situations experienced in the body. The findings of the study confirm the hypothesis that the achievements of middle school and high school students who learn probability through one movement lesson, will meet the requirement of a standard achievement test.

BACKGROUND AND THEORETICAL FRAMEWORK

Questioning the feasibility of learning abstract concepts in mathematics in middle school and high school was based on the complexity of the concept of probability and the limited achievements of the teachers to teach it by conventional means. The basic premise was derived from embodiment theory developed by Johnson and Lakoff in 1980, which implies that all learning begins with the body, from the theory of multiple intelligences, which emphasizes that one intelligence can be used to learn concepts in the field of another intelligence, and from the constructivist theory which states that an accurate level of teaching of concepts ensures optimal teaching. These theories explain complex processes that occur during the cultivation and refinement of neuronal networks in the brain.

Difficulties in teaching probability

Learning probability is based on familiarity with complex concepts, such as: variation, randomness, independence, inability to predict and uncertainty, but also chance, likelihood, or risk. These abstract terms do not have unequivocal definitions that can be explained in simple language or through an illustration, so dealing with creating meaning is not trivial and can only be reached after a continuous learning process.

Heitele (1975) points to the intuitive dimension in teaching mathematics and probability. In the case of randomness, he emphasizes, it is necessary to strengthen the intuitive understanding towards the formal teaching of the subject. Kuperman (2007) suggests teaching probability through discovery rather than as a formal set of rules. and offers diverse teaching methods for teaching probability, such as: translating problems from the real world into mathematical language, building mathematical models, discussing, and analysing results, discovering analogies, presenting paradoxes related

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to probability and developing probabilistic and combinatorial intuition. Batanero & Diaz (2012) attribute the reasons why the teaching of probability is difficult for mathematics teachers in that the specific training for teaching probability is far from sufficient.

It is evident, therefore, that there are inherent difficulties in teaching probability. These difficulties stem mainly from the formalistic teaching method that disconnect probability from everyday reality and ignore intuitive knowledge. To overcome these difficulties, one must first understand what alternatives can be offered.

Embodied Learning as a possible solution

The theory of embodiment is based on the premise that cognition is anchored in the body and depends on its physical experiences. This theory rejects the philosophical separation between the body and the soul/consciousness/intelligence. According to this approach, the mind is an activity based on the relationship between the body, the environment, society, and culture.

Research in the field of embodied cognition indicates that there is a strong correlation between physical movement and learning. That is, through perceptual processing and muscle control, the sensorimotor system can find solutions in the physical environment and understand specific learning tasks. Many studies have revealed that increased physical involvement during the learning process has the potential to positively affect cognitive ability, memory, and academic achievements.

Cope & Kalantzis (2004) state that embodied learning focuses on the knowledge that students acquire when they use the body as a tool to build it. When referring to "body" in this context, they include the entire learner - the physical body, the senses, the soul, and the mind. Lindgren & Johnson-Glenberg (2013) describe this as the sensorimotor activity relevant to the subject to be reproduced and the emotional involvement of the participant in the whole process. Shapiro & Stolz (2019) argue that the emerging research agenda of embodied cognition can greatly contribute to educators, researchers, and policy makers. Research in the field of embodied cognition provides thought-provoking recommendations on how to improve educational practice and lead to more effective learning. Their claims quote Nathan's (2012) position regarding the common mistake that control at the level of representations, in a specific knowledge domain, is necessary before it can be applied. Such a view, they claim, tends to reinforce a way of thinking rooted in dualistic views of knowledge that wrongly link intellectual work to the "mind" and practical work to the "body." A study by Glenberg (2008) examined these aspects in the teaching of mathematics and reading comprehension supporting the integrative approach to learning and emphasizing the importance of physical manipulation and abstract manipulation. In these two areas, the importance of the physical-tangible manipulation is evident before the importance of the abstract manipulation, because it is based on abstract symbols, i.e., letters and mathematical symbols.

Metaphors can act as a bridge to understanding

Lakoff & Johnson's (1980) conceptual metaphor theory deals with abstract concepts, how they are constructed and how they are formed, through the mapping of a structure from one concrete or sensory-motor domain to a more abstract domain. They claimed that many central cognitive processes, such as those concerning space and time, are expressed, and influenced by metaphors, and that many metaphors reflect the embodied experience of beings moving in the world. Most metaphors are a mapping of a concrete concept, that is, a description of an abstract concept using a clear concrete concept that lends its properties to the abstract concept. Conceptual metaphors rely on 'cross-domain mappings'. They carry the inferential structure of the first domain into the second domain and make it possible to understand. The goal domain is often more abstract and unclear. In terms of the field of origin, it becomes more precise and clearer. In education, metaphors can provide a powerful tool for teaching abstract concepts in terms of concrete models.

Metaphors, claims Duit (1991), are of central importance in the learning of conceptual change because they may help rebuild existing memory and prepare for new information. They may open new perspectives and even help to observe the familiar in entirely new ways. This "generative power" of metaphors makes them potentially valuable tools in learning conceptual change. They facilitate the reconstruction of the known and familiar. The use of exceptions and cognitive conflict, which is widely discussed in educational psychology, has great value as part of a conceptual change, which metaphors can produce and provoke. Metaphors usually involve a degree of imagination that helps to visualize abstract ideas and it seems that metaphors also link thinking and emotions and therefore may bridge the gap between the cognitive and emotional domains of learning.

In the field of mathematical education, there has been a growing recognition that metaphors are powerful cognitive tools that help to grasp or construct new mathematical concepts, as well as in solving problems effectively. Lakoff & Núñez (2000) view metaphors as an essential part of mathematical thinking; not just as auxiliary mechanisms used to visualize or facilitate understanding. When the full metaphorical nature of mathematical concepts is revealed, confusion and paradoxes disappear. Chiu (2000) determines that students can construct new mathematical concepts by metaphorical thinking that makes use of their intuitive knowledge. Also, students may use metaphorical reasoning to connect mathematical ideas, remember them, understand mathematical representations, and perform calculations.

According to this concept, it can be assumed that movement/dance activity which structures a concrete concept as a metaphor for complex concepts in probability may prevent the confusion in studying probability and establish a layer of information that can be relied on in the development of probabilistic literacy.

Research Hypothesis – An Embodied Learning lesson can improve probability comprehension.

The hypothesis of the study was that the achievements of middle school and high school students who learn probability through one movement lesson (2 academic hours), will meet the requirement of a standard achievement test. The evaluation of the students' achievements was carried out through an online multiple-choice test that included ten questions. The questions were collected from study materials approved by the Israeli Ministry of Education and validated matriculation questionnaires. The students answered the test questionnaire before the intervention, and at the end they were asked to answer it again. The difference between the scores before and after the intervention can be regarded as evidence of an improvement in the knowledge of the studied concepts

METHODOLOGY

The mathematics curriculum recommends teaching the chapter on probability in grades 8-9 in middle school. In practice, there are schools that give up probability studies altogether or those where the subject is taught through textbooks rich in tables into which data is poured, without understanding or connection to life and everyday contexts of probability calculations.

We distilled probabilistic concepts in the sequential order of learning as recommended by standard textbooks. Following conversations with math teachers, a preliminary stage of categorization and differentiation by properties was added. Seemingly simple, yet tricky, as students find it difficult to understand that an object may carry more than one property, which may pose an obstacle in the comprehension of probabilistic abstract concepts.

In this study, 38 students from three semi-private schools took part in one lesson composed of:

1. A conventional test delivered in schools that teach probability through an online test

2. Practical activities in movement:

A. Probability Concept studied - Categorization capabilities - (what is inside the group and what is outside the group) - perfecting the ability to categorize as a basis for model thinking. First part of the lesson used for "Breaking the ice" and creating a comfortable space for movement.

In a circle, the students were asked to preform free movement accompanied by music, in a circle. Next, they were asked to mirror a leader, isolating body parts – i.e., using only one body part (hand, leg, head etc.). The next stage was mirroring a leader moving in different directions – up, down, forward, sideways, diagonally. The following was mirroring an emotion conveyed by the leader (fear, disgust, happiness etc.)

In pairs students were asked to analyse a partner's gestures assembling two and then three of the categories. After that, verbally instructing the partner while assembling

two and then three categories. This phase assists the comprehension of the transition between concrete and verbal questions in math.

Summation of first part in a group game, making sure that everybody had a chance to experience categorization and adding a surprise dimension with a 'Simon Says' game raising focus level, having to determine whether to act on the leader's instructions (if he says 'Simon Says' or not).

B. Probability Concepts studied - Certainty, randomness, impossibility, and introduction of sample space.

The class was divided into groups of 3 students. Each member was asked to conceive a simple movement and teach the others, repeating each movement 5 times. Following this the students were asked to perform the movements, following leader no. 1 and then no.2. This phase represents a certainty on the part of the followers, as opposed to randomness. No. 3 leaders were called aside and asked to perform a different movement than the one previously practiced. This caused confusion and embarrassment, a predicted emotional response, which was later defined as uncertainty and randomness.

As the activity evolved, a sample space of 3 movements and later 4 was defined. The students were then asked to take turns leading a sequence of their choice of the 4 movements accumulated in each group. Each student was asked what the chances are of seeing one of his movements.

The final part of this section was a class circle game, learning accumulatively all of the groups' first 3 movements and realizing that as the sample space was enlarged, the chances of seeing each individual's movement decreased.

A short edit of this activity (subtitled in Hebrew, English to follow), click here.

C. Probability Concepts studied - Parallel independent events, complementary events.

The class practiced 4 different movements derived from an Israeli children's' chant: hands up, on the head, on the shoulders, clap. The whole class then practiced playing 'red light, green light' for which the stopping positions were one of the 4 practiced positions.

The class was divided into groups of 4, movements were limited to 2, one student was on the finish line, one was acting as a reviewer and 2 students were playing. The reviewer's role was to record the actions of the players (according to a simple structured index). The students were then handed a 2*2 table representing the different possible variations and probabilities. Later, the 3rd and 4th movements were added and respective tables of 3*3 and 4*4.

The tables that accompany the physical experience visually clarifies the many possibilities that exist in the body and enables a link to a concrete experience. A gap is revealed between the actual observation and the findings of the table. This gap allows

for a discussion of the difference between reality and mathematical calculation. While discussing the results, the concept of complimentary events was introduced.

3. Repetition of the conventional test from the beginning of the lesson.

DISCUSSION

In this study, we tested the feasibility of teaching an abstract and complex concept in statistical mathematics through teaching a structured program that included movement games and dance. The research question was whether it would be possible to achieve significant achievements by teaching a program of eight consecutive sessions. At the cliff of times (covid and closures) we had to shorten the experience that came to test whether it is possible to assimilate the abstract concepts: randomness, probability, certainty, uncertainty, sample space, event, independent probability, and conditional probability in a single meeting, built with great care and layered according to a graded construction of the concepts studied.

The existing research evidence for the construction of linguistic and quantitative concepts at an early age, based on movement and experience in the learner's living environment (Piaget, Montessori, Luria, Vygotsky, Dewey), is numerous. The questioning about the feasibility of learning abstract concepts in mathematics in middle school and high school was based on the complexity of the concept of probability and the limited achievements of the teachers to teach it by conventional means. The confidence in determining the research question stemmed from the knowledge of the theory of embodied learning, of conceptual development through metaphors, of the theory of multiple intelligences and the constructivist idea that new material will not be learned unless there is previous material that ensures its assimilation into the existing information networks.

The research hypothesis was that the achievements of students in the 8th grade, who learn probability through one movement lesson (2 academic hours), would meet the standard probability literacy test requirement. Due to logistical difficulties, students from higher and lower grades also participated in the study.

Findings indicate contribution of embodied learning activity to understanding of probability concepts

The assessment of the students' achievements was carried out through an online multiple-choice test, which included ten probability questions. The questions at the different age levels were taken from valid study materials and matriculation questionnaires. At the end of the activity, the students were asked to answer the same questionnaire again. The test checked students' achievements after participating in the workshop. The test scores were calculated by 10 points for each correct answer, for a total of 100 possible points. A complete data analysis was conducted for the results of 16 students: 8 11th grade students from a school in Zichron Yaakov and 8 10th-11th grade students from a school in Ashkelon.

The results were analysed twice, once using the t test, which assumes that the observations were sampled from a normal distribution, and a second time using the Wilcoxon test which does not assume this.

Assuming the test results reflect the level of experience of the students in physical/movement practices, an ANOVA test was performed, which due to the small number of observations did not yield unequivocal results and therefore a test comparing averages was conducted. The results indicate a lower starting point for students with a dance and movement background and a more noticeable improvement than the higher starting point for students without a dance and movement background, whose improvement in their achievements is less significant. These results confirm another premise that the embodied learning method is more suitable for students who have previous experience in dance and who feel comfortable moving freely in class.

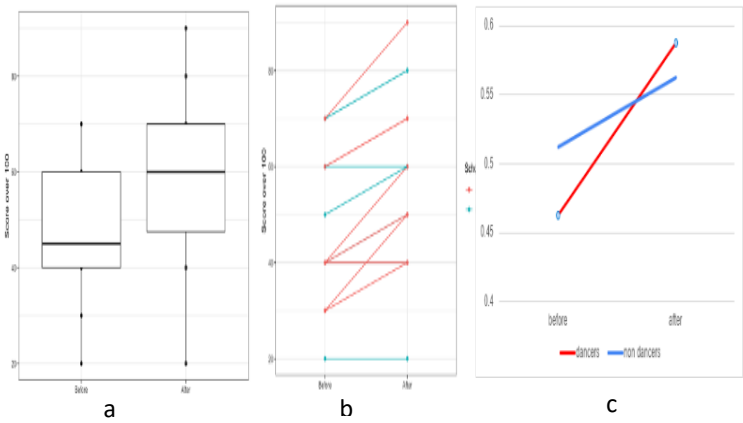

Figure 1: a- the average grades of the two classes increased between the first and second tests, scores are higher on average by 18 points out of 100; b- a noticeable difference between the two classes; c -a more pronounced improvement trend among students with a dance and movement background compared to students without it.

The results confirm the hypothesis that teaching based on a careful analysis of the studied concept allows learners to experiment with assimilating embodied learning and guarantees effective results in a short time. The selected movement games allowed the students to experience directly, the "wondering" associated with uncertainty, the difficulty of observing randomness and the imaginary scanning of the sample space. The somatic experience anchored in the muscles and sensory systems allowed the construction of new connections in the neurological network and ensured the ability to transfer what was learned and to apply acquired knowledge.

The didactic strategy was adapted to deal with intuitive models and was built on the basis of the experience of students who discover that what seems obvious to them, is not true. The cognitive dilemma was resolved by connecting the students' intuitive knowledge (concrete physical actions) to abstract concepts, based on the gap between probabilistic calculations and the realization of random situations. This recognition of perceptions and feelings created learning.

RECOMMENDATIONS FOR FUTURE RESEARCH

The focus of this study was to help students learn the concepts of theoretical and experimental probabilities using a structured context of play and movement. The research findings point to the advantages of this method, therefore it is appropriate that it be investigated in depth, to test how consistent and useful the findings of the research

experience may be in diverse contexts. It is recommended to examine in future studies whether it is possible to use embodied learning as an alternative in various highly abstract areas of knowledge such as mathematics, physics, and science.

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MAPPING THE EARLY ALGEBRAIC DISCOURSE OF SEVENTH-GRADE STUDENTS

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Algebra is an important pre-requisite for almost any advanced mathematical topic. The importance of learning this new topic entails the need to monitor students' advancement. In this work, we harness the commognitive theory of learning to analyze students' participation in algebraic discourse and asses their algebraic objects construction. We demonstrate the construction of a Realizations Mapping Tool (RMT) to map students' realizations of algebraic objects and visualize to which discourse they belong and the links between them. This paper exemplifies the affordances of the RMT by presenting the analysis of objectification of one seventh grade student, as he solves an algebraic task, and juxtaposes it with a second student's RMT representing the objectification of his discourse when solving the same task.

Algebra is one of the important domains studied in school, and mastering it is a critical requirement for success in advanced mathematics in high school and higher education. The significant part that algebra plays during an individual's mathematical life has led to the need for a careful diagnosis of students' performance as they learn algebra (e.g., Radford, Bardino & Sabena, 2007). While many studies have focused on examining specific skills in algebra (e.g., Humberstone & Reeve, 2008), this work focuses on the more general emergence of algebraic discourse when students begin learning algebra. For this goal we adopt commognition, which is a comprehensive theory that enables a micro-analytic scrutiny of students' performance while learning.

THEORETICAL BACKGROUND

A main concept of the socio-cultural theory of commognition is discourse, which is a form of communication with characteristics specific to a particular community (Sfard, 2008). Participation in a discourse reflects the proficiency of a person in that specific form of communication, and therefore, one can detect learning in the change of a student's discourse.

Commognition identifies mathematics as an autopoietic system, i.e., its objects are created by means of communication and the discourse revolves around them (Sfard, 2008). A historical growth of mathematical discourse begins when the meta-discourse of the present discourse itself develops. For example, the meta-arithmetic discourse emerges from the arithmetic discourse when, instead of communicating about specific numbers (five) and specific operation (2×4), we communicate about the general characteristics of numbers and operations (e.g., "when we multiply two even numbers, we get an even number"). The school algebraic discourse is the formalization and symbolization of the meta-arithmetic discourse (Sfard, 2008). Therefore, when examining its evolvement with students, it is expected to find characteristics of

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arithmetic, meta-arithmetic, and symbolic discourses that together incorporate the students' algebraic discourse (Caspi, 2014). This growth of mathematical discourses is further characterized by the emergence of new mathematical objects (objectification) via three main routes: *reification*, *saming*, and *encapsulation* (Sfard, 2008). Reification occurs when interlocutors move from communicating about processes using verbs (e.g., I multiply 5 and 6) to communicating about an object using nouns (e.g., the product of 5 and 6). The transformation from process to object is sometimes initiated when alienation occurs. This means that actions are performed with no performer (e.g., 5 times 6). In addition to the transformation from process to object, new objects may emerge via saming, when 5×6, 2×15 and 30 are recognized as the same mathematical entity. In this case we denote 5×6, 2×15 and 30 as realizations (discursive expressions) of one mathematical object, and one of these realizations (in this case 30) is denoted as the signifier of this specific object. The third route, encapsulation, occurs when interlocutors move from communicating about nouns in plural, to communicating about a singular noun (e.g., from (1,1), (2,4), etc. to $f(x)=x^2$).

Adopting commognitve terminology, in this work we are interested in mapping the students' extent of objectification, i.e., what algebraic realizations are present in their discourse, and what evidence can be found for the occurrence of reification, saming and encapsulation.

Since our aim is to characterize the objectification of algebraic realizations in the discourse of seventh graders, and since these students are yet in the early stages of participation in algebraic discourse, we expect to find that the students' algebraic discourse is not fully objectified (Caspi, 2014). Therefore, our analysis should be sensitive enough to reveal various intermediate stages of objectification.

To this end, we follow previous commognitive studies that generated a visual analytical tool for the purpose of mapping either discourses (Wallach, 2022) or the realizations of a single object in a whole class discussion (Weingarden, Heyd-Metzuyanim & Nachlieli, 2019). We aim to generate a visual analytical tool for mapping a single student's extent of objectification while identifying and characterizing the realizations of various objects in her algebraic discourse and the possible links between them (reification, saming, and encapsulation). This tool is called hereafter RMT (Realizations Mapping Tool). This leads us to the following research *question: How can students' objectification, while participating in algebraic discourse, be mapped by an RMT and what does this mapping afford?*

METHOD

The data for this work include think aloud interviews with 10 Hebrew speaking students from different schools and various achievement levels. To exemplify the objectification of students' algebraic discourse, we analyze the part of the interview where each student solved a specific task. The chosen task is taken from a protocol of tasks based on the Israeli curriculum of seventh grade mathematics, as follows: **"I thought of a specific number. If I multiply it by seven and subtract from the**

product fifty-four, I will get the number I was thinking of. What is the number?”. This task was selected since it is communicated mostly within the meta-arithmetic component of algebraic discourse, and therefore it calls for solutions within arithmetic, meta-arithmetic, or symbolic discourse.

A-priori stage of RMT construction

We developed the RMT (Realizations Mapping Tool) with two dimensions: horizontal and vertical. The horizontal dimension is unique to the task we focus on and includes the algebraic objects relevant for its solution. In this case, the objects relevant for the specific problem presented above, are *unknown*, *algebraic expression*, *sign of equality* and *equation*. The vertical dimension describes the sub-discourses that form the basis for the algebraic discourse: *arithmetic discourse*, *meta-arithmetic discourse*, and *symbolic discourse*. Since our participating students are novice to the algebraic discourse and may therefore use mostly processual and non-alienated talk (Caspi, 2014), we distinguish between alienated and non-alienated realizations for each of the vertical components.

RMT construction for each student

The transcripts and written documentation of the students’ solutions were meticulously read while marking realizations of mathematical objects present in them. Each realization is framed according to the following: When the realization is in line with a possible canonical (correct) task solution, then it is framed in a continuous line. Otherwise, if the realization is part of a non-canonical solution for the task it is framed in a dashed line. Each framed realization is placed in the appropriate row and column, characterizing what sort of object it is a realization of, and to which sub-discourse it belongs.

After placing the frames, the text is scrutinized for links between different realizations which are added to the RMT according to the following: Saming – continuous black line, reification - dashed black line, and encapsulation – dotted black line. When a link is mediated by the interviewer, the line is gray. Each link in the RMT is numbered according to the line in transcript it relies on.

RMTs are exemplified in figures 1 and 2. Note that the figures contain legends according to all the possible codes of an RMT, although not all of these codes appear in the specific RMTs presented herein.

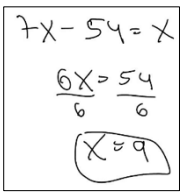
To answer our research question and demonstrate the construction and affordances of this tool, we selected two of the ten students (Alon and Gil), that both correctly solved the same task. Yet, the analysis of the extent of their objectification while solving the task yielded two very different RMTs.

FINDINGS

Following is an exemplification of the analysis that led to the construction of Alon’s RMT according to his written and spoken discourse.

Alon's transcript

1 Alon: ((Alon reads the question silently and writes))

2 Alon: 

3 Alon: The number is nine.

4 Int. Okay, can you explain it?

5 Alon Yes…mmm…I gave the number an unknown which is x,

6 Alon (I) multiplied it (x) by seven, seven x,

7 Alon and then I subtract from the product fifty-four

8 Alon and it says we get the number itself.

9 Int. Okay

10 Alon So, I made it (the number) is x,

11 Alon now I did… seven x…to move it (the x on the right side) to here (to the left side)

12 Alon so that's minus x and so it's six x, and then it is moved to here (fifty-four from left to right),

13 Alon so it's fifty-four.

14 Alon Six x equals fifty-four.

15 Alon I reduced both (nominator and denominator) by six

16 Alon and then it (the number) turned out nine

Alon started with silently reading the task, writing the appropriate equation, and solving it (turns 1-3). He arrived at the correct solution operating mostly within the symbolic discourse. His spoken discourse followed the written solution at the request of the interviewer (turns 5-16) and enabled unveiling the links between the written and spoken realizations in his discourse. The realizations and links are embedded in Alon's RMT in figure 1. In turns 5-7, Alon samed several meta-arithmetic realizations such as "the product", or "the number", presented in the task, with their symbolic realizations "7x" and "x" respectively. Therefore, these realizations were framed and located in the appropriate coordination in the RMT. Thus "the product" was located in row "meta-arithmetic/alienated" and column "algebraic expression"; "the number" was located in row "meta-arithmetic/alienated" and column "unknown"; "7x" and "x" were located in the same columns respectively and in row "symbolic discourse/alienated".

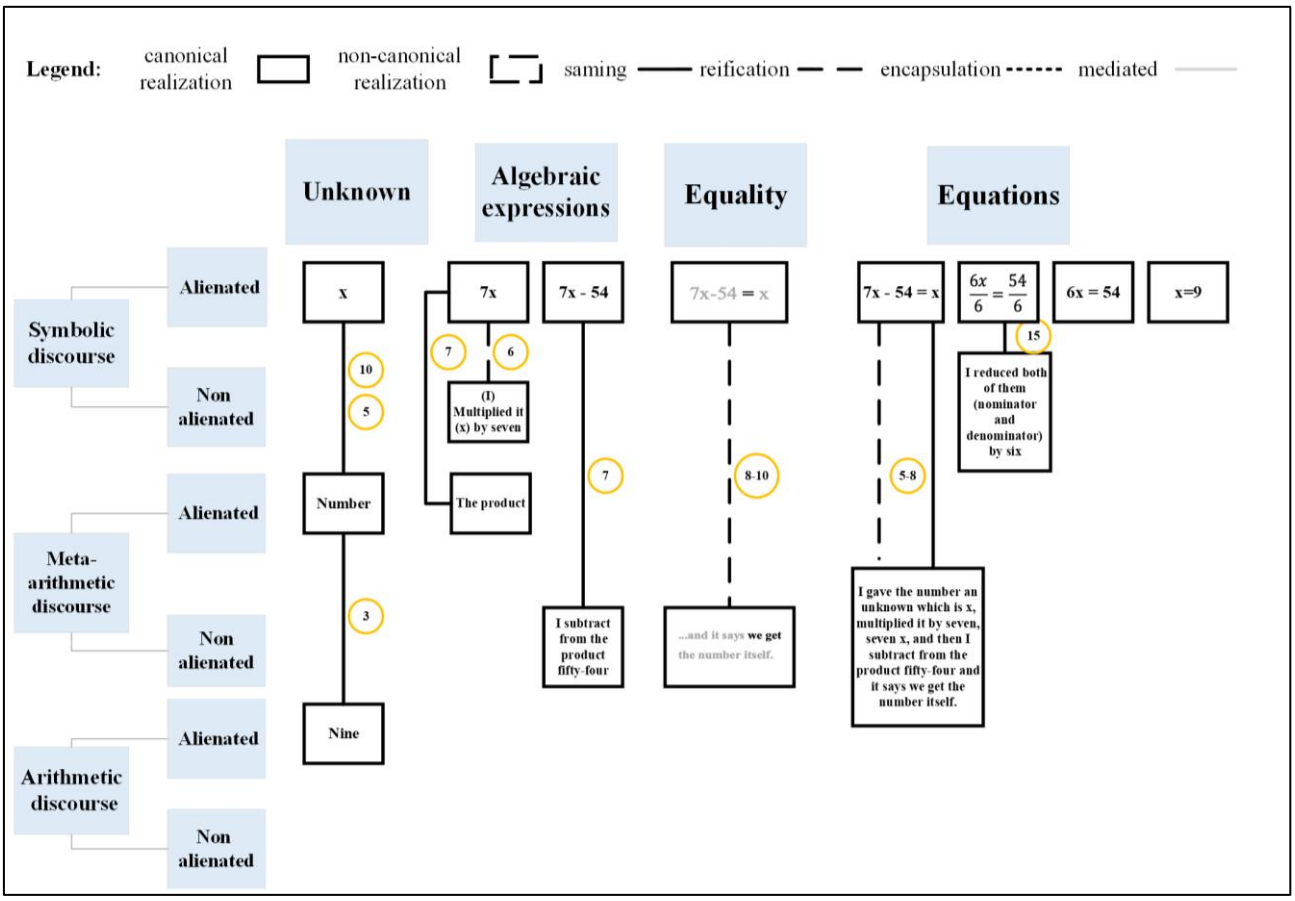


Figure 1: Alon's RMT

In addition, links of saming were drawn between "the product" and "7x" and "the number" and "x". Alon also demonstrated reification when he moved in the same sentence from saying "I multiplied it (x) by seven" and expressing the product as "seven x" (turn 6). This is apparent in the RMT when these two realizations ("I multiplied it (x) by seven" and "seven x" are linked with a reification link. In addition, we should notice that "equality" is articulated in the task and in Alon's meta-arithmetic discourse as an invitation to calculate ("I will get") and is reified (turns 8-10) by Alon to an equivalence of algebraic expressions. This stage enabled Alon to eventually same and reify the whole meta-arithmetic processual realization of an equation presented in the task with its symbolic equivalence of an equation (see turns 5-8). Furthermore, Alon moved from one realization of the equation to another (e.g., from $7x - 54 = x$ to $6x = 54$) and finally found its solution. Yet, in his discourse we did not find any expressions representing the equivalence/sameness of the various realizations of the equation. Therefore, no links were drawn between the frames that represent the different equations. The above analysis exemplified how Alon's extent of objectification while solving a specific task can be presented in the RMT.

We now turn to describing what can be "read off" an RMT to demonstrate its affordances when it stands alone as in image representing the student's talk. For this purpose, we present the RMT of Gil who is a student at the same grade level, solving

the same task. Gil's RMT (Figure 2) was generated from the transcript of his solution following the same procedure that generated Alon's RMT.

In general, Gil first tried to reverse the given instructions and present this procedure in an algebraic expression, yet this did not lead him to a solution. Then he made some educated choices of arithmetic calculations that helped him figure out that the number nine is the solution of the task.

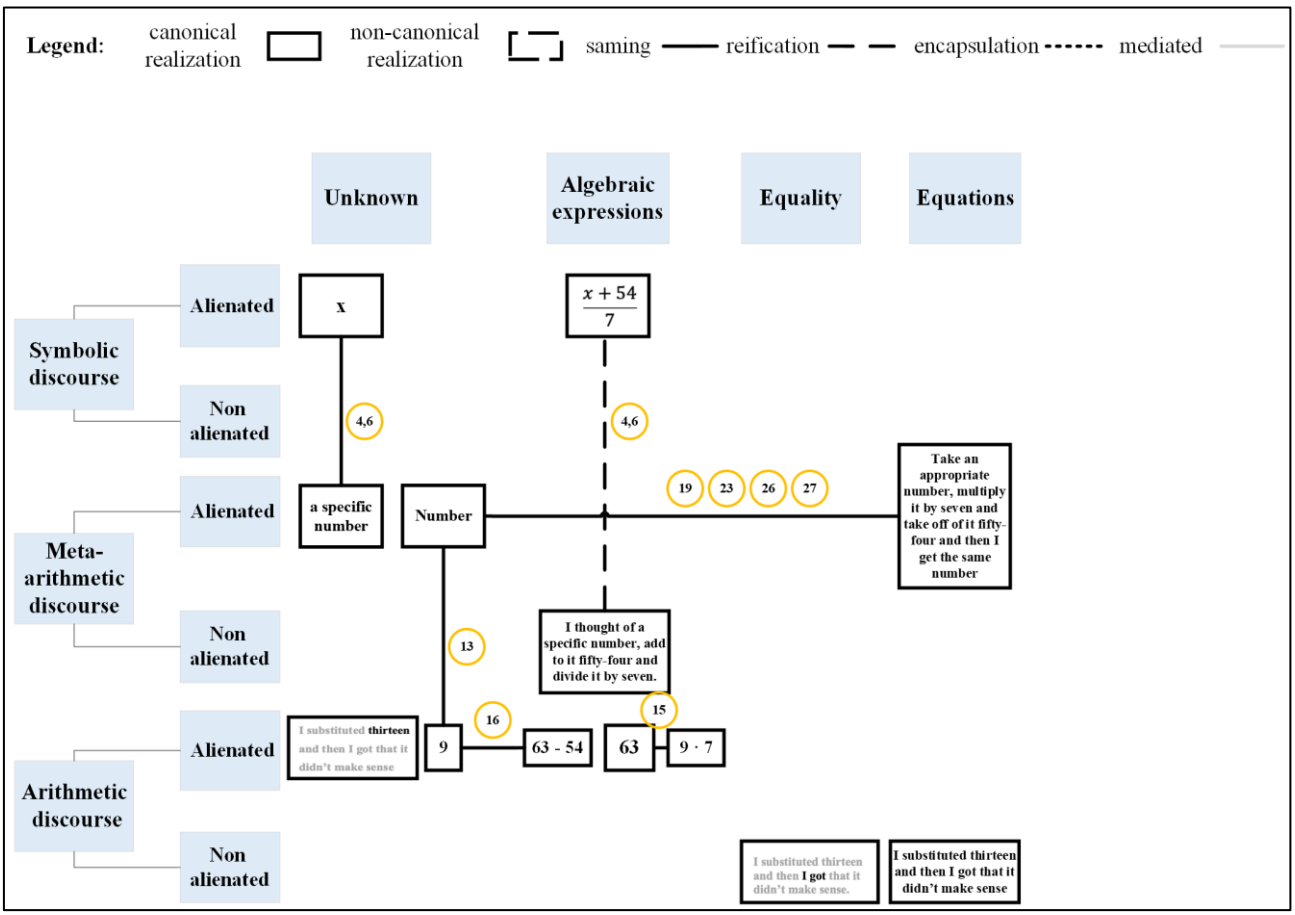


Figure 2: Gil's RMT

From the thirteen realizations in Gil's RMT four are in the meta-arithmetic discourse, only two are located in the symbolic discourse and all the rest are located in the arithmetic discourse. Moreover, when we focus on the links, we may notice that there are only three links that cross discourses (vertical lines). Two of them are links between meta-arithmetic and symbolic realizations (x and $\frac{x+54}{7}$ which are linked to meta-arithmetic realizations). These symbolic realizations are "left hanging" and are not connected to the solution of the task (the number 9). The third cross-discursive link is a saming link (turn13) connecting a meta-arithmetic realization of an "unknown" with the number "9" which is the solution of the task. There are three additional horizontal links, two of which are saming arithmetic realizations and the third is saming meta-arithmetic realizations. Thus, we see that most of Gil's solution is within the arithmetic discourse. Although there are links connecting meta-arithmetic realizations and symbolic realizations they are not developed into a solution of the task.

When comparing Alon's and Gil's RMTs, we conclude that Alon's algebraic discourse around this task is more objectified than that of Gil since most of his realizations were in the symbolic discourse and included multiple saming and reification-links between symbolic and meta-arithmetic realizations. In contrast, in Gil's discourse, most of the realizations were in the arithmetic and meta-arithmetic discourses and included only a few links mainly between arithmetic realizations and between arithmetic and meta-arithmetic realizations.

DISCUSSION

This work was led by the question: How can students' objectification, while participating in algebraic discourse, be mapped by a Realizations Mapping Tool and what does this mapping afford? We exemplified how the RMT is constructed, and how it illustrates qualitatively the extent of the students' objectification by mapping the students' realizations of the four algebraic objects, relevant for the task, to the sub-discourse they belong to (arithmetic, meta-arithmetic and symbolic), and marking the links between them. Furthermore, we juxtaposed two students' RMTs and inferred the similarities and differences in the extent of objectification of their participation in algebraic discourse for the purpose of solving the same specific task.

The limitations of this study mainly stem from its confinement to ten students and one task. Since this is a novel suggestion, further research is needed to learn about its limitations and affordances.

The possibility to compress one student's extent of objectification of a solution of a certain task into a single diagram (RMT) has several promising strands that can be followed. Firstly, the mapping of one student solving one task can be expanded to include mappings of the same student's solutions of various tasks. This will provide a more comprehensive picture of his extent of objectification while participating in algebraic discourse, and its dominant components. Secondly, since this work relies on previous commognitive research that harnessed various forms of diagrams for mapping purposes (Weingarden et al., 2019; Wallach, 2022), we now may learn the benefits of this kind of mappings from their recent evolvement. From Weingarden and Heyd-Metzuyanim (2023), we learn that realization trees that map the realizations of one mathematical object within a whole class discussion can be quantified via cluster analysis and enables comparisons of different groups of lessons. In a similar manner, RMTs representing the extent of objectification of groups of students (e.g., from different grade levels or classrooms) can be compared. In this way we may contribute to unveiling the complexity of the process of object construction in the early stages of students' learning algebra and thus may be helpful in suggesting appropriate interventions.

ACKNOWLEDGMENT

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CONTEXTS FOR ACCUMULATION

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The concept of accumulation, while central for integration and potentially significant, is often difficult for students to grasp. We examined the role of different contexts in expressing accumulative thinking. Initial results provide evidence that extra mathematical contexts facilitate students' thinking of accumulation and offers an opportunity to express their accumulative thinking. It seems that certain extra mathematical contexts are more helpful in this sense than others.

BACKGROUND AND RATIONALE

Integration as accumulation is at the core of understanding many ideas and applications. The accumulation function has the potential to serve as a model for various situations in continuous real-life processes, and in reverse, many continuous real-life processes have the potential to demonstrate accumulation.

According to Thompson and Silverman (2008), the concept of accumulation is central to the idea of integration. It goes hand in hand with a coherent understanding of rate of change (Thompson, 1994; Carlson et al., 2003): "When something changes, something accumulates" (Thompson & Silverman, 2008, p. 49). Although the concept of accumulation is familiar to students from everyday life, they have difficulty grasping the 'bits' accumulating (Thompson & Silverman, 2008). Thompson (2013) argues that a significant reason for this is a lack of reference to meaning in teaching and research. Considering this, it has been suggested to teach integration with an approach that places the concept of accumulation at the centre (Kouropatov & Dreyfus, 2013).

Researchers have shown that using extra mathematical context in the learning of mathematics may be helpful in different ways. Rubel & McCloskey (2021) studied what they called "contextualization of mathematics", which denotes a teacher's discursive turn that does not consist exclusively of mathematical objects. Among the rationales to contextualize mathematics, they state that it supports the learning of mathematics, claiming that familiar situations may act as "foundations" on which "mathematical skyscrapers" can be built, with scaffolding supplied by teachers (Carraher & Schliemann, 2002, also cited in Rubel & McCloskey, 2021).

Lakoff & Núñez (2000) speak of "grounding metaphors" which relate a target domain within mathematics to a source domain outside it, creating a conceptual relationship between a (familiar) initial domain and a (new or abstract) target domain.

Accumulation specifically has been found to be a concept which may be taught through extra mathematical contexts. Carlson et al. (2003) developed curricular materials aiming to promote students' understandings and reasoning abilities regarding the fundamental theorem of calculus, including tasks given in an extra mathematical

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context. The results showed that most students developed a strong understanding of the aspects regarding accumulation.

In this study we examined students' facility of expressing ideas related to accumulation in different intra- and extra mathematical contexts. We ask: In what contexts do students think of accumulation and express that accumulative thinking? What role does the context have in the process?

METHODOLOGY

This study is part of a larger project, in which we examine students' personal meanings for basic concepts of calculus. As part of the project, we developed various tasks, including one in which students are presented with a split-domain rate-of-change function, constant and positive in each subdomain, and then asked about its accumulation function (see Figure 1 below). The task was designed so that there is no need for technical work, but rather for understanding that if you know how fast a quantity is changing at every moment, you can know how much of that quantity has accumulated at every moment. The quantity accumulates continuously, and because of the positive rate of change, the accumulation function is monotonously increasing.

For examining the role of context, we designed 4 parallel versions of the task, based on different intra- and extra mathematical contexts: (1) Formal mathematical; (2) Area; (3) Motion; and (4) Pool. We present (4) in detail (Figure 1).

The accompanying sketch shows the graph of the function $f(x)$, which represents the flow of water into a pool. The flow is measured in units of litres per second. The pool was empty at the beginning.

Alona said that using these data it is possible to sketch the graph of the function $g(x) = \int_0^x f(t)dt, \quad x \geq 0$ which represents the amount of water in the pool at time x. The students discussed the meaning of her statement to them.

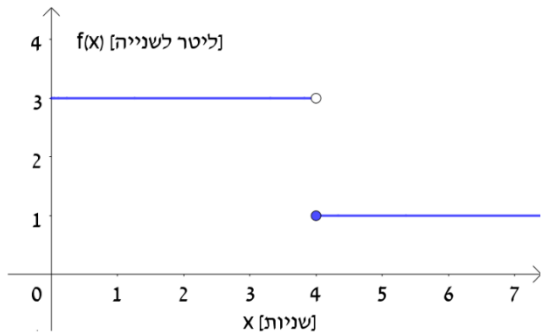

Figure 1: The pool context version of the task. The vertical axis of the graph is labelled: "$f(x)$ $[litres\ per\ second]$"; the horizontal axis is labelled "x $[seconds]$".

Table 2 (below) demonstrates the differences between the different contexts. Following Alona's claim, each version of the task presents five statements regarding the situation, supposedly made by five imaginary students discussing the meaning of Alona's claim. Each statement expresses a way of thinking about accumulation. In this report, we focus on the statement made by Paula: 'For me, $g(x)$ is an accumulation function. Since there is a jump in $f(x)$, one needs to relate to the accumulation separately in each domain. That is, the accumulation restarts at $x = 4$ and therefore also the graph of $g(x)$ also starts to rise again from 0 at $x = 4$.' Paula's statement contradicts the

continuity of the accumulating quantity. Accumulative thinking implies that the accumulation continues, albeit at a different rate, rather than restarts at $x = 4$.

Context	$f(x)$	$g(x)$	Alona said that using these data it is possible to sketch the graph of
Formal mathematical	Function	Integral of $f(x)$	the function $g(x) = \int_0^x f(t)dt, x \geq 0$.
Area	Function	The area that accumulated between the graph of $f(x)$ and the x-axis starting from $x = 0$	the function $g(x) = \int_0^x f(t)dt, x \geq 0$ which represents the area that accumulated betwen the graph of $f(x)$ and the x-axis starting from $x = 0$ up to $x = 7$.
Motion	The speed of a tortoise (meters per second)	The distance the tortoise passed at time x	the function $g(x) = \int_0^x f(t)dt, x \geq 0$ which represents the distance the tortoise passed as a function of the time x.
Pool	The rate of flow of water into a pool (litres per second)	The amount of water accumulated in the pool at time x	the function $g(x) = \int_0^x f(t)dt, x \geq 0$ which represents the amount of water in the pool at time x.

Table 1: The different versions of the task

Population and interview methodology

Twenty four high-school students learning advanced track mathematics in grades 11 or 12 were interviewed, after having learned about integration, 6 on each version of the task. Interviewees were asked to rate to what extent they identified with each of the five statements, including Paula's, on a scale of 1 to 4. They were asked to do this in writing before the conversation with the interviewer started. An additional interview using the motion context version was held with a student who was familiar with the pool context; more details about this student will be given below.

The students were told that the interviewer was not interested in right or wrong answers but was interested in their ways of thinking. The interviewer’s instructions were to discuss the students’ responses to the questionnaire with them, with the aim of clarifying the meanings they hold: The interviewer should ask about how the student interprets the statements in the questionnaire. The interviewer should note if there are inconsistencies between how students react to different statements and such inconsistencies should be addressed. The interviews were audio-recorded and transcribed. When analysing the interviews, we used the following criteria to identify utterances that are indicative of the interviewee's personal meanings:

1. Distinct language – the utterance contains concepts and language specific to the learner, which have not been used by the interviewer or in the task.
2. Repetition – the utterance contains concepts that repeat themselves.
3. Reasoning – the utterance is intended to explain or justify the mathematical concepts.
4. Unexpectedness - the utterance is surprising or unexpected to the researcher.
5. Statement of opinion – the utterance is explicitly qualified by the interviewee as their own belief, opinion, or interpretation. This includes, for example, utterances containing phrases such as “to me”, “in my opinion”.

FINDINGS

In this section we present the results of the interviews, according to the context.

Formal mathematical context

Four of the six interviewees stated that they do not understand Paula's statement since they don’t know what an accumulation function is, and another one said he agrees with Paula. Only Thomas, expressed some accumulative thinking:

> Thomas: "Yes, like I somehow, my initial thought of the function $g(x)$ was of a function that keeps going up. Continuous. When I started reading all their opinions, I understood that maybe it's not continuous, that there is some gap at $x = 4$ […] Because again, it's an accumulation function so like not at every point… okay, I'll rephrase it to myself, at every point on $g(x)$ the area, like doesn't matter if I start to count it from 0 or 4, the area will be bigger than 0."

While Thomas’ answer points to aspects of accumulation, he seems confused. While there was no mention of area accumulating, Thomas mentions area. This may be due to the close link between integral and area made in many Israeli high school classes.

Area

Two of the 6 students said that Paula is correct, and one said she doesn't know. The other three students expressed little accumulative thinking:

> Gal**:** "I'm saying that it's correct, you need to divide it into two accumulation segments. […]" The interviewer says: "I repeat what she's saying, up to the point 4 the graph rises, and then you start with zero area." Gal replies: "No, […] the graph doesn't reset, from her point of view you sum until 4, and then again from 4 to 7."

> Einat: "All the rest was correct except 'and thus the graph of the area $g(x)$ begins to rise again from 0 at $x = 4$.' Because it doesn't rise from 0. […] Like a graph that continues differently, okay, but it doesn't begin from 0."

> Later in the interview Einat said something that sounded contradictive and was asked about it. She replied: "If I look at it as an accumulation integral at this point specifically, at the first point of $f(x)$ I do think it starts at 0. […] I'm looking at area and at this [(4,1)] point specifically I don't have any area because nothing has accumulated."

> Tom: "The claim is completely incorrect, although I don't know what an accumulation graph is, the graph of g doesn't re-rise again from 0."

Of the six students, only Tom made a correct statement but did not explain it. Gal was able to say that the accumulation function doesn't reset, but his attention stayed on working on two sub-domains. Einat sounded sure of herself when referring to Paula, but then changed her mind.

Motion

Three students tended to agree with Paula, three others corrected Paula's thinking:

> Tori: "Paula was precise in the domain of each of them, and that it starts to rise at 0." Later: "I would change it… […] what I said about Paula. […] I don't think that the distance just rises from 0, because he [the tortoise] continues on the same course."
>
> Alan: "It's actually a count that starts over from the beginning, but it doesn't start over actually you need to add […] I just count like… count one speed, the quantity of meters he passed and then I just add the meters at the other speed."
>
> Niky: "I don't think so because the distance, he goes some distance, and the $g(x)$ represents distance, but it's not true that he goes back to the starting point and starts again from 0. He has gone some distance and from that distance he simply continues at a different speed. Maybe the slope will be smaller, but because he's moving slower, but it's incorrect to say that the distance is 0 again."

Alan, Tori and Niky reject Paula's claim that the accumulation restarts at $x = 4$ (although Tori first agrees with Paula, she later corrects herself). Two of them (Tori and Niky) talk about the figure continuing to walk. It seems that they use the walking figure as a tool to think about the continuity of the accumulation.

Pool

Five of the six interviewees disagreed with Paula and explained in their own words:

> Ron: "It's incorrect to do it like that. The water in the pool... like the pool isn't filled twice, it's filled once."
>
> Donna: "She says that it restarts again from zero so that like contradicts the whole point of accumulation. […] [The fact] that they changed the… the rate of the flow so they still… […] Like you continue to fill it, only differently. So, I think it needs to be a line that continues with no…. with no holes."
>
> Tami: "I agree with her that the accumulation is the amount of water. … don't agree with her that much with the issue of the graph. […] She calculates the amount of water separately […] but it's not something that I thought of. Because it's the same pool."
>
> Amir: "The accumulation doesn't start over, there's an amount of water in the pool, only the flow change changes."
>
> Omri: "It's an amount of water, it can't reset… it's not that they empty the pool."

In the pool context, five of the six students reject Paula's claim that the accumulation restarts at $x = 4$, most of them referring to the pool not being emptied at any point. The pool context enables the students to see the accumulation process as continuous.

Bonus interview: Mike

Mike was interviewed on the motion context version (in addition to the six above students). We present Mike's interview separately since his relevant mathematical background is unique. His high-school mathematics teacher is a researcher who studies developing accumulative thinking as an introduction to learning integral calculus. She designed a learning activity, based on a pool context. Her 11th grade class (Mike's class) carried out that activity as an introduction to integration (Falach, 2023).

During the interview, several weeks after the learning activity, Mike referred to accumulation before discussing Paula's statement, and said: "Like the accumulation function was presented to me, a certain quantity of water accumulates. In this case [the motion context] it's not… it's accumulating distance, the distance that he passed. Then, after 4 minutes he changes speed so his accumulation rate of the water, or in this case distance, is different. He basically changed speed." Later, when asked about the comparison to what he learned in the pool context he replied: "With the water, and here also, there is a quantity of accumulation, that's how I treat it. In the case of water, you accumulate water, for example in a pool, and in this case, he accumulates distance."

When asked about Paula's statement, Mike replied: "It's very correct and very close [to how I think]. Now that I read it again, it's exactly the way I think…I interpret it in the simplest way possible. She divides it, like I said before about the accumulation in the pool, into one part in which water accumulates in a pool, and a second part in which water accumulates.... In each part, there was a different quantity of accumulation."

Later in the interview, Mike referred to the graph of the accumulation function being continuous, in what may seem a contradiction to his agreement with Paula: "indeed he changed his rate, but he began from the same point […] I treat this change as the slope of the graph, not as a detachment of the point."

The main finding of this interview is that Mike fluently transfers what he learned in the pool context to the motion context. He makes the parallel between quantity of water and distance as well as between flow rate and speed. Although Mike agrees with Paula and identifies with her, he sees accumulation as a continuous process. The fact that he identifies with Paula demonstrates the fragility of his accumulative thinking. It is interesting that he uses the pool context to explain his thoughts, rather than the current motion context. This may be interpreted as more evidence to how intuitive it feels to talk about accumulation of water, even more than about distance in a motion context.

DISCUSSION

Students were presented with parallel tasks about accumulation in four different contexts. The context had a considerable influence on their ability to express accumulative thinking. Extra mathematical contexts typically facilitated accumulative thinking, especially the pool context. The area context facilitated accumulative thinking less than the motion and pool contexts but more than the formal mathematical context.

Some students used the extra mathematical pool or motion context as "foundations" to reason about the continuity of the process. For example, Niky says: "He has gone some distance and from that distance he simply continues at a different speed"; or Amir says: "The accumulation doesn't start over, there's an amount of water in the pool, only the flow change changes"; or Omri: "It's an amount of water, it can't reset… it's not that they empty the pool."

The language students use in the pool context is casual ("the pool isn't filled twice, it's filled once", "you continue to fill it, only differently", "it's the same pool"). The casual language suggests the reasoning using the pool is intuitive, and that filling a pool is a natural 'grounding metaphor' for accumulation. In Mike's interview we see that the pool context supports thinking in terms of accumulation, but his interview also demonstrates the difficulty and complexity of the accumulation concept. The pool context assisted Mike in expressing his thoughts but his exposure to the pool example did not prevent him from identifying with Paula.

In conclusion, we see evidence that certain contexts assist students to think about accumulation and to express their thoughts on it. Some contexts are more helpful in this sense than others. The continuance of the accumulation process was more evident in the pool context, than the motion context, and more in the motion context than in the area context. We highlight characteristics of the contexts which we think made them more (or less) helpful in this sense:

1. In the pool and motion contexts, the independent variable is time. This gives an inherent intuitive sense of what comes before what and connects to the smooth continuous conception of time. It is natural to perceive that filling a pool is a one-way process: Time will only go forward, and if you fill the pool, and do not empty it, the amount of water is going to grow.
2. In the pool context, the 'bits' that accumulate are visible. One can take a picture of them and mark the change in the picture. It is tangible. It is something that the change (described by the rate of change) created in the sense that it did not physically exist before the change: for motion – one can visualize the road and the current position of the tortoise, but one cannot see and touch the accumulated distance. This makes the accumulated change more approachable in the pool context than in the motion context, and in motion context more so than in the area context, which is theoretical (not from every-day life).

An interesting observation that might warrant further research is that in the extra mathematical contexts the students relate to the rate of change function in their answers:" I just count like… count one speed, the quantity of meters he passed and then I just add the meters at the other speed"; "Maybe the slope will be smaller, but because he's moving slower"; "they changed the… the rate of the flow"; "only the flow change changes". This demonstrates that the rate of change at which a quantity accumulates may be more approachable in extra mathematical contexts. This is of importance, since, as stated, the concept of accumulation goes hand in hand with a

coherent understanding of rate of change. This, and other findings, also call for future research in more directions: (1) Examine whether and how other extra mathematical contexts facilitate students' accumulative thinking; (2) Research students' processes of constructing knowledge about accumulation in a pool context; (3) Examine students' facility to express their thoughts about rate of change and derivative, in different intra- and extra mathematical contexts.

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CROSS-COMMUNITY COLLABORATIVE TASK DESIGN

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This study investigates processes of collaborative task design within a diverse group of stakeholders: elementary and secondary mathematics teachers, mathematics education researchers, teacher educators, and policymakers. Data were collected throughout one year out of a long-term research project, focusing on developing inquiry-based tasks for the elementary level under the pedagogical model of flipped-classrooms. In this paper, we explore how different focal points, raised by various stakeholders, may shape design decision-making processes. We focus on one particular decision and use an utterance map to represent the rich and complex interactions around this decision and the tensions between two central focal points: the discipline of mathematics, and the students. We discuss the potential and challenges of cross-community collaborative task design.

BACKGROUND, RATIONALE, AND RESEARCH QUESTIONS

In recent years we have witnessed dramatic changes in learning environments, as a result of the COVID-19 pandemic, which among other things forced – for considerable periods of time – a shift from classroom-based to home-based learning. These changes and the great challenges they posed for students, teachers, parents and the mathematics education community at large, urge the need to explore learning environments that go beyond the physical classroom, to serve both in future crises and in times of stability. In the ZEN-Math project (Zooming into Environments for Nurturing Mathematics), initiated in 2021 at the Weizmann Institute of Science, we aim to support greater flexibility in mathematical teaching and learning environments at the elementary school level. Specifically, the project draws on an adaptation of the emerging pedagogical model of the "flipped classroom" (FC). In this adapted model, students prepare for class by engaging in inquiry-based tasks designed for independent learning. This is then followed (in the classroom or remotely) by a teacher-facilitated inquiry into the material, grounded in students' prior work rather than in teacher presentations (Cevikbas & Kaiser, 2020).

A central premise of ZEN-Math is that the design, enactment, and dissemination of a hybrid pedagogy, integrating independent student learning with teacher-facilitated communal inquiry, requires multifaceted knowledge and expertise that does not lie within the confines of any sole stakeholder in mathematics education. It requires a productive collaboration between different stakeholders, such as researchers, teachers, teacher educators, and policy makers, who may have different aims and agendas that could be pertinent to the design (Margolinas et al., 2013). Thus, a key aspect of the ZEN-Math project is cross-community collaborative task design (Jones & Pepin, 2016): a diverse group of experts, representing different communities in mathematics education, jointly develop, test and refine inquiry-oriented FC tasks.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel., & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 299-306). PME 46.

Cross-community collaborative task design is far from being straightforward. A diverse design team brings not only multifaceted expertise but also different perspectives and values regarding the teaching and learning of mathematics. Differences may lead to conflicting opinions and to tensions that could hinder task design processes, thus preventing such teams from functioning as more than the 'sum of their parts' in developing novel ideas and insights (Pinto & Cooper, 2017, 2019). Research on cross-community collaborative task design is quite limited, particularly in relation to the questions of how a design team may draw on its multifaceted expertise, and how this multifaceted expertise may influence the resulting tasks. The study reported here addresses this gap by investigating the complex and layered dialogues among the different stakeholders in the ZEN-Math design team, in an attempt to understand how various perspectives and goals interrelate during the collaborative task design. For this, we employ an adapted version of the *practical rationality* framework (Herbst & Chazan, 2012, 2020), as discussed in the next section. Our investigation is guided by the following research question: What characterizes dialogues among different stakeholders in the process of collaborative task design?

THEORETICAL FRAMEWORK

Herbst and Chazan (2012, 2020) introduced the notion of practical rationality for studying the collective dispositions, norms and commitments underlying mathematics teaching, and possible tensions between them. They presented four *professional obligations for mathematics teachers*, which they named the disciplinary, individual, interpersonal, and institutional obligations. In broad terms, the *disciplinary obligation* implies that school mathematics needs to be a valid representation of the knowledge, practices, and applications of the discipline of mathematics. The i*ndividual obligation* refers to students' wellbeing and personal identity, taking into account diverse behavioral, cognitive, emotional, or social traits and needs, which a mathematics teacher may not ignore. The *interpersonal* obligation signifies the teacher's need to ensure a socially and culturally appropriate distribution of resources such as time, physical space, and symbolic space. Finally, the *institutional obligation* refers to the various ways by which teachers respond to their mathematics department, school, district, professional associations and unions, etc. When designing tasks and envisioning how these tasks may be enacted in practice, designers need to be attentive to these different teacher obligations. Moreover, designers' practical rationality may include parallel forms of obligations, for example for the wellbeing of individual teachers, or for policies and agendas of the Ministry of Education. Different stakeholders may identify different obligations in similar instructional settings and may feel less or more committed to them (Herbst & Chazan, 2020).

METHOD

Participants, setting, and data collection

The study focused on nine stakeholders, constituting the design team: three mathematics teachers, three mathematics teacher educators, two mathematics

education researchers, and one mathematics pre-service teacher (the latter was absent from the meeting analyzed in this report). We denote the teachers as T1, T2 and T3, teaching in elementary, middle and high schools, respectively. TE1, TE2 and TE3 denote the teacher educators (TE1 is a chief instructor and a policymaker in the Ministry of Education, TE2 and TE3 teach in a teacher education program). R1 and R2 denote the researchers. The team met face to face or via Zoom every three weeks, for four academic hours, throughout one year. Data for the study consisted of the recordings of all sessions; key episodes were transcribed.

The mathematical task and design dilemmas selected for this report

Out of the three tasks designed by the team, we focus here on a geometrical task for upper elementary students. In the independent part of the task, students are requested to use two line segments drawn on transparencies, to be considered as diagonals of quadrilaterals, and position them in various orientations to construct different quadrilaterals. Then, they are asked to sort the quadrilaterals into groups, according to characteristics of their choice. The aim of this task is to set the grounds for a teacher-facilitated inquiry into characterizations of quadrilaterals based on their diagonals. Two of the dilemmas discussed along the design process were the following: (1) should students be given only same-length diagonals or also different-length ones? (2) should students be given a thumbtack to be used as a hinge point around which the diagonals pivot? The seemingly minor detail in dilemma (2) turned out to entail tensions between different perspectives (as we show below), thus we selected the dialogue around it as the center of this report.

Data analysis

As a preliminary step for understanding the "thumbtack dialogue", we performed an exhaustive mathematical analysis to systematize all the possible solutions for the task (in the case of same-length diagonals). Then, we conducted a content analysis of the dialogue transcript. Firstly, we segmented the transcript into units of analysis, with each unit including consecutive utterances of the same speaker. Secondly, to identify the "designers' practical rationality" of different stakeholders, we coded the units according to six codes, inspired by the professional obligations discussed by Herbst and Chazan (2012, 2020). Rather than identifying obligations, we examined focal points, i.e., what designers are attentive to along the design process. Thus, our three first codes were the focal points of *the discipline*, *the students* (as individuals or as a whole class), and *the institution* (e.g., the school, the curriculum). To this we added the focal points of *teachers/teaching* and *task design* (e.g., making a design suggestion). Lastly, we added a code referring to speakers *inviting discussion*. The coding was validated by comparing codes obtained separately by the three authors, and discussing discrepancies until agreement was reached. In the final third step of analysis, we created an "utterance map" (Nurick, 2015), i.e., a visual representation of the different focal points in the dialogue.

FINDINGS

Figure 1 shows 12 different groups of quadrilaterals with same-length diagonals, obtained as solutions to the task, and arranged according to the diagonals' mutual relations. As can be seen, 4 of the groups consist of concave quadrilaterals whereas the other 8 consist of convex quadrilaterals. This distinction is central to our analysis: if students are requested to use a thumbtack when positioning the diagonals, the underlying premise is that the diagonals intersect, thus the concave quadrilaterals are a priory excluded from the inquiry.

We now turn to the analysis of the "thumbtack dialogue". Figure 2 (below) presents the utterance map of this

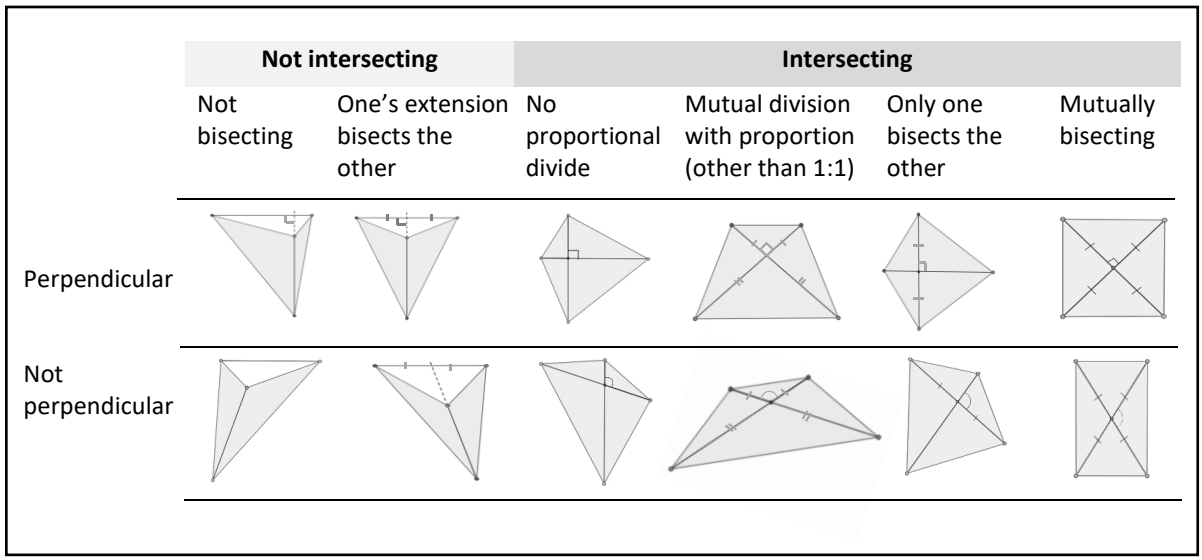

Figure 1: Different groups of quadrilaterals with same-length diagonals

dialogue, showing the coding of units by speakers (color legend included). Each rectangle represents a unit or consecutive units of the same speaker, with the notation of the speaker and the unit numbers. Longer units are represented by longer horizontal sides (without accurate proportionality), and the arrows indicate a response of a speaker to previous speakers (e.g., elaborating, opposing to, or supporting ideas raised earlier). The dialogue starts at the top right-hand side of the map, with TE1 repeating the dilemma ("*should we have students use a thumbtack when positioning the diagonals to create quadrilaterals*?", unit 25), raised earlier in the meeting within a small group discussion. Then, the map progresses downwards when speakers refer to this dilemma.

As an example of the coding represented in the map, we present below some of the utterances of T3 (part of the sequence of units 31-36) and their ensuing coding (by color):

31 T3 I raised this issue as a [design] question, it has a disadvantage and an advantage.

32 T3 The advantage is that it fixes the diagonals' intersection point, thus allowing a certain dynamic look […] so it creates a kind of dynamic geometry of some sort.

33 T3 […] Pedagogically, it allows [participation] of young children, who are inexperienced with higher concepts.

34 T3 [...] If the thumbtack is not in the midpoint of the diagonals, and the diagonals are correspondingly positioned […] then this center actually divides them proportionally […] and then when I move them I keep getting the same type of shapes and you can arrive at some conclusions. […] [It's different] to tack it in the middle because then I actually get rectangles, not equilateral trapezoids.

35 T3 […] The disadvantage is that […] pedagogically it fixates thinking […] and doesn't allow them to just throw the diagonals [as they wish], to liberate their creativity.

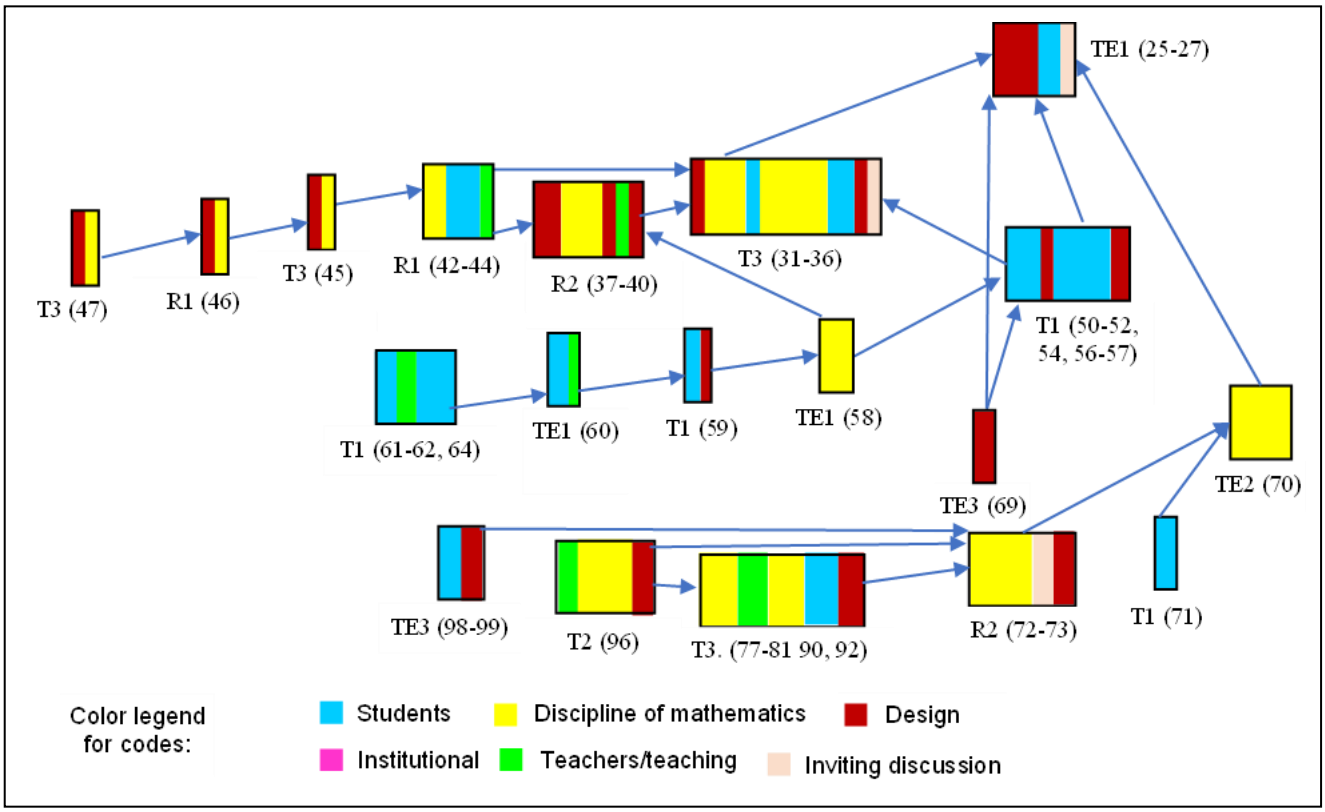

Figure 2: The utterance map illustrating the "thumbtack dialogue"

Another example, to demonstrate the links represented by arrows, is the response of T1 to both T3's and TE1's earlier utterances (part of the sequence of units 50-57):

51 T1 We can give [students] the thumbtack and ask, is it always possible to use it? Can you explain the reason why?

Here T1 answers TE1's query (unit 25) by suggesting to include a question in the design.

52 T1 […] It can stir [students'] thinking. In my opinion it's really a lovely activity, this issue of the thumbtack […] [with] guiding requests to explain the course of action, so that they can come to many conclusions. I really like this.

56 T1 I would however ask more open-ended questions about the thumbtack. You got a thumbtack […], do you need it? Does it add something? Give them thought-provoking questions. You have the thumbtack – you decide whether you use it.

In unit 52, T1 responds to T3's concern that using a thumbtack can "fixate thinking" (unit 35). She proposes that, on the contrary, it can provoke students' thinking. In unit 56 she elaborates guiding questions that may be incorporated in the design for achieving this.

What can be learned from the map? Firstly, it clearly shows that the conversation was collaborative and multifaceted. This is evident from the observations that (a) all eight team members present at the meeting were involved in the discussion; (b) the dialogue was interactive, as shown by the many arrows, i.e., speakers constantly responded to each other and were not simply presenting individual isolated ideas; and (c) the discussion branched out into three substantial directions, the upper branch (units 31-47), the middle branch (units 50-64), and the lower branch (units 70-99). Although not seen in the map, additional information about the content of these branches reveals that all of them developed the issue of the thumbtack: in the first one, advantages and drawbacks of using the thumbtack were discussed; in the second one the focus was on students' potential use of the thumbtack within their inquiry; and the third one centered on balancing the different goals of the task.

Secondly, the map shows that the prominent focal points of the participants in this dialogue were the discipline and the students, as evident from the dominance of the yellow and blue colors (appearing 13 times each). There was less reference to teachers/teaching (6 appearances), and no explicit references to the institutional focal point (we note however that it was found in other dialogues we analyzed). Another finding is that different stakeholders are not readily identified in the map by the color of their utterances. Not only did different stakeholders (e.g., T1, T2, T3, R1, R2, TE1) repeatedly refer to various focal points, but also different focal points were interweaved in a *single* unit. Thus, the dialogue can be characterized as complex in the sense that it is an intricate process in which each speaker simultaneously holds several perspectives.

The complexity of the process, however, goes beyond what is seen in the utterances map. A key characteristic of the dialogue concerns the tension between considerations focused on the nature of the mathematical inquiry, and considerations focused on the students' experiences. One instance of this tension can be seen in units 32-35 of T3, cited earlier, where there is a clear tension between orienting the design towards a rich, yet structured inquiry (e.g., recognizing the difference between the properties of diagonals in rectangles vs. equilateral trapezoids, unit 34), and orienting the design

towards a more open inquiry that appeals to students' creativity (unit 35). Such free inquiry may end up with students not reaching clear conclusions about properties of diagonals in various quadrilaterals, yet it gives students more agency over how the inquiry evolves and may invite questions or observations unanticipated by the designers. Additionally, a larger and more diverse corpus of student solutions may give teachers more flexibility in the follow-up lesson, allowing them to decide on which direction to pursue (e.g., convex or concave shapes).

Aspects of this tension were present in utterances by various stakeholders throughout the dialogue, for instance, "one drawback of the thumbtack is that definitely the option of the diagonals not meeting will not rise" (R1, unit 42); "if we want to focus on just the family of parallelograms, then the thumbtack can be actually very helpful" (TE2, unit 70); "in the most narrow case [equal-length diagonals, with a thumbtack] the inquiry can still yield 8 different families […], it clearly limits creativity, but we need to find a balance" (R2, unit 72); "do we want more inquiry and play, discovery work, or do we want an activity that builds the knowledge in the curriculum? [this decision] directs us if to use a thumbtack or not, equal diagonals or not" (T2, unit 96). Notably, although several design suggestions were brought up along the way, the conversation did not converge to concrete design decisions regarding both issues, the thumbtack and the same/different lengths of diagonals.

DISCUSSION

In this report we examined a dialogue among various stakeholders within a cross-community collaborative design team. Prior research points to the advantages of diversified teams, having multifaceted expertise in issues of mathematics education. However, existing literature does not clarify how the expertise of different stakeholders may become a resource in collaborative task design processes. Pinto and Cooper (2019) suggest that one common model of cross-community collaborative work is based on splitting the team into less diverse sub-teams to address particular issues, drawing on individual fields of expertise separately. Our findings suggest that when stakeholders are not confined to preconceived roles, their expressed designers' practical rationality goes beyond their zone of expertise. The resulting dialogues may thus become highly complex, as illustrated in the utterance map. Specifically, our findings highlight three characteristics of complex cross-community dialogues. First, they constantly revolve around various focal points. Second, different stakeholders refer to multiple focal points, demonstrating that they are not "flat characters", committed to their expected interests (e.g., the teacher is committed to students, the teacher educator - to teachers). Third, stakeholders respond to, and build on, suggestions and arguments of other stakeholders.

One consequence of highly complex dialogues is the rise of tensions between different perspectives. In the "thumbtack dialogue" we identified a tension between different modes of inquiry, which can be seen as representing broader agendas in mathematics education, for example about allowing students leeway to "play", vs. the commitment

to cover the curriculum; or about how important is it that students exhaust all the mathematical possibilities afforded in a certain topic. The tension we identified also touches upon questions of inclusiveness (e.g., a design that takes into account students' differential abilities), and professional power (e.g., how rigid or flexible should the design be, in order for teachers to use it effectively). Importantly, tensions may become a valuable resource or an insurmountable obstacle. On the one hand, addressing perspectives and values of different stakeholders from early stages of the design may increase the likelihood that the designed products are better aligned with diverse interests within the system, thus simplifying implantation and dissemination. On the other hand, addressing and resolving tensions comes with various costs and in extreme cases may even derail collaborations (Pinto & Cooper, 2017). A future direction for research is to investigate what allows for tensions to become productive in cross-community collaborative task design. One aspect of productiveness is the convergence of design processes into solid decisions. In this study, the "thumbtack dialogue" did not converge to a design decision, yet it did prepare the grounds to a consensus that represents a balance between different focal points, in a process of negotiation that will be reported elsewhere. Finally, we propose that the tool of the utterance map can assist in further exploring the practical rationality of diverse design teams, as well as other aspects of cross-community collaborative task design.

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FROM TEACHER PROFESSIONAL DEVELOPMENT TO TEACHER PERSONAL-PROFESSIONAL GROWTH: THE CASE OF EXPERT SCIENCE AND MATHEMATICS TEACHERS

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Scholars theorize that professional well-being is essential for teachers' personal-professional growth. However, teacher professional development programs (TPDPs) primarily focus on promoting student achievement and only partially and indirectly address teachers' professional well-being. The purpose of this study was to compare science and mathematics teachers' perceptions of their professional well-being needs with actual practices in effective TPDPs. Twenty interviews with expert teachers were conducted to identify well-being components (competence, relatedness, autonomy, and aspirations). The findings show that aspirations, emphasized by teachers as most important, were totally ignored in TPDPs. This finding may have practical and theoretical implications for designing and assessing TPDPs' effectiveness.

INTRODUCTION

Professional Development (PD) represents an essential part of teachers' professional lives, where their craft knowledge and professional orientation can be reached, guided, and moulded. Nonetheless, teacher professional development programs (TPDPs) primarily reflect policymakers' focus on the teacher's critical impact on student achievement while neglecting the role that the person plays in the profession (Intrator & Kunzman, 2007). Unsurprisingly, in their recent survey, Darling-Hammond et al. (2017) reveal an emphasis of educational systems on "effective programs" that lead to improved student achievement. They describe effective TPDPs as primarily aimed at promoting the quality of teaching, emphasizing relevance to the teaching content and the teacher's needs in this regard. Teachers are regarded as implementers expected to align their instruction with external and pre-determined goals and practices (Lindvall & Ryve, 2019). Moreover, TPDPs tend to overlook teachers' changing individual professional needs at various stages of their teaching careers and designate the same programs for all teachers (Huberman, 1995). Well-rooted in the general approach of holding teachers accountable for student achievement, TPDPs fail to address teachers' personality and do not regard teaching as a profession and teachers as specialists (Tucker, 2014). Despite a proliferation of effective TPDPs, this approach to teachers' PD turns out to be limited and insufficient, given teachers' dissatisfaction with their personal-professional growth (Santoro, 2021).

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Scholars have theoretically asserted that professional well-being (PWB), comprised of teachers' *competence*, *relatedness*, *autonomy*, and *aspirations* (Ryan et al., 2013), is an essential facet of teachers' personal-professional growth (Rubin & Brown, 2019). It has been argued that to advance education, one must move on to a model that maintains teachers' vocational vitality and emphasizes the development of teachers as individuals practicing professionals (Intrator & Kunzman, 2007), instead of just holding teachers accountable for their students' performance in exams (Tucker 2014). Despite these arguments, the literature on effective TPDPs does not intentionally address teachers' PWB (Lindvall & Ryve, 2019). Noticing that teachers' voices are absent from this discourse, this study aims at revealing how teachers perceive their PWB. The findings may redirect and improve PD beyond just enriching teaching quality (i.e., craft, content, and pedagogical knowledge) and contribute to teachers' professional growth by adding teachers' PWB to other desired TPDPs' effectiveness characteristics.

Specifically, addressing the case of expert Science and Mathematics (ESM) teachers, this study aimed to collect empirical evidence about the importance these teachers attribute to well-being in their professional lives. By confronting ESM teachers' perceptions about their well-being needs with how actual practices in effective TPDPs address these needs, we aimed to establish well-being (namely competence, relatedness, autonomy, and aspirations) as an additional factor that should be considered when assessing TPDPs' effectiveness. To this end, we formulated the following research question:

Whether and how are PWB components (competence, relatedness, autonomy, and aspirations) reflected in ESM teachers' perceptions of their personal-professional growth?

THEORETICAL BACKGROUND

Recent research, based on substantial evidence that among contented employees, who feel meaningfully engaged and connected to their profession, retention is higher, especially as career opportunities emerge (Hall, 2002; Ryan et al., 2013), has identified teacher PWB as a critical aspect of teachers' professional needs (Rubin & Brown, 2019). Consequently, our research focuses on teachers' PWB as determining teachers' professional needs.

What is Professional Well-Being (PWB)?

Humanistic theories (Maslow, 1970) assert that aside from physiological needs, safety needs, intimacy needs, and the need for status and esteem, there exist self-actualization needs that include the desire to become the most that one can be professionally. Self-Determination Theory point out that balancing four basic needs, namely competence, relatedness, autonomy, and aspirations, can foster PWB (Ryan et al., 2013). These four components determine the individual's sense of well-being and provide necessary conditions for growth in various areas of life in general and at work in particular (Koivu, 2013).

Competence refers to individuals' ability to influence and manage their internal and external environment, cope efficiently with problems, act in unfamiliar surroundings, and achieve accomplishments (Reeve, 2002; Ryan et al., 2013). *Relatedness* is concerned with the emotional needs of the individual, such as the need to belong, the need for affiliation, the need for acceptance, esteem, and approval (Vignoles et al., 2006). *Autonomy* refers to individuals' feelings and actions that relate to the extent to which they can exhibit independent, wilful, and consensual behaviour driven by a sense of free choice rather than by control and authority. In the context of PWB, *autonomy* is manifested in the person's participation in informed decision-making, organizational justice, and relations with superiors (Koivu, 2013). *Aspirations* refer to the level of motivation to advance oneself, to realize internal needs such as personal-professional growth, health, wealth, fame, image, and power. Consequently, an obvious question would be whether and to what extent do effective TPDPs support personal-professional growth and encompass PWB components?

Do Effective Science and Mathematics TPDPs Encompass PWB?

Attempting to characterize qualitative TPDPs, Darling-Hammond et al. (2017) defined teacher effective professional development as "structured professional learning that results in changes to teacher knowledge and practices and improvements in student learning outcomes". They identified seven characteristics of an effective TPDP: (1) is content focused on what is taught in class, (2) incorporates active learning and directly involves teachers in their instructional practices, (3) supports collaborative learning through various professional interactions, (4) uses models of effective practice that have proved to be efficient, (5) provides coaching and expert support by focusing directly on the personal instructional needs of teachers, (6) offers feedback and reflection towards the development of a more professional vision of instructional practices, and (7) Ensures a continuous and effective learning process.

We observed that these seven TPDP effectiveness characteristics, outlining most of the recent TPDPs, emphasize relevance to teaching content and the teachers' needs for promoting teaching quality (Darling-Hammond et al., 2017), but do not directly address teachers as persons in the profession (cf. Intrator & Kunzman, 2007). Consequently, the effectiveness characteristics mentioned above do not encompass all the characteristics that should be considered when designing effective PD programs. Alongside providing teachers with productive and applicable teaching tools that can enrich their teaching experience, efficient TPDPs should find sensitive ways to address various aspects of teachers' professional needs (Tucker, 2014).

To exemplify whether and to what extent do effective TPDPs address teachers' well-being needs, we focus on the literature describing two science and mathematics related TPDPs that, according to Darling-Hammond et al. (2017), demonstrate all seven effectiveness characteristics: 1. STeLLA (Science Teachers Learning from Lesson Analysis) (Roth et al., 2011; Taylor et al., 2017); 2. A biology TPDP integrating academic literacy and instruction (Greenleaf et al., 2011).

In both examples, searching for the four well-being components revealed that although Darling-Hammond et al. (2017) did not directly refer to teachers' well-being, three out of the four well-being components, namely competence, relatedness, and autonomy, were explicitly identified or could just be inferred from the text describing the seven characteristics of effective TPDPs. However, as a rule, the fourth component, namely aspirations, was missing and could not be inferred throughout our review. One could now ask whether there is a need at all for science and mathematics TPDPs to address teachers' PWB. Perhaps the current situation entirely provides the ESM teachers with their professional needs and makes any extra attention unnecessary? By pursuing this question, this study aims at gathering empirical evidence as to the necessity and importance that ESM teachers attach to their PWB for their personal-professional growth.

METHODOLOGY

Research Design

Intending to explore how PWB needs are reflected in ESM teachers' perceptions of their personal-professional growth, a qualitative case study methodology (Yin, 2014) was employed. A collective instrumental case study design was applied with individual ESM teachers from various secondary schools in Israel (Stake, 1995).

Research Participants

The participants in our study were twenty ESM teachers from six secondary schools in Israel. Participants were purposefully selected for their seniority in vocational education and academic training in science and mathematics teaching, with an experience ranging from 15 to 31 years. The participants provided a diverse group of teachers regarding gender and subjects taught: Mathematics, Physics, Chemistry, Biology and Computer-Science (CS).

Data Collection and Analysis

A semi-structured in-depth interview format involving the use of a carefully considered interview agenda (Willig, 2013) was employed within the present study. The interviews were conducted to reflect the teachers' perceptions regarding their PWB. Participants were interviewed about their professional lives and asked to relate to various aspects of their professional careers. We asked general questions about teachers' professional strengths, weaknesses, opportunities, and threats (SWOT), without explicitly addressing neither PWB nor any other aspects of teachers' professional lives (especially not PD). Therefore, interviewees could relate and elaborate on various issues on their agenda freely and unrestrictedly. The interviews were recorded and transcribed shortly after. The transcripts were used for thematic analysis, conducted vertically (for each interviewee) and horizontally (across the participants) to compare and contrast each interview.

RESULTS

All four well-being components were identified as basic teachers' professional needs in the analysis of the 20 interviews with ESM teachers (see excerpts in Table 1). Results are reported and detailed according to the four well-being components: *competence*, *relatedness*, *autonomy*, and *aspirations*. As expected from ESM teachers, they disclosed a high sense of teaching self-efficacy and competence. However, teachers expressed a professional need to sustain their competence, to learn, to be exposed to current and relevant scientific trends, to become aware of a variety of scientific and technological instructional approaches, and so to expand their craft knowledge. These feelings were sometimes accompanied by feelings of unrelatedness and professional loneliness, as some teachers seemed to lack the company of other expert colleagues in their school. In some science subjects, such as physics and chemistry, professional loneliness was embedded in school situations involving only very few professional teachers. In these cases, the professional loneliness seemed to have its own advantages, as some teachers felt a sense of freedom being responsible for themselves. This finding is in line with the *autonomy* component of PWB, asserting that an individual's wilful behaviour stems from a sense of free choice as opposed to control and authority (Ryan et al., 2013). However, the sense of independence described above is illusive, because it is not related to professional academic freedom, i.e., the freedom to choose and change the contents of teaching, the freedom to decide what is right, appropriate, and up to date to teach. ESM teachers asserted that curricula are dictated by the educational system, limiting the freedom to choose what to do. This is mostly because teachers are held accountable for students' achievements and they have to go through a lot of material. Teachers added that often curricular changes and cutbacks in teaching hours make things even worse. In computer science, teachers feel there is a special need to update the curriculum with new innovations in the field, so that learning does not become obsolete and meets outside daily used developments. In mathematics education, the subject coordinator usually structures a detailed yearly work plan for the mathematics team out of the current curriculum. Although this work-plan is meant to assist teaching, it further reduces academic freedom. In contrast to the three components of well-being, namely *competence*, *relatedness*, and *autonomy*, that were identified in the statements of some of the participants, the component of professional aspirations was indicated by *all* the 20 interviewed ESM teachers. All teachers highlighted the importance they attributed to the fulfilment of their professional *aspirations*, i.e., their internal need for personal-professional growth (Ryff & Keyes, 1995). Concurrently, they noted that ESM teachers are challenged in finding suitable direction for their personal-professional growth. Teachers have their own professional aspirations and blame Israeli educational reforms for putting an unbreakable glass ceiling above teachers' professional horizon. The limited opportunities for realizing teachers' aspirations are reflected by feelings of deep frustration and burnout resulting in a desire to elude confinement and restraint.

Table 1: Excerpts from the interviews

	Exemplifying Excerpts
Competence	*"…the only thing I take pleasure in is closing the door and delivering a good lesson… it is all about my experience and my disciplinary knowledge*" (Rachel, Biology) *"…even with a rag and a chalk I do well…"* (Shifi, Mathematics and CS).
	"…As a high school physics teacher, I feel that I am not expert enough to use a variety of teaching approaches…" (Zigi, Physics). *"…A teacher should be a polymath. I should be a person who knows a lot and learns a lot. I should also be an intellectual… to do so I must see different worlds, draw from them, and make syntheses…"* (Kelly, Chemistry).
Relatedness	*"…The mathematics professional coordinator is very energetic and work-oriented, the team is very professional, but there is very little collaboration and teamwork…"* (Shifi, Mathematics and CS) "…There are many problems here with the professional weaknesses of each of the team members… I really don't have much of a team to work with and I have to figure out how to minimize damages…" (Zigi, Physics) "…As the only physics teacher in my school, I'm busy over my head and I only enter the teachers' lounge maybe twice a month… When I look around, I find that I don't know about 80 percent of the teachers, so, if you ask what happens in school, I'm a loner. I've always been…" (Alex, Physics)
Autonomy	*"…I do what I want. I know I'm appreciated and trusted by one hundred percent. I mean, I get total freedom. They do not check on me and do not criticize me… I don't like to be dictated…"* (Barry, Physics). *"…when you're responsible for preparing students for the matriculation exams, you don't have much freedom…"* (Moti, Mathematics and CS). *"…I think it's very depressing that you're told exactly what to do every day. Even on the level of which exercise to do in class and which to do at home… it really doesn't let you develop any creativity… more freedom should be given to teachers in structuring their own teaching and professionalism…"* (Libi, Mathematics) *"..This [limited freedom] is also true for professional development programs that do not correspond to my experience level, but still try to teach me how handle my class… Instead of empowering me according to my own level, I feel they just waste my time, and I don't learn anything new…" (Libi, Mathematics)*

Aspirations	"*The development horizon for STEM teachers is very limited… this* it is *a critical point, the professional horizon… math and physics teachers are people who mostly don't go into management roles... a different way must be found. Things like doing some research, experiments"* (Miri, Physics)
	"…To do something beyond teaching... do things that promote... trying to do something to improve teaching methods and ways of thinking that are important to me... to go out to colleges, become a lecturer, or even head of department, these options are all very interesting… I would also like to do some research besides teaching the same subjects and chapters over and over again… this could offer an opportunity for me as a teacher… but there are no real opportunities…" (Paula, Biology)
	"…The main threat is that I'll get fed up … I'm getting a little tired of explaining the same thing… a little impatient from doing the same things over and over again… it's hard, going over and over the same lessons several times… Math teachers do not have many opportunities. Taking myself as an example, thanks to ICT, I found myself in a different place. Had I not found an opportunity to integrate ICT in mathematics teaching, I would not have been a teacher any more…" (Igor, Mathematics)

To conclude, although ESM teachers generally expressed an uncompromising need for the presence of the four well-being components in their professional lives, they particularly emphasized the need to realize their professional aspirations. Aspirations was precisely the well-being component that we found as missing from the effectiveness characteristics of TPDPs reviewed by Darling-Hammond et al. (2017).

DISCUSSION AND CONCLUSIONS

This research substantiates PWB as an additional and essential component in considering TPDPs' effectiveness (Rubin & Brown, 2019). Specifically focused on ESM teachers, this study captures teachers' views on the importance they place on their PWB. Regarding ESM teachers' aspirations, we found a controversy. On the one hand, teachers regard their professional development and the realization of their aspirations as most important. However, teachers' aspirations represent the only well-being component that effective TPDPs lack. Research participants substantiated that the oblivion of aspirations may have severe implications on ESM teachers' perception of their personal-professional growth. Consequently, TPDP designers should consider teachers' PWB, including aspirations for personal-professional growth. Observing PD through this extended prism may contribute to teachers perceiving their professional lives as meaningful, inspiring, and rewarding. However, this study has limitations concerning the small number of participants and the particular stage of their careers.

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‘LESS THAN NOTHING’ – A STUDY ON STUDENT’S LEXICAL MEANS FOR NEGATIVE NUMBERS

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This paper presents a qualitative study that analyzes lexical means used by 7th grade students when explaining the concept of negative numbers. For this purpose, 15 texts, that were written by students, were content-analyzed and the collected lexical means were contrasted with the vocabulary from various German mathematics textbooks. In total, three linguistic phenomena could be observed, which allowed conclusions to be drawn about possible mental representations that students might have about the concept of negative numbers. One of these linguistic phenomena – the use of relational interpretations in relation to the reference mark ‘nothing’ or with a missing reference marker – will be presented in this paper.

INTRODUCTION

The introduction of negative numbers is associated with great cognitive challenges for learners: On the one hand, established mental representations of the natural numbers need to be modified and, on the other hand, new sustainable conceptions of the negative numbers need to be built (Malle, 2007; vom Hofe & Hattermann, 2014). The cognitive activities are, on a linguistic level, accompanied by a complex and specific language vocabulary that learners need to draw on. The discourse practice of *explaining the meaning of concepts and operations* represents an important learning medium here because it unfolds so-called ‘epistemic power’. What is meant by this is that when *explaining* the mathematical concept of negative numbers – for example by giving constitutive properties or possible contexts in which they are being used – students also make the contents cognitively accessible to *themselve*s. At the same time, explaining meanings of concepts and operations can also serve as a medium for teachers to get an ‘access’ to students’ understanding of the concept of negative numbers, i.e., students’ individual conceptions (and potentially misconceptions). The study presented in this paper builds on this connection between language and thinking: The aim of it was to collect lexical means that 7th grade students use when explaining the concept of negative numbers and, via these means, to interpret students’ individual mental representations of negative numbers.

THEORETICAL BACKGROUND

The theoretical foundation of this paper is the approach of intertwining lexical and conceptual learning trajectories. It is an integrated, mathematics-specific approach that fosters the successive acquiring of vocabulary for developing conceptual understanding based on the cognitive demands of a given mathematical topic (Pöhler & Prediger, 2015). To investigate how content-related and lexical demands interact, it could be helpful to, following Reich (1989), adopt a functional perspective on language, which Prediger (2020) specifies for mathematics education as follows: Based

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on the content-related learning goals for a mathematical topic, relevant *discourse practices* for the mathematics classroom and the *lexical means* required for participating in discourse practices can be determined. In mathematics classrooms some of the most important discourse practices can be found in *reporting on procedures*, *arguing about the validity of a claim* and *explaining the meaning of concepts and operations* (Pöhler & Prediger, 2015). Various studies show that explaining the meaning of concepts and operations in particular is closely linked to the conceptual understanding needed to grasp a specific mathematical topic (e.g., Erath et al., 2018; Dohle & Prediger, 2020). And because it is the students' conceptual understanding of negative numbers (or rather their individual mental representations thereof) that will be investigated in this paper, the focus here will be on the discourse practice of *explaining the meaning of concepts and operations*. On the one hand, explaining meanings fosters and enhances (mathematical) knowledge and, on the other hand, unfolds 'epistemic power' (Morek et al., 2017). What is meant by this is that when explaining mathematical concepts and patterns, students also make these concepts and patterns cognitively 'accessible' to themselves. In turn, if a student is not able to *explain* the concept of negative numbers, the apparent difficulties may indicate a lack of *conceptual understanding* of negative numbers. This correlation relates to the role of language as a learning medium since language at the same time operates as a *medium of communication* and as a *tool for thinking* (Morek & Heller, 2012; Pimm, 1987).

For explaining the meaning of concepts and operations, specific lexical means are required. Prediger (2017) refers to lexical means as words, sentence components, syntactical constructions and graphical representations that need to be known – need to be mastered – to participate in discourse practices. These necessary lexical means comprise, on the one hand, topic-independent phrases for generally marking a discourse competence (e.g., syntactical structures as final clauses or causal clauses for explaining) and, on the other hand, topic-specific phrases linked to the concrete mathematical topic (e.g., formal technical vocabulary). The aim of the study presented in this paper was to collect lexical means that learners use when verbalizing their conceptions of negative numbers. Fabian (2022) already inventoried lexical means used to introduce negative numbers by drawing on the introductory chapters from four schoolbooks, that are frequently used in Berlin and Brandenburg (Germany) and analyzed them qualitatively. Some of these lexical means, namely the meaning-related vocabulary, are listed in Table 1 (translated from German to English by the author of this paper).

Table 1: Excerpt from the basic meaning-related vocabulary for the introduction of negative numbers

Mental representations (*Grundvorstellungen*) for the introduction of negative numbers	Basic meaning-related vocabulary
States	● ‘numbers with a minus sign as prefix’ ● ‘they are marked with a minus sign as prefix’
State changes	● ‘transition to a lower state (falling)’ ● ‘transition to a higher state (rising)’
Movements along the number line	● ‘move 3 to the right’ ● ‘steps in positive/negative direction’
Relative numbers regarding a fixed reference mark	● ‘to the left/right of the zero’ ● ‘states below/above a fixed normal state’ ● ‘3 units left/right of the zero point’ ● ‘below/above zero’ ● ‘before/behind zero’ ● ‘closer to zero’

Now it is precisely this meaning-related vocabulary used to introduce negative numbers, that is required to ‘perform’ the discourse practice of explaining the meaning of concepts and operations. The lexical means shown in Table 1 construct meanings for the concept of negative numbers and make it possible to communicate about them. In the table, the lexical means are assigned to the crucial mental representations (*Grundvorstellungen*) for the introduction of negative numbers in a way that using the respective lexical means can enhance the particular crucial mental representation. The inventoried vocabulary from the textbooks serves as a basis for comparison within the scope of this study. Thereby, the *formal language demands* (textbooks) can be contrasted with the *lexical resources enabled* by the students in order to answer this paper’s research question. It can be formulated as follows: Which similarities and differences occur between the lexical means that students use when explaining the concept of negative numbers and those lexical means that schoolbooks use to introduce the concept?

METHODICAL APPROACH

To get an access to students’ individual mental representations of the concept of negative numbers, the learners’ vocabulary was analyzed. With the aim of collecting lexical means that students use when explaining the concept of negative numbers, 15 texts written by 7th grade monolingual German students (aged 13-14) were content-analyzed. (In Germany negative numbers are usually introduced in the 7th grade.) The students were requested to explain in writing what they understand by the term ‘negative numbers’ and to give examples and possible contexts in which these numbers

are being used. Thus, the students were externally induced to perform the discourse practice of explaining the meaning of concepts and operations; the lexical means the students used to realise the discourse practice represent the object of the study. Examining learners' written language is insightful in this context, since it requires the mental sorting of thoughts in advance. According to Pimm (1987) thoughts – and along with it underlying conceptions, about which the use of language can provide clues – become more tangible in written form than in oral form. Since the interest of the study lies primarily in the conceptual understanding of the learners, the work assignment deliberately refrained from focusing the explanation on mathematical procedures. 10 texts were written by students attending a grammar school (*Gymnasium*) while the remaining 5 were written by students attending a comprehensive school. The assignment was digitally handwritten by the students on iPads. Subsequently, the resulting products were converted into typewritten texts and analyzed in three steps.

First, the lexical means used by the students were collected in a content-analytical procedure. In a second step, the found lexical means were compared with the inventoried vocabulary for introducing the concept of negative numbers from chapters of four mathematics schoolbooks frequently used in Berlin and Brandenburg (Germany). For this purpose, the findings of a study by Fabian (2022) were used. The comparison with lexical means appearing in the textbooks is legitimate insofar as the analyzed textbooks are devised for the 7th grade. To contrast them with the lexical means used by the students, they almost function as 'representatives' of the formal vocabulary. In this process, lexical means 'colliding' with the textbooks' vocabulary were identified – be it due to differing from it or because they expressed mathematical topic-specific mistakes. Thereby, three linguistic phenomena in learners' language could be identified, one of which will be presented in more detail in this paper. In order to explore the observed phenomena further, the third and last step of analysis consisted in applying the *functional method* (Luhmann, 1995; in the context of mathematics education research: Lensing, 2021) to the found linguistic phenomena. The functional method explores phenomena with problematic character by not assuming it being a *problem* but re-defining it being a *solution*. It then asks which problem might be solved by the phenomenon.

(AN EXCERPT OF THE) RESEARCH RESULTS

The empirically obtained results of the research consist in the collected lexical means that 7th grade students use when explaining the concept of negative numbers. The comparison with the inventoried lexical means used in mathematics textbook chapters introducing the concept made it possible to identify a total of three linguistic phenomena. These phenomena were heterogeneously distributed across the entire sample, regardless of the school type. Due to the limited scope of this paper, only one of these linguistic phenomena will be presented in detail below. Subsequently, I will try to find explanations for the occurrence of the observed phenomena and, using the functional method, try to get access to the underlying conceptions and mental representations of the students.

Relational interpretations

When analyzing the texts, it is noticeable that the students often use lexical means that suggest a relational interpretation of the negative numbers in relation to a certain reference mark. The occurring lexical means differ in the *type* of the named reference marker to which the negative numbers are relationally related. In some explanations, for instance, we find lexical means that express a relational interpretation in relation to the reference mark ‘zero’ (e.g., ‘negative numbers are always below zero’, ‘numbers that come after zero’) – a reference mark that also finds application in the analyzed textbook chapters. For other relational interpretations, no equivalent can be found in the textbooks. Among these is the use of lexical means that express a relational interpretation using ‘nothing’ as a reference mark. I would like to give some examples of this phenomenon from the students’ texts – translated from German into English – which make use of the reference mark ‘nothing’:

- ‘negative numbers are **less than nothing**’ (1)
- ‘so that you can name or specify this **less than nothing**’ (2)
- ‘they indicate the quantities **beyond nothingness**’ (3)

In all three examples, an attempt is made to characterize the concept of negative numbers by giving constitutive property – namely the positional relation to a reference mark. The reference mark itself, which in a relational interpretation of negative numbers is always chosen arbitrarily but fixed (e.g., ‘zero’ for intramathematical interpretations; ‘0 °C’ or ‘sea level’ for further contexts), is denoted by ‘nothing’. In examples (1) and (2) the indefinite pronoun ‘nothing’ is used for this purpose, which refers to a relative reference quantity that is *uncountable*, i.e., whose cardinality cannot be precisely determined. (For clarification: Countable reference quantities, on the other hand, can be referred to e.g., by the indefinite pronouns ‘a few’, ‘several’, or ‘many’). In example (3) ‘nothing’ occurs as a determined nominal phrase (‘nothingness’) and can most likely be interpreted as ‘general indefiniteness’. That the *fixed* reference mark to which the negative numbers are related is represented by an *indefinite quantity* in the above examples, generates a cognitive contradiction.

Some of the lexical means used in the students’ texts express a relational interpretation of the negative numbers without specifying *any* reference marker in the relational determination:

- ‘you need them to specify when something is **less**’ (4)
- ‘numbers stating that something is **not enough**’ (5)

While in the examples (4) and (5) a relation is determined (‘less’, ‘not enough’), the wording leaves room for interpretation, which is not filled in the explanatory context (‘less [than?]’, ‘not enough [compared to?]’). Both attempts of explanation fail to specify the explanatory object ‘negative numbers’ because of these remaining blanks. At this point, no assumption can be made as to whether the reference mark in (4) and (5) is ‘implicitly’ thought of or possibly cannot be specified at all.

DISCUSSION AND CONCLUSION

The observed linguistic phenomenon presented in examples (1) – (5) calls for further investigation since the documented lexical means do not match the vocabulary used in the mathematics textbook chapters for the introduction of negative numbers. This 'deviation' from content-related language norms should, for the following consideration, not be approached as a *problem*. Instead, in line with the functional method, a change of perspective shall allow us to re-define the lexical means (1) – (5) as a *solution* for a problem. Thus, to find explanations for the observed phenomenon, the following questions should be considered: For which problem may the used lexical means provide a viable solution from a student's point of view? Why would a student use these lexical means and not others – for instance, those offered by the textbooks – to solve the problem at hand? In other words: what function do the lexical means fulfill that the use of alternative means *cannot* or *not optimally* fulfill? One possible answer to these questions might be: That students use relational lexical means referring to the reference mark 'nothing' or to no reference mark at all indicates, that they hold on to the *cardinal aspect* as a conceptual basis. Thereby, students solve the 'problem of objectification', a problem that will now be discussed in more detail to conclude this paper.

The 'problem of objectification'

With the introduction of negative numbers, learners are confronted with the fact that previously established mental representations about the concept of numbers turn out not to be sustainable any longer and that well-known strategies lose their validity. In contrast to the natural numbers "non-positive integers are not representable concretely as manipulable objects" (Davidson, 1987, p. 431). To still be able to operate with the negative numbers, learners turn to the conceptions and mental representations of the natural numbers they have internalized so far, transferring them to the negative number range in order to solve the problem of objectification (on the difficulties of objectifying the concept of negative numbers in more detail: Malle, 1988). By this, they try to compensate for the "lack of a tangible, concrete, or realistic interpretation for negative numbers" (Pierson Bishop et al., 2010, p. 698). De Cruz (2006) describes the concept of negative numbers as "counterintuitive because they violate ontological expectations" (p. 317). To overcome this irritation, learners must understand that negative numbers are *theoretical* mathematical objects – an understanding for which they have to let go of the concept of cardinality.

The collected students' lexical means indicate that the idea of cardinality is a concept that still seems to be strongly anchored in the students' conceptual understanding. In contrast to the textbooks' meaning-related vocabulary, that enhances the mental representation of *relative numbers regarding a fixed reference mark* referring to the concept of negative numbers as being a *theoretical* one (e.g., 'left of the zero point', 'below zero'), the students' language is rather of an *empirical* character. Some lexical

means even include an explicit verbalization of cardinality ('they indicate the **quantities** beyond nothingness'). The use of these lexical means may serve us as an indication that the learners are making a 'cognitive compromise' when dealing with negative numbers: While explaining the concept, they take up the relational interpretation of negative numbers, however, via the lexical mean 'nothing' they simultaneously refer to a reference *quantity* and hence activate an idea that is only viable when dealing with natural numbers (counting numbers, comparing the cardinality of quantities). Furthermore, the absence of the reference mark in examples (4) and (5) can be read as an indication that the concept of negative numbers has not yet been (sufficiently) objectified. In the learners' conceptual understanding the negative numbers then appear as 'old numbers in special use'.

Due to the limited scope of this paper, the presentation of the research results will be concluded at this point and further observed linguistic phenomena along with the derived conclusions regarding students' conceptions of negative numbers cannot be touched upon. Despite its brevity, the analysis presented in this paper exemplified how to access to learners' presumable mental representations via their use of lexical means. Conversely, making students aware of relevant topic-specific vocabulary may help them overcome mental challenges and build viable conceptions. Even if the number of investigated students is still too small to generalize, the study complies with the "call for taking into account the epistemic function of language in the processes of knowledge constitution as a medium of thinking" (Erath et al., 2018, p. 162) in the context of mathematics education research. And although the study is restricted to the specific mathematical topic of negative numbers, it demonstrates a way to use language as a diagnostic tool for mathematical understanding processes. Especially when constructing meaning for mathematical concepts and communicating about those concepts requires a particularly extensive and complex vocabulary – as it is the case with negative numbers (Fabian, 2022) – "the discursive practice of explaining should be a more explicit learning goal in mathematics classrooms" (Erath et al., 2018, p. 177).

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DEVELOPING ACCUMULATIVE THINKING

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Accumulation is central to integration, and learning the integral concept based on the idea of accumulation has been found to be beneficial. In this study a learning activity on accumulation was designed using the context of a pool being filled, with the purpose of giving students opportunities to develop ways of thinking that support later studies of integration. Three pairs of 11th grade students carried out the activity. Our analysis shows which ways of thinking they developed, and how.

INTRODUCTION

Many students have difficulties with the integral concept (e.g., Bressoud, 2009; Orton, 1983; Rösken & Rolka, 2007). Researchers stress the significance of understanding integration through accumulation (e.g., Jones, 2015; Kouropatov, 2016; Thompson & Silverman, 2008). The idea of accumulation allows to naturally combine the concepts of the definite and indefinite integrals, as well as to lead to the Fundamental Theorem of Calculus (Kouropatov & Dreyfus, 2013). Constructing the integral concept based on the idea of accumulation has been shown to be beneficial (Carlson, et al., 2003; Kouropatov, 2016). Here we investigate an activity for students, who have not yet learned integration. The activity offers opportunities to develop *Accumulative Thinking*, that is ways of thinking useful for studying integration via an accumulation approach. Our research questions are: 1. What is the structure of Accumulative Thinking? 2. How do students construct the elements of Accumulative Thinking?

THEORETICAL BACKGROUND

The accumulation approach to integration is closely connected to Riemann's definition of the integral as the limit of a sum of products; if the variable of integration is time, the products are of the form time interval × rate of change (below: RoC) in that interval and give the 'bits' that accumulate. Hence, accumulation can be seen as derived from RoC. "When something changes, something accumulates. When something accumulates, it accumulates at some rate" (Thompson & Silverman, 2008, p. 49). And so, the amount added by a bit derives from the RoC in this bit.

According to Gravemeijer & Doorman (1999), context problems are problems that the student experiences as real. An everyday context enables students to act and reason in a meaningful way.

Abstraction in Context (AiC) is a theoretical framework proposed by Hershkowitz et al. (2001) for studying learners' construction of new (to them) abstract mathematical knowledge. The knowledge intended by the designer or teacher to be constructed is analysed a priori into knowledge elements that include concepts, procedures, and strategies. A posteriori, learners' processes of construction of these knowledge

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 323-330). PME 46.

elements is analysed by means of three observable epistemic actions: Recognizing (R) – the learner identifies a previous construct as relevant to the task at hand; Building-With (B) – the learner uses a recognized construct for achieving a local goal, and Constructing (C) – a new construct emerges for the learner by recognizing and building-with previous constructs. As R-actions are nested in B-actions and R- and B-actions are nested in C-actions, Hershkowitz et al. (2001) proposed the name "dynamically nested RBC-model".

In this research, we define *Accumulative Thinking* as (i) the knowledge elements related to the accumulation of bits (described in detail below); (ii) the dynamic nature of the accumulation process; and (iii) the ability to apply (i) and (ii).

METHODOLOGY

Our research tool is a learning activity designed with the purpose of developing Accumulative Thinking before learning integration. The activity consists of context problems set in the everyday context of a pool being filled with water. The activity presents three types of RoC in three consecutive parts: constant RoC, RoC constant in segments, and linear-decreasing RoC. The activity introduces students to Accumulative Thinking by considering bits of water that accumulate in the pool, and the effect the RoC function has on the corresponding accumulation function. The pool context was chosen as there is evidence that it helps students grasp RoC in an intuitive way (Elias et al., 2023), enabling them to act and reason meaningfully.

The a-priori analysis resulted in a list of 16 knowledge elements intended to be constructed by the students; together, these elements constitute Accumulative Thinking. Each of them has been given an operative definition, allowing the researcher to assess whether a student has constructed the knowledge element. In addition, we identified 15 preliminary knowledge elements, assumed to have been constructed earlier. For lack of space, we only present an overview of the relevant knowledge elements, followed by an example of a knowledge element with its operative definition.

In the case of a constant RoC, bits have a multiplicative nature since the amount added in a time interval is the product of the time interval by the rate at which the quantity accumulates (knowledge element M_nr: Multiplicativity - numeric representation). The ratio of the amounts of water flowing in two different time intervals equals the ratio of the lengths of the time intervals (preliminary knowledge element P6). When the RoC is constant, bits with the same time duration have the same amounts added (EB: Equal Bits); hence the accumulation function graph is a straight line whose slope equals the numeric value of the RoC (AFCSL: Accumulation Function of a Constant RoC is a Straight-Line function). Summing up consecutive bits within a given time interval results in the amount accumulated in that entire time interval (S: Summing consecutive bits). Conversely, the amount that accumulates in a time-interval can be split into tiny bits by splitting the time interval into small sub-intervals (TB_r: Tiny Bit reduction). In the case of a linear and decreasing RoC, the multiplicative connection rate × time does not give the exact amount added. The idea of instantaneous RoC is not

relevant for students at this stage of learning. Therefore, it becomes imperative to use the area under the graph as representing the amount added (A_dl: Area – decreasing and linear RoC). The bits that accumulate are graphically represented by the trapezoids, formed by the graph and the time axis within a given time interval. As time varies, the amounts added (the bits) with the same time duration are getting smaller (DB: Decreasing Bits). Hence, as the RoC decreases, the accumulation function in this case is concave downward (AFDL: Accumulation Function of a Decreasing Linear RoC).

We present one example of an operative definition, namely the one for S (Summing consecutive bits):

- We will say that students have constructed S if they express that the quantity of water added in a time interval equals the sum of the quantities added in its consecutive time sub-intervals; in case such sub-intervals are not given, this implies that they first split the given time interval into consecutive sub-intervals.

The participants were three pairs of grade 11 students studying mathematics at the advanced level: Roy and Don, Ana and Zoe, and Tim and Nic; the pairs of students were asked to carry out the learning activity in the presence of a researcher (the first author). The researcher moderately acted as interviewer, presenting questions to the students only in order to clarify their utterances and their mathematical meanings behind the course of action they took. At the time the interviews were conducted the students had learned the topic of differentiation but not yet the topic of integration. The pairs of students were audio-recorded during the learning activity. The interviews were transcribed and analysed using the RBC-model.

FINDINGS AND DISCUSSION

In this section, we present empirical evidence for 5 cases of students' construction processes that contribute to Accumulative Thinking. These cases are presented in chronological order.

Case 1

The first part of the activity deals with a pool that is being filled with water at the constant rate of 30 litres per minute. The students are asked:

Danny knows the rate at which a pool is being filled. He used the amount of water that accumulated until minute 1.4 to find the amount of water accumulated until minute 1.43. How do you think he did it? Write down the calculation.

At this point, the students have already calculated the amount of water that was accumulated up to a various given points in time, including 1.4. Roy says:

Roy: …he did 1.4 times 30 plus 0.03 times 30.

Roy and Don don't use the accumulated amount up to minute 1.4, which they have already found, but rather calculate this amount again by multiplying the time duration 1.4 minutes by the rate, which expresses the use of a previously constructed knowledge element (M_nr: Multiplicativity - numeric representation); to this amount, they add the

amount obtained by multiplying the time duration 0.03 by the rate. Splitting the time into 2 consecutive sub-intervals of duration of 1.4 and 0.03 and summing the amounts added in each to get the total amount accumulated expresses the construction of both TB_r and S.

At this point in time, Ana and Zoe haven't constructed M_nr yet. In order to find the amount added in the time interval of duration 0.03, they use quantity proportion (P6) (see Figure 1), thus bypassing the need for M_nr. Then they sum of the amount accumulated up to minute 1.4, which they have already calculated in the previous question, and the amount added during the 0.03 minutes, thus expressing the construction of S and TB_r but by using P6 rather than M_nr.

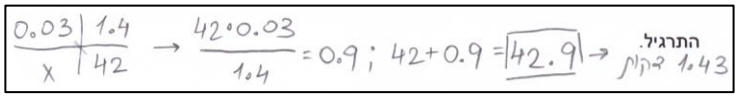

Figure 1. Ana's solution using quantities proportion (P6).

In summary, these two pairs constructed knowledge elements that contribute to the first component of Accumulative Thinking, namely, knowledge elements related to the accumulation of bits.

Case 2

The third part of the activity deals with a pool being filled with water at a linear and decreasing rate (Figure 2).

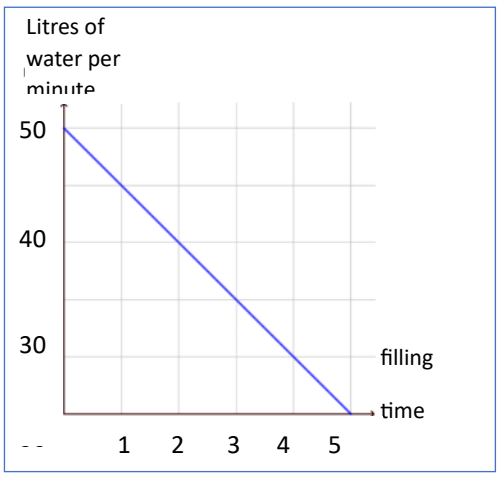

Figure 2. The rate of flow in the third part of the activity.

Students were asked: *What can be said about the amounts that are being added to the pool from the beginning until minute 5? How can this be seen from the given graph?*

Roy and Don answer that the amounts added are getting smaller, and that this can be seen from the graph since the rate is decreasing. When asked again, Don said:

Don: *By the area of the graph at any moment. In the first minute, the area is the largest and, in the 2nd minute the area starts to get smaller.*

In summary, Don expresses that the area represents the amount added in each bit (A_dl) and that the amounts are getting smaller since the areas are getting smaller (DB), which expresses an aspect of the second component of Accumulative Thinking, the dynamic nature of the process.

Case 3

The students were asked to draw the accumulation function for the given rate of flow graph, which is linear and decreasing (Figure 2). At this stage the students had already drawn the accumulation function for a RoC which is constant in segments (in the second part of the activity) by splitting the time interval into bits of time duration that correspond with the time interval of each segment, then calculating the amount added by each bit and summing up the bits to get the accumulated amounts. When the students discussed how to draw the accumulation function of a linear and decreasing RoC that is given graphically (Figure 2), Zoe and Don (the partners of Ana and Roy, respectively) referred to the rate in each bit of 1 minute duration as if the rate was constant and suggested to multiply the rate at the left border of the bit by the duration of 1 minute. However, Ana and Roy corrected their partners and said that this way was not applicable since the rate is not constant. The third pair (Tim and Nic) found the average rate of the rates at the beginning and at the end of each bit, calculating the amount in each bit by multiplying the average rate with time duration of 1 minute each.

Here, several students consider the accumulation process in a chunky manner, meaning they look at the bits that accumulate as chunks according to the time segmentation.

Case 4

Students also used another way to graph the accumulation function of a linear and decreasing RoC (Figure 2) by using an unexpected construct.

Ana	So you need to find the equation of the graph that describes the filling rate of the pool. If I know that this graph describes the slope at that point in the graph of the accumulation function, then I can determine that it is its derivative and then go from the filling rate graph equation, which is the derivative of the original function that is the accumulation, investigate it and draw it accordingly.

Here Ana expresses a new construct, namely that the given RoC is the derivative of the accumulation function. Ana previously constructed AFCSL (Accumulation Function of a Constant RoC is a Straight-Line function), and more specifically, that the slope of the accumulation function equals the numeric value of the RoC. Here Ana applies AFCSL to a linear and decreasing RoC. To actually draw the accumulation function, Ana first finds the algebraic expression $50 - 10x$ of the given RoC; next she finds an algebraic representation of a function whose derivative is $50 - 10x$; and then she draws the accumulation function by plotting 3 points and connecting them. Roy and Don have also expressed the same unexpected new construct; however, they did not follow up this approach.

In summary, Ana applied her strong connection between slope and derivative in the present situation, which is new to her: a RoC that is not piecewise constant. In doing so, she in fact overgeneralizes AFCSL but does succeed to obtain the correct answer. Ana solved the problem in a manner that was unexpected.

Case 5

After Ana and Zoe drew the accumulation function as described in case 4, the interviewer asked if they could do it in a different way. They used the area under the graph as representing the amount added in case of a linear and decreasing RoC. They first split the time into 1-minute intervals (S); to get the amount added in each bit, they calculated the area of the corresponding trapezoid (A_dl); to get the accumulated amounts, they then summed the amounts (S), marked the points in the empty coordinate system provided, and connected them (Figure 3).

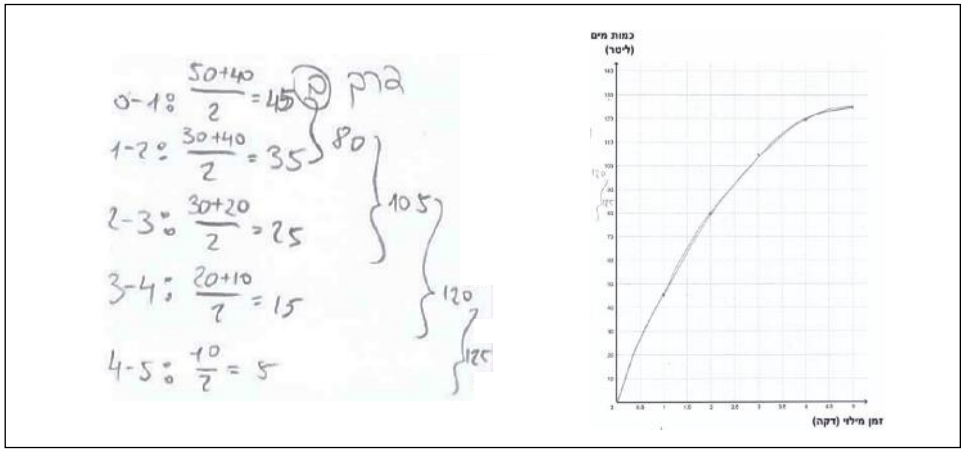

Figure 3. Ana's calculation and drawing.

After drawing the accumulation function for a linear and decreasing RoC, the students were asked to explain why the graph they drew is concave. Ana and Zoe answered that the amounts that are added (the bits) are gradually decreasing.

In summary, the students expressed knowledge about the bits, which were previously constructed (first component of Accumulative Thinking), described the accumulation process in a dynamic manner (second component of Accumulative Thinking) and applied these components in order to draw the graph of the accumulation function (third component of Accumulative Thinking).

CONCLUDING REMARKS

The activity in this research was designed as an introduction to integration via accumulation prior to studying integration. The everyday context of the pool allowed the students to act and reason in a meaningful way that helped them to develop strategies that are context dependant and to generalize from them (Gravemeijer and Doorman, 1999).

In the Findings section, we showed that the activity offered opportunities to the students for developing Accumulative Thinking, which possibly supports later studies

of integration. The students were able to use previously constructed knowledge elements to construct new knowledge elements. In the case of a constant RoC and a RoC constant in segments (not reported in this article), the students could multiply the constant rate and the time duration to get the accumulated amount. In the case of a linear and decreasing RoC, this way is not applicable; therefore, the students needed to overcome their chunky way of solving and construct the new knowledge element A_dl. Thus, students constructed knowledge about the elements of accumulation including bits, why these bits are changing the way they do, and how they accumulate by being summed up (cases 1, 4, 5). Grasping the accumulation process as dynamic (cases 2, 5) can help the student later when considering the bits that are accumulated as infinitesimal, resulting in a smooth graph of the accumulation function. According to Thompson and Silverman (2008), understanding covariation is necessary for understanding accumulation. The dynamic nature of the process of accumulation expressed by the students goes in that direction, in the sense that as time changes, the RoC changes and affects the accumulating bits, and the accumulation function changes accordingly.

Our research suggests but does not show that the approach used here may help students to apply their knowledge in other contexts as well, thus bypassing the difficulty of students, reported by researchers (e.g., Jones, 2015), in applying mathematical knowledge in different contexts. Further research, on a larger scale, should be conducted in order to investigate the ability of students to apply in other contexts Accumulative Thinking acquired in a pool context, also in other contexts. Similarly, the effect of an introductory learning activity such as the one used here on students learning processes about integration is an important issue for further research.

ACKNOWLEDGEMENT

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WHAT DO STUDENTS LEARN ABOUT THE DISCIPLINE OF MATHEMATICS IN UPPER-SECONDARY CLASSES?

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To acquire knowledge about the discipline of mathematics is an important goal of mathematics classes in upper-secondary schools. However, we do not know to what extent students achieve this goal. Therefore, in this study, we measure students' knowledge about the discipline of mathematics with a validated test in a paper-and-pencil-format and analyse which other students' characteristics and learning opportunities relate to this knowledge facet. In total, 116 upper-secondary school students participated in the study. The results show that knowledge about the discipline of mathematics depends on the kind of mathematics courses and extracurricular activities taken by students. These findings could contribute to a better understanding of how to foster students' knowledge about the discipline of mathematics.

INTRODUCTION

Knowledge about the discipline of mathematics (what we call meta-scientific knowledge) is an important component of mathematics education in upper-secondary schools. However, we have only limited insights into what upper-secondary school graduates know about the discipline of mathematics. One starting point for answering this question is to analyse meta-scientific knowledge of university students. For example, Ziegler (2012) complaints that beginning university students lack knowledge about what mathematics is and what it means to practice mathematics.

Research concerning meta-scientific knowledge about mathematics is multifaceted. The existing literature reports (1) on epistemological/meta-mathematical knowledge of teachers (e.g., Hoffmann & Even, 2021; Rott, 2021; Zazkis & Leikin, 2010) or (2) on epistemological beliefs of students and/or teachers (e.g., Schoenfeld, 1989; Xie & Cai, 2021) but there are only few studies that focus on students' meta-scientific knowledge about mathematics.

To contribute to this gap, the present paper proposes a framework for the theoretical conceptualisation of meta-scientific knowledge about mathematics. Our conceptualisation consists of students' knowledge about concepts to build up a mathematical theory (e.g., definitions, theorems, proofs) and practices (e.g., defining, proving) focussing mathematics as a deductive structure discipline. To learn more about this knowledge facet, we analyse which students' characteristics relate to this knowledge facet and analyse how learning opportunities are connected to this knowledge facet.

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THEORETICAL BACKGROUND

A framework for meta-scientific knowledge about mathematics

As mentioned above, meta-scientific knowledge about the discipline of mathematics is one goal of mathematics courses in upper-secondary schools because it is an important learning prerequisite for mathematics-related study programs as well as participating as a person of age in a scientific-oriented society. We understand meta-scientific knowledge about mathematics as knowledge "that refer[s] to cross-cutting themes that may appear within any mathematical content" (Zazkis & Leikin, 2010, p. 274). Therefore, it contains (epistemological) knowledge about concepts and practices, which is not bound to a specific content (e.g., algebra, geometry). For example, meta-scientific knowledge about concepts in mathematics contains knowledge about concepts like definitions, theorems, and proofs, and meta-scientific knowledge about mathematical practices contains knowledge about mathematical ways of thinking and working like defining objects or proving theorems. This differentiation between concepts (or products of mathematics) and practices (or processes in mathematics) is not new but known as the "dual nature of mathematical constructs" (Sfard, 1991, p. 5). When conceptualizing and operationalizing meta-scientific knowledge of mathematics, we consider both of these perspectives (see figure 1).

Meta-scientific knowledge about the discipline of mathematics	
Product-oriented perspective	**Process-oriented perspective**
• Knowledge about concepts and structures of the discipline mathematics (e.g., definition-theorem-proof structure) • Knowledge about principles of generating or validating mathematical findings	• Developing useful definitions and theorems • Understanding or constructing mathematical proofs • Evaluating definitions or proofs

Figure 1: Framework for students' meta-scientific knowledge about mathematics.

From a product-oriented perspective, knowledge about concepts, structures, and principles of mathematics can be seen as one important aspect of meta-scientific knowledge. It is essential to get an insight into the mathematical culture consisting of a specific, formal language and logical rules (Leviatan, 2008). That this knowledge is important for STEM (science, technology, engineering, mathematics) students is underlined by the study of Deeken et al. (2020). They investigated what university STEM educators expect from beginning STEM students. From their perspective, it is important that students have knowledge about features of mathematical definitions and proofs and their role for generating mathematical evidence.

Whereas the product-oriented perspective consists of knowledge about concepts of the discipline of mathematics, the process-oriented perspective focusses more on knowledge about the processes of how mathematics is done. For example, processes that play a key role in the discipline of mathematics are defining objects, generalizing, deducing, and proving (Leviatan, 2008).

Relationship between meta-scientific knowledge, learning opportunities, and students' characteristics

As previously mentioned, the acquisition of meta-scientific knowledge about mathematics is a primary goal of upper-secondary schools (school grades 11 and 12). In Germany, upper-secondary school students have to choose between studying mathematics on advanced (A) or basis (B) level. Both courses aim at a profound understanding of central concepts of Analysis, Analytical Geometry, and Stochastic – as the A-level course provides five hours per week and the B-level course three hours, the A-level course deepens the contents, e.g., the B-level course only deals with the product rule of derivation whereas the A-level course also deals with the chain rule of derivation. Therefore, meta-scientific knowledge is a more prominent learning aim in A-level courses than in B-level courses. Another opportunity to get to know mathematics as a deductive discipline are extracurricular activities like taking part in elective courses in mathematics or in mathematics competitions (e.g., International Mathematical Olympiad).

From the literature, we know that mathematical knowledge relates to cognitive variables like prior achievement in mathematics (e.g., Rach & Ufer, 2020) as well as to affective variables like interest and self-concept in mathematics (e.g., Marsh et al., 2005). Whereas the reported correlations between mathematical knowledge and other cognitive variables are on a rather high level, the reported results concerning the relation between mathematical knowledge and affective variables seems to differ quite a lot. Since there is only little research literature on meta-scientific knowledge about mathematics, it is yet unclear if the results between knowledge and other students' characteristics hold for meta-scientific knowledge as well.

RESEARCH QUESTIONS

The research questions for this study are as follows:

(RQ1) To what extent do students' characteristics predict meta-scientific knowledge about mathematics?

(RQ2) To what extent do learning opportunities predict meta-scientific knowledge about mathematics?

METHODS

Sample

We conducted a study with 116 students (53.4% female, mean age: 17.3 years, grade 12) from two different upper-secondary schools in Germany. Out of all 116 students, 61 students (52.6%) took mathematics at A-level and 55 students (47.4%) took mathematics at B-level. The sample was collected via convenience sampling, and participation was voluntary and anonymous.

Data collection and data analysis

To measure meta-scientific knowledge about mathematics, we used a self-developed test instrument consisting of 26 multiple-choice items. Figure 2 shows an example item which focusses on the concept of conjecture. This item was one of the easiest items with a solving rate of 76.7%.

A conjecture is...	□ ...a valid statement which does not need to be proved.
	□ ...a statement which is not proven to be true nor false.
	□ ...a statement in which something is presented as a fact.
	□ ...the assumption about the validity of an axiom.

Note: For this presentation, the original item was translated into English.

Figure 2: Example item “conjecture”.

The given instrument was validated in prior studies with beginning university students (Fesser & Rach, 2022). We assume that we can apply this instrument in this context because beginning university students and upper-secondary school students of grade 12 can be regarded as quite similar. Whereas the content validity was checked by a group discussion with university mathematicians, psychometric measures (e.g., item difficulties, reliability) were investigated in a quantitative study with over 300 university students (Fesser & Rach, 2022). To investigate the reliability of the scale in this study, we checked how the test items are correlated to the sum score of the knowledge scale. We found that one item was negatively correlated with the knowledge scale resulting in deleting the item. Therefore, the scale measuring meta-scientific knowledge about mathematics consists of 25 items. Having a look on Cronbach’s $\alpha = .50$, one can assume that meta-scientific knowledge is a formative construct rather than a reflective one. According to Stadler et al. (2021), an instrument measuring a formative construct often has a small Cronbach’s α.

Besides the knowledge test, students were also asked to fill out a questionnaire concerning other students’ cognitive and affective characteristics. The last grade in mathematics (ranging from 0 (worst) to 15 Points (best)) was collected as an indicator for academic achievement in mathematics. To analyse the validity of the instrument, we also collected students’ last grade in German. Because German as a school subject seems to differ quite a lot from mathematics, we expect that the knowledge scale and the last grade in German do not relate with each other. As proving is an important practice for the deductive discipline mathematics, we decided to collect data about interest and self-concept concerning proving. We used four items to measure interest in proving tasks (Ufer et al., 2017, Cronbach’s $\alpha = .80$) and three items to measure self-concept in proving tasks (Rach et al., 2017, Cronbach’s $\alpha = .77$). Students rated all items on a 4-point-likert scale (from 1 = disagree to 4 = agree). The descriptive analysis does not give hints for floor and ceiling effects, and the correlations between the analysed students’ characteristics are small to moderate (see table 1).

Table 1: Descriptive statistics and correlations between students' characteristics.

Variable	*M*	*SD*	Math	German	Interest	SC	CL
Last grade in mathematics (1 item)	9.23	3.61					
Last grade in German (1 item)	8.41	2.74	.36**				
Interest (4 items)	2.19	0.68	.44**	.08			
Self-concept (SC, 3 items)	2.60	0.64	.57**	.04	.56**		
Course level (CL, 1 item)	0.53	0.50	.24**	-.02	.36**	.45**	
Extracurricular activities (4 items)	0.72	0.70	.36**	-.12	.40**	.36**	.21*

Note: ** $p < .01$, * $p < .05$. Last grade in mathematics and German ranging from 0 (worst) to 15 points (best), interest and self-concept assessed with items on a likert-scale from 1 (disagree) to 4 (agree), course level 0 = B-level, 1 = A-level, extracurricular activities from 0 to 4.

The course level was measured with one item and the extracurricular activities with four items concerning participation in various extracurricular activities (e.g., mathematics competitions). We used the sum score that indicates the number of activities in which the students had participated. We also used the sum score for the knowledge scale resulting from the 25 test items and arithmetic means for the interest and self-concept scales. To answer the research questions, we used a multiple linear regression. All computations were performed by using *R* (version 4.2.0).

RESULTS

Descriptive statistics for the knowledge scale

Table 2 shows the descriptive statistics for the scale knowledge about the discipline of mathematics. Both the skewness and the kurtosis of the scale are close to zero suggesting that the distribution is normally distributed.

Table 2: Descriptive statistics for the knowledge scale.

Scale	*M*	*SD*	Skewness	Kurtosis
Meta-scientific knowledge	11.51	3.11	0.33	-0.37

Note: Possible values ranged from 0 to 25.

The mean values are similar to students at the beginning of their study program. Noticeable is that the students have high solving rates when answering items concerning concepts and structures of the discipline mathematics as well as items concerning principles of generating or validating mathematical findings. However, there is one exception, which is the concept "axiom". Students seem to have difficulties with items including the concept "axiom" resulting in rather low solving rates. In addition, students have difficulties evaluating definitions of mathematical objects.

For answering both research questions, we computed a multi linear regression (method blockwise). Model 1 consists of the achievement measures (in mathematics and

German), in model 2, we add the affective variables and finally model 3 additionally considers the learning opportunities. The results of the analyses can be seen in table 3.

Table 3: Multiple linear regression with knowledge as dependent variable.

Variable	model 1		model 2		model 3	
	β	*SE*	β	*SE*	β	*SE*
Last grade in mathematics	.33**	.08	.08	.10	.01	.09
Last grade in German	.03	.11	.09	.10	.16	.10
Interest in proving tasks			.30**	.46	.20*	.45
Self-concept in proving tasks			.16	.55	.05	.54
Course level[a]					.26**	.55
Extracurricular activities					.25**	.41
R^2	.11**		.23**		.33**	

Note: ** $p < .01$, * $p < .05$. [a] 0 = B-level, 1 = A-level.

(RQ1) Relationship between meta-scientific knowledge and other characteristics

Model 1 shows that the last grade in mathematics is a significant predictor of meta-scientific knowledge whereas the last grade in German does not seem to be relevant. The inclusion of the affective variables (while the last grades are still controlled) increases the explained variance slightly ($R^2 = .23$). In addition, it reveals a significant relation between interest in proving tasks and meta-scientific knowledge (β = .30).

(RQ2) Relationship between meta-scientific knowledge and learning opportunities

To check whether learning opportunities can explain differences in meta-scientific knowledge between students, we add the two variables in the regression analysis (model 3). The results show that the course level (β = .26) as well as extracurricular activities (β = .25) are significant predictors of the meta-scientific knowledge (while the individual characteristics are controlled). In total, 33% of the variance in meta-scientific knowledge is explained by model 3.

DISCUSSION AND OUTLOOK

Meta-scientific knowledge is an important aim of mathematics classes but there are only few approaches to conceptualize and operationalize this construct. The shortly presented theoretical framework and the test are one possibility to analyse which knowledge students have and which learning opportunities relate to this knowledge.

With the first research question, we investigated the relationship between meta-scientific knowledge and other students' characteristics. First, model 1 indicates that meta-scientific knowledge is a mathematics specific construct as it mainly relates to the last grade in mathematics. Model 2 suggested that only interest in proving tasks seems to have a significant impact on students' meta-scientific knowledge about

mathematics but contrary to prior studies (Marsh et al., 2005; Rach & Ufer, 2020), there are no hints that achievement in mathematics and self-concept predict meta-scientific knowledge besides interest. A possible explanation is that the achievement in mathematics (operationalized by the last grade in mathematics) does not measure how well students are acquainted with mathematics as a deductive discipline. That may be true because mathematics classes often focusses on the applicative side of mathematics whereas the deductive and proving side of the discipline seems to be neglected (see Sporn et al., 2022). Following this consideration, it would be interesting to investigate how both sides of the discipline can be made equivalently visible and accessible in mathematics classrooms. The second research question focusses on the relationship between learning opportunities and meta-scientific knowledge. In line with our expectation, we found that extracurricular activities as well as the course level are significant predictors of meta-scientific knowledge about mathematics besides individual characteristics. This result indicates that students in A-level courses get a better insight in the deductive nature of mathematics. It could be interesting to investigate what design principles of the learning opportunities lead to higher meta-scientific knowledge. This highlights the need for a design-based research approach (1) to identify important design principles for fostering students' meta-scientific knowledge about mathematics and (2) to implement those design principles.

Despite the interesting results, this study has its limitations. One limiting factor is the sample regarding its size and its collection. Another limiting aspect might be the used instrument for measuring meta-scientific knowledge about mathematics. As the instrument is validated for beginning university students (Fesser & Rach, 2022), it can be questioned whether the instrument can be applied to upper-secondary school students. Thus, future studies should investigate whether all items work for different samples (upper-secondary school vs. beginning university students).

Our findings suggest that the conceptual framework is valuable to measure meta-scientific knowledge about mathematics. However, this study can only be regarded as a first step towards research concerning meta-scientific knowledge about mathematics. For future research, it is important to expand this framework. In this regard, ideas like integrating (1) mathematics as an applicative discipline in the framework as well as (2) the ability to reflect on mathematics on a meta-scientific level can be put forward.

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AN INTERVIEW STUDY ON THE REVERSAL ERROR WITH PRIMARY SCHOOL STUDENTS

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The reversal error is a common and resilient error when constructing symbolic representations. It has been widely studied for older students and adults. Various explanations have been proposed and tested with inconsistent results. Since no relevant experience is yet available for the primary school age, an exploratory interview study on the reversal error was conducted with 28 fourth graders. It demonstrated that under certain conditions the reversal error occurred only very rarely, which is probably due to the intensive examination of examples.

INTRODUCTION

Algebra is not only an essential part of mathematics, it is also of great importance for mathematics education, especially in higher grades. This is justified by the importance of algebraic competencies in many professions or in postsecondary education (National Council of Teachers of Mathematics, 2000). Therefore, the difficulties that many students have with algebraic content, as demonstrated in numerous studies (e.g., Carraher & Schliemann, 2007; Kieran, 2007) are serious. Related to this, algebra also seen as a "gatekeeper", the mastery of algebraic requirements decides to a considerable extent on the overall success in mathematics education at secondary schools (Cai & Knuth, 2005).

Mathematical modelling is seen as one of the most powerful applications of algebra (González-Calero et al., 2015). However, the use of symbolic representations is also considered particularly demanding, especially for younger students. For example, Kieran (2004, p. 40) describes "a focus on both representing and solving a problem rather than on merely solving it" as an important and challenging transition to be mastered on the way into algebra. On this basis, it seems interesting to explore the extent to which students at the end of primary school are already able to symbolically represent relationships between and properties of numbers and quantities. This is the context of an interview study described in this article that focuses on the so-called reversal error, a difficulty that is frequently observed in higher grades and even in adulthood, but for which no experience is yet available for primary school age. Detailed analyses of primary school students' ways of thinking and proceeding relevant tasks will be used to find out whether and how the reversal error also arises at this school age, and if so, how the students succeed in avoiding it. The results of this and similar studies could, among other things, provide valuable information on the fostering of early algebraic abilities in mathematics teaching.

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THE REVERSAL ERROR – SOME THEORETICAL CONSIDERATIONS

For at least 40 years, the so-called reversal error has been studied again and again with older students and adults, for example using the now probably classic student-professor task:

> Write an equation for the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (Clement et al., 1981, p. 288; Rosnick & Clement, 1980, p. 4)

More generally, the task is to describe the relationship between two unknown quantities symbolically. If a reverse equation is written down for this purpose, e.g. $6S = P$, this is called a reversal error. In numerous studies in various countries with professionals and students from secondary school up to university, different but always comparatively high prevalence rates were found for the reversal error. Furthermore, it has been shown that it cannot be easily overcome. The high prevalence in different populations around the world and its resilience indicates deep cognitive roots of the reversal error (Jankvist & Niss, 2021).

Various hypotheses have been developed to explain the reversal error. Some studies suggested a *"word order matching approach"*, in which the equation is created by replacing the key words with algebraic symbols in the order in which they appear in the problem text. However, such a "syntactic translation" (MacGregor & Stacey, 1993) may not be the only cause, as the reversal error also occurs with other word orders or pictorial inputs (Jankvist & Niss, 2021).

In the *"static comparison approach"*, one does not proceed purely syntactically, but rather a situation model is developed. Two groups of objects ("students" and "professors") are related or compared to each other. The comparison of two groups, more generally of two quantities, is also called an "imagistic approach" (Goldin & Kaput, 1996), possibly encouraged by the easy imaginability of the quantities involved. The comparison result is noted as a "pseudo-algebraic equation" (Kaput & Sims-Knight, 1983), $6S = P$ then means something like "6 students correspond to one professor". Accordingly, the equals sign does not stand for numerical equivalence, S and P do not stand for numbers, but are identifiers or abbreviated names or units. The number 6 is noted next to the identifier belonging to the larger group and is used rather attributively. The strong influence of natural language is seen as an important reason for this approach (Kaput, 1987).

In contrast, a correct equation can be obtained with the *"operational approach"*. In this approach, described groups or quantities are no longer compared, but rather an operation is constructed or hypothetically executed that leads to a numerical equivalence. The result is written down as an equation. $6P = S$ then means something like "If you were to multiply the number of professors by six, it would be equal to the number of students." Among other things, this verbalisation indicates the particular demanding nature of the operational approach.

In their recent review of previous studies, Jankvist and Niss (2021, p. 201) state, "whilst some of the linguistic, syntactic and semantic explanations of the sources of the reversal error offered by researchers seems to be valid for some students in some contexts, they do not suffice for all students in all contexts and situations." While there are inconsistent results for older students and adults, there is so far no experience for the primary school age. This research gap will be worked on with a first interview study. This will investigate the extent to which fourth graders succeed in formulating equations with two unknowns for corresponding situations, how they proceed in doing so and what role the reversal error plays thereby.

DESIGN AND IMPLEMENTATION OF AN INTERVIEW STUDY

For primary school, there is hardly any experience with the occurrence of reversal error. Therefore, an exploratory study was planned in which fourth graders were asked to carry out several relevant tasks within the framework of semi-standardised clinical interviews (Hunting, 1997). The aim was to explore, to what extent students of this school age succeed in symbolically representing a textually described relation between two unknown quantities by means of an equation and how they approach corresponding tasks.

In the introductory phase of the interview, students should interpret given equations or "computations" such as $3 + 5 = 8$, $4 \cdot x = 8$ or $4 + a = b$ and formulate similar equations themselves. Subsequently, the students were asked to write down a "suitable computation" for various short texts presented on worksheets, whereby four texts described relations between two unknown numbers:

T1: The number a is equal to the triple of the number b.
T2: The number x is 12 less than the number y.
T3: There are six times as many chickens as geese on a farm.
T4: There are more girls than boys in a school class. The number of girls is 4 more than the number of boys.

In the texts T1 to T4 various aspects were varied which, according to earlier studies and explanations in the previous section, could have an influence on the difficulty and on the occurrence of the reversal error (cf. MacGregor & Stacey, 1993):

- An additive (T2, T4) or a multiplicative relationship is described (T1, T3).
- Numbers are used for this description (T2, T4) or it is less explicitly (T1, T3) formulated. In addition to the concrete arithmetic relationship, in T4 is explicitly stated that there are more girls than boys in the class.
- The (unknown) quantities are named in conformal (T1) or non-conformal order (T2, T3, T4) for the arithmetic operation suggested by the text.
- An arithmetic relation is described context-free (T1, T2) or as a simple factual situation (T3, T4) described.
- Related to this, unknowns and knowns can be numbers of the same kind (T1, T2). Otherwise, the unknowns are numbers of objects named in the text and

the knowns describe relations between or properties of these objects (T3, T4).

- The fact that in the second case symbolic representations are sought for the numbers (and not, for instance, for the objects themselves), makes T4 explicit, T3 does not.
- Both factual situations are easy to visualize. In T3 however, an assignment of objects (e.g. 1 goose ↔ 6 chickens) seems more obvious than in T4.

Based on this, students might find it particularly difficult to formulate a suitable equation for T3, which corresponds to the classic student-professor task.

Twenty-eight fourth graders with good to very good school performance in mathematics from ten classes in four schools took part in the study. The interviews were conducted and videotaped at the end of the school year in rooms of the respective school. Transcripts and worksheets were analyzed using qualitative content analysis based on deductive-inductive obtained categories (Mayring, 2014).

Since symbolic representation is considered very demanding and there is hardly any experience so far for primary school age, this first study focused on mathematically high-performing students (in terms of school grades and teachers' assessments). This must of course be considered in interpreting the results.

IMPORTANT RESULTS

The following table shows the types of solution and their frequency for the 28 participating students and the texts T1 to T4.

Type of solution	T1	T2	T3	T4
No processing of the text	1		1	1
No equation with two unknowns	8	8	10	10
False equation with two unknowns, without reverse error	5	9	4	3
False equation with reverse error	10	4	1	
Correct equation	4	7	12	14

Table 1: Results of the 28 participating fourth graders

Equations with unknowns and the algebraic representation of situations with unknown quantities or of relationships between them by symbolic terms play practically no role in primary school mathematics teaching in Germany. Nevertheless, depending on the text, up to half of the students succeed in writing down a correct equation with two unknowns. Even if these are rather high-achieving students, these results indicate that algebraic representations of this kind are already accessible in principle at this school age.

In addition, it is noticeable that the 26 correct equations for T3 and T4 are opposed by

only one equation with a reversal error. The student who makes this mistake writes down a correct equation for T2 and T4; the equation for T1 is wrong, but without reversal error. Accordingly, it is not a systematic error by the student. In total, only 4 of the 28 students make the reversal error twice; for all the other students it occurs at most once.
Another remarkable finding is the comparatively frequent occurrence of the reversal error for T1, despite the fact, that quantities, equal sign und less explicitly the operation are mentioned in conformal order in the text. Three of the four students with reversal errors for T1 and T2 are able to write down correct equations for T3 and T4. The fourth student at least notes a suitable equation for T4.
Overall, the fourth graders are obviously much more successful in correctly symbolically representing relationships between context-related unknown numbers (T3, T4).

Based on this quantitative data, it seems particularly interesting to explore *how* students find a symbolic representation of the relations described in the texts and what could be *reasons* for the frequent occurrence of the reversal error for T1.

Based on the qualitative analyses of the interview data, the various students' approaches that go beyond individual cases and finally lead to an equation with two unknowns are outlined below. These procedures of the students differ in particular in the use of and the handling of example solutions. Some of the cognitive approaches described above are embedded in them.

Student S2 looks at the worksheet for about 8 seconds, after which the following dialogue emerges (transcripts shortened):

1	S2:	Mathematically speaking, you couldn't calculate that because you don't know how many geese there are. …
2	I:	Can you write down a calculation like that again? *I refers to the previous worksheet*. Something like that?
3	S2:	*S2 thinks for about 5 seconds* So could I write now, geese times six equals that many chickens, or is that not possible?
6	I:	Well, yes, write it down.
7:	S2:	I'll write a G for geese now.
		Within about 15 seconds, S2 writes down an equation (see Fig. 1, left). She first writes down G. Then she hesitates for several seconds before writing down the rest from left to right.
8:	I:	Explain how you arrive at that.
9:	S2:	Well, because... there are six times as many chickens as geese. So now I have G times six. So, six geese, no, geese times six equals six, uh, chickens.

Without any recognizable prior involvement with examples, the student writes down a correct equation after thinking about it for several seconds. This procedure can be characterized as *algebraization based on the description of the situation*. It occurred very rarely in the study group.

Apparently, S2 quickly realizes that the number of geese and chickens cannot be calculated. She then formulates the idea of an equation, but may be unsure whether this corresponds to the interviewer's question formulated in 2. "Geese times 6" seems to refer to the animals, but it continues with "equals that many chickens", thus referring to a numerical equivalence and the number of chickens. Statement 9 indicates the "operative approach", although it seems that the interpretation of 6 as a factor is not yet entirely certain.

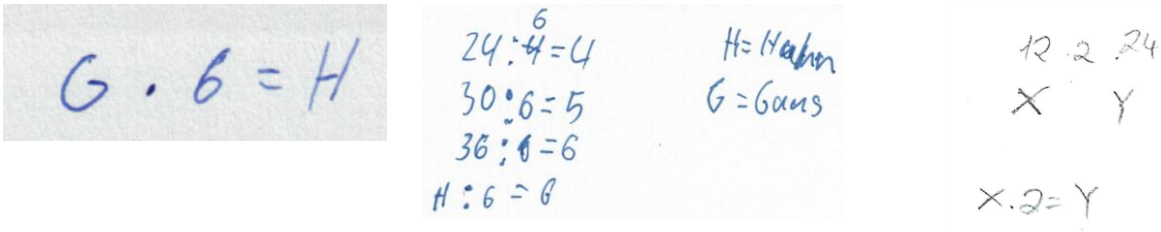

Fig. 1: Results by S2 (left), S26 (middle) and S9 (right)

S26 proceeds differently. After 8 seconds she notes the example $24:6=4$.

1	I:	What have you been thinking about?
2	S26:	Ehm that there are twenty-four chickens and then six geese. So, twenty-four would be six times four and that's why, yes. …
3	I:	And, ehm, is that the only possibility?
4	S26:	No, you could also, for example *In the next few seconds, S26 writes down two more examples.* So, you could write down the whole 6 times table.
5	I:	... could you also write it down somehow with letters?
6	S26:	*Within 12 seconds, S26 writes down an equation (see Fig. 1, middle).* H and G should still be written here. H is equal to chicken ... and G is equal to goose. And then you write here H divided by six is equal to G.

Examples obviously play a major role in the approach of S26; by dealing with them she succeeds in the operative approach. Overall, an *algebraization based on example solutions* takes place.

In 4, S26 calculates and notes down several examples. She is aware that these are possibilities and that there are many more. The calculation procedure, which is used several times, is then formulated for the unknown numbers G and H and noted in the form of the equation. However, the letters are introduced as abbreviations.

However, example solutions are not always used successfully, as the result of S9 on T2 shows (see Fig. 1, right). S9 first determines with (12,24) a suitable example. When asked, an equation is written down that is correct for the concrete example, but is

neither linked to the given text nor to the determination of the example solution. Based on this, the procedure of S9 could be described as *formalization of (random) arithmetic properties of example solutions*.

Based on the analysis of the students' working processes on T1, the use of number examples seems also to be a cause for the frequent occurrence of the reversal error for this text. The following example of S9 who makes the reversal error only on T1, demonstrates this:

1	S9:	I could now take, for example, the *(3sec pause)* 2 and then the 6. ... So, *a* is 2 and *b* is the 6. ...
2	I:	Is that the only option?
3	S9:	*(The student shakes his head)* I could also say three times three is nine or six times three or one times three.

Because the unknown *a* comes first in T1, many students as S9 first choose a number as an example for *a* (often the number 3) and then formalize the triple by 3·a. The sequence of unknown numbers, equals sign and operation in T1, which seems advantageous for a symbolic representation, thus proves to be unfavorable for example-based processing.

DISCUSSION AND OUTLOOK

Conducting an interview study has proven to be successful. On the one hand, the interviewer was able to use introductory tasks to gradually guide the students to work on the challenging texts T1 to T4. On the other hand, not only results but also working processes of the students can be accessed in this study design.

The presented results of this first exploratory study suggest that a stronger involvement of examples and their determination could facilitate the construction of a symbolic representation. Within the approach we called *algebraization based on example solutions*, many students succeed in the operational processing and the formulation of a correct equation. However, a step-by-step construction of the example solution along the descriptive text also involves risks, as the results for T1 suggest. Based on this, it could be worthwhile to explicitly address in this context the search for and use of example solutions in mathematics lessons.

In general, the results indicate that symbolic representations of relations between two unknown quantities are already attainable at a younger school age and therefore further, more detailed studies on this topic could be interesting.

In the study presented, T1 was chosen as the first text because of the solution-conforming order of its elements; furthermore, the challenging text T3 should be processed later in the interview. This could be changed or varied in a further study to test order effects. Whether representations of additive relations are easier to find than those of multiplicative relations cannot be decided on the basis of the available data.

This and, for example, the possible significance of the contextual reference, which is indicated in the presented results, could be investigated in larger studies.

In the fifth grade in Germany, equations and variables already play a greater role, so a corresponding interview study with fifth graders will be conducted this school year.

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GIOELE'S ATTEMPT TO INCORPORATE THE "SOLVE IT" RITUAL IN HIS MEANINGFUL DISCOURSE ON EQUATIONS

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Over the past decades research has highlighted many difficulties in the domain of algebra teaching and learning, especially for low achieving students. With the advent of new technologies, findings have highlighted ways of using technological tools to overcome some of such difficulties. This study, part of a greater research project, explores the case of Gioele, a low achieving student who participated to our intervention and developed meaningful narratives when using the Dynamic Interactive Mediators (DIMs) in the context of equations and inequalities. In particular, through a commognitive lens, we analyze how he attempts to incorporate his previously learned "solve it" ritual, into his DIM-based discourse on equations and their solutions.

INTRODUCTION

Over the past decades research has highlighted many difficulties in the domain of school algebra teaching and learning, that include giving meaning to algebraic symbols, unknown and variables; viewing the equal sign not necessarily as a signal to compute an answer but also as a relational symbol of equivalence, overcoming the transition to the letter-symbolic form of equations for which students need to interpret algebraic expressions as mathematical objects as well as computational processes, and accept unclosed expressions such as $2x+5$ as valid responses, without thinking that they should do something with them (e.g., Kieran, 2020; Arcavi et al., 2017). For low achieving students, algebra can be particularly daunting (Xin et al., 2022).

Research findings suggest that with appropriately designed tasks, digital means turn out to be particularly helpful to students with a history of low achievement in mathematics or "with special educational needs" (e.g., Baccaglini-Frank, 2021; Palmas et al., 2020). This study is part of a greater funded research project that, through a design-based methodology, is conducting case studies of second year high school students with a history of low achievement in mathematics. These students, volunteering from different Italian high schools, participate to an intervention conducted by researchers during which they engage in a set of newly designed digital activities in the context of algebra. In this study we explore the case of Gioele, a low achieving student who during the proposed activities developed meaningful narratives in the context of equations and inequalities, and that the researcher tried to push to incorporate his previously learned "solve it" ritual (for equations), into his new meaningful discourse on equations and their solutions.

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THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

We will take a commognitive perspective to mathematics teaching-learning (Sfard, 2008) and refer to the digital tools used by participants in their discourse as *Dynamic Interactive Mediators* (DIMs) (Baccaglini-Frank, 2021; Antonini et al., 2020). Previous studies have shown how DIMs can support secondary students' learning, if appropriately integrated into the teaching-learning of high school algebra, by offering "protagonists" for the development of meaningful narratives, specifically in the contexts of equations and inequalities (Baccaglini-Frank, 2021) and of functions and variables (Lisarelli, 2022; Antonini et al., 2020). In such studies, the *discursive*, or *commognitive*, approach allowed to capture sense-making processes through a fine-grained analysis of students' discourses, with particular attention to their *routines*.

A routine is composed of a *task* – as understood by a person in a given task situation (any setting in which a person considers herself bound to do something), is the set of all the characteristics of the precedent events (all that happened in a precedent task situation) that she considers as requiring replication – and a *procedure* – i.e., all the features of what was done in a previous task situation that the person believes should be replicated (Baccaglini-Frank, 2021; Lavie et al., 2019). Students' participation in mathematical discourse can be *ritualistic* if it consists mainly in the implementation of memorized routines for the sake of themselves, with the performer never attending to any product of this performance that could later be used independently of the procedure that produced it; or *explorative*, if it is aimed at constructing a *meaningful* narrative about *abstract objects*, in order to *make sense* of a particular task situation (Sfard, 2008). In discourse an abstract object is expressed through different *realizations* (e.g., an algebraic expression can be read as an indeterminate number or as a function). On the other hand, *unrealized symbols* are *concrete objects* that appear in the discourse alone and can only be manipulated in well-defined ways (Baccaglini-Frank, 2021). In the analyses we use these elements to capture the student's sense-making.

In this study we designed 3 digital artifacts (dynagraphs, two-pan balances with expressions, two-pan balances with weights – see Figure 1) that, for an expert, can be considered realizations of equations and inequalities, as well as their solutions. Used in students' discourse, these DIMs are designed to foster transitions between what an expert sees as different realizations of the same mathematical object; hence, they should foster students' construction of the mathematical objects equation, inequality and solution or set of solutions. We focus on the case of Gioele (pseudonym), who during the initial interview spoke of "solving" (an equation) and performed a(n incorrect) ritual involving symbolic manipulation of letters and numbers, and whom the researcher proposing the intervention (second author) tried to push to incorporate such ritual in the meaningful discourse he had developed in the context of the DIMs. To guide Gioele' s case study, we ask the following questions:

- RQ1: What are the characteristics of Gioele's discourse in the initial interview?

- RQ2: What are the characteristics of Gioele's DIM-based discourse by the end of the teaching intervention?
- RQ3: How does Gioele's discourse come to include the "solving an equation" routine recalled by the interviewer during the intervention?

METHODOLOGY

Gioele volunteered as a participant for the study, recognizing himself as a low achieving (in mathematics) student in 10th grade (15 years old). He was enrolled at a technical-professional high school and agreed to come to our research center 5 times in 2 months. During the first meeting he was interviewed by one of the researchers for 45 minutes. During the activity sessions he worked with two other researchers. He worked individually with the researchers in a quiet room with non-invasive recording devices. During the interview, Gioele had at his disposal a tablet where he could write, as with paper and pencil, and a computer displaying the questions as the interviewer asked them. During the activity sessions, Gioele had at his disposal one or two tablets showing the different digital artifacts and another tablet for writing. Since Gioele uses such digital artifacts as mediators of his discourse we will, for brevity, refer to them hereon as DIMs (Figure 1). DIM_A is a dynagraph with three arrows that realizes an independent and two dependent variables. The tick at the end of arrow *x* on the number line realizes the value "*x*" appearing in the expressions above and can be directly manipulated by dragging. The two arrows above realize the two expressions depending on *x*, and they move indirectly. DIM_B consists of a two-pan balance with expressions that "weigh" as much as the value of the draggable *x*-tick ("*x*=2" in Figure 1a,b). DIM_C has no symbolic inscriptions and it consists of a two-pan balance with weights (some known and some unknown, the triangles in Figure 1c) together with a dynagraph through which values can be assigned to the unknown weights.

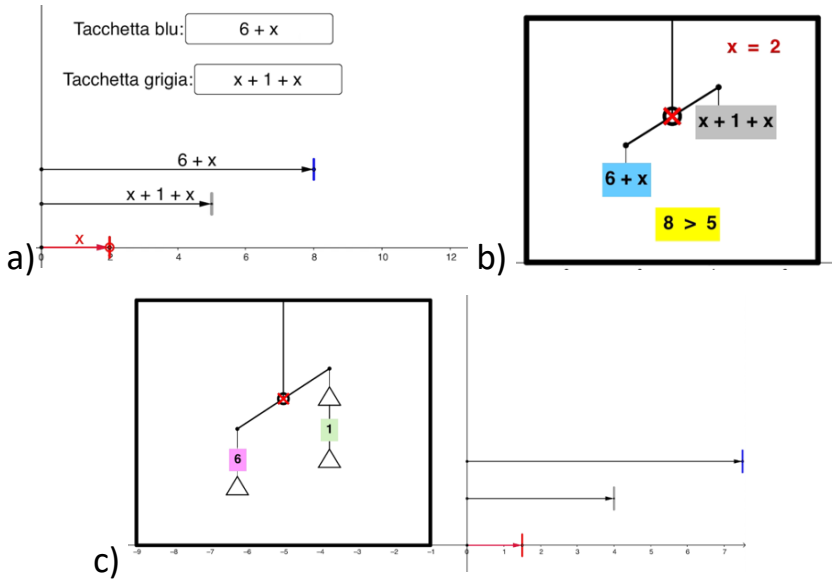

Figure 1: realizations of the inequality $6 + x > x + 1 + x$ with
a) dynagraphs, b) two-pan balance with expressions (for the value $x = 2$), and
c) two-pan balance with weights and associated dynagraphs (for the value $x = 1.5$).

In the activity we analyze Gioele uses a DIM that we call $DIM_{(A,B)}$ because it embeds DIM_A and DIM_B and later he also uses DIM_C . We note that the tick at the end of each arrow is not labeled, because we were interested in words students would use to speak of what for an expert is a "value of the unknown", "variable" or, in some positions, "solution". The symbols "x=2" and "8>5" in DIM_B change as the value of x changes and they change color (and the inequality changes to an equality) when the two-pan balance is balanced off.

During the activity sessions Gioele always had access to at least one DIM and he was asked to make predictions about when one expression would be greater, less than or equal to another, and then to manipulate the DIMs and explain his observations confirming or disproving his conjectures. The researcher sometimes would ask additional questions on-the-fly to gather more information about the student's reasoning. In the case of Gioele such questions often asked for more predictions related to changes of x's position, after an initial prediction and manipulation. The recordings of Gioele and the researcher were merged with the recordings of the screens of the tablets. They were then anonymized and transcribed by members of the research team. The analyses make use of the theoretical constructs introduced to reach answers to our RQS.

ANALYSIS OF SELECTED EXCERPTS AND ANSWERS TO THE RQS

Excerpt 1 - interview. This excerpt exemplifies Gioele's approach to solving equations during the interview before the activity sessions.

7 Int: What comes to mind if you see this, what would you do? [Shows the equation 13–A=13+11].

8 Stud: First I would find the A

[…]

12 Stud: It would occur to me to do… first group all the numbers together and then afterwards do like… do in parentheses… 13 minus 13 plus 11. In parentheses, A. [He writes on the tablet the expression (13–3+11)A]

13 Int: Ok.

14 Stud: And do everything, so 13 minus 13, zero, plus 11, 11 and that the result of that would be 11A. [He writes 11A]

15 Int: Ok, so if I ask do that have any solutions?

16 Stud: In my opinion yes, 11A.

17 Int: So, what can they be? 11A. Now the question is still the same, just change the writing. Three plus A equals A plus 3. Does it have any solutions? If any, what are they?

18 Stud: Yes, I mean, you have to group the numbers and on the other side group the letters, so... 3... 3... 3 plus 3, is equal to A plus A. And the result would be this. Although, being 3 plus 3, you could do like… Add 3 plus 3, that is 6, and A plus A raise it to the second power. [He writes 3+3=A+A. Then he writes in the line below $6=A^2$]

19 Int: So, would the solution be $6=A^2$?

20 Stud: Yes

Gioele's discourse here seems to be purely *ritualistic*, focused on performing (meaningless) procedures for their own sake. In this excerpt he performs his "find the A" ritual (in other excerpts he says "solve it" so we refer to this as his "solve it" ritual) twice. So, in front of the equation, without being asked to solve anything, he recognizes a familiar task situation, to which he responds to satisfy the interviewer. He uses verbs and impersonal forms like "do" (in [12] where it recurs 3 times, [14] where it seems to be synonym of "add up", [18] where it seems a synonym of "raise to the second power"), "group" (in [12], and "you have to" in [18]), "add" (in [18]). The objects of the discourse are mainly "numbers" and "letters" (in [12], [18]) and "A" ([8]) but there are no references to other realizations of these (concrete) objects, which therefore remain *unrealized symbols*. The only signifier that Gioele connects with different realizations is "result" (in [14], [18]), realized by 11A (in [16]) and $6=A^2$ (in [19]). Gioele performs manipulations solely to please the interviewer, without any apparent aim of creating meaningful (to him) stories. Moreover, Gioele's symbolic manipulation shows that he has no expectation about the outcome: he talks about "grouping" ([12], [18]) in the procedures he applies for both tasks, even if the two outcomes he obtained, for an expert, refer to two different mathematical objects, a literal expression and an equation. The "result" for Gioele thus seems to be whatever he finds at the end of his "solve it" ritual.

To answer RQ1, Gioele seems to recognize a familiar task concerning solving an equation; his discourse is characterized by ritualist manipulations of *unrealized symbols*; there are no references to other realizations of these objects and thus no transitions between realizations. In general, there is no evidence of sense-making concerning the "solution of an equation" in Gioele's discurse in the initial interview.

Excerpt 2 - last activity session. During this session Gioele's discourse always involves DIMs and the construction of meaningful narratives. In this excerpt, the interviewer asks Gioele to use a file with $DIM_{(A,B)}$ and to set the two-pan balance with the expressions $5+x$ on the left and $2x+1$ on the right (the default value of x is 2).

39	Int:	Ok. So now before you [...] imagine you put *x* on 4 – don't do that, wait a minute – and try to tell me everything that's going to change in this figure when you do that
40	Stud:	So, *x* plus 5 will change, which will be 4 plus 5 so 9, and 2 times 1... 2*x*... 2 times 4 so 8 plus 1
41	Int:	Yes
42	Stud:	9 and 9
43	Int:	Uh, ok
44	Stud:	So putting it [he refers to the arrow "*x*"] on 4 it should be in balance
[…]		
47	Int:	Great, what about these [pointing to the arrows in DIM_A] What do you think will change about these things here? If anything changes
48	Stud:	They will stretch, because *x* will stretch further by 5

In excerpt 2, Gioele's discourse includes DIM-based narratives, such as "putting it on 4 it should be in balance" in [44] referring to balance in DIM_B and "They will stretch" in [45] referring to arrows in DIM_A. Gioele also expresses narratives about objects, such as "*x*" or expressions involving *x*, such as "*x* will stretch" in [48] and "*x* plus 5 which will be 4 plus 5" in [40].

Excerpt 3 - last activity session. In one of the next tasks, the interviewer asks Gioele to reconstruct in a $DIM_{(A,B)}$ the two-pan balance shown in a DIM_C (realizing the inequality $7+2+x>3x$). Gioele solves the task rapidly and explains:

> Looking here [pointing to DIM_C], since there are blanks I have to add them. Like on $7+2+x$, I mean, the blank one [pointing to the white triangle under the "weights" 7 and 2], and instead here there is 3, 3 blanks, and therefore 3 unknowns, $3x$.

Now Gioele has linked the object "*x*" to at least 3 different realizations, namely the symbol "*x*", the "blank", and the term "unknown".

In response to RQ2, excerpts 2 and 3 show that Gioele's DIM-based discourse is characterized by objects (perhaps in the DIMs themselves) and by meaningful narratives around these objects. Indeed, these narratives make sense with respect to the new task situation Gioele has learned to make sense of. There are also several realizations of objects such as "*x*" (the arrow, the "blank" triangle and the "unknown"). These are indicators of an ongoing sense-making process related to mathematical objects "unknown" and "solution of an equation", albeit still in DIM-based contexts.

Excerpt 4 - last activity session. Since Gioele had come to set up what looked like equations using the expressions in DIM_B, the researcher decides to intervene, reminding him of the correct ritual for solving an equation through symbolic manipulation, so that he could "make better predictions of what x might work". Then she asks him to set up a $DIM_{(A,B)}$ with the expressions $6+x$ and $x+1+x$, and to try to predict what value of x will balance it off. Gioele correctly sets up the balance and makes a prediction that the solution will be 4, then without checking it on the DIMs he writes down $x=4$, the equation $6+x=x+1+x$, and carries out the solution procedure, obtaining $5=x$.

129	Int.:	So, you had imagined $x=4$, here you obtained $x=5$, ok? So, who do you think is right, let's say, this prediction of yours or your calculations [...]?
130	Stud.:	calculations
131	Int.:	calculations
132	Stud.:	Because the calculations, I mean… here they are saying a different thing from what I am saying
[…]		
143	Int:	Okay...but what do you think $x=5$ meant?
144	Stud:	That *x*… I don't know.
[…]		

146 Int: So, you predicted x=4 [and] actually still it didn't work, and instead calculations gave you x equals 5 [she points to the inscription "5=x"]

147 Stud: Ah, maybe it could be 5 then!

148 Int: Mm, how come?

149 Stud: Because here I did all this and it tells me that it is so

Gioele opens a new text file, starts writing 6+x again, then writes x=5 above, and completes the equality with 6+x=x+1+x, then solves it exactly as before.

155 Int: Excuse me, how come you wrote up here at the top your initial idea?

156 Stud: x equals 5?

157 Int: uhm

158 Stud: I mean, x=5 I wrote it here to remind me here that x equals 5

159 Int: Ah, and so like you wanted to "think it first" before you got it from here

160 Stud: Yeah, it's like… I mean, that is like, how can I say, test everything out

Now the object "x", previously the protagonist of *meaningful* DIM-based narratives, seems to return to being an *unrealized symbol*, like "A" in the initial interview – see in [144] when Gioele states that he does not know what x=5 means. The inscription "5=x" for Gioele now has an unclear relationship with his initial narrative "x=4", a conjecture about the balance position. Although he seems to be trying to make sense of it: in [132] he states "calculations […] are saying a different thing from what I am saying" However, Gioele does not conclude that the value of the unknown should be 5. "5=x" seems to just falsify his initial conjecture, as a test that failed (in [160]). Moreover, Gioele spontaneously constructs a narrative about "x" ("it could be 5 then!" in [147]) and performs the same procedure again ("all this [...] tells me that it is so" in [149]), constructing a new (for him) narrative for the hypothesis "x=5".

In response to RQ3, there is no evidence in Gioele's discourse indicating that the symbol "x" involved in the procedure recalled by the interviewer was recognized as a different realization of the object he was talking about in the interaction with DIMs. For Gioele, the inscriptions "x=4" and "x=5" seem to be meaningful narratives about x only in his DIM-discourse, but not when he is performing the "solve it" ritual. He seems to relate the symbolic equations he writes (invited by the researcher) to the situations realized by the two-pan balances, but not the output of the "solve it" ritual applied to such equations. The only element that may indicate a possible seed of a meaningful link that eventually could be established might be when he says "all this [...] tells me that it is so" in reference to the symbolic manipulation ending with "5=x".

CONCLUSION

Gioele's case highlights some still-problematic issues to be considered when teaching with DIMs such as the ones proposed in this study. Gioele's discourse on equations starts off as purely *ritualistic*, with no evidence that he has constructed the meaning of abstract objects such as "equation" or "solution of an equation". With the DIM-based activities he constructs meaningful narratives about "x" and the DIMs themselves, as

protagonists of the discourse, confirming previous findings (Baccaglini-Frank, 2021). However, this brief session of activities with DIMs was not sufficient for the researcher's attempt to meaningfully incorporate his previously learned "solve it" ritual into his DIM-based discourse on equations and their solutions to succeed. This finding points to obstacles to the eventually necessary transition from DIM-based discourse to formal mathematical discourse, especially when students have constructed extremely strong rituals detached from mathematical objects, and from any meaning. In Gioele's case, the researcher's attempt actually led him to interpret his revisited "solve it" ritual as a "test": the attempt to re-incorporate the symbolic manipulation into meaningful discourse led to a distortion from its endorsed use.

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ADDITIVE WORD PROBLEMS IN GERMAN 1ST AND 2ND GRADE TEXTBOOKS

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Primary school students' performance varies strongly between different types of additive word problems. One explanation is an imbalanced representation of different word problem types in textbooks, as it was observed in countries other than Germany. To investigate if this imbalance also occurs in German textbooks, all N = 652 additive word problems in German 1st and 2nd grade textbooks from five different series were analyzed. The imbalance regarding the semantic structure and the unknown set corresponds to other analyses. However, a balanced ratio of consistent and inconsistent word problems was unexpected. In addition, the imbalanced representation of additive and subtractive wording is reported and discussed.

Solving additive word problems is difficult for many primary school students. Additive word problems are mathematical problems embedded in a verbally described situation that can be solved with an arithmetic operation (addition or subtraction) and do not contain irrelevant information (Verschaffel & De Corte, 1997). Depending on student characteristics (e.g., language skills or basic arithmetic skills), there is interindividual variation in students' skills to solve additive word problems (Daroczy et al., 2015). On the other side, also task features influence the solution process. For example, many studies have found that features of the situation structure influence a problem's difficulty (e.g., Gabler & Ufer, 2021).

If students are exposed to different types of word problems to a different degree, this may explain the relation between students' different solution rates and the features. How often these types occur in textbooks is a frequently discussed indicator in this context (e.g., Vicente et al., 2022). However, previous textbook analyses have mostly focused on a restricted range of task features, and evidence from Germany is missing. This paper aims to fill this gap. First, we describe features of the situation structure that characterize different word problem types and report empirical findings how they influence the problems' difficulty. Based on prior studies, we argue that textbooks may influence students' learning. We report on studies that have examined the frequency of additive word problems in textbooks in other countries. Finally, we present our analysis of German textbooks to gain information on the frequency of different additive word problems in German textbooks.

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PRIOR RESEARCH AND FRAMING

Features of the situation structure and the difficulty of word-problem types

One feature of the situation structure in word problems is the semantic structure of the depicted problem. Additive word problems can describe situations referring to the increase or decrease of a set (change), the combination of two sets (*combine*), or the comparison of two sets (*compare*). In *equalize* problems, a less common type, two sets are introduced: "Susi has 3 marbles. Max has 8 marbles." The question is then, how one set can be changed, so that its cardinality is equivalent to the second set (e.g., adding 5 marbles to Susi's set): "How many marbles does Susi have to buy to have as many marbles as Max has?". Combine or change problems have been found to be easier than compare problems (Gabler & Ufer, 2021; Stern, 1993). One reason may be that for compare problems, numbers do not only describe concrete sets, but also the difference between the two concrete sets (Stern, 1993). Current models on number concept acquisition allocate compare structures in later phases of arithmetic development than combine structures, for instance ("relational number concept"; Fritz et al., 2018). Since equalize problems are addressed infrequently in studies, only few empirical data are available. While a prior study reported very high solution rates for this semantic structure for first graders (96 %, Stern, 1994), a more recent study reports rather moderate solution rates for second graders (71 %, Gabler & Ufer, 2021).

The difficulty of an additive word problem also varies depending on the *unknown set*: For combine problems, either one of the two parts can be unknown, or the whole set, which is formed by the parts. For change problems, either the start set, change set, or result set can be unknown. For compare problems, either one of the two quantities (reference set or compare set), or the difference between these two quantities can be unknown (difference set). For equalize problems, either one of the two quantities (start set or result set), or the change set can be unknown. In particular, compare problems with an unknown reference set, but also change problems with an unknown start set or combine problems with an unknown part are known as difficult word problem types (Gabler & Ufer, 2021; Stern, 1993).

A third feature of the situation structure is the additive/subtractive wording (*a/s wording*), which emphasizes different perspectives on a mathematical situation. For dynamic semantic structures, such as change and equalize problems, this can be expressed by action verbs referring to an increase (additive wording, e.g., "to get") or a decrease of a set (subtractive wording, e.g., "to give away"). For compare problems, a/s wording is expressed by relational terms (e.g., additive, "more than"; subtractive wording, "less than"). There is no variation of a/s wording for combine problems. From a linguistic perspective (Clark, 1969), additive relational terms (so-called "unmarked" terms) may be easier accessible than subtractive relational terms ("marked" terms), since unmarked terms are stored in memory in a less complex and more accessible form than their marked opposites. It is unclear yet if this is also relevant for additive and subtractive action verbs. Empirical findings on the difficulty depending on a/s

wording are inconsistent. While Hegarty et al. (1992) report that children need more time to solve word problems with subtractive wording, Gabler and Ufer (2021) could not identify an influence of a/s wording on task difficulty.

The a/s wording's influence on a problem's difficulty is also discussed together with the unknown set. The unknown set determines the mathematical structure of a word problem. A word problem is called *consistent* if it can be solved by adding the two numbers in the word problem and the wording is additive, or if it can be solved by subtracting the numbers and the wording is subtractive (e.g., "Susi has 8 marbles. Max has 3 marbles less. How many marbles does Max have?"; unknown result set). Otherwise, it is called *inconsistent* (e.g., "Susi has 8 marbles. She has 3 marbles less than Max. How many marbles does Max have?"; unknown reference set). Inconsistent word problems were found to be more difficult than consistent problems (e.g., Gabler & Ufer, 2021). This can be explained, for example, by the use of key word strategies: Using the a/s wording as a surface indicator to identify the required mathematical operation (e.g., "less than" as an indicator for subtraction) is only successful for consistent, but not for inconsistent word problems.

In summary, the difficulty of a word problem varies based on features of the situation structure presented in the problem text. It can be assumed that textbook authors take these effects into account to a certain extent, when they select or construct word problems. On the other hand, preferring or neglecting different types may also influence students' opportunities to gather experience with the corresponding types.

The influence of textbooks on students' performance

Teachers' use of textbooks for their lessons influences students' opportunities to learn in the classroom (e.g., Hiebert et al., 2003). There is empirical evidence that textbooks indirectly influence students' performance. For instance, Törnroos (2005) reports that students' performance was higher for topics, which are addressed more intensively in textbooks than those, which are addressed infrequently. Sievert et al. (2021) and Van den Ham and Heinze (2018) found that learning opportunities in textbooks predict students' learning gain, for example regarding knowledge of arithmetic concepts, or their understanding of compare problems. Furthermore, learners seem to benefit more from textbooks that encourage interleaved practice (different kinds of problems mixed within a unit) instead of dealing with blocks of problems addressing the same topic (Rohrer et al., 2020). Finally, an imbalance of task types may promote surface strategies, such as key word strategies for word problems (Siegler et al., 2020).

Prior textbook analyses focusing on additive word problems

Vicente et al. (2022) state that textbooks from high-performing countries (e.g., Singapore) contain a balanced and diversified distribution of additive word problems. Also textbooks from China (Xin, 2007), the Soviet Union (Stigler et al., 1986), and the USA (since the Common Core initiative 2010, Schoen et al., 2020) are named as positive examples. The total number of additive word problems in these textbooks is also higher than in other countries (Vicente et al., 2022). In contrast, textbooks from

Belgium (De Corte et al., 1985), Spain (Tárraga-Mínguez et al., 2021; Vicente et al., 2022), and older textbooks from the USA (Stigler et al., 1986) have been criticized for addressing only a limited range of word problem types, with an imbalance towards easier semantic structures (change, combination) and easier unknown sets (e.g., the result set). Inconsistent word problems rarely occur in these textbooks (Tárraga-Mínguez et al., 2021; Xin, 2007). The a/s wording has not been the focus of these studies, however. Moreover, we know of no similar analysis of German textbooks yet.

AIMS AND RESEARCH QUESTIONS

The main goal of this analysis was to investigate if the reported imbalances towards easier types of additive word problems also shows for German textbooks. Moreover, we intended to also consider a/s wording. We posed the following research questions:

RQ1: How frequent are different features of the situation structure of additive word problems in German textbooks? To what extent do these textbooks address empirically difficult word problem types?

Based on prior textbook analyses, we were particularly interested if textbooks would address more difficult types of additive word problems less frequently than easier types. Concerning the a/s wording, additive and subtractive wording could occur comparably often (due to their similar difficulty, Gabler & Ufer, 2021). It is also possible that the preference for unmarked terms also applies to word problem authors, which may show in a higher proportion of word problems with additive wording.

RQ2: How are the different semantic structures of additive word problems distributed over the span of grades 1 and 2, until the introduction of multiplication?

We expected that easier word problem types would occur earlier than harder ones. Regarding temporal distribution, we had no further hypotheses. We aimed to explore if single problem types would occur at discrete time points (a sign of blocked practice), or if they are distributed over longer periods (interleaved practice, Rohrer et al., 2020).

METHOD

We selected five common German textbook series and identified all additive word problems occurring in the 1st or 2nd grade editions until (excluding) the introduction of multiplication (around halftime of the second grade). Open word problems, which allow different interpretations of the semantic structure, were excluded, since no distinct situation structure could be assigned to such problems. To extend the sample, also word problems requiring more than one calculation were included and divided into their single parts. For example, “Maria buys chocolate for 3 €, apples for 2 €, and bread for 1 €. How much money did she spend?” counted as two combine problems with the whole set unknown. This resulted in $N = 652$ additive word problems.

All word problems were coded according to features of the situation structure (semantic structure, unknown set, a/s wording). Moreover, we recorded grade level, textbook series, and page number (relative to the number of pages of the book).

RESULTS

Regarding RQ1, the analyses revealed substantial differences in frequency of additive word problems between the different textbook series (49-128 additive word problems per textbook series). However, since open word problems were not considered for this analysis, this observation has to be interpreted cautiously.

Over all textbook series, we found a similar pattern for the frequency of semantic structures as in Spanish textbook analyses (e.g., Tárraga-Mínguez et al., 2021; Vicente et al., 2022). The most frequent semantic structure was the combination of sets, followed by change (see Fig. 1). As expected, comparison was scarce and equalization occurred even less frequent.

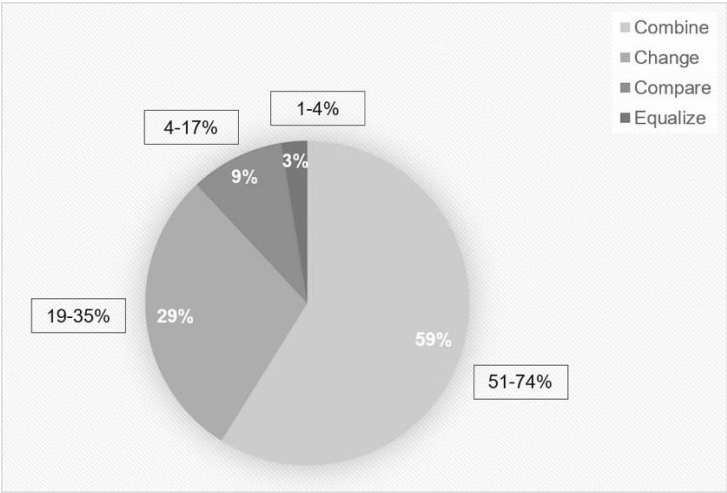

Fig. 1: Frequency of semantic structures and span for the different textbook series

For the unknown set, our findings match older analyses from the US and from Belgium, and newer analyses from Spain (e.g., De Corte et al., 1985; Stigler et al., 1986; Vicente et al., 2022). Difficult word problem types, such as combine problems with one part unknown (5 % of all word problems), change problems with unknown start (3 %) or change set (5 %), or compare problems with unknown reference set (1 %) were less frequent than other types (e.g., combine problems with the whole set unknown: 54 %). In almost all equalize problems, the change set was unknown (2 %) – the result set was unknown once, the start set was never unknown.

For the a/s wording, combine problems were excluded. In line with the assumption that authors prefer unmarked terms, additive wording is more frequent (59 % of all word problems) in the analysed textbooks than subtractive wording. Since unmarked relational terms are considered easier to understand, this also speaks for a bias towards more simple tasks. However, the a/s wording is not only expressed by relational terms, but also by actions verbs in change and equalize problems. Thus, we also analysed the a/s wording for each semantic structure. While additive wording occurs more frequently for compare and equalize problems (69 % for each), the a/s wording is more balanced for change problems (53 % additive wording). This suggests that the

preference for unmarked relational terms cannot completely be transferred to dynamic word problems.

Finally, we analysed if German textbooks also provide a balance of consistent and inconsistent word problems. Contrary to prior findings, almost half of the analysed word problems were inconsistent (45 %, combine problems excluded). To investigate this finding further, we examined how many of the word problems with additive or subtractive wording were inconsistent. Of the additively worded word problems, 70 % were inconsistent, while it was 9 % for subtractive wording. One possible explanation is that additive wording is favoured in general (59 % in our analysis; see also Clark, 1969).

To answer RQ2, we first examined the frequency of semantic structures for each grade. While the frequency of change problems declines from first to second grade, difficult semantic structures (compare, equalize) occur more often in second grade. However, also combine problems are addressed more often in second grade than in first grade. For a more detailed analysis, we divided the textbooks into ten consecutive sections per grade. Change problems are mostly addressed at specific times, with a peak in the middle to end of the first grade, and also in the middle of second grade. Combine problems peak towards the end of first grade and the middle of second grade. Compare and equalize problems are more equally distributed over the sections. All investigated textbooks contained blocks of problems addressing the combination of sets. For instance, there were whole pages focusing on the same context (e.g., purchase of tickets, etc.) and addressing almost exclusively combine problems. The observation that combine and change problems concentrate at specific times may also be due to their occurrence in word problems with more than one calculation step. Easier combine or change situations might be more frequent in such word problems than compare or equalize situations. Indeed, 76 % of the combine problems are part of word problems with more than one calculation step (change problems: 21 %).

DISCUSSION

In summary, our results are closer to prior results on textbooks that were criticized for addressing word problem types in an imbalanced way (e.g., Stigler et al., 1986; Tárraga-Mínguez et al., 2021; Vicente et al., 2022). Studies such as from Schoen et al. (2020) show that attempts to resolve this imbalance in response to empirical findings have been successful, for example in current US textbooks. Similar attempts might be fruitful in future revisions of German textbooks. Beyond only increasing the number of more difficult types, an explicit treatment of the corresponding semantic structures in the textbooks would be promising as well (Sievert et al., 2021).

With a focus on a/s wording and consistent vs. inconsistent problems, our study contributes a new perspective that was not covered in prior studies. The data indicate that word problems with additive wording are often inconsistent, while word problems with subtractive wording are mostly consistent. In future studies, it could be

investigated, which criteria textbook authors use to write or select additive word problems, and which problem features they focus on (e.g., unmarked or marked terms).

Concerning RQ2, it seems beneficial that difficult semantic structures occur continuously (as a sign of interleaved practice, Rohrer et al., 2020), and do not cumulate at single time points. Easier semantic structures are often addressed in blocks, but occur quite frequently. More balance combined with interleaved practice may improve the quality of the textbooks. It is unclear yet, if it makes sense that more difficult semantic structures are addressed later in accordance with models of number concept acquisition (Fritz et al., 2018) or if, for example, certain compare problem types could also be introduced earlier. It would also be of interest, when difficult task features (e.g., inconsistent problems) occur for the first time.

We excluded open word problems in our study, which may distort the analyses. These open tasks also offer learning opportunities concerning the situation structures of additive word problems. Also including word problems requiring more than one step has influenced the findings, since this raised the amount of combine problems substantially. Furthermore, the differences between the selected textbooks require further attention. Finally, we treated the frequency of word problems as one indicator for learning opportunities. Addressing situation structures explicitly can provide further learning opportunities, such as introducing compare situations (e.g., tasks on decomposed numbers with a missing part, the complementarity of addition and subtraction, or subtraction as a difference between numbers, Sievert et al., 2021).

In summary, this analysis adds to analyses showing an imbalance of different word problem types in textbooks. The US American textbooks indicate that this imbalance can indeed be resolved to a certain extent. It remains an important question in this context, how authors create and select word problems. Future studies may investigate this to understand why textbooks show this task distribution that seems suboptimal at first sight.

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STUDENTS' INTEREST WHEN COMBINING MODELLING AND EXPERIMENTATION – IS IT WORTH THE EFFORT?

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Mathematical modelling is a key competence but students are often less interested in modelling tasks than in other mathematical tasks. In a quantitative study with 82 tenth-grade students, we analysed students' interest in conducting experiments, in modelling processes that follow hands-on experiments, and in modelling processes without experiments, involving exponential functions. The results indicate that high achieving students are more interested in modelling processes without experiments than in the ones with experiments and in conducting experiments. In contrast, low achieving students are more interested in conducting experiments than in modelling processes afterwards or in modelling processes without experiments. Thus, it is questionable whether involving experiments is suitable to foster students' interest in modelling tasks.

INTRODUCTION

Modelling is considered a mathematical key competence with great relevance for other scientific disciplines, everyday life, and the society (Niss, 1994). Thus, modelling finds its place in many curricular documents as well as in the PISA framework (OECD, 2017). While modelling is undisputed important, it is also challenging for students as well as for teachers (Blum & Leiss, 2007).

Prior research indicates that many students do not value modelling tasks and are less interested in modelling tasks than in other mathematics tasks (e.g., Krawitz & Schukajlow, 2018). Thus, fostering students' interest in modelling tasks is a valuable educational goal. Our approach to foster students' interest is to combine modelling tasks with scientific hands-on experiments. In the reported study, we analyse which students profit from this approach.

THEORETICAL BACKGROUND

Mathematical Modelling

Mathematical modelling describes the whole process of solving real-world problems by transferring a real-world situation into a mathematical model. Contemporary conceptualisations usually describe modelling as a cyclical process. Following Blum and Leiss (2007), an idealized modelling process consists of seven steps (Figure 1).

A specific kind of modelling tasks are those combined with experiments. According to Halverscheid (2008, p. 226) "experiments related to mathematics find their natural place in the framework of mathematical modelling because they represent the 'rest of the world' for which mathematical models are built." Performing this kind of tasks, students first conduct hands-on experiments (e.g., physical experiments) and collect

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 363-370). PME 46.

experimental data. In the second step, students use their own experimental data for a subsequent modelling of the real situation that was represented by the experiment and solve a real problem based on the experimental data and their model.

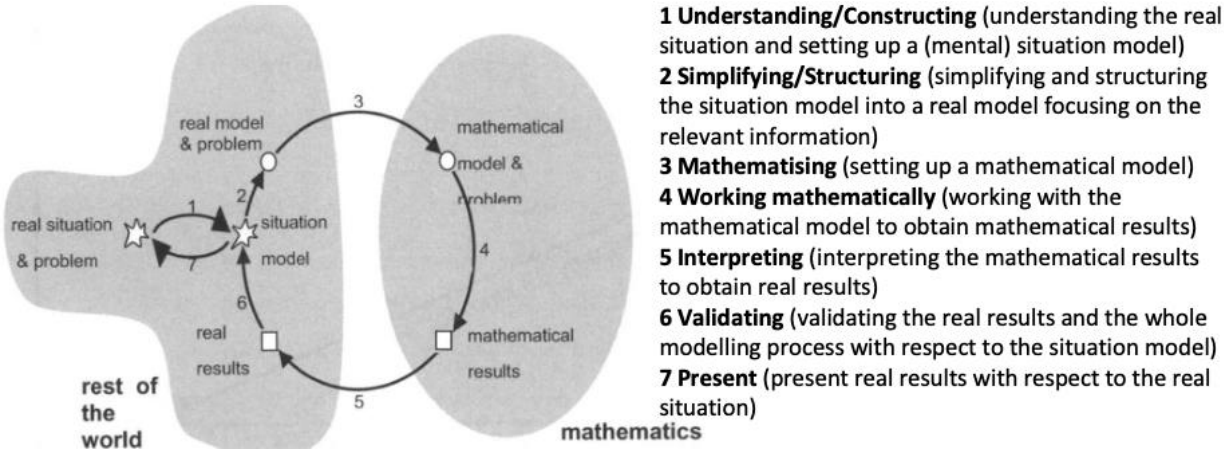

Figure 1: Modelling-Cycle following Blum & Leiß (2007)

Interest

Interest as a motivational variable is considered to play an important role in learning. Following the conceptualisation of Krapp (2007), interest is characterized as a special relationship between a person and an (abstract) idea, topic, etc. Interest comprises a feeling-related (feeling of joy) as well as a value-related valence (allocating a subjective high esteem to the object of interest) (Krapp, 2007).

Most conceptualisations distinguish between individual and situational interest. Individual interest is a longer lasting dispositional trait whereas situational interest is a fluctuating state that is mainly dependent on the interestingness of a specific (learning) situation. The frequent occurrence of situational interest in similar situations can lead to an internalization resulting in a stable individual interest (Krapp, 2007). As situational interest may influence individual interest and can be influenced by features of the learning situations, we analyse if a specific feature, in this case conducting experiments, influences students' situational interest in modelling processes.

Interest in modelling processes and in modelling processes with experiments

Theoretically, modelling tasks could enhance students' interest because students' interest in the real-world context may support students' interest in the mathematical task, which is embedded in the context (Schulze Elfringhoff & Schukajlow, 2021). Nevertheless, empirical studies report inconsistent results concerning students' motivation towards modelling tasks. A study by Parhizgar and Liljedahl (2019) revealed that students report slightly less engagement when working on modelling task than on mathematics tasks without a real-world context. However, the same study showed that an intervention with modelling tasks can lead to more positive attitudes towards mathematics in general. Krawitz and Schukajlow (2018) reported that students value modelling tasks less than other mathematical tasks. Likewise, students' state less

situational interest in modelling tasks than in other mathematical tasks (Krug & Schukajlow, 2013).

Given these results, previous studies tried to identify which features of learning situations are suitable to foster students' interest in modelling processes. Besides task characteristics, like the mathematical topic as well as the concrete real-world contexts (Krawitz & Schukajlow, 2018; Schulze Elfringhoff & Schukajlow, 2021), it is assumed that modelling tasks in combination with experiments are suitable to foster students' motivation (e.g., Ludwig & Oldenburg, 2007). Ganter (2013) showed in an intervention study that lessons involving experiments lead to more students' interest in mathematics than traditional lessons taught with textbooks. Likewise, students report a high situational interest in modelling tasks with experiments (Beumann, 2016; Carreira & Baioa, 2018). Geisler and Rach (in press) found that students experience higher situational interest in modelling tasks combined with experiments than in modelling tasks without experiments.

THE CURENT STUDY

The current study is part of the research project Mathematical Modelling with Experiments (MaMEx). The objectives of MaMEx are to design modelling tasks with experiments and to analyse the effects of these tasks on students' modelling competencies and their motivation. Within a first pilot study, explorative insights into students' validation processes when modelling with experiments (Geisler, 2021) and first results concerning motivational effects were gained (Geisler & Rach, in press). The current study enables a closer look on the effects on students' situational interest.

Research Questions

Prior studies indicate that students report high situational interest when working on modelling tasks combined with experiments (Beumann, 2016; Geisler & Rach, in press). However, it is yet unclear if students' interest refers to the whole modelling task with experiment (and especially the modelling process itself) or if students are mainly interested in conducting the experiment while the subsequent modelling process is less interesting. Thus, we differentiate between two learning situations constituting a modelling task combined with experiment: i) the conduction of experiments and ii) the subsequent modelling process with one's own experimental data. We compare these two situations to iii) the modelling process completely without experiment. In addition, we want to investigate if students' interests in these three learning situations depend on students' prior achievement in mathematics to adaptively support students in their learning process. In particular, we aim to answer the following questions:

1. Are there differences in students' situational interest between the following learning situations: i) conduction of experiment, ii) subsequent modelling process with experimental data, iii) modelling process without experiment?
2. Is students' situational interest in the aforementioned situations related to their prior achievement in mathematics?

METHODS

Sample

In order to answer the research questions, a quantitative study with 82 upper secondary students from three grammar schools (grade 10, *M*(age)=16, 51% girls) was conducted. In the study, modelling tasks (see next section) were used that involve exponential functions. Students had worked with exponential functions before and were familiar with characteristics of exponential growth and decrease.

Used Modelling Tasks

Two modelling tasks that both can be solved using exponential functions have been designed. Both tasks have the same structure and exist in one version combined with experiment and one version without experiment. The task "Cold Coffee" uses the context of cooling off a cup of coffee. The versions combined with and without experiment both begin with the same introduction of the context:

> *After brewing coffee needs some time to cool off in order to be conveniently drinkable. The desired drinking temperature differs from person to person. Model the temperature decrease and evaluate at which time the coffee can be delightfully drunken.*

In the version combined with experiment, students were asked to state a hypothesis concerning the temperature development and then conducted an experiment (following a given experimental guide) by measuring the temperature of freshly brewed coffee for 10 minutes. In the version without experiment, students were given a table with pre-existing data. In both versions, students were asked to model the cooling process using a function. However, no hint was given which type of function would be suitable.

In the equally structured task "Stale Beer" students model the decay of beer froth (for more information on the tasks, see Geisler, 2021).

Design and Instruments

During a 90 minutes lesson, all students worked on one modelling task combined with experiment (consisted of learning situations i) and ii)) and one modelling task without experiment (consisted of learning situation iii)). After each learning situation, students were asked to rate their situational interest on a short questionnaire using adapted items from Willems (2011) – 4 items, Cronbach's α = .85, for example "I liked conducting this experiment" for learning situation i). All items were structured in the same way for the three learning situations and students answered them on a six-point likert-scale (*1=totally disagree, 6=totally agree*). Furthermore, students' demographic data and their prior mathematics grade have been assessed.

To prevent effects of the order, type of task (combined with experiment and without experiment) and context (cooling of coffee and decay of beer froth) were permutated resulting in four versions. Students in version 1 started with working on the task "Cold Coffee" in combination with an experiment, following the task "Stale Beer" without

experiment whereas students in version 2 started with the task “Cold Coffee” without experiment, following the task “Stale Beer” in combination with experiment. The other two groups worked on the tasks in the opposite order (see Figure 2).

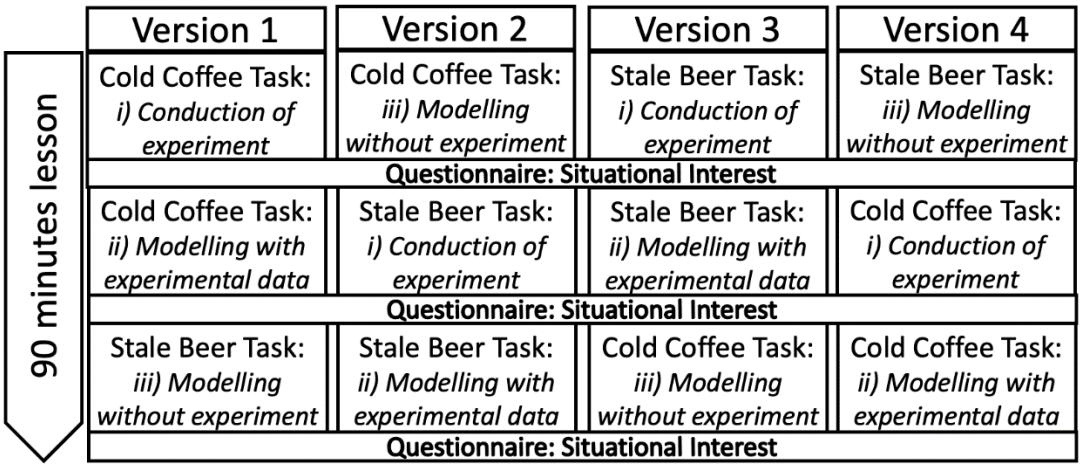

Figure 2: Design of the study

With regard to research question 2, students were divided in three groups based on their prior grade in mathematics: high achieving students (*n*=27, prior mark: *1=very good* or *2=good*), medium achieving students (*n*=27, prior mark: *3=satisfactory*) and low achieving students (*n*=28, prior mark: *4=sufficient*, *5=deficient* and *6=insufficient*). We used a mixed ANOVA with the factors learning situation and prior achievement to analyse students’ data.

RESULTS

Descriptive data concerning situational interest in the three learning situations of the whole sample and the groups (divided by prior achievement) can be found in table 1. Furthermore, results for the different achievement groups are visualized in Figure 3.

	Learning Situations		
	Modelling task combined with experiment		Modelling task without experiment
Sample	i) Conduction of experiments	ii) Modelling process with experimental data	iii) Modelling process without experiment
Whole sample (*N*=82)	3.78 (1.14)	3.46 (1.24)	3.73 (1.26)
High Achievers (*n*=27)	3.57 (1.27)	3.89 (1.25)	4.32 (1.17)
Medium Achievers (*n*=27)	4.02 (0.88)	3.44 (1.26)	3.51 (1.25)
Low Achievers (*n*=28)	3.79 (1.22)	2.97 (1.07)	3.34 (1.18)

Table 1: Situational interest (means and standard deviations) in the different learning situations, ratings from *1=totally disagree* to *6=totally agree.*

Research question 1 deals with students’ interests in the different learning situations. As visualized in table 1, data give us no hints that the modelling process after

experiment (M = 3.46) in the whole sample is more interesting than the modelling without experiment (M = 3.73). Surprising is that conducting experiments is similar interesting for students as the modelling process without experiment (M = 3.78).

Research question 2 yields at differences in interest between the learning situations dependent on students' prior mathematics achievement. The mixed ANOVA reveals no significant main effect of the factor prior achievement ($F(2,79)=3.83$, $p>.05$, $\eta^2=.06$) and no main effect of the factor learning situation ($F(2,79)=1.54$, $p>.20$, $\eta^2=.05$) but a significant interaction between achievement and learning situation with large effect size ($F(2,79)=5.60$, $p<.01$, $\eta^2=.17$). Indeed, in Figure 3 different interest patterns of the three groups of students can be seen.

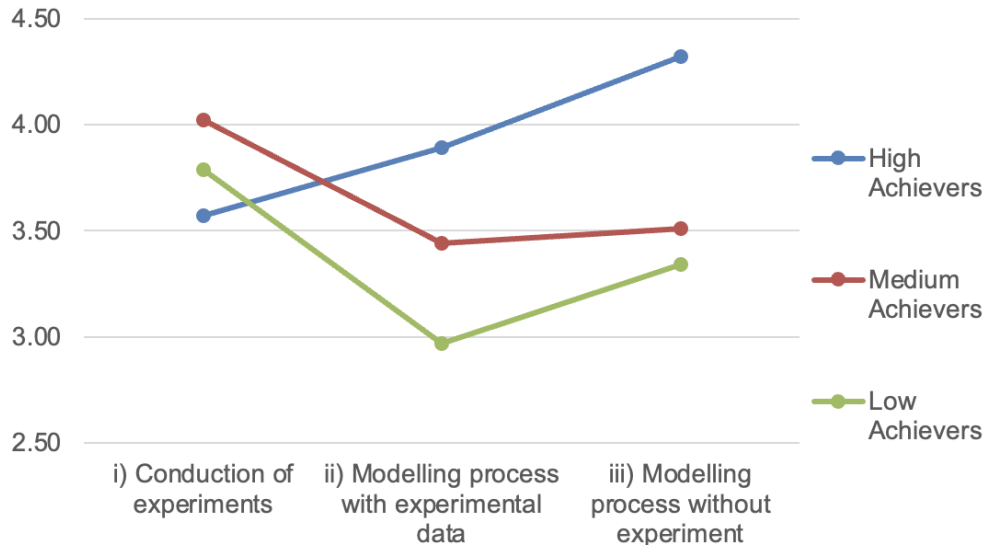

Figure 3: Interest patterns of the different achievement groups
(ratings from *1=totally disagree* to *6=totally agree*)

Within the three learning situations, high achieving students report the least interest in conducting experiments while they rate the subsequent modelling process with their experimental data as slightly more interesting and state most situational interest in the modelling process without experiments. In contrast, medium and low achieving students rate the conduction of experiments as the most interesting learning situation, while both modelling processes are less appealing to them. The low achieving students report even lower interest in the modelling process with their own experimental data than in the modelling process without experiments.

DISCUSSION

In this study, we focused on the potential of modelling tasks in combination with experiments to foster students' situational interest. Our results go beyond those of previous studies (e.g., Beumann, 2016; Geisler & Rach, in press) because we did not only compare the modelling process combined with and without experiments but enabled a more differentiated perspective by further distinguishing two learning situations that constitute modelling tasks combined with experiments: the conduction of the experiment and the subsequent modelling process with one's experimental data.

In contrast to the results of Beumann (2016), Ganter (2013) as well as Geisler and Rach (in press), we did not find a positive effect of experiments on students' interest in the modelling process in the whole sample. Students were similar interested in conducting the experiments as in the modelling process without experiments and slightly less interested in the modelling process with their own experimental data.

Besides results for the whole group, our study provides insights in the situational interest of three groups of students, which are homogenized according to their prior achievement in mathematics. High achieving students were least interested in conducting the experiments and most interested in the modelling process without experiments – even more than in the modelling process with their experimental data. It seems that in this case, experiments were neither necessary nor helpful to foster interest in the modelling process. While medium and low achieving students were most interested in conducting experiments, they were less interested in both modelling processes. Thus, their interest in the experiments was not helpful to induce interest in the subsequent modelling process with experimental data. Low achieving students were even less interested in the modelling process with experimental data than in the modelling process without experiments. It seems that experiments can even hinder the interest of low achieving students. One reason for this result could be that the modelling process with experimental data is more challenging for these students because the data contain measurement errors and irregularities (e.g., Geisler, 2021) and students' situational interest in modelling tasks is related to their experience of competence (Schulze Elfringhoff & Schukajlow, 2021). Although experiments are often promoted (e.g., Beumann, 2016; Ludwig & Oldenburg, 2007), our results indicate that not all students profit from experiments with regard to their interest in modelling activities.

A limitation of our study lies in the rather small sample. Instead of using a control group, all students in our sample worked on a modelling task combined with experiment and a similar task without experiment. Therefore, the results should be confirmed within a larger study with control group design. Furthermore, we only used modelling tasks and related experiments from the topic of exponential functions. As Krawitz and Schukajlow (2018) have shown that students' interest in modelling is also dependent on the mathematical topic, it remains an open question whether modelling tasks with experiments related to other mathematical topics are more suitable to foster students' situational interest.

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REPLICATION OF A POSITIVE PSYCHOLOGY INTERVENTION TO REDUCE MATHEMATICS RELATED SHAME

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Shame is an unpleasant and activating emotion that may affect mathematics learners' achievement, motivation, and identity development. Specifically, preservice primary teachers often experience shame in mathematics, which may impact their future teaching and their students' development of unpleasant emotions. In the current study, we investigated an intervention adapted from positive psychology to reduce preservice primary teachers' shame in mathematics during university education. Within a controlled experimental setting, the interventions' efficacy was analysed, and compared to a qualified control group. Findings suggest a positive effect of the adapted intervention on shame reduction. Practical implications are being discussed.

PRESERVICE PRIMARY TEACHERS' EMOTIONS IN MATHEMATICS

Mathematics learning is highly related to learners' emotions (Hannula, 2019). Several mathematic situations and contents can trigger pleasant or unpleasant emotions for mathematics learners, which can impact their achievement, motivation, and identity development (Göller & Gildehaus, 2021; Lutovac & Kaasila, 2014; Pekrun, 2006, 2021).

Preservice primary teachers, who often conceptualize themselves as learners more than teachers during their university education (Chen, 2017), are often confronted with unpleasant emotions during their studies, specifically in mathematics. In most countries, preservice primary teachers are trained as generalists, and their interest is more in the overall teaching aspect than specific subjects (Blömeke et al., 2012). However, at least some mathematics courses are obligatory for most of them and many are also obligated to later teach mathematics. Most of the preservice teachers' unpleasant emotions during their university education are often based on unpleasant emotions they experienced during their school education (Hodgen & Askew, 2007). Leaving university with unpleasant emotions towards mathematics is known to later affect their teaching. For example, Beilock et al. (2010) demonstrated that teachers with high mathematics anxiety could transmit their stereotypes to their students. However, Lutovac & Kaasila (2014) discussed that negative experiences with mathematics during school years may not in general prevent students from a successful identity work towards mathematics and the main reasons for different developments are pedagogical practices in mathematics courses. Similarly, Hodgen & Askew (2007) argued that emotional relationships with mathematics show both individual and social elements.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 371-378). PME 46.

Concretizing those different aspects around unpleasant emotions towards mathematics, many different operationalizations can be found in recent research. While many studies focus on anxiety as the primary unpleasant emotion, studies on other unpleasant emotions, such as frustration, hopelessness or shame are rare. Following the findings above, some recent research specifically focused on shame as an unpleasant emotion towards mathematics because it is usually emerging within a social context, based on upward comparison to others. According to control-value theory (Pekrun, 2006), shame can be seen as an unpleasant and activating emotion which is experienced after learning and achievement situations.

Shame in mathematics

Shame is experienced when difficulties in mathematics are subjectively attributed to a deficiency in one's general ability (e.g., "I am stupid"; Holm et al., 2017). Following Pekrun (2006), besides low control appraisals the extent to which an individual values mathematics is thus related to the intensity of shame that one may feel. Only when an individual highly values achievement in mathematics, mathematics in general as an important domain or when social values are perceived, they experience shame when not meeting mathematical requirements. In case of preservice primary teachers, it can be assumed that domain value is high because of the importance of mathematics knowledge for the acquisition of mathematics didactics knowledge in teacher education (Agathangelou & Charalambous, 2020). Furthermore, German preservice primary teachers especially might perceive their prospective responsibility for students' relative low achievement in mathematics at school in Germany.

Because shame "reflects the personal implication of the outcome of an event" (Turner et al., 2002, p.82), shame is referred to as a *self-conscious* emotion (Lewis, 2003). Thus, shame cannot be regulated easily, showing a strong connection to the self, with possible vicious cycles: Putting much effort into mathematics prior to experiencing failure, which is attributed to the self as an indicator of lack of ability, shame, and damage to self-worth are high (Turner et al., 2002). Individuals then tend to avoid further experiences of failure to protect the self, which may cause a lack of ability, e.g., within mathematics. In line with this, shame may disrupt students' self-regulation of learning processes and is strongly related to low-ability self-concept (Jenßen, 2021)

Additionally, preservice teachers often experience mathematics courses at universities as non-discursive places, where participation is almost impossible if one does not understand (Gildehaus & Liebendörfer, 2021). Such an environment may create various situations with low control appraisal for preservice teachers. Failures may thus be attributed to the self directly. In line with this, preservice primary teachers experience shame more often and intensely regarding mathematics than other subjects (Jenßen et al., 2021).

Recent studies revealed that preservice primary teachers' shame in mathematics shows a variety of negative effects: It affects their subject choices at university as they avoid mathematics courses (Jenßen, Möller, et al., 2022), it goes along with less achievement

in mathematics (Jenßen, Roesken-Winter, et al., 2022) and it negatively affects their intention to teach mathematics later at school (Jenßen, Roesken-Winter, et al., 2022). In light of these negative consequences of preservice primary teachers' shame in mathematics, Jenßen (resubmitted) developed an intervention to reduce shame during teacher education. The current study tries to replicate those findings by comparing the effects of the developed intervention to a similar but inherently different intervention.

THE "NAME THREE GOOD THINGS ABOUT YOURSELF IN MATHEMATICS" – INTERVENTION TO REDUCE SHAME

Focusing on the regulation of unpleasant emotions, as well as strengthening pleasant emotions, a variety of so-called positive psychology interventions (PPI in the following) have been developed and applied to the educational context (Carr et al., 2021; Seligman et al., 2005). A common PPI is the *three good things* technique that requests a systematic reflection from participants on three pleasant things they experienced during a specific time interval (Seligman et al., 2005). During the intervention, participants usually put greater conscious attention on possible positive situations, and thus during the PPI, their positive perception is being promoted and triggering cognitive change, strengthening the individual's resistance to unpleasant experiences (Seligman et al., 2005). Furthermore, increased effectiveness can be observed when the technique is performed over a longer period. Usually, the PPI is developed for general experiences in daily life, but it can be easily adapted to other context such as mathematics.

Jenßen (resubmitted) adapted the three good things technique to mathematics and preservice primary teachers as this intervention appears as economical, feasible, and able to be continued over a longer period and thus, may be easily implemented during teacher education at university. One main aim was to not address shame directly to minimize the risk of re-shaming but to focus on resources to increase resilience to shame experiences. Although, as discussed above, shame as an emotion is closely linked to the self, the intervention was developed to focus on concrete positive aspects of the self in mathematics situations. Thus, the following exercise has been formulated as central part of the intervention:

Name three good things you like about yourself in mathematics. For example, reflect on positive experiences in mathematics that you have personally brought about in the past few days. Name as many different things as possible over the time for this exercise.

Findings from the original study by Jenßen (resubmitted) revealed a small positive effect (*Cohen's d* = 0.36) regarding the reduction of preservice primary teachers' shame in mathematics when they participated twice a week over a period of five weeks. Duration of the intervention and dose per week has not been manipulated in this study although meta-analyses regarding PPIs revealed that these variables may affect the effectiveness of the intervention (Carr et al., 2021).

Evaluating the effectiveness of this exercise in an experimental setting (randomized controlled trial), participants were randomly assigned to either the intervention group or a control group.

Research Questions

The purpose of the current study was to replicate the findings by Jenßen (resubmitted) regarding the effectiveness of the developed PPI to reduce preservice primary teachers' shame in mathematics. Furthermore, the aim was not only to replicate the findings but to examine the effectiveness of the intervention in a different context: Not only did we implement the intervention to a different university, where the students of the course were more heterogenic, the focused lecture was in mathematics rather than in mathematics education and duration and dose of the intervention were also different to the design of the study by Jenßen (resubmitted). Accordingly, we pose the following research questions:

RQ1: Does participation in the PPI also reduce preservice primary teachers' shame in mathematics when the given context differs from that in the previous study?

RQ2: Does participation in the PPI reduce preservice primary teachers' shame in mathematics more effectively than in a control group?

METHODS

A total of *n*=99 students took part in the study. The exercise was integrated into a university education setting with 83 preservice primary teachers, 14 preservice teachers for special education, 2 without specification, at a medium-sized German university. Of these participants, 87 reported being female, 11 male, 0 non-binary, 1 without specification and 63 of them were in the last year of their master's program, 21 had just started their master's, 14 were in the last year of their bachelor's degree and 1without specification.

At the surveyed university, mathematics was obligatory for preservice primary teachers. However, they could voluntarily intensify courses in mathematics (8.2% reported to have chosen this option). The surveyed course was called *Elements of Mathematics*, which focused on basic ideas of formal mathematics, such as logic and (generic) proofs. It is usually connected with unpleasant feelings for some preservice teachers and is referred to as "the most challenging mathematics lecture" of their studies. Lectures and tutorials each took place once a week, and voluntarily, additional support was offered weekly in a learning support center. To participate in the exam at the end of the semester, students had to reach 50% of points in weekly homework (usually four short exercises). After the third week of the semester, one homework exercise was replaced either with the intervention or the control group exercise. Student's participation in the intervention was thus obligatory to be able to participate in the exam and complete the course. Within the intervention group, they worked on the exercise above weekly over a period of ten weeks (two times longer than in the study by Jenßen (resubmitted)) and once a week (compared to twice a week within the

previous intervention study). The control group worked on a content-based exercise ("Summarize the lecture's content in at least three sentences") weekly during the same time.

To assess preservice primary teachers' shame in mathematics, the SHAME-Q was used (Jenßen, Roesken-Winter, et al., 2022). The questionnaire showed good reliability in recent studies and the current one (Cronbach's $\alpha = .93$ at both time points) and was also comprehensively validated, making sure it is distinguishable from other unpleasant emotions, such as anxiety (Jenßen, Roesken-Winter, et al., 2022). The SHAME-Q used a 5-point-likert-scale from 1 (= strongly disagree) to 5 (= strongly agree), and was integrated into a wider questionnaire, thus the experimental setting was not too obvious for the students. The questionnaires were completed online during lecture time in the week before the intervention started (the third week of the semester) and the week after the intervention finished (11 weeks later). During the intervention students did not know, that they participated in an intervention, but they were informed afterwards.

To examine Research Question 1, a paired t-test was performed. A mixed ANOVA was applied to examine Research Question 2. Assumptions of these statistical techniques were analysed a priori. All analyses were done by using SPSS 29.0.

RESULTS

Means and standard deviations (in brackets) are shown in Table 1 differentiated for both intervention group and control group and also for T1 (pre-test) and T2 (post-test).

RQ1: Change in Intervention Group. A paired t-test was performed. There were no outliers in the data. The differences between the pre- and post-scores of the SHAME-Q were normally distributed, as assessed by the Shapiro-Wilk test ($p = .052$). SHAME-Q scores were significantly lower after the intervention, $t(50) = 3.85$, $p < .001$. The difference was $M = 2.18$ ($SD = 4.03$) with a medium effect size (*Cohen's d* = 0.54).

Group	T1	T2
Intervention (n=51)	15.00 (5.73)	12.82 (5.11)
Control (n=48)	14.35 (6.13)	13.83 (6.24)

Table 1: Descriptive results: means and standard deviations in brackets

RQ 2: Comparison of Intervention Group and Control Group. At T1 no outliers were in the date, at T2 three outliers were found for the intervention group. These remained in the analysis as they reflected theoretically possible values. SHAME-Q scores were only normally distributed for the intervention group at T1 as assessed by Shapiro-Wilk test ($p = .163$). However, mixed ANOVAs lead to robust results when violating this assumption as long the sample size is greater than 30 which was the case in the current application (Glass et al., 1972). There was homogeneity of the error variances, as assessed by Levene's test ($p < .05$). There was also homogeneity of covariances, as assessed by Box's test ($p = .356$). There was a statistically significant interaction

between time and group $F(1, 97) = 4.75$, $p = .032$. The effect size was medium with partial $\eta^2 = .05$.

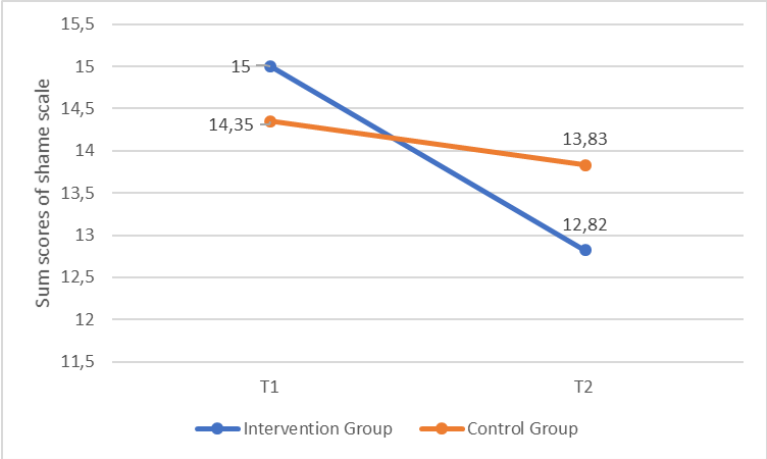

Figure 1: Change in means over time

DISCUSSION

The results showed a significant reduction in preservice primary teachers' shame in mathematics for those who participated in the intervention, with a medium-sized effect. We were able to replicate findings of the previous study regarding the effectiveness of the intervention. Furthermore, the PPI developed by Jenßen (resubmitted) was thus also effective in reducing preservice primary teachers' shame in mathematics when the context was slightly different (duration was set longer, dose was less frequent, students were more heterogenous, lecture content was mathematics). Compared to the previous study effect sizes were even *greater*. This validates findings from meta-analyses that duration and dose may affect the effectiveness of PPIs (Carr et al., 2021). A longer duration may show greater effects in this case. Furthermore, the different contents the students learned within the different settings may have had an influence as well. A formal mathematics learning environment may provide more challenging situations for shame, but also more opportunities to experience positive situations, one can attribute to the self. Compared to a mathematics education lecture the preservice primary teachers attended in the other setting, positive experiences may have also been more intense within the formal mathematics course.

Moreover, the PPI also occurred as effective in reducing preservice primary teachers' shame in mathematics compared to a control group. In comparison to the study by Jenßen (resubmitted) where controls were requested to name three good things about mathematics in general, we chose a more common control intervention as university students are usually assumed to summarize lecture's content at the end of the course. Nevertheless, the current study highlights once more the effectiveness of the developed PPI compared to a non-usual control group where no participation in any intervention is required.

This indicates that even though we successfully replicated the intervention, its efficacy seems closely connected to a specific social context and further research seems

desirable to investigate specific relations within different settings, e.g., settings in school as well as other mathematics study programs. Similarly, the discussed relations of shame with motivation, identity, and achievement need further investigation. To better understand shame in the context of individual identity-development qualitative insights about the individual's reflection during the intervention are desirable.

Unlike other interventions, such as *expressive writing* against performance anxiety in mathematics (Maloney et al, 2013), this intervention did not focus on a specific situation but on a general change within the self. Follow-up studies are needed to investigate this further, but it seems likely, that a longer-term effect could have been achieved.

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MATHEMATICS-SPECIFIC MOTIVATIONS FOR CHOOSING A MATHEMATICS TEACHING DEGREE STUDY PROGRAMME

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The transition from school to university in mathematics is accompanied by more abstract and formal modes of mathematical thinking, a focus on deductive proof and for many preservice teachers also by demotivation and frustration towards this kind of mathematics. The present study qualitatively analyses 13 preservice teachers' mathematics-specific motivation for choosing a mathematics teaching degree study programme before the beginning of their study. The results show that many preservice teachers enjoy schematic aspects of mathematics, which rather does not correspond to the type of mathematics taught at universities. These findings contribute to explaining demotivation and frustration of preservice teachers with the mathematical content of their university studies, what is discussed in more detail.

THE TRANSITION FROM SCHOOL TO UNIVERSITY IN MATHEMATICS

The transition from school to university in mathematics is associated with several well documented difficulties, comprising in particular the required more abstract and formal modes of mathematical thinking at university, the focus of deductive proof, as well as different institutional cultures regarding the teaching and learning of mathematics (e.g., Gueudet & Thomas, 2020). Upper secondary preservice teachers with mathematics as a subject, who in Germany typically attend courses on abstract university mathematics together with mathematics majors, often question the relevance of university mathematics (Gildehaus & Liebendörfer, 2021), feel frustrated (Göller & Gildehaus, 2021), report a higher interest in school mathematics and a lower interest in university mathematics as well as in proof and formal representations than mathematics majors (Ufer et al., 2017). Such an interest profile predicts study demotivation and dissatisfaction (Kosiol et al., 2019).

To better understand such interest profiles and preservice teachers' affect towards mathematics, the present paper aims at providing a qualitative insight into their mathematics-specific motivation for choosing a mathematics teaching degree study programme and especially into the intrinsic values they associate with mathematics.

THEORY

Study and career choice motivations of future teachers

Across disciplines, study and career choice motivations of future teachers have been frequently studied in recent years and are mostly categorised on the basis of an expectancy-value theory of motivation (Eccles & Wigfield, 2002; Watt et al., 2012), or distinguished in intrinsic, extrinsic, or altruistic motivations (Fray & Gore, 2018). Expectancy-value theory posits that students' motivation for an upcoming task or here

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study choice can be explained by their expectancy how well they will do on the tasks, their perceived enjoyment (intrinsic value), personal importance (attainment value), and usefulness for other goals (utility value) of the task as well as by perceived negative aspects (cost) associated with the task (Eccles & Wigfield, 2002). Table 1 shows typical study and career choice motivations identified and investigated this way.

Categorisation based on expectancy-value theory	Examples of typical study and career choice motivations (Fray & Gore, 2018; Watt et al., 2012)	Categorisation as intrinsic, extrinsic, or altruistic motivations
Expectancy	Perceived personal abilities (regarding study subject or teaching abilities)	-
Intrinsic value	Subject-specific interest, Pedagogical interest	Intrinsic
Attainment value	Make social contribution, Help other people	Altruistic
Utility value	High or secure salary, Time for family	Extrinsic
Cost	low difficulty of study, social influences (advice from others)	-

Table 1: Examples of typical study and career choice motivations of future teachers and their motivation-theoretical categorisation (cf. Göller & Besser, 2021).

Such study and career choice motivations of future teachers have proven to be an important predictor for individual success in university studies and profession: In particular intrinsic study and career choice motivations were found to correlate significantly with higher study strategy use, study satisfaction, learning goal orientation, career optimism, pedagogical knowledge, teaching quality, professional satisfaction, and with lower burnout symptoms (Hanfstingl & Mayr, 2007; König et al., 2018; McLean et al., 2019; Wach et al., 2016).

Mathematical world views

The before mentioned study and career choice motivations for a teaching degree programme apply to all study subjects and are not specific to mathematics. In the following, possible study and career choice motivations which take the specific nature of mathematics more strongly into account shall be identified. To do so, mathematical world views according to Grigutsch & Törner (2002) are introduced in more detail. They identified the following four aspects as central for peoples view and understanding of the nature of mathematics:

- The *process aspect* emphasises the constructive character of mathematics. Mathematics is understood as an active process of discovery, experimentation, and thinking about problems.

- The *application aspect* highlights the applicability and practical use of mathematics for society and everyday life.
- The *formalism aspect* identifies rigour, exactness, and precision as well as logical and objective thinking as essential for mathematics.
- The *schema aspect* identifies mathematics as a "toolbox" and set of formulas for solving tasks.

Process and application aspect are understood as *dynamic view* of mathematics. They identify mathematics as a problem-related discovery and understanding process associated to reality and real-world applications through which mathematical knowledge is constructed. Formalism and schema aspect are understood as *static view* of mathematics, in which mathematics is rather seen as a collection of existing knowledge and procedures.

Research questions

The present study aims at identifying mathematics-specific study and career choice motivations of future mathematics teachers, first focusing on possibly replicating and further elaborating general study choice motivations as shown in Table 1:

- RQ1: Which mathematics-specific study and career choice motivations do future mathematics teachers have at the start of their university studies?

For a deeper understanding of the specificity of mathematics, the intrinsic mathematics-related motivations shall be further elaborated with regard to the mathematical world views presented (Grigutsch & Törner, 2002):

- RQ2: To which characteristics of mathematics respectively to which mathematical world views do intrinsic study choice motivations refer?

METHODS

To answer these questions, interview data of 13 upper secondary preservice teachers (11 female, 2 male) were analysed, who were interviewed about three weeks before the beginning of their university studies. For this study answers referring to the following two questions were analysed: (1) How come you decided to study for a mathematics teaching degree? (2) Why mathematics, what do you like about it? The data were analysed mixing a concept-driven (deductive) and data-driven (inductive) coding approach (Kuckartz, 2019). As concept-driven categories, the study choice motivations listed in Table 1 as well as the four presented mathematical world views (process, application, formalism, schema aspect) were considered. Process aspect was coded for passages where interviewees stated they like mathematics when discovering new things and puzzling over problems, application aspect when interviewees stated they like mathematics for its real-world applications. Formalism aspect was coded when interviewees stated that they like mathematics because or when it is "logic" (for them) and precise e.g., in the sense that there is no discussion whether a solution is right or wrong, schema aspect when interviewees stated they like performing calculations and mathematical procedures. The deductive categories of study choice motivations were further refined inductively considering the specifics of mathematics.

RESULTS

Results for RQ1

Regarding the motivation for choosing mathematics as study subject, all but one interviewed preservice teacher named their perceived mathematical abilities. These perceived mathematical abilities were based on good school grades, on social comparisons with classmates, as well as on the sense of achievement when solving mathematical problems (see Table 2). Sense of achievement has been shown to be an important basis for the enjoyment (intrinsic value) of mathematics (see next section and quotation in Table 2).

Category	Example quotations
Perceived mathematical abilities	And I chose maths because, as I said, I didn't have any problems with maths at some point anymore.
- Good grades	Maths and sports were always the subjects where I always got the best grades. And then I thought, why don't you just study those subjects?
- Social comparisons	I was the only one in the class who got it and explained it to the others.
- Sense of achievement	The principle that you have so many experiences of achievement. That's quite appealing. The school grade, I think, also plays a role. That was decisive for me, I'd say.
Intrinsic value of mathematics	I've always liked doing mathematics. Maths has actually been my favourite subject since grade 11.
Utility value: Math teacher shortage	I checked, what are shortage subjects? Maths is also in high demand.
Cost: No rote learning	With other subjects it's just that you learn something by rote or follow some theories, and maths is just logical. It can be done just like that.
Perceived pedagogical abilities	my friends never complained when they didn't understand maths and I tutored them
Intrinsic value of teaching (mathematics)	I do private tutoring in maths and English, and I have to say that maths is twice, three times, five times the fun of English, because you can see success right away, and that's what I think is so enjoyable about it.
Attainment value: help other people	I like to impart my knowledge to other people. Especially when in maths, because many people find that difficult.

Table 2: Categories and example quotations of mathematics specific expectancies and values motivating the choice of a mathematics teaching degree study programme.

Besides mathematics-specific intrinsic values, which were mentioned by all interviewees, and which are elaborated in more detail in the following section, math teacher shortage (utility value) and the perceived absence of rote learning (low cost) were mentioned sporadically as mathematics-specific study choice motivations.

Regarding the motivation for choosing a teaching degree programme all but one interviewee named intrinsic or attainment values such as enjoying explaining, imparting one's knowledge, doing private tutoring, working with children, helping others, or taking responsibility for the development of learners. These are often mentioned in relation to mathematics however, the mathematics-specific characteristics are often not directly apparent. Nevertheless, there are statements in which the specificity of mathematics becomes more evident: For example, that successes are seen more immediately when teaching mathematics, or that mathematics teaching is attributed a special value because of learners' difficulties with mathematics (see Table 2). Perceived ability seems to play a lesser role for choosing a teaching degree programme than for choosing mathematics as a study subject. In the following interview excerpt the perceived ability for teaching takes a rather prominent role in comparison to other interviewees. It illustrates the interplay of working with children (intrinsic value), perceived mathematical abilities, perceived teaching abilities, and advice from others which motivated choosing a mathematics teaching programme:

> And I just notice that I enjoy working with children, also in maths, because I was always good at it at school. And my friends never complained when they didn't understand maths and I tutored them for a short time, they always said: "Wow, you explain so well, so I understand everything". And at some point, during my final year, I heard from various people who told me that my manner and my character, the way I am, might make me a good teacher. [...] And I'm the kind of person who, if I think something is great, I can get people excited about it. So to speak, I can promote it. Then I thought, I'll promote maths.

Results for RQ2

As mentioned before, the characteristics of mathematics and the mathematical world views associated with intrinsic study choice motivations were closely related to interviewed preservice teachers' sense of achievement. The following interview excerpt illustrates this close relation, in this case associated with the schema aspect indicated e.g., by "I like calculating":

> I've always liked doing it [mathematics]. As I said, only when I understood it. Because when I don't understand it, I'm always very frustrated and then I get a bit angry, and then I don't like it anymore. But as soon as I understand it again, I enjoy it again. […] when I know I can do it and I'm calculating the tasks without having any big problems, then I enjoy it, I like calculating. Yes, and that's how I ended up in the maths teaching programme. I just liked doing it [mathematics].

The schema aspect was found to be dominant in 7 of the 13 preservice teachers mathematical world views. Figure 1 gives an impression of the interviewees' mathematical world views coded in the data, depending on whether their personal intrinsic value of mathematics was rather related to calculating or to puzzling (schema

– process aspect, vertical axis), or rather to real-world applications of mathematics or to precise, logic inner-mathematics relationships with unambiguous right and wrong (application – formalism aspect, horizontal axis).

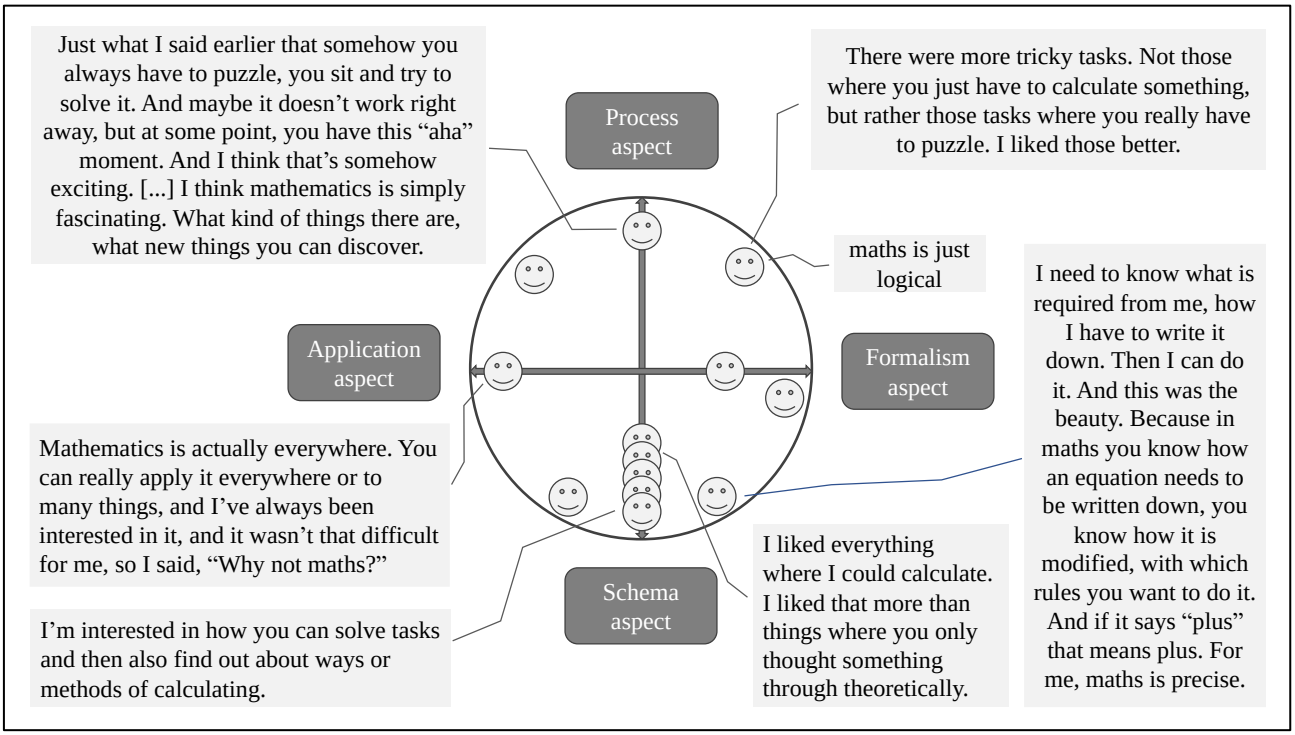

Figure 1: Illustration and example quotations of the 13 preservice teacher's mathematical world views. Vertical axis: schema – process aspect, horizontal axis: application – formalism aspect.

The close relation of preservice teachers' sense of achievement with their intrinsic values of mathematics is also reflected in the other aspects: For the process aspect, this is expressed, for example, through the "aha moment" in the quotation at the top left of Figure 1. For formalism aspect this relation is indicated by something being perceived as logical and thus comprehensible and controllable:

> I like its [mathematics] logic. If something is logical, then I find it nice. [...] when everything is so clear and you can somehow handle it really well in your head, even though it is so abstract, then I find that somehow a good feeling. I mean, when I understand it. And if you can apply it to something real, then it's even more interesting.

Interestingly, while "abstract" is positively connoted here by its reference to sense of achievement, it was often rather negatively connoted e. g., as being hard to understand in contrast to well understood procedures or to well imaginable mathematics content.

DISCUSSION

The paper gives a qualitative insight into preservice teachers' mathematics-specific motivation for choosing a mathematics teaching degree study programme. While general study and career choice motivations of preservice teachers, as investigated in other studies, were replicated, the present study additionally identifies mathematics-specific study choice motivations, such as sense of achievement, more visible success

when teaching or math teacher shortage (RQ1). All interviewees report intrinsic values of mathematics as a central study choice motivation, which was then examined in more detail (RQ2). Thereby, mathematical world views according to Grigutsch & Törner (2002) were found to be well suited for describing different mathematics-specific intrinsic values mentioned by preservice teachers.

It is noticeable how many preservice teachers enjoyed the schema aspect of mathematics. Process and formalism aspects were represented rather rarely, whereby the formalism aspect rather occurred here in the sense of mathematics being "logic" (for them), with unambiguous right and wrong, since the at the time of the interview preservice teachers did not yet experience axiomatic mathematics. Considering the characteristics of university mathematics as being abstract, formal and proof-based (Gueudet & Thomas, 2020) such findings can contribute to explaining demotivation (Gildehaus & Liebendörfer, 2021; Kosiol et al., 2019) and frustration (Göller & Gildehaus, 2021) of preservice teachers at the beginning of their studies: For many preservice teachers, the aspects they have enjoyed about mathematics in school and because of which they have chosen their study programme are likely to be less emphasised at university. Accordingly, for a positive appraisal of university mathematics, many students have to adapt their intrinsic values or develop new ones.

The found importance of the sense of achievement is likely to complicate the situation. Sense of achievement was closely linked here, e. g., with knowing what or how to calculate (schema aspect), or with experiencing something as logical (formalism aspect). Given the well-known difficulties in the transition from school to university (Gueudet & Thomas, 2020), it is to be expected that moments associated with a sense of achievement will rather decrease. Preservice teachers' mathematics-specific intrinsic values and the importance of sense of achievement should be considered when designing university mathematics courses to support students learning on as many levels as possible.

When interpreting the results, the small qualitative sample of preservice teachers from only one university must be taken into account. Accordingly, further studies are desirable that examine mathematics-specific study and career choice motivations and their significance for the further course of studies in more detail.

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SELECTING DIGITAL TECHNOLGY: A REVIEW OF TPACK INSTRUMENTS

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In this article, we present the results of a systematic literature review (n=196 articles) regarding instruments that address the digital competence of pre-service teachers. More precisely, we focus on existing instruments published during the time period of 2017 to 2022, pertaining to the TPACK framework and pre-service teachers and the skill of "selecting digital technology". The analysis shows, first, that the TPACK framework is frequently used across the world and teaching subjects for assessing digital competence. Second, three validated self-assessment instruments and two rubrics are predominantly used. The skill "selecting digital technology" is included in all, but one of them. Third, it can be conjectured that the distinction between skill and performance in assessments, less essential is as results are not reported individually.

INTRODUCTION

Digital technology (dT) and dT competence are important for educators, because of the prominence of dT in today's society (Teo et al., 2021) and because of the affordances of dT in (mathematics) education (Bakker et al., 2021). Thus, it is imperative to develop the digital competence of educators (Tabach & Trgalová, 2020); and it is important to have objective and reliable instruments to measure that competence, for example to evaluate development processes of pre-service teachers. To provide researchers with an overview of such instruments, it is our *goal* to conduct a systematic review of the recent literature. For this review, we used the following three specifications.

(1) As the frequently used TPACK (Technological Pedagogical And Content Knowledge) framework by Mishra and Koehler (2006) has become nearly synonymous with dT competence (McCulloch et al., 2021), we focus on studies specifically referring to TPACK. (2) Even though dT competence is important for both pre- and in-service teachers, we concentrate this review on pre-service teachers, as their developmental processes are distinct from in-service teachers (Tondeur et al., 2017, p. 3). (3) We also—because of the rapid development of dT and the number of dT available to educators (Handal et al., 2022, p. 200)—put an emphasis in the review on the skill of "selecting digital technology" (Gonscherowski & Rott, 2022).

With these specifications in mind, we pose the following two research questions:

RQ1: What are the current instruments, which utilize the TPACK framework, assessing the digital competence of pre-service teachers?

RQ2: How do existing instruments, that use the TPACK framework, assess the skill of pre-services teachers to "select digital technology for an instructional setting and specific learning content"?

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th* Conference *of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 387-394). PME 46.

THEORY

Digital competence entails knowledge, skills, beliefs or attitudes, and performance, or actions (OECD, 2003, p. 4). The TPACK framework by Mishra and Koehler (2006) is one of the frameworks—despite being a knowledge framework—which is used in instruments for assessing educators' digital knowledge, performance, and skills, and thus digital competence. The framework is also used in self-assessment instruments, assessing the self-efficacy of educators digital-competence, -knowledge, -performance and skill.

TPACK Framework

The framework by Mishra and Koehler (2006) has expanded the seminal work by Shulman (1986) by adding the dimension of technological knowledge (TK) to the existing dimensions of content knowledge (CK) and pedagogical knowledge (PK). Thereby creating the overlapping dimensions of technological content knowledge (TCK), technological pedagogical knowledge (TPK), and technological pedagogical content knowledge (TPCK) which are described by the authors as follows.

- "TK is knowledge about standard technologies, such as books, chalk and blackboard, and more advanced technologies, such as the Internet and digital video..." (Mishra & Koehler, 2006, pp. 1027–1028)
- "TCK is knowledge about the manner in which technology and content are reciprocally related. Although technology constrains the kinds of representations possible, newer technologies often afford newer and more varied representations and greater flexibility in navigating across these representations..." (ibid., p 1028)
- "TPK is knowledge of the existence, components, and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as the result of using particular technologies..." (ibid., p. 1028)
- "TPCK[...]is the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content..." (ibid., p. 1028-1029)

For the definition of the other dimensions we refer to Mishra and Kohler (2006, p. 1026-1027). Since its introduction, the framework has been further refined by adding the aspect of context. In addition, Mishra and Warr (2021) recently stated that more than just knowledge of tools, and instead knowledge, attitude, and skill (KAS) are required to successfully integrate technology in teaching, without providing further differentiating definitions of knowledge, skill, and attitude.

The notion of KAS is also embraced in the competence definition by Blömeke et al. (2015) suggesting that dispositions, such as cognition and affect/motivation, and actual performance are mediated by situation-specific skills. Performance is defined as actual observable in-situ behavior and situation-specific skills are the amalgamation of

interpretation and decision making. The latter does not necessarily require actual in-situ observable behavior, whereas the former does. Blömeke et al. (2015) further outline that for practical reasons a combination of assessment instruments—for example, knowledge-assessments in form of single/multiple choice, skill assessments using open-text in addition to video vignettes requiring interpretation and decision making—are needed to predict and develop real life in-situ observable behavior and, thus, competence.

METHODOLOGY

This systematic literature review extends the work of Scott (2021); following Scott's approach, we used the search terms ("tpck", "tpack", "technological pedagogical content knowledge"). Given our focus on pre-service teachers, we added specific search terms ("pre-service teachers", "teacher candidates", "preservice teachers", and/or "student teachers") as second condition. We limited the review to the time period from 2017 to October 2022, because there are existing reviews covering earlier time periods (Scott, 2021) and because of the changing nature of dT, which requires recent topicality (Teo et al., 2021; Valtonen et al., 2020). As data sources we used ERIC (n=447), eJournals (n=152) and PSYNDEX Literature with PSYNDEX Tests (n=2) within EBSCO, in addition to the dataset from Scott (2021) (n=45) and the RR/PP/OC of the PME proceedings (n=22 of 2003). We included in the review qualitative, quantitative and mix-methods studies not limited by teaching subject.

Of the qualifying records (n=668), we excluded dissertations (n=36) and records we could not retrieve (n=32). The remaining records (n=600) were reviewed; of those, records that did not include an instrument (n=194) or which solely focused on in-service teachers/teacher educators (n=146) were excluded, as were some records for other reasons (n=64). The remaining records (n=196) are the basis of the analysis.

We have grouped the studies based on the notion of KAS (Mishra & Warr, 2021) and the competence definition by Blömeke et al. (2015) as described in the previous section, into instruments which assess performances, skills, and knowledge, as well as self-assessment instruments of performances, skills, knowledge, and instruments assessing beliefs in context of TPACK.

- (P): Performance instruments, are assessments which evaluate real-life/in-situ behavior of participants.
- (S): Skill instruments, are assessments which require the interpretation of situations and involve decision making. These can be assessed by evaluating video-vignettes or text-based tasks using open-text answers or situational descriptive multiple-choice questions.
- (KN): Knowledge instruments, are assessments using single choice formats.

Whereas the above definitions describe external-assessments, the following definitions describe self-assessments.

- (SA): Self-assessment, requiring participants to assess their own level of proficiency. This can be self-assessment of skills, knowledge, performances, or competence.

- (B): Beliefs instruments, are assessments of attitude towards technology, content, curriculum, or pedagogical strategy—teacher or learner centric.

RESULTS

Overview

Figure 1 gives an overview of the records by publication date and study sample size. Each bubble represents one record, its size is proportional to the sample size of the study. Studies with a sample size greater 500 are depicted with the abbreviation for the assessment type—P, S, KN, B and SA and combination thereof. Please note that there is a time lag between the publication date and the time of the data collection in a study.

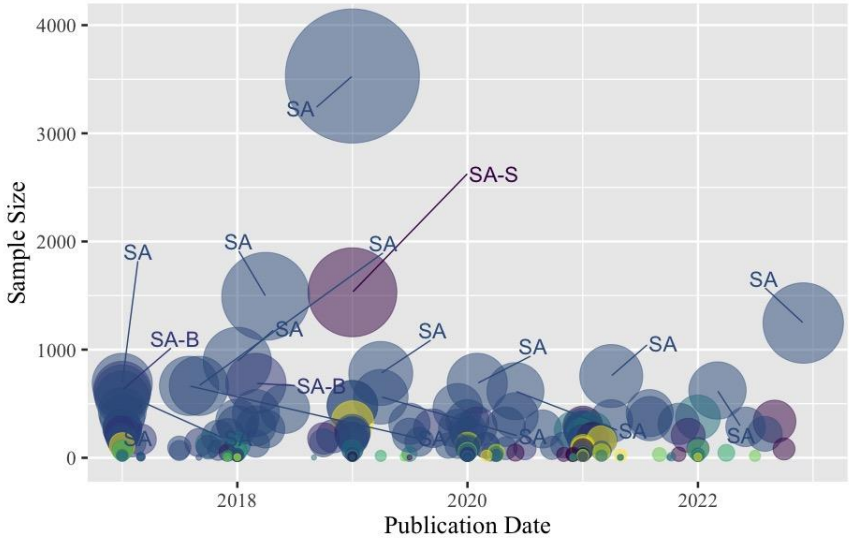

Figure 1: Distribution of the TPACK studies from 2017-2022, by sample size, publication date and self-, skill-, knowledge- and performance assessment

Following, we provide some descriptive perspectives of the dataset. Studies including a TPACK SA-instrument present the majority (n=134; 68.4%), followed by studies with S-assessment instruments (n=57; 29.1%) and studies with B-instruments (n=38; 19.3%). Studies with TPACK performance-assessments (n=21; 10.7%), and KN-assessment instruments (n=7; 3.6%) occur least frequently in the dataset. Some of the studies assessed digital S/KN/SA/P/B using multiple instruments for validation or to investigate educational variables—and thus the total number of applied instruments (n=257) is greater than the total number of records in the dataset (n=196). A slight upward trend year over year in the number of TPACK studies can be observed. Studies with SA-instruments have the largest sample-size (between 3 and 3530; median 179.5), and smaller sample sizes are seen in studies with S-assessments (between 2 and 1530; median 32), and studies with P-instruments have the smallest sample size (ranging from 1 to 74; median 18). The sample size of studies with B- (between 2 and 688; median 85.5) and with KN-instruments (between 4 and 117; median 49) are again small compared to studies with SA instruments.

The review has not been limited by the teaching subject; instruments were used in context of fourteen individual teaching subjects, whereby STEM pre-service teachers are included in more than half of the studies (n=122; 62.2%), and mathematics pre-service teachers being the largest subset (n=79; 40.3%). Although the great majority of the studies (n=102; 52.1%) do not report the school level of the pre-service teachers, instruments are used for pre-service teachers of different school levels. Pre-service teachers for secondary schools present the largest portion of the studies (n=47; 24%), followed by studies with pre-service teachers for primary schools (n=36; 18.4%), and pre-service teachers for early-childhood education (n=8; 4.1%). Instruments using the TPACK framework are implemented worldwide in the assessment of pre-service teachers, with a prominence in Turkey (n=68; 34.7%), the USA (n=26; 13.3%), and Indonesia (n=13; 6.6%). Eight of the studies are country comparison studies and one of them being categorized as world-wide, with participants from sixteen different countries. The number and the sample sizes of the TPACK studies highlight its popularity for the assessment of digital competence of pre-service teachers—especially STEM and mathematics. This is aligned with the frequent use of TPACK in the development processes of pre-service teachers (McCulloch et al., 2021).

Of the reviewed studies, the majority (n=134) used SA Likert-scale based TPACK instruments. In those studies, 39 different instruments could be identified, whereby the instrument by Schmidt et al. (2009) is the most frequently used (n=42), followed by the instruments by Kabakci Yurdakul et al. (2012) (n=14), and Sahin (2011) (n=7). Of the 39 different SA instruments only nine were developed in the review time period, and even some of those instruments refer for parts of their items, to the TPACK SA instrument by Schmidt et al. (2009). The studies with S-assessments (n=57) can be grouped by studies evaluating lesson plans (n=21), evaluating teaching artifacts (n=14), and studies using multiple-choice/open-text items (n=22). For the latter seven have published the items and the others refer to local standards or do not report their items. For the evaluation of lesson plans, the majority uses self-developed rubrics (n=12) and otherwise the rubric by Harris et al. (2010) is used multiple times (n=6) in addition to the rubric by Niess et al. (2009) (n=3). The latter two assess TPCK unidimensional on multiple performance levels. TPACK P-instruments for assessing microteaching activities and observations of pre-service teachers' interactions were used in only a few studies (n=21). These studies predominantly used research-specific rubrics (n=10). The rubric by Niess et al. (2009) and the instrument by Schmidt et al. (2009) were adapted for the assessment of TPACK performance in one study each.

Of the 196 studies, 53 used multiple instruments. The combination of SA&B presents the largest portion (n=17), and the combination of SA&S the next largest (n=15), and P&S assessments (n=9) the smallest sizeable group. The remaining studies (n=12) are various other combinations. The combination of P&S instruments, is of particular interest to us, because of the distinction of skill / performance, and competence (Blömeke et al., 2015; Tabach & Trgalová, 2020). The analysis of the nine P&S studies shows first, that four studies didn't report out different assessment results for the

evaluation of micro-teaching (P) and lesson plans (S). Second, three studies reported out separate assessment results but didn't analyze or reported out any findings in relation of the two. Third, one study reported separate assessment results for P&S and saw a positive correlation and another study had different research aims. To summarize, for the assessment of microteaching (P) and lesson plan evaluation (S) in this dataset a large portion of studies does not make a distinction between skill and performance, when assessing the digital-competence of pre-service teachers. This further supports the guidance by Blömeke et al. (2015) and Tabach and Trgalová (2020) as described in the beginning section of this paper. Instead of looking for distinct definitions of skill / performance, for the assessment of digital-competence, multiple instruments (and methodologies) in the development of pre-service teachers' digital competence should be applied.

Results regarding RQ2: How do existing instruments, that use the TPACK framework, assess the skill of pre-services teachers to "select digital technology for an instructional setting and specific learning content"?

For addressing RQ2, we focused on the most frequently used instruments and rubrics identified in RQ1. We considered all items and definitions in the instruments and

AX-Type	Instrument/Rubric	TK	TCK	TPK	TPCK
Self	Schmidt et al. (2009)	N	N	Y	Y
	Sahin, (2011)	N	N	Y	Y
	Kabakci Yurdakul et al. (2012	N	N	N	N
Skill	Harris et al. (2010)	NA	NA	NA	Y
	Niess et al. (2009)	NA	NA	NA	Y

Table 1: Instruments assessing "selecting dT" by type and TPACK dimension

rubrics which contained statements like, "I can identify" or "I can select" or "I can choose" in combination with technology/tool for this skill. Table 1 gives an overview of the instruments by assessment (AX) type. Since for performance assessments no particular rubric was used more than once, we did not include them in this overview. Table 1 shows that all but one of the frequently used SA-instruments and all of the frequently used S-assessment rubrics address the skill of "selecting dT". For the SA instruments, it is peculiar that none of them include the skill in the TCK dimension, which would address the selection of dT pertaining specific to a teaching subject (Mishra & Koehler, 2006, p. 1028). Instead, the selection of dT pertaining to learning content is part of the TPCK dimension.

CONCLUSION

Our results show first that the TPACK framework, despite being criticized for its lack of specificity, is frequently used for the assessment of digital knowledge, skills, performance, and the self-efficacy thereof across the world and teaching subjects. Second, under the label of TPACK, knowledge, skill and performance of pre-service

teachers are assessed, although TPACK nominally only addresses knowledge. Thus we can conjecture that in the research community TPACK stands synonymous for digital competence. Third, the self-efficacy of pre-service teachers TPACK knowledge/ skill/ performance is most frequently assessed using validated instruments by Kabakci Yurdakul et al. (2012); Sahin (2011) and Schmidt et al. (2009) and for S-assessments the rubrics by Harris et al. (2010) and Niess et al. (2009) next to localized and subject specific self-developed rubrics. Fourth, the skill of "selecting dT" is not consistently addressed in all the instruments, but it is included in all the major instruments and thus the majority of the studies within the dataset.

A limitation of the literature review stems inherently from the data sources used and the focus on TPACK as a framework for assessing digital competence, since there are other frameworks for assessing digital competence of pre-service teachers. A literature review and the results as presented here, are a summation and an abstraction of the complex and rich content provided in the reviewed studies. Lastly, we focused in RQ2 on the skill—selecting digital technology—which however important, is only one of the skills (pre-service) teachers require.

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REVEALING MODES OF KNOWING ABOUT DENSITY

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Rational number density has been investigated through open-ended question tasks and multiple-choice tasks or by asking to interpolate a number between two numbers. However, students' responses to these three types of tasks were not directly compared. The objective is to look for relationships between the three types of tasks in order to identify differences regarding students' density understanding depending on the type of knowledge elicited. Participants were 791 primary and secondary school students. Results show that most of the students believed that rational numbers are discrete. Differences between the modes of representation are also found. Finally, interpolating a number between two pseudo-consecutive ones is neither a necessary nor a sufficient condition for students to answer that there are infinitely many intermediate numbers.

THEORETICAL AND EMPIRICAL BACKGROUND

Understanding the density of rational numbers is considered a stumbling block for primary and secondary school students (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004), and even for undergraduates (Tirosh et al., 1999). Most of the difficulties have been attributed to the interference of the natural number based prior knowledge (Alibali & Sidney, 2015; Smith et al., 2005). While the natural number set is discrete (between two numbers there is a finite -possibly zero- number of numbers), the rational number set is dense (there is an infinite number of numbers between any two rational numbers).

Ample studies showed that the idea of discreteness is a "fundamental presupposition which constrains students' understanding of the structure of the set of rational numbers" (Vamvakoussi & Vosniadou, 2004, p. 457). Students believe that between two rational numbers there is only a finite (including zero) number of intermediate numbers. For instance, students believe that between the "pseudo-consecutive" fractions 5/7 and 6/7 there are no numbers, or that between 1/2 and 1/4 there is only one number, 1/3 (Merenluoto & Lehtinen, 2004). In decimal numbers, students think that there are no numbers between the "pseudo-consecutive" numbers 0.59 and 0.60, or that only 1.23 is between 1.22 and 1.24 (Moss & Case, 1999). Furthermore, students sometimes treat fractions and decimals numbers as unrelated sets of numbers, rather than as interchangeable representations of the same number (Khoury & Zazkis, 1994). Vamvakoussi and Vosniadou (2010) showed that some students believe that there are

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only decimals numbers between two decimals numbers and fractions between two fractions.

Previous studies by Vamvakoussi and Vosniadou have identified intermediate stages in secondary school students' density understanding. Some of these studies used open-ended question items (e.g., Vamvakoussi & Vosniadou, 2004). This type of task taps on students' available resources, at least the ones that they employ spontaneously (i.e., without any guidance). For instance, students who rely heavily on natural number knowledge might treat the given numbers as endpoints of a segment of the natural number sequence (responding that there is no other number between 2.1 and 2.2, or only three numbers between 2.1 and 2.5), or students might be able to interpolate more numbers, using transformation strategies, such as converting 2.1 and 2.2 to 2.10 and 2.20, respectively and they might or might not realize that this process is repeatable, thereby coming to understand density. It is also possible that some students have some experience with similar tasks and recall the correct answer ("there are infinitely many numbers in between"). Other studies used multiple-choice items (e.g., Vamvakoussi & Vosniadou, 2010). This type of task is typically facilitating for students, because they present the correct answer among a number of "naïve" and "less naïve" answers. This might prompt students to think of more sophisticated strategies or recall the correct answer. Finally, other studies asked students to interpolate (write) a number between two "pseudo-consecutive" numbers (e.g., Van Hoof et al., 2015). Such items require students to show whether they do have a strategy available to produce one number between two "pseudo-consecutive" numbers, if specifically asked to. Such strategies arguably constitute a possible necessary first step in the process of understanding the infinity of intermediates in an interval.

Although all these three types of tasks have been used to investigate students' understanding of density, to the best of our knowledge, no study has directly compared students' responses across tasks. This is the purpose of this study: We aimed at investigating whether the task format (open-ended or multiple-choice) makes a difference in students' responses; and whether being able to interpolate one number between two (pseudo-consecutive) ones is indeed a necessary condition for students to answer that there are infinitely many intermediates in an interval.

Identifying various ways of (not or not completely) understanding density

The current research aims at identifying differences in density understanding, by using the aforementioned variety of task types. It is part of a larger quantitative study with 953 Spanish primary and secondary school students.

The students in our sample answered a paper-and-pencil test composed of 13 items (González-Forte et al., 2022): three write items, six question items, and four multiple-choice items. In *write items* students had to write a number between two given rational numbers (between 3.49 and 3.50; 1/3 and 2/3; 1/8 and 1/9). In *question items* students had to answer the question how many numbers there are between two given rational numbers (1.42 and 1.43; 1.9 and 1.40; 2.3 and 2.6; 2/5 and 3/5; 2/5 and 4/5; 5/9 and

5/6). In *multiple-choice items* students had to answer the question how many numbers there are between two given rational numbers (3.72 and 3.73; 0.7 and 0.9; 1/3 and 2/3; 1/6 and 4/6), choosing one out of the seven answers offered, including the correct answer. Multiple-choice items were always at the end of each test since the word "infinite" appears and can help them to correctly solve the other items

Since the three types of tasks may elicit different response profiles in students, a cluster analysis per task was performed (González-Forte et al., 2022). In the write items, six profiles were identified: Students who considered that it was impossible to write a number between two pseudo-consecutive numbers (called *Naïve*). Students who considered that it was impossible to write a number between two pseudo-consecutive decimal numbers, but in fractions answered in a naïve consecutive way (i.e., 1/4 is between 1/3 and 2/3) (*Fraction consecutive*). Students who correctly wrote a number between two pseudo-consecutive decimals, and in fractions i) considered that it was impossible to write a number between two pseudo-consecutive fractions (*Correct decimals fraction naïve*); ii) answered in a naïve consecutive way (*Correct decimals fraction consecutive*); iii) correctly wrote a number between two pseudo-consecutive fractions with the same denominator, but in fractions with the same numerator, they considered that it was impossible to write a number (*Almost correct*). Students who correctly wrote a number between two pseudo-consecutive numbers (*Correct*).

In the question items, seven profiles were identified: Students who considered that there was no other number between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers (*Naïve*). Students who considered that there was no other number between two pseudo-consecutive fractions, and between two pseudo and non-pseudo-consecutive decimals i) considered that there was a finite number of numbers (*Decimal finiters*), ii) calculated the difference (*Decimal differencers*), iii) considered that there was an infinite number of numbers (*Correct decimals fraction naïve*). Students who considered that there was a finite number of numbers between two pseudo and non-pseudo-consecutive decimals and fractions (*Finiters*). Students who considered that there was an infinite number of numbers between two different fractions and two different decimals (*Correct*). Students with a generally low performance in all items who provided answers without any pattern (*Rest*).

In the multiple-choice items, nine profiles were identified: Students who considered that there were no numbers between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers (*Naïve*). Students who considered that there were no numbers between two pseudo-consecutive decimals, and a finite number of decimals between two non-pseudo-consecutive decimals, but in fractions considered that there was a finite number of fractions between two different fractions (*Decimal naïve fraction finiters*). Students who considered that there was a finite number of numbers between two different fractions and two different decimal numbers (*Finiters*). Students who considered that there was an infinite number of decimals between two different decimal numbers, and

between two different fractions i) considered that there were no numbers between two pseudo-consecutive fractions, and a finite number of fractions between two non-pseudo-consecutive fractions (*Decimal infiniters fraction naïve*), ii) considered that there was an infinite number of decimals (*Decimal infiniters*), iii) considered that there was an infinite number of fractions (*Infiniters*), iv) considered that there was an infinite number of numbers that could be represented by several different representations, such as decimals and fractions (*Decimal infiniters correct fractions*). Students who considered that between two different fractions and two different decimals there was an infinite number of numbers that can be represented by several different representations, such as decimals and fractions (*Correct*). Students with a generally low performance in all items who provided answers without any pattern (*Rest*).

RESEARCH GOAL

The three types of tasks elicit different knowledge about density: actively producing the statement that "there are infinite numbers" (*question items*), recognizing the correct answer (*multiple-choice items*) and having procedures to find an intermediate number (perhaps while knowing that there are infinitely many) (*write items*). However, so far students' responses to these three types of tasks were not directly compared. Therefore, the aim of this study is to look for relationships between the three types of tasks in order to identify differences regarding students' density understanding depending on the type of knowledge elicited.

METHOD

The final sample on which this study is based consists of 791 primary (5th and 6th grade) and secondary school students (from 7th to 10th grade). We did not include the 162 students who belonged to the "Rest" profile in question items (see above), as they performed so low in these items and their responses lacked any pattern, which made us conclude that it would be difficult and not meaningful to identify relationships with respect their performance on the other item types.

To compare among tasks, we categorized the profiles in broader categories looking for common characteristics (see Table 1). First, we categorized the profiles obtained in question and multiple-choice items looking at students' consistency in providing "infinitely many intermediates" or "a finite number, possibly zero, of intermediates" answers across items. Four categories were identified: *FIN*: students who consistently answered that there is a finite number (possibly zero) of intermediate numbers within task. *D-INF/F-FIN*: students who consistently answered that there are infinitely many intermediates between decimals, but a finite number between fractions. *FIN/INF*: students who provided "finite" and "infinite" responses within task, without following any recognizable pattern with respect to the type of the endpoints (i.e., decimals, or fractions). *INF*: students who consistently responded that there are infinitely many intermediates within task.

As far as the write items is concerned, we categorized profiles in four possible categories: *D & F correct*: to correctly interpolate a number between decimals as well

as fractions. *Only D correct*: to correctly interpolate a number only between decimals. *Only F correct*: to correctly interpolate a number only between fractions. This alternative was not present in our findings. *D & F none or incorrect*: to incorrectly interpolate a number, or not to be able to interpolate at all (by answering "impossible"), for decimals as well as for fractions.

Item	Profile	Category
Question	Naïve Decimal differencers Decimal finiters Finiters	FIN
	Correct decimals fraction naïve	D-INF/F-FIN
	Correct	INF
Multiple-choice	Naïve Decimal naïve fraction finiters Finiters	FIN
	Decimal infiniters fraction naïve	D-INF/F-FIN
	Rest	FIN/INF
	Infiniters Decimal infiniters Decimal infiniters correct fractions Correct	INF
Write	Naïve Fraction consecutive	D & F none or incorrect
	Correct decimals fraction naïve Correct decimals fraction consecutive Almost correct	Only D correct
	Correct	D & F correct

Table 1: Broader categories identified

RESULTS

Table 2 presents the four general groups identified.

Table 2: Summary of the four general groups identified

Item		Group 1	Group 2	Group 3	Group 4
Question		FIN	FIN	D-INF, F-FIN	INF
Multiple-choice		FIN	FIN/INF	FIN/INF	INF
Write	D & F correct	4 (1.62%)	8 (2.95%)	2 (2.56%)	56 (28.72%)
	Only D correct	72 (29.15%)	127 (46.86%)	71 (91.03%)	131 (67.18%)
	D & F none or incorrect	171 (69.23%)	136 (50.18%)	5 (6.41%)	8 (4.10%)
	Total	247	271	78	195

Group 1: Naïve (*n* = 247, 31.2%): Students who provided FIN answers consistently across multiple-choice and question items. The great majority of students in this group could not interpolate a number, neither between decimals nor between fractions. About one third of the students correctly interpolated a number between decimals; and four students correctly interpolated a number between decimals and fractions.

Group 2: Question items-Finiters (*n* = 271, 34.3%): Students who consistently provided FIN answers in question items and provided some "infinitely many intermediates" answers in some, but not all multiple-choice items. This group was better in interpolating (at least between decimals) than Group 1 students – still half of them were not able to interpolate neither between decimals, nor between fractions.

Group 3: Question items-Decimal Infiniters (*n* = 78, 9.9%): Students who gave INF answers for decimals but FIN answers for fractions in question items. Some of these students retained this response pattern in multiple-choice items, while others provided a mixture of FIN and INF answers. The great majority of Group 3 students were able to interpolate a number between decimals, but not between fractions.

Group 4: Advanced (*n* = 195, 24.6%): Students who provided INF answers in question items, and the majority of students (*n* = 178) also gave INF answers in all multiple-choice items. Within this (advanced) group, there are two interesting subgroups depending on how they solved the write items:

- **Group 4.1: Write items-Infiniters** (*n* = 56): Students who provided correct answers both in question and write items. These students answered that there are infinitely many numbers between two different fractions and two different decimal numbers and were able to correctly interpolate a number between them.
- **Group 4.2: Write items-Decimal infiniters** (*n* = 139): Students who provided correct answers in question items, but in write items, the majority (*n* = 131) could interpolate a number only between decimals. There are 8 students who, despite answering consistently that there are "infinitely many intermediates" in question as well as multiple-choice items, answered "impossible" in the write items.

DISCUSSION AND CONCLUSIONS

Comparing students' responses across three different tasks has provided some interesting findings beyond those found through assigning students to different profiles per task (González-Forte et al., 2022). Firstly, we could induce the existence of four different groups of students according to the way they have answered the three different tasks. A first major finding when looking at these groups' characteristics, is that the great majority of students consistently provided FIN answers in question items (Groups 1 and 2: n = 518). That is, when being asked in an open question how many numbers there are between two different fractions or decimals numbers, most of the primary and secondary school students answered that there is a finite (possibly zero) number of intermediates. There was a big group of students (Group 1, n = 247) who retained this response pattern also in the multiple-choice items, indicating that they were confident enough with their response to ignore the presence of more sophisticated options. A somewhat larger group (Group 2, *n* = 271) provided (some) "infinitely many intermediates" answers in the multiple-choice task, indicating that they recognized or recalled the correct answer. Group 1 and Group 2 students performed generally poorly in the write task. Still about one third and half of the students in Group 1 and 2, respectively, were able to interpolate a number between decimals, apparently without realizing that this process is repeatable, which could give them the insight about density.

It should be noted that being more competent with interpolating a number between decimals than between fractions was present across groups. This is clearly evident for Group 3 (*n* = 78). Interestingly, these students consistently answered "infinitely many intermediates" for decimals, but "a finite number or intermediates" for fractions in question items. Half of them persisted in answering the same in the multiple-choice task, despite having the correct answer as an option. It is possible that some students of Group 3 had realized that the process of interpolation can be repeated ad infinitum for decimals. In any case, these students had not realized that decimals and fractions are different representations of the same numbers, rather than different numbers (Vamvakoussi & Vosniadou, 2010).

Finally, students in Group 4 (*n* = 195) consistently answered that "there are infinitely many intermediates" in question and multiple-choice tasks and they also performed better in the write tasks, compared to all previous groups. Still, only a subgroup of these students (Group 4.1, *n* = 56) was able to interpolate a number between decimals as well as fractions.

It thus appears that the competence to interpolate one number between two given (pseudo-consecutive) ones is neither a necessary nor a sufficient condition for students to answer that there are infinitely many intermediate numbers (or choose this answer). The latter is not surprising, since interpolating one number does not lead to the realization that there are, in fact, infinitely many numbers, unless one also realizes that this process is repeatable. The interesting finding is that students may provide

"infinitely many" answers and still not have the necessary competencies to produce intermediate numbers. This could indicate that they merely recall (in open-ended tasks) or recognize (in multiple-choice tasks) the correct answer. Therefore, we should not overinterpret what learners really understand about density if they respond correctly to multiple-choice items or question items (items often used in previous research), as in other types of items that elicit other knowledge, they still show a lack of understanding.

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"THIS IS CLEARLY INCORRECT, WHY DOES IT WORK?": ON DIVISION OF FRACTIONS AND CONTINGENCY

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This paper reports a breaching experiment conducted to examine a teacher's spontaneous response to a hypothetical student's non-traditional solution strategy for dividing fractions. The student's work was presented to the teacher during a semi-structured interview. The teacher's response was analysed by drawing on components of the Knowledge Quartet. The findings suggest additional refinement of the framework in considering correct but unexpected solutions.

INTRODUCTION

Mathematical problems can be solved in different ways and curriculum documents often encourage teachers to promote students' use of multiple strategies. However, it is unclear how teachers follow this guidance when student-generated methods are unfamiliar to them. Researchers (e.g., Son & Crespo, 2009) highlight the importance of investigating this question. We continue this line of investigation by exploring how teachers respond to a student's alternative algorithm for dividing fractions.

There are various ways to carry out division by a fraction. In a common algorithm, division is replaced by multiplication and the divisor is replaced by its reciprocal. This strategy is often introduced to learners without a proper explanation (Borko et al., 1992) despite the fact that several justifications exist (see Son & Crespo, 2009, for a partial list). However, even when an explanation is provided, what remains in one's memory is the "invert and multiply" strategy, not the reason or explanation for it. As a result, this strategy is often perceived by learners only as a rule that must be followed.

However, division of fractions can also be carried out by dividing numerators and denominators separately, that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

This alternative method for division of fractions (AMDF), while easily verified as correct computationally and algebraically, often leaves people with a sense of disbelief. When the AMDF was first presented to prospective teachers, a majority of respondents incorrectly considered it to be an error (Tirosh, 2000). Even when this misconception was addressed via instructional interventions, participants still expressed a strong preference towards the "standard" algorithm. Further, Son and Crespo (2009) invited prospective teachers to respond to a scenario related to teaching division of fractions. In this scenario, the AMDF was presented by a student, but then rejected by another student on the grounds that it was not the way "we are supposed to divide fractions". The authors investigated how "teachers reason and respond to a particular student's

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non-traditional strategy for division of fractions," (p. 243) by analysing teachers' written responses related to the validity and generality of the AMDF.

We agree that responding to a teaching-scenario task might simulate "how mathematical work arises in the context of teaching" (Son & Crespo, 2009, p. 243). However, we anticipated that the teachers' written responses to prompts may not have captured elements of the teachers' spontaneous reaction. Therefore, we introduced the AMDF to an in-service teacher during an interview to elicit the teacher's spontaneous response to a student work.

THEORETICAL FRAMING: KNOWLEDGE QUARTET

Our analysis was guided by the Knowledge Quartet framework (Rowland et al., 2005) that categorizes the ways a teacher brings mathematical subject knowledge into play. The Knowledge Quartet (KQ) includes four elements: foundation, transformation, connection, and contingency. Foundational knowledge refers to a teacher's knowledge-in-possession, while the other elements refer to a teacher's knowledge-in-action—the specific ways in which teachers may leverage their foundational knowledge to prepare for and conduct teaching. Transformation and connection are a teachers' planned actions, contemplated prior to the lesson. In contrast, contingency refers to a teachers' spontaneous actions in response to an unexpected situation that concerns the teaching and learning of mathematics. These latter three elements of the KQ are especially relevant to our work and are elaborated upon below.

Contingency could be triggered by either students or by the teachers themselves (Rowland et al.,2015). Students' unexpected responses to a mathematical question or activity could require spontaneous teacher actions. Teachers might also deviate from their lesson plans when they realize that something in their instruction requires a change, for example, when they realize that a student's response that they initially evaluated as correct is actually wrong. According to Rowland et al. (2015), teachers may respond to an unexpected student response in three ways: (1) by ignoring the student response or by dismissing it as "wrong" immediately; (2) by attending to the student's work, but doing so in such a way that no clear conclusion about its validity can be made; (3) by integrating the student's idea into the main mathematical activity so that other students can also incorporate that idea into their mathematical knowledge.

Connection is revealed in teachers' practice when they aim for coherence within a lesson or across a series of lessons. Connection is demonstrated through linking different meanings, descriptions, and representations of mathematical concepts or procedures; anticipating the cognitive demand, complexity, or appropriateness of a concept; and ordering tasks and exercises to sequence instructions within and between lessons.

Transformation concerns the ways teachers make their own foundational knowledge accessible to students. This is often accomplished through explaining concepts, demonstrating procedures, confronting and resolving misconceptions, justifying or refuting mathematical conjectures, and engaging students with exercise/practice.

Transformation is revealed in teachers' deliberate choice and use of examples, analogies, and/or representations.

While the elements of the KQ are described separately, a specific teacher action may relate to more than one element. For example, while responding to a contingent event, a teacher can make a connection between the student's ideas and the previous topics. Considering this relationship, our investigation addressed the following research questions: *In a teacher's encounter with an unfamiliar algorithm, what triggers contingency? How does the teacher respond?*

METHODS

Breaching Experiment

A breaching experiment (Garfinkel & Sacks, 1986) is an ethnomethodological approach for making explicit the underlying norms of everyday social settings. In a breaching experiment, an individual engages in behaviors contrary to the anticipated norms of a particular setting in order to observe and analyse the reactions of affected social actors.

In teacher education, this method has been used to breach proving norms (Herbst et al., 2016) and solution methods of a linear equation (Chazan et al., 2018). Our study is a breaching experiment in that the AMDF is not a normative algorithm for solving fraction division. By presenting the AMDF to a teacher as part of a hypothetical students' work, we aimed to explore the teacher's spontaneous response to the unfamiliar non-traditional student strategy.

Participant and Interview Protocol

As part of a larger study, in this report we focus on a semi-structured interview with Valeria (pseudonym), an elementary school teacher with over five years of teaching experience. During the interview, the teacher was shown the same student's work at two different points in time. First, the student was depicted using only the AMDF to solve the 2nd, 5th and 6th problems (Fig. 1a). Valeria was asked how she would respond to this work, after teaching multiplication and division of fractions. Then, the student was shown using the standard algorithm on problem 1 (Fig. 1b). Valeria was asked again to explain her response to the student's work. Our aim in presenting these two situations was to explore how a teacher responds to the same student's work when additional information becomes available. Initially it is unclear whether the student knows the standard algorithm. Then student is depicted as knowing both the standard and the alternative algorithm.

Dividing Fractions	Dividing Fractions
1) $\frac{1}{2} \div \frac{8}{9} =$	1) $\frac{1}{2} \div \frac{8}{9} = \frac{1}{2} \times \frac{9}{8} = \frac{1 \times 9}{2 \times 8} = \frac{9}{16}$
2) $\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$	2) $\frac{15}{32} \div \frac{3}{4} = \frac{15 \div 3}{32 \div 4} = \frac{5}{8}$
3) $\frac{4}{6} \div \frac{2}{9} =$	3) $\frac{4}{6} \div \frac{2}{9} =$
4) $\frac{3}{4} \div \frac{11}{12} =$	4) $\frac{3}{4} \div \frac{11}{12} =$
5) $\frac{8}{9} \div \frac{2}{3} = \frac{8 \div 2}{9 \div 3} = \frac{4}{3}$	5) $\frac{8}{9} \div \frac{2}{3} = \frac{8 \div 2}{9 \div 3} = \frac{4}{3}$
6) $\frac{1}{9} \div \frac{1}{3} = \frac{1 \div 1}{9 \div 3} = \frac{1}{3}$	6) $\frac{1}{9} \div \frac{1}{3} = \frac{1 \div 1}{9 \div 3} = \frac{1}{3}$
(a)	(b)

Figure 1: Pages from a hypothetical student's notebook

Data Analysis

We read the transcribed interview several times, separately coded the teacher's responses by drawing on the elements of the KQ, and compared our codes until full agreement was reached.

Based on Rowland et al. (2015), we identified contingency as an unsettling event preventing a teacher from "making a surefooted and confident response at a time" (p. 76). To identify contingency, we focused on Valeria's explicit and implicit references to her experience of an unpredictable incident. For example, her statements that included words like "surprise" or that referred to her unfamiliarity were identified as evidence of a contingent situation. Valeria's evaluation of the algorithm first as wrong and then as right, or her reflections on her own thinking about fraction division, as well as long pauses and certain exclamations, also helped us identify Valeria's experience of contingency.

FINDINGS

This section presents the shift in the teacher's responses to a hypothetical student's work. We first highlight the trigger of contingency and then explicate how the transformation and connection dimensions of the KQ are revealed in the teachers' response to the contingent event.

At the beginning of the interview, Valeria interprets the student's work (Fig. 1a) as a misconception related to the meaning of fraction. According to her, the fact that "they just basically divide the top number by the top number and then bottom number by the

bottom number" is evidence that the student perceives fractions as two separate whole numbers which do not have a relationship. In response, Valeria plans on reteaching the meaning of fraction rather than focusing on division:

> I feel that, in this case, I would feel it would be too soon for that student, developmentally, to learn about division of fractions. Let alone – I feel like they're not even ready to do operations with fractions of any kind. Because they lack conceptual understanding of what a fraction is. So, I would go to the very beginning, and I would try to distinguish between a whole number and I would try to – just, explaining to students what fraction is, is already quite difficult.

In this episode, we did not identify any unsettling incident experienced by the teacher upon encountering the student's work. The teacher seems to be surefooted in her interpretation of the student's work and her choice of appropriate teacher moves. Therefore, this initial response—disregarding the student's algorithm as a product of misconception—did not reveal contingency.

After seeing the subsequent student work (Fig. 1b) Valeria initially attributes the student's work to the memorization of the standard algorithm and its application without understanding. Immediately after, her account indicates an insight about a different interpretation of the student work that she quickly disregards:

1	V:	Although... now that I'm thinking. Maybe actually it is the other way around. Maybe... [long pause]. Um. Nope. I think that's pretty much what it is. They've memorized the algorithm and they are good with multiplying whole numbers, so that's what they did.

After the interviewer prompts her to unpack her insight and solve the second division problem by using the standard algorithm, she concludes that the AMDF provides a correct result:

2	I:	You also said that now I'm thinking that maybe this is the way around. What were you thinking when you said that? What made you think...?
3	V:	For a quick second I wanted to double check that, is it possible that this student already knows the invert and multiply algorithm and just is applying it skipping a step?
4	I:	Can you show it?
		[…]
5	V:	That would be 60, and 32 times 3…Oh, did I say sixty...? [mumbling, long pause] Oh! Oh, what's going on here? [pause] I don't understand. This is clearly incorrect, why does it work? [pause]
6	I:	Is there a relationship between this one [the AMDF] and this one [the standard algorithm] here?
7	V:	Well, judging by the results—yes! […] if I invert and multiply here that would be 15 times 4. So I just uninvert? Is that's what's happening?

While examining the student's work, Valeria suspects that there is a relationship between the AMDF and the standard algorithm. She discovers that the AMDF, which she initially interpreted as a misconception, actually "works". This discovery puzzles

her. The long pauses, the questions she asks and her exclamatory reactions [line 5 and 7] suggest that Valeria experiences an unexpected and unsettling incident. The presented algorithm challenges her knowledge about fraction division, triggering a situation of contingency. The trigger seems to be the teacher's awareness that something she thought was wrong is actually correct.

This new algorithm also challenges her previous response to the student work. Valeria continues by questioning her conventional approach.

> If I saw a kid trying to do this, I would stop them! I would say, wait wait wait wait wait. You don't want to do that, you're just dividing whole numbers. You need to divide the whole fraction. But then again, it works! So why would I want to stop them? Why would I want to teach them an extra step that is completely unnecessary, that utilizes a different operation that I'm not even interested in teaching? I want to teach division, not multiplication right now. So why would I want to distract them from the division, into multiplication, when division works!? AUGH.

The questions in this excerpt indicate Valeria's uncertainty about how to respond to the AMDF. In her reflections, she first considers rejecting the student's response and teaching the standard algorithm. Immediately after, she problematizes her traditional approach to fraction division. This lack of confidence seems to be related to the coherence of her instruction. For her, the focus should be only on division while teaching division. Therefore, introducing another operation, multiplication, would be not only unnecessary but also inappropriate as it may distract the student's attention from the focus of the instruction. Thus, her account reveals a recognition of conceptual appropriateness which is an element of connection in the KQ.

After questioning the need for the standard algorithm, Valeria solves problem 1 using the AMDF. In doing so, Valeria confirms that the AMDF produces correct solutions. She also notices that, when the numerators or denominators are not divisible by each other, applying the AMDF can be more difficult than the standard algorithm. Then the interviewer asks Valeria what she would think if she saw a student using both methods, as in Fig. 1b. Her response indicates her intention to incorporate the student's response into her lesson and to dismiss the standard algorithm:

> I want to throw away the invert and multiply algorithm. Why do we need it at all? …If there is a reason of the way and the algorithm is just to save time, at the expense of understanding, why do we need it? Instead of giving them a worksheet of 20 questions, just work with this one [the first problem]. […] There's a whole lot of patterning happening here. I could probably spend an hour trying to figure out why this is so and this is that and where the connections and where the relationships and it would be a worthwhile lesson just to do that. Look at the relationships, look at the connections. Why is this, why is that and who needs the algorithm.

Valeria's response to the unexpected student work reveals two elements of the KQ: transformation and connection. We identified the element of connection in Valeria's attempt to prompt students to explore the relationship between the standard algorithm and the AMDF. We identified the element of transformation in Valeria's rethinking of

her instruction by changing the amount, content, and purpose of the examples she would introduce to students. Instead of assigning numerous division questions to students through a worksheet as individual practice, she intends to incorporate one specific question into the classroom discussion to help students relate the two algorithms to each other. Through this connection, she seems to justify the conjecture she poses [lines 3 and 7]: the standard algorithm and the AMDF are the same, the former showing extra steps. At the end, she refers back to her question about the need for using the standard algorithm indicating its dismissal once more.

DISCUSSION, CONTRIBUTIONS AND CONCLUSIONS

The findings demonstrate that Valeria's realization that the AMDF is correct triggered *contingency*. Then, her response to the contingent event revealed *transformation* and *connection*. Even though at the time of the interview Valeria did not explicitly explain the relationship between the standard algorithm and the AMDF, the specific ways she planned to integrate the AMDF into her teaching have potential to engage students with a meaningful mathematical activity which may eventually result in the discovery of the mathematical relationship.

This study contributes to the literature in three ways. Firstly, we expand the applicability of the KQ. Rowland et al. (2005, 2015) used this framework to analyse classroom events which unfold based on teachers' prior planning. We used it in the analysis of a teacher's potential teaching which spontaneously emerged as she described it an interview. Secondly, we expand Rowland et al.'s (2015) categorizations of triggers of contingency. Thirdly, we provide a further refinement related to teachers' responses to unexpected incidents. The last two contributions address the remark made in Rowland et al. (2015) that the provided categorization is not complete and thus has potential to be expanded.

Our findings show that Valeria's awareness that *a student's work is actually correct* unsettled her, challenged her previous response to the AMDF and triggered contingency. This type of trigger is different from Rowland et al. (2015), where the contingency resulted from teachers' awareness that a student's work was actually wrong. However, if Valeria had not seen the student work in Fig 1b, she may have not had such an insight. Therefore, we argue that Valeria's unfamiliarity with the AMDF, the extent of information she can access about student thinking, and this new type of teacher insight all together played a role in triggering contingency.

Valeria's realization of her initial misevaluation of the algorithm might be interpreted as the teacher's identification of a mistake about her teaching as in Rowland et al. (2015). However, we argue that it is important to distinguish the types of mistakes that trigger contingency. In Rowland et al. (2015), the teacher's mistake was agreeing with a mathematically wrong statement. In our findings, the mistake was disagreeing with a mathematically correct statement.

Valeria's specific response to the contingent event also illustrates a nuance in Rowland et al.'s (2015) categorization of teacher responses which included ignoring, putting

aside, and incorporating. Similar to the move of incorporating, Valeria considered integrating the unexpected student answer into her instruction; yet in doing so, she also dismissed teaching the standard algorithm. This finding is in discord with Tirosh (2000), which showed pre-service teachers' tendency to prefer the standard algorithm over the AMDF. This difference may be explained by the fact that teachers' professional obligations might influence their instructional decisions (Herbst et al., 2016), but this requires further investigation. Future research will focus on the role of teachers' knowledge and professional obligations in their responses to unexpected student work. We believe that the categorization of teachers' responses to student-triggered contingent events has potential to expand further according to the teachers' professional positions.

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INVESTIGATING THE ROBUSTNESS OF INTUITIVE CONCEPTIONS AMONG ADULTS AND TEACHERS THROUGH PRODUCTION TASKS

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INTRODUCTION

Intuitive conceptions refer to knowledge that is acquired with little or no effort, no teaching and which is not necessarily aligned with the culturally and scientifically accepted notions (Carey, 2000; Fischbein, 1987; Shtulman, 2017; Vosniadou, 2017). They can be entry points for acquiring new knowledge, but they also act as constraints that impose their entailments in newly encountered situations leading to errors (Hofstadter & Sander, 2013). In the current study, we explore how robust intuitive conceptions are, among adults and high-school teachers. By extensively documenting, through all four arithmetic operations, the scope to which intuitive conceptions are influential among the adult and teacher population valuable information can be provided for developing teacher training programs and school interventions that foster conceptual change among students.

Intuitive conceptions of elementary arithmetic operations

Even though mathematics constitutes a coherent body of scientific knowledge, it remains an essentially human activity in which formal mathematics can be distinguished from intuitive mathematics (Fischbein, 1993): some tacit models substitute for a complex, abstract notion, imposing their properties and constraints (Fischbein, 1989). Namely, when there is a conflict between intuitive and formal components, an epistemological obstacle emerges leading to misconceptions and systematic mistakes (Hofstadter & Sander, 2013; Shtulman, 2017).

Both Fischbein (1989) in his theory of tacit models and Lakoff and Núñez (2000) in their conceptual metaphor approach identify the prominent intuitive conception of subtraction as *taking away*. This can hinder young students when solving a problem such as '4 + ? = 17', because they will perceive it as a situation in which one can find the answer by looking for a missing addend and will not use a subtraction (De Corte & Verschaffel, 1996; Selter, et al., 2012; Usiskin, 2008; Brissiaud & Sander, 2010).

As for addition, the grounding metaphors of arithmetic proposed by Lakoff and Núñez (2000) entail an intuitive conception of addition as *putting together*. Yet, this conception of addition does not lead to appropriate inferences in all additive situations (Sophian, 2008; Usiskin, 2008). For instance, the word problems such as "Joe had some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?", whose answer should be found by using an addition, is particularly hard for students at the beginning of primary school. Indeed,

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when manipulating objects, it is only 28% of first grade students who succeed in solving such a problem (Riley et al., 1983)

The intuitive conception of multiplication as a *repeated addition* (Fischbein, 1987). This entails that intuitively multiplication is considered to 'make bigger'. Yet, when one multiplies with a decimal number smaller than one, then multiplication actually makes the initial quantity smaller. If one therefore sticks to the intuitive conception, multiplication involving a decimal smaller than one or a negative number has no intuitive meaning. It also constrains the possibility of multiplication to be understood as a commutative operation (Bell et al., 1981; Fischbein et al. 1985).

As for division, Fischbein et al. (1985) identified *sharing* as an intuitive conception, which considers that division splits an object into equal parts and what needs to be found is the size of each part. This partitive conception of division sees division as 'making smaller' since each fragment would be smaller than the initial quantity. If dividing is assimilated with partitioning, it entails that the divisor must be smaller than the dividend and must be a whole number, and that the result must be smaller than the dividend. However, the latter is not the case if the divisor is a positive number smaller than one.

Persistence of intuitive conceptions

Educational interventions should strive to take intuitive conceptions into account and achieve a conceptual change (Vosniadou, 2017). Interestingly, they continue to be influential across various domains among adults (Shtulman & Valcarel, 2012). When it comes to arithmetic operations, even adults can be influenced by the inferences entailed by the intuitive conceptions.

For instance, in one study, adults were presented with mathematical statements and their outcomes, either true (*e.g.* "$5+2x$ can be greater than 5"), or false (*e.g.* "$1+10t$ is always greater than 1") (Vamvakoussi et al., 2013). What was manipulated experimentally was the congruency of the response with the assumed intuitive conception. The statement "$5+2x$ can be greater than 5" was considered congruent since the response based on the intuition that *addition makes bigger* would lead to a correct judgment of the statement being true, while the statement "$1+10t$ is always greater than 1" was considered incongruent since the same intuition would lead to an incorrect judgment of the statement being true. The results showed that adults made fewer correct judgments on incongruent statements (75%-81%) than congruent statements (98%-99%), but also among the correct responses, participants took longer to make this correct judgment.

Another study presented pre-service teachers with quotative and partitive division word problems (Tirosh & Graeber, 1991). Half of the problems contained numerical values that respected the constraints imposed by the intuitive conception. The other half of the problems violated the constraint. Additionally, participants were presented with four numerical division statements (e.g. $6 \div 3$), half of which violated the constraints of the intuitive conception (e.g. $2 \div 6$). On average, pre-service teachers

had significantly higher performance when solving problems congruent with the intuitive conception (80.62%) than problems incongruent with the intuitive conception (56.5%), and overall they also had significantly more success on problems whose numerical values did not challenge the intuitive models (85.87%) than on problems that violated the constraint (51%). What is particularly concerning is that intuitive conceptions impact even teachers' diagnostic competence (Gvozdic & Sander, 2018). Indeed, when teachers are asked to explain the strategies students use to solve subtraction word problems, they can overlook certain empirically assessed difficulties on problems congruent with the intuitive conception.

Current study

The current study was conducted in order to provide insight into the extent to which intuitive conceptions prevail among adults and high-school teachers, including mathematics teachers. We propose that the usual performance tasks involving reaction times and solving procedures do not capture how robust intuitive conceptions are among the adult population. Indeed, production tasks have been used in categorization research as a way for participants to freely express their views and to the most prevalent conceptions (Rosch, 1978). They are gaining interest as a more ecological way of examining the conceptions spontaneously solicited by participants (Raynal, 2020; Dunbar & Blanchette, 2001). This therefore calls for a new methodological approach to capture the prevalence of intuitive conceptions long after the school notions have been acquired.

We studied the intuitive conceptions of the four elementary arithmetic operations, since they are among the first and crucial school notions taught in mathematics instruction. To create the conditions that would be suitable for assessing the robust persistence of intuitive conceptions, we gave participants problem posing tasks. Both problem solving and problem posing are widely used activities that promote conceptual understanding in students (Cai, et al. 2015). However, problem posing is an activity that requires solvers to go beyond problem solving procedures and reflect on the deeper conceptual structure on the goal of the task (Cai, J. et al. 2016). Furthermore, the semantic characteristics of word problems make it possible to determine if the problems posed by the participants are congruent or incongruent with the intuitive conception of the target notion.

We expect that when participants are asked to make up a word problem, for each arithmetic operation, there will be one dominant semantic category of problems that are posed, the one congruent with the intuitive conception. Furthermore, we expect that when participants are explicitly asked to transgress the entailments of the intuitive conception, participants will struggle to create problems because that will be incongruent with their intuitive conception.

METHOD

Participants. A total of 356 participants (mean age = 33.21, SD = 8.13, 204 female) took part in the study: 131 bachelor students and 225 high-school teachers. 57 of the

high-school teachers have already taught math in high-school. Participants were recruited from university classes in educational sciences as part of course credit.

Materials. The study contained two tasks. A **free problem posing task** consisted of 4 items, one regarding each arithmetic operation. The items consisted of an instruction for problem posing. For example, for subtraction the item stated: "Come up with a subtraction word problem that can be solved using the operation 8-5=3.". The numerical values were controlled and counterbalanced.

The second task was a **problem posing under constraint task** consisting of 4 items as well, one regarding each arithmetic operation. On each item, we first asked participants if it was possible to come up with an arithmetic word problem that is incongruent with the entailments based on the intuitive conception of the arithmetic operation:

- "Is it possible to come up with a subtraction word problem statement whose solution is found using a one-step subtraction and THAT DESCRIBES AN INCRESE/WIN/GAIN?"
- "Is it possible to come up with an addition word problem statement whose solution is obtained by one and only one addition operation and THAT DESCRIBES A LOSS?"
- "Is it possible to come up with a multiplication word problem whose solution is found using a one-step multiplication and WHICH MAKES IT SMALLER?"
- "Is it possible to come up with a division word problem whose solution is found using a one-step division and THAT MAKES IT BIGGER?"

If the participants responded that it was possible to pose such a problem, they were asked to come up with it, and if not, to justify their answer.

Coding. We first categorized the problems into semantic categories. For subtraction and addition, the problems were categorized into the widely used typology introduced by Riley et al. (1983). For multiplication, the word problems were categorized into those which depicted elements that solicit a repeated addition or those that solicit a product of the elements. For division, the problems were categorized into partitive or quotative problems. The second author proceeded with the coding of the data. A quarter of the data was then randomly selected and double-coded by the first author. The inter-rater reliability reached a perfect level of agreement.

We then coded if the semantic category was congruent or incongruent with the intuitive conception. For subtraction, the change problem with the unknown quantity bearing on the remainder (Change 2) was coded as congruent with the intuitive conception. For addition, Combine 1 and Change 1 problems were coded as congruent. For multiplication it was repeated addition that was coded as congruent with the intuitive conception. For division, partitive division was coded as congruent with the intuitive conception. The rest of the categories were coded as incongruent.

Procedure. The participants took part in the study online, on the Qualtrics platform. In the first task, each participant saw all four items with only one type of numerical value. After finishing the first task, the participants passed on to the second task. Items in both tasks were presented sequentially, in a random order.

RESULTS

Task 1: free problem posing. Table 1 presents the distribution of the overall coding of the results. Among the problems that it was possible to categorize based on the established coding, the results of the responses to the free problem posing task revealed that, 88.93% were problems congruent with the intuitive conception of the arithmetic operation. A Khi² comparison confirmed that the difference between congruent and incongruent problems was significant ($\chi 2(1, N = 356) = 739.75$, $p < .01$). The results in which participants' responses were not categorized into congruent vs. incongruent were not included in the further analysis.

Table 1: Overall performance on Task 1

Response coding	Absolute distribution
Congruent	76.19%
Incongruent	9.48%
No response	0.42%
Non-word problem	10.46%
Did not respect the instruction	3.44%

When looking at the performance among both populations, the same pattern of responses was observed (Table 2). The difference between congruent and incongruent word problems was significant both among bachelor student ($\chi 2(1, N = 131) = 314.56$, $p < .01$) and high-school teachers ($\chi 2(1, N = 225) = 425.58$, $p < .01$). While there were no significant differences between the two populations in their overall responses ($\chi 2(1, N = 356) = 425.58$, $p > .01$). Even among the math teachers, 89.34% of them proposed a problem congruent with the intuitive conception ($\chi 2(1, N = 57) = 121.95$, $p < .01$).

Table 2: Performance on Task 1 per participant groups.

	Congruent	Incongruent
Bachelor students	90.02%	9.98%
High-school teachers	88.20%	11.80%

We further explored the performance by taking the four arithmetic operations into consideration (Table 3). The difference between congruent and incongruent problems was significant when taking into account all four arithmetic operations ($\chi 2(3, N = 356) = 88.66$, $p < 0.01$).

Table 3: Performance on Task 1 per arithmetic operation.

	Congruent	Incongruent
Addition	98.03%	1.97%
Subtraction	76.45%	23.55%
Multiplication	95.11%	4.89%
Division	86.29%	13.71%

Task 2: problem posing with constraint.

We first looked at the distribution of participants' responses whether or not it is possible to make up a word problem which respects the given constraints, incongruent with the arithmetic operation in question. Four participants did not respond to the second task. Overall, 67.47% of the participants responded initially that it was possible to pose such a problem. However, the problems subsequently posed by the participants were coded following the same coding schema as in the first task. This revealed that it was only 34.63% of the time that participants did actually succeed to pose a problem compatible with the constraints and incongruent with the intuitive conception. Table 4 provides an overview of the distribution of the overall performance on the second task.

Table 4: Overall performance on Task 2.

Responded "No"	32.53%
Responded "Yes" but still posed a problem congruent with the intuitive conception	13.07%
Responded "Yes" but did not pose a word problem	31.04%
Responded Yes and posed a problem that respected the rules (incongruent with the intuitive conception)	23.37%

The results reveal that it was only 23.37% of the time that participants were able to actually propose a problem incongruent with the intuitive conception even when explicitly requested to do so.

Among bachelor students 25.97% manage to succeed to do so, and 21.86% of the high-school teachers (Table 5), with no significant difference among the two populations ($\chi 21$, N = 350) = 88.66, $p > 0.05$). Among the math teachers, it was only 31.28% who proposed a problem incongruent with the intuitive conception on the second task after responding that it was indeed possible.

Table 5: Performance on Task 2 after coding the posed problems

	Congruent	Incongruent
Bachelor students	25.97%	74.03%
High-school teachers	21.86%	78.14%

DISCUSSION

Findings from the current study revealed that intuitive conceptions of elementary arithmetic operations remain very robust late after instruction, among university students (about half of which are even future elementary school teachers), high-school teachers and even mathematics teachers. Previous findings using different investigation methods have suggested that intuitive conceptions persist in adults, but just how widespread they are has not been documented. The vast majority spontaneously of the participants posed problems congruent with intuitive conceptions. Moreover, the entailments of intuitive conceptions impaired the ability to find examples of situations that transgress these implicit rules. Participants were even

explicitly given the conditions to think of examples that are incongruent with the intuitive conception, the vast majority did not manage to succeed. These findings were consistently observed among all four arithmetic operations.

A particularly striking finding is the prevalence of intuitive conceptions even among high-school mathematics teachers. Infact, expertise is considered to make it possible for experts to rely on abstract principles or "deep structures" (Chi, Feltovich, & Glaser, 1981), this should aid teachers expert in the subject-matter to provide more diverse examples on problem posing tasks, especially when there are asked to provide incongruent problems. Our study of course does not question the mathematics expertise of the teachers, but it does go to show the persistence and robustness of intuitive conceptions even among experts. Furthermore, our findings inform lines of research which are interested in how teachers' own knowledge about the task impacts their diagnostic competencies of student performance. Indeed, highly proficient content knowledge teachers are subject to the *expert blind spot* leading them to overlook the difficulties certain content can pose for students (Nathan, Koedinger, & Alibali, 2001). On the other hand, when problems are congruent with the intuitive conception, a study has put forward that teachers consider them to be easier for children, just like non-teachers do, even when this diagnostic judgment about student performance is incorrect (Gvozdic & Sander, 2018). Thus, in a parallel way to the expert blind spot, teachers' PCK is overridden when the intuitive conception was involved – by their *intuitive blind spot*. The robustness of our findings therefore suggest that such a phenomenon can occur among teachers because of the great prevalence of intuitive conceptions among them.

These findings have important entailments for teacher trainings and provide insight into pedagogical phenomena observed among teachers. First, it stresses the importance of educating teachers about the shortcomings of intuitive conceptions and their development, not with the objective of eradicating them, but to favor knowledge acquisition principles that are aligned with students' conceptual development and propose activities that foster a conceptual change. For example, working on examples congruent with the intuitive conception can help highlight the limits of intuitive conception, and later attain a conceptual change through working with examples incongruent with the intuitive conception so that the students can apply the school notion in situations incongruent with it (Gvozdic & Sander, 2020; Scheibling-Seve et al. 2022).

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